Collisionless relaxation in gravitational systems: From violent relaxation to gravothermal collapse

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Theory and simulations are used to study collisionless relaxation of a gravitational N-body system. It is shown that when the initial one-particle distribution function satisfies the virial condition—potential energy is minus twice the kinetic energy—the system quickly relaxes to a metastable state described *quantitatively* by the Lynden-Bell distribution with a cutoff. If the initial distribution function does not meet the virial requirement, the system undergoes violent oscillations, resulting in a partial evaporation of mass. The leftover particles phase-separate into a core-halo structure. The theory presented allows us to quantitatively predict the amount and the distribution of mass left in the central core, without any adjustable parameters. On a longer time scale $\tau_G \sim N$, collisionless relaxation leads to a gravothermal collapse.

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Since the pioneering works of Boltzmann and Gibbs, systems with long-range interactions have been a major stumbling block to the development of statistical mechanics [1]. The difficulty was already well appreciated by Gibbs, who has noted that the equivalence between statistical ensembles breaks down when the interparticle potentials decay with exponents smaller than the dimensionality of the embedding space [2]. When this happens, systems exhibit some very unusual properties that appear to violate the second law of thermodynamics. For example, confined non-neutral plasmas are found to phase-separate into coexisting phases of *different* temperatures [3], while the self-gravitating systems, such as elliptical galaxies, are characterized by a *negative* specific heat [4]. The explanation for these counterintuitive results lies in the fact that when the interactions are long-ranged, thermodynamic equilibrium is never reached [5] and the laws of *equilibrium* thermodynamics do not apply.

In the limit in which the number of particles goes to infinity ($N \to \infty$), while the total mass and charge are kept fixed—the so called thermodynamic limit—the collision duration time diverges, and the dynamical evolution of non-neutral plasmas and gravitational systems is governed *exactly* by the collisionless Boltzmann, or as it is known in plasma physics, Vlasov equation [6]. This equation never reaches a stationary state—the spatiotemporal evolution continues *ad infinitum* on smaller and smaller length scales, while the one particle position and velocity distribution function evolves in time as an incompressible fluid. In practice, however, since there is always a minimum resolution, most systems do *appear* to evolve to a well defined stationary state. This state, however, is very different from the normal thermodynamic equilibrium characterized by the Maxwell-Boltzmann distribution—it depends explicitly on the initial distribution of the particle positions and velocities.

Forty years ago [7], Lynden-Bell argued that although the fine-grained distribution function of positions $\mathbf{r}$ and velocities $\mathbf{v}$, $f(t,\mathbf{r},\mathbf{v})$, never reaches equilibrium, the *coarse-grained* distribution function $\bar{f}(t,\mathbf{r},\mathbf{v})$, *averaged* on microscopic length scales, relaxes to a meta-equilibrium with $\bar{f}(\mathbf{r},\mathbf{v})$. Since in practice the very small length scales cannot be resolved experimentally, observations and simulations can only provide us with the information about the coarse-grained distribution function $\bar{f}(\mathbf{r},\mathbf{v})$. To obtain $\bar{f}(\mathbf{r},\mathbf{v})$, we divide the phase space into macrocells of volume $d^3\mathbf{r}d^3\mathbf{v}$, which are in turn subdivided into ν microcells, each of volume h^3. The initial distribution function $f_0(\mathbf{r},\mathbf{v})$ is discretized into a set of levels η_j, with $j=1,\ldots,p$. The incompressibility of the Vlasov dynamics requires that at any time t, each microcell contains at most one discretized level η_j and that the overall hypervolume of each level $\gamma(\eta_j)=\int \delta(f(t,\mathbf{r},\mathbf{v}) - \eta_j)d^3\mathbf{r}d^3\mathbf{v}$ be preserved by the dynamics. We denote the fraction of the volume of the *macrocell* at $(\mathbf{r},\mathbf{v})$ occupied by the level j as $\rho_j(\mathbf{r},\mathbf{v})$. Using a standard combinatorial procedure [3,7,8], it is now possible to associate a coarse-grained entropy S with the distribution of $\{\rho_j\}$. Lynden-Bell argued that the collisionless relaxation should lead to the density distribution of levels that is the most likely, i.e., the one that maximizes the *coarse-grained* entropy, consistent with the conservation of all the dynamical invariants—energy, momentum, angular momentum, and the hypervolumes $\gamma(\eta_j)$. In terms of the volume fractions $\{\rho_j\}$, the stationary distribution function becomes $\bar{f}(\mathbf{r},\mathbf{v})=\sum_j \eta_j \rho_j(\mathbf{r},\mathbf{v})$. If the initial distribution has only one level $p=1$ (is a water-bag),

$$f_0(\mathbf{r},\mathbf{v}) = \eta_1 \Theta(r_m - r)\Theta(v_m - v), \tag{1}$$

where $\Theta(x)$ is the Heaviside step function and $\eta_1 = 9/16\pi^2 r_m^3 v_m^3$—the maximization procedure is particularly simple, yielding a Fermi-Dirac distribution,

$$\bar{f}(\mathbf{r},\mathbf{v}) = \eta_1 \rho(\mathbf{r},\mathbf{v}) = \frac{\eta_1}{e^{\beta[\epsilon(\mathbf{r},\mathbf{v})-\mu]}+1}. \tag{2}$$

In the expression above, ϵ is the mean energy of particles at position $\mathbf{r}$ with velocity $\mathbf{v}$. β and μ are two Lagrange multipliers required by the conservations of energy and the number of particles,

$$\int d^3\mathbf{r}d^3\mathbf{v}\epsilon(\mathbf{r},\mathbf{v})\bar{f}(\mathbf{r},\mathbf{v}) = \epsilon_0,$$

$$\int d^3\mathbf{r} d^3\mathbf{v} \bar{f}(\mathbf{r},\mathbf{v}) = 1, \tag{3}$$

where ϵ_0 is the energy per particle of the initial distribution and the units are such that $h=1$. By analogy with the usual Fermi-Dirac statistics, we define the effective temperature of a stationary state T as $\beta=1/k_B T$. This temperature should not be confused with the standard definition of temperature in terms of the average kinetic energy, the latter being valid only for classical systems in thermodynamic equilibrium. In the thermodynamic limit, the gravitational potential ϕ of N particles with the total mass M satisfies the Poisson equation

$$\nabla^2 \phi = 4\pi G m n(\mathbf{r}), \tag{4}$$

where $m=M/N$ and $n(\mathbf{r})=N\int \bar{f} d^3\mathbf{v}$ is the particle number density. Equations (2)–(4) form the basis of Lynden-Bell's violent relaxation theory [7,9,10]. The idea is that the original distribution f_0—which is far from equilibrium, i.e., is statistically unlikely—will relax rapidly to $\bar{f}(\mathbf{r},\mathbf{v})$, thus maximizing the coarse-grained entropy. In practice, however, what is found is that self-gravitating systems usually relax to structures characterized by dense cores surrounded by dilute halos, the distribution functions of which are quite different from Lynden-Bell's $\bar{f}$. The failure of the theory was attributed to the fact that the violent relaxation occurs on a very fast dynamical time scale and the system does not have time to explore all of the phase space to find the most probable configurations [11]. Recent work on non-neutral plasmas [3], however, provides a very different picture. It has been found that confined non-neutral plasmas also relax to a core-halo structure. In that case, however, the halo production has been clearly shown to be the result of parametric resonances arising from the macroscopic bulk oscillations [12]. If the initial distribution is constructed in such a way as to suppress macroscopic oscillations, the resulting stationary state was found to be precisely the one predicted by the Lynden-Bell theory [3]. It is reasonable, therefore, to suppose that a similar mechanism will be at work for the self-gravitating systems as well. Strong oscillations will lead to parametric resonances—a form of a nonlinear Landau damping [13]—which will transfer a large amount of energy to some particles at the expense of the rest. These particles will either escape to infinity (evaporate) or will form a dilute halo that will surround the central core.

To test this theory, we first consider the case in which the macroscopic oscillations are suppressed. This can be achieved by forcing the original distribution to satisfy the virial condition $2K=-U$, where K is the total kinetic energy and U is the total potential energy. For such distributions the virial number defined as $\mathcal{R}\equiv -2K/U$ is one. To simplify the discussion, we will restrict our attention to the initial distributions of the water-bag form ($p=1$). For these distributions, the virial condition reduces to the requirement that $v_m=\sqrt{GM/r_m}$ and the average energy per particle is $\epsilon_0=\frac{3}{10}mv_m^2-\frac{3}{5}\frac{GMm}{r_m}$. We expect that under these conditions, f_0 will relax to the distribution given by Eq. (2), with $\epsilon(r,v)=mv^2/2+m\phi(r)$, subject to constraints of Eqs. (3). There is, however, one difficulty. Since the gravitational potential decays to zero at large distances, Eq. (2) requires that at any *finite* temperature there should be a nonvanishing particle density over *all* space. The normalization conditions (3), therefore, cannot be satisfied in an infinite space. However, if we confine our attention to sufficiently short times, before a significant number of particles has a chance to escape from the main clusters—in practice this time is very large when the virial condition is satisfied—the normalization problem can be avoided by artificially restricting the particle positions to lie within a sphere of radius R. The situation here is very similar to the one encountered in the theory of electrolyte solutions [14]. In that case, the canonical partition function of an ionic cluster is found to diverge and a cutoff has to be introduced to obtain finite results. The divergence is a natural consequence of the fact that at any *finite* temperature, ionic clusters are unstable and will fall apart after a sufficiently long time. On short time scales, however, the dynamics of ionic clusters is well described by a statistical theory with a cutoff. Furthermore, the thermodynamics of electrolyte solutions at low temperatures is found to be *completely* insensitive to the precise value of the cutoff used [15]. We find the same is true for the gravitational systems as well. In the infinite time limit, a gravitational cluster satisfying a virial condition is unstable and some particles will slowly evaporate. On "short" time scales, however, the cluster properties are well described by a statistical theory with a cutoff. The precise value of the cutoff is unimportant—as long as it is not too large. In our calculations, we have taken the cutoff to be at $R=10r_m$, but all the results remain visibly *unaffected* if we replace this by $5r_m$ or $100r_m$. The cutoff-Lynden-Bell distribution (cLB) [9] is then given by $\bar{f}_{\text{cLB}}(\mathbf{r},\mathbf{v})=\bar{f}(\mathbf{r},\mathbf{v})\Theta(R-r)$. It is also possible to use an energy cutoff [16], but for the purposes of the present calculation this is not necessary. We now iteratively solve the Poisson equation [Eq. (4)] with the distribution $\bar{f}_{\text{cLB}}$ subject to the conservation equations (3). Integrating the Fermi-Dirac distribution over all velocities and taking advantage of the radial symmetry of the distribution (1), Poisson's equation (4) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial \phi}{\partial r} = -16GM\pi^2\eta_1\sqrt{\frac{\pi}{2\tilde{\beta}^3}}\text{Li}_{3/2}(-e^{\tilde{\beta}[\tilde{\mu}-\phi(r)]}), \tag{5}$$

where $\tilde{\beta}=\beta m$, $\tilde{\mu}=\mu/m$, and $\text{Li}_n(x)$ is the nth polylogarithm function of x. This nonlinear equation is solved numerically with the boundary conditions $\phi'(r=0)=0$ and $\phi(r\to\infty)=0$. The gravitational potential $\phi(r)$ depends parametrically on $\tilde{\mu}$ and $\tilde{\beta}$, which are determined using the conservation equations (3). Integrating over the velocities, these become

$$-8\pi^2\eta_1\sqrt{\frac{\pi}{2}}\int_0^R\left(\frac{3}{\tilde{\beta}^{5/2}}\text{Li}_{5/2}(-e^{\tilde{\beta}[\tilde{\mu}-\phi(r)]}) + \frac{\phi(r)}{\tilde{\beta}^{3/2}}\text{Li}_{3/2}(-e^{\tilde{\beta}[\tilde{\mu}-\phi(r)]})\right)r^2 dr = \varepsilon_0,$$

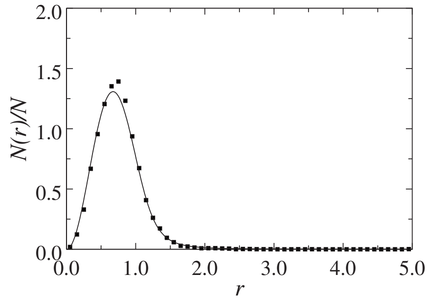

FIG. 1. Mass distribution for $\mathcal{R}=1$: solid curve is obtained using the cLB distribution and the points are the result of dynamics simulation. There is no halo; all mass is in the core.

$$16\pi^2\eta_1\sqrt{\frac{\pi}{2\tilde{\beta}^3}}\int_0^R r^2\mathrm{Li}_{3/2}(-e^{\tilde{\beta}[\tilde{\mu}-\phi(r)]})dr=1. \quad (6)$$

To compare the theory with the simulations, we calculate the number of particles inside shells located between r and $r+\mathrm{d}r$, $N(r)\mathrm{d}r=4\pi Nr^2\mathrm{d}r\int \mathrm{d}^3\mathbf{v}\bar{f}(\mathbf{r},\mathbf{v})$. In the simulations, 20 000 particles were initially distributed according to the water-bag distribution Eq. (1) and then allowed to evolve in an *infinite space* in accordance with Newton's equations of motion. To avoid the collisional effects and to speed up the simulations, the forces were calculated using the mean-field Gauss law. As discussed earlier, this procedure becomes exact in the thermodynamic limit. In Fig. 1, the solid lines are the values of $N(r)/N$ obtained using the theory above, while the points are the results of the dynamics simulation, the distances are measured in units of r_m, and the dynamical time scale is $\tau_D=\sqrt{r_m^3/GM}$. An excellent agreement is found between the theory and the simulations, *without any adjustable parameters*.

To further explore the dynamics of the relaxation process, we define the temporal deviation of the density distribution $N(r,t)$ from the stationary cLB value, $N_{\mathrm{cLB}}(r)$,

$$\chi(t)=\frac{1}{N^2}\int_0^\infty [N(r,t)-N_{\mathrm{cLB}}(r)]^2. \quad (7)$$

The inset of Fig. 2(a) shows that after a very short time interval of $\tau_R\approx\tau_D$, the original distribution f_0 quickly relaxes to the cLB form. Furthermore, the relaxation time τ_R is independent of the number of particles in the system—this is precisely Lynden-Bell's violent relaxation regime. The metastable cLB distribution persists until a finite fraction of particles evaporates from the main cluster. Following the violent relaxation, χ begins to increase again. The rate of this increase depends strongly on the number of particles in the system, Fig. 2(a). We define τ_G as the time at which the (violently) relaxed distribution begins to deviate from the cLB form by 1%, χ=0.01. This time depends on N as $\tau_G\approx 4N\tau_D$. Scaling the simulation time with τ_G gives an excellent collapse of the data on a single universal curve; see Fig. 2(b).

The fact that τ_G diverges with N implies that in the thermodynamic limit, the cLB distribution will last forever. We conclude that if the virial condition is satisfied and the macroscopic oscillations are suppressed, the phase-mixing—linear Landau damping [17]—mechanism is extremely efficient to produce a local ergodicity. For times larger than τ_G, the phase-space incompressibility condition $\rho_1(\mathbf{r},\mathbf{v})\leqslant 1$ is violated and the system undergoes a slow gravothermal collapse. We note, however, that since the time scale τ_G is larger than the Chandrasekhar time $\tau_{\mathrm{Ch}}\sim\tau_D N/\ln(N)$, the binary collisions omitted in our simulations must be explicitly taken into account to study this regime.

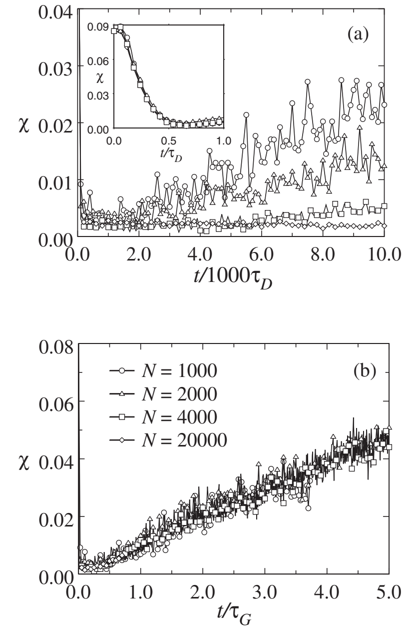

FIG. 2. (a) $\chi(t)$ for different number of particles in the system. Inset shows the violent relaxation regime, which occurs on a very short time scale τ_D and is independent of N. Following the violent relaxation, the system undergoes small oscillations that die out after about $100\tau_D$. At this time, the distribution is precisely of the cLB form. On a longer time scale τ_G, the system undergoes a gravothermal collapse. The rate of this collapse is a linear function of N. (b) When the time is scaled with τ_G, all the data in (a) collapse on one universal curve.

Small deviations from the virial condition $0.8<\mathcal{R}<1.2$ result only in weak oscillation that is not sufficient to produce significant parametric resonances. Thus, we find that for this range of virial numbers, the cLB theory remains in good agreement with the dynamics simulations. For larger deviations from $\mathcal{R}=1$, the situation changes dramatically. In these cases, the initial distribution undergoes violent oscillations resulting in a partial mass evaporation and a halo production. A fraction of the particles quickly gains enough energy from the resonances to completely escape from the main cluster (evaporate), while the other fraction gains only enough energy to move away from the core, remaining gravitationally bound to it. This latter class forms a dilute halo surrounding the dense central core. The evaporation and halo production progressively cool down the cluster until all the collective oscillations cease at $T=0$. The particles left in the core should then be in the energy ground state, with their distribution function given by that of a fully degenerate Fermi gas $\bar{f}_c(\mathbf{r},\mathbf{v})=\eta_1\Theta(\mu-\epsilon)$. This is precisely what is found when the cutoff R in the cLB distribution is extended to infinity. In this limit, the cLB distribution splits into two domains—a compact zero-temperature core described by $\bar{f}_c(\mathbf{r},\mathbf{v})$ plus an evaporated fraction of zero-energy particles at infinity. Integrating over velocities, the Poisson equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi}{\partial r}=\frac{32GM\pi^2\sqrt{2}\eta_1}{3}[\tilde{\mu}-\phi(r)]^{3/2}\Theta[\tilde{\mu}-\phi(r)], \tag{8}$$

where the value of $\tilde{\mu}$ is determined by the energy conservation,

$$-\frac{32\pi^2\sqrt{2}\eta_1}{5}\int_0^\infty r^2[\tilde{\mu}-\phi(r)]^{5/2}\Theta[\tilde{\mu}-\phi(r)]dr=\varepsilon_0. \tag{9}$$

The norm of $\bar{f}_c$ then gives the amount of mass left in the central core after the process of collisionless relaxation is completed. Figure 3 shows that the theoretically predicted $\bar{f}_c$ is in excellent agreement with the core mass distribution obtained using the dynamics simulations. However, in order to have a complete account of the halo mass distribution, a more detailed dynamical study is necessary. The work in this direction is now in progress. On a time scale larger than τ_G, the core is, once again, found to undergo a gravothermal collapse.

Traditionally the failure of Lynden-Bell's theory was attributed to the fact that for gravitational systems, relaxation occurs on a short dynamical time scale τ_D so that the system has no time to explore all of the phase space to find the most "probable" configuration. The present work, however, provides a very different picture. Strong oscillations lead to propagating density waves [18] and to parametric resonances which force some particles into statistically improbable regions of the phase space. These regions do not mix with the rest of the system. When the oscillations (parametric resonances) are suppressed, the mixing is very efficient and the predictions of the Lynden-Bell theory are verified quantitatively. Outside the virial condition, strong oscillations lead to a partial mass evaporation and a core-halo coexistence. The theory presented here gives a quantitatively accurate account of the core. It also allows us to predict the amount of mass that will be lost to the halo production and evaporation.

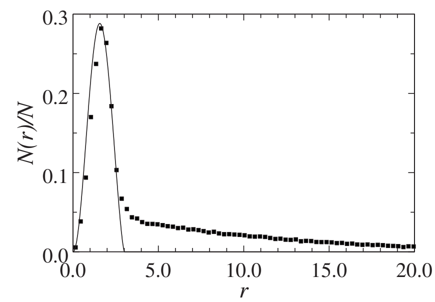

FIG. 3. Mass distribution for $\mathcal{R}=1.9$—points are the result of the simulation. Solid curve obtained using $\bar{f}_c$ gives the mass distribution inside the core. The theory predicts that 54% of the mass will be in the halo or will evaporate, which is in agreement with the simulations. There are no adjustable parameters

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[1] T. Padmanabhan, Phys. Rep. **188**, 285 (1990).
[2] J. W. Gibbs, *Collected Works* (Longmans, Green and Co., New York, 1928).
[3] Y. Levin, R. Pakter, and T. N. Teles, Phys. Rev. Lett. **100**, 040604 (2008).
[4] D. Lynden-Bell and R. M. Lynden-Bell, Mon. Not. R. Astron. Soc. **181**, 405 (1977); D. Lynden-Bell, Physica A **263**, 293 (1999).
[5] K. Michaelian and I. Santamaria-Holek, EPL **79**, 43001 (2007); D. Lynden-Bell and R. M. Lynden-Bell, *ibid.* **82**, 43001 (2008); K. Michaelian and I. Santamaria-Holek, *ibid.* **82**, 43002 (2008).
[6] W. Braun and K. Hepp, Commun. Math. Phys. **56**, 101 (1977); A. Antoniazzi, F. Califano, D. Fanelli, and S. Ruffo, Phys. Rev. Lett. **98**, 150602 (2007).
[7] D. Lynden-Bell, Mon. Not. R. Astron. Soc. **136**, 101 (1967).
[8] P.-H. Chavanis, Physica A **359**, 177 (2006).
[9] P.-H. Chavanis and J. Sommeria, Mon. Not. R. Astron. Soc. **296**, 569 (1998).
[10] A. Antoniazzi, D. Fanelli, J. Barre, P.-H. Chavanis, T. Dauxois, and S. Ruffo, Phys. Rev. E **75**, 011112 (2007).
[11] I. Arad and D. Lynden-Bell, Mon. Not. R. Astron. Soc. **361**, 385 (2005).
[12] R. L. Gluckstern, Phys. Rev. Lett. **73**, 1247 (1994); R. P. Nunes, R. Pakter, and F. B. Rizzato, Phys. Plasmas **14**, 023104 (2007).
[13] H. E. Kandrup, Ann. N.Y. Acad. Sci. **848**, 28 (1998).
[14] Y. Levin, Rep. Prog. Phys. **65**, 1577 (2002).
[15] Y. Levin and M. E. Fisher, Physica A **225**, 164 (1996).
[16] P.-H. Chavanis, Mon. Not. R. Astron. Soc. **300**, 981 (1998).
[17] D. Sagan, Am. J. Phys. **62**, 450 (1993).
[18] P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998); F. B. Rizzato, R. Pakter, and Y. Levin, Phys. Plasmas **14**, 110701 (2007).