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AN ASYMPTOTIC PROBLEM FOR A REACTION-DIFFUSION EQUATION WITH A FAST DIFFUSION COMPONENT

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AN ASYMPTOTIC PROBLEM FOR A REACTION–DIFFUSION EQUATION WITH A FAST DIFFUSION COMPONENT

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ABSTRACT. In this paper we consider an asymptotic problem for the propagation of wave front for the reaction-diffusion equation

$$\frac{\partial u^\varepsilon(t,x,y)}{\partial t} = \frac{1}{2\varepsilon}\frac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \frac{\varepsilon a(x,y)}{2}\frac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \frac{1}{\varepsilon} f(y,u^\varepsilon),$$

where $x, y \in \mathbb{R}$ and $\varepsilon > 0$ is a small parameter.

We analyze the asymptotic behavior as $\varepsilon \downarrow 0$ of the solution $u^\varepsilon(t,x,y)$ of the initial-boundary value problem with initial condition $u^\varepsilon(0,x,y) = g(x)$ and boundary condition $\frac{\partial u^\varepsilon(t,x,y)}{\partial y}|_{y=\pm b} = 0$ in the band $\{(x,y) \in \mathbb{R}^2 : |y| \le b\}$.

The Feynman-Kac formula provides an equation for the solution of the above problem in terms of a functional integral in the space of trajectories of the corresponding Markov process. To analyze the behavior of the solution $u^\varepsilon(t,x,y)$ as $\varepsilon \downarrow 0$ we use a Large Deviation Principle for certain family of random processes. This Large Deviation Principle is expressed through action functionals in space of continuous functions.

1. Introduction

Consider the following initial-boundary value problem

$$(1.1)\qquad \begin{cases} \dfrac{\partial u(t,x,y)}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u(t,x,y)}{\partial y^2} + \dfrac{a(\varepsilon x,y)}{2}\dfrac{\partial^2 u(t,x,y)}{\partial x^2} + f(y,u), \\ \text{for } x \in \mathbb{R}, |y| < b, t > 0 \\ u(0,x,y) = g(\varepsilon x) \\ \dfrac{\partial u(t,x,y)}{\partial y}|_{y=\pm b} = 0 \end{cases}$$

where $\varepsilon > 0$ is a parameter.

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The differential equation in (1.1) describes the evolution of the concentration $u(t,x,y)$ of particles as a result of diffusion of particles governed by the operator

$$L = \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{a(\varepsilon x, y)}{2}\frac{\partial^2}{\partial x^2} \tag{1.2}$$

and multiplication (killing) of particles governed by the nonlinear term.

We assume that for each $y \in [-b,b]$, f is differentiable in u, $f(y,0) = f(y,1) = 0$, $f(y,u) > 0$ for $u \in (0,1)$, $f(y,u) < 0$ for $u \notin [0,1]$, $\frac{\partial f(y,u)}{\partial u}|_{u=0} = \sup_{u\geq 0} \frac{f(y,u)}{u}$. Put $c(y,u) = \frac{f(y,u)}{u}$ for $u > 0$ and $c(y,0) = \lim_{u\downarrow 0} \frac{f(y,u)}{u}$. Assume that $c(y,u)$ is continuous in y for $y \in [-b,b]$ and Lipschitz continuous in u for $u \geq 0$. Let $c(y) \equiv c(y,0)$, i.e., $c(y) = \sup_{u\geq 0} c(y,u)$. We also assume that for some constants $\underline{c}$, $\bar{c}$, $0 < \underline{c} \leq c(y) \leq \bar{c}$ for every y with $|y| \leq b$.

The initial function $g(x)$ is supposed to be bounded, nonnegative, continuous in the interior of its support $G_0 = \{x : g(x) > 0\} \neq \mathbb{R}$, and $[G_0] = [(G_0)]$. Here $[A]$ denotes the closure of a set A and (A) its interior.

The existence and uniqueness of solution of problem (1.1) is ensured if there exist constants $\underline{a}$, $\bar{a}$ so that $0 < \underline{a} \leq a(x,y) \leq \bar{a}$ for every $x \in \mathbb{R}$, $|y| \leq b$ and if $a(x,y)$ is Lipschitz continuous in both variables. We assume that $a(x,y)$ satisfies the above conditions.

The diffusion coefficient in x-direction and the initial function in (1.1) are nonhomogeneous in x changing slowly with the small parameter $\varepsilon > 0$. To analyze the behavior of the solution of (1.1) in large time intervals (of order $\frac{1}{\varepsilon}$), a rescaling of the time and the space in x-direction is useful (see discussion in [2] and [4]). For $\varepsilon > 0$ define $u^\varepsilon(t,x,y) = u(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, y)$ where $u(t,x,y)$ is the solution of problem (1.1). So after going over to the new time and space scale the function u^ε is the solution of the following mixed problem:

$$\begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \dfrac{1}{2\varepsilon}\dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \dfrac{\varepsilon a(x,y)}{2}\dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon}f(y,u^\varepsilon) \\ \text{for } x \in \mathbb{R}, |y| < b, t > 0 \\ u^\varepsilon(0,x,y) = g(x) \\ \dfrac{\partial u^\varepsilon(t,x,y)}{\partial y}|_{y=\pm b} = 0. \end{cases} \tag{1.3}$$

Notice that now the diffusion coefficient of the variable y is of order $\frac{1}{\varepsilon}$. Then the y is called fast variable. The variable x is the slow variable.

It is known (see [7],[13]) that with the differential operator

$$L^\varepsilon = \frac{1}{2\varepsilon}\frac{\partial^2}{\partial y^2} + \frac{\varepsilon a(x,y)}{2}\frac{\partial^2}{\partial x^2}$$

acting on the space of bounded functions h, twice continuously differentiable with respect to x and y with bounded second derivatives, and satisfying $\frac{\partial h(x,y)}{\partial y}|_{y=\pm b} = 0$ is associated a random process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$ which, together with some process $\xi_t^\varepsilon = (\xi_t^{1,\varepsilon}, \xi_t^{2,\varepsilon})$, is the solution of the stochastic differential equation

$$(1.4)\qquad \begin{cases} d\widetilde{X}_t^\varepsilon = \sqrt{\varepsilon a(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon)}\, dW_t^{(1)} \\ dY_t^\varepsilon = \dfrac{1}{\sqrt{\varepsilon}} dW_t^{(2)} + d\xi_t^{1,\varepsilon} - d\xi_t^{2,\varepsilon} \\ \widetilde{X}_0^\varepsilon = x,\, Y_0^\varepsilon = y,\, \xi_0^{1,\varepsilon} = \xi_0^{2,\varepsilon} = 0 \end{cases}$$

where $W_t = (W_t^{(1)}, W_t^{(2)})$ is a Wiener process in $\mathbb{R}^2$ starting at zero, adapted to an increasing family of σ-fields $\mathcal{N}_t$ and, with probability one, $\xi_t^{1,\varepsilon}$ and $\xi_t^{2,\varepsilon}$ are nondecreasing processes respectively increasing only for $t \in \Gamma_1 = \{t : Y_t^\varepsilon = -b\}$ and $t \in \Gamma_2 = \{t : Y_t^\varepsilon = b\}$; further, Γ_1 and Γ_2 have Lebesgue measure zero a.s. As the solution of (1.4) the process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon, \xi_t^\varepsilon)$ has continuous components a.s. and is adapted to the underlying family of σ-fields $\mathcal{N}_t$. Furthermore, $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$ is a strong Markov process.

Sometimes we should refer to the Markov process $(Y_t; \bar{P}_y)$ as being the Wiener process in $[-b, b]$ starting at y governed by the operator $\frac{1}{2}\frac{\partial^2}{\partial y^2}$ in the interior of $[-b, b]$ with instantaneous reflection at the end points $-b$ and b. One can deduce from (1.4) that $Y_t^\varepsilon = Y_{\frac{t}{\varepsilon}}$.

The Feynman-Kac formula (see [2]) provides an equation for the solution of (1.3) in terms of a functional integral of the trajectories of the process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$:

$$(1.5)\qquad u^\varepsilon(t,x,y) = \widetilde{E}_{xy}^\varepsilon g(\widetilde{X}_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(Y_s^\varepsilon, u^\varepsilon(t-s, \widetilde{X}_s^\varepsilon, Y_s^\varepsilon))ds\right\}.$$

Taking into account that $c(y) = \sup_{u \geq 0} c(y,u)$ we have

$$(1.6)\qquad u^\varepsilon(t,x,y) \leq \widetilde{E}_{xy}^\varepsilon g(\widetilde{X}_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(Y_s^\varepsilon)ds\right\}.$$

Problem (1.3) is a generalization of a problem considered by Freidlin in [4] in which the small diffusion coefficient does not depend on the slow variable, i.e., $a(x,y) \equiv a(y)$. In this case the asymptotics of the solution as $\varepsilon \downarrow 0$ is described by the action functional for the two-dimensional process $(\int_0^t c(Y_{\frac{s}{\varepsilon}})\, ds, \int_0^t a(Y_{\frac{s}{\varepsilon}})\, ds)$. This action functional is expressed in terms of the first eigenvalue of the problem

$$(1.7)\qquad \begin{cases} \dfrac{1}{2}\phi''(y) + [\beta_1 a(y) + \beta_2 c(y)]\, \phi(y) = \lambda(\beta_1, \beta_2)\, \phi(y), \qquad \text{for } |y| < b \\ \phi'(b) = \phi'(-b) = 0 \end{cases}$$

with β_1, $\beta_2 \in \mathbb{R}$ (for details see Chapter 7 in [3] or Example 3.1 in this paper). The asymptotic velocity of the wave front is obtained by using this action functional.

When the small diffusion coefficient depends on the slow variable x the situation becomes more complicated. In this case to analyze the asymptotic behavior of $u^\varepsilon(t,x,y)$ as $\varepsilon \downarrow 0$ we shall use a Large Deviation Principle for the family of processes $(\widetilde{X}_t^\varepsilon, \int_0^t c(Y_s^\varepsilon)\,ds)$ which will be established by means of an action functional (see definition of action functional in [3]). To determine this action functional we shall use basically: the Large Deviation Principle connected with the averaging principle formulated by Freidlin (see Chapter 7 in [3]) , the fact that the process $\widetilde{X}_t^\varepsilon$ satisfies the equation

$$\widetilde{X}_t^\varepsilon = x + \sqrt{\varepsilon}\widetilde{W}_{\int_0^t a(\widetilde{X}_s, Y_s^\varepsilon)ds}$$

where $\widetilde{W}_t$ is a Wiener process in $\mathbb{R}$ starting at zero and independent of Y_t^ε (for the existence of such Wiener process see McKean [11]), and Theorem 3.3.1 in [3] which provides the relation between action functionals of two families of processes connected by a continuous operator. In Part 2 of this paper we deal with this problem.

In Part 3 we describe the limit behavior of the solution of (1.3) as $\varepsilon \downarrow 0$. Here we follow the ideas of Freidlin in [2] (chapter VI) and in [5] where he analyzes the wave front propagation for the generalized KPP (Kolmogorov-Petrovskii-Piskunov) equation. Under a suitable assumption (called Condition (N) by Freidlin [2]), we will be able to define a family of increasing sets $G_t \subset \mathbb{R}$ such that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ for $x \in G_t$, $|y| \le b$, and $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0$ for $x \in \mathbb{R} \setminus G_t$, $|y| \le b$. These sets are described by means of the action functional for the family of processes $(\widetilde{X}_t^\varepsilon, \int_0^t c(Y_s^\varepsilon)\,ds)$ and they determine the position of the wave front at time t.

The assumption that Condition (N) is fulfilled is a restriction. We also study the wave front of $u^\varepsilon(t,x,y)$ as $\varepsilon \downarrow 0$ in a more general situation, without Condition (N). We use the same approach as in Freidlin [5]. In the case of the generalized KPP equation there is no fast motion as in problem (1.3). But as we will see later, this difference between the two problems is managed by taking into account that the fast motion in (1.3) has a unique invariant probability measure. Again we use the action functional for $(\widetilde{X}_t^\varepsilon, \int_0^t c(Y_s^\varepsilon)\,ds)$ to define the sets G_t.

Problem (1.3) can be generalized in different ways. The fast and slow motions can be described by more general Markov processes. In Part 4 of this paper we just point out some ideas in this direction. More details will be published in a second paper.

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2. Large Deviations

Our goal here is to establish a Large Deviation Principle for the family of random processes $(\widetilde{X}_t^\varepsilon, \int_0^t c(Y_s^\varepsilon)\,ds)$ with $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon; \widetilde{P}_{xy}^\varepsilon)$ satisfying equation (1.4) and $c(y)$ the function considered in the introduction of this paper. We shall describe this Large Deviation Principle by means of an action functional on the space $(C_{[0,T]}(\mathbb{R}^2), \rho_{0T})$ where $C_{[0,T]}(\mathbb{R}^n)$ is the space of continuous functions on $[0,T]$ into $\mathbb{R}^n$; the metric ρ_{0T} is defined by

$$\rho_{0T}((\varphi^1,\varphi^2,...,\varphi^n),(\psi^1,\psi^2,...,\psi^n)) = \sum_{i=1}^{n} \|\varphi^i - \psi^i\|,$$

with $\|\cdot\|$ denoting the supremum norm in $C_{[0,T]}(\mathbb{R})$. Sometimes, to avoid ambiguities, we shall use $\|\cdot\|_{[0,T]}$ instead of $\|\cdot\|$. Observe that we are using the same notation ρ_{0T} for any n.

We have seen in the introduction of this paper that

$$\widetilde{X}_t^\varepsilon = x + \int_0^t \sqrt{\varepsilon a(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)}\, dW_s^{(1)}.$$

Besides, there exists a Wiener process $\widetilde{W}_t$ in $\mathbb{R}$ starting at zero and independent of Y_t^ε so that

$$\int_0^t \sqrt{a(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)}\, dW_s^{(1)} = \widetilde{W}_{\int_0^t a(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)ds}.$$

Therefore,

$$\widetilde{X}_t^\varepsilon = X^\varepsilon_{\int_0^t a(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)ds} \tag{2.1}$$

where X_t^ε is defined by

$$X_t^\varepsilon = x + \sqrt{\varepsilon}\,\widetilde{W}_t. \tag{2.2}$$

To simplify notation set

$$Z_t^\varepsilon = \int_0^t c(Y_s^\varepsilon)\, ds \tag{2.3}$$

and

$$\widetilde{\Upsilon}_t^\varepsilon = \int_0^t a(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\, ds. \tag{2.4}$$

Observe that $(\widetilde{X}_\cdot^\varepsilon, Z_\cdot^\varepsilon) = G(\widetilde{X}_\cdot^\varepsilon, \widetilde{\Upsilon}_\cdot^\varepsilon, Z_\cdot^\varepsilon)$ where G is the operator on $(C_{[0,T]}(\mathbb{R}^3);\rho_{0T})$ into $(C_{[0,T]}(\mathbb{R}^2);\rho_{0T})$ defined by $G(\varphi,\psi,\eta) = (\varphi,\eta)$. Clearly G is a continuous operator. We shall use Theorem 3.3.1 in [3] to get the action functional for $(\widetilde{X}_t^\varepsilon, Z_t^\varepsilon)$; this can be done if we know the action functional for $(\widetilde{X}_t^\varepsilon, \widetilde{\Upsilon}_t^\varepsilon, Z_t^\varepsilon)$. Let us now obtain the action functional for $(\widetilde{X}_t^\varepsilon, \widetilde{\Upsilon}_t^\varepsilon, Z_t^\varepsilon)$.

It is not difficult to show that the action functional for a n-dimensional family of random processes with independent components is the sum of the action functionals for each component. In the case of the three-dimensional family of processes $(\widetilde{X}_t^\varepsilon, \widetilde{\Upsilon}_t^\varepsilon, Z_t^\varepsilon)$ the components are not independent. We shall use the technique of freezing variables to be able to work with independent processes. Basically, we shall use two new families of processes which are obtained from $\widetilde{X}_t^\varepsilon$ and $\widetilde{\Upsilon}_t^\varepsilon$ by freezing variables.

First let us introduce for each $\varphi \in C_{[0,T]}(\mathbb{R})$ the family of processes

$$\Upsilon_t^{\varepsilon,\varphi} = \int_0^t a(\varphi_s, Y_s^\varepsilon)\, ds. \tag{2.5}$$

Note that the process (2.5) arises from (2.4) by freezing the slow variable. Secondly, let us consider for each $\psi \in F_{\bar{k}}$ the family of processes $X_{\psi_t}^\varepsilon$ where X_t^ε is defined in (2.2) and $F_{\bar{k}}$ is a set of the type

$$F_{\overline{k}} = \{\psi \in C_{[0,T]}(\mathbb{R}) : \psi_0 = 0, \exists \dot{\psi}_t \text{ a.e.}, 0 < \underline{k} \leq \dot{\psi}_t \leq \overline{k},\, t \in [0,T]\}; \tag{2.6}$$

it is easily seen that $F_{\overline{k}}$ is a compact set. Observe that $X_{\psi_t}^\varepsilon$ arises from $\widetilde{X}_t^\varepsilon$ by freezing the process $\widetilde{\Upsilon}_t^\varepsilon$. Clearly the families of processes $X_{\psi_t}^\varepsilon$ and $(\Upsilon_t^{\varepsilon,\varphi}, Z_t^\varepsilon)$ are independent. It is known (see [3]) that these families obey a Large Deviation Principle. Let us recall the main steps.

The action functional for the family of processes $(\Upsilon_t^{\varepsilon,\varphi}, Z_t^\varepsilon)$ is defined with the help of the eigenvalue problem

$$\begin{cases} \dfrac{1}{2}\phi''(y) + [\beta_1 a(x,y) + \beta_2 c(y)]\,\phi(y) = \lambda(x,\beta_1,\beta_2)\,\phi(y), & \text{for } |y| < b \\ \phi'(b) = \phi'(-b) = 0 \end{cases} \tag{2.7}$$

with $x \in \mathbb{R}$, $\beta_1, \beta_2 \in \mathbb{R}$. This problem has a discrete spectrum, the eigenvalue $\lambda(x,\beta_1,\beta_2)$ with the maximal real part is real, has multiplicity one, the corresponding eigenfunction is positive. Besides, $\lambda(x,\beta_1,\beta_2)$ is differentiable in β_1, β_2 (see Kato [10]).

One can prove (see [3]) that

$$\lambda(x,\beta_1,\beta_2) = \lim_{T\to+\infty} \frac{1}{T} \ln \bar{E}_y \exp\left\{\int_0^T [\beta_1 a(x,Y_s) + \beta_2 c(Y_s)]\, ds\right\} \tag{2.8}$$

uniformly in y. From relation (2.8) one can show that $\lambda(x,\beta_1,\beta_2)$ is jointly continuous in its variables and convex in (β_1,β_2). Let $L(x,\alpha^1,\alpha^2)$ be its Legendre transform:

$$L(x,\alpha^1,\alpha^2) = \sup_{(\beta_1,\beta_2)\in\mathbb{R}^2} \left\{\langle(\alpha^1,\alpha^2),(\beta_1,\beta_2)\rangle - \lambda(x,\beta_1,\beta_2)\right\}, \quad \alpha^1,\alpha^2 \in \mathbb{R}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$. This function is also convex in (α^1,α^2) and jointly lower semicontinuous in all variables; it assumes nonnegative values including $+\infty$. From the boundedness of the functions $a(x,y)$ and $c(y)$ it follows that $L(x,\alpha^1,\alpha^2) = +\infty$ outside some bounded set in the space of the variables (α^1,α^2).

Taking into account the properties of the functions $a(x,y)$ one can see that with probability one the trajectories of the processes (2.4) and (2.5) belong to $F_{\bar{a}}$; also, by considering the properties of the function $c(y)$ the trajectories of the process (2.3) belong to $F_{\bar{c}}$ a.s. Using the same proof of Theorem 7.4.1 in [3], one can show that for each $\varphi \in C_{[0,T]}(\mathbb{R})$, the normalized action functional on $(C_{[0,T]}(\mathbb{R}^2); \rho_{0T})$ for the family of random processes $(\Upsilon_t^{\varepsilon,\varphi}, Z_t^\varepsilon)$ is

$$
(2.9) \qquad S_{0T}^{\varphi}(\psi,\eta) = \begin{cases} \int_0^T L(\varphi_s, \dot{\psi}_s, \dot{\eta}_s)\, ds, & \text{if } \psi \in F_{\bar{a}}, \eta \in F_{\bar{c}} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbb{R}^2) \end{cases}
$$

with normalizing coefficient $\frac{1}{\varepsilon}$.

On the other hand, it is known (see [3]) that the normalized action functional for X_t^ε on $(C_{[0,T]}(\mathbb{R}); \rho_{0T})$ is given by

$$
S_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\dot{\varphi}_s|^2\, ds, & \text{if } \varphi \text{ is a.c.} \\ +\infty, & \text{for the rest of } C_{[0,T]}(\mathbb{R}) \end{cases}
$$

with normalizing coefficient $\frac{1}{\varepsilon}$. Let us consider for each $\psi \in F_{\bar{a}}$ the operator G_ψ on $(C_{[0,\psi_T]}(\mathbb{R}); \rho_{0\psi_T})$ into $(C_{[0,T]}(\mathbb{R}); \rho_{0T})$ defined by $G_\psi(\varphi) = \varphi_\psi$. Clearly G_ψ is a continuous operator. It follows from Theorem 3.3.1 in [3] that the action functional for the family of processes $X_{\psi_t}^\varepsilon$ is $\frac{1}{\varepsilon} S_{0T}^{\psi}(\varphi)$ with

$$
(2.10) \qquad S_{0T}^{\psi}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\, ds, & \text{if } \varphi \text{ is a.c.} \\ +\infty, & \text{for the rest of } C_{[0,T]}(\mathbb{R}). \end{cases}
$$

So far we have constructed for each $\psi \in F_{\bar{a}}$ and $\varphi \in C_{[0,T]}(\mathbb{R})$ two independent families of processes $X_{\psi_t}^\varepsilon$ and $(\Upsilon_t^{\varepsilon,\varphi}, Z_t^\varepsilon)$ each of them obeying a Large Deviation Principle. It turns out that the normalized action functional for the family of random processes $(\widetilde{X}_t^\varepsilon, \widetilde{\Upsilon}_t^\varepsilon, Z_t^\varepsilon)$ is given by

$$
(2.11) \quad \widetilde{S}_{0T}(\varphi,\psi,\eta) = \begin{cases} \frac{1}{2}\int_0^T \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\, ds + \int_0^T L(\varphi_s, \dot{\psi}_s, \dot{\eta}_s)\, ds, & \text{if } \varphi \text{ is a.c.} \\ & \psi \in F_{\bar{a}}, \eta \in F_{\bar{c}} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbb{R}^3) \end{cases}
$$

with normalizing coefficient $\frac{1}{\varepsilon}$. To prove this fact it suffices to verify the validity of the following conditions (see definition of action functional in [3]):

(A.0) Compactness of the level sets: The level sets of $\widetilde{S}_{0T}$

$$
\widetilde{\Phi}(s) = \{(\varphi,\psi,\eta) \in C_{[0,T]}(\mathbb{R}^3) : \ \widetilde{S}_{0T}(\varphi,\psi,\eta) \le s, \ \ \varphi_0 = x\}
$$

are compact sets for each $s \geq 0$.

(A.I) Lower bound: $\forall\delta > 0$, $\forall\gamma > 0$, and $\forall(\varphi,\psi,\eta) \in C_{[0,T]}(\mathbb{R}^3)$, $\exists\varepsilon_0 > 0$ such that

$$P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot),(\varphi,\psi,\eta)\right) < \delta\right\} \geq \exp\left\{-\frac{1}{\varepsilon}\left[\widetilde{S}_{0T}(\varphi,\psi,\eta)+\gamma\right]\right\}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

(A.II) Upper bound: $\forall\delta > 0$, $\forall\gamma > 0$, $\forall s > 0$, $\exists\varepsilon_0 > 0$ such that

$$P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot), \widetilde{\Phi}(s)\right) \geq \delta\right\} \leq \exp\left\{-\frac{1}{\varepsilon}(s-\gamma)\right\}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

Condition **(A.0)** can be split into two: joint lower semicontinuity of $\widetilde{S}_{0T}(\varphi,\psi,\eta)$ in all variables (which is equivalent to closedness of $\widetilde{\Phi}(s)$ for every $s > 0$) and relative compactness of $\widetilde{\Phi}(s)$. We shall use such a splitting to prove condition **(A.0)**.

A family $\mathcal{F}$ of functions is called *absolutely equicontinuous* if for any $\delta > 0$, there exists $\varepsilon > 0$ such that

$$\sum_i |\phi_{t_i} - \phi_{s_i}| < \delta \quad \text{for any } \phi \in \mathcal{F}$$

whenever the sum of lenghts of a finite number of nonoverlapping intervals $[s_i, t_i]$ is less than ε. It is easily seen that for absolute equicontinuity of some family of functions it is necessary and sufficient that these functions be absolutely continuous and that their derivatives be uniformly integrable.

Lemma 2.1. *For each $s > 0$ define*

$$\mathcal{F}(s) = \{\varphi \in C_{[0,T]}(\mathbb{R}) : \varphi_0 = x,\ S^\psi_{0T}(\varphi) \leq s \text{ for some } \psi \in F_{\bar{a}}\}$$

where $S^\psi_{0T}(\varphi)$ is defined in (2.10). The family of functions $\mathcal{F}(s)$ is absolutely equicontinuous.

Proof. Fix $s > 0$ and $\varphi \in \mathcal{F}(s)$. Then there exists $\psi \in F_{\bar{a}}$ such that $S^\psi_{0T}(\varphi) \leq s$. Hence φ is absolutely continuous and $\frac{1}{\bar{a}}\int_0^T |\dot{\varphi}_t|^2 dt \leq s$. Now, use the fact that $\frac{u^2}{|u|} \to \infty$ as $|u| \to \infty$ to conclude that $\dot{\varphi}$ is uniformly integrable. Therefore, $\mathcal{F}(s)$ is absolutely equicontinuous.

□

The main arguments in the proof of the following lemma were taken from Wentzell [17].

Lemma 2.2. *The functional $S^{\psi}_{0T}(\varphi)$ in (2.10) is jointly lower semicontinuous with respect to the uniform convergence.*

Proof. Let (φ^n, ψ^n) be any sequence in $C_{[0,T]}(\mathbb{R}) \times F_{\bar{a}}$ with $\varphi^n \to \varphi$ and $\psi^n \to \psi$ as $n \to \infty$ uniformly in $[0,T]$. We want to show that

$$S^{\psi}_{0T}(\varphi) \le \underline{\lim}_{n\to\infty} S^{\psi^n}_{0T}(\varphi^n).$$

It suffices to consider the case when $\underline{\lim}_{n\to\infty} S^{\psi^n}_{0T}(\varphi^n) = \bar{s}_\infty < \infty$. If $\bar{s}_\infty = \infty$, there is nothing to prove.

By assuming that $\bar{s}_\infty < \infty$ we may suppose that $S^{\psi^n}_{0T}(\varphi^n) < \bar{s}_\infty + 1$ for all n. This implies that φ is absolutely continuous. To see this note that $\varphi^n \in \mathcal{F}(\bar{s}_\infty + 1)$ for all n. By Lemma 2.1 the sequence $\{\varphi^n\}$ is absolutely equicontinuous. Thus for every $\delta > 0$, there exists $\varepsilon > 0$ such that $\sum_i |\varphi^n_{t_i} - \varphi^n_{s_i}| < \delta$ for all n whenever the sum of the lenghts of a finite number of nonoverlapping intervals $[s_i, t_i]$ is less than ε. Passing to the limit,

$$\lim_{n\to\infty} \sum_i |\varphi^n_{t_i} - \varphi^n_{s_i}| = \sum_i \lim_{n\to\infty} |\varphi^n_{t_i} - \varphi^n_{s_i}| = \sum_i |\varphi_{t_i} - \varphi_{s_i}| < \delta,$$

which means that φ is absolutely continuous. The function ψ is also absolutely continuous because the sequence $\{\psi^n\}$ is in $F_{\bar{a}}$; since $F_{\bar{a}}$ is compact, $\psi \in F_{\bar{a}}$. Besides, the functions in $F_{\bar{a}}$ are absolutely continuous.

Let us introduce the function $\bar{L}(a, \alpha) = \frac{1}{2}\frac{\alpha^2}{a}$ for $\alpha \in \mathbb{R}, a \in [\underline{a}; \bar{a}]$. This function is jointly continuous and downward convex in both variables. Note that $S^{\psi}_{0T}(\varphi) = \frac{1}{2}\int_0^T \bar{L}(\dot{\psi}_s, \dot{\varphi}_s)\, ds$. Now, using Jensen's inequality, the joint continuity of $\bar{L}$, and Fatou's Lemma one can show similarly to the proof of Theorem 3.1.(b) in [17] that $\int_0^T \bar{L}(\dot{\psi}_t, \dot{\varphi}_t)\, dt \le \bar{s}_\infty$.

□

Lemma 2.3. *The functional $\widetilde{S}_{0T}(\varphi, \psi, \eta)$ in (2.11) is jointly lower semicontinuous in all variables with respect to the uniform convergence.*

Proof. Let $(\varphi^n, \psi^n, \eta^n)$ be any sequence in $C_{[0,T]}(\mathbb{R}) \times F_{\bar{a}} \times F_{\bar{c}}$ with $\varphi^n \to \varphi$, $\psi^n \to \psi$, and $\eta^n \to \eta$ as $n \to \infty$ uniformly in $[0,T]$. We want to prove that

$$\widetilde{S}_{0T}(\varphi, \psi, \eta) \le \underline{\lim}_{n\to\infty} \widetilde{S}_{0T}(\varphi^n, \psi^n, \eta^n).$$

As in the proof of Lemma 2.2, it suffices to consider the case when

$$\underline{\lim}_{n\to\infty} \widetilde{S}_{0T}(\varphi^n, \psi^n, \eta^n) = s_\infty < \infty.$$

Assuming that $s_\infty < \infty$ we can consider that $\widetilde{S}_{0T}(\varphi^n, \psi^n, \eta^n) < s_\infty + 1$ for all n. Then $(\varphi^n, \psi^n, \eta^n) \in \widetilde{\Phi}(s_\infty + 1)$ where $\widetilde{\Phi}(s)$ was introduced in **(A.0)**. Hence, $S^{\psi^n}_{0T}(\varphi^n) < s_\infty + 1$ for all n. By Lemma 2.2 we conclude that $S^{\psi}_{0T}(\varphi) < s_\infty + 1$ and φ is absolutely continuous. Besides, the first derivative of φ^n and φ are integrable: this fact follows from the proof of Lemma 2.1 taking into account that $\int_0^T |\dot\varphi_s^n|^2\, ds \le 2\bar a(s_\infty + 1)$ and $\int_0^T |\dot\varphi_s|^2\, ds \le 2\bar a(s_\infty + 1)$.

We know that the functions in $F_{\bar k}$ are absolutely continuous with bounded first derivatives. Also the compactness of these sets implies that $\psi \in F_{\bar a}$ and $\eta \in F_{\bar c}$. Hence, ψ^n, η^n, ψ, and η are absolutely continuous with integrable first derivatives. We conclude that (φ,ψ,η) and $(\varphi^n,\psi^n,\eta^n)$, for all n, belong to $W^3_{1,1}[0,T]$, where $W^m_{1,1}[0,T]$ is the (Banach) space of absolutely continuous functions on $[0,T]$ into $\mathbb{R}^m$ with integrable first derivatives.

We have seen that it suffices to prove joint lower semicontinuity of $\widetilde S_{0T}(\varphi,\psi,\eta)$ only in $W^3_{1,1}[0,T]$. From the definition of the functional $\widetilde S_{0T}$ in (2.11) and Lemma 2.2 one can see that all we have to show is that

$$\int_0^T L(\varphi_t;\dot\psi_t,\dot\eta_t)\, dt \le \varliminf_{n\to\infty} \int_0^T L(\varphi_t^n;\dot\psi_t^n,\dot\eta_t^n)\, dt.$$

Let us recall some properties of the function $L(x,\alpha^1,\alpha^2)$. This function is the Legendre transform of $\lambda(x,\beta_1,\beta_2)$ in (2.8), it is nonnegative for all $x\in\mathbb{R}$ and $(\alpha^1,\alpha^2)\in\mathbb{R}^2$, convex in (α^1,α^2) for each x, and jointly lower semicontinuous in all variables. It is easily seen that $\lambda(x,\beta_1,\beta_2) \le \bar a\beta_1 + \bar c\beta_2 \equiv \bar\lambda(\beta_1,\beta_2)$ for all $x\in\mathbb{R}$. Notice that $\bar\lambda(\beta_1,\beta_2)$ is convex in its arguments and finite for all $(\beta_1,\beta_2)\in\mathbb{R}^2$. Let $\bar L(\alpha^1,\alpha^2)$ be the Legendre transform of $\bar\lambda(\beta_1,\beta_2)$. Then $L(x,\alpha^1,\alpha^2) \ge \bar L(\alpha^1,\alpha^2)$ for all $x\in\mathbb{R}$. The reader can verify that the above properties are sufficient to use Theorem 3- 9.1.4 in [8] to conclude that $\int_0^T L(\varphi_t;\dot\psi_t,\dot\eta_t)\, dt$ is jointly lower semicontinuous in $(\varphi,\psi,\eta)\in W^3_{1,1}[0,T]$.

□

We shall now prove conditions **(A.0)-(A.II)**.

Proposition 2.1. *Condition* **(A.0)** *holds, i.e, for any $s_0 > 0$, the set*

$$\widetilde\Phi(s_0) = \{(\varphi,\psi,\eta)\in C_{[0,T]}(\mathbb{R}^3):\ \ \widetilde S_{0T}(\varphi,\psi,\eta)\le s_0,\ \ \varphi_0 = x\}$$

is a compact set.

Proof. The closedness of $\widetilde\Phi(s_0)$ follows from Lemma 2.3. Hence it remains to prove that $\widetilde\Phi(s_0)$ is relatively compact.

Let $\mathcal{F}(s_0) = \{\varphi\in C_{[0,T]}(\mathbb{R}) : (\varphi,\psi,\eta)\in\widetilde\Phi(s_0)$ for some $(\psi,\eta)\in F_{\bar a}\times F_{\bar c}\}$. We shall use the following fact (see [12] or Lemma 3.2.1 in [3]): a function φ is absolutely continuous and its derivative is square integrable if and only if

$$\sup_{0\le t_0<t_1<\ldots<t_N\le T} \sum_{i=1}^N \frac{|\varphi_{t_i}-\varphi_{t_{i-1}}|^2}{t_i - t_{i-1}} < \infty$$

and in this case the supremum is equal to $\int_0^T |\dot\varphi_t|^2\, dt$. In particular, $|\varphi_t| \le |\varphi_0| + \sqrt{T\int_0^T |\dot\varphi_s|^2\, ds}$ for any $t\in(0,T]$.

Let $\varphi \in \mathcal{F}(s_0)$. Then $\varphi_0 = x$ and there exist $\psi \in F_{\bar{a}}$, $\eta \in F_{\bar{c}}$ such that $\widetilde{S}_{0T}(\varphi, \psi, \eta) \leq s_0$; hence, $\int_0^T |\dot{\varphi}_s|^2 \, ds \leq 2\bar{a}s_0$ and $|\varphi_t| \leq x + \sqrt{2T\bar{a}s_0}$; we conclude that the functions in $\mathcal{F}(s_0)$ are uniformly bounded. Similarly, for every $t, t+h \in [0,T]$, $|\varphi_{t+h} - \varphi_t| \leq \sqrt{h\int_0^T |\dot{\varphi}_s|^2 \, ds} \leq \sqrt{2\bar{a}s_0}\sqrt{h}$. This estimation implies the equicontinuity of the functions $\varphi \in \mathcal{F}(s_0)$. By Ascoli-Arzela's Theorem each sequence $\{\varphi^n\}$ in $\mathcal{F}(s_0)$ has a subsequence converging uniformly in $[0,T]$ to a continuous function φ.

Let $(\varphi^n, \psi^n, \eta^n)$ be any sequence in $\widetilde{\Phi}(s_0)$. Then $\{\varphi^n\}$ is a sequence in $\mathcal{F}(s_0)$. Let $\{\varphi^{n_k}\}$ be a subsequence of $\{\varphi^n\}$ that converges uniformly in $[0,T]$ to some continuous function φ. Since $\{\psi^{n_k}\}$ and $\{\eta^{n_k}\}$ are sequences from compact sets, there exist further subsequences $\{\psi^{n_{k_j}}\}$ and $\{\eta^{n_{k_j}}\}$ that converge uniformly in $[0,T]$ respectively to functions $\psi \in F_{\bar{a}}$ and $\eta \in F_{\bar{c}}$. Hence $\widetilde{\Phi}(s_0)$ is relatively compact.

□

The following lemma can be proved using the properties of the Wiener process and we omit its proof.

Lemma 2.4. *For any $\delta > 0$ and $\Delta > 0$,*

$$P\left\{\sqrt{\varepsilon}\sup_{A_\Delta}|\widetilde{W}_{t_1} - \widetilde{W}_{t_2}| \geq \delta\right\} \leq \frac{48T'}{\delta\sqrt{2\pi}}\frac{\sqrt{\Delta}}{\Delta}\sqrt{\varepsilon}\exp\left\{-\frac{\delta^2}{16\varepsilon\Delta}\right\}$$

where $T' = \bar{a}T + \Delta$ and $A_\Delta = \{(t_1, t_2) : 0 < t_1 < t_2 < T', 0 < t_2 - t_1 < \Delta\}$.

Proposition 2.2. *Condition* **(A.I)** *holds, i.e, $\forall\delta > 0$, $\forall\gamma > 0$, and $\forall(\varphi, \psi, \eta) \in C_{[0,T]}(\mathbb{R}^3)$, $\exists\varepsilon_0 > 0$ such that*

$$P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot), (\varphi, \psi, \eta)\right) < \delta\right\} \geq \exp\left\{-\frac{1}{\varepsilon}\left[\widetilde{S}_{0T}(\varphi, \psi, \eta) + \gamma\right]\right\}$$

for any $0 < \varepsilon \leq \varepsilon_0$.

Proof. Let $\delta > 0$, $\gamma > 0$, and $(\varphi, \psi, \eta) \in C_{[0,T]}(\mathbb{R}^3)$ with $\widetilde{S}_{0T}(\varphi, \psi, \eta) < +\infty$ be given. Then φ is a.c., $\psi \in F_{\bar{a}}$, $\eta \in F_{\bar{c}}$. To simplify notation set

$$P = P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot), (\varphi, \psi, \eta)\right) < \delta\right\}.$$

Choose $\delta' > 0$ and $\Delta > 0$ sufficiently small such that

$$P \geq P\left\{\|\widetilde{X}^\varepsilon_\cdot - \varphi\| < 2\delta', \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi\| < \Delta, \|Z^\varepsilon_\cdot - \eta\| < \delta'\right\}.$$

Then,

$$P \geq P\Big\{\|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi_\cdot}\| < \delta', \|X^\varepsilon_{\psi_\cdot} - \varphi\| < \delta', \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi\| < \Delta, \|\Upsilon^{\varepsilon,\varphi}_\cdot - \psi\| < \delta',$$
$$\|Z^\varepsilon_\cdot - \eta\| < \delta'\Big\}.$$

Taking into account that

$$(2.12)\qquad \left[\|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi\| < \Delta,\ \|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi_\cdot}\| < \delta'\right] \supseteq \left[\sqrt{\varepsilon}\sup_{A_\Delta}|\widetilde{W}_{t_1} - \widetilde{W}_{t_2}| < \delta'\right]$$

where A_Δ is the set introduced in Lemma 2.4, we obtain

$$(2.13)\qquad \begin{aligned} P \geq P&\left\{\|X^\varepsilon_{\psi_\cdot} - \varphi\| < \delta',\ \|\Upsilon^{\varepsilon,\varphi}_\cdot - \psi\| < \delta',\ \|Z^\varepsilon_\cdot - \eta\| < \delta',\right. \\ &\left.\ ,\sqrt{\varepsilon}\sup_{A_\Delta}|\widetilde{W}_{t_1} - \widetilde{W}_{t_2}| < \delta'\right\} \geq \\ \geq P&\left\{\|X^\varepsilon_{\psi_\cdot} - \varphi\| < \delta',\ \|\Upsilon^{\varepsilon,\varphi}_\cdot - \psi\| < \delta',\ \|Z^\varepsilon_\cdot - \eta\| < \delta'\right\} - \\ -P&\left\{\sqrt{\varepsilon}\sup_{A_\Delta}|\widetilde{W}_{t_1} - \widetilde{W}_{t_2}| \geq \delta'\right\}. \end{aligned}$$

Now, by choosing $0 < \Delta < \frac{\delta'^2}{16[\widetilde{S}_{0T}(\varphi,\psi,\eta)+\frac{\gamma}{2}]}$, $\varepsilon_0 > 0$ sufficiently small, and using Lemma 2.4,

$$\begin{aligned} P\left\{\sqrt{\varepsilon}\sup_{A_\Delta}|\widetilde{W}_{t_1} - \widetilde{W}_{t_2}| \geq \delta'\right\} &\leq \frac{48T'}{\delta'\sqrt{2\pi}}\frac{\sqrt{\Delta}}{\Delta}\sqrt{\varepsilon}\exp\left\{-\frac{\delta'^2}{16\varepsilon\Delta}\right\} \leq \\ &\leq \frac{1}{2}\exp\left\{-\frac{1}{\varepsilon}\left[\widetilde{S}_{0T}(\varphi,\psi,\eta) + \frac{\gamma}{2}\right]\right\},\quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned}$$

The processes $X^\varepsilon_{\psi_t}$ and $(\Upsilon^{\varepsilon,\varphi}_t, Z^\varepsilon_t)$ are independent. Thus the action functional of the corresponding three-dimensional random process is the sum of the functionals (2.10) and (2.9). Using the lower bound corresponding to this action functional we see that for all $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} P&\left\{\|X^\varepsilon_{\psi_\cdot} - \varphi\| < \delta',\ \|\Upsilon^{\varepsilon,\varphi}_\cdot - \psi\| < \delta',\ \|Z^\varepsilon_\cdot - \eta\| < \delta'\right\} \geq \\ &\geq \exp\left\{-\frac{1}{\varepsilon}\left[\widetilde{S}_{0T}(\varphi,\psi,\eta) + \frac{\gamma}{2}\right]\right\},\quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned}$$

The result follows by substituting the last two estimates into (2.13)

□

Proposition 2.3. *Condition* **(A.II)** *holds, i.e,* $\forall\delta > 0$, $\forall\gamma > 0$, $\forall s_0 > 0$, $\exists\varepsilon_0 > 0$ *such that*

$$P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot), \widetilde{\Phi}(s_0)\right) \geq \delta\right\} \leq \exp\left\{-\frac{1}{\varepsilon}(s_0 - \gamma)\right\}$$

for any $0 < \varepsilon \le \varepsilon_0$.

Proof. Let $\delta > 0$, $\gamma > 0$, and $s_0 > 0$ be given. The trajectories of $(\widetilde{\Upsilon}^\varepsilon_t, Z^\varepsilon_t)$ belong to $F_{\bar{a}} \times F_{\bar{c}}$ with probability 1. Let (ψ^i, η^i), $i = 1, \cdots, N$ be a finite δ'-net in the compact set $F_{\bar{a}} \times F_{\bar{c}}$. Then for any $\delta_0 > 0$,

$$
\begin{aligned}
&P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot)\ ,\ \widetilde{\Phi}(s_0)\right) \ge \delta\right\} \le \\
&\qquad \le \sum_{i=1}^N \Big[P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot), \widetilde{\Phi}(s_0)\right) \ge \delta,\ \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| < \delta',\right. \\
&\qquad\qquad \left.\|Z^\varepsilon_\cdot - \eta^i\| < \delta',\ \|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i}\| \ge \delta_0\right\} + \\
&\qquad\qquad + P\left\{\rho_{0T}\left((\widetilde{X}^\varepsilon_\cdot, \widetilde{\Upsilon}^\varepsilon_\cdot, Z^\varepsilon_\cdot), \widetilde{\Phi}(s_0)\right) \ge \delta, \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| < \delta',\right. \\
&\qquad\qquad \left.\|Z^\varepsilon_\cdot - \eta^i\| < \delta',\ \|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i}\| < \delta_0\right\}\Big] = \\
&\qquad = \sum_{i=1}^N \left[P(I_1^i) + P(I_2^i)\right].
\end{aligned} \tag{2.14}
$$

Using inclusion (2.12) we can see that

$$
\begin{aligned}
P(I_1^i) &\le P\left\{\|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| \le \delta', \|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i}\| \ge \delta_0\right\} \\
&\le P\left\{\sup_{A_{\delta'}} |\widetilde{W}_{t_1} - \widetilde{W}_{t_2}| \ge \delta_0\right\}.
\end{aligned}
$$

Choosing $0 < \delta' < \frac{{\delta_0}^2}{16(s_0 - \frac{\gamma}{8})}$ it follows from Lemma 2.4 that

$$
P(I_1^i) \le \exp\left\{-\frac{1}{\varepsilon}(s_0 - \frac{\gamma}{4})\right\},\ \ 0 < \varepsilon \le \varepsilon_0. \tag{2.15}
$$

Let $F^i = \left\{\varphi \in C_{[0,T]}(\mathbb{R}) : (\varphi, \psi^i, \eta^i) \in \widetilde{\Phi}(s_0)\right\}$ and $\Phi^{\psi^i}(s_0) = \{\varphi \in C_{[0,T]}(\mathbb{R}) : S^{\psi^i}_{0T}(\varphi) \le s_0, \varphi_0 = x\}$, $i = 1, \cdots, N$. Observe that $F^i \subseteq \Phi^{\psi^i}(s_0)$, F^i is compact (see proof of Proposition 2.1), and $F^i \times \{\psi^i\} \times \{\eta^i\} \subset \widetilde{\Phi}(s_0)$. Then, by choosing $\delta' > 0$ even smaller if necessary, one can see that there exists $\delta_1 > 0$ such that

$$
P(I_2^i) \le P\left\{\rho_{0T}(\widetilde{X}^\varepsilon_\cdot, F^i) \ge \delta_1,\ \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| < \delta',\ \|Z^\varepsilon_\cdot - \eta^i\| < \delta',\ \|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i}\| < \delta_0\right\}.
$$

Now, for any $\delta^* > 0$,

$$
\begin{aligned}
P(I_2^i) \le P\Big\{&\rho_{0T}(\widetilde{X}^\varepsilon_\cdot, F^i) \ge \delta_1,\ \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| < \delta',\ \|Z^\varepsilon_\cdot - \eta^i\| < \delta', \\
&\|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i_\cdot}\| < \delta_0,\ \rho_{0T}(\widetilde{X}^\varepsilon_\cdot, \Phi^{\psi^i}(s_0)) \ge \delta^*\Big\} + \\
+P\Big\{&\rho_{0T}(\widetilde{X}^\varepsilon_\cdot, F^i) \ge \delta_1,\ \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| < \delta',\ \|Z^\varepsilon_\cdot - \eta^i\| < \delta', \\
&\|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i_\cdot}\| \le \delta_0,\ \rho_{0T}(\widetilde{X}^\varepsilon_\cdot, \Phi^{\psi^i}(s_0)) < \delta^*\Big\} \\
= P(I_2^{i,1}) &+ P(I_2^{i,2}).
\end{aligned} \tag{2.16}
$$

Choose $0 < \delta_0 < \frac{\delta^*}{2}$ to get

$$
P(I_2^{i,1}) \le P\left\{\rho_{0T}\left(X^\varepsilon_{\psi^i_\cdot}, \Phi^{\psi^i}(s_0)\right) \ge \frac{\delta^*}{2}\right\}.
$$

Using the upper bound associated with the action functional (2.10) we obtain for $\varepsilon_0 > 0$ sufficiently small

$$
P(I_2^{i,1}) \le \exp\left\{-\frac{1}{\varepsilon}(s_0 - \frac{\gamma}{8})\right\}, \qquad 0 < \varepsilon \le \varepsilon_0. \tag{2.17}
$$

To estimate the second summand in (2.16), consider the compact set K obtained by omitting the δ_1-neighborhood of F^i from the compact set $\Phi^{\psi^i}(s_0)$. Let $\varphi^1, ..., \varphi^M$ be a finite Δ-net for K . Choose $0 < \Delta < \delta_1$ and $\delta^* > 0$ sufficiently small such that $K + \delta^*$ is covered by the finite Δ-net. Clearly F^i is contained in the complement of the Δ-net. Then

$$
P(I_2^{i,2}) \le \sum_{j=1}^{M} P\left\{\|\widetilde{X}^\varepsilon_\cdot - \varphi^j\| < \Delta,\ \|\widetilde{X}^\varepsilon_\cdot - X^\varepsilon_{\psi^i_\cdot}\| < \delta_0,\ \|\widetilde{\Upsilon}^\varepsilon_\cdot - \psi^i\| < \delta',\ \|Z^\varepsilon_\cdot - \eta^i\| < \delta'\right\}.
$$

From the Lipschitz continuity of the function $a(x,y)$ in the variable x we get the following inclusion:

$$
\left[\|\widetilde{X}^\varepsilon_\cdot - \varphi^j\| < \Delta\right] \subseteq \left[\|\widetilde{\Upsilon}^\varepsilon_\cdot - \Upsilon^{\varepsilon,\varphi^j}_\cdot\| \le (K+1)(T+1)\Delta\right]
$$

where $K > 0$ is the Lipschitz constant and $\Upsilon^{\varepsilon,\varphi^j}_t$ is the process introduced in (2.5). Then

$$
P(I_2^{i,2}) \le \sum_{j=1}^{M} P\left\{\|X^\varepsilon_{\psi^i_\cdot} - \varphi^j\| < \Delta + \delta_0,\ \|\Upsilon^{\varepsilon,\varphi^j}_\cdot - \psi^i\| < \bar{\delta},\ \|Z^\varepsilon_\cdot - \eta^i\| < \delta'\right\}
$$

for some $\bar{\delta} > 0$ which can be taken small whenever Δ, δ_0, and δ' are small.

The independence of the processes $X^{\varepsilon}_{\psi^i_\cdot}$ and $(\Upsilon^{\varepsilon,\varphi^j}_t, Z^{\varepsilon}_t)$ implies that for $\delta_2 = \max\{\Delta + \delta_0, \bar{\delta}, \delta'\}$,

$$P(I_2^{i,2}) \le \sum_{j=1}^{M} \left\{ P\left[\|X^{\varepsilon}_{\psi^i_\cdot} - \varphi^j\| < \delta_2 \right] \times P\left[\|\Upsilon^{\varepsilon,\varphi^j}_{\cdot} - \psi^i\| < \delta_2, \;\; \|Z^{\varepsilon}_{\cdot} - \eta^i\| < \delta_2 \right] \right\}.$$

Now, if we choose Δ, δ_0, $\bar{\delta}$ and δ' small enough (and hence δ_2 will be small) one can prove (see [3], Chapter 3) that there exists $\varepsilon_0 > 0$ such that

$$P(I_2^{i,2}) \le \sum_{j=1}^{M} \exp\left\{ -\frac{1}{\varepsilon}\left[\widetilde{S}_{0T}(\varphi^j, \psi^i, \eta^i) - \frac{\gamma}{16} \right] \right\}, \qquad 0 < \varepsilon \le \varepsilon_0.$$

Since $\varphi^j \in K$ then $(\varphi^j, \psi^i, \eta^i) \notin \widetilde{\Phi}(s_0)$. Hence, for $\varepsilon_0 > 0$ sufficiently small we get

$$\text{(2.18)} \qquad P(I_2^{i,2}) \le \exp\left\{ -\frac{1}{\varepsilon}(s_0 - \frac{\gamma}{8}) \right\}, \qquad 0 < \varepsilon \le \varepsilon_0.$$

Substituting the estimates (2.17) and (2.18) into (2.16), we obtain

$$\text{(2.19)} \qquad P(I_2^{i}) \le \exp\left\{ -\frac{1}{\varepsilon}(s_0 - \frac{\gamma}{4}) \right\}, \qquad 0 < \varepsilon \le \varepsilon_0.$$

The result follows putting together the estimates (2.15) and (2.19) into (2.14).

□

So far we have proved that the functional $\widetilde{S}_{0T}$ in (2.11) is the normalized action functional for the family of processes $(\widetilde{X}^{\varepsilon}_t, \widetilde{\Upsilon}^{\varepsilon}_t, Z^{\varepsilon}_t)$ with normalizing coefficient $\frac{1}{\varepsilon}$. Now, we are ready to get the action functional for the family of processes $(\widetilde{X}^{\varepsilon}_t, Z^{\varepsilon}_t)$.

Theorem 2.1. *The normalized action functional for the family of processes* $(\widetilde{X}^{\varepsilon}_t, Z^{\varepsilon}_t)$ *on* $(C_{[0,T]}(\mathbb{R}^2); \rho_{0T})$ *is*

$$\text{(2.20)} \quad S_{0T}(\varphi, \eta) = \begin{cases} \min\limits_{\psi \in F_a} \left\{ \frac{1}{2}\int_0^T \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\, ds + \int_0^T L(\varphi_s, \dot{\psi}_s, \dot{\eta}_s)\, ds \right\}, & \text{if } \varphi \text{ is a.c., } \eta \in F_{\bar{a}}, \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbb{R}^2) \end{cases}$$

with normalizing coefficient $\frac{1}{\varepsilon}$*. Moreover,* $S_{0T}(\varphi, \eta)$ *is jointly lower semicontinuous.*

Proof. Let us consider the operator G on $(C_{[0,T]}(\mathbb{R}^3); \rho_{0T})$ into $(C_{[0,T]}(\mathbb{R}^2); \rho_{0T})$ defined by $G(\varphi, \psi, \eta) = (\varphi, \eta)$. Clearly G is a continuous operator. Let $\{\mu^{\varepsilon}\}$ be the family of the distributions of $(\widetilde{X}^{\varepsilon}_\cdot, \widetilde{\Upsilon}^{\varepsilon}_\cdot, Z^{\varepsilon}_\cdot)$ on $(C_{[0,T]}(\mathbb{R}^3); \rho_{0T})$. If $\{\nu^{\varepsilon}\}$ is the family of the distributions of $(\widetilde{X}^\varepsilon_\cdot, Z^\varepsilon_\cdot)$ on $(C_{[0,T]}(\mathbb{R}^2); \rho_{0T})$ then $\nu^\varepsilon(A) = \mu^\varepsilon(G^{-1}(A))$ for any Borel subset A of $C_{[0,T]}(\mathbb{R}^2)$. Then, according to Theorem 3.3.1 in [3], the normalized action functional for $(\widetilde{X}^\varepsilon_t, Z^\varepsilon_t)$ is given by

$$S_{0T}(\varphi,\eta) = \inf\left\{\widetilde{S}_{0T}(\bar{\varphi},\bar{\psi},\bar{\eta}) : (\bar{\varphi},\bar{\psi},\bar{\eta}) \in G^{-1}\left((\varphi,\eta)\right)\right\}$$

and $+\infty$ if $G^{-1}\left((\varphi,\eta)\right) = \emptyset$. Further, the normalizing coefficient is $\frac{1}{\varepsilon}$. From this we obtain (2.20).

Recall that the functional $\widetilde{S}_{0T}$ in (2.11) is jointly lower semicontinuous in all variables (see Lemma 2.3). Since $F_{\bar{a}}$ is compact the infimum in (2.20) is attained. Therefore, $S_{0T}(\varphi,\eta)$ is also jointly lower semicontinuous.

□

3. Wave Front Propagation

In this part we shall use the same approach as in Freidlin in [2] (Chapter VI) and in [5]. For this reason most of the results here will not be proved in details.

Let us define

$$V(t,x) = \sup\left\{\eta_t - S_{0t}(\varphi,\eta) : \varphi \in C_{[0,t]}(\mathbb{R}),\, \varphi_0 = x,\, \varphi_t \in G_0,\, \eta \in F_{\bar{c}}\right\} \tag{3.1}$$

where S_{0t} given in (2.20) is the normalized action functional for $(\widetilde{X}^\varepsilon_t, Z^\varepsilon_t)$ and $G_0 = \operatorname{supp} g$. It turns out that under a suitable assumption (called Condition (N) by Freidlin in [2]), the solution $u^\varepsilon(t,x,y)$ of (1.3) converges to a step function $u^0(t,x,y)$ as $\varepsilon \downarrow 0$ given by

$$u^0(t,x,y) = \begin{cases} 0, & \text{if} \quad V(t,x) < 0, \quad |y| \le b \\ 1, & \text{if} \quad V(t,x) > 0, \quad |y| \le b. \end{cases}$$

In other words, the set $\{(t,x,y) : V(t,x) = 0,\, |y| \le b\}$ describes the position of the wave front as $\varepsilon \downarrow 0$.

The fact that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0$ in the region where $V(t,x) < 0$ is a consequence of the following Laplace-type asymptotic formula:

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln\, \widetilde{E}_{xy} g(\widetilde{X}^\varepsilon_t) \exp\left\{\frac{1}{\varepsilon} Z^\varepsilon_t\right\} = V(t,x) \tag{3.2}$$

where Z^ε_t is defined in (2.3). This formula is obtained by using the properties : compactness of the level sets, lower bound, and upper bound corresponding to the action functional S_{0T} in (2.20). The proof is similar to the proof of Lemma 6.2.1 in [2]. Using (3.2) and (1.6) we get $\lim\limits_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0$ if $V(t,x) < 0$. In fact one can prove that the convergence is uniform in any compact subset of $\{(t,x,y) : V(t,x) < 0,\, |y| \le b\}$.

To prove that $\lim\limits_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ in the region where $V(t,x) > 0$ we shall assume the following condition (see [2]):

Condition (N): For all (t,x) such that $V(t,x)=0$,

$$V(t,x)=\sup\{\eta_t - S_{0t}(\varphi,\eta) : \varphi\in C_{[0,t]}(\mathbb{R}),\ \varphi_0=x,\ \varphi_t\in G_0,\ V(t-s,\varphi_s)<0 \\ \text{for } s\in(0,t),\ \eta\in F_{\bar c}\}$$

As in Theorem 6.2.1 in [2] one can prove that under Condition (N), $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y)=1$ uniformly in (t,x,y) belonging to any compact subset of $\{(t,x,y) : V(t,x)>0,\ |y|\le b\}$. This set may be interpreted as the domain occupied by the excitation. The set $G_t = \{x\in\mathbb{R} : V(t,x)>0\}\times[-b,b]$ represents the excited region at time t. Notice that $\{(t,x) : V(t,x)=0\}$ is the graph of some continuous function of x and $\{(t,x,y) : V(t,x)=0,\ |y|\le b\}$ describes the position of the wave front.

Remark 3.1. The nonlinear term in (1.3) may depend on the slow variable, i.e, $f\equiv f(x,y,u)$. In this case the process Z_t^ε in (2.3) becomes $\widetilde{Z}_t^\varepsilon = \int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon)\,ds$ where $c(x,y)\equiv c(x,y,0)$.

The action functional for $(\widetilde{X}_t^\varepsilon,\widetilde{Z}_t^\varepsilon)$ is given by (2.20) if we understand $L(x,\alpha^1,\alpha^2)$ as the Legendre transform of the first eigenvalue of problem (2.7) with $c(x,y)$ instead of $c(y)$. Relation (1.5) becomes

$$u^\varepsilon(t,x,y)=\widetilde{E}^\varepsilon_{xy} g(\widetilde{X}_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(\widetilde{X}_s^\varepsilon, Y_s^\varepsilon, u^\varepsilon(t-s,\widetilde{X}_s^\varepsilon,Y_s^\varepsilon))ds\right\}. \tag{3.3}$$

One can verify that assuming Condition (N), the asymptotic behavior of $u^\varepsilon(t,x,y)$ in (3.3) as $\varepsilon\downarrow 0$ is described in the same way as in the case of the nonlinear term be independent of x.

The following example was considered by Freidlin in [4].

Example 3.1. Consider problem (1.3) with the small diffusion coefficient and the nonlinear term independent of the slow variable x, i.e., $a(x,y)\equiv a(y)$ and $c(x,y)\equiv c(y)$. In this case the function $V(t,x)$ in (3.1) is given by

$$V(t,x)=\sup\left\{\eta_t-\int_0^t\frac{|\dot\varphi_s|^2}{\dot\psi_s}\,ds-\int_0^t L(\dot\psi_s,\dot\eta_s)\,ds : \varphi_0=x,\ \varphi_t\in G_0,\ \psi\in F_{\bar a},\ \eta\in F_{\bar c}\right\}$$

where $L(\alpha^1,\alpha^2)$ is the Legendre transform of the first eigenvalue $\lambda(\beta_1,\beta_2)$ of problem (1.7).

To simplify the solution we will assume that $G_0=\{x : x<0\}$. From the Euler-Lagrange equation (see [1]),

$$V(t,x)=\sup_{\gamma_1,\gamma_2,z}\left\{\gamma_2 t - L(\gamma_1,\gamma_2)t-\frac{(z-x)^2}{t^2}\frac{t}{2\gamma_1}\right\}= \\ = t\mathcal{L}\left(\frac{x}{t}\right)$$

where $0 < \underline{c} \leq \gamma_1 \leq \bar{c},\ 0 < \underline{a} \leq \gamma_2 \leq \bar{a},\ z \in \mathbb{R}$ and $\mathcal{L}(\alpha) = \sup_{\gamma_1, \gamma_2} \left\{ \gamma_2 - L(\gamma_1, \gamma_2) - \alpha^2 \frac{1}{2\gamma_1} \right\}$.

Using the properties of the functions $\lambda(\beta_1, \beta_2)$ and $L(\alpha^1, \alpha^2)$ one can see that there exists a unique positive root α^* of the equation $\mathcal{L}(\alpha) = 0$. Thus $V(t,x) > 0$ if and only if $x < \alpha^* t$. The position of the wave front is described by the set $\{(t,x,y) : x = \alpha^* t,\ |y| \leq b\}$. Clearly Condition (N) is satisfied and the asymptotic velocity is constant (equal to α^*).

Now we shall construct an example showing that Condition (N) is not always fulfilled.

Example 3.2. Take $a(x,y) \equiv a(y)$, $G_0 = \{x : x < 0\}$, and $c(x,y) \equiv c(x) = c_1 \mathcal{X}_{[x \leq h]} + c_2 \mathcal{X}_{[x > h]}$ for fixed $h > 0$ and $c_1 > c_2$.

Let

$$R_{0t}(\varphi, \psi) = \int_0^t c(\varphi_s)\, ds - \frac{1}{2} \int_0^t \frac{|\dot{\varphi}_s|^2}{\dot{\psi}_s}\, ds - \int_0^t L(\dot{\psi}_s)\, ds$$

where $L(\alpha)$ is the Legendre transform of the first eigenvalue of the problem

$$\begin{cases} \frac{1}{2}\phi''(y) + \beta a(y)\, \phi(y) = \lambda(\beta)\, \phi(y), & \text{for } |y| < b \\ \phi'(b) = \phi'(-b) = 0 \end{cases}$$

with $\beta \in \mathbb{R}$. The function $V(t,x)$ in (3.1) is transformed into

$$V(t,x) = \sup_{\varphi, \psi} \{ R_{0t}(\varphi, \psi) : \varphi \in C_{[0,t]}(\mathbb{R}),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \psi \in F_{\bar{a}} \}.$$

Notice that inside each domain $(0,h)$ and $(h, +\infty)$ the Euler equations imply that $\ddot{\varphi} = 0$ and $\ddot{\psi} = 0$. Hence, the extremals φ of $R_{0t}(\varphi, \psi)$ are straight lines or broken lines with vertices on $x = h$ and the extremals ψ are straight lines trought the origin.

First consider the case $x \leq h$. It is easily verified that the extremal φ is a segment of line connecting the points $(0,0)$ and (t,x), and $V(t,x) = t\mathcal{L}_1(\frac{x}{t})$ where

$$\mathcal{L}_1(\alpha) = \sup_{\underline{a} \leq \gamma \leq \bar{a}} \left\{ c_1 - \frac{1}{2\gamma}\alpha^2 - L(\gamma) \right\}.$$

Let α_1 be the unique positive root of $\mathcal{L}_1(\alpha) = 0$. Then, $V(t,x) = 0$ if and only if $x = t\alpha_1$. Set $T_0 \equiv \frac{h}{\alpha_1}$.

Secondly, let $x > h$. One can verify that the extremal φ is a broken line starting at $(0,0)$ with vertice (t_1, h) for some $0 < t_1 < t$ satisfying $\frac{h}{t_1} < \frac{x-h}{t-t_1}$ which means that the broken line is upwards convex. The function $V(t,x)$ is given by

$$V(t,x) = \sup_{\substack{0 < t_1 < t \\ 0 < \underline{a} \leq \gamma \leq \bar{a}}} \left\{ t_1 \left[c_1 - \frac{1}{2\gamma} (\frac{h}{t_1})^2 - L(\gamma) \right] + (t - t_1) \left[c_2 - \frac{1}{2\gamma} (\frac{x-h}{t-t_1})^2 - L(\gamma) \right] \right\}.$$

Let

$$\mathcal{L}_2(\alpha) = \sup_{\underline{a}\leq\gamma\leq\bar{a}} \left\{ c_2 - \frac{1}{2\gamma}\alpha^2 - L(\gamma) \right\}$$

and α_2 be the unique positive root of $\mathcal{L}_2(\alpha) = 0$. Taking into account that $c_1 > c_2$ one can see that $\alpha_2 < \alpha_1$.

Define

$$\widetilde{V}(t,x) = \sup_{T_0 < t_1 < t} \left\{ (t-t_1)\mathcal{L}_2\left(\frac{x-h}{t-t_1}\right) \right\}.$$

Since $t\,\mathcal{L}_2\left(\frac{x}{t}\right)$ is increasing in t we have $\widetilde{V}(t,x) = (t-T_0)\mathcal{L}_2\left(\frac{x-h}{t-T_0}\right)$. Then $\widetilde{V}(t,x) = 0$ if and only if $x = h + \alpha_2(t - T_0)$.

It is not difficult to verify that $V(t,x) \geq \widetilde{V}(t,x)$ for $x > h$. Therefore, the region $\{(t,x) : V(t,x) = 0\}$ is contained in the region $\{(t,x) : \widetilde{V}(t,x) \leq 0\}$. This means that the velocity of the wave front in the region $x > h$, $t > T_0$, is not less than α_2. Moreover, using the relation for $V(t,x)$, it is possible to show that in the region where $x > h$ and $t > T_0$, the velocity of the wave front is close to α_2.

Taking into account the shape of the extremal φ and the fact that $\alpha_2 < \alpha_1$ one can see that Condition (N) is not fulfilled.

Example (3.2) shows that Condition (N) is a restriction. We shall now analyze the wave front of $u^\varepsilon(t,x,y)$ in (3.3) as $\varepsilon \downarrow 0$ in a general situation, without Condition (N). We shall consider only the case when the small diffusion coefficient is independent of the slow variable, i.e, $a(x,y) \equiv a(y)$ and the nonlinear term depends on x and y. In the rest of this section the reader should be aware of such assumptions when we refer to $u^\varepsilon(t,x,y)$ as the solution of problem (1.3).

From the assumptions on the function $a(y)$ we know that for each $\varepsilon > 0$ the differential operator

$$\frac{\partial}{\partial_t} - \frac{1}{2\varepsilon}\frac{\partial^2}{\partial y^2} - \frac{\varepsilon a(y)}{2}\frac{\partial^2}{\partial x^2} \tag{3.4}$$

is uniformly parabolic. Besides, it is possible to prove (see [5]) that $0 \leq u^\varepsilon(t,x,y) \leq 1 \wedge \|g\|$, i.e., the solution $u^\varepsilon(t,x,y)$ of problem (1.3) is bounded. The following properties are consequences of the maximum principle for linear uniformly parabolic equations :

(M.1) If $u_1^\varepsilon(t,x,y)$ and $u_2^\varepsilon(t,x,y)$ are the solutions of (1.3) for $g = g_1(x)$ and $g = g_2(x)$ respectively and $g_1(x) \geq g_2(x)$ for $x \in \mathbb{R}$ then $u_1^\varepsilon(t,x,y) \geq u_2^\varepsilon(t,x,y)$ for $t \geq 0$, $x \in \mathbb{R}$, $|y| \leq b$.

(M.2) If $u_1^\varepsilon(t,x,y)$ and $u_2^\varepsilon(t,x,y)$ are the solutions of (1.3) for respectively $f = f_1(x,y,u)$ and $f = f_2(x,y,u)$ with the same initial function g and $f_1(x,y,u) \geq f_2(x,y,u)$ for $|y| \leq b$, $x \in \mathbb{R}$, $0 \leq u \leq 1 \vee \|g\|$, then $u_1^\varepsilon(t,x,y) \geq u_2^\varepsilon(t,x,y)$ for $t \geq 0$, $x \in \mathbb{R}$, $|y| \leq b$.

Let us introduce the functional $\tau = \tau_F(t, \varphi^1, \varphi^2)$ on $(-\infty, +\infty) \times C_{[0,+\infty)}(\mathbb{R}) \times C_{[0,+\infty)}([-b, b])$ with values in $[0, +\infty]$ defined by

$$\tau_F(t, \varphi^1, \varphi^2) = \inf\left\{s : (t-s, \varphi^1_s, \varphi^2_s) \in F \times [-b, +b]\right\},$$

where F is any closed subset of $(-\infty, +\infty) \times \mathbb{R}$. Denote by Θ the set of all such functionals. Notice that $\tau_F(t, \varphi^1, \varphi^2)$ does not depend on φ^2.

We can see that $\tau_F(t, \widetilde{X}^\varepsilon_\cdot, Y^\varepsilon_\cdot)$ is the first time when the process $(t-s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)$ reaches $F \times [-b, +b]$; τ_F is a Markov time with respect to the family of σ-fields $\{\mathcal{F}_s : s \geq 0\}$ with $\mathcal{F}_s$ being the minimal σ-field in the probability space such that $(\widetilde{X}^\varepsilon_{s_1}, Y^\varepsilon_{s_1})$ is $\mathcal{F}_s$-measurable for any $s_1 \leq s$.

Define a function $V^*(t, x)$ by

$$V^*(t,x) = \inf_{\tau \in \Theta} \sup_{\varphi, \eta} \{\eta_{t\wedge\tau} - S_{0,t\wedge\tau}(\varphi, \eta) : \varphi \in C_{[0,t]}(\mathbb{R}),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \eta \in F_{\bar{c}}\} \tag{3.5}$$

where S_{0t} is defined in (2.20) and it is the normalized action functional for $\left(\widetilde{X}^\varepsilon_t, \widetilde{Z}^\varepsilon_t\right)$ (see Remark 3.1). Since $\tau \equiv t$ and $\tau \equiv 0$ belong to Θ we have $V^*(t,x) \leq (0 \wedge V(t,x)) \leq 0$ where $V(t,x)$ is the function (3.1).

Let us consider for each $t > 0$ the family of processes $(V^\varepsilon_s; \widetilde{P}^\varepsilon_{txy}) = (t_s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s; \widetilde{P}^\varepsilon_{txy})$ on the set $\mathcal{H} = (-\infty, +\infty) \times \mathbb{R} \times [-b, b]$ where $t_s = t - s$ is a deterministic process with velocity -1 and $(\widetilde{X}^\varepsilon_t, Y^\varepsilon_t)$ satisfies (1.4). This process is governed by the operator (3.4) in the interior of $\mathcal{H}$ and subject to the reflection along the normal of its boundary. Using the strong Markov property of the process $(V^\varepsilon_s; \widetilde{P}^\varepsilon_{txy})$ we derive from (3.3) that

$$0 \leq u^\varepsilon(t,x,y) = \widetilde{E}^\varepsilon_{txy} u^\varepsilon(V^\varepsilon_{t\wedge\tau}) \exp\left\{\frac{1}{\varepsilon}\int_0^{t\wedge\tau} c(\widetilde{X}^\varepsilon_s, Y^\varepsilon_s, u^\varepsilon(V^\varepsilon_s))\, ds\right\}. \tag{3.6}$$

The above equality holds for any Markov time τ with respect to the family of σ-fields $\mathcal{F}_s$. The use of equality (3.5) instead of (3.3) is the main difference between the approachs for the general case and the case when Condition (N) holds.

As in Lemma 1 in [5] one can prove that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0$ uniformly in (t,x,y) from any compact subset of $\{(t,x,y) : t > 0,\ V^*(t,x) < 0,\ |y| \leq b\}$. Taking into account that $V^*(t,x) \leq 0$ for every $t \geq 0$, $x \in \mathbb{R}$, we can see that the region where $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ must be contained in the set

$$M = \{(t,x,y) : t > 0,\ |y| \leq b,\ V^*(t,x) = 0\}.$$

We shall prove that the interior of M is contained in the region above cited. To prove this fact we shall use a kind of "cone" argument, i.e., we shall identify a set of points (t_0, x_0, y_0) such that for each such a point there exists a positive constant A such that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ uniformly in (t,x,y) from any compact subset of

$$K^A_{t_0,x_0} = \{(t,x,y) : t > t_0,\ |y| \leq b,\ |x - x_0| < A(t - t_0)\}.$$

It turns out that the points $(t_0, x_0, y_0) \in (M)$ satisfy the above condition. It is also possible to prove that if $(t_0, x_0, y_0) \in M^c$ then there exists a positive constant B such that $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x,y) = 0$ uniformly in any compact subset of

$$D^B_{t_0,x_0} = \{(t,x,y) : t < t_0,\ |y| \le b,\ |x - x_0| < B(t_0 - t)\}\,.$$

Hence the frontier ∂M of the set M determines the position of the limit wave as $\varepsilon \downarrow 0$.

The main result in this part is established in the following theorem:

Theorem 3.1. *Let $u^\varepsilon(t,x,y)$ be the solution of problem (1.3) with $a(x,y) \equiv a(y)$ and the nonlinear term depending also on the slow variable x. Then,*
(a) $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x,y) = 0$ uniformly in (t,x,y) belonging to any compact subset of $\{(t,x,y) : V^*(t,x) < 0,\ |y| \le b\}$.
(b) $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x,y) = 1$ uniformly in (t,x,y) from any compact subset of (M) where $M = \{(t,x,y) : V^*(t,x) = 0,\ |y| \le b\}$.

Part (a) of the above theorem can be proved in the same way as Lemma 1 in [5]. For convenience we will split the proof of Part (b) in some lemmas and propositions.

Let us introduce for each $x_0 \in \mathbb{R}$, $y_0 \in [-b,b]$, $\delta > 0$, and $k > 0$ a function $g^{\varepsilon,\delta}$ given by

$$g^{\varepsilon,\delta}(x,y) = \begin{cases} e^{-\frac{\delta}{\varepsilon}}, & \text{if } |x - x_0| \le e^{-\frac{k\delta}{\varepsilon}},\ \ |y - y_0| \le e^{-\frac{k\delta}{\varepsilon}} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tilde f(v) = \inf_{x,y} f(x,y,v)$ and $\tilde f(v) = \tilde c(v) v$. The function $\tilde f$ satisfies the same conditions of $f(x,y,v)$ given in the introduction of this paper. Let $\tilde c = \tilde c(0) = \sup_{v \ge 0} \tilde c(v)$. Let $v^{\varepsilon,\delta}(t,x,y)$ be the solution of the problem

$$(3.7)\qquad \begin{cases} \dfrac{\partial v^{\varepsilon,\delta}(t,x,y)}{\partial t} = \dfrac{1}{2\varepsilon}\dfrac{\partial^2 v^{\varepsilon,\delta}(t,x,y)}{\partial y^2} + \dfrac{\varepsilon a(y)}{2}\dfrac{\partial^2 v^{\varepsilon,\delta}(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon}\tilde f(v^{\varepsilon,\delta}), \\ \text{for } |y| < b, t > 0, x \in \mathbb{R} \\ v^{\varepsilon,\delta}(0,x,y) = g^{\varepsilon,\delta}(x,y) \\ \dfrac{\partial v^{\varepsilon,\delta}(t,x,y)}{\partial y}\Big|_{y=\pm b} = 0 \end{cases}$$

The Feynman-Kac formula implies that

$$(3.8)\qquad 0 \le v^{\varepsilon,\delta}(t,x,y) = \widetilde{E}^\varepsilon_{xy} g^{\varepsilon,\delta}(\widetilde{X}^\varepsilon_t, Y^\varepsilon_t) \exp\left\{\frac{1}{\varepsilon}\int_0^t \tilde c\left(v^{\varepsilon,\delta}(t-s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\}$$

where $\widetilde{X}^\varepsilon_t$ satisfies the stochastic differential equation

$$d\widetilde{X}^\varepsilon_t = \sqrt{\varepsilon a(Y^\varepsilon_t)}\, dW_t, \quad \widetilde{X}^\varepsilon_0 = x$$

with W_t being a Wiener process independent of Y^ε_t. The action functional for the family of processes $\widetilde{X}^\varepsilon_t$ is $\frac{1}{\varepsilon} S_{0T}(\varphi)$ with

(3.9)
$$S_{0T}(\varphi) = \begin{cases} \min\limits_{\psi \in F_a} \left\{ \frac{1}{2} \int_0^T \frac{|\dot\varphi_s|^2}{\psi_s}\, ds + \int_0^T L(\dot\psi_s)\, ds \right\}, & \text{if } \varphi \text{ is a. c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbb{R}) \end{cases}$$

where $L(\alpha)$ is the same function as in Example 3.2.

Let $p_\varepsilon(t,x,z)$ be the transition probability density of $x + \sqrt{\varepsilon K} W_t$ where $0 < \underline{a} \le K \le \bar a$ and W_t is a Wiener process. It is easily seen that $\forall \delta_1 > 0,\, t > 0,\, \exists \varepsilon_0 > 0,\, \delta_0 > 0$ such that

(3.10)
$$p_\varepsilon(t,x,z) > e^{-\frac{\delta_1}{\varepsilon}}, \text{ for } |x-z| < \delta_0, \quad 0 < \varepsilon \le \varepsilon_0$$

and ε_0 is independent of K.

Proposition 3.1. *$\forall \delta_2 > 0$, $\forall s_1 \in (0, \frac{\delta_2}{8})$, $\exists \delta, \varepsilon_0, \delta_3 > 0$ such that*

(3.11)
$$v^{\varepsilon,\delta}(s_1,x,y) > e^{-\frac{\delta_2}{\varepsilon}} \quad \textit{if} \quad |x-x_0| < \delta_3,\ |y| \le b,\ 0 < \varepsilon \le \varepsilon_0.$$

Proof. One can prove (see [5]) that $0 \le v^{\varepsilon,\delta}(t,x,y) \le 1 \vee \|g^{\varepsilon,\delta}\| \le 1$; since $\tilde c(v) \ge 0$ if $0 \le v \le 1$ then $\tilde c(v^{\varepsilon,\delta}) \ge 0$. From (3.8) we have

(3.12)
$$\begin{aligned} v^{\varepsilon,\delta}(t,x,y) &\ge \widetilde E^\varepsilon_{xy} g^{\varepsilon,\delta}(\widetilde X^\varepsilon_t, Y^\varepsilon_t) = \\ &= e^{-\frac{\delta}{\varepsilon}} \widetilde P^\varepsilon_{xy}(|\widetilde X^\varepsilon_t - x_0| < e^{-\frac{k\delta}{\varepsilon}}, |Y^\varepsilon_t - y_0| < e^{-\frac{k\delta}{\varepsilon}}) = \\ &= e^{-\frac{\delta}{\varepsilon}} E\left[\widetilde P^\varepsilon_{xy}(|\widetilde X^\varepsilon_t - x_0| < e^{-\frac{k\delta}{\varepsilon}}, |Y^\varepsilon_t - y_0| < e^{-\frac{k\delta}{\varepsilon}} / Y^\varepsilon_t)\right]. \end{aligned}$$

But the conditional distribution of $\widetilde X^\varepsilon_t$ given $Y^\varepsilon_t = \bar y$ is the distribution of the process $(X^{\varepsilon,\bar y}_t, P^\varepsilon_x)$ defined by $X^{\varepsilon,\bar y}_t = x + \sqrt{\varepsilon a(\bar y)} W_t$. Then, the corresponding conditional probability in (3.12) is equal to $\mathcal{X}_{[|\bar y - y_0| < e^{-\frac{k\delta}{\varepsilon}}]} P^\varepsilon_x \left(|X^{\varepsilon,\bar y}_t - x_0| < e^{-\frac{k\delta}{\varepsilon}}\right)$.

Let $\delta_2 > 0$ be given. Using (3.10) one can see that there exist $\delta > 0$, $\delta_3 > 0, \varepsilon_0 > 0$ such that

$$P^\varepsilon_x(|X^{\varepsilon,\bar y}_t - x_0| < e^{-\frac{k\delta}{\varepsilon}}) > e^{-\frac{\delta_2}{4\varepsilon}}, \quad \text{if } |x - x_0| < \delta_3,\ 0 < \varepsilon \le \varepsilon_0.$$

On the other hand, it is well known (see [14] or Theorem 1.7.1 in [2]) that the normalized Lebesgue measure in $[-b,b]$ is the unique invariant probability measure for the process Y_t. Besides, for any bounded and measurable function f,

$$\left| \bar E^\varepsilon_y f(Y^\varepsilon_t) - \frac{1}{2b} \int_{-b}^{b} f(y)\, dy \right| \le c_1 \|f\| \exp\left\{-\frac{c_2 t}{\varepsilon}\right\}$$

where c_1 and c_2 are some positive constants (independent of f); note that c_2 can be taken less or equal to 1. Then $\bar{P}_y^\varepsilon(|Y_t^\varepsilon - y_0| < e^{-\frac{k\delta}{\varepsilon}}) > c_1 e^{-\frac{t}{\varepsilon}}(\frac{1}{bc_1}e^{\frac{c_2 t - k\delta}{\varepsilon}} - 1)$. Now, by choosing $\delta < c_2 t/k$ and $t \in (0, \frac{\delta_2}{8})$ we can find $\varepsilon_0 > 0$ sufficiently small such that

$$\bar{P}_y^\varepsilon(|Y_t^\varepsilon - y_0| < e^{-\frac{k\delta}{\varepsilon}}) > e^{-\frac{\delta_2}{4\varepsilon}}, \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Using the above estimates into (3.12) we get (3.11).

□

Let us define the function $\mathcal{L}(\alpha)$ by

$$\mathcal{L}(\alpha) = \sup_{0<\underline{a}\leq\gamma\leq\bar{a}} \left\{ \tilde{c} - \frac{1}{2\gamma}\alpha^2 - L(\gamma) \right\}.$$

Using the properties of the functions $\lambda(\beta)$ and $L(\alpha)$ one can verify that there exists an unique positive root α^* of the equation $\mathcal{L}(\alpha) = 0$.

Proposition 3.2. *Let $v^{\varepsilon,\delta}(t,x,y)$ be the solution of problem (3.7). Then*
(a) $\lim_{\varepsilon\downarrow 0} v^{\varepsilon,\delta}(t,x,y) = 0$ uniformly in (t,x,y) from any compact subset of $\widetilde{Q}_- \times [-b;b]$ where $\widetilde{Q}_- = \{(t,x,y) : |x - x_0| > t\alpha^*\}$.
(b) $\lim_{\varepsilon\downarrow 0} v^{\varepsilon,\delta}(t,x,y) = 1$ uniformly in (t,x,y) from any compact subset of $\widetilde{Q}_+ \times [-b;b]$ where $\widetilde{Q}_+ = \{(t,x,y) : |x - x_0| < t\alpha^*$.

Proof. Define $m(t,x) = \inf\{S_{0t}(\varphi) : \varphi \in C_{[0,t]}(\mathbb{R}), \varphi_0 = x, \varphi_t = x_0\}$ where S_{0t} is given in (3.9). Using the Euler equations we get

$$m(x,t) = t \inf_{0<\underline{a}\leq\gamma\leq\bar{a}} \left\{ \frac{1}{2\gamma}\left(\frac{x - x_0}{t}\right)^2 + L(\gamma) \right\}.$$

Since $\tilde{c}(v) \leq \tilde{c}$ we obtain from (3.8)

$$0 \leq v^{\varepsilon,\delta}(t,x,y) \leq e^{\frac{\tilde{c}t}{\varepsilon}} \widetilde{E}_{xy}^\varepsilon g^{\varepsilon,\delta}(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon).$$

Notice that the initial function $g^{\varepsilon,\delta}$ depends on ε. This fact does not affect the proof of part (a): it is similar to the proof of Lemma 6.2.1 in [2] and we omit it. But the proof of part (b) is sligthly different.

First of all we shall prove that if (t,x,y) belongs to the set $\{(t,x,y) : m(t,x) = \tilde{c}t, |y| \leq b\}$ then for all $\delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that $v^{\varepsilon,\delta}(t,x,y) > \exp\{-\frac{\delta_2}{\varepsilon}\}$ for $0 < \varepsilon \leq \varepsilon_0$. To see this take (t,x,y) such that $m(t,x) = \tilde{c}t$. Then $x = x_0 \pm t\alpha^*$. Fix $x = x_0 + t\alpha^*$. Define

$$\hat{\phi}_s = \begin{cases} x_0 + \alpha^* t, & \text{if } s \in [0,\theta] \\ x_0 + \alpha^* t + (x_0 - x + \theta\alpha^*)\frac{s-\theta}{t-(\theta+\sqrt{\theta})}, & \text{if } s \in (\theta, t - \sqrt{\theta}] \\ x_0 + \alpha^*(t - s), & \text{if } s \in (t - \sqrt{\theta}, t]. \end{cases}$$

where c_1 and c_2 are some positive constants (independent of f); note that c_2 can be taken less or equal to 1. Then $\bar{P}^\varepsilon_y(|Y^\varepsilon_t - y_0| < e^{-\frac{k\delta}{\varepsilon}}) > c_1 e^{-\frac{t}{\varepsilon}}(\frac{1}{bc_1} e^{\frac{c_2 t - k\delta}{\varepsilon}} - 1)$. Now, by choosing $\delta < c_2 t/k$ and $t \in (0, \frac{\delta_2}{8})$ we can find $\varepsilon_0 > 0$ sufficiently small such that

$$\bar{P}^\varepsilon_y(|Y^\varepsilon_t - y_0| < e^{-\frac{k\delta}{\varepsilon}}) > e^{-\frac{\delta_2}{4\varepsilon}}, \quad \text{for } 0 < \varepsilon \le \varepsilon_0.$$

Using the above estimates into (3.12) we get (3.11).

□

Let us define the function $\mathcal{L}(\alpha)$ by

$$\mathcal{L}(\alpha) = \sup_{0<\underline{a}\le\gamma\le\bar{a}} \left\{ \tilde{c} - \frac{1}{2\gamma}\alpha^2 - L(\gamma) \right\}.$$

Using the properties of the functions $\lambda(\beta)$ and $L(\alpha)$ one can verify that there exists an unique positive root α^* of the equation $\mathcal{L}(\alpha) = 0$.

Proposition 3.2. *Let $v^{\varepsilon,\delta}(t,x,y)$ be the solution of problem (3.7). Then*
(a) $\lim_{\varepsilon\downarrow 0} v^{\varepsilon,\delta}(t,x,y) = 0$ uniformly in (t,x,y) from any compact subset of $\widetilde{Q}_- \times [-b;b]$ where $\widetilde{Q}_- = \{(t,x,y) : |x - x_0| > t\alpha^*\}$.
(b) $\lim_{\varepsilon\downarrow 0} v^{\varepsilon,\delta}(t,x,y) = 1$ uniformly in (t,x,y) from any compact subset of $\widetilde{Q}_+ \times [-b;b]$ where $\widetilde{Q}_+ = \{(t,x,y) : |x - x_0| < t\alpha^*$.

Proof. Define $m(t,x) = \inf\{S_{0t}(\varphi) : \varphi \in C_{[0,t]}(\mathbb{R}), \varphi_0 = x, \varphi_t = x_0\}$ where S_{0t} is given in (3.9). Using the Euler equations we get

$$m(x,t) = t \inf_{0<\underline{a}\le\gamma\le\bar{a}} \left\{ \frac{1}{2\gamma}\left(\frac{x - x_0}{t}\right)^2 + L(\gamma) \right\}.$$

Since $\tilde{c}(v) \le \tilde{c}$ we obtain from (3.8)

$$0 \le v^{\varepsilon,\delta}(t,x,y) \le e^{\frac{\tilde{c}t}{\varepsilon}} \widetilde{E}^\varepsilon_{xy} g^{\varepsilon,\delta}(\widetilde{X}^\varepsilon_t, Y^\varepsilon_t).$$

Notice that the initial function $g^{\varepsilon,\delta}$ depends on ε. This fact does not affect the proof of part (a): it is similar to the proof of Lemma 6.2.1 in [2] and we omit it. But the proof of part (b) is sligthly different.

First of all we shall prove that if (t,x,y) belongs to the set $\{(t,x,y) : m(t,x) = \tilde{c}t, |y| \le b\}$ then for all $\delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that $v^{\varepsilon,\delta}(t,x,y) > \exp\{-\frac{\delta_2}{\varepsilon}\}$ for $0 < \varepsilon \le \varepsilon_0$. To see this take (t,x,y) such that $m(t,x) = \tilde{c}t$. Then $x = x_0 \pm t\alpha^*$. Fix $x = x_0 + t\alpha^*$. Define

$$\hat{\phi}_s = \begin{cases} x_0 + \alpha^* t, & \text{if } s \in [0,\theta] \\ x_0 + \alpha^* t + (x_0 - x + \theta\alpha^*)\frac{s-\theta}{t-(\theta+\sqrt{\theta})}, & \text{if } s \in (\theta, t-\sqrt{\theta}] \\ x_0 + \alpha^*(t-s), & \text{if } s \in (t-\sqrt{\theta}, t]. \end{cases}$$

The piece-wise linear function $\hat{\phi}_s$ connects the points (x_0, t) and $(x, 0)$. Besides $(t - s, \hat{\phi}_s, y) \in \widetilde{Q}_-$ for $s \in [\theta, t - \sqrt{\theta}]$. Using the Markov property of the process $(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon)$ and Proposition 3.1 we conclude that for every $\delta_2 > 0$ and $s_1 \in (0, \frac{\delta_2}{16})$ there exist δ, ε_0, $\delta_3 > 0$ such that

$$v^{\varepsilon,\delta}(t,x,y) = \widetilde{E}^\varepsilon_{xy} v^{\varepsilon,\delta}\left(s_1, \widetilde{X}^\varepsilon_{t-s_1}, Y^\varepsilon_{t-s_1}\right) \exp\left\{\frac{1}{\varepsilon}\int_0^{t-s_1} \tilde{c}\left(v^{\varepsilon,\delta}(t-s_1-s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\} \geq$$

$$\geq e^{-\frac{\delta_2}{2\varepsilon}} \widetilde{E}^\varepsilon_{xy} \mathcal{X}_{[|\widetilde{X}^\varepsilon_{t-s_1} - x_0| < \delta_3]} \times$$

$$\times \exp\left\{\frac{1}{\varepsilon}\int_0^{t-s_1} \tilde{c}\left(v^{\varepsilon,\delta}(t-s_1-s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

For $s_1 > 0$ sufficiently small we get

$$v^{\varepsilon,\delta}(t,x,y) \geq e^{-\frac{\delta_2}{2\varepsilon}} \widetilde{E}^\varepsilon_{xy} \mathcal{X}_{[\|\widetilde{X}^\varepsilon_\cdot - \hat{\phi}\| \leq \frac{\delta_3}{2}]} \times$$

$$\times \exp\left\{\frac{1}{\varepsilon}\int_0^{t-s_1} \tilde{c}\left(v^{\varepsilon,\delta}(t-s_1-s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\}.$$

Using the Lipschitz continuity of $\tilde{c}(v)$, part (a) of this proposition, and the lower bound corresponding to the action functional (3.9) one can deduce that for $\theta > 0$ and $\varepsilon_0 > 0$ sufficiently small

$$v^{\varepsilon,\delta}(t,x,y) \geq e^{-\frac{\delta_2}{4\varepsilon}} \widetilde{E}^\varepsilon_{xy} \mathcal{X}_{[\|\widetilde{X}^\varepsilon - \hat{\phi}\| \leq \frac{\delta_3}{2}]} \geq e^{-\frac{\delta_2}{2\varepsilon}} \exp\left\{\frac{1}{\varepsilon}(\tilde{c}t - S_{0t}(\hat{\phi}))\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Now, by choosing $\theta > 0$ even smaller if necessary and recalling that $m(t,x) = \tilde{c}t$ we get $\tilde{c}t - S_{0t}(\hat{\phi}) > -\frac{\delta_2}{2}$. Hence there exists $\delta > 0$ and $\varepsilon_0 > 0$ such that $v^{\varepsilon,\delta}(t,x,y) > e^{-\frac{\delta_2}{\varepsilon}}$ if $0 < \varepsilon \leq \varepsilon_0$.

From this result one can prove that

$$\lim_{\varepsilon \downarrow 0} v^{\varepsilon,\delta}(t,x,y) = 1 \qquad \text{for } (t,x,y) \in \widetilde{Q}_+$$

by using arguments similar to the ones in the proof of Theorem 6.2.1 in [2].

□

Lemma 3.1. *Suppose that* $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t_0, x_0, y_0) = 0$, $t_0 > 0$. *Then there exists a constant* $A > 0$ *such that* $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ *uniformly in* (t,x,y) *from any compact subset of*

$$K^A_{t_0,x_0} = \{(s,x,y) : s > t_0, |y| \leq b, |x - x_0| < A(s - t_0)\}.$$

Proof. The proof of this lemma is similar to Lemma 2 in [5].

Assume that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^{\varepsilon}(t_0, x_0, y_0) = 0$. Then for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that $u^{\varepsilon}(t_0, x_0, y_0) > e^{-\frac{\delta}{2\varepsilon}}$, $0 < \varepsilon \le \varepsilon_0$. Using the a priori bound for the Hölder norm of $u^{\varepsilon}(t, x, y)$ we derive that

$$u^{\varepsilon}(t_0, x, y) > e^{-\frac{\delta}{\varepsilon}} \quad \text{for } |x - x_0| \le e^{-\frac{k\delta}{\varepsilon}},\ |y - y_0| \le e^{-\frac{k\delta}{\varepsilon}}$$

for some positive constant k. Then $u^{\varepsilon}(t_0, x, y) > v^{\varepsilon,\delta}(0, x, y)$ for $x \in \mathbb{R}$ and $|y| \le b$. Using properties **(M.1)-(M.2)** we conclude that

$$u^{\varepsilon}(t, x, y) \ge v^{\varepsilon,\delta}(t - t_0, x, y) \qquad \text{for } t > t_0,\ \ x \in \mathbb{R}, |y| \le b.$$

By Proposition 3.2 (b), $\lim_{\varepsilon\downarrow 0} v^{\varepsilon,\delta}(t - t_0, x, y) = 1$ if $|x - x_0| < (t - t_0)\alpha^*$, $|y| \le b$ uniformly in any compact subset of $\{(t, x, y) : |x - x_0| < (t - t_0)\alpha^*,\ |y| \le b\}$. Take $A = \alpha^*$ and the result follows.

□

The following lemma is analogous to Lemma 3 in [5].

Lemma 3.2.
(a) Assume that $\lim_{\varepsilon'\downarrow 0} u^{\varepsilon'}(t_0, x_0, y_0) = 0$ *for some sequence* $\varepsilon' \downarrow 0$. *Then there exists* $A > 0$ *such that* $\lim_{\varepsilon'\downarrow 0} u^{\varepsilon'}(t, x, y) = 0$ *uniformly in* (t, x, y) *belonging to compact subsets of* $D^A_{t_0,x_0} = \{(t, x, y) : 0 < t < t_0,\ |x_0 - x| < A(t_0 - t),\ |y| \le b\}$.
(b) Let $\mathcal{E}^{(\varepsilon')} = \{(t, x, y) : \lim_{\varepsilon'\downarrow 0} u^{\varepsilon'}(t, x, y) = 0,\ t > 0\}$. *For every compact* F *belonging to the interior* $(\mathcal{E}^{(\varepsilon')})$ *of* $\mathcal{E}^{(\varepsilon')}$, $\lim_{\varepsilon'\downarrow 0} u^{\varepsilon'}(t, x, y) = 0$ *uniformly in* F.

Proof. The proof of part (a) follows from Lemma 3.1 by contradiction argument. The uniformity of the convergence is a consequence of the uniformity of the bounds in Proposition 3.2 (b). Part (b) follows by observing that the compact set F can be covered by a finite number of sets $D^A_{t_k,x_k}$ with vertices $(t_k, x_k) \in (\mathcal{E}^{(\varepsilon')}) \setminus F$.

□

Remark 3.1. It follows from Lemma 3.2 (a) that if $(t, x, y) \in \mathcal{E}^{(\varepsilon')}$ then $(t - h, x, y) \in (\mathcal{E}^{(\varepsilon')})$ for any $0 < h \le t$. Thus, $\mathcal{E}^{(\varepsilon')} \subseteq \overline{(\mathcal{E}^{(\varepsilon')})}$.

Remark 3.2. Taking into account Lemma 3.2 and Remark 3.1 we conclude that the set $\mathcal{E}^{(\varepsilon')}$ has the form $\{(s, x) : x \in \mathbb{R},\ 0 \le s \le s(x)\} \times [-b, b]$, where $s(x)$ is some function of x.

The proof of the following lemma is similar to the proof of Lemma 4 in [5].

Lemma 3.3. *Let* $M = \{(t, x, y) : t > 0,\ x \in \mathbb{R},\ |y| \le b,\ V^*(t, x) = 0\}$. *Let* F *be a compact subset of (M). Then* $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^{\varepsilon}(t, x, y) = 0$ *uniformly in* $(t, x, y) \in F$.

Proof. From Remark 3.2 we can see that the functional

$$\tau = \tau(t, \varphi^1, \varphi^2) = \inf\left\{s : (t - s, \varphi^1_s, \varphi^2_s) \notin (\mathcal{E}^{(\varepsilon')})\right\}$$

is independent of the third argument φ^2. The reader can check the proof of Lemma 4 in [5] to realize that this fact allows us to use the same arguments used by Freidlin in that lemma. □

Now we can see that Theorem 3.1 (b) follows from Lemma 3.1 and Lemma 3.3. Notice that the position of the wave front is described by the frontier ∂M of the set $M = \{(t,x,y) : V^*(t,x) = 0, |y| \leq b\}$ and, by Remark 3.2, the boundary of the set $\{(t,x) : V^*(t,x) = 0\}$ is the graph of some function of x. Notice that the definition of $V^*(t,x)$ and $V(t,x)$ imply that $\{(t,x) : V(t,x) < 0\} \subseteq \{(t,x) : V^*(t,x) < 0\}$. Hence, if Condition (N) is fulfilled these two sets are the same.

4. Remarks and Generalizations

Remark 4.1. Suppose that the initial function g in (1.3) depends also on the fast variable y. Let $H_0 = \text{supp } g \subset \mathbb{R} \times [-b,b]$ and $[H_0] = [(H_0)]$. Let G_0 be the projection of H_0 over the x-axis. Clearly $[G_0] = [(G_0)]$. Let us assume that $[G_0] \neq \mathbb{R}$. In this case the asymptotic behavior of the solution of (1.3) is the same as before: exactly as in Theorem 6.2.1 [2] one can prove that for any $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that

$$\widetilde{E}_{xy} g(\widetilde{X}_t^\varepsilon, Y_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(Y_s^\varepsilon)\, ds\right\} \leq \exp\left\{\frac{1}{\varepsilon}[V(t,x)+\gamma]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Hence, $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0$ if $V(t,x) < 0$, where $V(t,x)$ is given in (3.1). However, the fact that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ in the region $V(t,x) > 0$ is obtained in a slightly different way.

For any $\delta_2 > 0$ choose functions $\hat{\varphi}$, $\hat{\eta}$ such that $\hat{\varphi} \in C_{[0,t]}(\mathbb{R})$, $\hat{\varphi}_0 = x$, $\hat{\varphi}_t \in G_0, \hat{\eta} \in F_{\bar{c}}$, and $R_{0t}(\hat{\varphi},\hat{\eta}) > V(t,x) - \frac{\delta_2}{2}$ where $R_{0t}(\hat{\varphi},\hat{\eta}) = \hat{\eta}_t - S_{0t}(\hat{\varphi},\hat{\eta})$ and $S_{0t}(\hat{\varphi},\hat{\eta})$ is defined in (2.20). The upper semicontinuity of R_{0t} and the fact that $[G_0] = [(G_0)]$ allow us to choose $\hat{\varphi}$ such that dist $(\hat{\varphi}_t, \mathbb{R}\setminus G_0) > 0$.

Take $y^* \in [-b,b]$ and a sufficiently small positive constant K such that $g(x,y) > 0$ for $|x - \hat{\varphi}_t| \leq K$ and $|y - y^*| \leq K$ (we may assume $y^* \in (-b,b)$). Let $\min_{[\,|x-\hat{\varphi}_t|\leq K,\, |y-y^*|\leq K\,]} g(x,y) = c_1 > 0$.

Define

$$\bar{g}(x,y) = \begin{cases} c_1, & \text{if } |x - \hat{\varphi}_t| \leq K, \quad |y - y^*| \leq K \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{u}^\varepsilon(t,x,y)$ be the solution of (1.3) with initial function $\bar{g}$ and nonlinear term $\tilde{f}(u) = \inf_y f(y,u)$. Then,

$$\bar{u}^\varepsilon(s,x,y) \geq \widetilde{E}^\varepsilon_{xy}\bar{g}(\widetilde{X}^\varepsilon_s, Y^\varepsilon_s) = c_1 \widetilde{P}^\varepsilon_{xy}\left\{|\widetilde{X}^\varepsilon_s - \hat{\varphi}_s| \leq K, |Y^\varepsilon_t - y^*| \leq K\right\}.$$

Recall that $\widetilde{X}^\varepsilon_s = x + \sqrt{\varepsilon}\,\widetilde{W}_{\widetilde{\Upsilon}^\varepsilon_s}$ where $\widetilde{\Upsilon}^\varepsilon_s = \int_0^s a(\widetilde{X}^\varepsilon_v, Y^\varepsilon_v)\, dv$ and $\widetilde{W}_s$ is a $\mathbb{R}$-Wiener process starting at zero and independent of Y^ε_s. Then, the conditional distribution of $\widetilde{X}^\varepsilon_s$ given $\widetilde{\Upsilon}^\varepsilon_s = \psi_s$ with $\psi \in F_{\bar{a}}$, is the distribution of

$$X^\varepsilon_{\psi_s} = x + \sqrt{\varepsilon}\,\widetilde{W}_{\psi_s}.$$

Clearly $X^\varepsilon_{\psi_s}$ and Y^ε_s are independent. Moreover, the transition probability density of $X^\varepsilon_{\psi_s}$ satisfies a relation similar to (3.10). Then, one can see (as in the proof of Proposition 3.1) that $\forall\delta_2 > 0$, $\forall s > 0$, $\exists\delta_1 > 0$, $\exists\varepsilon_0 > 0$ (ε_0 independent of ψ) such that

$$\widetilde{P}_{xy}\left\{|\widetilde{X}^\varepsilon_s - \hat{\varphi}_t| \le K/\widetilde{\Upsilon}^\varepsilon_s = \psi_s\right\} > \exp\left\{-\frac{\delta_2}{4\varepsilon}\right\}$$

if $|x - \hat{\varphi}_t| < \delta_1$ and $0 < \varepsilon \le \varepsilon_0$.

On the other hand, one can show (see the proof of Proposition 3.1) that for $s \in (0, \frac{\delta_2}{8})$, $\exists\varepsilon_0 > 0$ sufficiently small such that

$$\bar{P}_y\left\{|Y^\varepsilon_s - y^*| \le K\right\} > \exp\left\{-\frac{\delta_2}{4\varepsilon}\right\}, \quad \text{for } 0 < \varepsilon \le \varepsilon_0.$$

Therefore, $\forall\delta_2 > 0$, $\forall s_1 \in (0, \delta_2/8)$, $\exists\delta_1 > 0$, $\varepsilon_0 > 0$ such that

$$\widetilde{P}^\varepsilon_{xy}\left\{|\widetilde{X}^\varepsilon_{s_1} - \hat{\varphi}_t| \le K, |Y^\varepsilon_{s_1} - y^*| \le K\right\} = E\left[\widetilde{P}^\varepsilon_{xy}\left\{|\widetilde{X}^\varepsilon_{s_1} - \hat{\varphi}_t| \le K, |Y^\varepsilon_{s_1} - y^*| \le K/\widetilde{\Upsilon}^\varepsilon_\cdot\right\}\right] \ge$$
$$\ge \exp\left\{-\frac{\delta_2}{4\varepsilon}\right\} \times \bar{P}_y\left\{|Y^\varepsilon_{s_1} - y^*| \le K\right\} > \exp\left\{-\frac{\delta_2}{2\varepsilon}\right\}$$

if $|x - \hat{\varphi}_t| < \delta_1$, $|y| \le b$, and $0 < \varepsilon \le \varepsilon_0$ and then , for those values of (x, y), $\bar{u}^\varepsilon(s_1, x, y) > e^{-\frac{\delta_2}{\varepsilon}}$ for $s_1 \in (0, \frac{\delta_2}{8})$.

Now, using the strong Markov property and properties **(M.1)-(M.2)** we obtain for any $0 < s_1 < t$

$$u^\varepsilon(t, x, y) = \widetilde{E}^\varepsilon_{xy} u^\varepsilon(s_1, \widetilde{X}^\varepsilon_{t-s_1}, Y^\varepsilon_{t-s_1}) \exp\left\{\frac{1}{\varepsilon}\int_0^{t-s_1} c\left(Y^\varepsilon_s, u^\varepsilon(t - s_1 - s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\} \ge$$
$$\ge \widetilde{E}^\varepsilon_{xy} \bar{u}^\varepsilon(s_1, \widetilde{X}^\varepsilon_{t-s_1}, Y^\varepsilon_{t-s_1}) \exp\left\{\frac{1}{\varepsilon}\int_0^{t-s_1} c\left(Y^\varepsilon_s, u^\varepsilon(t - s_1 - s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\},$$

for $0 < \varepsilon \le \varepsilon_0$. Choose $s_1 \in (0, \frac{\delta_2}{8})$ to get

$$u^\varepsilon(t, x, y) > e^{-\frac{\delta_2}{\varepsilon}} \widetilde{E}^\varepsilon_{xy} \mathcal{X}_{[|\widetilde{X}^\varepsilon_{t-s_1} - \hat{\varphi}_t| < \delta_1]} \exp\left\{\frac{1}{\varepsilon}\int_0^{t-s_1} c\left(Y^\varepsilon_s, u^\varepsilon(t - s_1 - s, \widetilde{X}^\varepsilon_s, Y^\varepsilon_s)\right) ds\right\}.$$

Since s_1 can be chosen arbitrarily small, the rest of the proof is the same as in Theorem 6.2.1 in [2].

Remark 4.2. Assume that the initial function in (1.3) depends on $\varepsilon > 0$ in the following way: $\lim_{\varepsilon\downarrow 0} \varepsilon \ln g^\varepsilon(x) = \mu(x)$ uniformly in x, $\mu(x)$ is uniformly continuous, $\mu(x) < 0$ for $x > 0$, $\mu(0) = 0$, and $\mu(x) \le \bar{\mu}$ for $x < 0$.

Define $\widetilde{V}(t,x)$ as:

$$\widetilde{V}(t,x) = \sup\left\{\eta_t + \mu(\varphi_t) - S_{0t}(\varphi,\eta) : \varphi \in C_{[0,t]}(\mathbb{R}),\ \varphi_0 = x,\ \eta \in F_{\bar{c}}\right\}$$

where S_{0t} is given in (2.20).

One can prove that the position of the wave front of $u^{\varepsilon}(t,x,y)$ as $\varepsilon \downarrow 0$ is determined by $\widetilde{V}(t,x) = 0$. The main point here is to see that the Laplace-type asymptotic formula (3.2) still holds with $\widetilde{V}(t,x)$ instead of $V(t,x)$. The proof is the same of Lemma 6.2.1 [2] if we take into account the assumptions made on $g^{\varepsilon}(x)$.

Notice that if $\mu_1(x) < \mu_2(x)$ for $x > 0$ then the wave front corresponding to μ_2 reaches some fixed value x faster than the wave corresponding to μ_1. The minimum asymptotic velocity is obtained when $\mu(x) = -\infty$, i.e, when $g(x) = 0$ for $x > 0$.

Remark 4.3. Using results from [5] and [6] one can show that $\bar{V}(t,x) = V^*(t,x)$ for $t > 0$, $x \in \mathbb{R}$ where $V^*(t,x)$ is defined in (3.5) and

$$\bar{V}(t,x) = \sup_{\varphi,\eta}\left\{\min_{0\le a\le t}\left[\eta_a - S_{0a}(\varphi,\eta) : \varphi \in C_{[0,t]}(\mathbb{R}),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \eta \in F_{\bar{c}}\right]\right\}.$$

Remark 4.4. Consider a weakly coupled R.D.E. with equations of the type as in (1.3):

$$(4.1)\qquad \begin{cases} \dfrac{\partial u_k^{\varepsilon}(t,x,y)}{\partial t} = \dfrac{1}{2\varepsilon}\dfrac{\partial^2 u_k^{\varepsilon}(t,x,y)}{\partial y^2} + \dfrac{\varepsilon a(x,y)}{2}\dfrac{\partial^2 u_k^{\varepsilon}(t,x,y)}{\partial x^2} + \\ \qquad + \dfrac{1}{\varepsilon}\left[f_k(y,u_k^{\varepsilon}) + \displaystyle\sum_{j=1}^{n} d_{kj}(u_k^{\varepsilon} - u_j^{\varepsilon})\right], \quad x \in \mathbb{R}, |y| < b, t > 0 \\ u_k^{\varepsilon}(0,x,y) = g_k(x) \\ \dfrac{\partial u_k^{\varepsilon}(t,x,y)}{\partial y}\Big|_{y=\pm b} = 0 \end{cases}$$

for $k = 1,\cdots,n$ and $d_{ij} \ge 0$ for $i,j \in \{1,\cdots,n\}$ and $i \ne j$.

The probabilistic approach allows us to analyze the behavior of the solution of the problem (4.1) as $\varepsilon \downarrow 0$ by considering the right continuous strong Markov process $(\widetilde{X}_t^{\varepsilon}, Y_t^{\varepsilon}, \nu_t^{\varepsilon}; \widetilde{P}_{xyk}^{\varepsilon})$ in the phase space $\mathbb{R}\times[-b,b]\times\{1,\cdots,n\}$ corresponding to the infinitesimal operator

$$\mathcal{A}^{\varepsilon}h(x,y,i) = \frac{1}{2\varepsilon}\frac{\partial^2 h(x,y,i)}{\partial y^2} + \frac{\varepsilon a_i(x,y)}{2}\frac{\partial^2 h(x,y,i)}{\partial x^2} + \frac{1}{\varepsilon}\sum_{j=1}^{n} d_{ji}\left[h(x,y,j) - h(x,y,i)\right]$$

where h is bounded, has uniformly continuous bounded first and second order derivatives in x and y up to the boundary, and $\frac{\partial h(x,y,i)}{\partial y}|_{y=\pm b} = 0$. For the existence and properties of such processes see, for example, Skorokhod [13].

As in Part 3 of this paper, the propagation of the wave front for (4.1) is analyzed by considering the form of the action functional for the family of processes $(\widetilde{X}_t^\varepsilon, \int_0^t c_{\nu_s^\varepsilon}(Y_s^\varepsilon)\,ds)$.

Remark 4.5. The space of the fast variable can be any compact subset D of $\mathbb{R}^r$ (with smooth enough boundary). The fast motion can be described by any diffusion process with coefficients independent of the slow variable, with reflection along the inward normal to the boundary ∂D of D. For a construction of such processes, see for example Freidlin [2]. More general boundary conditions may also be considered (see Wentzell [15]).

Remark 4.6. Let $(Y_t, \bar{P}_y)$ be a Markov family in the phase space $(D, \mathcal{B}(D))$ where $D \subset \mathbb{R}^r$ is compact and $\mathcal{B}(D)$ is the σ-field generated by the Borel subsets of D in the topology inherited from the Euclidean norm in $\mathbb{R}^r$. Let $\{T_t\}_{t\geq 0}$ be the semigroup of operators given by $T_t g(y) = \bar{E}_y g(Y_t)$, g a bounded and measurable function, and $\mathcal{A}^1$ its infinitesimal generator. Problem (1.3) can be generalized to

$$
(4.2) \qquad \begin{cases} \dfrac{\partial u^\varepsilon(t,x,y)}{\partial t} = \mathcal{A}^{1,\varepsilon} u^\varepsilon(t,x,y) + \dfrac{\varepsilon a(x,y)}{2} \dfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \dfrac{1}{\varepsilon} f(y, u^\varepsilon), \\ \text{for } x \in \mathbb{R},\, y \in D,\, t > 0 \\ u^\varepsilon(0,x,y) = g(x) \end{cases}
$$

where $\mathcal{A}^{1,\varepsilon}$ is the infinitesimal generator of $(Y_{\frac{t}{\varepsilon}}; \bar{P}_y)$.

Under some general assumptions on $(Y_t; \bar{P}_y)$ of the Feller type and of stochastic continuity, a Large Deviation Principle for the family of processes $Z_t^\varepsilon = \int_0^t c(Y_{\frac{s}{\varepsilon}})\,ds$ can be established.

Again, the limit behavior of the solution of problem (4.2) is analyzed by means of the action functional for $(\widetilde{X}_t^\varepsilon, Z_t^\varepsilon)$.

Remark 4.7. As the slow motion, Markov processes belonging to the class of the locally infinitely divisible processes can be considered. These processes are extentions of processes with independent increments. Wentzell, in [17], established a Large Deviation Principle for locally infinitely divisible processes by assuming suitable conditions on the cumulant of such processes.

The slow variable in (4.2) can be an infinitely divisible process with frequent jumps whose infinitesimal generator is given by

$$
\begin{aligned} \mathcal{A}^{2,\varepsilon}_{(y)} f(x) = {} & b(x,y) \frac{d}{dx} f(x) + \frac{\varepsilon}{2} a(x,y) \frac{d^2}{dx^2} f(x) + \\ & + \frac{1}{\varepsilon} \int\limits_{\mathbb{R}\setminus\{0\}} \left[f(x+\varepsilon\beta) - f(x) - \varepsilon\beta \frac{df(x)}{dx} \right] \mu_{x,y}(d\beta), \end{aligned}
$$

where $\mu_{x,y}$ is a measure on $\mathbb{R} \setminus \{0\}$ such that $\int\limits_{\mathbb{R}\setminus\{0\}} \beta^2\, \mu_{x,y}(d\beta) < \infty$. The subscript (y) means that the above operator depends also on the fast variable. For more details about this process, the reader may consult [3] or [17].

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