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[13]In this respect, the present result differs from the conclusion of the EP theory which asserts that good agreement with the experimental FID shape can be obtained by considering $G_0(t)$ and $G_1(t)$ only. Detailed evaluations of Eq. (4) have shown, however, that this apparent agreement arises from the fact that $G_1(t)$ for the [100] direction reported by EP is about 25% smaller than that given in Fig. 1.

[14]The FID shape at zero time, which is needed to normalize the experimental points, cannot be observed directly because of the instrumental limitations, but has to be obtained by extrapolations using analytical functions to which the experimental points are well fitted (Refs. 7 and 10). Such curve-fitting procedures cause unavoidable arbitrariness of the normalized amplitudes.

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EXACT PHOTOCOUNT DISTRIBUTIONS FOR LASERS NEAR THRESHOLD

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Photocount fluctuation problems, treated previously for Gaussian statistics only, or in the limit of short measurement times, have been extended to arbitrary times (compared with the intensity correlation time) and to general Markoffian processes. In particular, we calculate the distribution function of the time-integrated light intensity for a laser operated near threshold.

The statistical distribution $W(\Omega)$ of Ω, the light intensity integrated over a time T, is studied experimentally in terms of the probability[1]

$$p(m,T) = \int \frac{\Omega^m}{m!} e^{-\Omega} W(\Omega) d\Omega \tag{1}$$

of achieving m photocounts in a time T. These distributions, even for small m, are interesting and difficult to compute because they contain information concerning general multitime correlations of intensity fluctuations. For short times compared with an intensity correlation time, the probability density $W(\Omega)$ is determined by the steady-state distribution of light intensity, and is relatively uninteresting. For larger times, only the case of a Gaussian distribution of light amplitudes has been calculated.[2-4] These results are appropriate to a laser well below threshold. For a laser near threshold, whose behavior is clearly nonlinear, the vacuum of available theory will be filled by this Letter. Our procedure applies to general Markoffian processes, of which our laser model is a special case.

The distribution $p(m,T)$ depends on three parameters: $s = \Lambda_a T$, the linewidth of the spectrum of intensity fluctuations times measurement time; p, the net pumping rate, which defines the operating point of the laser; and $\langle m \rangle$, the mean counting rate, which depends on counter efficiency and geometry. The probability density $W(\Omega)$ is a more favorable quantity to deal with in that it can be scaled to eliminate the dependence on $\langle m \rangle = \langle \Omega \rangle$:

$$W(\Omega, s, p, \langle\Omega\rangle) = \langle\Omega\rangle^{-1} V(\omega, s, p);$$
$$\omega = \Omega / \langle\Omega\rangle. \tag{2}$$

Our method consists of calculating the Laplace transform of $V(\omega)$ and then inverting this transform.

We start from the work by Kelley and Kleiner[5] who showed that, for conditions that can be well approximated experimentally, the probability $p(m,T)$ of measuring m photocounts in a time T with an absorption detector is given by the multitime quantum-mechanical average

$$p(m,T) = \langle T_N (\Omega_{op}{}^m e^{-\Omega_{op}}/m!) \rangle,$$
$$\Omega_{op} = \epsilon \int_0^T b^\dagger(t) b(t) dt, \tag{3}$$

where T_N is the time and normally ordering operator,[6] $b^\dagger(t)$ and $b(t)$ are the creation and annihilation operators of the (single-mode) electromagnetic field, and ϵ takes account of the efficiency of the detector and the geometry of the experiment. Lax and Louisell[7] showed that for lasers for which the atomic rate constants are fast compared with the photon-decay constants, $b^\dagger$ and b have a random Markoffian behavior and obey equations of motion of the form of a rotating-wave Van der Pol (RWVP) oscillator, with quantum noise sources that can be calculated from first principles. A dynamic correspondence was also established[6] between the average of time and normally ordered multitime quantum-mechanical Markoffian operators, and the classical average of the classical random variables corresponding to these operators. Defining $\beta(t)$ and its complex conjugate $\beta^*(t)$ as the classical variables corresponding to $b(t)$ and $b^\dagger(t)$, the complicated Eq. (3) reduces to Eq. (1), the familiar Mandel[1] formula, where $W(\Omega)$ now stands for

$$W(\Omega) = \langle \delta(\Omega - \epsilon \int_0^T \rho(t)dt)\rangle; \quad \rho(t) = |\beta(t)|^2. \qquad (4)$$

Instead of calculating $W(\Omega)$, we shall find it more convenient to calculate

$$M(\lambda') = \int e^{-\lambda'\Omega} W(\Omega) d\Omega \qquad (5)$$

and then invert the Laplace transform. Scaling $W(\Omega)$ according to Eq. (2), Eq. (5) becomes, writing $\lambda' = \lambda T\langle\rho\rangle$,

$$M(\lambda, s, p) = \int \exp(-\lambda T\langle\rho\rangle\omega) V(\omega, s, p) d\omega = \langle \exp[-\lambda \int_0^T \rho(t)dt]\rangle. \qquad (6)$$

Eq. (1) becomes

$$p(m, T) = \int \frac{(\langle m\rangle\omega)^m}{m!} \times \exp(-\langle m\rangle\omega) V(\omega, s, p) d\omega. \qquad (7)$$

Following techniques used by Kac and Siegert,[8] Lax[9] showed that for random Markoffian processes in a set of variables $\vec{a} = [a_1, a_2, \cdots]$, where ρ is an arbitrary function of the set $\vec{a}$, Eq. (6) can be calculated from

$$M(\lambda, s, p) = \int d\vec{a}_0 P_0(\vec{a}_0) \int d\vec{a} \hat{P}(\vec{a}, t|\vec{a}_0, t_0), \qquad (8)$$

where P_0 and $\hat{P}$ satisfy the equations

$$LP_0 = 0, \quad \partial\hat{P}/\partial t = -(L + \lambda\rho)\hat{P}, \qquad (9)$$

subject to the condition

$$\hat{P}(\vec{a}, t_0|\vec{a}_0, t_0) = \delta(\vec{a} - \vec{a}_0). \qquad (10)$$

Here L is the generalized Fokker-Planck differential operator of the original random process and $\hat{P}$ is a conditional probability for a modified process in which "particles" disappear at a rate $\lambda\rho$. In our case of the RWVP oscillator, $a_1 = \beta$, $a_2 = \beta^*$, and $\beta = \rho^{1/2}\exp(-i\varphi)$. We will assume that L, generally a non-Hermitian operator in the Sturm-Liouville sense, and its adjoint $L^\dagger$ (and likewise $L + \lambda\rho$ and its adjoint) have complete sets of biorthogonal eigenfunctions:

$$LP_j = \Lambda_j P_j; \quad L^\dagger\varphi_i = \Lambda_i^*\varphi_i;$$
$$(L + \lambda\rho)\hat{P}_j = \hat{\Lambda}_j\hat{P}_j; \quad (L^\dagger + \lambda^*\rho^*)\hat{\varphi}_i = \hat{\Lambda}_i^*\hat{\varphi}_i. \qquad (11)$$

The Green's function solution of Eqs. (9) and (10) is[10]

$$\hat{P}(\vec{a}, t|\vec{a}_0, t_0) = \sum_n e^{-\hat{\Lambda}_n T}\hat{P}_n(\vec{a})\hat{\varphi}_n^*(\vec{a}_0), \quad t - t_0 = T \geq 0. \qquad (12)$$

For ordinary Fokker-Planck processes (which include our RWVP oscillator) it was shown[7] that time reversibility in the form of detailed balance is sufficient to guarantee that

$$\varphi_n^* = P_n/P_0; \quad \hat{\varphi}_n^* = \hat{P}_n/P_0. \qquad (13)$$

[In fact, it is possible to show that for generalized Fokker-Planck processes, detailed balance[11] is both a necessary and sufficient condition for Eqs. (13) to be valid.] Substituting Eqs. (12) and (13) in Eq. (8),

$$M(\lambda, s, p) = \sum_n \exp(-\hat{\Lambda}_n s/\Lambda_a)[\int \hat{P}_n(\vec{a})d\vec{a}]^2. \qquad (14)$$

Our Fokker-Planck operator can be written in the form[7]

$$-L = \frac{\partial}{\partial\rho}(2\rho^2 - 2p\rho - 4) + \frac{\partial^2}{\partial\rho^2}(4\rho) + \frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}. \qquad (15)$$

Although L is not separable, its eigenfunctions have the form $f(\rho)\exp(il\varphi)$. The integral over φ in Eq. (14) assures that only the $l = 0$ or pure amplitude fluctuation modes contribute to $p(m, T)$. Writing these $l = 0$ eigenfunctions of L as $P_j(\rho)$, we can solve for the corresponding $\hat{P}_j(\rho)$ by setting $\hat{P}_n(\rho) = \sum_j P_j(\rho) C_{jn}$ to obtain the secular equations for C_{jn} and the associated eigenvalues $\hat{\Lambda}_n$:

$$\sum_j [(\Lambda_i - \hat{\Lambda}_n)\delta_{ij} + \lambda\rho_{ij}] C_{jn} = 0, \qquad (16)$$

$$\rho_{ij} = \int d\rho\, P_i(\rho)\rho P_j(\rho)/P_0(\rho). \qquad (17)$$

The use of the biorthogonality of $P_j(\rho)$ against $\varphi_0(\rho) = 1$ permits one to reduce Eq. (14) to the simple form

$$M(\lambda, s, p) = \sum_n \exp\{-s\hat{\Lambda}_n(\lambda, p)/\Lambda_a(p)\} \times [C_{0n}(\lambda, p)]^2. \qquad (18)$$

$\{\Lambda_i\}$ and Λ_a were given by Risken and Vollmer[12] and by Hempstead and Lax.[13] The latter also computed ρ_{ij} for $i, j = 0, 1, \cdots, 10$.[14]

The standard inversion of the Laplace transform requires knowledge of $M(\lambda)$ for $\lambda \to \pm i\infty$, and for large λ the 11×11 matrix procedure of Eq. (16) is no longer valid. We therefore tried the real-axis techniques of Bellman et al.[15] in which the Laplace integral is replaced by a summation formula involving $V(\omega_i)$, $i = 1, 2, \cdots, n$. Evaluating $M(\lambda)$ at $\lambda = \lambda_j$, $j = 1, 2, \cdots, n$, leads to n simultaneous equations for the $V(\omega_i)$. Since we are performing an "unsmoothing" operation, the matrix to be inverted will be ill conditioned (i.e., significant figures are lost on inversion) unless the points λ_j and ω_i are chosen judiciously. We found the Gauss-Laguerre integration formula to work well, with the λ_j at carefully chosen uniform spacing.

To obtain a continuous function $V(\omega)$ rather than a discrete set of values $V(\omega_i)$, we introduced the new procedure of representing $V(\omega)$ by a judiciously chosen function times a polynomial. The n polynomial coefficients were determined to satisfy n values of the Laplace transform. Continuous solutions consistent with the previously obtained discrete results were achieved in the region $-10 \leq p \leq 1$ with

$$V(\omega, s, p) = [\Gamma(a)]^{-1} a^a \omega^{a-1} e^{-a\omega} \textstyle\sum_n b_n \omega^n, \quad (19)$$

and in the region $1 < p \leq 10$ with

$$V(\omega, s, p) = N(\alpha/\beta) \exp\{\alpha\omega - (\beta\omega/2)^2\} \textstyle\sum_n d_n \omega^n. \quad (20)$$

The functions multiplying the polynomials were themselves initially normalized and of unit mean, and their second moment was set to be nearly equal to that of $V(\omega)$ itself.[16] Five terms in the polynomials were usually sufficient. Equation (7) can now be expressed as a sum of simple functions with appropriate coefficients.

Figure 1 summarizes some of our results for the density function $V(\omega, s, p)$ for various values of s and p. Laser threshold is defined by $p = 0$.

In the past, the special case of a Gaussian random Markoffian variable β has been studied in detail. The exact $W(\Omega)$ was computed by Slepian[2] for real β, and by Jakeman and Pike[4] for complex β. The exact $p(m, T)$ for complex β was calculated by Bédard[3] and also computed in Ref. 4. We use this Gaussian case as a check on the accuracy of our techniques by computing $V(\omega)$ in two ways: (1) We omit the ρ^2 term in

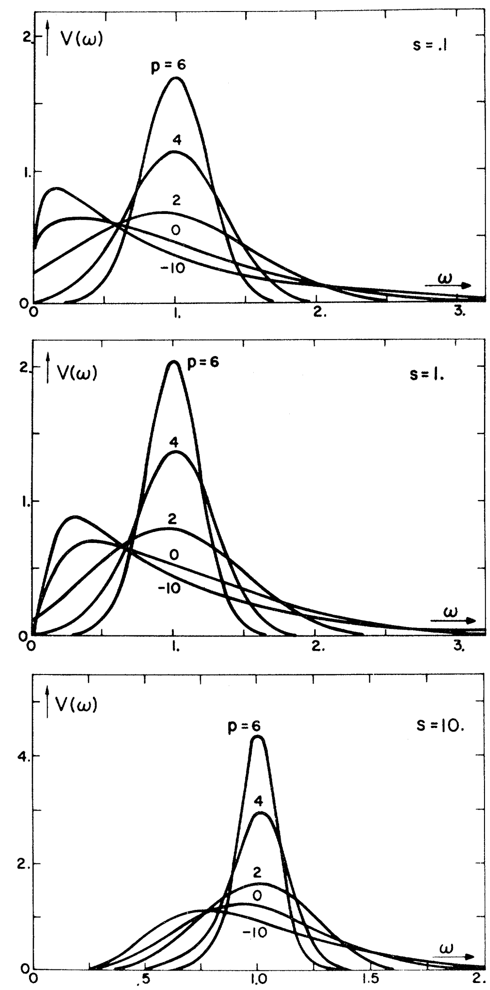

FIG. 1. Probability density $V(\omega, s, p)$ vs ω of the normalized time-integrated light intensity $\omega = \int_0^T \rho(t)dt / T\langle\rho\rangle$ of a laser operating near threshold. Here $s = \Lambda_a T$, $\Lambda_a = \langle(\Delta\rho)^2\rangle / \int_0^\infty \langle\Delta\rho(t)\Delta\rho(0)\rangle dt$ is the linewidth of the spectrum of intensity fluctuations, and p defines the operation point of the rotating-wave Van der Pol oscillator assumed as the laser model. Threshold is given by $p = 0$.

Eq. (15) and obtain closed-form solutions for P_n, Λ_n, and ρ_{ij}. We then follow the procedure of this paper using Eq. (19). (2) We start from the closed-form expression for $M(\lambda)$ [see Eq. (8.63) of Ref. 9, and Ref. 4] and invert the Laplace transform by the usual residue methods. These two procedures were found to agree to about one percent.

Experiments[17] support the RWVP oscillator model for a laser in single-mode operation near threshold. A few factorial moments of $p(m, T)$ for finite T were measured[18] at $p \approx -10$ and -3.5, but detailed measurements of $p(m, T)$ in the threshold region have not yet been reported.

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CHARACTERISTIC LENGTHS λ AND ξ OF BCS THEORY

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The calculations of Eilenberger and Büttner are misleading in the regime where λ and ξ are complex. We study the regime of validity of linear-response theory in calculating the asymptotic properties of plane boundary and vortex structure. On this basis, it is not possible to predict if a sufficiently clean low-κ type-II superconductor such as niobium or vanadium will exhibit field reversal at low temperatures.

One of the most exciting of recent developments in BCS theory is the creation of formalism that renders feasible the calculation of fluxoid and plane-boundary structure throughout the entire H, T plane.[1,2] As a first step in a rigorous, self-consistent BCS calculation of vortex structure, Eilenberger and Büttner have calculated the asymptotic behavior of the magnetic field and order parameter far from the vortex axis.[3] They obtain a functional form, basically exponential, characterized by a length r_0. Similar behavior is exhibited in the plane-boundary problem with occurrence of an identical r_0.

Linear-response theory predicts that the vector potential $\vec{A}$ and order-parameter perturbation $\delta\Delta$ satisfy the equations

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r}) = \int K(\vec{r}-\vec{r}\,')\vec{A}(\vec{r}\,')d\vec{r}\,', \tag{1}$$

$$\delta\Delta(\vec{r}) = \int D(\vec{r}-\vec{r}\,')\delta\Delta(\vec{r}\,')d\vec{r}\,', \tag{2}$$

a solution of which is simply e^{-sz} if the integration $d\vec{r}\,'$ is carried out over all space. In Eq. (1), $s=1/\lambda$, and in Eq. (2), $s=1/\xi$. It is a simple matter to show, using conventional linear-response theory for a BCS superconductor containing a random array of impurities, that λ and ξ satisfy Eqs. (2.17) and (2.18) of Ref. 3 identically.[4]

We limit ourselves to the magnetic case for a pure superconductor at $T=0$. From BCS, we have

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r}) = -\frac{3}{c^2\Lambda\xi_0}\int\frac{\vec{R}[\vec{R}\circ\vec{A}(\vec{r}\,')]J(R,0)d\vec{r}\,'}{R^4}, \tag{3}$$