Renormalization and phase transitions in Potts ϕ^3-field theory with quadratic and trilinear symmetry breaking

Marcia C. Barbosa, M. A. Gusmão,* and W. K. Theumann*
Instituto de Física, Universidade Federal do Rio Grande do Sul, 90049 Porto Alegre, Rio Grande do Sul, Brazil
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Renormalized perturbation theory with generalized minimal subtraction is used as an appropriate renormalization-group procedure for the study of crossover behavior in the continuum version of the p-state Potts model with quadratic and trilinear symmetry breaking, within the representation of Priest and Lubensky, by means of a two-loop-order calculation in $d=6-\epsilon$ dimensions. The boundary between first- and second-order phase transitions is studied for longitudinal and transverse ordering as a function of p. A fixed-point runaway for longitudinal ordering is consistent with a mean-field interpretation of a first-order transition for $p>p^*$, where $p^*\lesssim 2$ but not with a second-order transition for $p<p^*$. Finite and stable fixed points are obtained for transverse ordering, one that follows by crossover from the symmetric fixed point for $2<p<\frac{13}{3}$, in consistency with the usual mean-field interpretation of a second-order transition for $2<p<3$.

I. INTRODUCTION

Crossover phenomena involving first- and second-order phase transitions in the continuum version of the p-state Potts model have been considered in recent works.[1–4] Physical realizations of particular importance are the percolation problem, described by the limit $p=1$, and structural phase transitions represented by the three-state model.[5,6] It has been shown that quadratic symmetry breaking (QSB) describes the crossover from the continuous percolation threshold to the critical line in random ferromagnets,[1,3] in support of earlier scaling arguments,[7] and that linear and QSB perturbations either change the first-order phase transition in the three-state Potts model into a second-order transition at a tricritical point or terminate the first-order transition at a critical point.[2]

Recent renormalization-group (RG) works show that QSB leads to a break in trilinear symmetry.[3,4] To see the relevance of this break, it has been recently[8] pointed out that the mean-field effects of trilinear symmetry breaking (TSB) on the (three-state) Potts-model transition of uniaxially stressed $SrTiO_3$ are large enough to mask fluctuation corrections to the shift of the tricritical stress parameter in $d=4-\epsilon$ dimensions.[9] This suggests that one may have to consider the RG in $d=6-\epsilon$ dimensions to study fluctuation corrections induced by QSB in the continuum Potts model.

The aim of this paper is to obtain fluctuation corrections to mean-field theory in crossover behavior due to spin anisotropy in the p-state Potts model, which amounts to QSB in the continuum version. Rather than studying the detailed crossover for a particular state p, we are interested in the dependence of the boundary between first- and second-order transitions both on general p and for different forms of spin anisotropy. It is also our aim to obtain the effect of fluctuation corrections to two-loop order in renormalized perturbation theory (RPT), and in the course of doing this, there appear a number of interesting aspects of the theory which are worth discussing and such discussion is also presented here.

Indeed, to calculate crossover behavior due to spin anisotropy by means of RPT,[10,11] one has to obtain a finite theory for *all* values of a dimensionless *noncritical* mass which acts as an effective QSB parameter. Amit and Goldschmidt[12] showed how to absorb logarithmic mass divergences by means of dimensional regularization and generalized minimal subtraction (GMS), in an n-vector ϕ^4-field theory to one-loop order, in extension of the usual minimal subtraction of dimensional poles.[13,14] Despite logarithmic momentum-dependent terms that arise in calculating the massless diagrams for the symmetric theory in ordinary critical phenomena,[15], an exact cancellation of such terms takes place order-to-order in perturbation theory in the course of renormalization with dimensional regularization and minimal subtraction of poles, in accordance with what one would expect from an appropriate RG procedure.[11] We show that new momentum-dependent terms appear in RPT for Potts ϕ^3-field theory with quadratic and trilinear symmetry breaking in a two-loop-order calculation in $d=6-\epsilon$ dimensions. It must be shown that such terms can be correctly disposed of in RPT.

Despite the presence of instanton solutions that dominate the high-order behavior of ϕ^3-field theory for $p>1$,[16] we argue that a number of conclusions may still be drawn from a low-order RG calculation on such a theory, particularly in the case of transverse ordering, where a finite and stable fixed point can be associated with a second-order transition predicted by mean-field theory.

The outline of the paper is the following. In Sec. II the continuum-field model is introduced. In Sec. III we discuss the renormalization; first, for the massive symmetric theory in order to point out particular features of mass renormalization in ϕ^3-field theory which also appear in our calculations and, second, we show there the new logarithmic momentum-dependent terms in the diagrams to two-loop order. An exact cancellation takes place when GMS is used to calculate the renormalization functions that appear in this work. The complete list of the two-loop-order diagrams and other calculational details can be found elsewhere.[17] However, a check on the RG procedure that yields singularity-free beta functions, for longitudinal and transverse ordering is presented in Secs. IV and V, respectively, which also contain the main results of our work. In the case of longitudinal ordering, there is a fixed-point runaway, in addition to the unstable Gaussian fixed point, for all values of p. Although for large p this may be interpreted as a first-order transition [consistent with a mean-field analysis on the free energy carried out in Sec. IV, which shows that a first-order transition should be expected when $p \geq 2-\delta(g)$, with a small $\delta(g) \geq 0$, for small QSB parameter $g > 0$], the runaway for small p is inconsistent with a mean-field prediction of a second-order transition for $p < 2+\delta(g)$, indicating that a ϕ^4-field term may be needed to restore a stable and accessible fixed point. The RG calculations for transverse ordering in Sec. V reveal the presence of three finite and nontrivial fixed points in the asymptotic crossover limit, one of which can be reached through crossover from the symmetric fixed point for $2 < p < \frac{13}{3}$ with a fixed-point runaway for other p, in an extension of a mean-field calculation also shown in Sec. V that predicts a second-order transition $2 < p < 3$. A further discussion appears in Sec. VI.

II. THE MODEL AND THE CONTINUUM THEORY

The discrete p-state Potts model on a lattice, with a ferromagnetic nearest-neighbor spin-anisotropy exchange interaction that breaks the permutation-group symmetry S_{n+1}, for $p = n+1$, may be described by the Hamiltonian

$$H = -\sum_{\langle \mathbf{r},\mathbf{r}'\rangle}\sum_{\nu=1}^{p-1} J_\nu(|\mathbf{r}-\mathbf{r}'|)S_\nu(\mathbf{r})S_\nu(\mathbf{r}') , \tag{2.1}$$

for classical ν-component ($\nu = 1, \ldots, p-1$) "spins" $\mathbf{S}(\mathbf{r})$ which can be in p states given by the Potts vectors $\mathbf{a}_i$, $i = 1, 2, \ldots, p$ in $(p-1)$-dimensional space, satisfying the relations,[18]

$$\begin{aligned} &\sum_{i=1}^{p} a_i^\alpha = 0 , \\ &\sum_{\alpha=1}^{p-1} a_i^\alpha a_j^\alpha = \delta_{ij} - \frac{1}{p} , \\ &\sum_{i=1}^{p} a_i^\alpha a_i^\beta = \delta_{\alpha\beta} . \end{aligned} \tag{2.2}$$

For physical reasons, it is convenient to break the equivalence between the Potts-state vectors with a J_ν that favors either one or the remaining $(n-1)$-spin components. By means of a rotation of the $\{a_i\}$, $\mathbf{a}_1$ can be made to coincide with the single spin component as in the representation of Priest and Lubensky,[19] in which

$$a_i^\alpha = \left[\frac{p-\alpha}{p-\alpha+1}\right]^{1/2} \times \begin{cases} 0 & \text{if } i < \alpha , \\ 1 & \text{if } i = \alpha , \\ -1/(p-\alpha) & \text{if } i > \alpha . \end{cases} \tag{2.3}$$

A different way of breaking the symmetry can be implemented through the representation of Wallace and Young,[20] discussed in Ref. 21. The latter is useful for crossover in random ferromagnets,[1,3] for which a two-loop-order calculation will be reported in separate work.

The effective Hamiltonian of the continuum theory in momentum space follows in the standard way as

$$\begin{aligned} \mathcal{H} = &-\tfrac{1}{4}\int [k^2+m_0^2(1)]A_1(k)A_1(-k)+[k^2+m_0^2(2)]A_q(k)A_q(-k) \\ &+\frac{1}{3!}\kappa^{\epsilon/2}\hat{u}_1 D_{111}\int A_1(k)A_1(k')A_1(-k-k')+\frac{3}{3!}\kappa^{\epsilon/2}\hat{u}_2 D_{1qr}\int A_1(k)A_q(k')A_r(-k-k') \\ &+\frac{1}{3!}\kappa^{\epsilon/2}\hat{u}_3 D_{qrs}\int A_q(k)A_r(k')A_s(-k-k')+O(A^4) , \end{aligned} \tag{2.4}$$

in the representation of Priest and Lubensky, assuming a single longitudinal and $(p-1)$ transverse field components, $A_1(k)$ and $(A_q(k)$, respectively, with summation over repeated transverse components q, r, and s, and with integrations over all momenta in $d = 6-\epsilon$ dimensions. With an arbitrary momentum scale parameter κ, the bare dimensionless coupling constants are denoted by $\hat{u}_i$. The tensorial coefficients, which depend explicitly on the representation, are given by

$$D_{\alpha\beta\gamma} \equiv \sum_i a_i^\alpha a_i^\beta a_i^\gamma = \frac{1}{[(p-\alpha)(p-\alpha+1)]^{1/2}} \times \begin{cases} -1 & \text{if } \alpha < \beta = \gamma \\ p-\alpha-1 & \text{if } \alpha = \beta = \gamma \\ 1 & \text{otherwise} \end{cases} \tag{2.5}$$

with only three nonzero terms: $D_{111}, D_{1qq}, D_{qqq}$. The squared, $m_0^2(1)$ and $m_0^2(2)$ bare longitudinal and transverse masses are proportional to the reduced mean-field critical temperatures for longitudinal and transverse ordering, $t = (T-T_0)/T$ and $t' = (T-T_0')/T$, respectively.

The bare, one-particle irreducible, dimensionless two- and three-point vertex functions, $\Gamma^{(2)}$ and $\Gamma^{(3)}$, and the two-point vertex function with a ϕ^2 insertion,[11] $\Gamma^{(2,1)}$, expanded to two-loop order may be written as

$$\Gamma^{(2)}_{\alpha\alpha}(k;\{\hat{u}_j\},m_0(i)) = k^2+m_0^2(s) - \sum_{l,m,n}{}' B^{(0)}_{lmn}\hat{u}_1^{\,l}\hat{u}_2^{\,m}\hat{u}_3^{\,n} , \tag{2.6}$$

with $s=1$ for $\alpha=1$ and $s=2$ for $\alpha=q$ (>1), the summation being over $l+m+n=2,4$;

$$\Gamma^{(3)}_{\alpha\beta\gamma}(k_s;\{\hat{u}_j\},m_0(i))=D_{\alpha\beta\gamma}\sum_{l,m,n}{}' A^{(0)}_{lmn}\hat{u}_1^l\hat{u}_2^m\hat{u}_3^n , \qquad (2.7)$$

with the summation over $l+m+n=1,3,5$, and

$$\Gamma^{(2,1)}_{\alpha\alpha}(k,q;\{\hat{u}_j\},m_0(i))=1+\sum_{l,m,n}{}' C^{(0)}_{lmn}\hat{u}_1^l\hat{u}_2^m\hat{u}_3^n , \qquad (2.8)$$

with $l+m+n=2,4$. Moreover, k and $m_0(i)$ denote dimensionless momenta and masses, scaled with the parameter κ. Without giving the full details for the diagrams and the singular parts of the expansion coefficients in Eqs. (2.6)–(2.8), which can be found elsewhere,[17] we will discuss in the following section some of the mass-renormalized expressions.

The singular parts of the vertex functions, which are dimensional poles in ϵ, logarithmic mass divergences, and combinations of both, are taken care of by renormalization functions and coupling-constant renormalization given by[10,11]

$$\hat{u}_i=\sum_{l,m,n}{}' a^{(i)}_{lmn}u_1^l u_2^m u_3^n,\quad i=1,2,3 , \qquad (2.9)$$

$$Z^{(s)}_{\phi}(\{u_i\})=1+\sum_{l,m,n}{}' b^{(s)}_{lmn}u_1^l u_2^m u_3^n,\quad s=1,2 \qquad (2.10)$$

$$Z^{(s)}_{\phi^2}(\{u_i\})=1+\sum_{l,m,n}{}' c^{(s)}_{lmn}u_1^l u_2^m u_3^n,\quad s=1,2 \qquad (2.11)$$

with the summations over $l+m+n=1,3,5$ in the first one, and over $l+m+n=2,4$ in the other two. Because of the presence of a *noncritical* mass, which also becomes renormalized, and due to particular features of ϕ^3-field theory, we postpone a discussion concerning this point until the following section.

Renormalized two- and three-point vertex functions are then obtained, as usual, according to

$$\Gamma^{(2)}_{R\alpha\alpha}=\begin{cases} Z^1_{\phi}\Gamma^{(2)}_{\alpha\alpha}, & \alpha=1 \qquad (2.12a)\\ Z^{(2)}_{\phi}\Gamma^{(2)}_{\alpha\alpha}, & \alpha>1 , \qquad (2.12b)\end{cases}$$

$$\Gamma^{(3)}_{R\alpha\beta\gamma}=(Z^{(1)}_{\phi})^{(1/2)n_L}(Z^{(2)}_{\phi})^{(1/2)n_T}\Gamma^{(3)}_{\alpha\beta\gamma} , \qquad (2.13)$$

for n_L longitudinal and n_T transverse points, and

$$\Gamma^{(2,1)}_{R\alpha\alpha}=\begin{cases} Z^{(1)}_{\phi^2}\Gamma^{(2,1)}_{\alpha\alpha} , & \alpha=1 \qquad (2.14a)\\ Z^{(2)}_{\phi^2}\Gamma^{(2,1)}_{\alpha\alpha} , & \alpha>1 . \qquad (2.14b)\end{cases}$$

The renormalized theory has to be finite for an asymptotically large dimensionless noncritical mass[12] and since there are features of mass renormalization in ϕ^3-field theory in this limit which deserve attention, we consider these next.

III. RENORMALIZATION

The relevant points about mass renormalization that appear in our work can already be discussed in the massive symmetric theory. We do this first and then discuss the case of quadratic and TSB.

A. Massive symmetric theory

The bare effective Hamiltonian with a single mass m_0 and coupling constant u_0 leads to a bare, dimensionless, two-point vertex function to two-loop order,

$$\begin{aligned}\Gamma^{(2)}(k;u_0,m_0)=k^2+m_0^2-B_1\bar{I}(k,m_0)u_0^2\\ -[B_2^{(1)}\bar{I}_2^{(1)}(k,m_0)\\ +B_2^{(2)}\bar{I}_2^{(2)}(k,m_0)]u_0^4 ,\end{aligned} \qquad (3.1)$$

in which $B_1=\frac{1}{2}\alpha_1$, $B_2^{(1)}=\frac{1}{2}\alpha_1^2$, and $B_2^{(2)}=\frac{1}{2}\alpha_1\beta_1$ contain the now standard tensorial coefficients α_1 and β_1 of the one-loop and the two topologically different two-loop diagrams that can be found in the literature for the massless symmetric theory.[15,22] In contrast to the latter,[15] which is made finite by means of minimal subtraction of dimensional poles, it is now necessary to reduce first the degree of (quadratic) divergence of Eq. (3.1), without resorting to "partial p,"[14] in order to be able to apply the GMS procedure which takes care of only logarithmic mass divergences. Reduction of the degree of divergence is achieved, as usual, by means of an intermediate mass renormalization in which[23]

$$m_1^2=\Gamma^{(2)}(k=0;u_0,m_0) . \qquad (3.2)$$

Equation (3.1) then becomes

$$\Gamma^{(2)}(k;u_0,m_1)=m_1^2+k^2\left[1-B_1I(k,m_1)u_0^2-\left[B_2^{(1)}I_2^{(1)}(k,m_1)+B_2^{(2)}I_2^{(2)}(k,m_1)+B_1^2\bar{I}(0,m_1)\frac{\partial}{\partial m_1^2}I(k,m_1)\right]u_0^4\right] , \qquad (3.3)$$

where the expressions for the singular parts, in the small-ϵ and large-m_1 limit,

$$I(k,m_1)\equiv k^{-2}[\bar{I}(k,m_1)-\bar{I}(0,m_1)]$$

$$\cong -\frac{1}{3\epsilon}(1-\tfrac{1}{2}\epsilon M_1) , \qquad (3.4)$$

$$I_2^{(1)}(k,m_1)+\tfrac{1}{2}\bar{I}(0,m_1)\frac{\partial}{\partial m_1^2}I(k,m_1)$$

$$\cong\frac{1}{18\epsilon^2}(1-\tfrac{7}{12}\epsilon-\epsilon M_1+\tfrac{7}{12}\epsilon^2M_1+\tfrac{1}{2}\epsilon^2M_1^2) , \qquad (3.5)$$

$$I_2^{(2)}(k,m_1)\cong-\frac{1}{3\epsilon^2}(1-\tfrac{7}{6}\epsilon-\epsilon M_1+\tfrac{7}{6}\epsilon^2M_1+\tfrac{1}{2}\epsilon^2M_1^2) , \qquad (3.6)$$

in which

$$I_2^{(i)}(k,m_1) \equiv k^{-2}[\bar{I}_2^{(i)}(k,m_1) - \bar{I}_2^{(i)}(0,m_1)], \quad i=1,2 \tag{3.7}$$

contain only logarithmic mass divergences through[24]

$$M_1 \equiv \ln(1+m_1^2) \,. \tag{3.8}$$

In distinction to the massless theory, one must now keep terms with $m_1 \neq 0$, even when $k=0$. Indeed, these terms become singular in the limit $m_1 \to \infty$ (also with the final mass renormalization below); for instance, $\bar{I}(0,m) \propto (m^2)^{1-\epsilon/2}$ which will be needed later. Note also that mass renormalization yields the mass-derivative term in Eq. (3.3) which is *not* canceled by a mass insertion diagram, as in the case of ϕ^4-field theory.[11]

With coupling constant, field, and final mass renormalization introduced by means of

$$u_0 = u(1+a_1u^2+a_2u^4) \,, \tag{3.9}$$

$$Z_\phi(u) = 1+b_1u^2+b_2u^4 \,, \tag{3.10}$$

$$m^2 = (1+b_1u^2+b_2u^4)m_1^2 \,, \tag{3.11}$$

the renormalized two-point vertex function,

$$\Gamma_R^{(2)}(k;u,m) = Z_\phi(u)\Gamma^{(2)}(k;u_0(u),m_1(u,m)) \,, \tag{3.12}$$

requires that the coefficients that absorb the singular parts in GMS be given by

$$b_1 = B_1[I(k,m)]_{\text{sing}} \,, \tag{3.13}$$

$$b_2 = \Big[(b_1+2a_1)B_1I(k,m) + B_2^{(1)}I_2^{(1)}(k,m) + B_2^{(2)}I_2^{(2)}(k,m) + B_1[B_1\bar{I}(0,m) - b_1m^2]\frac{\partial}{\partial m^2}I(k,m) \Big]_{\text{sing}} \,, \tag{3.14}$$

with a new mass-derivative term that also behaves as the right-hand side of Eq. (3.8), in the large-mass limit. In taking the singular parts, the coefficients in Z_ϕ become momentum independent, as they should. The fact that the mass-derivative term does not vanish shows that mass renormalization in the two-point vertex function is *not* fully accounted for by subtraction at zero momentum, to two-loop order. This should still be true to all-loop order.

Consider next the dimensionless three-point vertex function, to two loop, given by

$$\Gamma_{\alpha\beta\gamma}^{(3)}(\{k_j\};u_0,m_0) = D_{\alpha\beta\gamma}u_0 \left[1 + G_1L(\{k_j\},m_0)u_0^4 + \sum_{i=1}^{3} G_2^{(i)}L_2^{(i)}(\{k_j\},m_0)u_0^4 \right] , \tag{3.15}$$

in which $G_1=\beta_1$, $G_2^{(1)}=3\beta_1$, $G_2^{(2)}=\frac{3}{2}\alpha_1\beta_1$, and $L, L_2^{(i)}$ are the one- and two-loop diagrams that appear in the literature for the massless symmetric theory, here taken with a finite mass.

Full mass renormalization is now needed only to next-to-leading order in the form

$$m_0^2 = m^2 - [b_1m^2 - B_1\bar{I}(0,m)]u^2 \,. \tag{3.16}$$

Renormalization of the three-point vertex function by means of GMS then yields

$$-a_1 = G_1[L(\{k_j\},m)]_{\text{sing}} \,, \tag{3.17}$$

$$-a_2 = \Big[\tfrac{3}{8}b_1^2 + \tfrac{3}{2}b_2 + \tfrac{3}{2}b_1[a_1+G_1L(\{k_j\},m)] + 3a_1G_1L(\{k_j\},m) + \sum_{i=1}^{3} G_2^{(i)}L_2^{(i)}(\{k_j\},m) + G_1[B_1\bar{I}(0,m) - b_1m^2]\frac{\partial}{\partial m^2}L(\{k_j\},m) \Big]_{\text{sing}} \,, \tag{3.18}$$

again, with a contribution coming from the last term, and momentum-independent coefficients are obtained in taking the singular parts. The details, given by a lengthy expression for a_2, need not be presented here. Similarly, the coefficients in the renormalization function, Z_{ϕ^2}, for the two-point vertex with a ϕ^2 insertion, are given by explicit momentum-independent expressions that we omit here. The two-loop order coefficient has also a mass-derivative contribution similar to the one in Eq. (3.18). In the three vertex functions discussed so far, it is easy to check that the known results for the coefficients of the massless symmetric theory are obtained in the limit $m=0$, as one would expect from GMS.[15]

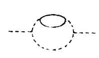
B. Quadratic and trilinear symmetry breaking

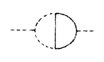
In order to obtain results applicable to crossover behavior when either the longitudinal or the transverse components of the fields become critical, we consider the renormalization with either a noncritical transverse or longitudinal mass, respectively.

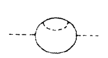
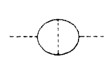
When the singular parts of the diagrams for the theory with quadratic and TSB introduced in Sec. II are calculated following the mass-renormalization scheme discussed above, we find new interesting features that are worth pointing out.

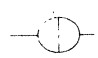
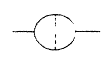
With quadratic and TSB, there are a number of diagrams which are either momentum independent or have the momentum dependence of the symmetric theory. All these diagrams, of which there are plenty, contribute to the calculation of the renormalization functions and will not be presented here for the sake of brevity. There are other diagrams, however, which have *new* momentum-dependent terms and we exhibit these in Tables I and II. These are diagrams for the coefficients in Eqs. (2.6)–(2.8), with mass renormalization which are now written as A_{lmn}^i, B_{lmn}^i, where $i=L$ and T for longitudinal and transverse ordering and C_{lmn} only for the latter. Dotted and

TABLE I. Mass-renormalized coefficients of the two-point vertex functions that contain new momentum-dependent terms. The diagrams are described in the text. The superscripts L and T denote longitudinal and transverse ordering. When a given coefficient is shown for both, the first line corresponds to longitudinal ordering; $c^2 \equiv [p(p-1)]^{-1}$ and $M \equiv \ln(1+m^2)$. The coefficients of the following diagrams when combined with the integrals for the three-point vertex functions shown in Table II, with the appropriate symmetry factors, yield C_{lmn} for the two-point vertex with a ϕ^2 insertion.

$B^{L}_{220} =$  $= \frac{1}{2\epsilon^2} c^4 (p-2)^3 I^{(1)}_{22}(M)$

$B^{L,T}_{130} =$  $= \frac{-1}{\epsilon^2} c^4 (p-2)^2 \times \begin{cases} I^{(2)}_{22}(M) \\ I^{(2)}_{22} - \frac{1}{24}\epsilon^2 M \end{cases}$

$B^{T}_{040} =$  $+$  $= \frac{1}{\epsilon^2} c^4 (p-2)[I^{(1)}_{22}(M) + \frac{1}{2} I^{(2)}_{24}(M) + \frac{5}{216}\epsilon^2 M]$

$B'^{T}_{022} =$  $+$  $= \frac{1}{\epsilon^2} c^4 p(p-3) + [2 I^{(2)}_{22}(M) + \frac{1}{2} I^{(2)}_{24}(M) - \frac{5}{18}\epsilon + \frac{1}{36}\epsilon^2 M]$

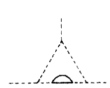
$$I^{(1)}_{22}(M) = \frac{1}{18}(1 + \frac{25}{12}\epsilon - \epsilon \ln k^2 - \frac{5}{4}\epsilon^2 M - \frac{1}{4}\epsilon^2 M^2 + \frac{1}{2}\epsilon^2 M \ln k^2)$$
$$I^{(2)}_{22}(M) = -\frac{1}{3}(1 + \frac{1}{6}\epsilon - \frac{1}{2}\epsilon \ln k^2 - \frac{1}{2}\epsilon M + \frac{1}{8}\epsilon^2 M^2 + \frac{1}{4}\epsilon^2 M \ln k^2)$$
$$I^{(2)}_{24}(M) = -\frac{1}{3}(1 + \frac{3}{2}\epsilon - \epsilon \ln k^2 - \frac{5}{12}\epsilon^2 M - \frac{1}{4}\epsilon^2 M^2 + \frac{1}{2}\epsilon^2 M \ln k^2)$$

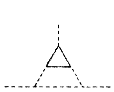
solid lines indicate longitudinal and transverse propagators, respectively, each of which may be massless or carry a noncritical mass, depending on the kind of ordering. For transverse ordering, for instance, the longitudinal propagators are massive and the transverse propagators are massless.

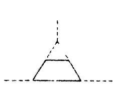
When the RG procedure outlined above for the symmetric theory is extended to allow for quadratic and TSB, we found an exact cancellation between all the momentum-dependent terms that come from the diagrams: those that are already present in the symmetric theory and the new diagrams discussed here.

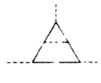
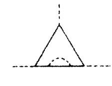
In addition to the cancellation of momentum-dependent terms, there are two further checks on the RG procedure that were carried out in this work. First, summation of all the coefficients in Eqs. (2.9)–(2.11) to a given order in a specific renormalization function for either longitudinal or transverse ordering yields, as it must, the known results

TABLE II. Some of the mass-renormalized coefficients of the three-point vertex functions that contain new momentum-dependent terms. There are other coefficients, not shown here, that do not involve new diagrams.

$A^{L}_{320} =$  $= \frac{3}{2\epsilon^2} c^4 (p-2)^3 L^{(2)}_{22}(M)$

$A^{L}_{230} =$  $= -\frac{3}{\epsilon^2} c^4 (p-2)^2 L^{(1)}_{22}(M)$

$A'^{T}_{140} =$  $= \frac{3}{\epsilon^2} c^4 (p-2)[L^{(1)}_{22}(M) + \frac{1}{8}\epsilon^2 M]$

$A^{T}_{050} =$  $+$  $= -\frac{3}{\epsilon^2} c^4 [L^{(1)}_{22}(M) + L^{(2)}_{22}(M) + \frac{11}{36}\epsilon^2 M]$

$$L^{(1)}_{22}(M) = \frac{1}{2}(1 - \frac{5}{4}\epsilon - 2\epsilon \hat{L} - \frac{1}{4}\epsilon^2 M - \frac{1}{4}\epsilon^2 M^2 + \epsilon^2 M \hat{L})$$
$$L^{(2)}_{22}(M) = -\frac{1}{6}(1 - \frac{11}{12}\epsilon - 2\epsilon \hat{L} - \frac{1}{12}\epsilon^2 M - \frac{1}{4}\epsilon^2 M^2 + \epsilon^2 M \hat{L})$$
$$\hat{L} \equiv \hat{L}(k_1,k_2) = \int_0^1 dx \int_0^{1-x} dy \ln[x(1-x)k_1^2 + y(1-y)k_2^2 + 2xyk_1k_2]$$

for the massless symmetric theory in the limit $m \to 0$.[15] Second, the calculated Wilson β functions turn out to be free of singularities, as they should. This is shown in the following section for longitudinal ordering and in Sec. V for transverse ordering.

IV. FIXED POINTS AND RUNAWAY FOR LONGITUDINAL ORDERING

The effective crossover to asymptotic critical behavior, if any, may be obtained by studying the dependence on the scaled mass m/ρ, ρ being the vanishingly small flow parameter, of the roots of the Wilson β functions, defined as[10,11]

$$\beta_s \left(\kappa \frac{\partial u_s}{\partial \kappa} \right)_{\{\lambda_i\}}, \quad s=1,2,3 \tag{4.1}$$

with fixed bare dimensional couplings λ_i, $i=1,2,3$. The results for the singularity-free β functions, to two-loop order, are given for longitudinal ordering by

$$\begin{aligned}\beta_1 = -\frac{\epsilon}{2} \Bigg[& u_1 + \frac{3}{2\epsilon}(p-2)^2 c^2 u_1^3 - \frac{1}{2\epsilon}(p-2)c^2 \frac{u_1 u_2^2}{1+m^2} - \frac{2}{\epsilon} c^2 \frac{u_2^3}{1+m^2} + \frac{125}{72\epsilon}(p-2)^4 c^4 u_1^5 \\ & + \frac{1}{6\epsilon}(p-2)^3 c^4 \left[\frac{7}{4} - \frac{13}{3(1+m^2)} - \frac{5}{2}\frac{M}{1+m^2} \right] u_1^3 u_2^2 \\ & + \frac{1}{2\epsilon}(p-2)^2 c^4 \left[\frac{51}{6} - \frac{61}{6(1+m^2)} - 5\frac{M}{1+m^2} \right] u_1^2 u_2^3 + \frac{161}{36\epsilon}(p-2)c^4 \frac{u_1 u_2^4}{1+m^2} \\ & - \frac{13}{72\epsilon} p(p-2)(p-3)c^4 \frac{u_1 u_2^2 u_3^2}{1+m^2} - \frac{1}{3\epsilon}c^4 \frac{u_2^5}{1+m^2} - \frac{23}{12\epsilon} p(p-3)c^4 \frac{u_2^3 u_3^2}{1+m^2} \Bigg] ,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\beta_2 = -\frac{\epsilon}{2} \Bigg\{ & u_2 - \frac{1}{6\epsilon}(p-2)^2 c^2 u_1^2 u_2 + \frac{2}{3\epsilon} c^2 [2 - \tfrac{1}{4}(p-2)] \frac{u_2^3}{1+m^2} + \frac{5}{3\epsilon} p(p-3)c^2 \frac{u_2 u_3^2}{1+m^2} - \frac{2}{\epsilon}(p-2)c^2 \frac{u_1 u_2^2}{1+m^2} \\ & - \frac{13}{216\epsilon}(p-2)^4 c^4 u_1^4 u_2 - \frac{1}{9\epsilon}(p-2)^3 c^4 (1-6M)\frac{1}{1+m^2} u_1^3 u_2^2 \\ & + \frac{1}{9\epsilon}(p-2)^2 c^4 \left[-\frac{7}{6} + \frac{82}{3}\frac{1}{1+m^2} + \frac{M}{1+m^2} + (p-2)\left[\frac{1}{8} + \frac{1}{3(1+m^2)} + \frac{1}{4}\frac{M}{1+m^2} \right] \right] u_1^2 u_2^3 \\ & - \frac{1}{3\epsilon}(p-2)c^4 \left[5\frac{1}{1+m^2} + (p-2)\left[\frac{1}{4} - \frac{25}{12}\frac{1}{1+m^2} - \frac{1}{2}\frac{M}{1+m^2} \right] \right] u_1 u_2^4 \\ & - \frac{83}{36\epsilon} p(p-2)(p-3)c^4 \frac{1}{1+m^2} u_1 u_2^2 u_3^2 + \frac{1}{27\epsilon} c^4 [67 + \tfrac{43}{4}(p-2)] \frac{u_2^5}{1+m^2} \\ & + \frac{1}{36\epsilon} p(p-3)c^4 [239 - \tfrac{13}{6}(p-2)] \frac{u_2^3 u_3^2}{1+m^2} + \frac{2}{9\epsilon} p^2 (p-3) c^4 [8(p-4) + \tfrac{1}{12}(p-3)] \frac{u_2 u_3^4}{1+m^2} \Bigg\} ,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\beta_3 = -\frac{\epsilon}{2} \Bigg[& u_3 + \frac{5}{\epsilon} c^2 \frac{u_2^2 u_3}{1+m^2} - \frac{2}{\epsilon} p c^2 [\tfrac{1}{4}(p-3) - (p-4)] \frac{u_3^3}{1+m^2} \\ & + \frac{1}{3\epsilon}(p-2)^2 c^4 \left[-\frac{4}{3} + \frac{1}{24(1+m^2)} + \frac{5}{4}\frac{M}{1+m^2} \right] u_1^2 u_2^2 u_3 - \frac{13}{3\epsilon}(p-2)c^4 \frac{u_1 u_2^3 u_3}{1+m^2} \\ & + \frac{1}{36\epsilon} c^4 [389 - \tfrac{31}{2}(p-2)] \frac{u_2^4 u_3}{1+m^2} + \frac{5}{24\epsilon} p c^4 (55p - 217) \frac{u_2^2 u_3^3}{1+m^2} \\ & + \frac{1}{72\epsilon} c^4 p^2 (125p^2 - 1044p + 2259) \frac{u_3^5}{1+m^2} \Bigg] ,\end{aligned} \tag{4.4}$$

in which $c^2 \equiv [p(p-1)]^{-1}$ and m is the transverse renormalized mass and

$$M \equiv \ln(1+m^2) . \quad (4.5)$$

The terms to one-loop order are those of previous work,[4(b)] while the new two-loop-order terms are either mass independent or have a nonsingular mass dependence, both in the limits $m^2=0$ and $m^2 \to \infty$. With $u_s = O(\epsilon^{1/2})$ for $s=1,2,3$, the low-order β functions are regular both in ϵ and in m^2, as one would expect.[11] The cancellation of singularities that appear in intermediate steps discussed in the preceding section thus provides a strong test of renormalization with GMS, to two-loop order, and to our knowledge, this is the first time that a result concerning this point is obtained. Other finite Wilson functions, which will not be discussed here, may also be derived in the same way.

A further result of the two-loop-order calculation reported here is that the explicit m^2 dependence of the terms in Eqs. (4.2)–(4.4) is of the form: $\text{const}+f(M)/(1+m^2)$, in which $f(M)$ is either a constant or a constant plus a linear term in M, a feature already present to one-loop order. From the way in which this result is derived, we expect a similar mass dependence in higher-loop-order terms, except for positive powers of M in $f(M)$. This is relevant for the asymptotic behavior of the fixed-point structure discussed next.

The symmetric fixed point

$$(u_s^*)^2 = \frac{2p}{10-3p}\epsilon \left[1 + \frac{125p^2-794p+1340}{18(10-3p)^2}\epsilon \right] , \quad s=1,2,3 , \quad (4.6)$$

is recovered in the limit $m^2=0$ from the solutions to the equations $\beta_s=0$, $s=1,2,3$, together with the trivial one $u_1^*=u_2^*=u_3^*=0$. For any large, but finite m^2, there are three sets of roots for the β functions: (i) $u_1^* \neq 0$, $u_2^* \neq 0$, and $u_3^* \neq 0$; (ii) $u_1^* \neq 0$, $u_2^* \neq 0$, and $u_3^*=0$; (iii) $u_1^*=u_2^*=0$ with $u_3^* \neq 0$. These appear already to one-loop order,[4(b)] but the appropriate crossover behavior with asymptotically large m^2 has not been discussed before. Indeed, there is a fixed-point runaway in this limit, in addition to the trivial fixed point. This can be seen either directly, by studying the flow of the coupling constants which follows by solving numerically the equations

$$\rho \frac{du_s(\rho)}{d\rho} = \beta_s(\{u_i(\rho)\}, m/\rho), \quad s=1,2,3 \quad (4.7)$$

in which the flow parameter ρ starts with $\rho=1$ and becomes vanishingly small in the asymptotic critical region. The solution of Eqs. (4.7) and the fixed-point runaway for $p=3$ is shown in Fig. 1.

Alternatively, from inspecting the β functions, it follows that terms involving $(u_s^*)^2/(1+m^2)$, or with two different u_s^*, vanish in the limit of large m^2 if u_s^* remains finite (or zero) or does not grow as fast as $(1+m^2)^{1/2}$. If this is the case, the trivial fixed point u_s^*, $s=1,2,3$ is the only solution of the equations $\beta_s=0$. Otherwise, for $(u_s^*)^2/(1+m^2)$ to remain finite there must be a fixed-point runaway in $u_s^*(m^2)$ which is also a solution of $\beta_s=0$. It is this solution which is the outcome of the coupling-constant flow.

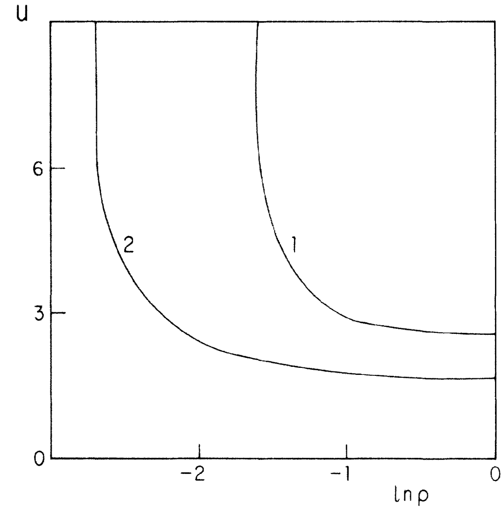


FIG. 1. Coupling-constant flow of $u \equiv u_1$ for longitudinal ordering when $p=3$, showing a runaway at asymptotic crossover for $\rho \to 0$, with $m^2=10^{-2}$, $\kappa=1$, and $\epsilon=1$ (curve 1) and $\epsilon=0.5$ (curve 2).

The fixed-point runaway appears for all p. Although normally this would mean a first-order phase transition, one should be cautious with a pure ϕ^3-field-theory calculation like that of the present work. Indeed, a mean-field analysis indicates that one may expect a second-order transition for a range of $p<2-\delta(g)$, where $\delta(g) \geq 0$ for $g \geq 0$, as will be shown next.

Mean-field analysis. The analysis based on the mean-field (MF) Hamiltonian[19,25] follows from Eq. (2.4) keeping only uniform, momentum-independent fields A, and we include now explicit quartic terms: (i) the rotationally invariant $(\sum_\alpha A_\alpha^2)^2$ and (ii) $E_{\alpha\beta\gamma\delta}A_\alpha A_\beta A_\gamma A_\delta$ in which $E_{\alpha\beta\gamma\delta}=a_i^\alpha a_i^\beta a_i^\gamma a_i^\delta$, both with summation over repeated indices[4(a),19] that correspond to the cubic terms in the diagonal traceless tensors Q_{ii} of Priest and Lubensky. We denote now with u_i, $i=1,2,3$, the dimensional couplings of the trilinear terms, factors of 1/3! being absorbed, and $i=4,5$ for the quartic terms. In order to obtain relationships which also serve for transverse ordering, as well as for longitudinal, assuming both to be uniaxial, we write

$$A_1 = Q_1 + \mathscr{L}_1, \quad A_2 = Q_2 + \mathscr{L}_2 , \quad A_q = \mathscr{L}_q, \quad q \geq 2 , \quad (4.8)$$

in which $Q_1 \equiv \langle A_1 \rangle$ and $Q_2 \equiv \langle A_2 \rangle$ are the longitudinal and transverse order parameters, respectively, and $\mathscr{L}_\alpha$ are the fluctuating parts. In what follows, $q \geq 2$:

$$\mathscr{H}_{\rm MF} = H_{\rm MF1} + \mathscr{H}_{\rm MF2} + \mathscr{H}_{\rm MF12} , \quad (4.9)$$

$$\mathscr{H}_{\rm MF1} = -\tfrac{1}{4} r_L Q_1^2 + u_1 D_{111} Q_1^3 - (u_4 + E_{1111} u_5) Q_1^4 , \quad (4.10a)$$

$$\mathscr{H}_{\rm MF2} = -\tfrac{1}{4} r_T Q_2^2 + u_3 D_{222} Q_2^3 - (u_4 + E_{2222} u_5) Q_2^4 , \quad (4.10b)$$

$$\mathscr{H}_{\rm MF12} = 3u_2 D_{122} Q_1 Q_2^2 - 2(u_4 + 3E_{1122} u_5) Q_1^2 Q_2^2 , \quad (4.10c)$$

in which $r_L \equiv m_0^2(1) = r - g$ and $r_T \equiv m_0^2(2) = r + g$, where $r \equiv m_0^2$ of the symmetric theory and g is the QSB parameter. The order parameters Q_1 and Q_2 follow then as minima of $\mathscr{H}_{MF}$ given by

$$\frac{\partial \mathscr{H}_{MF}}{\partial Q_1} = 0, \quad \frac{\partial \mathscr{H}_{MF}}{\partial Q_2} = 0 , \tag{4.11}$$

together with the positive coefficients of the quadratic terms in the fluctuating parts,

$$\hat{r}_L = r_L - 12u_1 D_{111} Q_1 + 24(u_4 + E_{1111} u_5) Q_1^2 + 8(u_4 + 3E_{1122} u_5) Q_2^2 , \tag{4.12}$$

$$\hat{r}_T = r_T - 12u_2 D_{122} Q_1 - 12u_3 D_{222} Q_2 + 8(u_4 + 3E_{1122} u_5) Q_1^2 + 24(u_4 + E_{2222} u_5) Q_2^2 , \tag{4.13}$$

$$\hat{r}'_T = r_T - 12u_2 D_{1qq} Q_1 - 12u_3 D_{2qq} Q_2 + 8(u_4 + 3E_{11qq} u_5) Q_1^2 + 8(u_4 + 3E_{22qq} u_5) Q_2^2 , \tag{4.14}$$

where $\hat{r}_L$ corresponding to $\mathscr{L}_1^2$, $\hat{r}_T$ to $\mathscr{L}_2^2$, and $\hat{r}'_T$ to $\mathscr{L}_q^2$ are inverse susceptibilities that serve to establish the stability of the small and large order-parameter solutions that follow from Eqs. (4.11).

For longitudinal ordering, the case we are interested in this section, $r_L < r_T$ and the ordered region is reached when $r_L < 0$. In the representation of Priest and Lubensky, longitudinal ordering means $Q_1 \neq 0$ and $Q_2 = 0$. Although one may also have a nonzero Q_1 when $Q_2 \neq 0$, this should be recognized as transverse ordering since there is no stable solution with $Q_2 \neq 0$ and $Q_1 = 0$.

To explore the possibility of a second-order transition, we look for the small order-parameter solution and find, in standard way,

$$\hat{r}_L \cong -r_L , \tag{4.15a}$$

$$\hat{r}_T = 2g + \frac{r_L}{p-2}\left[p - 2 + 2\frac{u_2}{u_1} \right] . \tag{4.15b}$$

Since $r_L < 0$, the first one yields $\hat{r}_L > 0$ for all p, while the second equation with $u_1 \lesssim u_2$ for small g, yields a positive $\hat{r}_T$ only for $p \leq 2 + \delta_1(g)$, with a small $\delta_1 \geq 0$ depending on g (≥ 0). As will be seen below, however, the actual range of values of p for which a second-order phase transition is to be expected is $p \leq 2 - \delta(g)$, $\delta \geq 0$, because of a competing first-order transition. Thus, with neglect of fluctuation corrections, one may expect a second-order transition with longitudinal ordering for small p. If the RG calculation described above could be trusted (a point that requires further investigation, including a ϕ^4 field), one would then conclude that the effect of fluctuations is to change the nature of the phase transition for low values of p. However, at present, we cannot reach a definite conclusion with regard to this point.

On the other hand, analysis of the large order-parameter solution that determines a first-order transition making use of $\mathscr{H}_{MF}(Q_1 = 0) = \mathscr{H}_{MF}(Q_1 \neq 0)$—the condition for the two free-energy minima being of equal depth—yields

$$\hat{r}_L = \frac{(p-2)^2 c^2 u_1^2}{u_4 + E_{1111} u_5}, \quad Q_c = \frac{\hat{r}_L}{2u_1(p-2)c} , \tag{4.16}$$

with $E_{1111} = (p^2 - 3p + 3)c^2$, where Q_c is the discontinuity of the order parameter Q_1, and

$$\frac{2g + (p-2)c^2[(p-2)u_1^2 + 6u_1 u_2]}{u_4 + E_{1111} u_5} > 0 \tag{4.17}$$

is sufficient for $\hat{r}_T > 0$. This can be satisfied for $p > 2 - \delta(g)$, $\delta(g) \geq 0$ for $g \geq 0$ which limits the range of p values for a second-order transition, referred to above. One may thus conclude that for $p > 2 - \delta(g)$, the RG calculation on the pure ϕ^3-field theory is in agreement with the mean-field expectation.

V. FIXED POINTS AND CROSSOVER FOR TRANSVERSE ORDERING

The singularity-free Wilson β functions for transverse ordering are given by

$$\begin{aligned}
\beta_1 = -\frac{\epsilon}{2}\Bigg[& u_1 + \frac{3}{2\epsilon}(p-2)^2 c^2 \frac{u_1^3}{1+m^2} - \frac{1}{2\epsilon}(p-2)c^2 u_1 u_2^2 - \frac{2}{\epsilon}c^2 u_2^3 + \frac{125}{72\epsilon}(p-2)^4 c^4 \frac{u_1^5}{1+m^2} \\
& - \frac{1}{4\epsilon}(p-2)^3 c^4 \left[\frac{5}{2} - \frac{7}{9(1+m^2)} - \frac{5M}{3(1+m^2)} \right] u_1^3 u_2^2 \\
& - \frac{1}{4\epsilon}(p-2)^2 c^4 \left[15 - \frac{35}{3(1+m^2)} - 12\frac{M}{1+m^2} \right] u_1^2 u_2^3 \\
& - \frac{1}{3\epsilon}(p-2)c^4 \left[\frac{59}{6} - \frac{93}{4(1+m^2)} - 8\frac{M}{1+m^2} \right] u_1 u_2^4 - \frac{13}{72\epsilon}p(p-2)(p-3)c^4 u_1 u_2^2 u_3^2 \\
& + \frac{1}{3\epsilon}(p-2)c^4 u_2^5 \left[1 - \frac{2}{(m^2+1)} - \frac{2M}{(m^2+1)} \right] - \frac{23}{12\epsilon}p(p-3)c^4 u_2^3 u_3^2 \Bigg] ,
\end{aligned} \tag{5.1}$$

$$\beta_2=-\frac{\epsilon}{2}\left\{u_2-\frac{1}{6\epsilon}(p-2)^2c^2\frac{u_1^2u_2}{1+m^2}+\frac{c^2}{3\epsilon}\left[\frac{4}{1+m^2}-\frac{(p-2)}{2}\right]u_2^3+\frac{5}{3\epsilon}p(p-3)c^2u_2u_3^2-\frac{2}{\epsilon}(p-2)c^2\frac{u_1u_2^2}{1+m^2}\right.$$
$$-\frac{13}{216\epsilon}(p-2)^4c^4\frac{u_1^4u_2}{1+m^2}-\frac{1}{9\epsilon}(p-2)^3c^4\frac{u_1^3u_2^2}{1+m^2}$$
$$+\frac{1}{54\epsilon}(p-2)^2c^4\left[157+(p-2)\left[\frac{9}{4}+\frac{1}{2(1+m^2)}-\frac{3}{2}\frac{M}{1+m^2}\right]\right]u_1^2u_2^3$$
$$-\frac{1}{3\epsilon}(p-2)c^4\left[5-(p-2)\left[\frac{7}{4}+\frac{1}{12(1+m^2)}-\frac{M}{1+m^2}\right]\right]u_1u_2^4$$
$$-\frac{1}{6\epsilon}p(p-2)(p-3)c^4\left[5+\frac{53}{6(1+m^2)}-\frac{5M}{1+m^2}\right]u_1u_2^2u_3^2$$
$$+\frac{1}{27\epsilon}c^4\left[67+(p-2)\left[\frac{51}{2}-\frac{59}{4(1+m^2)}-\frac{27M}{1+m^2}\right]\right]u_2^5$$
$$-\frac{1}{\epsilon}\frac{p(p-3)}{216}c^4\left[13(p-2)+344-\frac{1778}{1+m^2}-\frac{660M}{1+m^2}\right]u_2^3u_3^2$$
$$\left.+\frac{1}{\epsilon}p^2(p-3)c^4[8(p-4)+\tfrac{1}{12}(p-3)]u_2u_3^4\right\}, \tag{5.2}$$

$$\beta_3=-\frac{\epsilon}{2}\left\{u_3+\frac{5}{\epsilon}c^2\frac{u_2^2u_3}{1+m^2}-\frac{1}{2\epsilon}pc^2(13-3p)u_3^3-\frac{31}{72\epsilon}c^4(p-2)^2\frac{u_1^2u_2^2u_3}{1+m^2}-\frac{13}{3\epsilon}(p-2)c^4\frac{u_1u_2^3u_3}{1+m^2}\right.$$
$$+\frac{1}{36\epsilon}c^4\left[389+(p-2)\left[2-\frac{35}{2(1+m^2)}+15\frac{M}{1+m^2}\right]\right]u_2^4u_3$$
$$-\frac{1}{\epsilon}pc^4\left[(p-4)\left[\frac{1}{3}-\frac{67}{6(1+m^2)}-\frac{5M}{1+m^2}\right]\right.$$
$$\left.+(p-3)\left[\frac{34}{18}-\frac{181}{72(1+m^2)}+\frac{65M}{12(1+m^2)}\right]\right]u_2^2u_3^3$$
$$\left.+\frac{1}{72\epsilon}c^4p^2(125p^2-1044p+2259)u_3^5\right\}. \tag{5.3}$$

These β functions have not been calculated before to even one-loop order. It can be seen that they are nonsingular, both in ϵ and in m^2, in the limits $m^2=0$ and $m\to\infty$. Here again, the cancellation of singularities that appear in intermediate steps of the calculation is a check on renormalization with GMS, to two-loop order. Also, for the m^2 dependence of the higher-loop order terms, we expect the same behavior as that for the longitudinal ordering pointed out above.

In addition to the symmetric fixed point (FP) for $m^2=0$ and to the Gaussian fixed point, there are now three further nontrivial stable fixed points in the limit $m^2\to\infty$. Indeed, solving Eqs. (5.1)–(5.3) to one-loop order, it turns out that in this limit,

$$u_1^*=\mp\frac{(p+9)}{(p-1)(p-2)}\left[\frac{2p(p-1)(p+9)}{(p-2)(13-3p)}\epsilon\right]^{1/2}, \tag{5.4a}$$

$$u_2^*=\pm\left[\frac{2p(p-1)(p+9)}{(p-2)(13-3p)}\epsilon\right]^{1/2}, \tag{5.4b}$$

$$u_3^*=\pm\left[\frac{2(p-1)}{13-3p}\epsilon\right]^{1/2}, \tag{5.4c}$$

for one of the fixed points, with a runaway only when p is outside the range $2<p<\frac{13}{3}$.[26] The stability of the solution given by Eq. (5.4) follows from the positivity of the eigenvalues λ_i of the matrix with elements

$$\Lambda_{s\sigma}=\left[\frac{\partial\beta_s}{\partial\sigma}\right]_{\rm FP}, \quad \sigma=u_1,u_2,u_3, \tag{5.5}$$

given by

$$\lambda_1=2(p-1)\epsilon/(13-3p),$$
$$\lambda_2=(p+9)\epsilon/3(13-3p), \tag{5.6}$$
$$\lambda_3=\epsilon.$$

For the second fixed point, $u_1^*=0=u_2^*$ while u_3^* is given by Eq. (5.4c), and this is stable for $1<p<\frac{13}{3}$. The third fixed point with $u_3^*=0$, has

$$u_1^* = \mp \frac{6}{(p-2)} \left[\frac{6p(p-1)}{(p-2)} \epsilon \right]^{1/2} , \tag{5.7a}$$

$$u_2^* = \pm \left[\frac{6p(p-1)}{(p-2)} \epsilon \right]^{1/2} . \tag{5.7b}$$

Only the first fixed point can be reached through crossover from the symmetric one and the solutions of Eqs. (4.7) for the coupling-constant flow are shown in Figs. 2–4 for the case $p=3$.

We resort next to the mean-field analysis which yields support to the results of the RG calculation for transverse ordering done here with a pure ϕ^3-field theory.

In the case of transverse ordering with $Q_2 \neq 0$, there is also a $Q_1 \neq 0$ and now $r_T < 0$ in the ordered phase. For the second-order transition with small order parameter, it turns out that replacement of Eq. (4.11) into (4.12) yields

$$\hat{r}_L \cong -6c(p-2)u_1 Q_1 - 6cu_2 \frac{Q_2^2}{Q_1} , \tag{5.8}$$

which is positive if $Q_1 < 0$ and $p > 2$.[27] Similarly, Eq. (4.13) yields

$$\hat{r}_L = -r_T - 6u_2 c Q_1 , \tag{5.9}$$

also positive when $Q_1 < 0$, while Eq. (4.14) leads to

$$\hat{r}'_T = \frac{p-1}{p-3}(r_T + 12cu_2 Q_1) , \tag{5.10}$$

and this is positive only if $1 < p < 3$. Thus, the small order-parameter solution in mean-field theory that corresponds to a second-order transition is stable when $2 < p < 3$. Note that the quartic terms in Eqs. (4.9) and (4.10) do not enter into this analysis and that the lower limit for p is the same as that obtained above with the RG calculation.

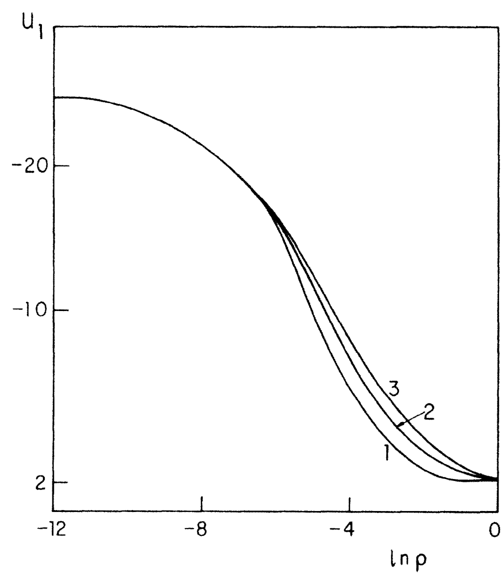


FIG. 2. Flow of u_1 for the $p=3$ state model with transverse ordering when $m^2 = 10^{-2}, 10^{-1}, 1$ (curves 1,2,3, respectively), $\kappa = 1$, and $\epsilon = 0.5$.

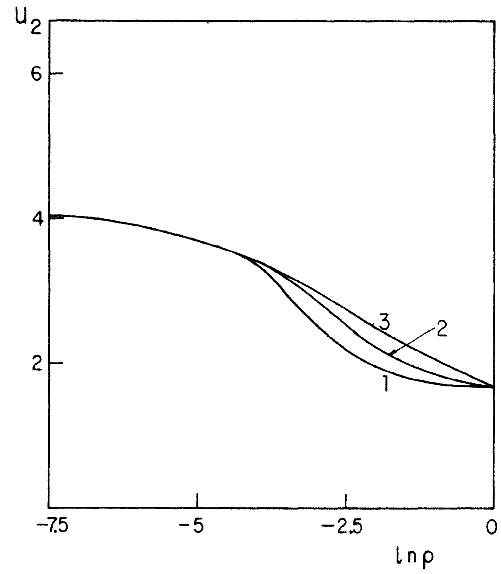


FIG. 3. Flow of u_2 for the $p=3$ state model with transverse ordering when $m^2 = 10^{-2}, 10^{-1}, 1$ (curves 1,2,3, respectively), $\kappa = 1$ and $\epsilon = 0.5$.

A word of caution is in order here. Indeed, to follow the mean-field prediction of a second-order transition for $2 < p < 3$, one has to show that there is no deeper, large (Q_1, Q_2) minimum of the free energy that could account for a first-order phase transition within the same range of p values. Actually, further calculations (not shown here) indicate a deeper minimum that appears with large $Q_1 < 0$ and $Q_2 > 0$ for small values of g with a negative $\hat{r}'_T$ for the remaining transverse components. Although this means that the large (Q_1, Q_2) minimum is unstable, the fact that it is the deeper minimum leaves us in a situation similar to that encountered by Pytte[25] for the symmetric theory in analyzing the second-order transition for $p < 2$. Indeed, he also found a deeper though unstable minimum, which suggests that one may have to keep higher-order terms in the order-parameter expansion for the free energy. It is possible that the large (Q_1, Q_2) minimum disappears in the free energy calculated to all orders in (Q_1, Q_2) leaving just the second-order transition for $2 < p < 3$.

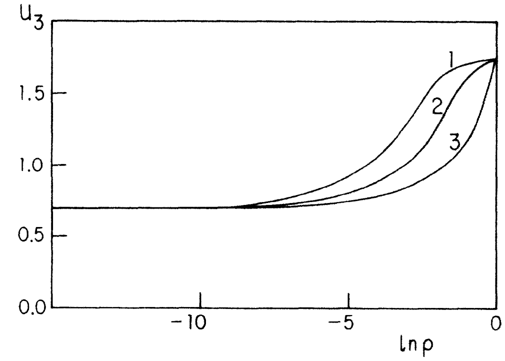


FIG. 4. Flow of u_3 for the $p=3$ state model with transverse ordering when $m^2 = 10^{-2}, 10^{-1}, 1$ (curves 1,2,3, respectively), $\kappa = 1$ and $\epsilon = 0.5$.

Though we cannot reach a definite picture from mean-field theory in this case, in the same way that Pytte could not reach one for the symmetric theory with $p<2$, we interpret the results of our mean-field analysis of the region $2<p<3$ as favoring a second-order transition. This is in accordance with the observation that QSB which favors transverse ordering yields a $(p-1)$-state symmetric Potts model. In agreement with this, our calculations also indicate the presence of a first-order transition for transverse ordering when $p>3$.

The RG result for the upper limit $p=\frac{13}{3}$ to the continuous transition presumably does not mean a fluctuation correction to the mean-field result, but rather a continuation to a spinodal line, similar to the RG result for the upper limit of $p=\frac{10}{3}$ in the symmetric theory.[25]

Finally, the critical exponents for the second-order transition that occurs when $2<p<3$ can be directly obtained from the shift $p \rightarrow p-1$ in the symmetric theory.

VI. CONCLUDING DISCUSSION

The work presented in this paper contains the first systematic study of renormalization, as far as we know, to account for fluctuation corrections to mean-field theory in critical phenomena with two-length scales, associated with the masses of longitudinal and transverse components of the fields, to two-loop order, taking into account coupling-constant symmetry breaking (TSB, in the present case) due to a break in quadratic symmetry. Previous work on ϕ^4-field theory only deals with some general features on coupling-constant renormalization, in a single coupling-constant theory.[12]

Both the results of mean-field theory and the RG calculations that account for fluctuation corrections can be used on systems of physical interest through the relationship between symmetry breaking in the components of the fields and in the original discrete model, discussed in more detail in Ref. 21. In particular, it is seen that the three-state model is a boundary case between first- and second-order transition under QSB that favors transverse ordering.

The present study is limited in various ways. One is the absence of an external field and the other is the neglect of stabilizing ϕ^4-field terms in our RG calculation which is thus restricted to the disordered phase. Indeed, extension into the ordered phase is desirable for a proper analysis of the Potts-model transition in uniaxially stressed $SrTiO_3$, as pointed out in recent work.[8] It would also be of interest in itself, in order to obtain universal amplitude ratios induced by QSB.

The neglect, for simplicity, of ϕ^4-field terms should not invalidate the main conclusions of this work: (a) there is a first-order phase transition for pure uniaxial longitudinal ordering induced by QSB, for all but low p; (b) there is a continuous transition for transverse ordering with p in the range $2<p<\frac{13}{3}$ (and possible exclusion to account for a spinodal point when $3<p<\frac{13}{3}$; (c) one may use RPT with GMS as a proper RG scheme. The effect of stabilizing ϕ^4-field terms could be a second-order transition for longitudinal ordering with low p and a proper first-order transition for transverse ordering with $p>3$, or $p>3+O(\epsilon)$.[28] Of course, ϕ^4-field terms will also be relevant for a proper description of the ordered phase.

In the work presented here we were only concerned with crossover induced by QSB to whatever asymptotic critical (or first-order transition) behavior there may be, and we did not discuss the "soft" expansion[12] in ϕ^3-field theory *with* TSB, a matter of separate interest, that could yield generalized crossover exponents to account for QSB perturbations. Previous works on crossover exponents in ϕ^3-field theory induced by QSB assume a single trilinear coupling, whether in the percolation problem[20] or in the general p-state Potts model.[21]

It should also be of interest to apply the present study to crossover behavior in the percolation problem. As pointed out before,[21] this requires separate calculations in the representation of Wallace and Young.[20] We expect to report on these extensions in future work.

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*On leave of absence at Department of Physics and Astronomy, Rutgers University, New Jersey.

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[24]Note that in the massless limit Eqs. (3.5) and (3.6) do *not* check with the resulting diagrams for the massless symmetric theory (Ref. 15), because of missing logarithmic momentum-dependent terms in the expressions calculated here. This is not surprising, nor is there anything wrong with it, since M_1 replaces a momentum dependence only in the large-mass limit. This does not affect quantities, like the renormalization functions, which should check in the appropriate limit with results of the symmetric theory, because of an exact cancellation of those momentum-dependent terms.

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[28]In similarity to the symmetric theory: A. Aharony and E. Pytte, Phys. Rev. B **23**, 362 (1981).