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CHARGE COMMUTATOR FOR ANY MOMENTUM

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Abstract: The nucleon matrix elements of the charge commutation relations are considered for arbitrary momentum. The resulting expression is exact. The high-momentum limit reduces to the Alder-Weisberger sum rule. For zero momentum one obtains the known low-energy result together with a closed expression for the correction.

1. INTRODUCTION

The commutation relations of the SU3 × SU3 (or SU2 × SU2) charges [1] have been used successfully to derive sum rules and low-energy theorems. The well-known Adler-Weisberger sum rule is most easily obtained by directly employing the commutation relation taken between proton states at infinite momentum [2]. Low-energy theorems, on the other hand, are conventionally derived by considering off-shell amplitudes of time-ordered products in the limit of vanishing four-momentum of the π-meson. The off-shell amplitude is obtained using an interpolating π-meson field defined by

$$\phi^{\alpha}(x) = \frac{1}{f_{\pi} m_{\pi}^2} \partial^{\lambda} A_{\lambda}^{\alpha}(x) , \tag{1}$$

where $A^{\alpha}(x)$ ($\alpha = 1, 2, 3$) is the axial current operator and numerically $f_{\pi} \simeq \frac{1}{10} M_{\mathrm{p}}$. The commutation relations then provide the desired generalized Ward-identities. The low energy results obtained in this fashion must, of course, be connected with the quantities taken at the physical value of the pion momentum, as is described in an enlightening paper by Fubini and Furlan [3]. A close look at the relevant formulae then reveals that by such a procedure one simply comes back to the charge commutator, but taken between states of *finite* (or zero) space momentum [4,5]. The entire physical content of the charge algebra should be directly obtainable from the matrix elements of the commutators.

In this note we study the SU2 × SU2 charge commutator between nucleon states of arbitrary momentum. Fubini considered this matrix element under

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the assumption of commuting current divergences. We shall show that it is possible to avoid this assumption. It turns out that the disconnected contributions are decisive for small space momentum. As expected from the above discussion we find a close connection with Weinberg's formula [2] for the scattering lengths for pions on protons and - of course - with the Adler-Weisberger result for G_A/G_V.

2. THE CHARGE COMMUTATOR

We discuss the commutator relation

$$\lim_{\boldsymbol{p}'\to\boldsymbol{p}} \langle p'|[\int d^3x\, A_0^{1+i2}(\boldsymbol{x},0), A_0^{1-i2}(0)]|p\rangle = 2\langle p|V_0^3(0)|p\rangle \tag{2}$$

where p denotes a proton state of space momentum $\boldsymbol{p}$. The limiting procedure $\boldsymbol{p}'\to\boldsymbol{p}$ is necessary since strictly speaking the space integrated charges $\int d^3x\, A_0^{\alpha}(\boldsymbol{x},t)$ do not exist but may be used between wave packet states.

It is practical to separate the axial current operator into a longitudinal and a transversal part:

$$\begin{aligned} A_\mu^\alpha(x) &= A_\mu^{\alpha,\mathrm{L}}(x) + A_\mu^{\alpha,\mathrm{T}}(x) , \\ A_\mu^{\alpha,\mathrm{L}}(x) &= \frac{1}{\Box}\partial_\mu\, \partial^\lambda A_\lambda^\alpha(x) , \\ A_\mu^{\alpha,\mathrm{T}}(x) &= A_\mu^\alpha(x) - \frac{1}{\Box}\partial_\mu \partial^\lambda A_\lambda^\alpha(x) . \end{aligned} \tag{3}$$

The longitudinal part carries zero angular momentum and its matrix elements contain the π-meson pole. The transversal part carries angular momentum one and the matrix elements of $A_0^{\alpha,\mathrm{T}}(x)$ vanish in the limit of zero space momentum transfer. The singularity at four-momentum transfer zero causes no difficulty. Only the neutron intermediate state in eq. (2) contributes to it and can, of course, be treated separately. For simplicity we take the mass of the neutron slightly larger than the one of the proton.

Defining the meson source function $J_\pi^\alpha(x)$ by

$$J_\pi^\alpha(x) = (\Box + m_\pi^2)\phi_\pi^\alpha(x) , \tag{4}$$

we may write from definitions (1) and (3)

$$A_\mu^{\alpha,\mathrm{L}}(x) = -f_\pi \partial_\mu(\tfrac{1}{2}\phi_{\mathrm{in}}^\alpha(x) + \tfrac{1}{2}\phi_{\mathrm{out}}^\alpha(x)) + \frac{f_\pi m_\pi^2}{\Box}\, \mathrm{P}\frac{1}{\Box + m_\pi^2}\, \partial_\mu J_\pi^\alpha(x) . \tag{5}$$

The free-field part of eq. (5) will appear in eq. (2) multiplied with the principal value term containing the π-meson pole. Only the $\pi^\pm p'$ and $\pi^\pm p$ intermediate states contribute to these special "disconnected diagrams" which we denote by D.

$$D = -\tfrac{1}{2}\sqrt{2}\, f_\pi^2 m_\pi^2 (2\pi)^{\frac{3}{2}} \lim_{\boldsymbol{k}\to 0} \left[\frac{m_\pi + \Delta E}{(m_\pi + \Delta E)^2 - \boldsymbol{k}^2} \frac{\langle p'\pi^-(\boldsymbol{0})|J_\pi^{1-i2}|p\rangle}{(m_\pi + \Delta E)^2 - \boldsymbol{k}^2 - m_\pi^2} \right.$$

$$+ \frac{1}{k_0 - \Delta E} \frac{\langle p'|J_\pi^{1+i2}|p\pi^-(\boldsymbol{k})\rangle}{(k_0 - \Delta E)^2 - m_\pi^2} - \frac{1}{k_0 + \Delta E} \frac{\langle p'\pi^+(-\boldsymbol{k})|J_\pi^{1+i2}|p\rangle}{(k_0 + \Delta E)^2 - m_\pi^2}$$

$$\left. - \frac{m_\pi - \Delta E}{(m_\pi - \Delta E)^2 - \boldsymbol{k}^2} \frac{\langle p'|J_\pi^{1-i2}|p\pi^+(\boldsymbol{0})\rangle}{(k_0 - \Delta E)^2 - \boldsymbol{k}^2 - m_\pi^2} \right] . \quad (6)$$

We use the abbreviations and normalizations

$$\boldsymbol{k} = \boldsymbol{p}' - \boldsymbol{p} , \qquad k_0 = \sqrt{m_\pi^2 + \boldsymbol{k}^2} , \qquad p_0 = \sqrt{M_p^2 + \boldsymbol{p}^2} ,$$

$$\Delta E = p_0' - p_0 , \qquad \phi^{1\pm i2} = \phi_1 \pm i\phi_2 , \qquad \langle 0|\phi^{1\pm i2}|\pi^\mp\rangle = \sqrt{2}(2\pi)^{-\frac{3}{2}} ,$$

$$\bar{u}u = 2M_p , \qquad \langle p'|p\rangle = 2p_0\delta^3(\boldsymbol{p}' - \boldsymbol{p}) , \qquad \langle S - 1\rangle = i\langle T\rangle\delta^4(p_f - p_i) . \quad (7)$$

The scattering states in eq. (6) are the averages of "in" and "out" states. Time reversal invariance can be used to bring the one-particle states to one side of the operator J_π.

Evaluating the limit in eq. (6) one sees that it exists and is independent of the direction in which $\boldsymbol{k}$ goes to zero. To describe the result we introduce the non spin-flip "scattering" matrix elements averaged over spin

$$\mathrm{Re}\, T_{\pi\pm} = \frac{(2\pi)^4}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2}} \langle p'\pi^\pm(\boldsymbol{q})|J_\pi^{1\pm i2}(0)|p\rangle$$

$$= \tfrac{1}{2} \sum_{S=-\frac{1}{2}}^{+\frac{1}{2}} \mathrm{Re}\, \{A_\pm(s,t,u)(\bar{u}_p^S, u_p^S) + B_\pm(s,t,u)(\bar{u}_p^S, \gamma q u_p^S)\} , \quad (8)$$

where s, t, u are *independent* variables:

$$s = (p' + q)^2 , \qquad t = (p' - p)^2 , \qquad u = (p - q)^2 , \quad (9)$$

with

$$p'^2 = p^2 = M_p^2 , \qquad q^2 = m_\pi^2 .$$

The limiting procedure (6) leads to two parts. The first part (D_1) is simply proportional to the difference of the forward scattering amplitudes:

$$D_1(p) = \frac{f_\pi^2}{2\pi} \frac{3}{2m_\pi} \mathrm{Re}\,(T_{\pi^-} - T_{\pi^+})_{p'=p,\boldsymbol{q}=0} . \quad (10)$$

The second part (D_2) contains derivatives obtained from the variation of the proton and meson momenta as prescribed in eq. (6) and therefore it is not a true scattering matrix element.

$$D_2(p) = -\frac{f_\pi^2}{2\pi} 2M_p \left\{ \left[(p_0 + m_\pi) \frac{\partial}{\partial s} - (p_0 - m_\pi) \frac{\partial}{\partial u} \right] \mathrm{Re}(A_- + A_+) + m_\pi \frac{p_0}{M} \left[(p_0 + m_\pi) \frac{\partial}{\partial s} - (p_0 - m_\pi) \frac{\partial}{\partial u} + \frac{1}{m_\pi} \right] \mathrm{Re}(B_- - B_+) \right\}_{\boldsymbol{p}'=\boldsymbol{p},\boldsymbol{q}=0.} \quad (11)$$

In the differentiation the meson mass $q^2 = m_\pi^2$ is, of course, kept fixed. Interestingly, the right-hand side of eq. (11) can also be written as the derivative of the "forward scattering" matrix elements $T_{\pi\pm}(p,q) = T_{\pi\pm}(\boldsymbol{p}'=\boldsymbol{p}, \boldsymbol{q}=0, q_0)$ with respect to the "variable" q_0 occurring implicitly in

$$s = M_p^2 + 2p_0 q_0 + q_0^2, \qquad u = M_p^2 - 2p_0 q_0 + q_0^2 ,$$

and explicitly in the covariant $(\bar{u}_p \gamma q u_p)$:

$$D_2(p) = -\frac{f_\pi^2}{2\pi} \frac{1}{2} \frac{\partial}{\partial q_0} \mathrm{Re}\Big(T_{\pi^-}(p,q) - T_{\pi^+}(p,q)\Big)_{q_0=m_\pi} . \quad (12)$$

Thus, the wave packet approach $\boldsymbol{p}' \to \boldsymbol{p}$ with fixed $q^2 = m_\pi^2$ may be replaced by a definite infinitesimal continuation in the (kinematical) π-meson mass. From eqs. (10) and (12) we obtain

$$D(p) \doteq \frac{f_\pi^2}{\pi} \left[\left(1 - m_\pi^2 \frac{\partial}{\partial q_0^2}\right) \frac{\mathrm{Re}\,(T_{\pi^-}(p,q) - T_{\pi^+}(p,q))}{2q_0} \right]_{q_0 = m_\pi} . \quad (13)$$

Let us now turn to the remaining terms of eq. (2). Inserting eq. (5) without the free fields this contribution can be written

$$C(p) = \lim_{\boldsymbol{k}\to 0} \frac{f_\pi^2 m_\pi^4}{2\pi} \mathrm{P} \int_{-\infty}^{+\infty} \frac{\mathrm{d}q_0}{q_0} \frac{q_0 + \Delta E}{(q_0 + \Delta E)^2 - \boldsymbol{k}^2} \times \frac{M_{\pi^-}(q_0, \boldsymbol{q}=0, \boldsymbol{p}', \boldsymbol{p}) - M_{\pi^+}(-q_0 - \Delta E, \boldsymbol{q}=-\boldsymbol{k}, \boldsymbol{p}', \boldsymbol{p})}{(q_0^2 - m_\pi^2)\,[(q_0 + \Delta E)^2 - \boldsymbol{k}^2 - m_\pi^2]} . \quad (14)$$

where

$$M_{\pi\pm}(q,p',p) = \int \mathrm{d}^4x\, \mathrm{e}^{iqx} \langle p' | j_\pi^{1\mp i2}(x)\, j_\pi^{1\pm i2}(0) | p \rangle . \quad (15)$$

The principal value integral (14) contains two singularities which are separated as long as $\boldsymbol{k} \neq 0$ and in which one may put $\boldsymbol{p}' = \boldsymbol{p}$ (i.e. $\boldsymbol{k} = 0$, $\Delta E = 0$) everywhere except in the last denominator. It can also be written as the real part of a quantity having an integrand with a double pole slightly shifted off the real axis *. Thus

$$C(p) = \frac{f_\pi^2 m_\pi^4}{2\pi} \operatorname{Re} \int_0^\infty \frac{dq_0}{q_0^2} \frac{1}{(q_0^2 - m_\pi^2 + i\epsilon)^2} \{M_{\pi^-}(q,p,p) - M_{\pi^+}(q,p,p)\}_{\boldsymbol{q}=0} . \quad (16)$$

Using the Lorentz-invariant variables $\nu = p \cdot q$ and q^2 this result takes the form

$$C(p) = \frac{f_\pi^2}{2\pi} m_\pi^4 \, p_0 \operatorname{Re} \int_0^\infty \frac{d\nu}{\nu^2} \frac{dq^2}{(q^2 - m_\pi^2 + i\epsilon)^2}$$

$$\times \delta\left(q^2 - \left(\frac{\nu}{p_0}\right)^2\right)\left(M_{\pi^-}(q^2,\nu) - M_{\pi^+}(q^2,\nu)\right) . \quad (17)$$

We are now able to write the charge commutation relation (2) in a form valid for all space momenta of the outside proton state

$$(2\pi)^2 f_\pi^2 \left[\left(1 - m_\pi^2 \frac{\partial}{\partial q_0^2}\right) \operatorname{Re} \frac{T_{\pi^-}(p,q) - T_{\pi^+}(p,q)}{2\nu}\right]_{\boldsymbol{q}=0,\, q_0 = m_\pi}$$

$$+ \frac{(2\pi)^2}{2} f_\pi^2 m_\pi^4 \operatorname{Re} \int_{\nu_0}^\infty \frac{d\nu}{\nu^2} \frac{dq^2 \, \delta\left(q^2 - \left(\frac{\nu}{p_0}\right)^2\right)}{(q^2 - m_\pi^2 + i\epsilon)^2} \left(M_{\pi^-}(q^2,\nu) - M_{\pi^+}(q^2,\nu)\right)$$

$$+ \left|\frac{G_A}{G_V}\right|^2 \frac{\boldsymbol{p}^2}{p_0^2} = 1 . \quad (18)$$

We extracted the neutron intermediate state from the integral; the threshold is now $\nu_0 = p_0(\sqrt{(M_p + m_\pi)^2 + \boldsymbol{p}^2} - \sqrt{M_p^2 + \boldsymbol{p}^2})$. The weak coupling constant ratio G_A/G_V and f_π are connected by the Goldberger-Treiman formula which with our normalization reads

$$\frac{G_A}{G_V} = \frac{f_\pi g(0)}{M_p} , \quad (19)$$

where $g^2(m_\pi^2)/4\pi = 14.6$ and $g(0) \simeq 12$. According to its definition in eq. (15), $M_{\pi^\pm}(q^2,\nu)$ can formally be expressed as a fictitious total cross section for a π-meson with kinematical mass q^2 and laboratory energy ν/M_p scattered on a proton:

$$M_{\pi^\pm}(q^2,\nu) = \frac{\sqrt{\nu^2 - q^2 M_p^2}}{\pi^3} \sigma_{\pi^\pm}^{tot}(q^2,\nu) . \quad (20)$$

This 'total cross section' includes however disconnected contributions - different from the π-meson part - at least for finite proton momenta.

* Since $M(q_0)$ has a square root dependence at $q_0 = m_\pi$ for $\boldsymbol{p} = 0$ this statement holds only as long as $\boldsymbol{p} \neq 0$. In the subsequent formula $\boldsymbol{p} = 0$ can, however, be reached by the simple limiting procedure $\boldsymbol{p} \to 0$.

The first term in eq. (18) vanishes for $p_0 \to \infty$ and thus the integral should exist in this limit. Naively taking the limit under the integral we obtain the Adler-Weisberger formula

$$\frac{2}{\pi} f_\pi^2 \int_{Mm_\pi + \frac{1}{2}m_\pi^2}^{\infty} \frac{d\nu}{\nu} [\sigma_{\pi^-}^{\text{tot}}(0, \nu) - \sigma_{\pi^+}^{\text{tot}}(0, \nu)] + \left|\frac{G_A}{G_V}\right|^2 = 1. \tag{21}$$

We now consider the opposite extreme, i.e., $\boldsymbol{p} = 0$. In this case only intermediate states of total spin $\frac{1}{2}$ and negative parity contribute to the integral in eq. (18). Since the integrand goes as ν^{-6} a very rapid convergence is to be expected. More important, the small meson mass further suppresses the value of the integral. An appreciable contribution to it could only come from the integration region in which q_0/m_π is of order one. Neglecting at present this part and also the derivative part in eq. (18) we obtain for $\boldsymbol{p} = 0$ Weinberg's formula

$$\frac{a_{\pi^-} - a_{\pi^+}}{2m_\pi} \simeq \frac{1}{8\pi f_\pi^2} \frac{1}{1 + m_\pi/M_p}, \tag{22}$$

where we have introduced the scattering lengths

$$a_{\pi^\pm} = \frac{\pi}{2(M_p + m_\pi)} T_{\pi^\pm}(\boldsymbol{p}=0, q_0 = m_\pi). \tag{23}$$

Eq. (22) is in fair agreement with experiment. The π-meson disconnected contributions saturate the sum rule almost completely! For zero π-meson mass this saturation would be exact provided no $\frac{1}{2}^-$ baryon state were degenerate with the nucleon in the same limit.

To obtain a rough estimate of the correction to the low energy result (22) one may calculate the integral contribution by substituting for $M(q_0)$ its functional form near threshold [3]

$$M_-(\boldsymbol{p} \to 0, q_0) - M_+(\boldsymbol{p} \to 0, q_0) \approx \frac{8}{3} \frac{M_p}{\pi^2} (a_{\frac{1}{2}}^2 - a_{\frac{3}{2}}^2) \sqrt{q_0^2 - m_\pi^2 + O(\boldsymbol{p}^2)}. \tag{24}$$

The quantitives $a_{\frac{1}{2}}$ and $a_{\frac{3}{2}}$ are the scattering lengths for isospin quantum numbers $\frac{1}{2}$ and $\frac{3}{2}$ respectively. With this form for $M(q_0)$ the integral in eq. (18) takes the value (for $\boldsymbol{p} \to 0$)

$$-\frac{8}{3}\, 2f_\pi^2 (a_{\frac{1}{2}}^2 - a_{\frac{3}{2}}^2)$$

giving a 10% correction in the right direction. One more correction is due to the derivative part in eq. (18). It may be estimated by simple models and it is expected to be small.

3. CONCLUDING REMARKS

To see the connection of formula (18) with the usual low energy limit one may consider the function

$$X(q_0^2) = X(q^2, \nu)\Big|_{\boldsymbol{q}=0} = (-q^2 + m_\pi^2)^2 (2\pi)^2 \frac{f_\pi^2}{2\nu} \tag{25}$$

$$\times \pi \lim_{\boldsymbol{p}'\to\boldsymbol{p}} \mathrm{Re} \int \mathrm{d}^4x\, \mathrm{e}^{iqx} i\langle p' | T\{\phi^{1+i2}(x)\,\phi^{1-i2}(0) - \phi^{1-i2}(x)\,\phi^{1+i2}(0)\} | p\rangle \Big|_{\boldsymbol{q}=0} .$$

Using the same procedure which led to eq. (18) one obtains

$$X(q_0^2) = X(m_\pi^2) + (q_0^2 - m_\pi^2) \frac{\partial X}{\partial q_0^2}\Bigg|_{m_\pi^2}$$

$$+ (q_0^2 - m_\pi^2)^2 \frac{(2\pi)^2 f_\pi^2}{2p^0} \mathrm{Re} \int_0^\infty \frac{\mathrm{d}q_0'}{q_0'^2 - q_0^2} \frac{M_{\pi^-}(q',p) - M_{\pi^+}(q',p)}{(q_0'^2 - m_\pi^2 + i\epsilon)^2}\Bigg|_{\boldsymbol{q}=0} . \tag{26}$$

The quantity $X(q_0^2)$ is thus the Fourier transform of the time ordered product of the meson source functions $j_\pi^\alpha(x)$, twice subtracted at the point $q_0^2 = m_\pi^2$. The subtraction constants $X(m_\pi^2)$ and $(\partial/\partial q_0^2)X|_{m_\pi^2}$ are proportional to the first and second term in D [eq. (13)]. They arise from the disconnected π-nucleon intermediate states.

Comparing then eq. (26) with eq. (18) one has

$$X(0) = 1 , \tag{27}$$

which is the low-energy theorem also obtainable from eq. (25) by partial integrations.

Eq. (27) expresses in a covariant form the content of the charge algebra. The insertion of a complete set of intermediate states gives eq. (18) where the choice of the frame enhances or suppresses the contribution from the on-shell strong interaction scattering amplitude to the sum rule.

We have seen that the straightforward use of Gell-Mann's commutation relation between states of arbitrary momenta leads to a single formula combining the low- and high-energy results of the charge algebra. Additional commutation relations involving divergences are not needed; if postulated, they may, of course, be treated in an analogous way.

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