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ENSAIOS EM MODELAGEM DE DEPENDÊNCIA EM SÉRIES FINANCEIRAS MULTIVARIADAS UTILIZANDO CÓPULAS

Porto Alegre
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Orientador: Prof. Dr. Flávio A. Ziegelmann
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Ao meu noivo, Douglas.

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RESUMO

O presente trabalho foi motivado pela forte demanda por modelos de dependência mais precisos e realistas para aplicações a dados financeiros multivariados. A recente crise financeira de 2007-2009 deixou claro quão importante é uma modelagem precisa da dependência para a avaliação correta do risco financeiro: percepções equivocadas sobre dependências extremas entre diferentes ativos foram um elemento importante da crise do *subprime*. O famoso teorema de Sklar (1959) introduziu as cópulas como uma ferramenta para se modelar padrões de dependência mais sofisticados. Ele estabelece que qualquer função de distribuição conjunta n-dimensional pode ser decomposta em suas n distribuições marginais e uma cópula, sendo que a última caracteriza completamente a dependência entre as variáveis. Enquanto existe uma variedade de famílias de cópulas bivariadas que podem descrever um amplo conjunto de dependências complexas, o conjunto de cópulas com dimensão mais elevada era bastante restrito até recentemente. Joe (1996) propôs uma construção de distribuições multivariadas baseada em *pair-copulas* (cópulas bivariadas), chamada *pair-copula construction* ou modelo de *vine* cópula, que reverteu esse problema. Nesta tese, desenvolvemos três ensaios que exploram a teoria de cópulas para obter modelos de dependência multivariados muito flexíveis para aplicações a dados financeiros. Patton (2006) estendeu o teorema de Sklar para o caso de distribuições condicionais e tornou o parâmetro de dependência da cópula variante no tempo. No primeiro ensaio, introduzimos um novo enfoque para modelar a dependência entre retornos financeiros internacionais ao longo do tempo, combinando cópulas tempo-variantes e o modelo de mudança Markoviana. Aplicamos esses modelos de cópula e também os modelos propostos por Patton (2006), Jondeau e Rockinger (2006) e Silva Filho et al. (2012a) aos retornos dos índices FTSE 100, CAC 40 e DAX. Comparamos essas metodologias em termos das dinâmicas de dependência resultantes e das habilidades dos modelos em prever Valor em Risco (VaR). Interessantemente, todos os modelos identificam um longo período de alta dependência entre os retornos começando em 2007, quando a crise do *subprime* teve início oficialmente. Surpreendentemente, as cópulas elípticas mostram melhor desempenho na previsão dos quantis extremos dos retornos dos portfólios. No segundo ensaio, estendemos nosso estudo para o caso de $n > 2$ variáveis, usando o modelo de *vine* cópula para investigar a estrutura de dependência dos índices CAC 40, DAX, FTSE 100, S&P 500 e IBOVESPA, e, particularmente, checar a hipótese de dependência assimétrica nesse caso. Com base em nossos resultados empíricos, entretanto, essa hipótese não pode ser verificada. Talvez a dependência assimétrica com caudas inferiores mais fortes ocorra apenas temporariamente, o que sugere que a incorporação de variação temporal ao modelo de *vine* cópula pode melhorá-lo como ferramenta para modelar dados financeiros internacionais multivariados. Desta forma, no terceiro ensaio, introduzimos dinâmica no modelo de *vine* cópula permitindo que os parâmetros de dependência das *pair-copulas* em uma decomposição *D-vine* sejam potencialmente variantes no tempo, seguindo um processo ARMA(1,m) restrito como em Patton (2006). O modelo proposto é avaliado em simulações e também com respeito à acurácia das previsões de Valor em Risco (VaR) em períodos de crise. Os experimentos de Monte Carlo são bastante favoráveis à cópula *D-vine* dinâmica em comparação a uma cópula *D-vine* estática. Adicionalmente, a cópula *D-vine* dinâmica supera a cópula *D-vine* estática em termos de acurácia preditiva para os nossos conjuntos de dados.

Palavras-chave: Dependência assimétrica. Modelo de *pair-copulas*. *Vine* regular. Cópula tempo-variante. Cópula-GARCH. Modelo de mudança Markoviana. Valor em Risco.

ABSTRACT

This work was motivated by the strong demand for more precise and realistic dependence models for applications to multivariate financial data. The recent financial crisis of 2007-2009 has made it clear how important is a precise modeling of dependence for the accurate assessment of financial risk: misperceptions about extreme dependencies between different financial assets were an important element of the subprime crisis. The famous theorem by Sklar (1959) introduced the copulas as a tool to model more intricate patterns of dependence. It states that any n-dimensional joint distribution function can be decomposed into its n marginal distributions and a copula, where the latter completely characterizes the dependence among the variables. While there is a variety of bivariate copula families, which can match a wide range of complex dependencies, the set of higher-dimensional copulas was quite restricted until recently. Joe (1996) proposed a construction of multivariate distributions based on pair-copulas (bivariate copulas), called pair-copula construction or vine copula model, that has overcome this issue. In this thesis, we develop three papers that explore the copula theory in order to obtain very flexible multivariate dependence models for applications to financial data. Patton (2006) extended Sklar's theorem to the conditional case and rendered the dependence parameter of the copula time-varying. In the first paper, we introduce a new approach to modeling dependence between international financial returns over time, combining time-varying copulas and the Markov switching model. We apply these copula models and also those proposed by Patton (2006), Jondeau and Rockinger (2006) and Silva Filho et al. (2012a) to the return data of FTSE 100, CAC 40 and DAX indexes. We compare these methodologies in terms of the resulting dynamics of dependence and the models' abilities to forecast Value-at-Risk (VaR). Interestingly, all the models identify a long period of high dependence between the returns beginning in 2007, when the subprime crisis was evolving. Surprisingly, the elliptical copulas perform best in forecasting the extreme quantiles of the portfolios returns. In the second paper, we extend our study to the case of $n > 2$ variables, using the vine copula model to investigate the dependence structure of the broad stock market indexes CAC 40, DAX, FTSE 100, S&P 500 and IBOVESPA, and, particularly, check the asymmetric dependence hypothesis in this case. Based on our empirical results, however, this hypothesis cannot be verified. Perhaps, asymmetric dependence with stronger lower tails occurs only temporarily, what suggests that incorporating time variation into the vine copula model can improve it as a tool to model multivariate international financial data. So, in the third paper, we introduce dynamics into the vine copula model by allowing the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, following a nonlinear restricted ARMA(1,m) process as in Patton (2006). The proposed model is evaluated in simulations and further assessed with respect to the accuracy of Value-at-Risk (VaR) forecasts in crisis periods. The Monte Carlo experiments are quite favorable to the dynamic D-vine copula in comparison with a static D-vine copula. Moreover, the dynamic D-vine copula outperforms the static D-vine copula in terms of predictive accuracy for our data sets.

Keywords: Asymmetric dependence. Pair-copula constructions. Regular vine. Time-varying copula. Copula-GARCH. Markov switching model. Value-at-Risk.

SUMÁRIO

1 Introdução

Em teoria de portfólio, em geral, assume-se que os retornos financeiros têm uma distribuição Normal multivariada e o coeficiente de correlação linear é amplamente utilizado em Finanças como uma medida de dependência. Entretanto, é bem sabido que os retornos financeiros exibem assimetria e excesso de curtose nas distribuições univariadas. Existem também fatos estilizados observados na estrutura de dependência de dados financeiros multivariados que não são capturados por distribuições elípticas. Uma característica peculiar é a dependência caudal, que diz respeito à dependência em valores extremos. Além disso, estudos em Finanças empíricas têm encontrado evidências de que os retornos financeiros internacionais tendem a exibir dependência assimétrica, no sentido de que retornos negativos são mais dependentes entre si do que retornos positivos (veja, por exemplo, Longin e Solnik (2001), Ang e Bekaert (2002), Ang e Chen (2002), Hong et al. (2007)). Dois tipos de dependência assimétrica têm sido reportados: um é dependência mais forte entre retornos de diferentes ativos durante períodos de crise e de alta volatilidade nos mercados financeiros do que em outros momentos, como, por exemplo, em Ang e Bekaert (2002); e o segundo tipo é dependência caudal mais forte na cauda inferior (perdas) em comparação à cauda superior (ganhos) para pares de ativos, como em Hong et al. (2007).

Desta forma, resultados baseados em normalidade podem não ser apropriados para a construção de portfólios ótimos e gerenciamento de risco (SANCETTA; SATCHELL, 2001). A recente crise financeira de 2007-2009 deixou isso claro: a avaliação incorreta do risco financeiro na ocorrência de eventos extremos nos mercados foi um elemento importante da crise do *subprime*. Portanto, há uma forte demanda por procedimentos mais precisos e realistas para a modelagem de distribuições multivariadas de retornos de ativos e para a mensuração da dependência entre os mesmos.

O teorema de Sklar (1959) introduziu as funções cópulas. De acordo com esse teorema, é possível decompor qualquer função de distribuição conjunta n-dimensional em suas n distribuições marginais e uma cópula, sendo que a última descreve completamente a dependência entre as variáveis. Essa decomposição permite uma maior flexibilidade na construção de distribuições multivariadas, o que explica o papel de destaque que as cópulas têm desempenhado em modelagem multivariada ultimamente.

Como as cópulas representam a estrutura de dependência entre variáveis aleatórias, elas fornecem uma forma natural de se estudar e medir a dependência entre essas variáveis. As medidas de dependência baseadas em cópulas têm a propriedade desejável de invariância sob transformações estritamente crescentes (NELSEN, 2006).

Enquanto existe uma variedade enorme de famílias de cópulas bivariadas capazes de capturar os mais diversos padrões de dependência, o conjunto de cópulas com dimensão mais elevada era bastante restrito até recentemente. A chamada *pair-copula construction* (PCC) ou modelo de cópula *vine* regular reverteu esse problema. Joe (1996) propôs uma construção probabilística de distribuições multivariadas baseada em *pair-copulas* (cópulas bivariadas), mais tarde estendida e organizada sistematicamente por Bedford e Cooke (2001, 2002) por meio da especificação de um modelo gráfico chamado *vine* regular. O modelo de *vine* cópula consiste em decompor uma densidade multivariada em uma cascata de (densidades de) *pair-copulas*, além das densidades marginais. É um método mais flexível de se modelar distribuições multivariadas, uma vez que as cópulas bivariadas podem pertencer a qualquer família e várias famílias podem ser combinadas em uma *vine* cópula, capturando toda estrutura de dependência possível. Aas et al. (2009) introduziram a inferência para dois casos especiais de cópulas *vine* regulares, as chamadas cópulas *C-vine* e *D-vine*.

Nesta tese, nós exploramos a teoria de cópulas para obter modelos de dependência suficientemente flexíveis para aplicações a dados financeiros multivariados. O trabalho é composto por três ensaios, descritos brevemente a seguir.

Patton (2006) estendeu o teorema de Sklar (1959) para o caso de distribuições condicionais, permitindo tornar a estrutura de dependência variante no tempo. No primeiro ensaio, Capítulo 2 desta tese, nós introduzimos um novo enfoque para modelar a dependência entre retornos financeiros internacionais ao longo do tempo, combinando a metodologia proposta por Patton (2006) e o modelo de mudança Markoviana. Como já mencionado anteriormente, vários estudos recentes têm encontrado evidências de dependência assimétrica nos mercados financeiros internacionais que sugerem a existência de dois regimes nesses mercados: um regime de alta dependência, com retornos baixos e voláteis, e um regime de baixa dependência, com retornos elevados e estáveis.

Nós assumimos que não apenas o grau de dependência entre os retornos varia de acordo com os regimes, como também o tipo de dependência. Mais especificamente, assumimos que os regimes dos mercados financeiros são caracterizados por funções cópulas com diferentes comportamentos caudais, com a cópula Normal caracterizando um regime e uma cópula assimétrica o outro. Adicionalmente, permitimos que os parâmetros de dependência das cópulas variem deterministicamente ao longo do tempo, seguindo um processo ARMA(1,10) restrito como em Patton (2006).

Nós aplicamos o modelo proposto e também os modelos de cópulas tempo-variantes propostos por Patton (2006), Jondeau e Rockinger (2006) e Silva Filho et al. (2012a) aos dados de retornos dos índices europeus FTSE 100 (Reino Unido), CAC 40 (França) e DAX (Alemanha), de 04 de janeiro de 1999 a 28 de abril de 2011. Nosso interesse é comparar essas metodologias em termos das dinâmicas de dependência resultantes e das habilidades dos modelos em prever possíveis perdas de capital. Como os riscos relacionados à ocorrência de eventos extremos são importantes para o gerenciamento de risco, escolhemos comparar e selecionar os modelos com base em previsões de Valor em Risco (VaR).

Interessantemente, todos os modelos identificam um longo período de alta dependência entre os retornos dos índices FTSE e CAC e entre FTSE e DAX começando em 2007, quando a crise do *subprime* teve início oficialmente. Inesperadamente, os modelos com mudança de regime também indicam períodos de alta dependência entre os anos de 2003 e 2006, quando as condições nos mercados financeiros eram favoráveis. Na previsão dos quantis extremos dos retornos dos portfólios igualmente ponderados dos índices FTSE e CAC e dos índices FTSE e DAX, as cópulas elípticas mostram melhor desempenho. Isso contrasta com as evidências que esperávamos encontrar para dados de retornos financeiros e também com as evidências encontradas na literatura de cópulas estáticas, em que, geralmente, as cópulas que apresentam dependência caudal e assimetria têm melhores ajustes às caudas das distribuições conjuntas. Parece que parte da assimetria pode ser gerada por parâmetros tempo-variantes. A falta de dependência caudal também pode ser parcialmente compensada pela possibilidade de uma forte dependência mais geral, o que explicaria por que a cópula Gaussiana ajusta-se tão bem aos dados.

No segundo ensaio, Capítulo 3, estendemos nosso estudo para o caso de $n > 2$ variáveis, usando o modelo de *vine* cópula para investigar a estrutura de dependência dos índices de mercados de ações amplos da Alemanha (DAX), França (CAC 40), Reino Unido (FTSE 100), Estados Unidos (S&P 500) e Brasil (IBOVESPA), e, particularmente, checar a hipótese de dependência assimétrica entre os mesmos. Com base em descobertas de Joe et al. (2010) a respeito da dependência caudal em *vine* cópulas, ajustamos ao nosso conjunto de dados cópulas *C-vine* com *linking* cópulas bivariadas de dois parâmetros que podem ter dependência caudal inferior diferente da superior (BB1, BB7 e *Symmetrized Joe-Clayton*), o que nos fornece um meio de checar se há alguma assimetria nas caudas conjuntas da distribuição dos retornos dos índices financeiros. Adicionalmente, para levar em conta a possibilidade de dependência caudal simétrica, ajustamos uma cópula *C-vine* com todas as *pair-copulas* t-Student.

Com base em nossos resultados empíricos, não podemos verificar a hipótese de dependência assimétrica, uma vez que a decomposição *C-vine* com todas as *pair-copulas* t-Student tem o melhor ajuste aos dados, segundo o critério de verossimilhança. Nikoloulopoulos et al. (2010) consideram que, talvez, a dependência assimétrica com cauda inferior mais forte ocorra apenas temporariamente, por alguns períodos de tempo, o que sugere que a incorporação de dinâmica ao modelo de *vine* cópula pode melhorá-lo como instrumento para modelagem de dados financeiros internacionais multivariados. Isso serve como motivação adicional para o desenvolvimento do terceiro ensaio.

A pesquisa em modelagem de dependência de dados financeiros usando *vine* cópulas está concentrada principalmente no caso de estruturas de dependência estáticas. Entretanto, isso vai contra evidências encontradas na literatura de que a dependência entre retornos financeiros não é constante no tempo. Heinen e Valdesogo (2009, 2011) foram os primeiros e os únicos até então a introduzir variação temporal no contexto de *vine* cópula por meio da especificação de uma lei de movimento para os parâmetros das *pair-copulas*, baseada nas equações DCC. Outra direção de pesquisa combina o modelo de cópula *vine* regular com o modelo de mudança Markoviana para permitir que a estrutura de dependência como um todo, representada pela cópula, varie de acordo com os regimes que caracterizam os mercados financeiros internacionais (veja, por exemplo, Chollete et al. (2009), Garcia e Tsafak (2011), Stöber e Czado (2012)).

No terceiro ensaio, Capítulo 4, nós incorporamos dinâmica ao modelo de *vine* cópula especificando uma equação de evolução diretamente para os parâmetros das *pair-copulas*, para obter um modelo de dependência bastante flexível para aplicações a dados de retornos financeiros multivariados. Nós permitimos que os parâmetros de dependência das *pair-copulas* em uma decomposição *D-vine* sejam potencialmente variantes no tempo, seguindo um processo ARMA(1,m) restrito como em Patton (2006). Chamamos o modelo de cópula *D-vine* dinâmica ou tempo-variante. O ganho em termos de flexibilidade com essa extensão do modelo de *vine* cópula não tem como contrapartida a perda de tratabilidade do modelo, que é estimado utilizando-se um procedimento sequencial muito rápido e ainda assintoticamente eficiente.

A dinâmica especificada nesse ensaio evita certas limitações inerentes à dinâmica do modelo de cópula *vine* dinâmica de Heinen e Valdesogo (2009, 2011). Como a última, baseada nas equações DCC, envolve obter o coeficiente de correlação em cada período t, convertê-lo para o tau de Kendall e transformá-lo no parâmetro da *pair-copula*, pelo menos duas dificuldades surgem. Primeiro, como a transformação não-linear do tau de Kendall para o parâmetro da cópula não pode ser feita de forma fechada para todas as cópulas, a estimação torna-se uma tarefa difícil quando os parâmetros precisam ser obtidos numericamente. Segundo, só é possível adotar famílias de cópulas de um parâmetro como *building blocks* na construção da *vine* e, de acordo com Joe (2011), é importante ter cópulas com dependências caudais inferior e superior flexíveis, como as cópulas de dois parâmetros BB1 e BB7, para fazer inferência sobre as probabilidades caudais conjuntas, que podem representar riscos conjuntos.

O modelo proposto é primeiramente avaliado em um estudo de simulação. Os experimentos de Monte Carlo são bastante favoráveis à cópula *D-vine* dinâmica em comparação a uma cópula *D-vine* estática. Quando o processo gerador dos dados (PGD) é a cópula *D-vine* dinâmica, a cópula *D-vine* estática tende a viesar as estimativas dos parâmetros das *pair-copulas* e o viés não parece diminuir conforme o número de observações nas amostras aumenta. As estimativas feitas a partir da cópula *D-vine* dinâmica, nesse caso, são muito superiores às estimativas do modelo estático tanto em termos dos erros médios como em termos da raiz quadrada dos erros quadráticos médios. Quando as amostras são retiradas da cópula D-vine estática, ambos os modelos têm desempenho similar em termos dos erros médios, com estimativas não viesadas.

O modelo dinâmico falha apenas em termos da raiz quadrada dos erros quadráticos médios, quando o PGD é estático, o que sugere que suas estimativas apresentam uma variabilidade mais elevada do que as estimativas do modelo estático nesse caso.

A cópula *D-vine* dinâmica também é avaliada em um estudo empírico com relação à acurácia das previsões de VaR em períodos de crise. Modelamos a dependência entre os retornos dos índices DAX, CAC 40, FTSE 100, S&P 500 e IBOVESPA, empregando tanto a cópula *D-vine* dinâmica como uma cópula *D-vine* estática. Consideramos dois períodos distintos, de 03 de janeiro de 2003 a 28 de dezembro de 2007, bem como de 02 de janeiro de 2008 a 04 de maio de 2012, que denominamos "período de não-crise" e "período de crise", respectivamente. Além de permitir investigar os padrões de dependência que caracterizam tais períodos, os modelos estimados no período de 03 de janeiro de 2003 a 28 de dezembro de 2007 são usados para prever o VaR diário para um portfólio igualmente ponderado dos índices mencionados acima no período de 02 de janeiro de 2008 a 19 de agosto de 2008 (150 dias), e os modelos estimados no período de 02 de janeiro de 2008 a 04 de maio de 2012 são usados para previsão de VaR no período de 08 de maio de 2012 a 06 de setembro de 2012 (79 dias). A cópula *D-vine* dinâmica mostra bom desempenho fora da amostra, em períodos de crise, geralmente superando a cópula *D-vine* estática, mesmo na situação adversa em que nós utilizamos o modelo de cópula estimado no período de não-crise para fazer previsões no contexto de crise.

2 A Comparison Study of Copula Models for European Financial Index Returns

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Abstract. In this paper, we introduce a new approach to modeling dependence between international financial returns over time, combining time-varying copulas and the Markov switching model. We apply these copula models and also those proposed by Patton (2006), Jondeau and Rockinger (2006) and Silva Filho et al. (2012) to the return data of FTSE, CAC-40 and DAX indexes. We are particularly interested in comparing these methodologies in terms of the resulting dynamics of dependence and the models' abilities to forecast possible capital losses. Because risks related to extreme events are important for risk management, we compare and select the models based on VaR forecasts. Interestingly, all the models identify a long period of high dependence between the returns beginning in 2007, when the subprime crisis was evolving. Surprisingly, the elliptical copulas perform best in forecasting the extreme quantiles of the portfolios returns.

Keywords and phrases: Time-varying copulas; Markov switching model; Copula-GARCH; IFM method; Value at Risk.

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2.1 Introduction

In portfolio theory, multivariate financial returns are often assumed to be normally distributed in order to draw simple results, but, in practice, this assumption does not hold (Sancetta and Satchell, 2001). Financial returns are found to be leptokurtic and they also show skewness in univariate distributions. Additionally, a number of recent studies have also reported asymmetry in the dependence amongst international financial returns, in the sense that they exhibit greater dependence during market downturns than during market upturns (see, for example, Longin and Solnik, 2001, Ang and Bekaert, 2002, Ang and Chen, 2002). Therefore, it is necessary to find a different approach to modeling multivariate distributions of asset returns and measuring their dependence to reach optimum portfolio construction. Modern risk management requires an understanding of dependence that goes beyond simple linear correlation (Embrechts et al., 2002), since it is no longer sufficient to describe the dependence among the variables of interest when their joint distribution is not elliptical.

A theorem due to Sklar (1959) introduced the copula functions. According to this theorem, it is possible to decompose any n-dimensional joint distribution function into its n marginal distributions and a copula, where the latter completely describes the dependence amongst the variables. This decomposition allows for more flexibility in the construction of multivariate distributions, which explains the major role that copulas have played in multivariate modeling lately.

Since copulas represent the dependence structure amongst random variables, they provide a natural way of studying and measuring the dependence amongst these variables. In addition, copula-based measures of dependence have the desirable property of invariance under strictly increasing transformations (Nelsen, 2006). In particular, the tail dependence, a copula-based measure which indicates the dependence in extreme values, should be of interest to risk management practitioners.

Patton (2006) extended Sklar's theorem to the conditional case, defining the conditional copula, and rendered the dependence parameter conditional and time varying. Allowing for time variation in the conditional dependence among economic time series seems natural, since time variation in the conditional mean and variance of such series has been widely reported

(Patton, 2006).

As already mentioned above, several recent studies investigated the asymmetry in dependence structures in international equity markets, observing that dependence tends to be high in both highly volatile markets and bear markets. These findings suggest the existence of two regimes in international equity markets: a high dependence regime, with low and volatile returns, and a low dependence one, with high and stable returns.

Following this conjecture, a modest literature combines the theory of conditional copulas and the Markov switching model, resulting in a sufficiently flexible framework which enables one to introduce further asymmetries in a very natural way. Jondeau and Rockinger (2006) and Silva Filho et al. (2012) adopt copulas with Markov switching dependence parameters. The latter allows the dependence parameter to vary according to a restricted ARMA(1,10) process, as in Patton (2006), plus an intercept term that follows a first order Markov chain with two regimes. The former accepts that the dependence parameter assumes only two different values according to the regime. Okimoto (2008) considers more general models with Markov switching copulas, i.e., different copula functions in each regime. It is assumed therein that the copulas dependence parameters are static.

One of the aims of this paper is to introduce an approach to modeling dependence between international financial returns over time, combining the conditional copula theory and the Markov switching model in a way that, to our knowledge, has not been explored yet. We assume different copula functions across the two regimes characterizing international equity markets, with observation driven time-varying dependence parameters. We employ these copula models and also those proposed by Patton (2006), Jondeau and Rockinger (2006) and Silva Filho et al. (2012) to model the dependence structures between the FTSE returns and the CAC-40 and DAX returns, respectively. We are particularly interested in comparing these methodologies in terms of the resulting dynamics of dependence and the models' abilities to forecast possible capital losses. Because risks related to extreme events are important for portfolio construction and risk management, we compare and select the best model based on VaR forecasts for equally-weighted portfolios composed by the returns of the CAC-40 and DAX indexes in pairs with the FTSE returns.

The remainder of the paper is as follows. In the next section, we briefly introduce the conditional copula theory by Patton (2006) and present the competing methodologies under consideration in this study. Because we are interested in comparing the different dynamics of dependence resulting from the fitted models, we focus on scalar measures of dependence based on copulas, described on Section 2.3. Following, in Section 2.4, we describe the estimation procedure in copula modeling in general as well as the particularities of the inferential procedures of each methodology we adopt. In Section 2.5, we describe the goodness-of-fit tests based on VaR forecasts. In Section 2.6, we first analyze the univariate return data, then, we present the estimation results and investigate the dynamics of the dependence structures, finishing with the presentation of the goodness-of-fit tests and the selection of the best models. We bring some concluding remarks in Section 2.7.

2.2 Methodology: Conditional Copulas

As observed by Patton (2006), the case of random variables of interest conditioned on some pre-determined variables is important in economic time series analysis. Particularly, the dynamics of financial time series and their dependence structures may be captured by conditional distributions given past observations. For this reason, the extension of Sklar's theorem to the conditional case has proved to be very useful. Patton extended the Sklar's theorem as follows:

Theorem 1 *(Sklar's Theorem - two-dimensional conditional case) Let F_1 be the conditional distribution of $X_1|W$, F_2 be the conditional distribution of $X_2|W$, and H be the joint conditional distribution of $(X_1, X_2)|W$, where W is the conditioning variable with support Ω. For exposition purposes, it is assumed that W has dimension 1. Then there exists a conditional copula C such that, for any $(x_1, x_2) \in \bar{\mathbb{R}}^2$ and each $w \in \Omega$,*

$$H(x_1, x_2|w) = C(F_1(x_1|w), F_2(x_2|w)|w). \tag{2.2.1}$$

If F_1 and F_2 are continuous, then C is unique; otherwise, it is uniquely determined on $\text{Ran}F_1 \times \text{Ran}F_2$*.Conversely, if we let F_1 be the conditional distribution of $X_1|W$, F_2 be the conditional distribution of $X_2|W$, and C be a conditional copula, then the function H as defined above is a conditional bivariate distribution function with conditional marginal distributions F_1 and F_2.*

It is clear from the theorem above that copulas are functions that bind the multivariate distributions to their marginal distributions. They contain all information from the joint distribution that is not contained in the marginal distributions, which means that they contain all the information on the dependence among the variables. It is for this reason that copulas are alternatively called "dependence functions".

According to Patton, it is the converse of the Sklar's theorem that is most interesting for multivariate modeling, since it implies that we may link together any univariate distributions with any copula and a valid multivariate distribution will be defined.

The conditional density function corresponding to the distribution function in equation (2.2.1) can be easily recovered, provided that F_1 and F_2 are differentiable and H and C are twice differentiable:

$$\begin{aligned} h(x_1, x_2|w) &\equiv \frac{\partial^2 H(x_1, x_2|w)}{\partial x_1 \partial x_2} \\ &= \frac{\partial F_1(x_1|w)}{\partial x_1} \cdot \frac{\partial F_2(x_2|w)}{\partial x_2} \cdot \frac{\partial^2 C(F_1(x_1|w), F_2(x_2|w)|w)}{\partial u_1 \partial u_2} \\ &= f_1(x_1|w) \cdot f_2(x_2|w) \cdot c(u_1, u_2|w), \end{aligned} \tag{2.2.2}$$

where $u_1 = F_1(x_1|w)$ and $u_2 = F_2(x_2|w)$. This result is useful for maximum likelihood analysis, as we will see later on Section 2.4.

In this paper, we analyze and compare four distinct methodologies based on the conditional copula theory and applied to multivariate financial data. They are described in the next two subsections.

2.2.1 Observation Driven Time-Varying Copulas

The first methodology we adopt is due to Patton (2006). He allows for time-variation in the conditional copula by assuming that the dependence parameter, θ_c, evolves through time according to an equation that follows a kind of restricted ARMA(1,10) process, with an autoregressive component, to capture any persistence in the dependence parameter, and a forcing variable, to capture any variation in dependence. The evolution equation of the dependence parameter may be written as

$$\theta_{ct} = \Lambda(\omega + \beta\theta_{ct-1} + \alpha\psi_t), \tag{2.2.3}$$

where Λ is a logistic transformation used to keep the parameter in its interval at all times and ψ_t is the forcing variable, defined by Patton as the mean absolute difference between the transformed marginals u_{1t} and u_{2t} over the past ten observations[1]. The idea is to use this measure as an indication of how far the data were from comonotonicity: if X_1 and X_2 are comonotonic, $|u_{1,t} - u_{2,t}|$ is close to zero.

We use the following copulas to analyze the dependence between the indexes considered: the elliptical copulas Normal and Student-t, and a few Archimedean copulas, namely Gumbel, Rotated-Gumbel, Clayton, Symmetrized Joe-Clayton and BB1[2]. The Normal and Student-t are the copulas most frequently used in financial literature, probably because the dependence structure associated with them, the linear correlation coefficient, is very easy to be computed and some basic theories in finance are based on linear correlation between different financial instruments. The Student-t copula has shown to be generally superior to the Normal copula and the reason is that it has tail dependence. A drawback regarding these copulas, however, is that they describe only symmetric dependence, and it has been widely reported in literature that asymmetries are expected in financial returns, meaning that dependence in lower tail can be larger than dependence in upper tail and vice-versa. For this reason, it is also important to adopt asymmetric copulas, such as the Symmetrized Joe-Clayton and the BB1 (with lower tail dependence different from upper tail dependence), the Gumbel copula (with only upper tail dependence), and the Rotated-Gumbel and Clayton copulas (with only lower tail dependence).

2.2.2 Copulas with Markov Switching

We also adopt other three methodologies, which combine copula theory with regime switching. In Markov switching models, introduced into econometrics by Hamilton (1989), the evolution of a time series is influenced by the different states of the world or the economy. Recently, numerous empirical studies have found evidences that financial returns tend to exhibit different patterns of dependence according to the different states characterizing the international equity

[1]This is the definition for non-elliptical copulas. For the elliptical ones, the forcing variable is defined as the mean of the product $\Phi^{-1}(u_{1,t-j}) \cdot \Phi^{-1}(u_{2,t-j})$, for the Gaussian copula, and the mean of $T_\nu^{-1}(u_{1,t-j}) \cdot T_\nu^{-1}(u_{2,t-j})$, for the Student-t, over the previous 10 lags, where Φ^{-1} is the inverse of the standard Normal *c.d.f.* and T_ν^{-1} is the inverse of the Student-t *c.d.f.* with ν degrees of freedom. If data is positively dependent, the inverse of the marginal transforms of both variables will have the same sign, thus, α is expected to be positive.

[2]Their functional forms as well as the evolution equations of their dependence parameters following Patton (2006) are described in Appendix A.

markets. More precisely, such returns tend to be more dependent during crisis periods and periods of high volatility in international equity markets than otherwise. This behavior may be described by the specification of a copula model with switching regimes.

It is possible to assume that the dependence structure of international financial returns is influenced by a hidden Markov chain with two states. For simplicity, it is assumed that this Markov chain is homogeneous and of first order such that it can be completely characterized by its transition matrix $\Pr(S_t = i | S_{t-1} = j) = P_{ij}$, with $i, j = 0, 1$, where S_t denotes the unobserved regime at time t. Jondeau and Rockinger (2006) consider that the parameter θ_c pertaining to the dependence structure, i.e., to the copula, is driven by the following equation:

$$\theta_{ct} = \theta_0(1 - S_t) + \theta_1 S_t, \tag{2.2.4}$$

where θ_0 is the value assumed by the dependence parameter during a low-volatility/low-dependence regime and θ_1 is the value assumed during a high-volatility/high-dependence regime and S_t is as defined above.

Besides the stochastic influence on the dependence parameter, it may also be conditioned on past observations as in Silva Filho et al. (2012). The authors allow the dependence parameter to follow an ARMA(1,10) process, with the intercept term changing according to a Markov chain of first order and with two regimes, as described following:

$$\theta_{ct} = \Lambda((\omega_0(1 - S_t) + \omega_1 S_t) + \beta\theta_{ct-1} + \alpha\psi_t), \tag{2.2.5}$$

where S_t is the regime at time t and ψ_t is the forcing variable according to Patton (2006).

We now introduce a more general approach combining time-varying copulas and regime switching. We assume here that not only the degree of dependence changes according to the regimes characterizing international equity markets, as the type of dependence also changes, which is not the case for the two aforementioned methodologies. It means that we assume different copula functions across the two regimes. Additionally, we allow for time-variation in the dependence parameter following Patton (2006), which differentiates our methodology from the one proposed by Okimoto (2008), for example, where the copula is only state dependent.

The Markov switching copula model, in this case, is specified as follows:

$$C_{S_t,t}(u_{1t}, u_{2t} | S_t, \theta_{cS_t,t}) = \sum_{k=0}^{1} \mathbf{1}_{\{S_t = k\}} \cdot C_{kt}(u_{1t}, u_{2t} | \theta_{ckt}), \tag{2.2.6}$$

with

$$\theta_{ckt} = \Lambda(\omega_k + \beta_k \theta_{ckt-1} + \alpha_k \psi_t). \tag{2.2.7}$$

$C_{kt}(\cdot)$ is the copula function characterizing regime k, with time-varying dependence parameter θ_{ckt}, $k = 0, 1$.

Because dependence in tails is one of the properties that discriminates between the different copulas and is relevant for both the phenomenon of asymmetric dependence and risk management, we assume copulas with different tail behaviors in each regime. Particularly, we assume that the Normal copula, which has no tail dependence, characterizes one regime, whereas an asymmetric copula prevails in the other one.

2.3 Dependence Measures

Because one of our interests is to analyze and compare the evolutions of the dependence between the indexes captured by each model studied, it is interesting to focus on a scalar measure of dependence. Widely known concordance measures such as Kendall's tau, τ, and Spearman's rho, ρ_S, as well as tail dependence, λ, are copula-based measures of dependence and have the useful property of being invariant under strictly increasing transformations of the random variables.

Tail dependence measures the dependence in extreme values, i.e., it measures the dependence in the upper or lower tail of a bivariate distribution. For this reason, it is an important measure for risk management and we explore it in our study. Its formal definition is given below[3]:

Definition 1 *If the limit* $\lim_{\varepsilon \to 0} \Pr[U_1 \leq \varepsilon | U_2 \leq \varepsilon] = \lim_{\varepsilon \to 0} \Pr[U_2 \leq \varepsilon | U_1 \leq \varepsilon] = \lim_{\varepsilon \to 0} C(\varepsilon, \varepsilon)/\varepsilon = \lambda_L$ *exists, then the copula* C *has lower tail dependence if* $\lambda_L \in (0, 1]$ *and no lower tail dependence if* $\lambda_L = 0$. *If the limit* $\lim_{\delta \to 1} \Pr[U_1 > \delta | U_2 > \delta] = \lim_{\delta \to 1} \Pr[U_2 > \delta | U_1 > \delta] = \lim_{\delta \to 1} (1 - 2\delta + C(\delta, \delta))/(1 - \delta) = \lambda_U$ *exists, then the copula* C *has upper tail dependence if* $\lambda_U \in (0, 1]$ *and no upper tail dependence if* $\lambda_U = 0$.

In other words, the lower (upper) tail dependence is the probability that one variable takes an extremely large negative (positive) value, given that the other variable took an extremely large negative (positive) value.

[3] For more details on tail dependence and other copula-based measures of dependence, see Chapter 2 of Joe (1997) and Chapter 5 of Nelsen (2006).

While this measure exists for some copulas, there are families that do not allow tail dependence. The Normal copula, for example, has $\lambda_L = \lambda_U = 0$. Hence, for comparison purposes, we also adopt a dependence measure that can be calculated based on every copula. Besides that, it is interesting to study dependence from alternative aspects.

The Kendall's tau and the Spearman's rho measure a particular dependence known as concordance. They are reasonable alternatives to the linear correlation as measures of dependence for non-elliptical distributions. Because both of them can be computed for all the parametric families of copulas, we adopt one of them to measure dependence from another perspective than the dependence in the tails. We choose the Kendall's tau arbitrarily. It is defined as the difference of the probability of two random concordant pairs and the probability of two random discordant pairs for two iid random vectors (X_1, Y_1) and (X_2, Y_2) of continuous random variables. Formally:

Definition 2 *Let X and Y be continuous random variables whose copula is C. Then the population version of Kendall's tau for X and Y is given by*

$$\begin{aligned} \tau = \tau_{X,Y} &= P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0] \\ &= 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1. \end{aligned}$$

For some copulas, the Kendall's tau has a closed form, for others, it has to be computed numerically (see the Appendix for the cases of the copulas adopted in this paper).

2.4 Estimation Procedure: The Copula-GARCH Model

In a time series context, we can rewrite the joint density function given in equation (2.2.2), in Section 2.2, as

$$h_t(x_{1t}, x_{2t}|w; \theta) = f_{1t}(x_{1t}|w; \theta_1) \cdot f_{2t}(x_{2t}|w; \theta_2) \cdot c_t(u_{1t}, u_{2t}|w; \theta_c), \qquad (2.4.1)$$

where $u_{it} = F_{it}(x_{it}|w; \theta_i), i = 1, 2$, and $\theta = [\theta_1', \theta_2', \theta_c']$ is the vector of all parameters of both the marginals and the copula.

The decomposition in equation (2.4.1) suggests that, in general, the modeling of the joint density function $h(\cdot)$ can proceed in two steps: (i) identify the univariate marginal distributions for X_1 and X_2, and (ii) define the appropriate copula function.

The expression for the log-likelihood function is as follows:

$$\sum_{t=1}^{T} \log h_t(x_{1t}, x_{2t}|w;\theta) = \sum_{t=1}^{T} \log f_{1t}(x_{1t}|w;\theta_1) + \sum_{t=1}^{T} \log f_{2t}(x_{2t}|w;\theta_2) + \sum_{t=1}^{T} \log c_t(u_{1t}, u_{2t}|w;\theta_c)$$

$$\ell(\theta) = \ell_{f_1}(\theta_1) + \ell_{f_2}(\theta_2) + \ell_c(\theta_c), \tag{2.4.2}$$

with the maximum likelihood estimator defined as $\hat{\theta}_{MLE} = \max_{\theta \in \Theta} \ell(\theta)$.

Maximum likelihood of the joint log-likelihood function $\ell(\theta)$ is possible, but this is computationally very intensive and time-consuming, especially when introducing dynamics and stochastic dependency in the model. If we look at the joint log-likelihood function, we will note that it can be decomposed into two parts, one part involving the marginals and their parameters, and the other one involving the copula density, its parameters and also the parameters of the marginals. When the dependence is not too strong, the Inference Function for Margins (IFM) method (Joe and Xu, 1996) can efficiently estimate the model parameters. Joe (1997) proves that, under regular conditions, the IFM estimator verifies the property of asymptotic normality

$$\sqrt{T}\left(\hat{\theta} - \theta_0\right) \to \mathbf{N}\left(0, \mathcal{G}^{-1}(\theta_0)\right),$$

with $\mathcal{G}(\theta_0)$ the Godambe information matrix, $\mathcal{G}(\theta_0) = D^{-1}V(D^{-1})'$, where $D = \mathbf{E}\left[\partial s(\theta)/\partial\theta\right]$, $V = \mathbf{E}\left[s(\theta)s(\theta)'\right]$ and $s(\theta)$ is the score function of the maximization problem. This method consists of a two-step approach: in a first step, the parameters of the marginals are estimated, and, in a second step, the copula log-likelihood is maximized over the copula parameter, taking the parameters of the marginals as fixed at the estimated values from the first step. This approach disseminated the so called copula-GARCH model in copula modeling.

Following, we describe in details the two steps of the copula-GARCH model. This is the estimation procedure adopted by the authors of the competing methodologies studied here, with some particularities that will be emphasized.

2.4.1 The Models for the Marginal Distributions

The estimation procedure in copula modeling begins with the identification of the marginal distributions and the estimation of their parameters via maximum likelihood. For financial return data, a univariate ARMA(p,q)-GARCH(m,n)[4] specification is usually chosen to model

[4]Extensions of the GARCH model, such as EGARCH, TARCH, among others, are also fitted to data in order to find out the best model for the marginals.

the marginal distributions. It can be described by the following equations:

$$x_t = \phi_0 + \sum_{i=1}^{p} \phi_i x_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j} = \mu_t + a_t \tag{2.4.3}$$

$$a_t = h_t^{1/2} \varepsilon_t, \tag{2.4.4}$$

$$h_t = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{n} \beta_j h_{t-j}, \tag{2.4.5}$$

where μ_t and h_t are the conditional mean and variance given past information, respectively. ε_t is the innovation process and, in this paper, we assume that it may have a standard Normal distribution, $\varepsilon_t \sim Normal(0,1)$, a Student-t distribution, $\varepsilon_t \sim Student - t(\nu)$, or a Skewed-t distribution proposed by Hansen (1994), $\varepsilon_t \sim Skewed - t(\nu, \lambda)$.

If, for example, $\varepsilon_t \sim Skewed - t(\nu, \lambda)$, then the conditional distribution function of X_t is given by $F_t(x_t|\mu_t, h_t) = Skewed - t_{\nu,\lambda}((x_t - \mu_t)h_t^{-1/2})$. Thus, if the marginal distribution is well specified, the probability integral transform (PIT) of the standardized residuals will have a uniform distribution in [0,1]. This is a necessary result to identify the conditional copula in the second step, since it is a conditional joint distribution function defined over $F_{it}(x_{it}|\mu_{it}, h_{it}) = u_{it}, i = 1, 2$, with u_{it} uniform in [0,1]. To test whether the PIT of the standardized residuals has distribution $U[0,1]$, we use the Kolmogorov-Smirnov test of goodness-of-fit.

Summing up, in the first step of the estimation, the ARMA-GARCH filter is applied to the return data to obtain the univariate parameter estimates and the transformed data, i.e., the uniforms.

2.4.2 Establishing a Functional Form for the Copula and Estimating the Dependence Parameter

In the second step of the estimation procedure, the joint log-likelihood function, $\ell(\theta)$, can be reduced to the copula log-likelihood, $\ell_c(\theta_c)$, since the log-likelihoods related to the marginal distributions, $\ell_{f1}(\theta_1)$ and $\ell_{f2}(\theta_2)$, are fixed. The problem now is to maximize the copula log-likelihood over the copula parameter, taking the parameters of the marginals as fixed at the estimated values from the first step, that is:

$$\max_{\theta_c} \ell_c(\theta_c) = \sum_{t=1}^{T} \log c_t(u_{1t}, u_{2t}|\hat{\theta}_1, \hat{\theta}_2; \theta_c),$$

with $\hat{\theta}_1$ and $\hat{\theta}_2$, the estimates of the parameters from step one.

The methodologies under analysis in this paper differ in terms of their hypotheses about the dependence structure between the financial returns, therefore, they have distinct copula log-likelihoods. For the time-varying copulas in Patton (2006), ℓ_c may be written as

$$\ell_c(\theta_{ct}) = \sum_{t=1}^{T} log c_t(u_{1t}, u_{2t}|\hat{\theta}_1, \hat{\theta}_2; \theta_{ct}), \tag{2.4.6}$$

where θ_{ct} is as defined in equation (2.2.3), and the log-likelihood function can be maximized directly over the dependence parameter.

For the copulas with Markov switching parameters in Jondeau and Rockinger (2006) and Silva Filho et al. (2012), the copula log-likelihoods have the forms

$$\ell_c(\theta_{cSt}) = \sum_{t=1}^{T} log c_t(u_{1t}, u_{2t}|\hat{\theta}_1, \hat{\theta}_2; \theta_{cSt}) \tag{2.4.7}$$

$$\ell_c(\theta_{ct,St}) = \sum_{t=1}^{T} log c_t(u_{1t}, u_{2t}|\hat{\theta}_1, \hat{\theta}_2; \theta_{ct,St}), \tag{2.4.8}$$

respectively, being θ_{cSt} as defined in equation (2.2.4) and $\theta_{ct,St}$ as in equation (2.2.5).

Finally, the Markov switching copula model that we propose in this work has the following form for the copula log-likelihood:

$$\ell_c(\theta_{cS_t,t}) = \sum_{t=1}^{T} log c_{St,t}(u_{1t}, u_{2t}|\hat{\theta}_1, \hat{\theta}_2; S_t, \theta_{cS_t,t}), \tag{2.4.9}$$

where it is the functional form of the copula density function itself that depends directly on S_t.

The log-likelihoods in (2.4.7), (2.4.8) and (2.4.9) cannot be maximized directly over θ_c. In order to draw inference for copula models with Markov switching, we need to overcome the challenge of having unobserved latent variables. To do so, we decompose the copula density of u_{1t}, u_{2t} and the unobserved variable S_t into the product of conditional and marginal densities:

$$c_t(u_{1t}, u_{2t}, S_t|\omega_{t-1}) = c_t(u_{1t}, u_{2t}|S_t, \omega_{t-1}) \cdot \Pr(S_t|\omega_{t-1}),$$

where ω_{t-1} is all information available up to time *t-1*. And, then, integrate the S_t variable out of the joint density by summing over all possible values of S_t:

$$\begin{aligned} c_t(u_{1t}, u_{2t}|\omega_{t-1}) &= \sum_{S_t=0}^{1} c_t(u_{1t}, u_{2t}, S_t|\omega_{t-1}) \\ &= \sum_{S_t=0}^{1} c_t(u_{1t}, u_{2t}|S_t, \omega_{t-1}) \cdot \Pr(S_t|\omega_{t-1}). \end{aligned}$$

Thus, the copula log-likelihoods can be rewritten as[5]

$$\ell_c = \sum_{t=1}^{T} \log \left(\sum_{S_t=0}^{1} c_t(u_{1t}, u_{2t}|S_t, w_{t-1}) \cdot \Pr(S_t|w_{t-1}) \right). \tag{2.4.10}$$

To compute the conditional probabilities $\Pr(S_t|w_{t-1}), S_t = 0, 1$, we apply Kim's filter, described in Kim and Nelson (1999). Being able to calculate these probabilities, we can evaluate and maximize the copula log-likelihood, obtaining the estimates of the copula parameters.

2.5 Goodness-of-Fit Based on VaR Forecasts

Information criteria, such as AIC and BIC, suggest which copula model may be a good choice. However, they do not bring much information about how good is the model fit to data. For this purpose, formal goodness-of-fit tests may be used (see Fermanian (2005) and Dobrié and Schmid (2007) for implementation of some of these tests in copula literature).

Because risks associated with rare (or extraordinary) events are important for the composition of portfolios, we instead choose to compare and select the best model based on VaR (Value at Risk) forecasts. The VaR over the time horizon h with probability α, $0 < \alpha < 1$, can be defined as $\text{VaR}_{\alpha,h} = \inf_x \{x : F_h(x) \geq \alpha\}$, where F_h is the cumulative distribution function of the portfolio returns and $\text{VaR}_{\alpha,h}$ is the 100α-th quantile of F_h[6]. In other words, the VaR is the maximal loss associated with the portfolio, during a certain time period (in our case, we choose a one-day period), for a given significance level α. Although F_h can be theoretically computed from the marginal conditional distributions, it is not easy to reach a closed analytical form for the joint distribution. For this reason, we obtain the parametric distributions of equally-weighted portfolio returns of the form $X_{Pt} = 0.5X_{1t} + 0.5X_{2t}$ via Monte Carlo simulations and find the extreme quantiles, using the following algorithm:

For each point in time, $t = 1, ..., T$:

1. Simulate K samples of uniforms from the fitted copula model, $(u_{1t}^{(k)}, u_{2t}^{(k)}), k = 1, ..., K$. To simulate random vectors from copulas, use the conditional sampling method as follows[7]:

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[5]Just to keep the notation in accordance with the one used up to this point, note that, in the case that the functional form of the copula density changes according to the regime, the copula density function should be denoted by $c_{S_t,t}(\cdot)$.

[6]Since the VaR is calculated here from the distribution of the portfolio daily log-returns, it is expressed in percentage.

[7]We suppress the subscript t here to simplify the notation.

Let the conditional distribution of U_2 *given* U_1 *be*

$$F_{U_2|U_1}(u_2|u_1) = P(U_2 \leq u_2|U_1 = u_1) = \frac{\partial}{\partial u_1}C(u_1, u_2) = c_{u_1}(u_2)$$

where $c_{u_1}(u_2)$ *is the partial derivative of the copula function in relation to* u_1.

It allows us to generate K pairs (u_1, u_2) in the following manner:

(a) Generate $k = 1, ..., K$ realizations of independent *Uniform*(0,1) random variables, u_1 and v;

(b) Note that as $c_{u_1}(v^{(k)}) = F_{U_2|U_1}(v^{(k)}|u_1^{(k)})$ and $v^{(k)}$ is an observation of a standard uniform random variable, $u_2^{(k)} = c_{u_1}^{(-1)}(v^{(k)})$, with $c_{u_1}^{(-1)}$ the generalized inverse of c_{u_1}, is an observation of the random variable $U_2|U_1 = u_1$. Hence we have that the pair $(u_1^{(k)}, u_2^{(k)})$ is an observation of the random vector (U_1, U_2) which has the joint distribution C.

2. For $k = 1, ..., K$, convert $u_{it}^{(k)}$ to $\varepsilon_{it}^{(k)}$, $i = 1, 2$, using the inverse of the marginal distribution F_{it} with the estimated parameters from the first step of the IFM method, $\varepsilon_{it}^{(k)} = F_{it}^{-1}(u_{it}^{(k)})$.

3. For $k = 1, ..., K$, convert $\varepsilon_{it}^{(k)}$ to the log-return $x_{it}^{(k)} = \hat{\mu}_{it} + \sqrt{h_{it}} \cdot \varepsilon_{it}^{(k)}$, where $\hat{\mu}_{it}$ and h_{it} are the conditional mean and variance values, as obtained in the first step of the IFM method.

4. For $k = 1, ..., K$, compute the portfolio return as $x_{Pt}^{(k)} = 0.5x_{1t}^{(k)} + 0.5x_{2t}^{(k)}$.

5. Calculate the one-day 100α-th percentile of $x_{Pt}^{(k)}$, $k = 1, ..., K$, which corresponds to $\widehat{VaR}_{\alpha,1}$. If the observed value of X_{Pt} for day t is less than $\widehat{VaR}_{\alpha,1}$, then we say that a violation (or exceedance) occurs.

To evaluate the VaR forecasts (and their underlying models), we initially use the likelihood ratio tests proposed by Kupiec (1995) and Christoffersen (1998). Based on the previous algorithm, it is possible to construct an indicator sequence of violations $I_t, t = 1, ..., T$. The unconditional coverage test, introduced by Kupiec, is a test of the null hypothesis that the indicator function I_t, which is assumed to follow an i.i.d. Bernoulli process, has a constant "success" probability equal to the significance level of the VaR_α, where success corresponds to the portfolio losing more than the VaR. The test statistic is a likelihood ratio statistic given by

$$LR_{uc} = -2\,ln\left(\frac{\pi_{exp}^{n_1}(1-\pi_{exp})^{n_0}}{\pi_{obs}^{n_1}(1-\pi_{obs})^{n_0}}\right) \sim_{H_0} \chi^2_1, \tag{2.5.1}$$

where π_{exp} is the expected proportion of violations, $\pi_{exp} = \alpha$, π_{obs} is the observed proportion of violations, n_1 is the observed number of violations and $n_0 = T - n_1$ is the number of returns with indicator 0, called "good returns". Note that $\pi_{obs} = n_1/T$.

It may happen that the VaR violations come in clusters, which suggests that the VaR model is not sufficiently responsive to changes in the market circumstances. In this case, a model may pass the unconditional coverage test, i.e., the observed number of violations is close to the expected number, but we can still reject the VaR model if the violations are not independent. Christoffersen (1998) proposed a combined test for both unconditional coverage and independence, with H_0 being serial independence and a violation rate of α. The test statistic is the conditional coverage statistic given by

$$LR_{cc} = -2\,ln\left(\frac{\pi_{exp}^{n_1}(1-\pi_{exp})^{n_0}}{\pi_{01}^{n_{01}}(1-\pi_{01})^{n_{00}}\pi_{11}^{n_{11}}(1-\pi_{11})^{n_{10}}}\right) \sim_{H_0} \chi^2_2, \tag{2.5.2}$$

where n_{ij} is the number of returns with indicator i followed by returns with indicator j, with $i, j = 0, 1$, and $\pi_{ij} = \Pr(I_t = j|I_{t-1} = i) = \left(n_{ij}/\sum_j n_{ij}\right)$. Under H_0, $\pi_{01} = \pi_{11} = \alpha$, meaning that, if the exceedances are independent, the probability of observing a new violation of the VaR at t is the same, no matter if a violation occurred or not in the earlier instant.

According to Lopez (1999), the statistical tests proposed by Kupiec and Christoffersen to evaluate the accuracy of VaR models can have relatively low power against inaccurate VaR models, which means that the chances of misclassifying inaccurate models as accurate, in this case, can be high. For this reason, he proposed an alternative methodology based not on a statistical testing framework, but instead on standard forecast evaluation techniques: the accuracy of the VaR forecasts is determined by how well they minimize a certain regulatory loss function. Simulation results indicated that this methodology is less susceptible to model misclassification and there is also the advantage of specifying the loss function according to particular interests. In light of these facts, we implement this additional procedure proposed by Lopez and we apply a test for superior predictive ability (SPA) proposed by Hansen (2005) to determine which model significantly minimizes the expected loss function.

The Basel II Accord, still in process of implementation by several countries, proposed that the capital requirements for bank's credit risk exposure were determined according to its daily VaR estimates with a 1% significance level. The capital requirement loss function (CR) is based on the larger out of the current VaR estimate and a multiple of the average estimate over the past 60 days as follows[8]:

$$CR_t = \max\left[\frac{(3+\delta)}{60}\sum_{i=0}^{59} VaR_{\alpha,t-i}, VaR_{\alpha,t}\right], \delta = \begin{cases} 0, & \text{if } \zeta \leq 4; \\ 0.3 + 0.1(\zeta - 4), & \text{if } 5 \leq \zeta \leq 6; \\ 0.65, & \text{if } \zeta = 7; \\ 0.65 + 0.1(\zeta - 7), & \text{if } 8 \leq \zeta \leq 9; \\ 1, & \text{if } \zeta \geq 10. \end{cases} \quad (2.5.3)$$

where δ is a multiplicative factor that depends on the number of violations of the VaR in the previous 250 trading days (ζ). We adopt this regulatory loss function to evaluate the VaR forecasts.

To compare the models performances in minimizing the CR loss function, we use the SPA test statistic proposed by Hansen. A superior predictive ability test is applied whenever the interest is to test whether a particular forecasting procedure is outperformed by alternative forecasts. We are interested to know whether any of the models, $k = 1, ..., m$, are better than a benchmark, *bch*, in terms of minimizing the expected loss function. So we test the null hypothesis that the best model is not better than the benchmark. The relative performance of a model to the benchmark may be defined as $d_{k,t} = CR_{bch,t} - CR_{k,t}$. Provided that $E(d_{k,t}) = \mu_{k,t}$ is well defined, we can formulate the null hypothesis of interest as

$$H_0 : \max_{k=1,...,m} \mu_k \leq 0,$$

whereas the alternative hypothesis is that the best model is superior to the benchmark. A k model is better than the benchmark if and only if $E(d_{k,t}) > 0$. The test statistic is given by

$$T^{SPA} \equiv \max\left[\max_{k=1,...,m} \frac{T^{1/2}\bar{d}_k}{\hat{\omega}_k}, 0\right],$$

where $\bar{d}_k \equiv T^{-1}\sum_{t=1}^{T} d_{k,t}$ and $\hat{\omega}_k^2$ is some consistent estimator of $\omega_k^2 \equiv var(T^{1/2}\bar{d}_k)$. The test is implemented via stationary bootstrap of Politis and Romano (1994).

[8]Note that, since the VaR is a negative value, to compute the loss function, it will be calculated here as minus the (100α-th percentile) of the c.d.f. of the returns.

2.6 Data Description, Estimations and Forecasts

In this section, we model the dependence between the returns of the FTSE index and the returns of the CAC-40 and DAX indexes, respectively, according to the methodologies presented in Section 2.2, analyze the dependence time dynamics and verify the goodness-of-fit on the basis of the accuracy of the VaR forecasts. We begin with the description of our data set and its main features. Then, we proceed to the implementation of the first step of the IFM method, i.e., we model the marginal distributions of the returns, and, following, we advance to the estimation and analysis of the dependence over time. Finally, the estimated copula models are used for the forecasting of extreme quantiles of the portfolios constructed, and these forecasts are evaluated in order to determine which copula model has the best fit to the data when it comes to inference involving tails.

2.6.1 Return Data and Descriptive Statistics

To perform our comparative study, we use 3017 observations of daily log-returns of the stock indexes FTSE, CAC-40 and DAX from January 04, 1999, to April 28, 2011. We use close-to-close returns, meaning that the daily returns are those observed for trading days occurring simultaneously in all the three stock markets considered. The rationale for using only European indexes is to guarantee that the trading times overlap the most possible in order to obtain a synchronism of the returns.

The period covered by our data sample comprises two main stock market crashes worthy to be mentioned: the one of 2000 till 2002, which was a ramification of the "dot-com bubble", that burst on March 2000, and the subprime crisis, from Fall 2007 to June 2009. These are periods when the markets were in a downturn trend, so we expect that the copula models capture greater dependence then. The period 2003 to 2006 is considered a period when the markets performed well.

A few descriptive statistics of the returns are provided in Table 2.1. We can notice that data usually shows negative asymmetry, except for CAC log-returns, suggesting that the presence of negative extreme values is more common, i.e., the left tail of the distribution is heavier. Data also presents excess kurtosis, especially the FTSE log-returns. Also, according to the Jarque-

Bera test statistics, it is possible to reject the null hypothesis of normality of the returns for all indexes. All these statistics are in accordance with what is reported in the financial literature.

2.6.2 Modeling the Marginal Distributions

Here we parameterize the marginal distributions of the returns according to the ARMA-GARCH model, as described in Subsection 2.4.1. Since the return data has signs of asymmetry and excess kurtosis, we assume that the errors may have a Student-t or a Skewed-t distribution, although we also consider a Normal distribution. To capture the *leverage effect*, present in financial time series, we also consider asymmetric GARCH specifications, such as the TARCH, EGARCH and GJR models.

Taking into account the information criteria AIC and BIC, the diagnostic (autocorrelation) tests of the residuals and the goodness-of-fit test that the PIT of the residuals are uniforms, we choose as the best specifications for the marginals the following models: AR(3)-GARCH(1,1) with Skewed-t errors for the FTSE and CAC-40 returns, and AR(4)-GARCH(1,1) with Skewed-t errors for the DAX returns, where the Skewed-t density is given by:

$$g(z|\nu,\lambda) = \begin{cases} bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(\nu+1)/2} & z < -a/b \\ bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(\nu+1)/2} & z \geq -a/b \end{cases}$$

with the constants a, b and c defined as

$$a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right), b^2 = 1+3\lambda^2 - a^2, c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}$$

and the parameters ν and λ representing the degrees of freedom and asymmetry, respectively.

The estimates from the ARMA-GARCH fits[9], the diagnostic checks and the results of the goodness-of-fit test are presented in Table 2.2. If the ARMA-GARCH specifications are successful at modeling the serial correlation in the conditional mean and variance, there should be no autocorrelation left in the standardized residuals and squared standardized residuals. Table 2.2 provides the p-values of the Ljung-Box test of autocorrelation in the standardized and squared standardized residuals with 15 lags, $Q(15)$ and $Q^2(15)$, respectively. For all series, the null hypothesis of no autocorrelation left cannot be rejected at the 5% level. Additionally, if the marginal distributions are well specified, the probability integral transforms of the standardized

[9] All marginals were estimated using the Oxford MFE Toolbox by Kevin Shepard.

residuals should be $U[0,1]$. The p-values of the Kolmogorov-Smirnov test, reported in Table 2.2, suggest that the PIT of the standardized residuals have the correct distribution.

2.6.3 Modeling the Dependence Structure and Analyzing its Dynamics

Having estimated the marginal distributions, the next step is the modeling of the dependence structures between the FTSE returns and the returns of CAC-40 and DAX. The estimations[10] are performed in pairs between FTSE and each one of the other two indexes. In this subsection, we present the copulas estimates and we are interested in investigating the dynamics of the dependence structures. For this purpose, we analyze the estimates of the dependence parameters and make an initial selection of the copula model that may be a good choice to represent the dependence structure of each pair of indexes based on the maximum log-likelihood (LogL)[11], checking the statistical significance of the estimated coefficients of the models. Additionally, we compare the different evolutions of the dependence measured by the Kendall's tau and the tail dependence, computed based on the different copulas, to provide further insight on the dependence dynamics through time.

We begin analyzing the estimates of the observation driven time-varying copulas of Patton (2006) in Table 2.3. For the pair FTSE-CAC, the Student-t copula is the best one out of these copulas based on the LogL and taking into account the fact that the BB1 copula does not have all estimated coefficients considered statistically significant. The estimate of β_{1T}, 4.8885, indicates a high persistence in the linear correlation between FTSE and CAC. Notice that the LogLs of the Student-t and the Rotated-Gumbel copulas are very close, 2059.6 and 2058.8, respectively, so we should not just discard the Rotated-Gumbel copula. Analyzing the estimates for FTSE-DAX, we choose the Rotated-Gumbel as the best copula, according to the same criterion of highest log-likelihood amongst the models with all coefficients considered statistically significant. The Normal copula, in this case, also captures a significant persistence in the estimate of the correlation coefficient, with the estimated β_N equal to 3.5251.

[10]The copula models were estimated using the Copula Toolbox provided by Andrew Patton and some functions written by the authors for the Matlab 7® software. The standard errors were computed numerically using the functions "hessian_2sided" and "MyFuncScores" from Dynamic Copula Toolbox 3.0 provided by Manthos Vogiatzoglou.

[11]Since our data sample is quite long, the information criteria AIC and BIC choose the same model as the LogL. For lack of space, these information criteria are not presented here, but they are available from the authors upon request.

To observe the dependence time dynamics captured by these copulas, we focus on the evolutions of the Kendall's tau and the tail dependence. The evolution of the dependence measured by the Kendall's tau for the pair FTSE-CAC is shown in Figure 2.1. We have the time path of the dependence obtained from the Student-t copula in panel (a), and we also show the time paths obtained from the other copulas with all coefficients statistically significant. It is possible to notice that, from both the Student-t and the Normal copulas, the Kendall's tau evolves through time following a path that oscillates around the constant values estimated from the static versions of such copulas (0.6557 and 0.6496, respectively) all along the sample. From the Rotated-Gumbel, in panel (b), the Kendall's tau remains under the estimated constant value, 0.6376, till the end of 2001, when it reaches another level and begins oscillating around this value, and, finally, it rises above this baseline from 2007 till the second semester of 2010. We expected to observe this increase in dependence after 2007 because of the subprime crisis. The subtle increase in dependence by the end of 2001 may be associated with the market crash due to the "dot-com bubble", but notice that its influence on dependence is much weaker than the effect of the subprime crisis. The evolution of the tail dependence for this same pair of indexes can be observed in Figure 2.2. The tail dependence path calculated based on the Student-t copula (panel (a)) is quite erratic and not very informative, remaining a little under the constant value, 0.4763, most of the time, with abrupt jumps in 2008 and 2010 associated with sudden disturbances in the estimated degrees of freedom. The lower tail dependence measured based on the Rotated-Gumbel copula, in panel (b), is greater than the one measured by the Student-t, probably because of the symmetry imposed by the elliptical functional forms, and it increases after 2007, corroborating again the expectations of greater dependence during crisis periods.

The Kendall's tau paths computed based on all statistically significant copulas for the pair FTSE-DAX are presented in Figure 2.3. From the Rotated-Gumbel copula, in panel (a), the Kendall's tau remains under the estimated constant value, 0.5744, almost all the time from 1999 to the end of 2003, oscillates around this value in the period 2004 to 2006, and, from 2007 to the end of 2010, it rises above it. The same behavior is captured by the Symmetrized Joe-Clayton copula (panel (b)). In panel (c), the time path of Kendall's tau captured by the Normal copula is always oscillating around the constant value, 0.5891. The time evolution of the tail dependence in Figure 2.4 is rather similar to the Kendall's tau evolution in the sense that it begins lower than its constant value, oscillates around it in the period 2004 to 2006 and finally rises above it from 2007 on. The lower tail dependence computed based on the Rotated-Gumbel is quite the same as if computed from the SJC, as can be seen from their time evolutions in panels (a) and (b), respectively. About the dependence in the upper tail computed based on the SJC, its constant value is 0.5747, lower than the constant value for the lower tail dependence, as expected due to the typical behavior of higher dependence in lower tails than upper tails in financial markets. However, the evolution of the upper tail dependence, in panel (c), is more volatile.

From the estimation of the observation driven time-varying copulas of Patton (2006), we find strong evidences of dependence between extreme values (tail dependence) for the pairs FTSE-CAC and FTSE-DAX, since the Normal copula has the lowest value for the log-likelihood amongst all copulas. These models also capture an increase in dependence beginning in 2007, when the subprime crisis was evolving, and it reaches the highest level during this period.

Table 2.4 reports the estimates of the Markov switching dependence parameters following Jondeau and Rockinger (2006). We denote the probabilities $\Pr(S_t = 1|S_{t-1} = 1)$ by p and $\Pr(S_t = 0|S_{t-1} = 0)$ by q. Examining the table, we select the Normal copula as the best model for the pair FTSE-CAC, noting that, though the Student-t has a higher LogL, the estimate of the degrees of freedom in regime 1 is not significant. The increase in the estimated correlation coefficient from 0.7659 in regime 0 to 0.9253 in regime 1 is strongly significant. It is also possible to conclude that the regimes are rather balanced, with very close expected durations, of $(1 - \hat{p})^{-1} = 98$ days for the high dependence regime and $(1 - \hat{q})^{-1} = 83$ days for the low dependence regime, suggesting that both of them are relevant to capture the dynamics in the dependence structure between FTSE and CAC. In fact, all the estimated copulas with significant coefficients present this same persistence in the regimes and strong significance of the dependence parameters. For the pair FTSE-DAX, we choose the Student-t copula as the best model, although with some reservations, since $\hat{p}$ and $\hat{q}$ are significant only at the 10% level. The estimated linear correlation increases from 0.7093 to 0.8818, and the degrees of freedom do not change much, they go from 13.4763 to 14.5128. The regimes here are very persistent, regime 1 lasts about 270 days, whereas regime 0 lasts 244.

We analyze now the evolutions of the dependence measures τ and λ through time to provide further insight on the dynamics of the dependence captured by these copulas with regime dependent parameters. As in Jondeau and Rockinger (2006), the paths are calculated based on the ex-ante probabilities and defined as $\tau_t = \tau_0 p_{0t} + \tau_1(1 - p_{0t})$ and $\lambda_t = \lambda_0 p_{0t} + \lambda_1(1 - p_{0t})$, where $p_{0t} = \Pr(S_t = 0|w_{t-1})$. In Figure 2.5, we plot $\hat{\tau}_t$ for FTSE-CAC and it reveals that the dependence between these indexes is characterized essentially by three subperiods. The first one, that goes from 1999 to the first semester of 2001, is mainly associated with the low dependence regime, with two short-lasting increases in dependence in 1999 and 2001. In the second period, beginning by the end of 2001 and going till 2006, the two regimes are intercalated. From the second semester of 2001 to the end of 2002, the higher dependence can be explained by the market crash due to the "dot-com bubble", besides of the terrorist attack to the Twin Towers on September, 11, 2001. Unexpectedly, these copulas capture increases in dependence during the period 2003 to 2006. This is a surprise because it is a period when the international financial markets performed well. Finally, the third period, from 2007 on, is associated with the high dependence regime, when the two markets are strongly dependent. This period of high dependence coincides with the subprime crisis till the first semester of 2009 and goes beyond it till the end of 2010, and then the dependence decreases in the beginning of 2011. During this period, there is a short-lasting decrease in dependence in the first semester of 2008, probably associated with an increase in the economic activity of the US identified by the Business Cycle Dating Committee of the National Bureau of Economic Research. Figure 2.6 displays the evolution of the tail dependence, whose pattern is similar to the one described above for the Kendall's tau. From the BB1 copula, in panel (a), $\hat{\lambda}_L$ is lower than $\hat{\lambda}_U$ during regime 0, and higher than this during regime 1. This behavior is reverted when the tail dependence is computed based on the Symmetrized Joe-Clayton copula (panel (c)).

In Figure 2.7, we have the Kendall's tau evolution for the pair FTSE-DAX. There we can identify two long subperiods characterizing the dependence between these two indexes. From 1999 to January, 2004, the dependence is mainly low, with a significant increase by the end of 2002 and beginning of 2003. The second subperiod, from 2004 on, is essentially associated with the high dependence regime. During this period, an important decrease in dependence occurs in the second semester of 2005 and it rises again from 2006 to the end of 2010. Just as it happened for FTSE-CAC, the dependence between FTSE and DAX increased during the crises of 2002 and 2007-2009. Tail dependence through time is displayed in Figure 2.8. The lower and upper tail dependence parameters calculated based on the Student-t copula (panel (a)), assume the value 0.1389 in regime 0 and 0.3367 in regime 1, suggesting that both the dependence between extreme negative values and the dependence between extreme positive values increase during turbulent periods such as 2002 and 2007 to 2009. The asymmetric copulas indicate much higher extreme dependence, ranging from 0.4 to 0.55 in regime 0 and 0.6 to 0.75 in regime 1. Based on the BB1 copula (panel (b)), $\hat{\lambda}_L$ is lower than $\hat{\lambda}_U$ in regime 0, and higher than this in regime 1. Based on the Symmetrized Joe-Clayton (panel (d)), $\hat{\lambda}_L$ is higher than $\hat{\lambda}_U$ all along their paths.

The estimated dependence parameters following Silva Filho et al. (2012) are provided in Table 2.5. According to the log-likelihood criterion, we choose the Student-t copula as the best model for both the pairs FTSE-CAC and FTSE-DAX. Note that, differently from the other copulas with two parameters, we assume, in the case of the Student-t, that the degrees of freedom are only regime dependent, they do not follow an ARMA(1,10) process. The reason is that the estimation of the Student-t copula is very time-consuming and, with this assumption, we intend to make this task a little easier. The estimates of the correlation coefficient for both pairs are very persistent, with the estimated β_{1T} equal to 3.3971 and 2.3508 for FTSE-CAC and FTSE-DAX, respectively. The intercept term of the estimated correlation coefficient and the estimated degrees of freedom increase significantly from regime 0 to regime 1 for both pairs of indexes. The regimes are balanced and very persistent: regime 1 lasts 128 days and regime 0 lasts 122, for FTSE-CAC, whereas the expected durations are 555 and 500 days for regimes 1 and 0, respectively, for FTSE-DAX.

To scrutinize the dynamics of the dependence structure between FTSE and CAC captured by these models, we look at Figure 2.9. Interestingly, we notice that the Kendall's tau path estimated based on the Student-t copula (panel (a)) displays a rather similar pattern to the Kendall's tau path computed based on the Normal copula following Jondeau and Rockinger (Figure 2.5, panel (a)). This corroborates the close values of the LogLs obtained for the two models (2147.4 and 2128.8, respectively). The Student-t copula does not indicate some short-lasting increases in dependence in 1999, beginning of 2001 and second semester of 2003, as the Normal copula does, but the models do coincide in determining the regimes for the rest of the sample. Still examining Figure 2.9, $\hat{\tau}_t$ in panel (c), related to the Symmetrized Joe-Clayton copula, behaves similarly to the $\hat{\tau}_t$ in panel (a), related to the Student-t, but the evolution of the dependence measured based on the Rotated-Gumbel, in panel (b), suggests more unbalanced regimes. The patterns of evolution of the tail dependence parameters from the copulas Student-t, Rotated-Gumbel and SJC in Figure 2.10 mimic their counterparts in Figure 2.9. Once again, the tail dependence captured by the asymmetric copulas is higher than that captured by the Student-t, and $\hat{\lambda}_L$ is higher than $\hat{\lambda}_U$ in the regime of high dependence, based on the SJC copula.

For FTSE and DAX, the evolution of the Kendall's tau computed based on the Student-t copula, in Figure 2.11, panel (a), suggests that the dependence between these indexes is characterized by two subperiods, one of low dependence, from 1999 to January, 2004, and another of high dependence, from 2004 on, just as the Student-t copula with only regime dependent parameters, in Figure 2.7, panel (a), indicates. Although, this time, the regimes are even more persistent: the current copula does not indicate a significant increase in dependence by the end of 2002 and beginning of 2003, besides of other short-lasting increases during the period 1999 to 2004, and it does not capture a decrease in dependence in the first semester of 2005, as the other model does. This behavior of the dependence confirms the close values of the LogLs of the two models, 1683.8 for the Student-t as in Silva Filho et al. (2012), and 1671.1 for the Student-t copula following Jondeau and Rockinger (2006). Differently from the results obtained from the elliptical copulas, the asymmetric copulas BB1 and Rotated-Gumbel in Figure 2.11 suggest that regime 0 is more persistent than regime 1. The tail dependence computed based on the current Student-t copula, Figure 2.12, panel (a), does not differ substantially from that based on the Student-t copula with only regime dependent parameters: on average, it measures 0.1490 in regime 0 and 0.3285 in regime 1. Curiously, the tail dependence parameters based on the BB1 copula, panel (b), are more volatile in the low dependence regime than in the high dependence one.

The estimation results of the copulas with regime switching dependence parameters indicate, for both pairs of indexes, the prevalence of a long period of high dependence beginning in 2007, for FTSE-CAC, and 2006, for FTSE-DAX, till the end of 2010. This period coincides with or slightly precedes the beginning of the subprime crisis and goes beyond it. For FTSE and CAC, these copulas also capture significant increases in dependence by the end of 2001 and 2002, whereas for FTSE and DAX, not all copulas following Silva Filho et al. (2012) capture increases in dependence during the crash of 2000 to 2002. Surprisingly, these copulas also suggest increases in dependence during the period 2003 to 2006, when the conditions in the international financial markets were favorable. In addition, we can notice that, for these models, the elliptical copulas have a better fit to the data. The tail dependence, when it exists, is probably symmetrical, given the better fit of the Student-t copula.

Finally, Table 2.6 presents the estimates of the Markov switching copulas combining the time-varying Normal copula in one regime with an asymmetric time-varying copula in the other one. For the pair FTSE-CAC, the only model with all coefficients statistically significant is the combination Normal and Clayton. In the estimation of the combinations with the SJC, Rotated-Gumbel and Gumbel copulas, all coefficients related to the asymmetric copulas are statistically not significant, except for the estimate of w_{RG}. For the model Normal and BB1, the estimates of β_{2bb1} and α_{1bb1} are not significant, indicating lack of autocorrelation in the parameter $\hat{\kappa}_t$ and no influence of the combined movement of the marginal probability transforms in the path of $\hat{\gamma}_t$. For the pair FTSE-DAX, none of the combinations has all coefficients statistically significant: for the Normal-Clayton and the Normal-SJC, coefficients related to the Normal copula are not significant, whereas for Normal and Gumbel and Normal and Rotated-Gumbel, coefficients related to the asymmetric copula have no statistical significance. The Normal-BB1 combination has almost all coefficients statistically not significant.

Figure 2.13 displays, in panel (a), the evolution of the Kendall's tau for the pair FTSE-CAC, computed based on the Normal-Clayton Markov switching copula, and the smoothed probabilities of the high and low dependence regimes, in panels (b) and (c), respectively. As before, we can identify here three subperiods characterizing the dependence between FTSE and CAC. Till the first semester of 2001, we have mainly low dependence between the indexes. From the end of 2001 to the second semester of 2006, the regimes are intercalated, however, now, the low dependence periods are much shorter than before. The low dependence regime has expected duration of 20 days, in contrast with the expected duration of 98 days of the high dependence regime. From 2007 onwards, high dependence prevails. Notice that the two regimes are characterized by different tail behaviors, with no tail dependence at all in the high dependence regime, and lower tail dependence in the low dependence one, which is, actually, opposite to what we expected to find.

2.6.4 Forecasting VaR for Goodness-of-Fit Check and Model Selection

We proceed now to the evaluation of the VaR forecasts based on the estimated copula models. For the VaR computation, we simulate 1000 replications of the portfolio returns for each point in time from the estimated copulas with all coefficients statistically significant. We first evaluate the forecasts by applying the likelihood ratio tests proposed by Kupiec and Christoffersen. Then, we compute the (capital requirement) losses based on the 1% daily VaR forecasts from those models that passed the first tests, and apply the SPA test to determine which model has a superior predictive ability. To facilitate the analysis, we evaluate the forecasts based on the copula models within each methodology separately, in a first moment. After that, the SPA test is applied to the best models chosen in this first trial, to finally find the one that has the best fit to the data in the lower tail of the joint distribution of the returns.

In Table 2.7, the results of the Kupiec and Christoffersen tests for the daily VaR forecasts based on the observation driven time-varying copulas of Patton (2006) are presented. One can see from this table that, for both pairs of indexes, these copulas have good performances in terms of forecasting, since their 1% and 5% VaR forecasts pass both likelihood ratio tests of unconditional coverage and combination of coverage and independence. Notice that the VaR forecasts at the 1% significance level are aggressive, meaning that the forecasted quantile is more extreme than the theoretical VaR, and, at the 5% level, they are more conservative, except for the Rotated-Gumbel forecasts for the pair FTSE-CAC. The average losses computed based on the 1% daily VaR forecasts and the results of the SPA test[12] for the observation driven time-varying copulas are presented in Table 2.8. We implement the SPA test considering each copula at a time as the benchmark and the remaining ones as the competing alternatives. For both

[12] For the stationary block bootstrap, we use 10000 re-samples and select the optimum block length in accordance with Politis and White (2004). We use the Matlab code "opt_block_length_REV_dec07" compiled by Andrew Patton to implement the automatic block length selection.

pairs of indexes, it is not possible to reject the null hypothesis that the best competing copula cannot outperform the Normal copula.

Table 2.9 reports the results of the likelihood ratio tests for the VaR forecasts from the copulas with Markov switching dependence parameters following Jondeau and Rockinger (2006). With the exception of the BB1 copula, with too aggressive 1% daily VaR forecasts, the other copulas pass the Kupiec and Christoffersen tests of the VaR forecasts accuracy. Also for these models, the 1% VaR forecasts are more aggressive than the 5% VaR forecasts, considered quite conservative, except for the Rotated-Gumbel forecasts. In terms of minimizing the regulatory loss function, the results in Table 2.10 suggest that no alternative copula can outperform the Normal copula for both the FTSE-CAC and FTSE-DAX pairs.

The results of the likelihood ratio tests for the forecasts based on the copulas with regime switching parameters following Silva Filho et al. (2012) are presented in Table 2.11, from which is possible to state that the BB1 copula is the only one that cannot appropriately approximate the joint distribution of the FTSE and DAX returns, at least, not in its tails. The results of the SPA test, provided in Table 2.12, indicate that the Student-t performs best in terms of forecasting for the pair FTSE-CAC, whereas the Normal copula is the best choice for FTSE and DAX.

Having selected, for both pairs of indexes, the models with the best performances in forecasting out of each methodology, we now apply the SPA test to them. In the case of FTSE-CAC, we also consider the Normal-Clayton Markov switching copula, since it is the only model remaining from this methodology and it passes the Kupiec and Christoffersen tests, with p-values 0.6048 and 0.5848, respectively, for the VaR forecasts at the 1% level, and 0.3483 and 0.5506, for the 5% VaR forecasts. Table 2.13 displays the results of the SPA test. It is not possible to reject the null hypothesis that the best alternative model cannot outperform the Normal copula with only regime dependent correlation coefficient in the forecasting of the first percentile of the FTSE-CAC portfolio returns. At the same time, one should notice that, at a lower significance level, of 1%, it is not possible to reject the null that the best alternative model is not better than the time-varying Normal copula of Patton(2006), when this one is assumed as the benchmark. In this case, we could assert that both models provide statistically similar predictions of the extreme (negative) quantiles of the FTSE-CAC portfolio. However, the last result is not robust to alternative choices of the block length for the bootstrap. So, we conclude that the Normal copula with regime switching parameter following Jondeau and Rockinger (2006) provides the best VaR forecasts for the portfolio composed by FTSE and CAC, noticing that, amongst the best forecasts, these are the most aggressive ones. For the portfolio formed by FTSE and DAX, the SPA test indicates that no alternative model has a better performance in terms of forecasting the extreme quantiles than the observation driven time-varying Normal copula. Although, at the 1% significance level, we could accept that the Normal copula with the correlation coefficient following an ARMA(1,10) process and intercept changing according to a two-regime Markov chain provides similar forecasts. Nevertheless, this result changes when we assume different block lengths. Hence, according to our results, the time-varying Normal copula of Patton (2006) performs best in forecasting the VaR for the portfolio composed by FTSE and DAX, with less aggressive predictions.

2.7 Concluding Remarks

In this paper, we introduce a new approach to modeling dependence between financial return data over time, combining time-varying copulas and the Markov switching model, and we employ this methodology and also those proposed by Patton (2006), Jondeau and Rockinger (2006) and Silva Filho et al. (2012) to model the dependence structures between the FTSE returns and the returns of the indexes CAC-40 and DAX, respectively, over the period 1999 to April 2011. We also use the copula estimates to carry out tail inferences and find out the copula model with the best performance in forecasting the extreme quantiles for each portfolio composed by the pairs of indexes aforementioned.

Based on the log-likelihood criterion, we find strong evidences of tail dependence for the pairs FTSE-CAC and FTSE-DAX from the estimation of the observation driven time-varying copulas, since the Normal copula has the lowest value for the log-likelihood amongst all copulas. For FTSE and DAX, the tail dependence is asymmetric, with stronger lower tail. On the other hand, analyzing the estimation results of the copula models with regime switching dependence parameters, the elliptical copulas have a better fit to the data according to the same criterion, suggesting that, the tail dependence, when it exists, is probably symmetrical, given the better fit of the Student-t copula. We barely can find empirical evidences of time-varying copulas with distinct tail behaviors characterizing the dependence between the returns according to the international equity markets regimes. The combination of a high dependence regime characterized by the time-varying Normal copula and a low dependence regime associated with the time-varying Clayton copula for FTSE and CAC, though statistically significant, does not show a good fit to the data.

Both the observation driven time-varying copulas and the copula models with Markov switching identify a long period of high dependence from 2007 to the end of 2010, for both the pairs FTSE-CAC and FTSE-DAX. This period coincides with or slightly precedes the beginning of the subprime crisis and goes beyond it. The copula models with regime switching also capture significant increases in dependence during the period 2001 to 2002, especially between FTSE and CAC, which can be explained by the market crash due to the "dot-com bubble", besides of the so called September 11 effect. Unexpectedly, these models also indicate increases in dependence during the period 2003 to 2006, when the financial markets did well.

Because in risk management the main concern has to do with tail risks, we compare and select the copula models based on VaR forecasts. To evaluate the accuracy of these predictions, we apply the traditional Kupiec and Christoffersen tests, besides of an alternative methodology based on standard forecast evaluation techniques, i.e, the accuracy of the VaR forecasts is determined by how well they minimize a certain regulatory function. We also apply the SPA test by Hansen (2005) to determine which copula model significantly minimizes the expected loss function. Based on the results of the SPA test, we cannot reject the null hypothesis that the best alternative model cannot outperform the Normal copula with only regime dependent correlation coefficient in the forecasting of the extreme (negative) quantiles of the FTSE-CAC portfolio returns. This result suggests that the high persistence in the linear correlation coefficient between FTSE and CAC captured by the observation driven time-varying Student-t and Normal copulas is inappropriate, and the long-memory feature of this dependence parameter is, in fact, a consequence of a model with large but infrequent breaks, as the Markov switching model. Also according to the results of the SPA test, no alternative copula model can outperform the time-varying Normal copula of Patton (2006) in the forecasting of the VaR losses for the portfolio composed by FTSE and DAX returns. Surprisingly, the Normal copula has the best fit to the tails of the joint distributions of the returns, which is in contrast to the evidences we expected to find for financial return data and also to the evidences found in the literature of static copulas, where usually copulas that feature tail dependence and asymmetry show better fits. It seems that part of the asymmetry may be generated by time-varying parameters. Also, the lack of tail dependence may be partially compensated by the possibility of large overall dependence, which would explain why the Gaussian copula fits the data so well.

2.8 References

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2.9 Appendix A: Copula Functions

Normal copula: the Normal copula, extracted from the bivariate Normal distribution, is defined as follows:

$$C_N(u_1,u_2|\rho)=\int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)}\frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{\frac{-(r^2-2\rho rs+s^2)}{2(1-\rho^2)}\right\}drds, \rho\in(-1,1),$$

where the dependence parameter, ρ, is the linear correlation coefficient. Its dynamic equation may be written as[13]

$$\rho_t=\Lambda\left(\omega_N+\beta_N\rho_{t-1}+\alpha_N\cdot\frac{1}{10}\sum_{j=1}^{10}\Phi^{-1}(u_{1,t-j})\cdot\Phi^{-1}(u_{2,t-j})\right).$$

The Normal copula is symmetric and has no tail dependence, that is, $\lambda_L=\lambda_U=0$. The Kendall's tau may be computed based on the correlation coefficient as $\tau=(2/\pi)\arcsin\rho$.

Student-t copula: it is associated with the bivariate Student-t distribution and has the following functional form:

$$C_T(u_1,u_2|\rho,\nu)=\int_{-\infty}^{t_\nu^{-1}(u_1)}\int_{-\infty}^{t_\nu^{-1}(u_2)}\frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{r^2-2\rho rs+s^2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}drds,$$

where the parameters ρ and ν are the linear correlation coefficient and the degrees of freedom, respectively. In addition, their evolution equations are given by

$$\rho_t=\Lambda\left(\omega_{1T}+\beta_{1T}\rho_{t-1}+\alpha_{1T}\cdot\frac{1}{10}\sum_{j=1}^{10}T_\nu^{-1}(u_{1,t-j})\cdot T_\nu^{-1}(u_{2,t-j})\right)$$

and

$$\nu_t=\tilde{\Lambda}\left(\omega_{2T}+\beta_{2T}\nu_{t-1}+\alpha_{2T}\cdot\frac{1}{10}\sum_{j=1}^{10}T_\nu^{-1}(u_{1,t-j})\cdot T_\nu^{-1}(u_{2,t-j})\right).$$

The Student-t copula has symmetrical tail dependence, with $\lambda_L=\lambda_U=2T_{\nu+1}(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}})$, where $T_{\nu+1}$ is the Student-t *c.d.f.* with $(\nu+1)$ degrees of freedom. The Kendall's tau is given by $\tau=(2/\pi)\arcsin\rho$.

Gumbel copula: it has the form of

$$C_G(u_1,u_2|\theta)=\exp\left(-\left((-\log u_1)^\theta+(-\log u_2)^\theta\right)^{1/\theta}\right),\ \theta\in[1,\infty).$$

The dynamics is given by the following equation governing the dependence parameter evolution:

$$\theta_t=\Lambda\left(\omega_G+\beta_G\theta_{t-1}+\alpha_G\cdot\frac{1}{10}\sum_{j=1}^{10}|u_{1,t-j}-u_{2,t-j}|\right).$$

[13] $\Lambda(\cdot)$ and $\tilde{\Lambda}(\cdot)$, which appear hereafter, are logistic transformations to keep the parameters in their intervals.

The Gumbel copula exhibits only upper tail dependence, with $\lambda_U = 2 - 2^{1/\theta}$. It can be shown that the Kendall's tau is given by $\tau = 1 - \theta^{-1}$.

Rotated-Gumbel copula: or Survival Gumbel copula, which is the complement ("Probability of survival") of the Gumbel copula. It has the following form:

$$C_{RG}(u_1, u_2|\theta) = u_1 + u_2 - 1 + C_G(1 - u_1, 1 - u_2|\theta),$$

where C_G corresponds to the Gumbel copula. The dependence parameter, θ, follows the process

$$\theta_t = \Lambda \left(\omega_{RG} + \beta_{RG}\theta_{t-1} + \alpha_{RG} \cdot \frac{1}{10} \sum_{j=1}^{10} |u_{1,t-j} - u_{2,t-j}| \right).$$

The Rotated-Gumbel copula has only lower tail dependence, given by $\lambda_L = 2 - 2^{1/\theta}$, and the Kendall's tau may be computed as $\tau = 1 - \theta^{-1}$.

Clayton copula: or Kimeldorf-Sampson copula, has the following distribution function:

$$C_C(u_1, u_2|\delta) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-1/\delta},\ \delta \in (0, \infty).$$

The evolution equation of the dependence parameter is

$$\delta_t = \Lambda \left(\omega_C + \beta_C\delta_{t-1} + \alpha_C \cdot \frac{1}{10} \sum_{j=1}^{10} |u_{1,t-j} - u_{2,t-j}| \right).$$

This copula exhibits only lower tail dependence, $\lambda_L = 2^{-1/\delta}$. The Kendall's tau has the form $\tau = \delta/(\delta + 2)$.

Symmetrized Joe-Clayton copula: this copula was defined by Patton (2006) and takes the form of

$$C_{SJC}(u_1, u_2|\lambda_U, \lambda_L) = 0.5 \cdot \left(C_{JC}\left(u_1, u_2|\lambda_U, \lambda_L\right) + C_{JC}\left(1 - u_1, 1 - u_2|\lambda_U, \lambda_L\right) + u_1 + u_2 - 1\right),$$

where C_{JC} is the Joe-Clayton copula, also called BB7 copula (Joe, 1997), given by

$$C_{JC}\left(u_1, u_2|\lambda_U, \lambda_L\right) = 1 - \left(1 - \left\{[1 - (1 - u_1)^\kappa]^{-\gamma} + [1 - (1 - u_2)^\kappa]^{-\gamma} - 1\right\}^{-1/\gamma}\right)^{-1/\kappa},$$

with $\kappa = 1/\log_2\left(2 - \lambda_U\right),\ \gamma = -1/\log_2\left(\lambda_L\right)$ and $\lambda_U, \lambda_L \in (0, 1)$.

The SJC copula has upper and lower tail dependence and its dependence parameters are the upper and lower tail dependence parameters, λ_U and λ_L, respectively. Furthermore, λ_U and λ_L range freely and are not dependent on each other. Since this copula nests symmetry as a special case, it is a more interesting specification than the BB7 copula. The evolution equations for the parameters λ_U and λ_L are

$$\lambda_{Ut} = \Lambda\left(\omega_U + \beta_U \lambda_{Ut-1} + \alpha_U \cdot \frac{1}{10}\sum_{j=1}^{10} |u_{1,t-j} - u_{2,t-j}|\right)$$

and

$$\lambda_{Lt} = \Lambda\left(\omega_L + \beta_L \lambda_{Lt-1} + \alpha_L \cdot \frac{1}{10}\sum_{j=1}^{10} |u_{1,t-j} - u_{2,t-j}|\right).$$

The Kendall's tau, in this case, has no closed form, so we compute it numerically.

BB1 copula (Joe, 1997): it has the following functional form:

$$C_{bb1}(u_1, u_2, \kappa, \gamma) = \{1 + [(u_1^{-\kappa} - 1)^{\gamma} + (u_2^{-\kappa} - 1)^{\gamma}]^{1/\gamma}\}^{-1/\kappa},\ \kappa \in (0, \infty), \gamma \in [1, \infty).$$

The dynamic equations of the dependence parameters are

$$\kappa_t = \Lambda\left(\omega_{1bb1} + \beta_{1bb1}\kappa_{t-1} + \alpha_{1bb1} \cdot \frac{1}{10}\sum_{j=1}^{10} |u_{1,t-j} - u_{2,t-j}|\right)$$

$$\gamma_t = \tilde{\Lambda}\left(\omega_{2bb1} + \beta_{2bb1}\gamma_{t-1} + \alpha_{2bb1} \cdot \frac{1}{10}\sum_{j=1}^{10} |u_{1,t-j} - u_{2,t-j}|\right).$$

The BB1 copula has upper and lower tail dependence given by $\lambda_U = 2 - 2^{1/\gamma}$ and $\lambda_L = 2^{-1/\gamma\kappa}$, respectively. The Kendall's tau may be calculated based on κ and γ as $\tau = 1 - (2/(\gamma(\kappa + 2)))$.

2.10 Appendix B: Tables

Table 2.1: Summary statistics of the FTSE, CAC-40 and DAX daily log-returns.

	FTSE	CAC-40	DAX
Mean	1.1098E-05	1.2715E-05	1.3427E-04
Median	3.3649E-04	2.3667E-04	8.7989E-04
Maximum	0.0938	0.1059	0.1080
Minimum	-0.0926	-0.0947	-0.0958
Std. Deviation	0.0131	0.0157	0.0164
Asymmetry	-0.0675	0.0509	-0.0048
Kurtosis	8.6249	7.9411	7.4391
Jarque-Bera	3968.9 (0.000)	3061.8 (0.000)	2470.1 (0.000)

Note: Jarque-Bera corresponds to Jarque-Bera test statistics with p-values in parentheses.

Table 2.2: Estimates from the univariate ARMA-GARCH models.

	Conditional Mean Equation		
Parameter	FTSE	CAC-40	DAX
ϕ_1	-0.0577	-0.0369	...
	(0.0002)	(0.0013)	...
ϕ_2	-0.0637	-0.0440	...
	(0.0005)	(0.0002)	...
ϕ_3	-0.0731	-0.0626	...
	(0.0006)	(0.0021)	...
ϕ_4	...	...	0.0444
	...	...	(0.0024)
	Conditional Variance Equation		
Parameter	FTSE	CAC-40	DAX
α_0	1.1588E-06	1.5820E-06	1.7875E-06
	(0.0000)	(0.0000)	(0.0000)
α_1	0.0933	0.0761	0.0821
	(0.0117)	(0.0101)	(0.0114)
β_1	0.9012	0.9181	0.9127
	(0.0119)	(0.0102)	(0.0112)
ν	21.6701	13.2143	10.9277
	(7.5025)	(3.0347)	(2.2477)
λ	-0.1430	-0.1229	-0.1112
	(0.0247)	(0.0258)	(0.0225)
$Q(15)$	0.5504	0.5880	0.5372
$Q^2(15)$	0.2378	0.5044	0.4330
K-S Test	0.8464	0.9835	0.2673

Note: Standard errors in parentheses. $Q(15)$, $Q^2(15)$ and K-S Test are p-values.

Table 2.3: Estimates from the observation driven time-varying copulas of Patton (2006).

	FTSE-CAC	FTSE-DAX		FTSE-CAC	FTSE-DAX		FTSE-CAC	FTSE-DAX
Coefficient						Coefficient		
	Normal						*Rotated-Gumbel*	
ω_N	-1.3084	-0.7882				ω_{RG}	0.9170	0.6591
	(0.3294)	(0.0002)					(0.0502)	(0.0623)
β_N	4.2331	3.5251				β_{RG}	0.2327	0.2872
	(0.4598)	(0.0001)					(0.0080)	(0.0138)
α_N	0.3189	0.2397				α_{RG}	-1.9806	-1.2652
	(0.0768)	(0.0017)					(0.2421)	(0.2145)
LogL	2013.5	1570.5				LogL	2058.8	**1594.3**
	FTSE-CAC	FTSE-DAX		FTSE-CAC	FTSE-DAX		FTSE-CAC	FTSE-DAX
Coefficient			Coefficient			Coefficient		
	BB1			*Student-t*			*Symmetrized Joe-Clayton*	
ω_{1bb1}	0.8677	0.9711*	ω_{1T}	-1.7873	-0.8011	ω_U	1.4412	3.5787
	(0.1764)	(0.9019)		(0.2373)	(0.0008)		(0.2052)	(0.1787)
ω_{2bb1}	0.6240	0.5411*	ω_{2T}	-3.0778	-2.2559	ω_L	1.6559	0.7553
	(0.0872)	(0.3916)		(0.4748)	(0.4064)		(0.5029)	(0.2006)
β_{1bb1}	0.1360*	-0.2172*	β_{1T}	4.8885	3.5476	β_U	0.0546*	-2.2294
	(0.1935)	(0.6566)		(0.3172)	(0.0064)		(1.6770)	(0.5639)
β_{2bb1}	0.2915	0.3194	β_{2T}	-0.0797	-0.0475	β_L	0.4801*	0.8895
	(0.0194)	(0.1076)		(0.0308)	(0.0015)		(0.8199)	(0.1121)
α_{1bb1}	-1.1909**	-0.3940*	α_{1T}	0.1858	0.1987	α_U	-7.1533	-14.4149
	(0.7205)	(3.5840)		(0.0350)	(0.0517)		(0.6647)	(1.2824)
α_{2bb1}	-1.5880	-1.5886*	α_{2T}	0.7947	-0.0273*	α_L	-9.4762	-5.4483
	(0.4010)	(1.4258)		(0.2479)	(0.2528)		(2.7984)	(1.5493)
LogL	2125.5	1643.3	LogL	**2059.6**	1609.7	LogL	2032.0	1575.4

Note: Standard errors in parentheses. (*) stands for statistically not significant, whereas (**) stands for significant only at the 10% level. The notation here is as presented in Section 2.2 and in Appendix A.

Table 2.4: Estimates from the copulas with regime switching parameters as in Jondeau and Rockinger (2006).

Coefficient	FTSE-CAC	FTSE-DAX				Coefficient	FTSE-CAC	FTSE-DAX
	Normal						*Rotated-Gumbel*	
ρ_0	0.7659	0.6928				θ_0	2.1110	1.8827
	(0.0186)	(0.0385)					(0.0938)	(0.0737)
ρ_1	0.9253	0.8988				θ_1	3.7159	3.1997
	(0.0059)	(0.0153)					(0.1589)	(0.2039)
p	0.9898	0.9785				p	0.9899	0.9887
	(0.0733)	(0.1045)					(0.0771)	(0.1535)
q	0.9880	0.9775				q	0.9871	0.9886
	(0.0892)	(0.1431)					(0.0839)	(0.1358)
LogL	**2128.8**	1666.0				LogL	2045.0	1601.2
Coefficient	FTSE-CAC	FTSE-DAX	Coefficient	FTSE-CAC	FTSE-DAX	Coefficient	FTSE-CAC	FTSE-DAX
	BB1			*Student-t*			*Symmetrized Joe-Clayton*	
κ_0	0.5547	0.4812	ρ_0	0.7793	0.7093	λ_{U0}	0.5434	0.4673
	(0.1038)	(0.0837)		(0.0192)	(0.0252)		(0.0278)	(0.0381)
κ_1	0.8168	0.8197	ρ_1	0.9275	0.8818	λ_{U1}	0.8011	0.6556
	(0.1223)	(0.1378)		(0.0076)	(0.0194)		(0.0000)	(0.0054)
γ_0	1.7324	1.5566	ν_0	11.5834	13.4763	λ_{L0}	0.6167	0.5031
	(0.0819)	(0.0718)		(3.6496)	(6.1923)		(0.0178)	(0.0425)
γ_1	2.6850	2.1590	ν_1	41.3768*	14.5128	λ_{L1}	0.7804	0.7539
	(0.2337)	(0.0862)		(54.9352)	(6.8262)		(0.0001)	(0.0743)
p	0.9910	0.9953	p	0.9906	0.9963**	p	0.9868	0.9931
	(0.0736)	(0.2303)		(0.0827)	(0.5380)		(0.0796)	(0.1811)
q	0.9900	0.9948	q	0.9906	0.9959**	q	0.9883	0.9924
	(0.0837)	(0.2269)		(0.0814)	(0.5286)		(0.0984)	(0.1945)
LogL	2102.8	1637.4	LogL	2138.1	**1671.7**	LogL	2003.8	1570.0

Note: Standard errors in parentheses. (*) stands for statistically not significant, whereas (**) stands for significant only at the 10% level. The notation here is as presented in Section 2.2 and in Appendix A.

Table 2.5: Estimates from the copulas with regime switching parameters as in Silva Filho et al. (2012).

Coefficient	FTSE-CAC	FTSE-DAX				Coefficient	FTSE-CAC	FTSE-DAX
	Normal						*Rotated-Gumbel*	
ω_{N0}	0.4226*	1.3001				ω_{RG0}	-1.0715	-0.4675
	(0.6034)	(0.1364)					(0.0916)	(0.0867)
ω_{N1}	1.2766**	2.3473				ω_{RG1}	0.9764	2.5377
	(0.7531)	(0.1020)					(0.0538)	(0.0740)
β_N	1.7286**	0.1549				β_{RG}	0.2222	-0.3209
	(0.9405)	(0.0657)					(0.0081)	(0.0142)
α_N	0.3493	0.3737				α_{RG}	-2.0588	0.6656**
	(0.1084)	(0.0921)					(0.2819)	(0.3521)
p	0.9922	0.9900				p	0.9978	0.9910
	(0.0826)	(0.2225)					(0.0794)	(0.1584)
q	0.9888	0.9864				q	0.9894	0.9945
	(0.1076)	(0.2934)					(0.2109)	(0.1605)
LogL	2145.1	1675.2				LogL	2079.7	1604.2
	FTSE-CAC	FTSE-DAX		FTSE-CAC	FTSE-DAX		FTSE-CAC	FTSE-DAX
Coefficient			Coefficient			Coefficient		
	BB1			*Student-t*			*Symmetrized Joe-Clayton*	
ω_{10bb1}	0.2880**	-0.4544	ω_{10T}	-0.6644	-0.0229	ω_{U0}	1.1802	2.1219
	(0.1668)	(0.2262)		(0.0588)	(0.0085)		(0.1087)	(0.1156)
ω_{11bb1}	0.5221	0.9416	ω_{11T}	-0.0393**	0.5144	ω_{U1}	1.7125	2.1413
	(0.0482)	(0.2045)		(0.0236)	(0.0526)		(0.0045)	(0.2027)
ω_{20bb1}	-1.4688	-0.4549	ν_0	11.9186	12.1589	ω_{L0}	0.1471	-0.0631*
	(0.2986)	(0.1928)		(3.7720)	(4.0464)		(0.0783)	(0.1197)
ω_{21bb1}	0.7128	1.5930	ν_1	15.0000	15.0000	ω_{L1}	0.9387	0.6713
	(0.0663)	(0.2089)		(2.4045)	(3.7565)		(0.0019)	(0.0682)
β_{1bb1}	0.4861	0.2808	β_{1T}	3.3971	2.3508	β_U	-0.2633	-0.6172
	(0.0404)	(0.0849)		(0.1194)	(0.0561)		(0.0001)	(0.1901)
β_{2bb1}	0.2714	-0.3478	α_{1T}	0.1114	0.1492	β_L	0.6745	1.0390
	(0.0153)	(0.0609)		(0.0271)	(0.0351)		(0.0008)	(0.0480)
α_{1bb1}	-0.1265*	-2.1677	...	...	...	α_U	-5.4357	-10.7196
	(0.1232)	(0.9771)		...	...		(0.0214)	(0.6474)
α_{2bb1}	-2.1346	1.4201	...	...	...	α_L	-1.6781	-2.7932
	(0.3201)	(0.5684)		...	...		(0.0050)	(0.0344)
p	0.9965	0.9952	p	0.9922	0.9982	p	0.9946	0.9931
	(0.1681)	(0.2316)		(0.0793)	(0.2217)		(0.0614)	(0.1428)
q	0.9780	0.9957	q	0.9918	0.9980	q	0.9914	0.9923
	(0.3329)	(0.1968)		(0.0764)	(0.2067)		(0.0794)	(0.1647)
LogL	2143.2	1657.2	LogL	**2147.4**	**1683.8**	LogL	2037.4	1599.2

Note: Standard errors in parentheses. (*) stands for statistically not significant, whereas (**) stands for significant only at the 10% level. The notation here is as presented in Section 2.2 and in Appendix A.

Table 2.6: Estimates from the Markov switching copulas.

Coefficient	FTSE-CAC	FTSE-DAX	Coefficient	FTSE-CAC	FTSE-DAX	Coefficient	FTSE-CAC	FTSE-DAX
	Normal/Clayton			*Normal/Gumbel*			*Normal/Rotated-Gumbel*	
ω_N	-3.4314	1.5731*	ω_N	-3.4070	5.1152	ω_N	-3.6888	0.7360
	(0.1346)	(8.5125)		(0.3598)	(0.2004)		(0.1479)	(0.0223)
ω_C	-0.5596	-1.0128	ω_G	1.0634*	0.3575*	ω_{RG}	1.0596	-1.6510
	(0.0705)	(0.3585)		(5.5451)	(0.3557)		(0.2923)	(0.1510)
β_N	6.9297	0.9382*	β_N	6.8881	-3.1589	β_N	7.2237	1.9599
	(0.1616)	(9.9745)		(0.4349)	(0.2944)		(0.1739)	(0.0391)
β_C	-0.4039	0.4156	β_G	0.0730*	0.2976	β_{RG}	0.0524*	0.4867
	(0.0290)	(0.0426)		(2.1589)	(0.0858)		(0.0948)	(0.0431)
α_N	0.1779	0.2852**	α_N	0.2403	0.3732**	α_N	0.2256	0.2995
	(0.0252)	(0.1540)		(0.0482)	(0.1919)		(0.0300)	(0.0598)
α_C	-0.2491	-1.8673*	α_G	-1.3699*	-0.2412*	α_{RG}	-0.9710*	-0.3152*
	(0.1009)	(1.5211)		(7.7933)	(0.9439)		(0.7403)	(0.7471)
p	0.9898	0.9856	p	0.9944	0.9891	p	0.9937	0.9900
	(0.0988)	(0.0889)		(0.0635)	(0.0915)		(0.0930)	(0.0655)
q	0.9512	0.9474	q	0.9876	0.9740	q	0.9875	0.9793
	(0.2973)	(0.1948)		(0.1012)	(0.2020)		(0.1095)	(0.1144)
LogL	2095.7	1638.4	LogL	2119.9	1634.7	LogL	2130.9	1659.9

Coefficient	FTSE-CAC	FTSE-DAX	Coefficient	FTSE-CAC	FTSE-DAX
	Normal/BB1			*Normal/Symmetrized Joe-Clayton*	
ω_N	-3.9135	1.6258**	ω_N	-3.4995	1.4217
	(0.1430)	(0.8553)		(1.0684)	(0.4894)
ω_{1bb1}	0.4021	1.2382	ω_U	-0.0842*	0.2897
	(0.0438)	(0.4464)		(3.1934)	(0.1237)
ω_{2bb1}	0.8370	0.3780*	ω_L	0.5731*	1.4937
	(0.4011)	(0.6279)		(4.2548)	(0.0061)
β_N	7.4839	-0.3579*	β_N	6.9979	0.1849*
	(0.1666)	(1.4047)		(1.2537)	(0.5423)
β_{1bb1}	0.5597	-0.1331*	β_U	1.3566*	1.9657
	(0.0597)	(0.2992)		(8.2579)	(0.0392)
β_{2bb1}	0.1587*	0.3600**	β_L	-0.1108*	-0.0790
	(0.1678)	(0.1857)		(0.2036)	(0.0258)
α_N	0.2413	0.5031	α_N	0.2389	0.4739
	(0.0348)	(0.1864)		(0.0619)	(0.2082)
α_{1bb1}	0.1784*	-1.7946*	α_U	-3.6760*	-8.0732
	(0.2448)	(4.6244)		(12.0692)	(1.8708)
α_{2bb1}	-1.6811	-0.7594*	α_L	-1.7287*	-1.8739
	(0.7918)	(2.1270)		(6.5093)	(0.0897)
p	0.9945	0.9859	p	0.9945	0.9822
	(0.0805)	(0.2587)		(0.1822)	(0.0598)
q	0.9923	0.9894	q	0.9887	0.9757
	(0.1263)	(0.2219)		(0.1320)	(0.0384)
LogL	2151.8	1667.4	LogL	2137.0	1633.5

Note: Standard errors in parentheses. (*) stands for statistically not significant, whereas (**) stands for significant only at the 10% level. The notation here is as presented in Section 2.2 and in Appendix A.

Table 2.7: Results of the Kupiec and Christoffersen tests for the daily VaR forecasts from the time-varying copulas of Patton (2006).

	FTSE-CAC			
Copula	α	n_1/T	Kupiec	Christoffersen
Student-t	1%	0.0096	0.8351	0.7310
	5%	0.0531	0.4389	0.5157
Rotated-Gumbel	1%	0.0080	0.2446	0.4157
	5%	0.0475	0.5191	0.7318
Normal	1%	0.0093	0.6931	0.7047
	5%	0.0501	0.9767	0.9891
	FTSE-DAX			
Copula	α	n_1/T	Kupiec	Christoffersen
Rotated-Gumbel	1%	0.0083	0.3332	0.2887
	5%	0.0504	0.9103	0.9881
Symmetrized Joe-Clayton	1%	0.0090	0.5596	0.6548
	5%	0.0521	0.5980	0.8229
Normal	1%	0.0093	0.6931	0.7047
	5%	0.0524	0.5420	0.8240

Note: Kupiec and Christoffersen correspond to the p-values of the respective tests.

Table 2.8: Average losses computed based on the 1% daily VaR forecasts from the time-varying copulas of Patton (2006) and the results of the SPA test.

FTSE-CAC		FTSE-DAX	
Benchmark	Average Loss (%)	Benchmark	Average Loss (%)
Student-t	9.9677	Rotated-Gumbel	10.2728
	(0.0000)		(0.0000)
Rotated-Gumbel	10.0320	Symmetrized Joe-Clayton	10.2461
	(0.0000)		(0.0000)
Normal	**9.7856**	**Normal**	**9.9492**
	(**1.0000**)		(**1.0000**)

Note: In parentheses, we have the p-value of the SPA test.

Table 2.9: Results of the Kupiec and Christoffersen tests for the daily VaR forecasts from the copulas with regime switching dependence parameters following Jondeau and Rockinger (2006).

	FTSE-CAC			
Copula	α	n_1/T	Kupiec	Christoffersen
Normal	1%	0.0086	0.4387	0.5857
	5%	0.0514	0.7174	0.8280
BB1	1%	0.0056	**0.0088**	**0.0293**
	5%	0.0451	0.2134	0.3892
Rotated-Gumbel	1%	0.0090	0.5596	0.6548
	5%	0.0498	0.9566	0.9297
Symmetrized Joe-Clayton	1%	0.0083	0.3332	0.5037
	5%	0.0531	0.4389	0.6097
	FTSE-DAX			
Copula	α	n_1/T	Kupiec	Christoffersen
Student-t	1%	0.0076	0.1731	0.3287
	5%	0.0528	0.4889	0.4923
Normal	1%	0.0090	0.5596	0.4225
	5%	0.0518	0.6565	0.6772
BB1	1%	0.0043	**0.0004**	**0.0018**
	5%	0.0465	0.3679	0.5328
Rotated-Gumbel	1%	0.0086	0.4387	0.3497
	5%	0.0498	0.9566	0.8054
Symmetrized Joe-Clayton	1%	0.0093	0.6931	0.4899
	5%	0.0511	0.7802	0.8893

Note: Kupiec and Christoffersen correspond to the p-values of the respective tests.

Table 2.10: Average losses computed based on the 1% daily VaR forecasts from the copulas with regime switching dependence parameters following Jondeau and Rockinger (2006) and the results of the SPA test.

FTSE-CAC		FTSE-DAX	
Benchmark	Average Loss (%)	Benchmark	Average Loss (%)
Normal	**9.7691**	Student-t	9.9250
	(**1.0000**)		(0.0002)
Rotated-Gumbel	9.9397	**Normal**	**9.7691**
	(0.0000)		(**1.0000**)
Symmetrized Joe-Clayton	9.9931	Rotated-Gumbel	10.1896
	(0.0000)		(0.0000)
...	...	Symmetrized Joe-Clayton	10.3284
	...		(0.0000)

Note: In parentheses, we have the p-value of the SPA test.

Table 2.11: Results of the Kupiec and Christoffersen tests for the daily VaR forecasts from the copulas with regime switching dependence parameters following Silva Filho et al. (2012).

	FTSE-CAC			
Copula	α	n_1/T	Kupiec	Christoffersen
Student-t	1%	0.0090	0.5596	0.4225
	5%	0.0561	0.1321	0.2991
Rotated-Gumbel	1%	0.0080	0.2446	0.4157
	5%	0.0485	0.6961	0.4849
Symmetrized Joe-Clayton	1%	0.0083	0.3332	0.5037
	5%	0.0538	0.3483	0.5506
	FTSE-DAX			
Copula	α	n_1/T	Kupiec	Christoffersen
Student-t	1%	0.0103	0.8740	0.6067
	5%	0.0524	0.5420	0.7824
Normal	1%	0.0086	0.4387	0.3497
	5%	0.0508	0.8446	0.8400
BB1	1%	0.0043	**0.0004**	**0.0018**
	5%	0.0455	0.2469	0.4643
Rotated-Gumbel	1%	0.0083	0.3332	0.2774
	5%	0.0495	0.8901	0.7806

Note: Kupiec and Christoffersen correspond to the p-values of the respective tests.

Table 2.12: Average losses computed based on the 1% daily VaR forecasts from the copulas with regime switching dependence parameters following Silva Filho et al. (2012) and the results of the SPA test.

FTSE-CAC		FTSE-DAX	
Benchmark	Average Loss (%)	Benchmark	Average Loss (%)
Student-t	**9.8933**	Student-t	10.0754
	(1.0000)		(0.0010)
Rotated-Gumbel	9.9747	**Normal**	**9.9681**
	(0.0001)		**(1.0000)**
Symmetrized Joe-Clayton	9.9952	Rotated-Gumbel	10.2184
	(0.0012)		(0.0000)

Note: In parentheses, we have the p-value of the SPA test.

Table 2.13: Average losses computed based on the 1% daily VaR forecasts and the results of the SPA test for the best models out of each methodology for the pairs FTSE-CAC and FTSE-DAX.

FTSE-CAC		FTSE-DAX	
Benchmark	Average Loss (%)	Benchmark	Average Loss (%)
Normal (Patton (2006))	9.7856	**Normal (Patton(2006))**	**9.9493**
	(0.0339)		**(1.0000)**
Normal (J&R(2006))	**9.7691**	Normal (J&R(2006))	9.9814
	(1.0000)		(0.0000)
Student-t (SFZ&D(2012))	9.8933	Normal (SFZ&D(2012))	9.9681
	(0.0000)		(0.0338)
Normal-Clayton (MS copula)	9.9922	...	...
	(0.0000)		...

Note: In parentheses, we have the p-value of the SPA test.

2.11 Appendix C: Figures

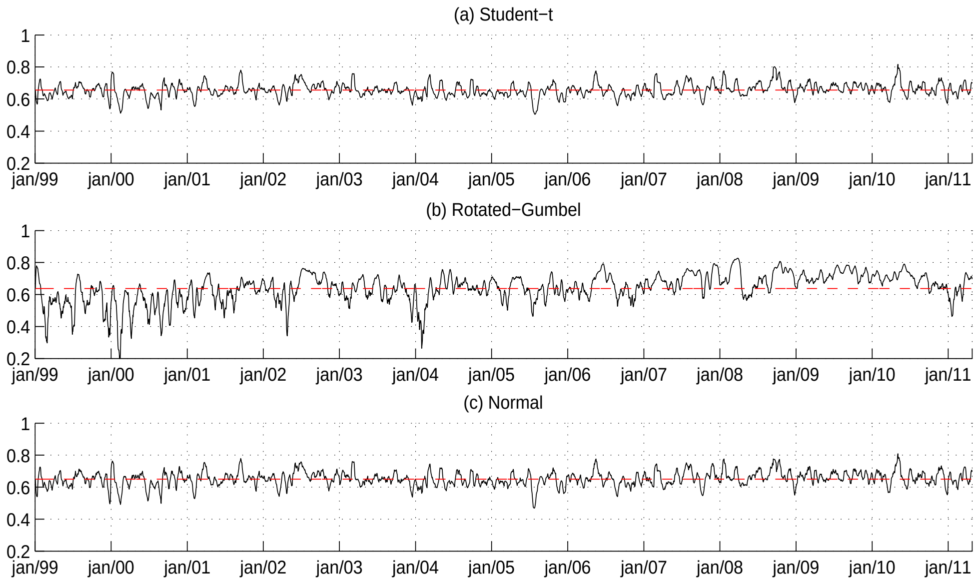

Figure 2.1: The evolution of the Kendall's tau following Patton (2006), for the FTSE-CAC pair.

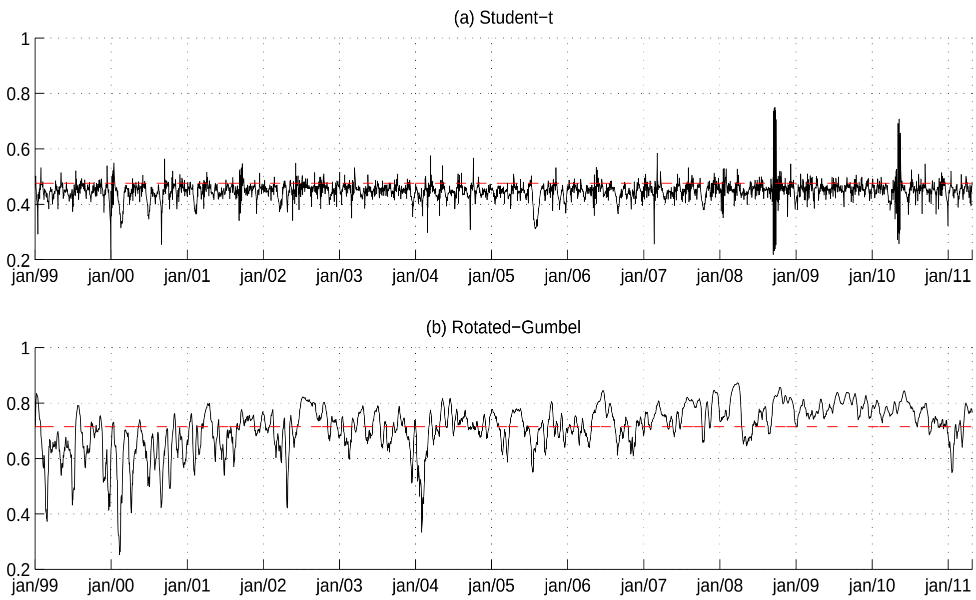

Figure 2.2: The evolutions of the tail dependence parameters following Patton (2006), for the FTSE-CAC pair.

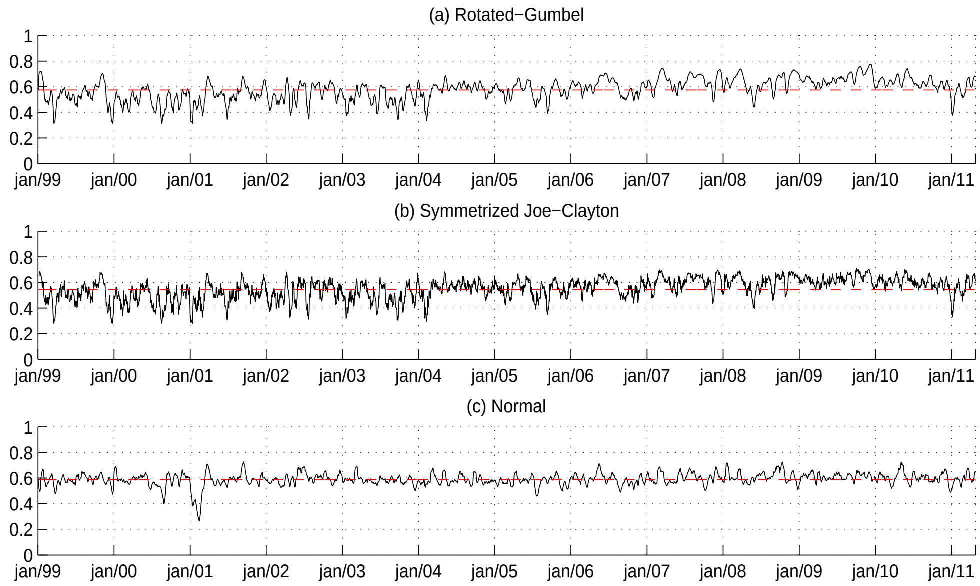

Figure 2.3: The evolution of the Kendall's tau following Patton (2006), for the FTSE-DAX pair.

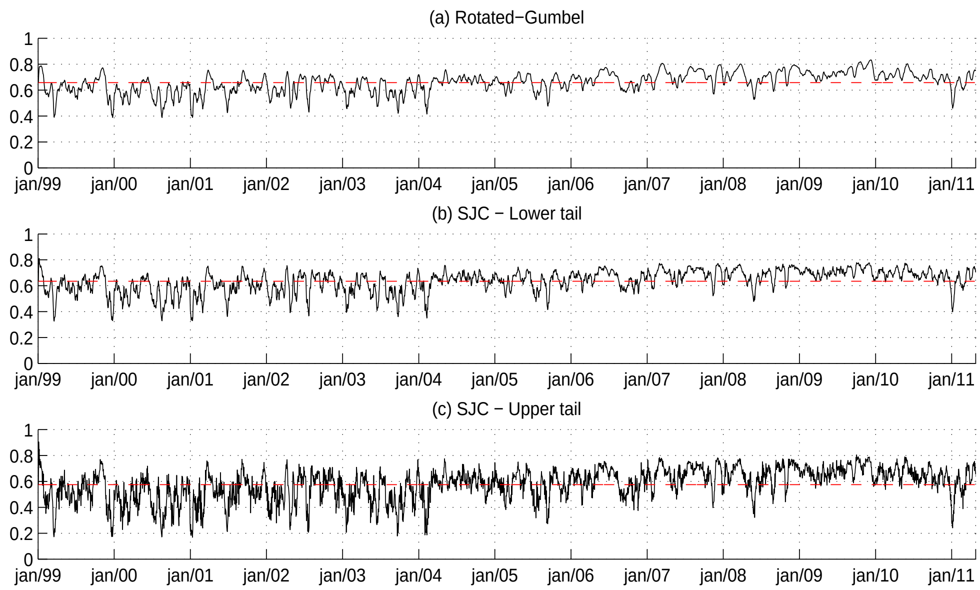

Figure 2.4: The evolutions of the tail dependence parameters following Patton (2006), for the FTSE-DAX pair.

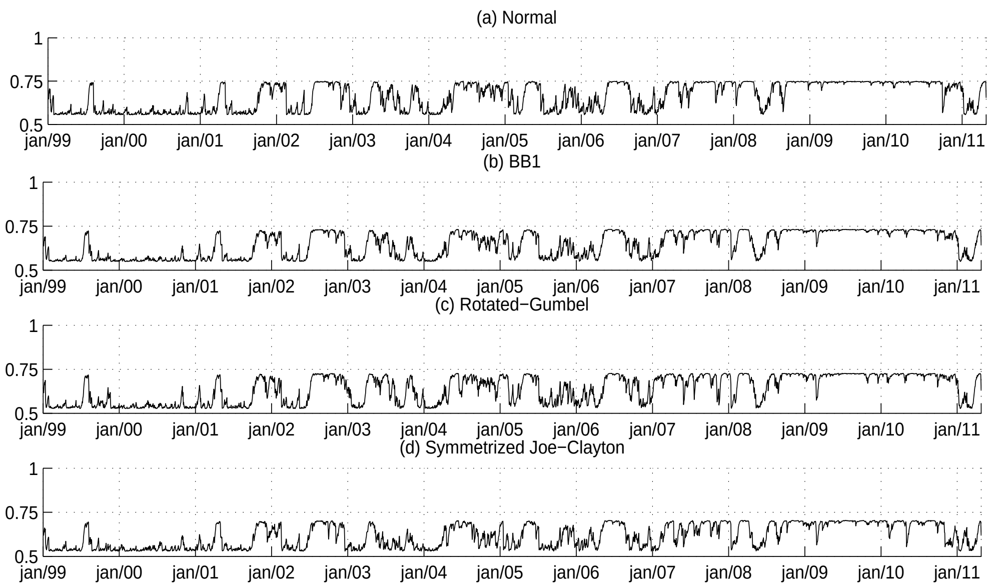

Figure 2.5: The evolution of the Kendall's tau following Jondeau and Rockinger (2006), for the FTSE-CAC pair.

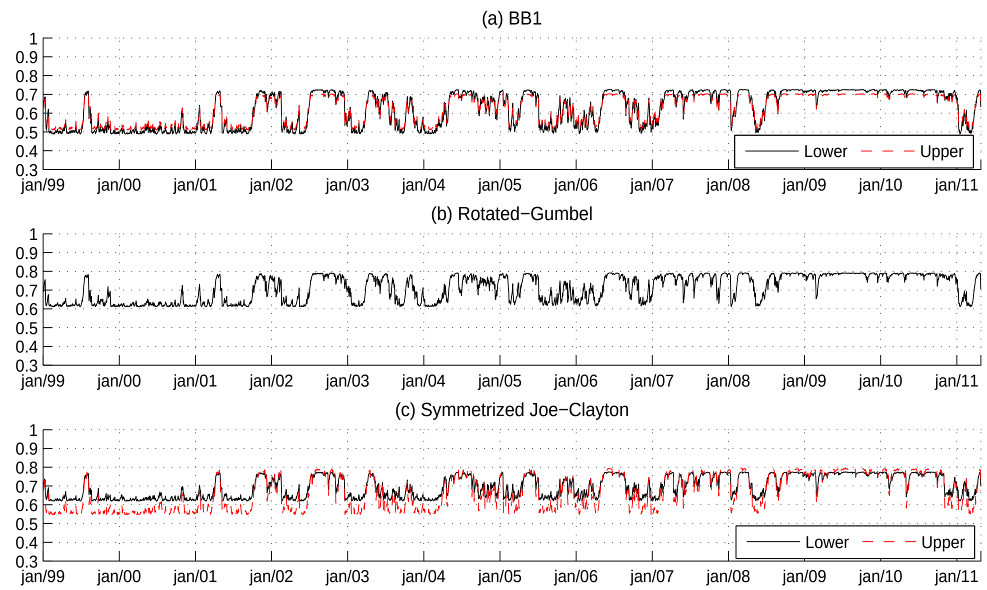

Figure 2.6: The evolutions of the tail dependence parameters following Jondeau and Rockinger (2006), for the FTSE-CAC pair.

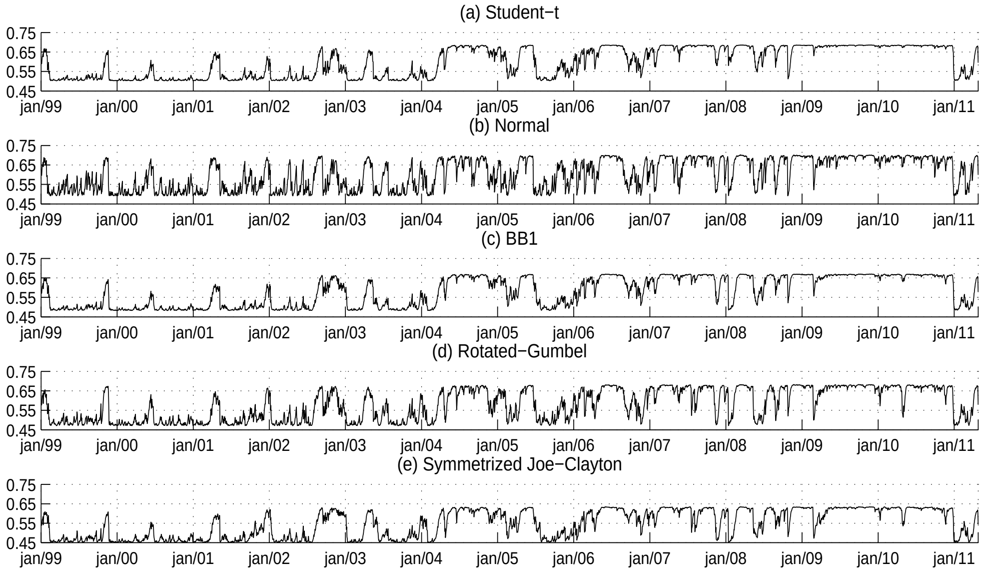

Figure 2.7: The evolution of the Kendall's tau following Jondeau and Rockinger (2006), for the FTSE-DAX pair.

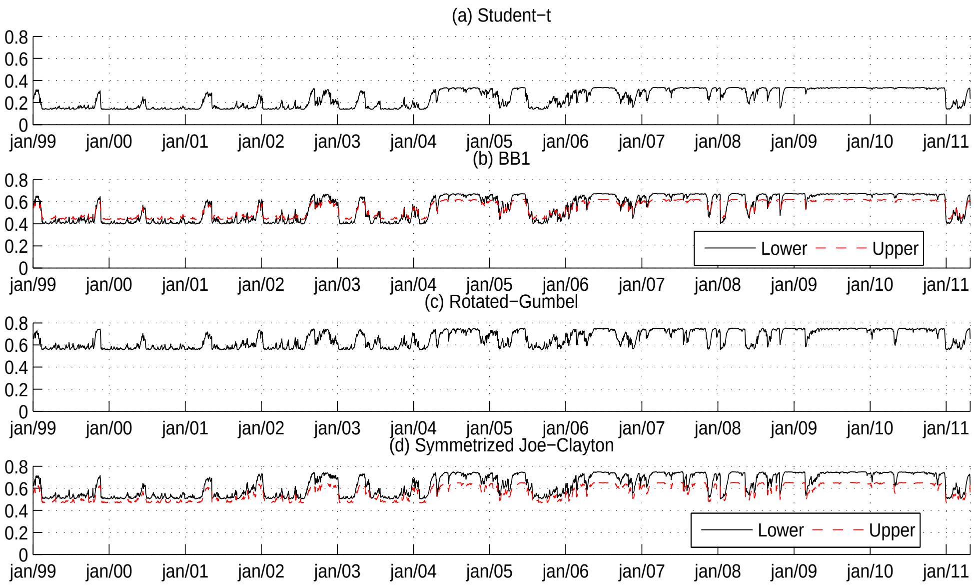

Figure 2.8: The evolutions of the tail dependence parameters following Jondeau and Rockinger (2006), for the FTSE-DAX pair.

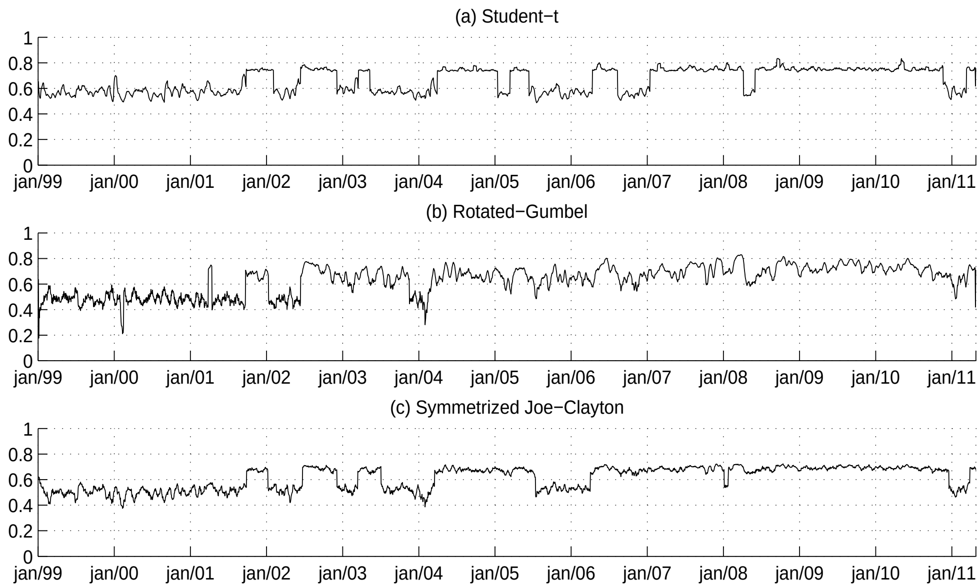

Figure 2.9: The evolution of the Kendall's tau following Silva Filho et al. (2012), for the FTSE-CAC pair.

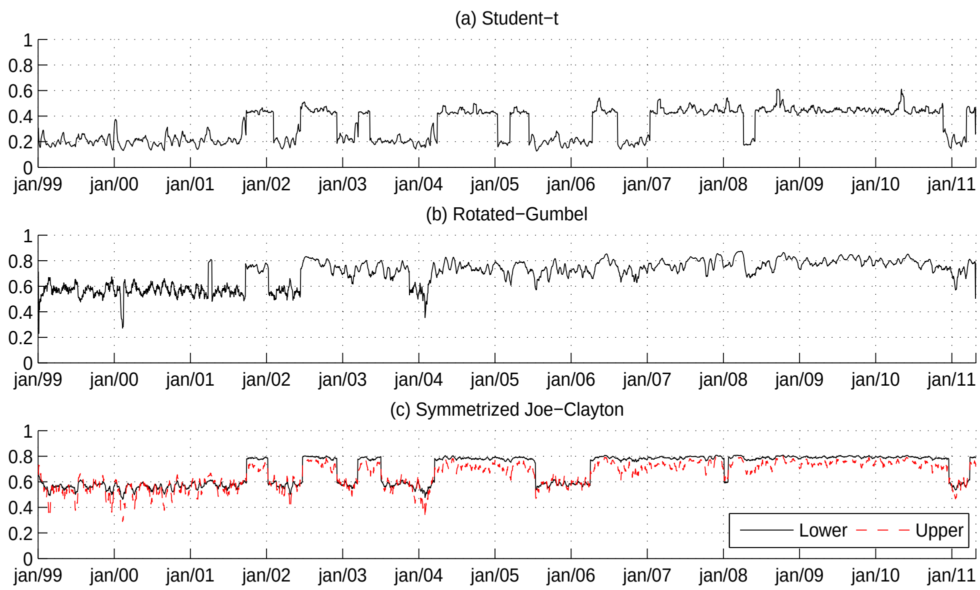

Figure 2.10: The evolutions of the tail dependence parameters following Silva Filho et al. (2012), for the FTSE-CAC pair.

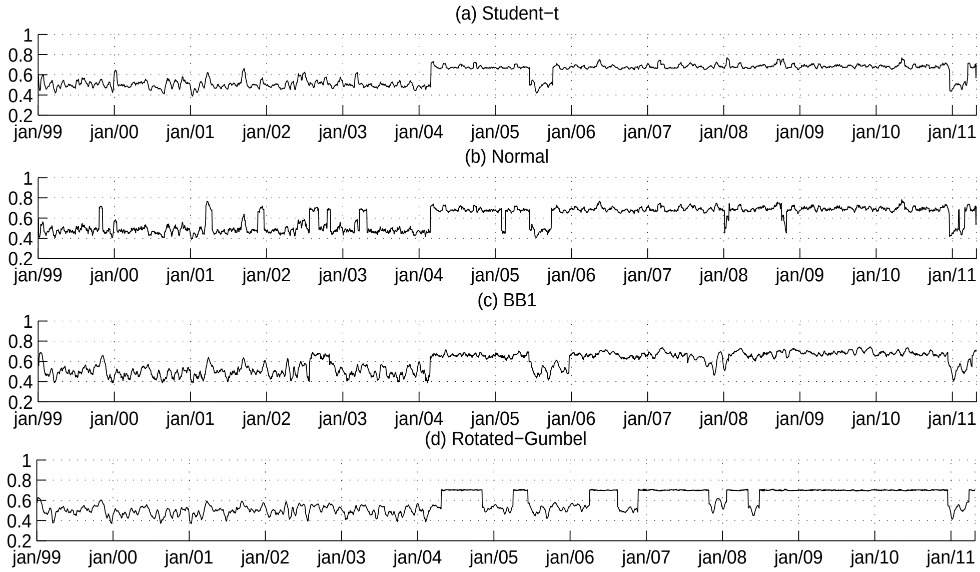

Figure 2.11: The evolution of the Kendall's tau following Silva Filho et al. (2012), for the FTSE-DAX pair.

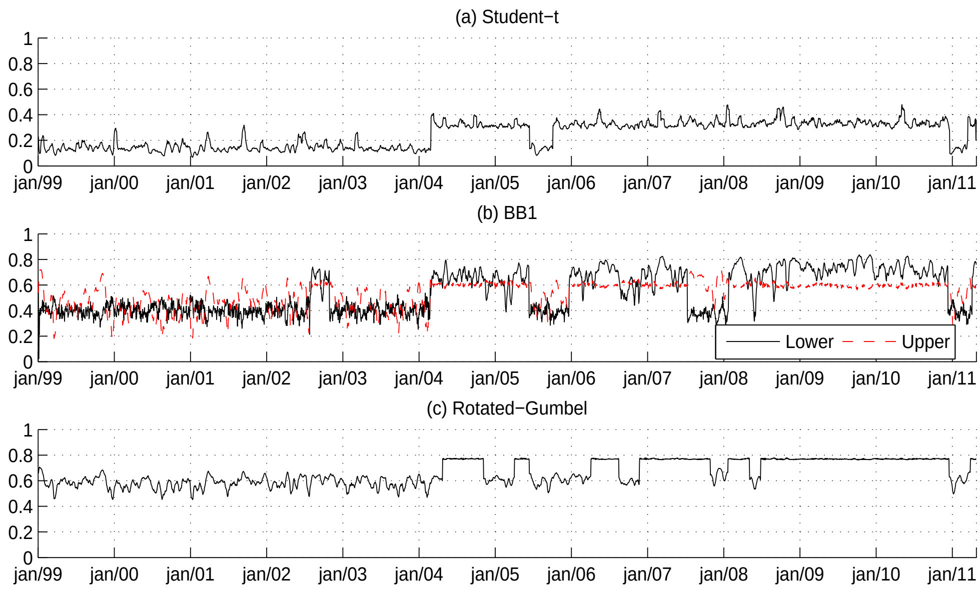

Figure 2.12: The evolutions of the tail dependence parameters following Silva Filho et al. (2012), for the FTSE-DAX pair.

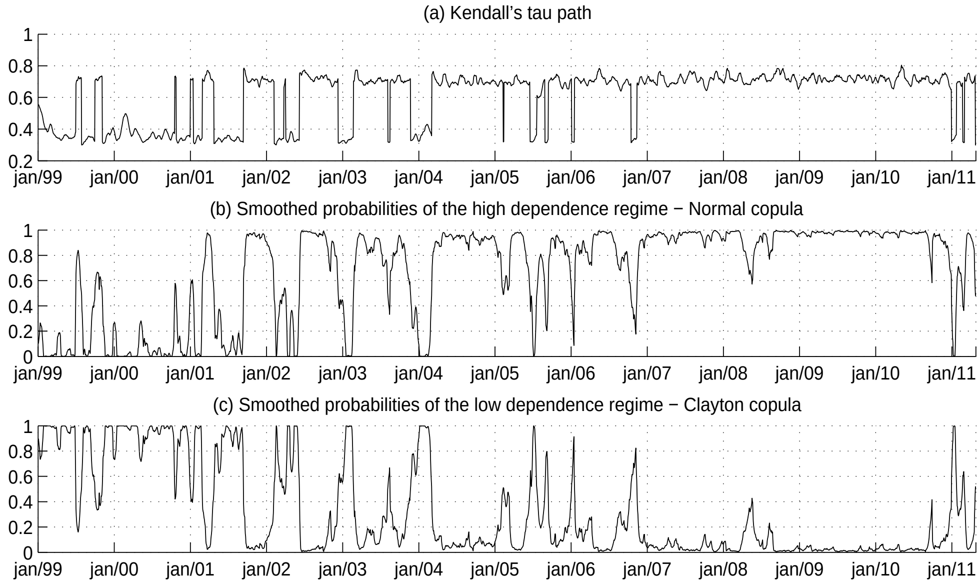

Figure 2.13: The evolution of the Kendall's tau computed based on the Normal-Clayton Markov switching copula and the smoothed probabilities of the high and low dependence regimes, for the FTSE-CAC pair.

3 Modeling International Financial Returns Using Vine Copulas

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Abstract. In this paper, we use the vine copula model to study the dependence structure amongst broad stock market indexes from Germany, France, Britain, the United States and Brazil, and check the asymmetric dependence hypothesis in this case. Nevertheless, based on our empirical results, this hypothesis cannot be verified, since a canonical vine copula with all Student-t bivariate copulas seems to characterize the dependence properly. This means that the analyzed international financial returns possibly exhibit a symmetrical dependence, at least under our copula specifications.

Keywords and phrases: Dependence structure; Asymmetric dependence; Pair-copula; Regular vine.

JEL Classification: C13; C46; G15.

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3.1 Introduction

It is well known that financial data has some peculiar features, called stylized facts, such as (i) heavy tails in the unconditional distribution of asset returns, (ii) volatility clustering, i.e., the clustering of large and small moves in asset prices time series, (iii) asymmetry, the tendency for changes in stock prices to be differently correlated with changes in stock volatility depending whether prices are increasing or decreasing, amongst other particular characteristics. Besides that, a number of recent studies have found evidence that international financial returns tend to exhibit asymmetric dependence (see, e.g., Longin and Solnik (2001), Ang and Bekaert (2002), Ang and Chen (2002), Hong et al. (2007)), which means that negative returns tend to be more dependent between themselves than positive returns.

Despite these facts, it is a common practice to assume multivariate normality when it comes to portfolio analysis, and the linear correlation coefficient is widely used as a measure of dependence in the finance world. It is widely known though that the Gaussian distribution is not able to capture the stylized facts and to reproduce the extreme events of financial markets. Moreover, the linear correlation in this context is not a suitable measure of dependence (Embrechts et al., 2002). Hence, results based on normality may not be appropriate for optimum portfolio construction and management (Sancetta and Satchell, 2001).

Recently, the copula functions have emerged as an important tool for flexible and more realistic modeling of multivariate distributions of asset returns and their dependence. Copula theory goes back to the work of Sklar (1959), who showed that an n-dimensional joint distribution function can be decomposed into its n marginal distributions and a copula, which fully characterizes the dependence among the variables. Examples of copula applications in finance can be found at Mendes and Moretti (2002), Hurlimann (2004), Dias and Embrechts (2004), Rodriguez (2007), Silva Filho et al. (2012), to name a few.

Although the number of bivariate copulas is rather large, the range of higher-dimensional copulas was somehow limited until recently. The classes of elliptical and Archimedean copulas have attracted particular interest, however, they are very restrictive. If one n-dimensional Archimedean copula is assumed to represent the dependence amongst some variables, the dependence structures between all possible pairs of variables are expressed by the same specified copula. For example, if one n-dimensional Clayton copula with parameter $\theta = 1$ is fitted to a data set, all pairs of random variables will have their dependence expressed by a bivariate Clayton copula with parameter $\theta = 1$. This problem occurs to any Archimedean copula. With regard to the multivariate Gaussian and Student-t copulas, they can only capture symmetric dependencies.

The so-called vine copula model or pair-copula construction (PCC) overcomes this issue. Joe (1996) proposed a probabilistic construction of multivariate distributions based on simple building blocks called pair-copulas, later extended and systematically organized by Bedford and Cooke (2001, 2002) through the specification of a graphical model called regular vine. The vine copula model consists in decomposing a multivariate density into a cascade of pair-copulas (bivariate copulas) and the marginal densities. It is a more flexible method to model multivariate distributions, since the pair-copulas do not have to be the same neither have to belong to the same family.

The goal of this paper is to use the vine copula model to analyze the dependence structure amongst broad stock market indexes from Germany (DAX), France (CAC 40), Britain (FTSE 100), the United States (S&P 500) and Brazil (IBOVESPA), and check the asymmetric dependence hypothesis in this case. Two types of asymmetric dependence have been reported: one is stronger dependence in returns of different assets during bear markets than bull markets, as in e.g. Ang and Bekaert (2002); and the second one is stronger tail dependence in the lower tail (losses) in comparison with the upper tail (gains) for pairs of assets, see, e.g., Hong et al. (2007). We will investigate the presence of the second type of asymmetry in our data set. Such asymmetries imply that investors might lose the benefits of portfolio diversification when they are most valuable.

The remainder of the paper is structured as follows. We present a brief introduction to the vine copula model and summarize the inferential procedure in Section 3.2, and define tail dependence in Section 3.3. In Section 3.4, we describe the data, the statistical models for the marginal distributions, the vine copulas considered. We also report the parameters estimates. Lastly, in Section 3.5, we bring some final remarks.

3.2 Vine Copulas

Sklar's Theorem (Sklar, 1959) states that every multivariate cumulative probability distribution function F with marginals $F_1, \ldots, F_n$ may be written as

$$F(x_1, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \tag{3.2.1}$$

for some appropriate n-dimensional copula C. In terms of the joint probability density function f, for an absolutely continuous F with strictly increasing continuous marginals $F_1, \ldots, F_n$, we have

$$f(x_1, \ldots, x_n) = c_{12\ldots n}(F_1(x_1), \ldots, F_n(x_n)) \cdot f_1(x_1) \ldots f_n(x_n). \tag{3.2.2}$$

Consider now, for example, a trivariate random vector $X = (X_1, X_2, X_3)$. Its density can be factorized as

$$f(x_1, x_2, x_3) = f(x_3) \cdot f(x_2|x_3) \cdot f(x_1|x_2, x_3). \tag{3.2.3}$$

According to equation (3.2.2), we can write

$$f(x_2|x_3) = c_{23}(F_2(x_2), F_3(x_3)) \cdot f_2(x_2). \tag{3.2.4}$$

Similarly, it is possible to decompose the conditional density of X_1 given X_2 and X_3 as

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot f(x_1|x_2). \tag{3.2.5}$$

Now, decomposing $f(x_1|x_2)$ in (3.2.5) further, we have

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1). \tag{3.2.6}$$

Finally, from equations (3.2.4) and (3.2.6), the joint density function for the trivariate case can be written as

$$\begin{aligned} f(x_1, x_2, x_3) = f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot \\ \cdot c_{23}(F_2(x_2), F_3(x_3)) \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)). \end{aligned} \tag{3.2.7}$$

That is, the trivariate density can be factorized as a product of the marginals, two bivariate copulas, c_{12} and c_{23}, and a third copula $c_{13|2}$ named conditional, whose arguments are conditional distributions.

The previous results for the trivariate case can be generalized for an n-dimensional vector, using the following formula:

$$f(x|\boldsymbol{v}) = c_{xv_j|\boldsymbol{v}_{-j}}(F(x|\boldsymbol{v}_{-j}), F(v_j|\boldsymbol{v}_{-j})) \cdot f(x|\boldsymbol{v}_{-j}), \tag{3.2.8}$$

for a vector $\boldsymbol{v}$ with dimension d. Here v_j is an arbitrarily chosen component of $\boldsymbol{v}$ and $\boldsymbol{v}_{-j}$ corresponds to the vector $\boldsymbol{v}$ excluding this component. It follows that the multivariate density function with dimension n can be decomposed into its marginal densities and a set of iteratively conditioned bivariate copulas.

The pair-copula decomposition of a multivariate density involves marginal conditional distributions of the form $F(x|\boldsymbol{v})$, computed using a formula of Joe (1996):

$$F(x|\boldsymbol{v}) = \frac{\partial C_{x,v_j|\boldsymbol{v}_{-j}}(F(x|\boldsymbol{v}_{-j}), F(v_j|\boldsymbol{v}_{-j}))}{\partial F(v_j|\boldsymbol{v}_{-j})}. \tag{3.2.9}$$

As the number of variables grows, the different possibilities of decomposition in pair-copulas also increase. To organize these possibilities, Bedford and Cooke (2001, 2002) introduced a graphical model called regular vine, with two special cases, the drawable vine (D-vine) and the canonical vine (C-vine). Each of these graphical models provides a specific way of decomposing the density $f(x_1, \ldots, x_n)$. The n-dimensional density $f()$ associated with a C-vine may be written as

$$\prod_{k=1}^{n} f(x_k) \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|1,\ldots,j-1}(F(x_j|x_1, \ldots, x_{j-1}), F(x_{j+i}|x_1, \ldots, x_{j-1})). \tag{3.2.10}$$

In a C-vine, there are $n-1$ hierarchical trees with increasing conditional sets and a key variable located at the root of the tree, and there are $n(n-1)/2$ bivariate copulas. For a detailed description, see Aas et al. (2009). Figure 3.1 shows a C-vine decomposition for $n = 5$. It consists of 4 nested trees, where tree T_j has $6 - j$ nodes and $5 - j$ edges corresponding to a pair-copula.

In a time series context, the parameters of the canonical vine density can be estimated by maximum likelihood. The log-likelihood associated with a C-vine may be written as

$$\begin{aligned}
\ell(\boldsymbol{\alpha}, \boldsymbol{\theta}; \mathbf{x}) &= \sum_{t=1}^{T} \sum_{k=1}^{n} \log(f(x_{k,t}; \boldsymbol{\alpha}_k)) \\
&+ \sum_{t=1}^{T} \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \log(c(F(x_{j,t}|x_{1,t}, \ldots, x_{j-1,t}), F(x_{j+i,t}|x_{1,t}, \ldots, x_{j-1,t})); \boldsymbol{\theta}_{j,j+i|1,\ldots,j-1}) \\
&= \ell_M(\boldsymbol{\alpha}; \mathbf{x}) + \ell_C(\boldsymbol{\alpha}, \boldsymbol{\theta}; \mathbf{x}),
\end{aligned} \tag{3.2.11}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ denote the parameters of the marginal distributions and the C-vine copula, respectively. The Inference Function for Margins (IFM) method (Joe and Xu, 1996) can efficiently estimate the model parameters, so estimation can proceed in two steps. In a first step, an ARMA-GARCH model is applied to the return data and the estimates of the parameters of the marginal distributions are derived. In a second step, the vine copula log-likelihood, ℓ_C, is maximized over the copula parameters, taking the parameters of the marginals as fixed at the estimated values from the first step. Likelihood evaluation for the C-vine copula is performed using the algorithm in Aas et al. (2009).

3.3 Tail Dependence

In order to investigate tail asymmetry, we first define tail dependence. Loosely speaking, it is a copula-based dependence measure and refers to the degree of dependence in extreme values. Formally, if the limit

$$\lim_{\varepsilon \to 0} \Pr\left[U_1 \leq \varepsilon | U_2 \leq \varepsilon\right] = \lim_{\varepsilon \to 0} \Pr\left[U_2 \leq \varepsilon | U_1 \leq \varepsilon\right] = \lim_{\varepsilon \to 0} C(\varepsilon, \varepsilon)/\varepsilon = \lambda_L$$

exists, the copula C has lower tail dependence if $\lambda_L \in (0, 1]$ and no lower tail dependence if $\lambda_L = 0$. Similarly, if the limit

$$\lim_{\delta \to 1} \Pr\left[U_1 > \delta | U_2 > \delta\right] = \lim_{\delta \to 1} \Pr\left[U_2 > \delta | U_1 > \delta\right] = \lim_{\delta \to 1} \left(1 - 2\delta + C(\delta, \delta)\right) / (1 - \delta) = \lambda_U$$

exists, the copula C has upper tail dependence if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. In other words, the lower (upper) tail dependence is the probability that one variable takes an extremely large negative (positive) value, given that the other variable took an extremely large negative (positive) value.

Joe et al. (2010) proved that if all the pair-copulas in a vine are reflection symmetric[1], then the resulting vine copula is reflection symmetric. In this case, whenever reflection symmetric bivariate copulas, such as the Student-t copula, are used throughout the construction of the vine, each bivariate margin (j, k) has both upper and lower tail dependence and $\lambda_{jk,L} = \lambda_{jk,U}$ for any $j < k$. So, in order to obtain a multivariate copula with asymmetric upper and lower tails for each bivariate margin, one can use reflection asymmetric pair-copulas in the vine construction.

[1]Let $c_{1 \ldots m}$ be the copula density, then reflection symmetry implies $c(u_1, \ldots, u_m) = c(1 - u_1, \ldots, 1 - u_m)$.

In particular, they have shown that vine copulas can have asymmetric tail dependence when asymmetric bivariate copulas with different upper/lower tail dependence are used in level 1 of the vine.

3.4 Empirical Application

The aim here is to model the dependence amongst CAC 40, DAX, FTSE 100, S&P 500 and IBOVESPA indexes using the vine copula model. We begin describing our data set and analyzing its main features. Then, in the first stage of the modeling process, we estimate the marginal distributions of the five indexes returns, considering both their conditional mean and variance. Finally, in the second stage, we study which vine copula is the most appropriate to characterize the dependence structure.

3.4.1 Data Description

Data was collected from *Bloomberg* and the daily log-returns were used for the period from January 07, 1999 to November 23, 2010, with a total of 2735 return observations. Figure 3.2 shows the log-returns of the five indexes in the period considered.

Table 3.1 shows some descriptive statistics of the log-returns. We can notice that data usually shows signs of both negative asymmetry and excess kurtosis, except for IBOVESPA and CAC 40 log-returns, which show positive asymmetry. Also there is evidence that the log-returns are not normally distributed according to the Jarque-Bera test statistics. These statistics are in line with what is described in the literature on financial data.

3.4.2 Modeling the Marginal Distributions

We studied various models for the marginals[2] and, comparing the optimum values of the log-likelihood functions and the information criteria AIC and BIC, we chose an AR(1)-GARCH(1,1) with Skewed-t[3] errors as the most appropriate model for CAC 40, DAX, FTSE 100 and IBOVESPA returns. For S&P 500, an AR(1)-EGARCH(2,1) with Skewed-t errors was necessary to model the serial correlation in the conditional mean and variance. The rationale for having included a skew in the marginal distributions was to ensure that any asymmetry we found in the depen-

[2] All marginals were estimated using the Oxford MFE Toolbox provided by Kevin Shepard.

[3] The Skewed-t GARCH model was defined by Hansen (1994).

dence structure truly reflected dependence and could not be attributed to poor modeling of the marginals.

Let $r_{i,t}$ denote the log-return of the *i-th* index at time t, the model AR(1)-GARCH(1,1) with Skewed-t errors can be described as

$$\begin{aligned} r_{i,t} &= c_{0i} + c_{1i} r_{i,t-1} + h_{i,t}^{1/2} \varepsilon_{i,t}, \\ h_{i,t} &= \omega_i + \beta_{1i} h_{i,t-1} + \alpha_{1i} \varepsilon_{i,t-1}^2, \\ \varepsilon_{i,t} &\sim Skewed - t(\nu_i, \lambda_i), \end{aligned} \tag{3.4.1}$$

where $h_{i,t}$ is the conditional variance of $r_{i,t}$, whereas ν and λ are, respectively, the degrees of freedom and the asymmetry of the the Skewed-t distribution. The conditional variance of an EGARCH(2,1) process can be modeled as

$$\ln(h_{i,t}) = \omega_i + \sum_{k=1}^{2} \alpha_{ki}(|\varepsilon_{i,t-k}| - \mathbb{E}|\varepsilon_{i,t-k}|) + \gamma_{1i}\varepsilon_{i,t-1} + \beta_{1i} \ln(h_{i,t-1}). \tag{3.4.2}$$

The results of the estimation of the marginal models are presented in Table 3.2. Because the estimated parameter λ_i is significantly negative for all series analyzed, it is possible to assert that all marginals are asymmetric. Additionally, the estimation output suggests that the marginal distributions are adequately specified: the p-values of the Ljung-Box test indicate that there is no autocorrelation left in the standardized residuals nor in the squared standardized residuals of the marginal models. Moreover, the models passed the Kolmogorov-Smirnov test of goodness-of-fit. If the marginal distribution is well specified, the probability integral transform (PIT) of the standardized residuals will have uniform distribution in the interval $[0, 1]$. This is a necessary condition for copula modeling. Hence, to test whether the PIT of the residuals had such distribution, the Kolmogorov-Smirnov test was applied.

3.4.3 Copula Modeling

A canonical vine copula was fitted to the probability integral transform (PIT) of the ARMA-GARCH residuals. We followed Aas et al. (2009) for the specification of the copula. Using the sample Kendall's taus, reported in Table 3.3, we first ordered the variables by decreasing dependencies, choosing the one with the largest dependence as the first variable to condition on. This led us to place CAC 40 Index returns at the root of the tree, followed by DAX, FTSE 100, S&P 500 and IBOVESPA indexes returns. By doing so, we intended that most of the dependence structure in the copula would be captured in the first stages of the canonical vine.

As we expected to observe mainly lower tail dependence, we started estimating a model with all bivariate Clayton copulas[4], since the Clayton copula exhibits only lower tail dependence. We also estimated C-vines built on the very flexible BB1, BB7 and SJC (Symmetrized Joe-Clayton) copulas, with lower tail dependence different from upper tail dependence, to check the hypothesis of asymmetric dependence. The estimates of the pair-copulas parameters and the C-vine copulas log-likelihoods are provided in Table 3.4[5]. Based on the likelihood criterion, the C-vine with all Clayton copulas is outperformed by those models composed by two-parameter pair-copulas, suggesting that the dependence structure of the investigated indexes is characterized by both lower and upper tail dependence. Additionally, to account for the possibility of symmetrical tail dependence, we fitted a canonical vine copula with all bivariate Student-t copulas, which feature same upper and lower tail dependence. The latter construction is the best model based on log-likelihood. Table 3.5 presents the model-based tail dependence parameters and Kendall's taus. One can notice that the copula models have the strongest dependence in the baseline copulas and weak conditional dependence. Moreover, the tail dependencies of the bivariate margins measured by the Student-t pair-copulas are lower than the tail dependencies measured by the other two-parameter pair-copulas, probably because of the symmetry imposed by the elliptical copulas.

So, based on our empirical results, it is not possible to conclude for an asymmetry in the dependence amongst the stock markets considered, since a canonical vine copula with all Student-t pair-copulas seems to characterize their dependence structure properly, which means that the international financial returns analyzed possibly exhibit a symmetrical dependence.

Since the n-dimensional Student-t copula has been widely used in finance to model multivariate return data, it is worth comparing the results obtained from the estimation of the vine copula model with all Student-t pair-copulas with the results from the estimation of a 5-dimensional Student-t copula. In order to estimate the latter, a method similar to that in

[4]In order to facilitate coding, we used pair-copulas of the same family throughout the vine.

[5]The estimation of the pair-copulas parameters was performed using some functions from the Dynamic Copula Toolbox 3.0 provided by Manthos Vogiatzoglou as well as some functions written by the authors for the Matlab 7® software.

Mashal and Zeevi (2002) was employed, according to which the estimated correlation matrix of the Student-t copula is taken to be equal to the sample correlation of the transformed standardized residuals, $\widehat{\Sigma} = \sin(\pi\hat{\tau}/2)$, where $\hat{\tau}$ is the Kendall's tau estimator, and only the degree of freedom parameter is estimated. We obtained an estimate of 8.81 degrees of freedom in this case. There is only one parameter for modeling tail dependence, so we believe that, if the tail dependencies of distinct pairs of variables are very different, the pair-copula decomposition will describe the dependence structure better. The computed AIC criterion for the vine copula model is -11035, whereas, for the 5-dimensional Student-t copula, it is -11002. Therefore, based on the AIC criterion, the vine copula structure is more appropriate than the 5-dimensional copula.

To illustrate the difference between these two structures in terms of tail dependence, we computed the the tail dependence parameters for the four bivariate margins CAC40-DAX, CAC40-FTSE100, CAC40-S&P500 and CAC40-IBOVESPA, based on both structures. For the Student-t copula, the tail dependence parameter is given by (Embrechts et al., 2002):

$$\lambda_L(X,Y) = \lambda_U(X,Y) = 2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{(1-\rho)/(1+\rho)}), \tag{3.4.3}$$

where $t_{\nu+1}$ is the cumulative distribution function of a univariate Student-t with $\nu + 1$ degrees of freedom. Table 3.6 presents the tail dependence coefficients for the four margins and both structures. We can observe that the values for the vine copula model are higher than the corresponding ones for the Student-t copula, except for the last margin. The practical implication of this difference in tail dependence is that the probability of observing a large loss in a portfolio of each of these pairs of indexes is much higher if based on the pair-copula decomposition than on the 5-dimensional copula.

3.5 Final Remarks

In the present work, we used the vine copula model to study the dependence structure amongst the log-returns of some important stock market indexes, especially the pattern of dependence in the tails of their joint distribution, to verify the asymmetric dependence hypothesis.

We could not conclude for an asymmetry in the dependence amongst such markets based on our empirical study, since we found that a C-vine with all Student-t pair-copulas describes their dependence structure properly. Nikoloulopoulos et al. (2010) consider that, perhaps, asymmetric tail dependence with stronger lower tails occurs only temporarily, for some periods of time, what suggests that incorporating time variation into the vine copula model could improve it as a tool to model multivariate international financial data.

If we look at the graphics of the log-returns in Figure 3.2, we can identify at least two clusters of greater volatility in all the series: one during the year of 2002, when there were ramifications of the "dot-com bubble", which had its climax on March 10, 2000; and another one during 2008, when the subprime crisis was still occurring. When new clusters of volatility emerge in two or more time series at the same time, especially with greater volatility, the dependence amongst the series probably increases, and this further suggests that dynamic copulas should be considered.

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3.7 Appendix A: Tables

Table 3.1: Descriptive statistics of the log-returns of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA indexes.

	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
Mean	8.9170E-05	-4.4007E-05	-2.3894E-05	-1.9315E-05	8.2530E-04
Median	0.00085	0.00019	0.00039	0.00045	0.00121
Maximum	0.1080	0.1059	0.0938	0.1096	0.2882
Minimum	-0.0979	-0.0947	-0.0927	-0.0947	-0.1228
Std. Deviation	0.0171	0.0163	0.0137	0.0142	0.0217
Asymmetry	-0.0170	0.0271	-0.0807	-0.1151	0.7467
Kurtosis	7.0876	7.6107	8.2476	9.5857	17.2255
Jarque-Bera	1898	2415.3	3131.6	4934.5	23257

Table 3.2: Estimated parameters of the marginal models.

Coefficients	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
c_0	...	...	...	...	8.193E-04
	...	...	...	...	(0.0000)
c_1	-0.0268	-0.0361	-0.0585	-0.0917	0.0173
	(0.0001)	(0.0002)	(0.0004)	(0.0028)	(0.0019)
ω	1.872E-06	1.861E-06	1.386E-06	-0.1408	9.529E-06
	(0.0000)	(0.0000)	(0.0000)	(0.0291)	(0.0000)
α_1	0.0816	0.0855	0.1018	-0.1414	0.0697
	(0.0111)	(0.0116)	(0.0131)	(0.0431)	(0.0108)
α_2	...	...	...	0.2588	...
	...	...	...	(0.0460)	...
γ_1	...	...	...	-0.1206	...
	...	...	...	(0.0117)	...
β_1	0.9132	0.9085	0.8922	0.9840	0.9056
	(0.0107)	(0.0113)	(0.0130)	(0.0033)	(0.0148)
ν	10.7752	14.5104	23.0364	10.0372	9.4347
	(2.3307)	(3.7768)	(10.0162)	(1.8751)	(1.6389)
λ	-0.1185	-0.0949	-0.1124	-0.1283	-0.0845
	(0.0229)	(0.0277)	(0.0263)	(0.0235)	(0.0277)
$Q(20)$	0.3132	0.3506	0.2553	0.2928	0.4923
$Q^2(20)$	0.8951	0.5702	0.7764	0.3152	0.9927
K-S Test	0.2395	0.9798	0.9395	0.5611	0.8934

Note: Standard errors are in parenthesis. $Q(20)$ and $Q^2(20)$ correspond to the p-values of the Ljung-Box test for the autocorrelation in the residuals and in the squared residuals, respectively. K-S Test is the Kolmogorov-Smirnov test p-value.

Table 3.3: Matrix of the sample Kendall's taus computed based on the PIT of the ARMA-GARCH residuals.

	CAC 40	DAX	FTSE 100	S&P 500	IBOVESPA
CAC 40	1.0000				
DAX	0.7196	1.0000			
FTSE 100	0.6633	0.6047	1.0000		
S&P 500	0.3862	0.4044	0.3658	1.0000	
IBOVESPA	0.2767	0.2887	0.2821	0.4223	1.0000

Table 3.4: Estimated C-Vine copula parameters and log-likelihood.

Block	LogL		Estimated C-vine copula parameters									
			Level 1				Level 2			Level 3		Level 4
			c_{12}	c_{13}	c_{14}	c_{15}	$c_{23\|1}$	$c_{24\|1}$	$c_{25\|1}$	$c_{34\|12}$	$c_{35\|12}$	$c_{45\|123}$
Clayton	4395.1	$\hat{\theta}$'s	3.25	2.47	0.80	0.52	0.15	0.17	0.12	0.09	0.09	0.61
			(0.11)	(0.08)	(0.04)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)
BB1	5462.2	$\hat{\kappa}$'s	0.76	0.63	0.34	0.34	0.12	0.11	0.09	0.06	0.07	0.26
			(0.07)	(0.06)	(0.04)	(0.04)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.04)
		$\hat{\delta}$'s	2.44	2.15	1.38	1.18	1.05	1.08	1.04	1.03	1.03	1.27
			(0.08)	(0.06)	(0.03)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.03)
BB7	5243.9	$\hat{\kappa}$'s	2.82	2.44	1.48	1.24	1.06	1.10	1.04	1.03	1.02	1.32
			(0.08)	(0.06)	(0.04)	(0.03)	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.03)
		$\hat{\delta}$'s	2.48	1.85	0.62	0.46	0.16	0.17	0.12	0.07	0.09	0.45
			(0.09)	(0.08)	(0.04)	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.03)
SJC	5263.5	$\hat{\lambda}_U$'s	0.71	0.66	0.37	0.18	0.03	0.08	0.02	0.00	0.00	0.27
			(0.00)	(0.01)	(0.03)	(0.05)	(0.00)	(0.04)	(0.00)	(0.00)	(0.00)	(0.03)
		$\hat{\lambda}_L$'s	0.76	0.69	0.37	0.28	0.06	0.04	0.02	0.00	0.00	0.26
			(0.00)	(0.01)	(0.03)	(0.04)	(0.02)	(0.01)	(0.01)	(0.00)	(0.00)	(0.04)
t	5537.5	ρ's	0.90	0.86	0.57	0.43	0.16	0.22	0.14	0.10	0.12	0.48
			(0.00)	(0.00)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)
		ν's	5.04	6.12	8.10	9.65	9.13	19.14	25.54	59.63	42.61	10.30
			(0.50)	(0.81)	(1.44)	(2.14)	(1.66)	(7.20)	(11.87)	(19.02)	(12.94)	(2.17)

Note: The estimated C-Vine copulas are built on all Clayton, BB1, BB7, SJC and Student-t pair-copulas (blocks). For Clayton, $\theta > 0$; for BB1, $\kappa > 0$ and $\delta \geq 1$; for BB7, $\kappa \geq 1$ and $\delta > 0$; for SJC, $0 < \lambda_U, \lambda_L < 1$. Variable 1 = CAC 40, 2 = DAX, 3 = FTSE 100, 4 = S&P 500, 5 = IBOVESPA.

Table 3.5: Estimated tail dependence parameters and Kendall's taus.

Block		Estimated C-vine copula parameters									
		Level 1				Level 2			Level 3		Level 4
		c_{12}	c_{13}	c_{14}	c_{15}	$c_{23\|1}$	$c_{24\|1}$	$c_{25\|1}$	$c_{34\|12}$	$c_{35\|12}$	$c_{45\|123}$
Clayton	$\hat{\tau}$'s	0.62	0.55	0.28	0.21	0.07	0.08	0.06	0.04	0.04	0.23
	$\hat{\lambda}_L$'s	0.81	0.76	0.42	0.26	0.01	0.02	0.00	0.00	0.00	0.32
BB1	$\hat{\tau}$'s	0.70	0.65	0.38	0.28	0.10	0.13	0.08	0.05	0.06	0.30
	$\hat{\lambda}_U$'s	0.67	0.62	0.35	0.21	0.07	0.11	0.06	0.04	0.03	0.27
	$\hat{\lambda}_L$'s	0.69	0.60	0.22	0.18	0.00	0.00	0.00	0.00	0.00	0.13
BB7	$\hat{\tau}$'s	0.66	0.61	0.36	0.26	0.10	0.13	0.08	0.05	0.05	0.29
	$\hat{\lambda}_U$'s	0.72	0.67	0.40	0.25	0.08	0.12	0.06	0.04	0.03	0.31
	$\hat{\lambda}_L$'s	0.76	0.69	0.32	0.22	0.01	0.02	0.00	0.00	0.00	0.22
SJC	$\hat{\tau}$'s	0.66	0.61	0.36	0.26	0.11	0.13	0.08	0.05	0.06	0.28
	$\hat{\lambda}_U$'s	0.71	0.66	0.37	0.18	0.03	0.08	0.02	0.00	0.00	0.27
	$\hat{\lambda}_L$'s	0.76	0.69	0.37	0.28	0.06	0.04	0.02	0.00	0.00	0.26
t	$\hat{\tau}$'s	0.71	0.66	0.39	0.28	0.10	0.14	0.09	0.06	0.07	0.32
	$\hat{\lambda}$'s	0.59	0.49	0.15	0.06	0.02	0.00	0.00	0.00	0.00	0.07

Note: $\hat{\tau}$'s corresponds to the estimated Kendall's taus, whereas $\hat{\lambda}_U$'s and $\hat{\lambda}_L$'s refer to the estimated upper and lower tail dependence parameters, respectively. Variable 1 = CAC 40, 2 = DAX, 3 = FTSE 100, 4 = S&P 500, 5 = IBOVESPA.

Table 3.6: Tail dependence parameters computed based on the C-Vine with all Student-t pair-copulas and on the 5-dimensional Student-t copula.

Margins	Vine Copula Model	5-dimensional Student-t
CAC 40, DAX	0.5946	0.4996
CAC 40, FTSE 100	0.4875	0.4164
CAC 40, S&P 500	0.1509	0.1328
CAC 40, IBOVESPA	0.0647	0.0740

3.8 Appendix B: Figures

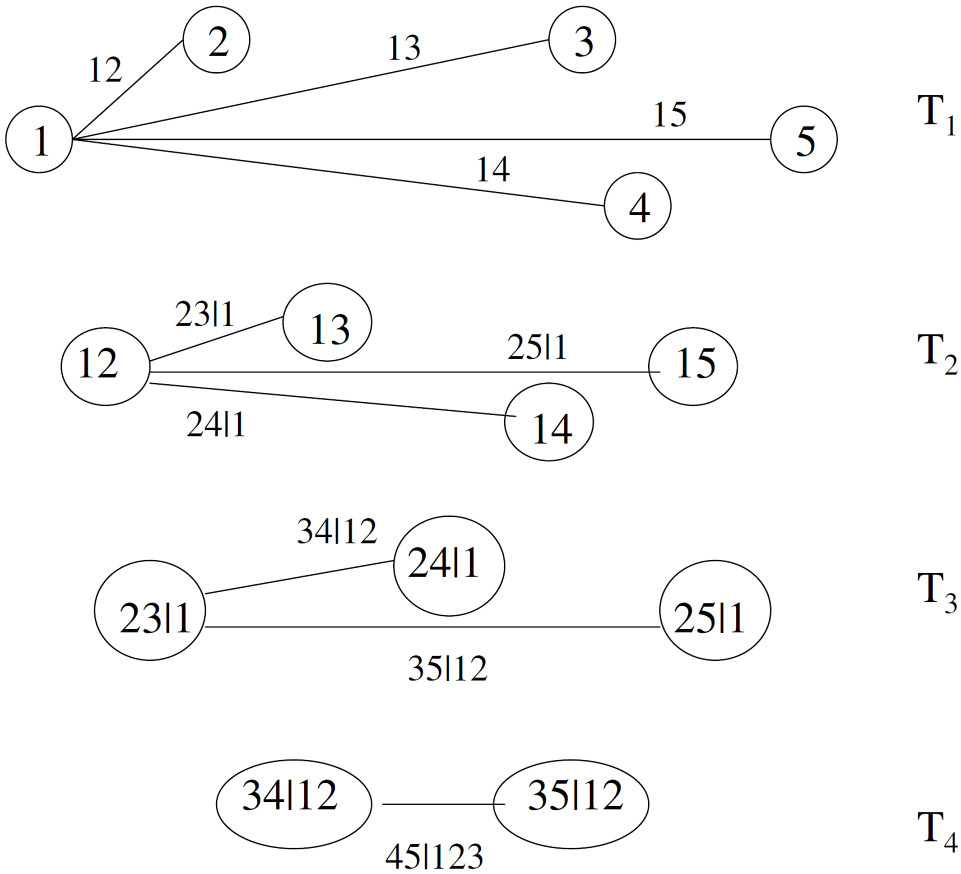

Figure 3.1: A C-vine with 5 variables, 4 trees and 10 edges. Each edge is associated with a pair-copula density.

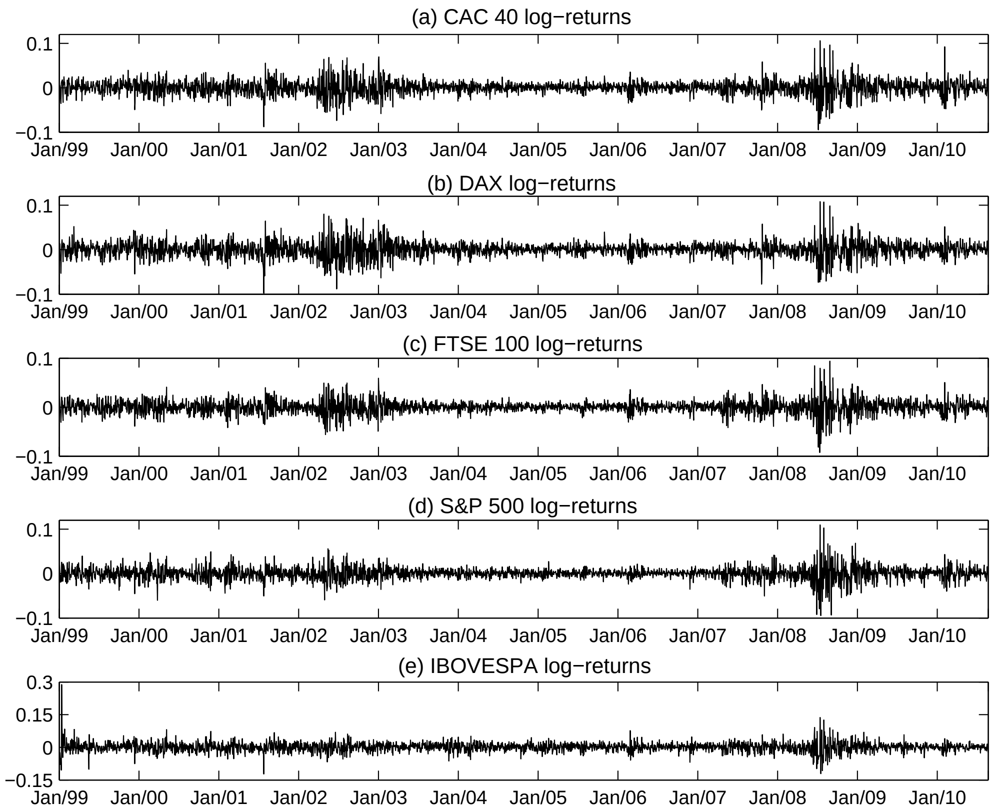

Figure 3.2: Daily log-returns of CAC 40, DAX, FTSE 100, S&P 500 and IBOVESPA indexes for the period from January, 1999 to November, 2010.

4 Dynamic D-Vine Copula Model with Applications to Value-at-Risk (VaR)

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Abstract. Regular vine copulas constitute a very flexible class of multivariate dependence models. Built on bivariate copulas, they can match any possible dependence structure. Research on vine copulas applied to financial data is focused mostly on the case of time-homogeneous dependence structures, however it goes against evidence found in the literature that dependence among financial returns is not constant over time. In this paper, we allow the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, following a nonlinear restricted ARMA(1,m) process as in Patton (2006), in order to obtain a very flexible dependence model for applications to multivariate financial data. The proposed model is evaluated in simulations and further assessed with respect to the accuracy of Value-at-Risk (VaR) forecasts in crisis periods. The Monte Carlo experiments are quite favorable to the dynamic D-vine copula in comparison with a static D-vine copula. Our findings in the empirical application illustrate that time variation is present in the dependence structure of multivariate financial returns. Moreover, the dynamic D-vine copula outperforms the static D-vine copula in terms of predictive accuracy for our data sets.

Keywords and phrases: Regular vine; Pair-copula constructions; Time-varying copulas; Value-at-Risk.

JEL Classification: C13; C51; C52; C58; G15.

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4.1 Introduction

A realistic modeling of the dependence structure of multivariate financial return data is fundamental in finance for the accurate computation of Value-at-Risk, the construction of optimal portfolios, the pricing of financial products with multiple underlying assets, amongst other applications. In particular, it is essential for the correct assessment of financial risk, and the subprime crisis has made it clear.

The linear correlation coefficient has long time been adopted as a measure of dependence in finance. However, it is a suitable measure of dependence only in the elliptical context, in other situations, it may be misleading (Embrechts et al., 2002). It is widely documented that financial returns are characterized by asymmetry and excess kurtosis. There are also stylized facts observed in the dependence structure of multivariate financial returns which are not captured by elliptical distributions. One such feature is asymmetric dependence, meaning that negative returns tend to be more dependent between themselves than positive returns. Another one is tail dependence, which refers to the dependence in extreme values. Also tail dependencies exhibit asymmetries, i.e. lower tail dependence can be larger than upper tail dependence and vice versa. Some authors have found that there is stronger tail dependence in the joint lower tail than upper tail, which means that different stocks are more likely to crash than to thrive together (see, for instance, Jondeau et al. (2007)).

The famous theorem by Sklar (1959) introduced the copulas as a tool to model more intricate patterns of dependence. It states that any n-dimensional joint distribution function can be decomposed into its n marginal distributions and a copula, where the latter completely characterizes the dependence among the variables. While there is a variety of bivariate copula families, which can match a wide range of complex dependencies, the set of higher-dimensional copulas was quite restricted until recently.

Latter developments in the construction of high dimensional copulas tend towards hierarchical structures. Joe (1996) proposed a probabilistic construction of multivariate distributions based on simple building blocks called pair-copulas, later extended and systematically organized by Bedford and Cooke (2001, 2002) through the specification of a graphical model called regular vine. The regular vine copula model (R-vine copula), also called pair-copula constructions

(PCC), is hierarchical in nature and consists in decomposing a multivariate density into a cascade of pair-copulas (bivariate copulas) and the marginal densities. It is a more flexible method to model multivariate distributions, since the bivariate copulas may belong to any family and several families may be mixed in a vine copula, matching any possible dependence structure. Inference of two special cases of regular vine copulas, the canonical vine (C-vine) and the drawable vine (D-vine) copulas, was introduced by Aas et al. (2009).

Research in multivariate dependence modeling using copulas is focused mostly on the case of time-homogeneous dependence structures, however promising approaches for allowing time variation in dependence have been put forth (see Manner and Reznikova (2012) for a recent survey of time-varying copula models with focus on the bivariate case). The dependence among variables can be rendered time-varying by allowing either the dependence parameter or the copula function to vary over time. The first line of research includes fully parametric models, as the one proposed by Patton (2006), who allows the dependence parameters of bivariate copulas to follow a kind of restricted ARMA(1,10) process. There are also semi-parametric models (Hafner and Reznikova, 2010) and adaptive approaches (Giacomini et al., 2009), both applied to the bivariate case. The copulas parameters can also be influenced by a Markov chain, as in Jondeau and Rockinger (2006) and Silva Filho et al. (2012). To our knowledge, Heinen and Valdesogo (2009, 2011) were the first ones (and the only ones so far) to introduce time variation in the vine copula context by specifying a law of motion for the pair-copulas parameters, based on the DCC equations. The other direction of research combines copula models with regime switching to allow for changes in the whole dependence structure, represented by the copula function, according to the regimes characterizing the international financial markets. Chollete et al. (2009), Garcia and Tsafak (2011) and Stöber and Czado (2012) are examples of publications combining the regular vine copula model with the Markov switching model.

In this paper, we propose to introduce dynamics into the vine copula model according to the first approach above-mentioned, specifying an evolution equation directly for the pair-copulas parameters, in order to obtain a very flexible dependence model for applications to multivariate financial return data. We allow the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, following a nonlinear restricted ARMA(1,m) process as in Patton (2006)[1]. Hereafter we will call this time-varying or dynamic D-vine copula model.

The dynamics specified here circumvents certain limitations inherent in the dynamic vine copula model of Heinen and Valdesogo (2009, 2011). Because their specification, based on the DCC equations, involves obtaining the correlation coefficient at each period t, converting it to the Kendall's tau and transforming it into the parameter of the pair-copula, at least two difficulties arise. First, since the non-linear transformation from the Kendall's tau to the copula parameter cannot be done in closed form for all copulas, estimation becomes a difficult task when the parameters have to be obtained by solving numerically for the solution. Second, it is only possible to adopt one-parameter copula families as building blocks in the vine construction, and, according to Joe (2011), it is important to have copulas with flexible lower and upper tail dependencies, such as the two-parameter copulas BB1 and BB7 (see Joe (1997)), for making inferences on joint tail probabilities, which might represent joint risks.

We first evaluate the performance of the proposed model in a simulation study. The overall findings of the Monte Carlo experiments are quite favorable to the dynamic D-vine copula model in comparison with a static D-vine copula. When the data generating process is the dynamic D-vine copula, the dependence parameters estimates from this same model are far superior to those from the static model in terms of both the mean errors and the root mean squared errors. When the samples are drawn from the static D-vine copula, both models have similar performance in terms of the mean errors, nonetheless the dynamic D-vine copula performs worse in terms of the root mean squared errors.

We also investigate both the static and the dynamic D-vine copula models in an empirical study, using two data sets of daily log-returns of the broad stock market indexes from Germany (DAX), France (CAC 40), Britain (FTSE 100), the United States (S&P 500) and Brazil (IBOVESPA), one comprising the period from January 03, 2003 to December 28, 2007 and another one from January 02, 2008 to May 04, 2012, which we denominate "non-crisis period" and "crisis period", respectively. Besides of analyzing the different patterns of dependence characterizing these periods, the intention is to further evaluate the dynamic D-vine copula model

[1] Abbara (2009) uses a trivariate vine decomposition, with time-varying dependence parameters following Patton (2006), in a contagion analysis. He is only interested in investigating the dependence between two Latin American or European index returns, given information on the North American market. He does not explore this dynamic vine decomposition more closely, as it is done in this paper.

with respect to Value-at-Risk (VaR) forecasting accuracy in crisis periods. In an out-of-sample exercise, the estimated models are used to forecast one-day VaR for an equally weighted portfolio of the aforementioned indexes in the period from January 02, 2008 to August 19, 2008 (150 days) and also from May 08, 2012 to September 06, 2012 (79 days). Based on the results of the superior predictive ability (SPA) test of Hansen (2005), the dynamic D-vine copula model outperforms the static D-vine copula in terms of predictive accuracy, especially in the second testing period.

The remainder of this paper is organized as follows. In the next section, we introduce the dynamic D-vine copula model by first providing necessary background on regular vine copulas in Section 4.2.1 and then specifying the dynamic structure of dependence in Section 4.2.2. In Section 4.3, we focus on inference of the dynamic D-vine copula, describing the sequential estimation procedure. The proposed model is first evaluated in a simulation study in Section 4.4 before we turn to the empirical application in Section 4.5, where the dependence structure of the above-mentioned indexes is investigated both in the non-crisis and in the crisis period and the dynamic D-vine copula is validated out-of-sample based on VaR forecasting accuracy. Section 4.6 provides some concluding remarks and an outlook to future research.

4.2 The Dynamic D-Vine Copula Model

In this section, we present the dynamic D-vine copula model. We first provide a brief account of the regular vine copula theory and, then, we give details on the specification of the dynamic structure of dependence.

4.2.1 Regular Vine Copulas

Sklar's Theorem (Sklar, 1959) states that every multivariate cumulative probability distribution function F with marginals $F_1, \ldots, F_n$ may be written as

$$F(x_1, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \tag{4.2.1}$$

for some appropriate n-dimensional copula C. In terms of the joint probability density function f, for an absolutely continuous F with strictly increasing continuous marginals $F_1, \ldots, F_n$, we have

$$f(x_1, \ldots, x_n) = c_{12\ldots n}(F_1(x_1), \ldots, F_n(x_n)) \cdot f_1(x_1) \ldots f_n(x_n). \tag{4.2.2}$$

Consider now, for example, a trivariate random vector $X = (X_1, X_2, X_3)$. Its density can be factorized as

$$f(x_1, x_2, x_3) = f(x_3) \cdot f(x_2|x_3) \cdot f(x_1|x_2, x_3). \tag{4.2.3}$$

According to equation (4.2.2), we can write

$$f(x_2|x_3) = c_{23}(F_2(x_2), F_3(x_3)) \cdot f_2(x_2). \tag{4.2.4}$$

Similarly, it is possible to decompose the conditional density of X_1 given X_2 and X_3 as

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot f(x_1|x_2). \tag{4.2.5}$$

Now, decomposing $f(x_1|x_2)$ in (4.2.5) further, we have

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1). \tag{4.2.6}$$

Finally, from equations (4.2.4) and (4.2.6), the joint density function for the trivariate case can be written as

$$\begin{aligned} f(x_1, x_2, x_3) = f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot \\ \cdot c_{23}(F_2(x_2), F_3(x_3)) \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)). \end{aligned} \tag{4.2.7}$$

That is, the trivariate density can be factorized as a product of the marginals, two bivariate copulas, c_{12} and c_{23}, and a third copula $c_{13|2}$ named conditional, whose arguments are conditional distributions.

The previous results for the trivariate case can be generalized for an n-dimensional vector, using the following formula:

$$f(x|\boldsymbol{\upsilon}) = c_{x\upsilon_j|\boldsymbol{\upsilon}_{-j}}(F(x|\boldsymbol{\upsilon}_{-j}), F(\upsilon_j|\boldsymbol{\upsilon}_{-j})) \cdot f(x|\boldsymbol{\upsilon}_{-j}), \tag{4.2.8}$$

for a vector $\boldsymbol{\upsilon}$ with dimension d. Here υ_j is an arbitrarily chosen component of $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon}_{-j}$ corresponds to the vector $\boldsymbol{\upsilon}$ excluding this component. It follows that the multivariate density function with dimension n can be decomposed into its marginal densities and a set of iteratively conditioned bivariate copulas.

The pair-copula decomposition of a multivariate density involves marginal conditional distributions of the form $F(x|\boldsymbol{\upsilon})$, computed using a formula of Joe (1996):

$$F(x|\boldsymbol{\upsilon}) = \frac{\partial C_{x,\upsilon_j|\boldsymbol{\upsilon}_{-j}}(F(x|\boldsymbol{\upsilon}_{-j}), F(\upsilon_j|\boldsymbol{\upsilon}_{-j}))}{\partial F(\upsilon_j|\boldsymbol{\upsilon}_{-j})}. \tag{4.2.9}$$

As the number of variables grows, the different possibilities of decomposition in pair-copulas also increase. To organize these possibilities, Bedford and Cooke (2001, 2002) introduced a graphical model called regular vine (R-vine). The R-vines are a sequence of nested trees that facilitate the identification of the needed pairs of variables and their corresponding set of conditioning variables (we refer the reader to Bedford and Cooke (2001, 2002) for more details on general R-vines and to Dißmann et al. (2013) for inference of R-vines). Two boundary cases, popularized by Aas et al. (2009), are the canonical vine (C-vine) and the drawable vine (D-vine). Canonical vines resemble factor models, with a particular variable playing the role of pivot (factor) in every tree. Because we are not interested in exploring any factor structure of financial data sets here, we will focus our attention on the D-vine.

An n-dimensional D-vine consists of $n-1$ hierarchical trees (or levels), with path structures in their sequences and increasing conditional sets, and $n(n-1)/2$ edges corresponding to a pair-copula (for a more detailed description, see Aas et al. (2009)). Define the index sets $\upsilon_{ij} = \{i+1, \ldots, i+j-1\}$, with $\upsilon_{i1} = \varnothing$, and $w_{ij} = \{i, \upsilon_{ij}, i+j\}$, for $1 \leq i \leq n-j, 1 \leq j \leq n-1$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ denote the parameters of the marginals and the n-dimensional copula, respectively, and $\boldsymbol{\theta}_{i,i+j|\upsilon_{ij}}$ be the parameters of the copula density $c_{i,i+j|\upsilon_{ij}}$. Finally, define $\boldsymbol{\theta}_{i\to i+j} = \{\boldsymbol{\theta}_{s,s+t|\upsilon_{st}} : (s, s+t) \in w_{ij}\}$, with $\boldsymbol{\theta}_{i\to i} = \varnothing$, and $\boldsymbol{\theta}_j = \{\boldsymbol{\theta}_{s,s+t|\upsilon_{st}} : |\upsilon_{st}| = j-1\}$, where $|\cdot|$ denotes the cardinality, i.e. $\boldsymbol{\theta}_j$ gathers all parameters at level j of the structure. The density $f(x_1, \ldots, x_n; \boldsymbol{\alpha}, \boldsymbol{\theta})$ associated with a D-vine may be written as[2]

$$
\begin{aligned}
f(x_1, \ldots, x_n; \boldsymbol{\alpha}, \boldsymbol{\theta}) = &\\
&\prod_{k=1}^{n} f(x_k; \boldsymbol{\alpha}_k)\\
&\cdot \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,i+j|\upsilon_{ij}} (F_{i|\upsilon_{ij}}(x_i|\mathbf{x}_{\upsilon_{ij}}; \boldsymbol{\alpha}_{w_{i,j-1}}, \boldsymbol{\theta}_{i\to i+j-1}),\\
&F_{i+j|\upsilon_{ij}}(x_{i+j}|\mathbf{x}_{\upsilon_{ij}}; \boldsymbol{\alpha}_{w_{i+1,j-1}}, \boldsymbol{\theta}_{i+1\to i+j}); \boldsymbol{\theta}_{i,i+j|\upsilon_{ij}}),
\end{aligned} \tag{4.2.10}
$$

where index j identifies the trees, whereas i runs over the edges in each tree. The whole decomposition is given by the $n(n-1)/2$ pair-copulas and the marginal densities of each variable.

Figure 4.1 depicts a five-dimensional D-vine. The density $f(x_1, x_2, x_3, x_4, x_5; \boldsymbol{\alpha}, \boldsymbol{\theta})$ is decomposed in a simple manner by multiplying the edges of the nested set of trees and the marginal

[2]Here we use the notation of Haff (2010).

densities $f(\cdot)$, as indicated below

$$
\begin{aligned}
& f_1(x_1;\boldsymbol{\alpha}_1)\cdot f_2(x_2;\boldsymbol{\alpha}_2)\cdot f_3(x_3;\boldsymbol{\alpha}_3)\cdot f_4(x_4;\boldsymbol{\alpha}_4)\cdot f_5(x_5;\boldsymbol{\alpha}_5)\\
&\quad \cdot c_{12}(F_1(x_1;\boldsymbol{\alpha}_1),F_2(x_2;\boldsymbol{\alpha}_2);\boldsymbol{\theta}_{12})\cdot c_{23}(F_2(x_2;\boldsymbol{\alpha}_2),F_3(x_3;\boldsymbol{\alpha}_3);\boldsymbol{\theta}_{23})\\
&\quad \cdot c_{34}(F_3(x_3;\boldsymbol{\alpha}_3),F_4(x_4;\boldsymbol{\alpha}_4);\boldsymbol{\theta}_{34})\cdot c_{45}(F_4(x_4;\boldsymbol{\alpha}_4),F_5(x_5;\boldsymbol{\alpha}_5);\boldsymbol{\theta}_{45})\\
&\quad \cdot c_{13|2}(F_{1|2}(x_1|x_2;\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\theta}_{12}),F_{3|2}(x_3|x_2;\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\theta}_{23});\boldsymbol{\theta}_{13|2})\\
&\quad \cdot c_{24|3}(F_{2|3}(x_2|x_3;\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\theta}_{23}),F_{4|3}(x_4|x_3;\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{34});\boldsymbol{\theta}_{24|3})\\
&\quad \cdot c_{35|4}(F_{3|4}(x_3|x_4;\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{34}),F_{5|4}(x_5|x_4;\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5,\boldsymbol{\theta}_{45});\boldsymbol{\theta}_{35|4})\\
&\quad \cdot c_{14|23}(F_{1|23}(x_1|x_2,x_3;\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\theta}_{12},\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{13|2}),\\
&\quad F_{4|23}(x_4|x_2,x_3;\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{24|3});\boldsymbol{\theta}_{14|23})\\
&\quad \cdot c_{25|34}(F_{2|34}(x_2|x_3,x_4;\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{24|3}),\\
&\quad F_{5|34}(x_5|x_3,x_4;\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5,\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{45},\boldsymbol{\theta}_{35|4});\boldsymbol{\theta}_{25|34})\\
&\quad \cdot c_{15|234}(F_{1|234}(x_1|x_2,x_3,x_4;\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{12},\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{13|2},\boldsymbol{\theta}_{24|3},\boldsymbol{\theta}_{14|23}),\\
&\quad F_{5|234}(x_5|x_2,x_3,x_4;\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5,\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{45},\boldsymbol{\theta}_{24|3},\boldsymbol{\theta}_{35|4},\boldsymbol{\theta}_{25|34});\boldsymbol{\theta}_{15|234}).
\end{aligned}
\tag{4.2.11}
$$

Copula-based Dependence Measures and Tail Dependence in Regular Vine Copulas

Because copulas describe the dependence structure among random variables, it is natural to think of dependence measures expressible in terms of the copula function. The Kendall's tau and the tail dependence[3] are useful copula-based dependence measures.

The Kendall's tau relies on the notion of concordance. Informally, a pair of random variables are concordant whenever large values of one tend to be associated with large values of the other and small values of one with small values of the other. Let (x_1, y_1) and (x_2, y_2) be two observations from a vector (X, Y) of continuous random variables, then we say that the pairs are *concordant* whenever $(x_1 - x_2)(y_1 - y_2) > 0$, and *discordant* whenever $(x_1 - x_2)(y_1 - y_2) < 0$. The population version of Kendall's tau for X and Y is given by

$$
\begin{aligned}
\tau = \tau_{X,Y} &= P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]\\
&= 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1,
\end{aligned}
$$

[3]For more details on these and other copula-based measures of dependence, see Chapter 2 of Joe (1997) and Chapter 5 of Nelsen (2006).

where C is the copula of X and Y.

Tail dependence measures the dependence in extreme values, for this reason it is an important measure for risk management. If the limit

$$\lim_{\varepsilon \to 0} \Pr\left[U_1 \leq \varepsilon | U_2 \leq \varepsilon\right] = \lim_{\varepsilon \to 0} \Pr\left[U_2 \leq \varepsilon | U_1 \leq \varepsilon\right] = \lim_{\varepsilon \to 0} C(\varepsilon, \varepsilon)/\varepsilon = \lambda_L$$

exists, then the copula C has lower tail dependence if $\lambda_L \in (0, 1]$ and no lower tail dependence if $\lambda_L = 0$. Similarly, if the limit

$$\lim_{\delta \to 1} \Pr\left[U_1 > \delta | U_2 > \delta\right] = \lim_{\delta \to 1} \Pr\left[U_2 > \delta | U_1 > \delta\right] = \lim_{\delta \to 1} \left(1 - 2\delta + C(\delta, \delta)\right) / (1 - \delta) = \lambda_U$$

exists, then the copula C has upper tail dependence if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. In other words, the lower (upper) tail dependence is the probability that one variable takes an extremely large negative (positive) value, given that the other variable took an extremely large negative (positive) value.

Recently, Joe et al. (2010) have found interesting results concerning tail dependence in vine copulas. They have a main theorem which states that if the supports of the pair-copulas in a vine are the entire $(0, 1)^2$ and all the pair-copulas in level 1 have lower (upper) tail dependence, then the vine copula C has lower (upper) tail dependence. If a copula C has multivariate lower (upper) tail dependence, then all bivariate and lower-dimensional margins have lower (upper) tail dependence. Another important finding is concerned with tail asymmetry of the vine copulas. They show that vine copulas can have different upper and lower tail dependence for each bivariate margin when asymmetric bivariate copulas with upper/lower tail dependence are used in level 1 of the vine.

4.2.2 Introducing Dynamics into the Vine Copula

Most of the works on vine copulas applied to financial data are focused on time-homogeneous models, however it goes against evidence found in the literature that dependence among returns is not constant over time (see, e.g., Longin and Solnik (2001), Ang and Bekaert (2002), Ang and Chen (2002)). Therefore we propose to introduce dynamics into the D-vine copula model, by allowing the dependence parameters of the pair-copulas to be potentially time-varying, evolving through time according to an equation that follows a nonlinear restricted ARMA(1,m) process as in Patton (2006). The evolution equation of the dependence parameter $\theta_{i,i+j|\upsilon_{ij}}$ of the pair-copula $c_{i,i+j|\upsilon_{ij}}$, with $\upsilon_{ij} = \{i+1, \dots, i+j-1\}$ and $\upsilon_{i1} = \varnothing$, for $1 \leq i \leq n-j,\ 1 \leq j \leq n-1$, may be written as

$$\theta_{i,i+j|\upsilon_{ij},t} = \Lambda(\omega + \beta\theta_{i,i+j|\upsilon_{ij},t-1} + \alpha\psi_t), \tag{4.2.12}$$

where Λ is a logistic transformation used to keep the parameter in its interval at all times and ψ_t is a forcing variable. The latter is defined as the mean absolute difference between the transformed data $u_{i|\upsilon_{ij},t} = F_{i|\upsilon_{ij}}(x_{i,t}|\mathbf{x}_{\upsilon_{ij},t})$ and $u_{i+j|\upsilon_{ij},t} = F_{i+j|\upsilon_{ij}}(x_{i+j,t}|\mathbf{x}_{\upsilon_{ij},t})$ over the past m observations, $\frac{1}{m}\sum_{k=1}^{m} |u_{i|\upsilon_{ij},t-k} - u_{i+j|\upsilon_{ij},t-k}|$. The idea is to use this measure as an indication of how far the data was from comonotonicity: if $X_i|\mathbf{X}_{\upsilon_{ij}}$ and $X_{i+j}|\mathbf{X}_{\upsilon_{ij}}$ are comonotonic, $|u_{i|\upsilon_{ij},t} - u_{i+j|\upsilon_{ij},t}|$ is close to zero. For the Normal copula, the cross-product $\frac{1}{m}\sum_{k=1}^{m} \Phi^{-1}(u_{i|\upsilon_{ij},t-k}) \cdot \Phi^{-1}(u_{i+j|\upsilon_{ij},t-k})$, where Φ^{-1} is the inverse of the standard Normal *c.d.f.*, is used as the forcing variable. For the Student-t, the forcing variable is $\frac{1}{m}\sum_{k=1}^{m} T_{\nu}^{-1}(u_{i|\upsilon_{ij},t-k}) \cdot T_{\nu}^{-1}(u_{i+j|\upsilon_{ij},t-k})$, where T_{ν}^{-1} is the inverse of the Student-t *c.d.f.* with ν degrees of freedom. If data is positively dependent, the inverse of the transforms of both variables will have the same sign, thus, α is expected to be positive. Patton restricted m to be equal to 10, but here we do allow for different window lengths for the forcing variable. We assume $m = 5, 10$ or 15 in order to investigate the comovements over three different periods, which correspond to the last 1, 2 and 3 weeks, respectively, for daily returns.

4.3 Inference Procedure: Sequential Estimation

According to equation (4.2.10), the log-likelihood function corresponding to a D-vine is given by

$$\begin{aligned}
\ell(\boldsymbol{\alpha}, \boldsymbol{\theta}; \mathbf{x}) = \\
&\sum_{t=1}^{T} \log(f(x_{1,t}, \dots, x_{n,t}; \boldsymbol{\alpha}, \boldsymbol{\theta})) \\
&= \sum_{t=1}^{T}\sum_{k=1}^{n} \log(f_k(x_{k,t}; \boldsymbol{\alpha}_k)) \\
&+ \sum_{t=1}^{T}\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} \log(c_{i,i+j|\upsilon_{ij}}(F_{i|\upsilon_{ij}}(x_{i,t}|\mathbf{x}_{\upsilon_{ij},t}; \boldsymbol{\alpha}_{w_{i,j-1}}, \boldsymbol{\theta}_{i\to i+j-1}), \\
&F_{i+j|\upsilon_{ij}}(x_{i+j,t}|\mathbf{x}_{\upsilon_{ij},t}; \boldsymbol{\alpha}_{w_{i+1,j-1}}, \boldsymbol{\theta}_{i+1\to i+j}); \boldsymbol{\theta}_{i,i+j|\upsilon_{ij}})) \\
&= \ell_M(\boldsymbol{\alpha}; \mathbf{x}) + \ell_C(\boldsymbol{\alpha}, \boldsymbol{\theta}; \mathbf{x}).
\end{aligned} \tag{4.3.1}$$

In particular, for the five-dimensional case, based on the density (4.2.11), we have

$$
\begin{aligned}
&\sum_{t=1}^{T}\Big(\log f_1(x_{1,t};\boldsymbol{\alpha}_1)+\log f_2(x_{2,t};\boldsymbol{\alpha}_2)+\log f_3(x_{3,t};\boldsymbol{\alpha}_3)+\log f_4(x_{4,t};\boldsymbol{\alpha}_4)+\log f_5(x_{5,t};\boldsymbol{\alpha}_5)\Big)\\
&\quad+\sum_{t=1}^{T}\Big(\log(c_{12}(F_1(x_{1,t};\boldsymbol{\alpha}_1),F_2(x_{2,t};\boldsymbol{\alpha}_2);\boldsymbol{\theta}_{12}))+\log(c_{23}(F_2(x_{2,t};\boldsymbol{\alpha}_2),F_3(x_{3,t};\boldsymbol{\alpha}_3);\boldsymbol{\theta}_{23}))\\
&\quad+\log(c_{34}(F_3(x_{3,t};\boldsymbol{\alpha}_3),F_4(x_{4,t};\boldsymbol{\alpha}_4);\boldsymbol{\theta}_{34}))+\log(c_{45}(F_4(x_{4,t};\boldsymbol{\alpha}_4),F_5(x_{5,t};\boldsymbol{\alpha}_5);\boldsymbol{\theta}_{45}))\Big)\\
&\quad+\sum_{t=1}^{T}\Big(\log(c_{13|2}(F_{1|2}(x_{1,t}|x_{2,t};\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\theta}_{12}),F_{3|2}(x_{3,t}|x_{2,t};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\theta}_{23});\boldsymbol{\theta}_{13|2}))\\
&\quad+\log(c_{24|3}(F_{2|3}(x_{2,t}|x_{3,t};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\theta}_{23}),F_{4|3}(x_{4,t}|x_{3,t};\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{34});\boldsymbol{\theta}_{24|3}))\\
&\quad+\log(c_{35|4}(F_{3|4}(x_{3,t}|x_{4,t};\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{34}),F_{5|4}(x_{5,t}|x_{4,t};\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5,\boldsymbol{\theta}_{45});\boldsymbol{\theta}_{35|4}))\Big)\\
&\quad+\sum_{t=1}^{T}\Big(\log(c_{14|23}(F_{1|23}(x_{1,t}|x_{2,t},x_{3,t};\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\theta}_{12},\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{13|2}),\\
&\quad F_{4|23}(x_{4,t}|x_{2,t},x_{3,t};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{24|3});\boldsymbol{\theta}_{14|23}))\\
&\quad+\log(c_{25|34}(F_{2|34}(x_{2,t}|x_{3,t},x_{4,t};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{24|3}),\\
&\quad F_{5|34}(x_{5,t}|x_{3,t},x_{4,t};\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5,\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{45},\boldsymbol{\theta}_{35|4});\boldsymbol{\theta}_{25|34}))\Big)\\
&\quad+\sum_{t=1}^{T}\Big(\log(c_{15|234}(F_{1|234}(x_{1,t}|x_{2,t},x_{3,t},x_{4,t};\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\theta}_{12},\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{13|2},\boldsymbol{\theta}_{24|3},\boldsymbol{\theta}_{14|23}),\\
&\quad F_{5|234}(x_{5,t}|x_{2,t},x_{3,t},x_{4,t};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\alpha}_5,\boldsymbol{\theta}_{23},\boldsymbol{\theta}_{34},\boldsymbol{\theta}_{45},\boldsymbol{\theta}_{24|3},\boldsymbol{\theta}_{35|4},\boldsymbol{\theta}_{25|34});\boldsymbol{\theta}_{15|234}))\Big). \qquad (4.3.2)
\end{aligned}
$$

The hierarchical structure of the vine copula allows us to adopt a very fast yet asymptotically efficient sequential estimation procedure (Haff, 2010). It is clear from the decomposition (4.3.2) above that the joint log-likelihood function of a five-dimensional D-vine can be separated in, at least, five parts: the one corresponding to the log-likelihoods of the marginal distributions, and those corresponding to the log-likelihoods associated with the different levels of the vine, i.e. with the pair-copulas with increasing conditioning sets. In this case, estimation can proceed in five steps, and, at each step, it is carried out conditionally on the parameters estimated in earlier steps. In the first step, the parameters of the marginal distributions are estimated via maximum likelihood and the log-returns are transformed into uniforms, which become inputs for the pair-copulas in the first level of the D-vine. In the second step, the parameters of the pair-copulas in the first tree are estimated via maximum likelihood, taking the parameters of the marginals as fixed at the estimated values from the first step. Additionally, the transformed data (i.e. conditional distributions) necessary for tree 2 are computed using (4.2.9). In the sequel, the parameters of the pair-copulas in level 2 are estimated, given the estimates of the parameters in the previous levels. We proceed like this till we have the parameters estimates for all trees. It means that the whole procedure can be decomposed into a series of optimizations for the marginals and for iteratively conditioned bivariate copulas. The levelwise estimation improves significantly the computational efficiency, which is very important because the number of parameters to be estimated increases rapidly with the number of variables. Following, we give details on the model specification.

4.3.1 GARCH Models for the Univariate Distributions

A univariate ARMA(p,q)-GARCH(m,n)[4] specification is usually chosen to model the marginal distributions of return data. It can be described by the following equations:

$$\begin{aligned} x_t &= \phi_0 + \sum_{i=1}^{p} \phi_i x_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j} = \mu_t + a_t & (4.3.3) \\ a_t &= h_t^{1/2} \varepsilon_t, & (4.3.4) \\ h_t &= \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{n} \beta_j h_{t-j}, & (4.3.5) \end{aligned}$$

where μ_t and h_t are the conditional mean and variance given past information, respectively. ε_t is the innovation process and, in this paper, we assume that it may have a standard Normal distribution, $\varepsilon_t \sim Normal(0,1)$, a Student-t distribution, $\varepsilon_t \sim Student-t(\nu)$, or a Skewed-t distribution proposed by Hansen (1994), $\varepsilon_t \sim Skewed-t(\nu,\lambda)$.

If, for example, $\varepsilon_t \sim Skewed-t(\nu,\lambda)$, then the conditional distribution function of X_t is given by $F(x_t|\mu_t,h_t) = Skewed-t_{\nu,\lambda}((x_t-\mu_t)h_t^{-1/2})$. Thus, if the marginal distribution is well specified, the probability integral transform (PIT) of the standardized residuals will have a uniform distribution in [0,1]. This is a necessary result to identify the copulas in the second step of the estimation procedure, since they are joint distribution functions defined over $u_{i,t} = F_i(x_{i,t}|\mu_{i,t},h_{i,t})$ and $u_{i+1,t} = F_{i+1}(x_{i+1,t}|\mu_{i+1,t},h_{i+1,t})$, $i=1,\ldots,n-1$, with $u_{i,t}$ and $u_{i+1,t}$ uniforms in [0,1]. To test whether the PIT of the standardized residuals has distribution $U[0,1]$, we use the Kolmogorov-Smirnov test of goodness-of-fit.

[4]Extensions of the GARCH model, such as EGARCH, TARCH, among others, are also fitted to data in order to find out the best model for the marginals.

In summary, in the first step of the sequential estimation procedure, the ARMA-GARCH filter is applied to the return data to obtain the univariate parameters estimates and the transformed data (uniforms) necessary for the first level of the D-vine.

4.3.2 Building Blocks: Bivariate Copulas

For fitting multivariate financial data with flexible lower/upper tail dependence, possibly with stronger tail dependence in the joint lower tail than upper tail, we consider copula families with different strengths of tail behavior in the estimation of the D-vine copula:

1. BB1, BB7 and Symmetrized Joe-Clayton (SJC) copulas, which have different upper and lower tail dependence.

2. Gumbel copula, with only upper tail dependence.

3. Rotated-Gumbel and Clayton copulas, with only lower tail dependence.

4. Student-t copula, with reflection symmetric[5] upper and lower tail dependence.

5. Normal copula, also reflection symmetric, but with no tail dependence at all.

For each pair of transformed data, we estimate both a static and a dynamic version of such copulas[6] and we use the AIC to choose the best of all of them. In addition, to verify whether the dependence structure between the data was appropriately modeled, we apply the Kolmogorov-Smirnov and the Anderson-Darling goodness-of-fit tests.

4.4 Simulation Study

In this section, we carry out a simulation study in order to evaluate the performance of the proposed model. We consider two data generating processes (DGPs) in the Monte Carlo study: (i) the dynamic D-vine copula model and (ii) a static D-vine copula. In both cases, we choose a decomposition with all Rotated-Gumbel pair-copulas to account for the evidence of asymmetric tail dependence in financial data. Using the simulation algorithm for a D-vine in Aas et

[5] Let $c_{1 \ldots m}$ be the copula density, then reflection symmetry implies $c(u_1, \ldots, u_m) = c(1 - u_1, \ldots, 1 - u_m)$.

[6] Their functional forms as well as the evolution equations of their dependence parameters following Patton (2006) are described in Appendix A.

al. (2009)[7], we replicate 1000 four-dimensional time series, with $T = 1000$, $T = 2000$ and $T = 5000$ observations.

The dependence dynamics of the time-varying D-vine copula model is as set below, with the following evolution equations for the pair-copulas dependence parameters:

$$\begin{aligned}
\theta_{12,\,t} &= \Lambda\Big(0.9539 + 0.2217\,\theta_{12,\,t-1} - 1.9303 \cdot \frac{1}{10}\sum_{k=1}^{10} |u_{1,\,t-k} - u_{2,\,t-k}|\Big) \\
\theta_{23,\,t} &= \Lambda\Big(2.2046 - 0.0290\,\theta_{23,\,t-1} - 5.0000 \cdot \frac{1}{10}\sum_{k=1}^{10} |u_{2,\,t-k} - u_{3,\,t-k}|\Big) \\
\theta_{34,\,t} &= \Lambda\Big(1.1043 + 0.1831\,\theta_{34,\,t-1} - 2.1552 \cdot \frac{1}{10}\sum_{k=1}^{10} |u_{3,\,t-k} - u_{4,\,t-k}|\Big) \\
\theta_{13|2,\,t} &= \Lambda\Big(0.2185 + 0.2399\,\theta_{13|2,\,t-1} - 0.6637 \cdot \frac{1}{10}\sum_{k=1}^{10} |u_{1|2,\,t-k} - u_{3|2,\,t-k}|\Big) \\
\theta_{24|3,\,t} &= \Lambda\Big(-0.5610 + 0.9133\,\theta_{24|3,\,t-1} - 0.5804 \cdot \frac{1}{10}\sum_{k=1}^{10} |u_{2|3,\,t-k} - u_{4|3,\,t-k}|\Big) \\
\theta_{14|23,\,t} &= \Lambda\Big(0.2515 + 0.2122\,\theta_{14|23,\,t-1} - 0.5546 \cdot \frac{1}{10}\sum_{k=1}^{10} |u_{1|23,\,t-k} - u_{4|23,\,t-k}|\Big)
\end{aligned}$$

where $\theta_{i,i+j|\upsilon_{ij},t}$ corresponds to the parameter of the copula density $c_{i,i+j|\upsilon_{ij}}$ in t, with $\upsilon_{ij} = \{i+1,\ldots,i+j-1\}$ and $\upsilon_{i1} = \varnothing$, for $1 \leq i \leq 4-j, 1 \leq j \leq 3$. Additionally, $u_{i|\upsilon_{ij},t} = F_{i|\upsilon_{ij}}(x_{i,t}|\mathbf{x}_{\upsilon_{ij},t})$ and $u_{i+j|\upsilon_{ij},t} = F_{i+j|\upsilon_{ij}}(x_{i+j,t}|\mathbf{x}_{\upsilon_{ij},t})$ are the arguments of $c_{i,i+j|\upsilon_{ij}}(\cdot)$. This dynamic structure comprises better defined time paths for the dependence parameters of the pair-copulas in the first level of the vine and noisier time paths in higher levels. Furthermore, the degree of dependence is higher in the first tree than in the second and third ones. These characteristics are intended to reproduce evidences found for real data.

For the parameters of the static D-vine copula, we fix the following values: $\theta_{12} = 3.0581$, $\theta_{23} = 3.7837$, $\theta_{34} = 3.1028$, $\theta_{13|2} = 1.21$, $\theta_{24|3} = 1.17$ and $\theta_{14|23} = 1.1$. Again, in this case, we have greater dependence in the first level of the vine.

Having simulated the data sets, for each time series replicated from both DGPs, we estimate models (i) and (ii)[8] above-mentioned. To simplify the exercise, in the identification we model the

[7] The algorithm uses the conditional inversion method described in e.g. Embrechts et al. (2003). Given the D-vine structure, all the conditional distribution functions involved are of the form (4.2.9), so, in order to be computed, only the first partial derivative of a bivariate copula is required. A numerical inversion is necessary for the Rotated-Gumbel copula.

[8] To obtain the estimates of the vine copula parameters in the former case, we adopt the sequential estimation procedure. For the second model, as it is usual in the literature of static vine copulas, we first estimate the parameters of the D-vine copula using the sequential estimation procedure and, then, we maximize the copula

correct pair-copula function, i.e. we estimate the models with all Rotated-Gumbel pair-copulas as in the source models. We then compute the mean errors (ME) and root mean squared errors (RMSE) based on the difference between the estimates and the true values of the pair-copulas dependence parameters over time. The ME_i and $RMSE_i$, $i = 1, \ldots, 1000$, corresponding to the estimates from models (i) and (ii) for the pair-copulas dependence parameters are displayed in box plots, in Figures 4.2 to 4.13. Additionally, a few statistics of the Monte Carlo experiments are presented in Tables 4.1 to 4.4.

Figures 4.2 to 4.7 show the box plots of the mean errors. One can easily notice that, when the DGP is the dynamic D-vine copula (Figures 4.2, 4.3 and 4.4), the static D-vine copula tends to underestimate the dependence parameters. It can be inferred from the negative medians of the mean errors associated with the estimates from the static model. Though the negative bias is greater in the estimates of the pair-copulas parameters in the first tree (Figure 4.2 and first row of Figure 4.3), which is expected, given that the dynamics is better defined at this level in the source model, it also occurs in the estimates of the pair-copulas parameters in higher levels (Figure 4.3, second row, and Figure 4.4). Moreover, for all pair-copulas parameters, the mean errors median barely changes given increases in the sample size, although the variability of the mean errors decreases, which means that the bias does not diminish as the number of observations in the sample increases. On the other hand, when the samples drawn from the dynamic D-vine model are estimated with the same model, for the parameters of all pair-copulas, the mean errors have median closer to zero and their variability decreases as the sample size increases. All these evidences are also found in Table 4.1, which reports the mean, median and standard deviation of the mean errors when the data is simulated from the dynamic D-vine model. Considering, now, that the static D-vine copula is the DGP (Figures 4.5, 4.6 and 4.7), we can observe from the box plots that, for either the dynamic or the static model estimates, the mean errors have medians close to zero, which means that the estimates from both models are unbiased, and their variabilities tend to decrease as the sample size increases. This information is also summarized in Table 4.2. So, the previous results suggest that the dynamic D-vine copula model outperforms the static D-vine copula in terms of the mean errors, or bias.

log-likelihood over all dependence parameters, using as starting values the parameters obtained from the stepwise procedure.

The box plots of the root mean squared errors are displayed in Figures 4.8 to 4.13. When the underlying data generating model is the dynamic D-vine copula, regarding the parameters of the pair-copulas in the first level of the vine, note, in Figure 4.8 and in the first row of Figure 4.9, that the medians of the errors associated with the static D-vine copula estimates are far much higher than those related to the errors of the dynamic D-vine copula. Furthermore, they do not seem to decrease as the number of observations in the samples increases, although the errors variabilities diminish. It suggests that, even though the estimates from the static model show less variability as the sample size increases, the bias, previously analyzed, does not allow for precise estimates. Contrary to what happens to the static model, both the median and dispersion of the errors from the dynamic model decrease as the number of observations in the samples increases, which indicates that the estimates become more precise. Concerning the parameters of the pair-copulas in the second and third levels of the vine (Figure 4.9, second row, and Figure 4.10), both models display smaller values for the errors medians and these are closer to each other, if compared with the first level. Despite that, the errors medians of the static model still remain above those of the dynamic model, and they barely change with the different sample sizes, which may be explained by the persistent bias. On the other hand, the errors from the static model show less dispersion than those from the dynamic model, for all sample sizes. Table 4.3 presents summary statistics of the root mean squared errors from both estimated models, considering the dynamic D-vine copula as the DGP, which confirm the aforesaid. When the samples are drawn from the static D-vine copula (Figures 4.11, 4.12 and 4.13), both the median and the variability of the errors from the dynamic D-vine are higher than those of the static model, for all pair-copulas parameters. Consequently, although the dynamic D-vine copula provides unbiased estimates when the DGP is static, the variability of its estimates is higher than that of the static D-vine estimates. One should take note, though, that both the median and the variability of the errors from the dynamic model decrease as the sample size increases. Table 4.4 gathers the statistics related to the root mean squared errors, regarding the static D-vine copula as the DGP. In summary, in terms of the root mean squared errors, the dynamic D-vine copula is far superior to the static D-vine when the data comes from the dynamic model, especially with regard to the pair-copulas parameters in the first tree.

Nevertheless, the dynamic D-vine is not able to outperform the static D-vine when the latter is the DGP, given the higher variability of the former's estimates in this case.

Overall, the findings of the Monte Carlo experiments are quite favorable to the dynamic D-vine copula. Notedly, when the DGP is the time-varying model, the static model tends to underestimate the pair-copulas dependence parameters. Furthermore, the negative bias does not seem to diminish as the number of observations in the samples increases. The estimates from the dynamic D-vine copula, in this case, are far superior to the estimates from the static D-vine, both in terms of the mean errors and the root mean squared errors. When the data comes from the static D-vine copula, both models have similar performance in terms of the mean errors, with unbiased estimates. However, the dynamic D-vine copula performs worse in terms of the root mean squared errors, what suggests that its estimates display higher variability, though it diminishes as the sample size increases.

4.5 Empirical Application: Dependence Modeling and VaR Backtesting

In this section, we model the dependence among the returns of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA indexes, using both the dynamic D-vine copula model and a static D-vine copula. We consider two distinct periods, one from January 03, 2003 to December 28, 2007 and another one from January 02, 2008 to May 04, 2012, which we denominate "non-crisis period" and "crisis period", respectively. Besides of investigating the different patterns of dependence characterizing these periods, the intention here is to further evaluate the proposed model concerning the accuracy of the VaR forecasts in crisis periods.

4.5.1 Return Data

In our empirical study, we use two data sets of daily log-returns of the indexes DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA: one comprising the period from January 03, 2003 to December 28, 2007, which we call "non-crisis period" because the financial markets were in an upturn trend till September, 2007, with a total of 1178 observations; and another one spanning the period from January 02, 2008 to May 04, 2012, considered a "crisis period" because it coincides with the subprime crisis till June, 2009, and with the European sovereign debt crisis from early 2010 and thereafter, with 1029 observations. We use close-to-close returns, meaning that the daily returns are those observed for trading days occurring simultaneously in all five stock markets considered.

Table 4.5 provides a few descriptive statistics of our data sets. We can see from the table that the average returns of all indexes become negative in the crisis period and the standard deviations increase. It is also possible to notice that both data sets present signs of non-normality. All returns series have kurtosis above 3, and the excess kurtosis is higher in the crisis period. FTSE 100 and S&P 500 returns display negative skewness in both periods, whereas DAX, CAC 40 and IBOVESPA returns change from negative to positive skewness. Also, according to the Jarque-Bera test statistics, it is possible to reject the null hypothesis of normality for all indexes returns in both periods.

4.5.2 Marginal Models

We first proceed to the modeling of the marginal distributions using the ARMA-GARCH specification. To account for the leverage effect, present in financial time series, we also consider asymmetric GARCH specifications, such as the EGARCH, TGARCH and GJR models[9].

We choose the best specifications for the marginals based on the information criteria AIC and BIC. In the non-crisis period, we choose an AR(1)-EGARCH(1,1) for S&P 500, an AR(1)-GARCH(1,1) for FTSE 100, a GARCH(1,1) for both CAC 40 and IBOVESPA, and, finally, an EGARCH(1,1) for DAX. In the crisis period, we choose an AR(1)-EGARCH(2,1) for S&P 500, an AR(3)-EGARCH(1,1) for IBOVESPA, and a GARCH(1,1) for FTSE 100, CAC 40 and DAX, with conditional means modeled by an AR(2) in the first two cases. Because our data sets display clear signs of asymmetry and excess kurtosis, we assume $Skewed - t(\nu, \lambda)$ distributed iid innovations[10]. The estimates from the ARMA-GARCH fits[11] are presented in Tables 4.6 and 4.7. We can observe that the estimated asymmetry coefficient, $\hat{\lambda}$, is negative and statistically

[9]The conditional variance, h_t, of an EGARCH(m,o,n) process can be modeled as follows:

$$\ln(h_t) = \alpha_0 + \sum_{i=1}^{m} \alpha_i(|\varepsilon_{t-i}| - \mathbb{E}|\varepsilon_{t-i}|) + \sum_{k=1}^{o} \gamma_k \varepsilon_{t-k} + \sum_{j=1}^{n} \beta_j \ln(h_{t-j}).$$

For lack of space, we do not present here the other asymmetric specifications. For a survey on GARCH models, see Bollerslev (2008).

[10]Although not before also testing for the symmetric Student-t and Normal distributions.

[11]All marginals were estimated using the Oxford MFE Toolbox by Kevin Shepard.

significant for all series in both periods, suggesting a heavy tail to the left for the marginal distributions, i.e., large negative returns are more likely than large positive returns. The estimated degrees of freedom, $\hat{\nu}$, range from 11.5421 for S&P 500 to 18.8979 for FTSE 100 in the non-crisis period, and from 8.7683 for DAX to 14.3925 for FTSE 100 in the crisis period, suggesting heavier tails in the second period, which is in accordance with the descriptive statistics of the unconditional distributions of the returns series. Regarding the conditional variances, in some cases, we choose the EGARCH model as the best specification, suggesting the presence of some sort of leverage effect. This is the case, for example, of the S&P 500 returns, which display an asymmetric effect over the volatility, with greater impact induced by big negative returns, captured by the estimates of γ_1, -0.0820 in the non-crisis period and -0.1642 in the crisis period. Note that the leverage effect is intensified in the crisis period. We also find evidence of leverage effect for IBOVESPA returns in the crisis period, with $\hat{\gamma}_1 = -0.0963$, and for DAX returns in the non-crisis period, with $\hat{\gamma}_1 = -0.1127$. Tables 4.6 and 4.7 also provide the p-values of the Ljung-Box test of autocorrelation in the standardized and squared standardized residuals with 15 lags, $Q(15)$ and $Q^2(15)$, respectively. For all series, the null hypothesis of no autocorrelation left cannot be rejected at the 5% level, indicating that the ARMA-GARCH specifications are successful at modeling the serial correlation in the conditional mean and variance. Additionally, these tables report the p-values of the Kolmogorov-Smirnov test of uniformity of the PIT of the standardized residuals. For all series, there is no evidence against uniformity, so all marginal distributions seem to be well specified, which is very important, since, otherwise, the copula estimation would be affected.

4.5.3 Copula Structure

Having chosen the D-vine decomposition of the multivariate copula, we still have to match the indexes returns to the labels $1, \ldots, 5$, since there are $5!/2$ possible distinct permutations. A rule for selecting the best permutation for D-vines, according to Nikoloulopoulos et al. (2010), consists of choosing and connecting the most dependent pairs in the first tree. Using the sample Kendall's taus computed based on the PIT of the ARMA-GARCH residuals, reported in Tables 4.8 and 4.9, we choose as the best permutation for the first level of the D-vines of both periods under analysis (1, 2, 3, 4, 5) = (FTSE 100, CAC 40, DAX, S&P 500, IBOVESPA), since it comprises the largest possible dependencies.

Tables 4.10 and 4.11 report the estimates of the pair-copulas chosen to compose the dynamic D-vine copula in the non-crisis period and in the crisis period, respectively[12]. We can observe that time-varying pair-copulas are selected only in the first tree[13]. The dependence between FTSE and CAC during the non-crisis period is characterized by the BB1 copula, with the estimated parameter $\hat{\gamma_t}$ following an ARMA(1,10), whereas $\hat{\kappa}$ remains constant, equal to 0.7813. The same copula characterizes their dependence during the crisis period too, however, in this case, both estimated parameters $\hat{\gamma_t}$ and $\hat{\kappa_t}$ evolve through time according to an ARMA(1,5). We also choose a BB1 copula for the pair CAC-DAX: in the non-crisis period, $\hat{\gamma_t}$ follows a MA(10) and $\hat{\kappa}$ is constant, equal to 0.7252, whereas in the crisis period, $\hat{\gamma_t}$ evolves according to an ARMA(1,10) and $\hat{\kappa}$ remains constant, equal to 0.9915. The reflection symmetric Student-t copula is selected for both pairs DAX-S&P500 and S&P500-IBOVESPA in both periods. With regard to the dynamics, for the pair DAX-S&P500, during the non-crisis period, there is evidence of time variation for the correlation coefficient, with $\hat{\rho_t}$ following an ARMA(1,10), whereas the estimated degrees of freedom remain constant, equal to 8.7969; in the crisis period, we find no dynamics at all for both estimated parameters, $\hat{\rho} = 0.7088$ and $\hat{\nu} = 18.8746$. For S&P500-IBOVESPA, $\hat{\rho_t}$ follows an ARMA(1,10) and $\hat{\nu} = 6.9966$, in the non-crisis period, whereas $\hat{\rho_t}$ follows an ARMA(1,15) and $\hat{\nu} = 9.5523$, in the crisis period.

The dynamics of the dependencies in the first level of the estimated D-vines can be observed in Figures 4.14 to 4.17, which display the evolutions of the Kendall's tau and the tail dependence parameters computed based on the pair-copulas of the first tree. In Figure 4.14, panel (a), the dependence between FTSE and CAC measured by the Kendall's tau oscillates around 0.6372 from January 03, 2003 to December 28, 2007, when it increases and begins to oscillate around 0.7359 from January 02, 2008 to May 04, 2012. Note that the Kendall's tau path is a bit noisier during the crisis period. Also the tail dependence parameters, in panel (b), increase from the non-crisis period to the crisis period. Interestingly, the lower tail dependence steadily fluctuates

[12]Remind that, for each pair of transformed data, we fit both a static and a dynamic version of the copulas listed in Section 4.3.2, whose functional forms as well as the evolution equations of their dependence parameters following Patton (2006) are described in Appendix A. Using the AIC, we choose the best of all of them. Regarding the dynamics, for two-parameter copulas, it may happen that only one of the estimated parameters displays time variation. It may also happen that not all estimated coefficients of the evolution equation are statistically significant. In this case, the copula is re-estimated omitting the non-significant coefficient.

[13]Heinen and Valdesogo (2009, 2011) also find evidence of time variation especially in level 1.

above the path of the upper tail dependence all over the non-crisis period, becoming quite volatile during the crisis period. For the pair CAC-DAX, in Figure 4.15, panel (a), the Kendall's tau follows a path that fluctuates around 0.7031 till the end of 2007, when it reaches a higher level and begins oscillating closely to 0.7743 during the crisis period. In panel (b), the lower and upper tail dependence parameters evolve near each other during the non-crisis period, fluctuating around 0.6785, and move apart from 2008 on, with the upper tail dependence oscillating around 0.7382, whereas the lower tail dependence oscillates around a higher level, 0.7873. Figure 4.16 presents the evolution of the dependence between DAX and S&P500. In panel (a), the Kendall's tau varies over time along the first period, moving around 0.3609, but changes to a constant path in the crisis period, assuming the value 0.5015, estimated from the static Student-t copula. In panel (b), the tail dependence oscillates around 0.1180 till the end of 2007, when it, surprisingly, experiences a decrease, assuming a constant value of 0.0807 during the crisis period. Finally, the dependence between S&P500 and IBOVESPA, when measured by the Kendall's tau, in panel (a) of Figure 4.17, experiences an increase from the non-crisis to the crisis period: it oscillates around 0.4328 over the first period, whereas, in the second one, it fluctuates around 0.5207. On the other hand, although the tail dependence parameters, in panel (b), experience a meaningful increase during the end of 2008 and 2009, on average, they do not reach much higher a baseline during the crisis period in comparison with the previous period. From 2003 to 2007, the tail dependence oscillates close to 0.2134, and, from 2008 on, it oscillates around 0.2387, under the estimated value from the static Student-t, 0.3038.

Looking at Tables 4.10 and 4.11 once more, for higher levels of the D-vine, we choose mainly symmetric copulas in the non-crisis period, and asymmetric copulas in the crisis period. For example, the conditional copula of FTSE, DAX|CAC changes from the Student-t in the non-crisis period to the Rotated-Gumbel in the crisis period, with an implied increase in the lower tail dependence from 0.0083 to 0.1345. Also the type of dependence between the French and the North American markets, given information on the German market, captured by the conditional copula of CAC, S&P500|DAX, changes from symmetrical and with no tail dependence at all, in the non-crisis period, to asymmetrical featuring upper tail dependence of 0.0579 and lower tail dependence equal to 0.0204, in the crisis period.

Important features of the joint dependence among the indexes can be inferred from the preceding estimation results, based on the findings of Joe et al. (2010). Because the pair-copulas in the first level of the estimated D-vines have upper and lower tail dependence, both multivariate copulas also have upper and lower tail dependence. Moreover, since two of these pair-copulas are tail asymmetric, the range of upper/lower tail dependence for the bivariate (and lower-dimensional) margins is quite flexible. These characteristics are in accordance with empirical evidence found in the literature that financial data tends to exhibit tail dependence and asymmetries. The previous findings suggest that the overall dependence structure of the indexes does not change dramatically from the non-crisis to the crisis period, although the predominance of asymmetric pair-copulas in higher levels of the estimated D-vine for the crisis period may create a little more asymmetric dependence structure.

Notedly, the estimated dynamic D-vines differ in terms of the dependence strength that they describe, with stronger overall as well as tail dependencies captured by the estimated D-vine for the crisis period.

For the purpose of comparison, we also estimate a static D-vine copula for the two investigated data sets. To obtain the estimates of the pair-copulas parameters in this case, as it is usual in the literature of static vine copulas, we first estimate the parameters using the sequential estimation procedure[14] and, then, we maximize the D-vine copula log-likelihood over all dependence parameters, using as starting values the parameters obtained from the stepwise procedure. It corresponds to applying the two-step estimation procedure of Joe and Xu (1996), the Inference Function for Margins (IFM) method. The estimates of the pair-copulas composing the static D-vine copula in the non-crisis period and in the crisis period are presented in Tables 4.12 and 4.13, respectively. These tables also report the estimated Kendall's tau and tail dependence parameters, computed based on the estimated pair-copulas. The estimation results suggest that both D-vines display lower and upper tail dependence, since the pair-copulas in the first tree of both constructions are all Student-t. Furthermore, the dependence structure characterizing the crisis period is more asymmetric than the one of the non-crisis period, given the prevalence of asymmetric pair-copulas in the second level of the estimated D-vine. Concerning the degree

[14]For each pair of transformed data, we estimate a static version of the copulas listed in Section 4.3.2 and choose the best of them based on the AIC criterion.

of dependence described by the estimated D-vines, it is clear from the dependence measures reported in the tables that the estimated D-vine for the crisis period captures a stronger dependence among the indexes. In comparison with the estimated dynamic D-vines, the range of upper/lower tail dependencies of the margins, in this case, is less flexible, since only symmetric copulas are selected in the first level of the D-vines.

4.5.4 VaR Backtesting

We are interested in comparing the static and the dynamic D-vine copula models' abilities to forecast capital losses in the occurrence of extreme events, more specifically, crisis. For this purpose, we compare their performance in an out-of-sample exercise. The estimated models for the period from January 03, 2003 to December 28, 2007 are used to forecast one-day VaR at the 1%, 5% and 10% significance levels for an equally weighted portfolio of the indexes DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA in the period from January 02, 2008 to August 19, 2008 (150 days). Additionally, the estimated models for the period from January 02, 2008 to May 04, 2012 are used for VaR forecasting from May 08, 2012 to September 06, 2012 (79 days). Notice that both testing periods belong to the crisis period. In the former case, there is an additional motivation regarding the models' abilities to forecast extremal losses in bear markets, given that the copulas parameters were estimated in a different context, of bull markets. Given the estimation set of $\{1, \ldots, T\}$ daily observations for the copula model and the testing set $\{T+1, \ldots, T+h\}$, the exercise is done as follows:

1. For $k = 1, \ldots, 1000$:
   (a) From the fitted copula model, we simulate a sample $u_{1,t}^{(k)}, \ldots, u_{5,t}^{(k)}, t = 1, \ldots, h$.
   (b) For $j = 1, \ldots, 5$, we convert $u_{j,t}^{(k)}$ to $\hat{\varepsilon}_{j,t}^{(k)}$, $t = 1, \ldots, h$, using the inverse Skewed-t cdf's, i.e., $\hat{\varepsilon}_{j,t}^{(k)} = F_j^{-1}(u_{j,t}^{(k)})$.
   (c) For $j = 1, \ldots, 5$, we convert $\hat{\varepsilon}_{j,t}^{(k)}$ to the return forecasts as

$$\hat{x}_{j,T+t}^{(k)} = \hat{\mu}_{j,T+t} + \sqrt{\hat{h}_{j,T+t}} \cdot \hat{\varepsilon}_{j,t}^{(k)}, t = 1, \ldots, h,$$

   where $\hat{\mu}_{j,T+t}$ and $\hat{h}_{j,T+t}$ correspond to the one-step ahead forecasts of the conditional mean and variance, respectively[15].

[15] We re-estimate the parameters of the ARMA-GARCH specifications in a recursive scheme, using an expanding

(d) Then we compute the portfolio return forecasts as $\hat{x}^{(k)}_{P,T+t} = \sum_{j=1}^{5} \hat{x}^{(k)}_{j,T+t}/5$, $t = 1, \ldots, h$.

2. For significance levels $\alpha \in \{0.01, 0.05, 0.1\}$, we compute the one-day $\text{VaR}_{\alpha,1}$ forecast for the day $T+t$ as the 100αth-percentile of $\hat{x}^{(k)}_{P,T+t}, k = 1, \ldots, 1000$. If the observed value of the portfolio return for the day $T+t$, $x_{P,T+t}$, is less than $\widehat{VaR}_{\alpha,1}$, then a violation (or exceedance) is said to occur.

To evaluate the VaR forecasts, we initially use the likelihood ratio tests proposed by Kupiec (1995) and Christoffersen (1998). Based on the previous procedure, it is possible to construct an indicator sequence of violations $I_t, t = 1, ..., h$, called hits. If the forecasts are accurate, the hit sequence should exhibit two properties. First, the proportion of violations should approximately equal the VaR significance level α. The unconditional coverage test of Kupiec is a test of the null hypothesis that the expected violation rate is equal to the theoretical rate α of the VaR and the test statistic is defined as

$$LR_{uc} = -2\ln[\alpha^n(1-\alpha)^{h-n}] + 2\ln[(n/h)^n(1-n/h)^{h-n}] \sim_{H_0} \chi^2_1,$$

where n is the number of VaR violations, h is the size of the testing sample and n/h is the observed proportion of violations. Second, the exceedances should occur independently, i.e., not in clusters. Christoffersen (1998) proposed a combined test for both unconditional coverage and serial independence. He considers a binary first-order Markov chain for the hits, with transition probability matrix $\pi_{ij} = \Pr(I_t = j | I_{t-1} = i)$, with $i, j = 0, 1$ ("0" means no VaR violation and "1" means VaR violation), $\hat{\pi}_{ij} = \left(n_{ij}/\sum_j n_{ij}\right)$, where n_{ij} is the number of hits with indicator i followed by hits with indicator j. Under $H_0 : \pi_{01} = \pi_{11} = \alpha$, the test statistic is the conditional coverage statistic given by

$$LR_{cc} = -2\ln[\alpha^n(1-\alpha)^{h-n}] + 2\ln[\hat{\pi}_{01}^{n_{01}}(1-\hat{\pi}_{01})^{n_{00}}\hat{\pi}_{11}^{n_{11}}(1-\hat{\pi}_{11})^{n_{10}}] \sim_{H_0} \chi^2_2.$$

According to Lopez (1999), the statistical tests proposed by Kupiec and Christoffersen to evaluate the accuracy of VaR models can have relatively low power against inaccurate VaR models. For this reason, he proposed an alternative methodology based not on a statistical

window up to $T+t-1$, and use these estimates to obtain one-step ahead forecasts of the conditional mean and variance in $T+t$.

testing framework, but instead on standard forecast evaluation techniques: the accuracy of the VaR forecasts is determined by how well they minimize a certain regulatory loss function. We implement this additional procedure proposed by Lopez, adopting the capital requirement loss function (CR) defined at the Basel II Accord[16]:

$$CR_t = \max\left[\frac{(3+\delta)}{60}\sum_{i=0}^{59} VaR_{\alpha,t-i}, VaR_{\alpha,t}\right], \delta = \begin{cases} 0, & \text{if } \zeta \leq 4; \\ 0.3 + 0.1(\zeta - 4), & \text{if } 5 \leq \zeta \leq 6; \\ 0.65, & \text{if } \zeta = 7; \\ 0.65 + 0.1(\zeta - 7), & \text{if } 8 \leq \zeta \leq 9; \\ 1, & \text{if } \zeta \geq 10. \end{cases}$$

where δ is a multiplicative factor that depends on the number of violations of the VaR in the previous 250 trading days (ζ).

To compare the VaR models performance by using the CR loss function, we apply the superior predictive ability (SPA) test statistic proposed by Hansen (2005). Testing for SPA is to test whether a particular forecasting procedure is outperformed by alternative forecasts. The relevant question is whether an observed excess performance by an alternative model is significant or not. In Hansen's framework, the interest is to know whether any of the alternative models, $k = 1, ..., m$, are better than the benchmark, bch, in terms of expected loss L. So he tests the null hypothesis that the best alternative model is not better than the benchmark. The performance of model k relative to the benchmark at time t may be defined as $d_{k,t} = L_{bch,t} - L_{k,t}$. Provided that $E(d_{k,t}) = \mu_{k,t}$ is well defined, the null hypothesis of interest can be formulated as

$$H_0 : \max_{k=1,\ldots,m} \mu_k \leq 0,$$

whereas the alternative hypothesis is that the best alternative model is superior to the benchmark. A k model is better than the benchmark if and only if $E(d_{k,t}) > 0$. The test statistic is given by

$$T^{SPA} \equiv \max\left[\max_{k=1,\ldots,m} \frac{T^{1/2}\bar{d}_k}{\hat{\omega}_k}, 0\right],$$

where $\bar{d}_k \equiv T^{-1}\sum_{t=1}^{T} d_{k,t}$ and $\hat{\omega}_k^2$ is some consistent estimator of $\omega_k^2 \equiv var(T^{1/2}\bar{d}_k)$. The test is implemented via stationary bootstrap of Politis and Romano (1994).

The actual portfolio returns and the VaR forecasts from the static and dynamic D-vine copulas for both testing periods are displayed in Figures 4.18 and 4.19. Apparently, the VaR

[16] Note that, since the VaR is a negative value, to compute the loss function, it will be calculated here as minus the (100α-th percentile) of the c.d.f. of the returns.

forecasts from both models are quite similar, however, we would like to evaluate them using the above tests.

For the testing period from January 02, 2008 to August 19, 2008 (150 days), the results of the Kupiec and Christoffersen tests are reported in Table 4.14. For a testing period of 150 days and significance levels of 10%, 5% and 1%, we expect 15, 7.5 and 1.5 exceedances, respectively. Both estimated copula models produced the same hit sequences. For the 1% and 5% significance levels, there is a (non-significant) lack of coverage, since the numbers of exceedances are slightly increased in comparison with the expected ones. For the 10% significance level, the VaR forecasts are too conservative, and the null hypotheses of the Kupiec and Christoffersen tests are rejected using a 5% level for the LR_{uc} and LR_{cc} statistics. To some extent, the increased number of violations was expected, since we used the estimated models for the non-crisis period, with lower degree of dependence and less asymmetric dependence structure, to forecast VaR into the crisis period, with higher degree of dependence and more asymmetric dependence structure. Table 4.16 reports the average (capital requirement) losses computed based on the VaR forecasts and the results of the SPA test[17]. The latter is implemented considering each copula model at a time as the benchmark. According to the test results, based on the 1%-VaR forecasts, both models display similar performance in terms of predictive accuracy, however, with regard to the 5%-VaR, the static D-vine copula model performs worse than the dynamic D-vine copula.

The results of the VaR backtests for the period from May 08, 2012 to September 06, 2012 (79 days) are provided in Table 4.15. For a testing period of 79 days and significance levels of 10%, 5% and 1%, we expect 7.9, 3.95 and 0.79 exceedances, respectively. The tests results suggest that the forecasts of all three quantiles from both models are accurate, since the null hypotheses of unconditional and conditional coverage cannot be rejected. Nevertheless, it is worthy noticing that the observed numbers of exceedances of the 5%-VaR and 10%-VaR forecasts from the dynamic D-vine copula are closer to the expected numbers. Further, the results of the SPA test in Table 4.17 indicate that the forecasting performance of the static D-vine copula is inferior to the dynamic D-vine copula performance for all three quantiles.

[17] To compute Hansen's consistent p-value, we use the "bsds" function from the Oxford MFE Toolbox by Kevin Shepard, along with the Matlab code "opt_block_length_REV_dec07" compiled by Andrew Patton to implement the automatic optimum block length selection in accordance with Politis and White (2004). For the stationary block bootstrap, we use 10000 re-samples.

Overall, the dynamic D-vine copula seems to work very well out-of-sample, in crisis periods, usually outperforming the static D-vine copula, with more accurate VaR forecasts. It is true even in the adverse situation when we use the estimated copula corresponding to the non-crisis period to forecast VaR in a crisis context.

4.6 Concluding Remarks and Outlook

In this paper, we introduce dynamics into the state-of-the-art model for multivariate dependencies, the vine copula model. We allow the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, evolving through time according to an equation that follows a nonlinear restricted ARMA(1,m) process as in Patton (2006). Our contribution is towards providing a very flexible dependence model for applications to multivariate financial return data, which accounts for possible time variation in the dependence structure. The proposed model has proved to be superior to the time-homogeneous D-vine copula, both in a simulation and in an empirical study.

The overall findings of the Monte Carlo study are quite favorable to the dynamic D-vine copula. When the data generating process is the time-varying model, the static model tends to underestimate the pair-copulas dependence parameters. Furthermore, the negative bias does not seem to diminish as the number of observations in the samples increases. The estimates from the dynamic D-vine copula, in this case, are far superior to the estimates from the static D-vine, both in terms of the mean errors and the root mean squared errors. When the samples are drawn from the static D-vine copula, both models have similar performance in terms of the mean errors, with unbiased estimates. The dynamic D-vine copula fails only in terms of the root mean squared errors, when the data comes from the static model, what suggests that its estimates display higher variability in this case.

In the empirical study, we model the dependence among the returns of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA indexes, using both the dynamic D-vine copula model and a static D-vine copula. We consider two distinct periods, one from January 03, 2003 to December 28, 2007 and another one from January 02, 2008 to May 04, 2012, which we call non-crisis and crisis period, respectively. Our findings illustrate that time variation is present in the dependence structure of multivariate financial returns. In particular, time-varying pair-copulas are selected in the first level of the estimated dynamic D-vine copulas. They provide accurate description of variations in the unconditional dependencies all over the non-crisis and crisis periods, as well as from one period to the other. Overall, both estimated static and dynamic D-vine copulas capture stronger dependence during the crisis period. The estimated dynamic D-vine copulas give insightful information about the joint dependence among the above-mentioned indexes: there is evidence of joint upper and lower tail dependence, with some degree of flexibility, in both periods. The estimated static D-vines, on the other hand, suggest that the range of upper and lower tail dependencies of the margins is less flexible. In an out-of-sample exercise, the estimated models are used to forecast one-day VaR for an equally weighted portfolio of the investigated indexes in the period from January 02, 2008 to August 19, 2008 (150 days) and also from May 08, 2012 to September 06, 2012 (79 days). Both testing periods belong to the crisis period. Based on the results of the superior predictive ability (SPA) test of Hansen (2005), the dynamic D-vine copula model outperforms the static D-vine copula in terms of predictive accuracy, especially in the second testing period.

Further research is to be done on improving the dynamic D-vine copula model by extending it to the general case of regular vine copulas and investigating it more closely in higher-dimensional applications. Additionally, in future, we can assume that the pair-copulas dependence parameters not only follow an ARMA(1,m) process, but they are also influenced by a Markov chain, since we found evidence of change in the degree of dependence among the returns from the non-crisis to the crisis period.

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4.8 Appendix A: Copula Functions

Normal copula: the Normal copula, extracted from the bivariate Normal distribution, is defined as follows:

$$C_N(u_1, u_2|\rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{\frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} drds, \rho \in (-1, 1),$$

where the dependence parameter, ρ, is the linear correlation coefficient. Its dynamic equation may be written as[18]

$$\rho_t = \Lambda\left(\omega_N + \beta_N \rho_{t-1} + \alpha_N \cdot \frac{1}{m}\sum_{j=1}^{m} \Phi^{-1}(u_{1,t-j}) \cdot \Phi^{-1}(u_{2,t-j})\right).$$

The Normal copula is symmetric and has no tail dependence, that is, $\lambda_L = \lambda_U = 0$. The Kendall's tau may be computed based on the correlation coefficient as $\tau = (2/\pi) \arcsin \rho$.

Student-t copula: it is associated with the bivariate Student-t distribution and has the following functional form:

$$C_T(u_1, u_2|\rho, \nu) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{r^2 - 2\rho rs + s^2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} drds,$$

where the parameters ρ and ν are the linear correlation coefficient and the degrees of freedom, respectively. In addition, their evolution equations are given by

$$\rho_t = \Lambda\left(\omega_{1T} + \beta_{1T}\rho_{t-1} + \alpha_{1T} \cdot \frac{1}{m}\sum_{j=1}^{m} T_\nu^{-1}(u_{1,t-j}) \cdot T_\nu^{-1}(u_{2,t-j})\right)$$

and

$$\nu_t = \tilde{\Lambda}\left(\omega_{2T} + \beta_{2T}\nu_{t-1} + \alpha_{2T} \cdot \frac{1}{m}\sum_{j=1}^{m} T_\nu^{-1}(u_{1,t-j}) \cdot T_\nu^{-1}(u_{2,t-j})\right).$$

The Student-t copula has symmetrical tail dependence, with $\lambda_L = \lambda_U = 2T_{\nu+1}(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}})$, where $T_{\nu+1}$ is the Student-t *c.d.f.* with $(\nu + 1)$ degrees of freedom. The Kendall's tau is given by $\tau = (2/\pi) \arcsin \rho$.

Gumbel copula: it has the form of

$$C_G(u_1, u_2|\theta) = \exp\left(-\left((-\log u_1)^\theta + (-\log u_2)^\theta\right)^{1/\theta}\right), \ \theta \in [1, \infty).$$

The dynamics is given by the following equation governing the dependence parameter evolution:

$$\theta_t = \Lambda\left(\omega_G + \beta_G\theta_{t-1} + \alpha_G \cdot \frac{1}{m}\sum_{j=1}^{m} |u_{1,t-j} - u_{2,t-j}|\right).$$

[18] $\Lambda(\cdot)$ and $\tilde{\Lambda}(\cdot)$, which appear hereafter, are logistic transformations to keep the parameters in their intervals.

The Gumbel copula exhibits only upper tail dependence, with $\lambda_U = 2 - 2^{1/\theta}$. It can be shown that the Kendall's tau is given by $\tau = 1 - \theta^{-1}$.

Rotated-Gumbel copula: or Survival Gumbel copula, which is the complement ("Probability of survival") of the Gumbel copula. It has the following form:

$$C_{RG}(u_1, u_2|\theta) = u_1 + u_2 - 1 + C_G(1 - u_1, 1 - u_2|\theta),$$

where C_G corresponds to the Gumbel copula. The dependence parameter, θ, follows the process

$$\theta_t = \Lambda\left(\omega_{RG} + \beta_{RG}\theta_{t-1} + \alpha_{RG} \cdot \frac{1}{m}\sum_{j=1}^{m}|u_{1,t-j} - u_{2,t-j}|\right).$$

The Rotated-Gumbel copula has only lower tail dependence, given by $\lambda_L = 2 - 2^{1/\theta}$, and the Kendall's tau may be computed as $\tau = 1 - \theta^{-1}$.

Clayton copula: or Kimeldorf-Sampson copula, has the following distribution function:

$$C_C(u_1, u_2|\delta) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-1/\delta},\ \delta \in (0, \infty).$$

The evolution equation of the dependence parameter is

$$\delta_t = \Lambda\left(\omega_C + \beta_C\delta_{t-1} + \alpha_C \cdot \frac{1}{m}\sum_{j=1}^{m}|u_{1,t-j} - u_{2,t-j}|\right).$$

This copula exhibits only lower tail dependence, $\lambda_L = 2^{-1/\delta}$. The Kendall's tau has the form $\tau = \delta/(\delta + 2)$.

Symmetrized Joe-Clayton copula: this copula was defined by Patton (2006) and takes the form of

$$C_{SJC}(u_1, u_2|\lambda_U, \lambda_L) = 0.5 \cdot \left(C_{JC}\left(u_1, u_2|\lambda_U, \lambda_L\right) + C_{JC}\left(1 - u_1, 1 - u_2|\lambda_U, \lambda_L\right) + u_1 + u_2 - 1\right),$$

where C_{JC} is the Joe-Clayton copula, also called BB7 copula (Joe, 1997), given by

$$C_{JC}\left(u_1, u_2|\lambda_U, \lambda_L\right) = 1 - \left(1 - \left\{[1 - (1 - u_1)^{\kappa}]^{-\gamma} + [1 - (1 - u_2)^{\kappa}]^{-\gamma} - 1\right\}^{-1/\gamma}\right)^{-1/\kappa},$$

with $\kappa = 1/\log_2\left(2 - \lambda_U\right),\ \gamma = -1/\log_2\left(\lambda_L\right)$ and $\lambda_U, \lambda_L \in (0, 1)$.

The SJC copula has upper and lower tail dependence and its dependence parameters are the upper and lower tail dependence parameters, λ_U and λ_L, respectively. Furthermore, λ_U and λ_L range freely and are not dependent on each other. Since this copula nests symmetry as a special case, it is a more interesting specification than the BB7 copula. The evolution equations for the parameters λ_U and λ_L are

$$\lambda_{Ut} = \Lambda\left(\omega_U + \beta_U \lambda_{Ut-1} + \alpha_U \cdot \frac{1}{m}\sum_{j=1}^{m} |u_{1,t-j} - u_{2,t-j}|\right)$$

and

$$\lambda_{Lt} = \Lambda\left(\omega_L + \beta_L \lambda_{Lt-1} + \alpha_L \cdot \frac{1}{m}\sum_{j=1}^{m} |u_{1,t-j} - u_{2,t-j}|\right).$$

The Kendall's tau, in this case, has no closed form, so it has to be computed numerically.

BB1 copula (Joe, 1997): it has the following functional form:

$$C_{bb1}(u_1, u_2|\kappa, \gamma) = \{1 + [(u_1^{-\kappa} - 1)^{\gamma} + (u_2^{-\kappa} - 1)^{\gamma}]^{1/\gamma}\}^{-1/\kappa}, \ \kappa \in (0, \infty), \gamma \in [1, \infty).$$

The dynamic equations of the dependence parameters are

$$\kappa_t = \Lambda\left(\omega_{1bb1} + \beta_{1bb1}\kappa_{t-1} + \alpha_{1bb1} \cdot \frac{1}{m}\sum_{j=1}^{m} |u_{1,t-j} - u_{2,t-j}|\right)$$

$$\gamma_t = \tilde{\Lambda}\left(\omega_{2bb1} + \beta_{2bb1}\gamma_{t-1} + \alpha_{2bb1} \cdot \frac{1}{m}\sum_{j=1}^{m} |u_{1,t-j} - u_{2,t-j}|\right).$$

The BB1 copula has upper and lower tail dependence given by $\lambda_U = 2 - 2^{1/\gamma}$ and $\lambda_L = 2^{-1/\gamma\kappa}$, respectively. The Kendall's tau may be calculated based on κ and γ as $\tau = 1 - (2/(\gamma(\kappa + 2)))$.

4.9 Appendix B: Tables

Table 4.1: Statistics of the mean errors of the Monte Carlo study - DGP dynamic D-vine copula.

	T = 1000			T = 2000			T = 5000		
	M	Md	SD	M	Md	SD	M	Md	SD
					Tree 1				
ME12,s	-0.1321	-0.1338	0.0905	-0.1372	-0.1392	0.0633	-0.1408	-0.1423	0.0400
ME12,d	0.0025	-0.0013	0.0761	0.0031	0.0048	0.0542	0.0016	-0.0004	0.0346
ME23,s	-0.1043	-0.0942	0.1281	-0.1173	-0.1161	0.0877	-0.1165	-0.1153	0.0557
ME23,d	0.0106	0.0126	0.1002	-0.0006	-0.0016	0.0656	0.0016	0.0013	0.0430
ME34,s	-0.1029	-0.1047	0.0872	-0.1050	-0.1030	0.0656	-0.1096	-0.1103	0.0408
ME34,d	0.0043	-0.0012	0.0771	0.0037	0.0028	0.0562	-0.0004	-0.0018	0.0358
					Tree 2				
ME13|2,s	-0.0073	-0.0079	0.0225	-0.0075	-0.0078	0.0152	-0.0086	-0.0089	0.0098
ME13|2,d	0.0026	0.0021	0.0221	0.0013	0.0011	0.0153	5.86E-05	-4.0E-06	0.0096
ME24|3,s	-0.0055	-0.0057	0.0201	-0.0066	-0.0069	0.0144	-0.0065	-0.0069	0.0090
ME24|3,d	0.0042	0.0029	0.0193	0.0015	0.0009	0.0140	0.0007	0.0004	0.0091
					Tree 3				
ME14|23,s	-0.0092	-0.0100	0.0237	-0.0103	-0.0104	0.0164	-0.0111	-0.0111	0.0103
ME14|23,d	0.0022	0.0014	0.0235	0.0010	0.0007	0.0167	-3.3E-05	-0.0001	0.0104

Note: ME$i, i+j|v_{ij}, d$ and ME$i, i+j|v_{ij}, s$ refer to the mean errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i,i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the mean errors, respectively.

Table 4.2: Statistics of the mean errors of the Monte Carlo study - DGP static D-vine copula.

	T = 1000			T = 2000			T = 5000		
	M	Md	SD	M	Md	SD	M	Md	SD
					Tree 1				
ME12,s	-0.0009	-0.0007	0.0809	0.0016	0.0045	0.0571	0.0006	-0.0011	0.0365
ME12,d	0.0054	0.0043	0.0821	0.0046	0.0070	0.0573	0.0018	-6.4E-05	0.0368
ME23,s	0.0079	0.0107	0.1041	-0.0007	0.0008	0.0673	0.0016	0.0014	0.0442
ME23,d	0.0153	0.0191	0.1047	0.0031	0.0041	0.0686	0.0030	0.0027	0.0444
ME34,s	0.0014	-0.0025	0.0799	0.0027	0.0017	0.0588	-0.0014	-0.0025	0.0360
ME34,d	0.0080	0.0031	0.0822	0.0052	0.0036	0.0590	-1.4E-05	-0.0013	0.0366
					Tree 2				
ME13|2,s	0.0013	0.0002	0.0288	0.0004	0.0001	0.0203	-0.0003	-0.0001	0.0127
ME13|2,d	0.0030	0.0015	0.0297	0.0012	0.0007	0.0206	0.0001	0.0002	0.0128
ME24|3,s	0.0003	-0.0001	0.0262	-0.0001	-0.0007	0.0192	0.0004	0.0003	0.0124
ME24|3,d	0.0020	0.0013	0.0268	0.0007	0.0001	0.0193	0.0009	0.0007	0.0124
					Tree 3				
ME14|23,s	0.0011	0.0012	0.0233	8.34E-05	-0.0004	0.0162	-0.0002	-0.0002	0.0103
ME14|23,d	0.0021	0.0013	0.0238	0.0007	-2.2E-05	0.0167	-2.6E-05	-0.0002	0.0104

Note: ME$i, i+j|v_{ij}, d$ and ME$i, i+j|v_{ij}, s$ refer to the mean errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i,i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the mean errors, respectively.

Table 4.3: Statistics of the root mean squared errors of the Monte Carlo study - DGP dynamic D-vine copula.

	T = 1000			T = 2000			T = 5000		
	M	Md	SD	M	Md	SD	M	Md	SD
					Tree 1				
RMSE12,s	0.6008	0.5977	0.0645	0.6070	0.6055	0.0449	0.6079	0.6071	0.0277
RMSE12,d	0.1281	0.1237	0.0579	0.0900	0.0835	0.0448	0.0570	0.0528	0.0276
RMSE23,s	0.5779	0.5595	0.0865	0.5783	0.5700	0.0626	0.5757	0.5731	0.0388
RMSE23,d	0.1399	0.1327	0.0716	0.0953	0.0906	0.0478	0.0587	0.0547	0.0288
RMSE34,s	0.5299	0.5218	0.0540	0.5312	0.5272	0.0408	0.5311	0.5301	0.0255
RMSE34,d	0.1252	0.1193	0.0554	0.0900	0.0861	0.0398	0.0552	0.0512	0.0254
					Tree 2				
RMSE13|2,s	0.0380	0.0352	0.0084	0.0349	0.0334	0.0050	0.0333	0.0323	0.0030
RMSE13|2,d	0.0356	0.0322	0.0193	0.0247	0.0223	0.0130	0.0156	0.0147	0.0069
RMSE24|3,s	0.0400	0.0388	0.0073	0.0375	0.0364	0.0050	0.0357	0.0353	0.0027
RMSE24|3,d	0.0358	0.0318	0.0187	0.0255	0.0227	0.0136	0.0159	0.0148	0.0074
					Tree 3				
RMSE14|23,s	0.0379	0.0344	0.0098	0.0350	0.0330	0.0061	0.0331	0.0321	0.0037
RMSE14|23,d	0.0373	0.0330	0.0205	0.0253	0.0230	0.0129	0.0162	0.0153	0.0073

Note: RMSE$i, i{+}j|v_{ij}, d$ and RMSE$i, i{+}j|v_{ij}, s$ refer to the root mean squared errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i,i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the root mean squared errors, respectively.

Table 4.4: Statistics of the root mean squared errors of the Monte Carlo study - DGP static D-vine copula.

	T = 1000			T = 2000			T = 5000		
	M	Md	SD	M	Md	SD	M	Md	SD
					Tree 1				
RMSE12,s	0.0646	0.0540	0.0488	0.0460	0.0398	0.0338	0.0288	0.0244	0.0224
RMSE12,d	0.1297	0.1252	0.0597	0.0889	0.0859	0.0414	0.0559	0.0533	0.0254
RMSE23,s	0.0831	0.0699	0.0631	0.0535	0.0443	0.0407	0.0352	0.0303	0.0268
RMSE23,d	0.1600	0.1489	0.0800	0.1107	0.1049	0.0513	0.0693	0.0661	0.0314
RMSE34,s	0.0631	0.0524	0.0490	0.0475	0.0403	0.0347	0.0284	0.0238	0.0222
RMSE34,d	0.1298	0.1251	0.0631	0.0911	0.0879	0.0430	0.0561	0.0526	0.0270
					Tree 2				
RMSE13|2,s	0.0229	0.0192	0.0176	0.0162	0.0134	0.0122	0.0102	0.0086	0.0076
RMSE13|2,d	0.0446	0.0422	0.0225	0.0309	0.0289	0.0148	0.0194	0.0183	0.0091
RMSE24|3,s	0.0209	0.0174	0.0159	0.0150	0.0123	0.0119	0.0100	0.0088	0.0074
RMSE24|3,d	0.0432	0.0404	0.0225	0.0305	0.0286	0.0178	0.0185	0.0178	0.0082
					Tree 3				
RMSE14|23,s	0.0181	0.0149	0.0147	0.0130	0.0116	0.0095	0.0082	0.0069	0.0063
RMSE14|23,d	0.0386	0.0350	0.0207	0.0260	0.0237	0.0135	0.0158	0.0149	0.0076

Note: RMSE$i, i{+}j|v_{ij}, d$ and RMSE$i, i{+}j|v_{ij}, s$ refer to the root mean squared errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i,i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the root mean squared errors, respectively.

Table 4.5: Summary statistics of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA log-returns.

	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
			Non-Crisis Period		
Mean	8.1810E-04	4.8050E-04	4.0711E-04	4.1290E-04	0.0014
Median	0.0013	9.0553E-04	7.8933E-04	8.4515E-04	0.0020
Maximum	0.0661	0.0700	0.0590	0.0348	0.0516
Minimum	-0.0579	-0.0583	-0.0492	-0.0359	-0.0686
Std. Deviation	0.0123	0.0112	0.0093	0.0085	0.0168
Asymmetry	-0.2246	-0.1134	-0.1191	-0.2174	-0.3382
Kurtosis	5.8964	6.4352	6.7222	4.6685	3.6245
Jarque-Bera	418.1 (0.0000)	577.1 (0.0000)	677.5 (0.0000)	144.3 (0.0000)	41.1 (0.0000)
			Crisis Period		
Mean	-2.0078E-04	-5.6018E-04	-1.3186E-04	-7.4701E-05	-4.7780E-05
Median	4.1623E-04	-2.6411E-04	-4.8268E-05	9.4609E-04	7.1406E-04
Maximum	0.1080	0.1059	0.0938	0.1096	0.1368
Minimum	-0.0774	-0.0947	-0.0926	-0.0947	-0.1210
Std. Deviation	0.0187	0.0195	0.0163	0.0177	0.0215
Asymmetry	0.1161	0.1458	-0.0381	-0.2247	0.0723
Kurtosis	7.3303	7.0833	8.2463	8.9963	9.1444
Jarque-Bera	799.4 (0.0000)	712.3 (0.0000)	1170.8 (0.0000)	1538.3 (0.0000)	1607.1 (0.0000)

Note: Jarque-Bera corresponds to the Jarque-Bera test statistics, with p-values in parentheses.

Table 4.6: Estimates from the univariate ARMA-GARCH models for the non-crisis period.

			Conditional Mean Equation		
Parameter	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
ϕ_0	0.0008	0.0005	...	0.0005	0.0014
	(0.0003)	(0.0002)	...	(0.0002)	(0.0005)
ϕ_1	...	...	-0.1162	-0.1039	...
	...	...	(0.0230)	(0.0005)	...
			Conditional Variance Equation		
Parameter	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
α_0	-0.1586	2.3943E-06	1.9933E-06	-0.1036	8.5316E-06
	(0.0520)	(0.0000)	(0.0000)	(0.0668)	(0.0000)
α_1	0.1182	0.0797	0.1052	0.0702	0.0437
	(0.0201)	(0.0165)	(0.0225)	(0.0207)	(0.0106)
γ_1	-0.1127	...	...	-0.0820	...
	(0.0243)	...	...	(0.0175)	...
β_1	0.9825	0.8978	0.8705	0.9893	0.9258
	(0.0057)	(0.0206)	(0.0265)	(0.0069)	(0.0173)
ν	11.8568	14.8531	18.8979	11.5421	16.3456
	(4.3112)	(5.4413)	(8.5510)	(4.4127)	(6.8297)
λ	-0.1736	-0.1421	-0.1907	-0.1485	-0.1670
	(0.0352)	(0.0403)	(0.0384)	(0.0301)	(0.0439)
$Q(15)$	0.5760	0.5948	0.5433	0.1916	0.9503
$Q^2(15)$	0.8239	0.9945	0.6798	0.1666	0.8767
K-S Test	0.4648	0.9996	0.2570	0.3906	0.8364

Note: Standard errors in parentheses. $Q(15)$, $Q^2(15)$ and K-S Test are p-values.

Table 4.7: Estimates from the univariate ARMA-GARCH models for the crisis period.

	Conditional Mean Equation				
Parameter	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
ϕ_1	...	...	...	-0.1336	...
	...	...	...	(0.0249)	...
ϕ_2	...	-0.0763	-0.0933	...	...
	...	(0.0038)	(0.0020)	...	...
ϕ_3	...	...	...	...	-0.0667
	...	...	...	...	(0.0025)
	Conditional Variance Equation				
Parameter	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
α_0	3.8372E-06	5.5846E-06	3.1461E-06	-0.2446	-0.1092
	(0.0000)	(0.0000)	(0.0000)	(0.0650)	(0.0742)
α_1	0.0872	0.0945	0.0971	-0.1913	0.1565
	(0.0192)	(0.0237)	(0.0222)	(0.0680)	(0.0471)
α_2	...	...	...	0.3375	...
	...	...	...	(0.0730)	...
γ_1	...	...	...	-0.1642	-0.0963
	...	...	...	(0.0235)	(0.0285)
β_1	0.9020	0.8900	0.8901	0.9709	0.9861
	(0.0193)	(0.0240)	(0.0227)	(0.0076)	(0.0093)
ν	8.7683	11.1004	14.3925	8.9613	10.4290
	(2.4751)	(3.6827)	(6.5502)	(2.4611)	(3.0912)
λ	-0.0730	-0.0914	-0.1111	-0.2012	-0.0975
	(0.0329)	(0.0394)	(0.0361)	(0.0337)	(0.0377)
$Q(15)$	0.9581	0.9914	0.7119	0.6389	0.9749
$Q^2(15)$	0.4818	0.3397	0.4298	0.1873	0.8453
K-S Test	0.3417	0.2214	0.7452	0.4970	0.7881

Note: Standard errors in parentheses. $Q(15)$, $Q^2(15)$ and K-S Test are p-values.

Table 4.8: Matrix of the sample Kendall's taus computed based on the PIT of the residuals for the non-crisis period.

	CAC 40	DAX	FTSE 100	S&P 500	IBOVESPA
CAC 40	1.0000				
DAX	0.7322	1.0000			
FTSE 100	0.6591	0.5955	1.0000		
S&P 500	0.3444	0.3597	0.3249	1.0000	
IBOVESPA	0.2604	0.2587	0.2673	0.4349	1.0000

Table 4.9: Matrix of the sample Kendall's taus computed based on the PIT of the residuals for the crisis period.

	CAC 40	DAX	FTSE 100	S&P 500	IBOVESPA
CAC 40	1.0000				
DAX	0.7858	1.0000			
FTSE 100	0.7501	0.7053	1.0000		
S&P 500	0.5074	0.5030	0.4905	1.0000	
IBOVESPA	0.3833	0.3629	0.3871	0.5140	1.0000

Table 4.10: Estimation results of the dynamic D-vine copula for the non-crisis period.

	Bivariate Copula	θ_1	θ_2	ω_1	α_1	β_1	ω_2	α_2	β_2
			Tree 1						
FTSE, CAC	BB1 tvp	0.7813					0.3303	-0.7987	0.3795
		(0.0863)					(0.0635)	(0.2189)	(0.0195)
CAC, DAX	BB1 tvp	0.7252					1.6422	-4.3897	–
		(0.0905)					(0.0790)	(0.6589)	
DAX, S&P500	Student-t tvp		8.7969	1.7175	0.2465*	-1.2680*			
			(2.2623)	(0.3564)	(0.1343)	(0.7013)			
S&P500, IBOVESPA	Student-t tvp		6.9966	-0.3757	0.0584	2.8970			
			(1.4740)	(0.0016)	(0.0041)	(0.0005)			
			Tree 2						
FTSE, DAX|CAC	Student-t	0.0723	10.6398						
		(0.0314)	(3.3715)						
CAC, S&P500|DAX	Normal	0.0847							
		(0.0297)							
DAX, IBOVESPA|S&P500	Rotated-Gumbel	1.0568							
		(0.0187)							
			Tree 3						
FTSE, S&P500|CAC, DAX	Normal	0.0978							
		(0.0293)							
CAC, IBOVESPA|DAX, S&P500	Normal	0.0660							
		(0.0290)							
			Tree 4						
FTSE, IBOVESPA|CAC, DAX, S&P500	Normal	0.0754							
		(0.0300)							

Note: Estimates obtained using the sequential estimation procedure. Standard errors in parentheses. (*) stands for significant only at the 10% level. In the second column, we have the pair-copula selected based on the AIC criterion. The copula name followed by "tvp" means that at least one of the copula's parameters is time-varying. The columns labeled θ_i, $i = 1, 2$ (considering two-parameter copulas), present the estimates of the constant parameters. ω_i, α_i and β_i, refer to the coefficients of the evolution equation of the time-varying parameter, $\theta_{i,t} = \Lambda(\omega_i + \beta_i \theta_{i,t-1} + \alpha_i \psi_t)$, $i = 1, 2$, as defined in Section 4.2.2 and presented in Appendix A for the time-varying parameters of the different copula functions adopted in this paper. With regard to the BB1 tvp copula selected for both FTSE-CAC and CAC-DAX, only the estimated parameter $\hat{\gamma}_t$ displays time variation, following an ARMA(1,10) for FTSE-CAC and MA(10) for CAC-DAX, whereas $\hat{\kappa}$ remains constant. Regarding the Student-t tvp chosen for both DAX-S&P500 and S&P500-IBOVESPA, only the estimated correlation coefficient is time-varying, following an ARMA(1,10), whereas the estimated degrees of freedom remain constant.

Table 4.11: Estimation results of the dynamic D-vine copula for the crisis period.

	Bivariate Copula	θ_1	θ_2	ω_1	α_1	β_1	ω_2	α_2	β_2
			Tree 1						
FTSE, CAC	BB1 tvp			0.7046	-1.3185*	0.3499	0.6366	-0.9050	0.2801
				(0.1020)	(0.7325)	(0.0602)	(0.0980)	(0.4269)	(0.0220)
CAC, DAX	BB1 tvp	0.9915					2.4714	-8.0118	-0.1519*
		(0.1265)					(0.3390)	(2.4776)	(0.0784)
DAX, S&P500	Student-t	0.7088	18.8746*						
		(0.0137)	(10.5248)						
S&P500, IBOVESPA	Student-t tvp		9.5523	-0.4945	0.2145	3.0149			
			(3.1905)	(0.1019)	(0.0583)	(0.1991)			
			Tree 2						
FTSE, DAX|CAC	Rotated-Gumbel	1.1117							
		(0.0253)							
CAC, S&P500|DAX	BB1	0.1706	1.0443						
		(0.0494)	(0.0221)						
DAX, IBOVESPA|S&P500	Gumbel	1.0776							
		(0.0216)							
			Tree 3						
FTSE, S&P500|CAC, DAX	Gumbel	1.0607							
		(0.0198)							
CAC, IBOVESPA|DAX, S&P500	Normal	0.1142							
		(0.0296)							
			Tree 4						
FTSE, IBOVESPA|CAC, DAX, S&P500	Normal	0.1118							
		(0.0315)							

Note: Estimates obtained using the sequential estimation procedure. Standard errors in parentheses. (*) stands for significant only at the 10% level. In the second column, we have the pair-copula selected based on the AIC criterion. The copula name followed by "tvp" means that at least one of the copula's parameters is time-varying. The columns labeled θ_i, $i = 1, 2$ (considering two-parameter copulas), present the estimates of the constant parameters. ω_i, α_i and β_i, refer to the coefficients of the evolution equation of the time-varying parameter, $\theta_{i,t} = \Lambda(\omega_i + \beta_i \theta_{i,t-1} + \alpha_i \psi_t), i = 1, 2$, as defined in Section 4.2.2 and presented in Appendix A for the time-varying parameters of the different copula functions adopted in this paper. Here, both estimated parameters of the BB1 tvp copula chosen for the pair FTSE-CAC follow an ARMA(1,5). As far as the BB1 tvp copula of CAC-DAX is concerned, only the estimated parameter $\hat{\gamma}_t$ displays time variation, following an ARMA(1,10), whereas $\hat{\kappa}$ remains constant. With regard to the Student-t tvp selected for the pair S&P500-IBOVESPA, only the estimated correlation coefficient is time-varying, following an ARMA(1,15), whereas the degrees of freedom remain constant.

Table 4.12: Estimation results of the static D-vine copula for the non-crisis period.

	Bivariate Copula	θ_1	θ_2	τ	λ_L	λ_U
	Tree 1					
FTSE, CAC	Student-t	0.8562	14.8356	0.6544	0.2846	0.2846
		(0.0066)	(6.3513)			
CAC, DAX	Student-t	0.9035	8.9643	0.7180	0.4935	0.4935
		(0.0049)	(2.1594)			
DAX, S&P500	Student-t	0.5370	8.7557	0.3609	0.1180	0.1180
		(0.0204)	(2.2666)			
S&P500, IBOVESPA	Student-t	0.6287	7.0128	0.4328	0.2134	0.2134
		(0.0177)	(1.6477)			
	Tree 2					
FTSE, DAX|CAC	Student-t	0.0669	9.5239	0.0426	0.0119	0.0119
		(0.0317)	(2.4388)			
CAC, S&P500|DAX	Normal	0.0819		0.0522		
		(0.0294)				
DAX, IBOVESPA|S&P500	Rotated-Gumbel	1.0580		0.0548	0.0746	
		(0.0187)				
	Tree 3					
FTSE, S&P500|CAC, DAX	Normal	0.1053		0.0672		
		(0.0288)				
CAC, IBOVESPA|DAX, S&P500	Normal	0.0630		0.0402		
		(0.0292)				
	Tree 4					
FTSE, IBOVESPA|CAC, DAX, S&P500	Normal	0.0854		0.0544		
		(0.0305)				

Note: Estimates obtained using the IFM method. Standard errors in parentheses. In the second column, we have the pair-copula selected based on the AIC criterion. The columns labeled θ_i, $i = 1, 2$ (considering two-parameter copulas), present the estimates of the copula parameters. τ corresponds to the estimate of the Kendall's tau, whereas λ_L and λ_U refer to the estimates of the lower and upper tail dependence parameters, respectively.

Table 4.13: Estimation results of the static D-vine copula for the crisis period.

	Bivariate Copula	θ_1	θ_2	τ	λ_L	λ_U
	Tree 1					
FTSE, CAC	Student-t	0.9231	13.9509	0.7487	0.4515	0.4515
		(0.0038)	(5.5628)			
CAC, DAX	Student-t	0.9434	5.4932	0.7848	0.6777	0.6777
		(0.0035)	(1.4046)			
DAX, S&P500	Student-t	0.7088	18.8746*	0.5015	0.0807	0.0807
		(0.0137)	(10.5248)			
S&P500, IBOVESPA	Student-t	0.7297	6.7582	0.5207	0.3038	0.3038
		(0.0143)	(1.6322)			
	Tree 2					
FTSE, DAX|CAC	BB1	0.1156	1.0494	0.0990	0.0033	0.0642
		(0.0440)	(0.0223)			
CAC, S&P500|DAX	BB1	0.1983	1.0354	0.1213	0.0342	0.0468
		(0.0481)	(0.0211)			
DAX, IBOVESPA|S&P500	Gumbel	1.0763		0.0709		0.0959
		(0.0215)				
	Tree 3					
FTSE, S&P500|CAC, DAX	Normal	0.1239		0.0791		
		(0.0306)				
CAC, IBOVESPA|DAX, S&P500	Normal	0.1146		0.0731		
		(0.0302)				
	Tree 4					
FTSE, IBOVESPA|CAC, DAX, S&P500	Student-t	0.1140	17.8143*	0.0728	0.0011	0.0011
		(0.0323)	(10.0107)			

Note: Estimates obtained using the IFM method. Standard errors in parentheses. (*) stands for significant only at the 10% level. In the second column, we have the pair-copula selected based on the AIC criterion. The columns labeled θ_i, $i = 1, 2$ (considering two-parameter copulas), present the estimates of the copula parameters. τ corresponds to the estimate of the Kendall's tau, whereas λ_L and λ_U refer to the estimates of the lower and upper tail dependence parameters, respectively.

Table 4.14: Results of the VaR backtests for the testing period from January 02, 2008 to August 19, 2008 (150 days).

	Dynamic D-Vine Copula		
α	1%	5%	10%
n	2	9	23
n/h	0.0133	0.0600	0.1533
Kupiec	0.6962	0.5854	0.0417
Christoffersen	0.8897	0.4539	0.0144
	Static D-Vine Copula		
α	1%	5%	10%
n	2	9	23
n/h	0.0133	0.0600	0.1533
Kupiec	0.6962	0.5854	0.0417
Christoffersen	0.8897	0.4539	0.0144

Note: Kupiec and Christoffersen correspond to the p-values of the respective tests.

Table 4.15: Results of the VaR backtests for the testing period from May 08, 2012 to September 06, 2012 (79 days).

	Dynamic D-Vine Copula		
α	1%	5%	10%
n	0	3	7
n/h	0.0000	0.0380	0.0886
Kupiec	0.2076	0.6092	0.7312
Christoffersen	0.4520	0.7485	0.8822
	Static D-Vine Copula		
α	1%	5%	10%
n	0	2	6
n/h	0.0000	0.0253	0.0759
Kupiec	0.2076	0.2678	0.4587
Christoffersen	0.4520	0.5004	0.4254

Note: Kupiec and Christoffersen correspond to the p-values of the respective tests.

Table 4.16: Average losses computed based on the VaR forecasts for the testing period from January 02, 2008 to August 19, 2008 (150 days) and the results of the SPA test.

1%-VaR		5%-VaR	
Benchmark	Average Loss (%)	Benchmark	Average Loss (%)
Dynamic D-Vine	10.1303	Dynamic D-Vine	7.2424
	(0.4583)		(1.0000)
Static D-Vine	10.1266	Static D-Vine	7.3551
	(1.0000)		(0.0000)

Note: In parentheses, we have the p-value of the SPA test.

Table 4.17: Average losses computed based on the VaR forecasts for the testing period from May 08, 2012 to September 06, 2012 (79 days) and the results of the SPA test.

1%-VaR		5%-VaR		10%-VaR	
Benchmark	Average Loss (%)	Benchmark	Average Loss (%)	Benchmark	Average Loss (%)
Dynamic D-Vine	10.4407	Dynamic D-Vine	6.5108	Dynamic D-Vine	5.3403
	(1.0000)		(1.0000)		(1.0000)
Static D-Vine	10.6093	Static D-Vine	6.7058	Static D-Vine	5.4557
	(0.0000)		(0.0000)		(0.0000)

Note: In parentheses, we have the p-value of the SPA test.

4.10 Appendix C: Figures

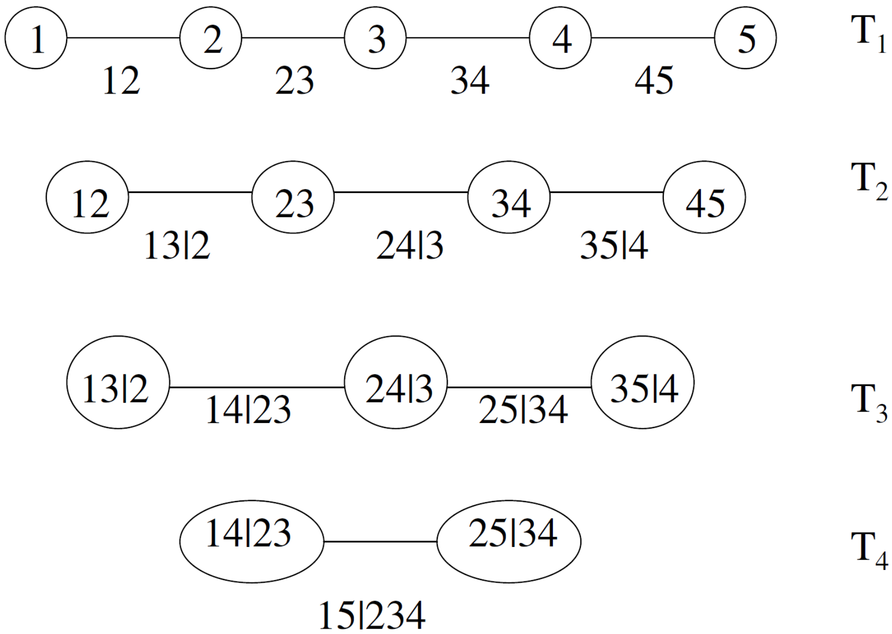

Figure 4.1: A D-vine with 5 variables, 4 trees and 10 edges. Each edge is associated with a pair-copula density.

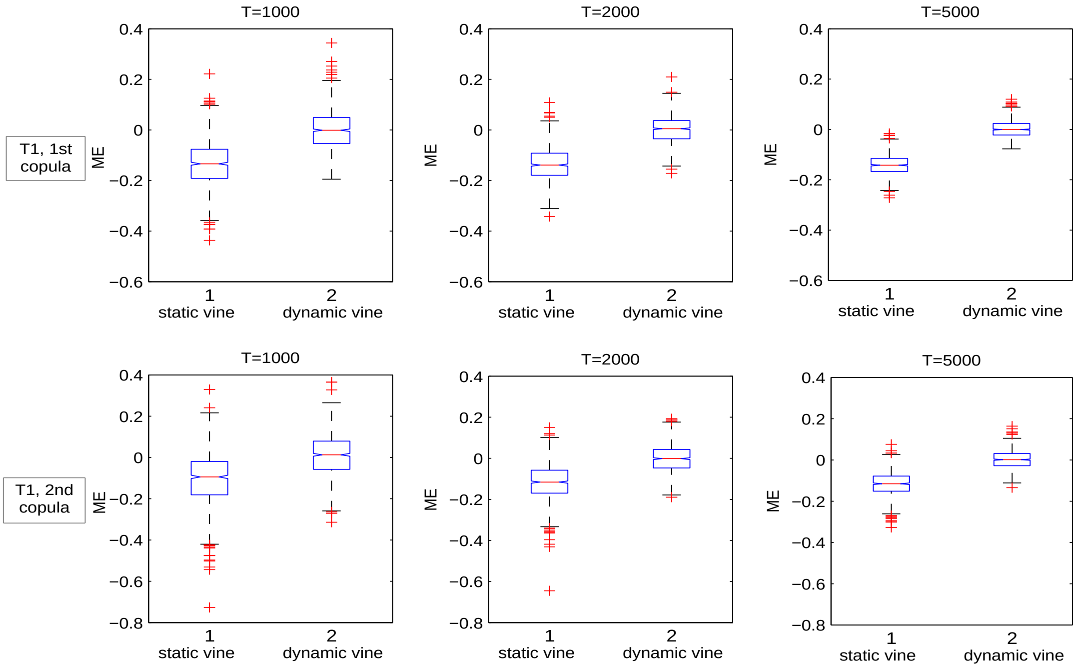

Figure 4.2: Mean errors for the parameters of c_{12} and c_{23} - DGP dynamic D-vine.

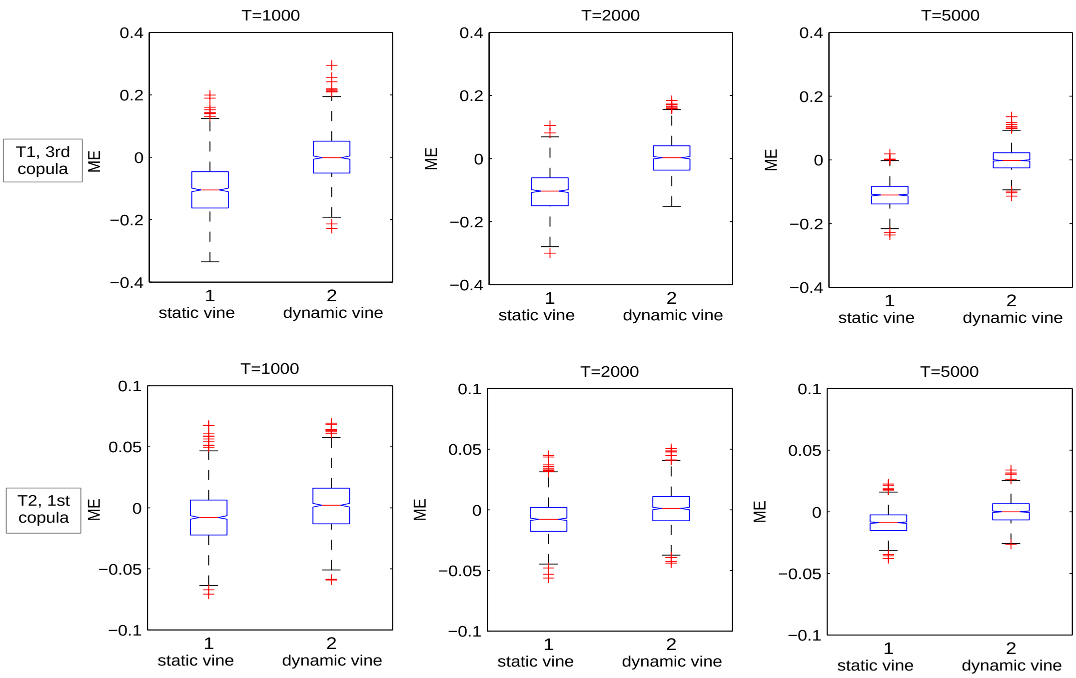

Figure 4.3: Mean errors for the parameters of c_{34} and $c_{13|2}$ - DGP dynamic D-vine.

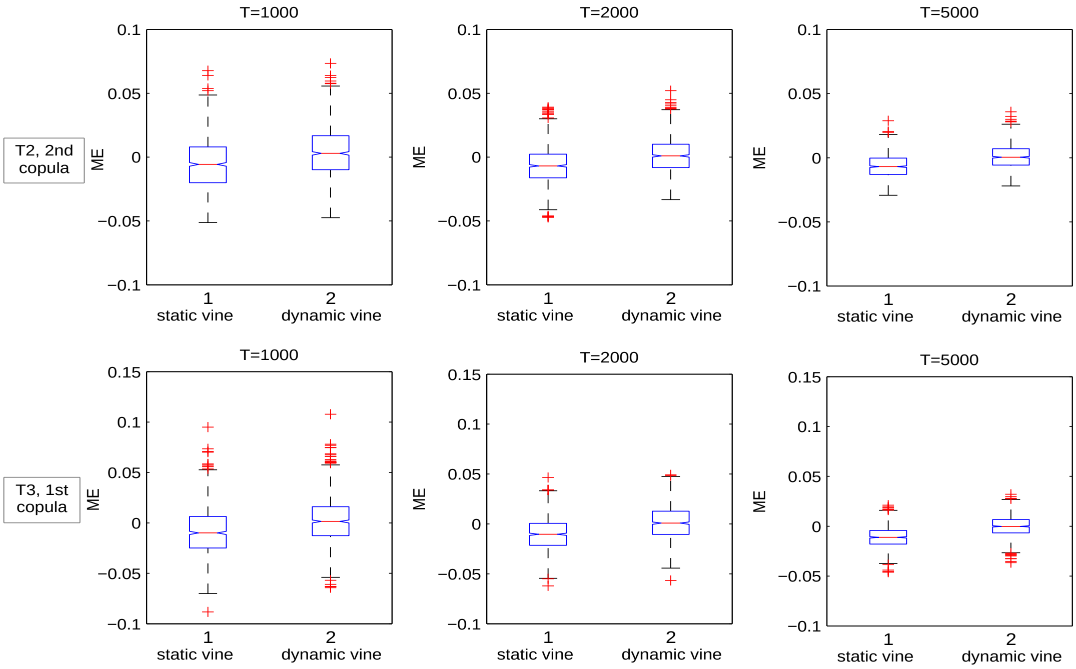

Figure 4.4: Mean errors for the parameters of $c_{24|3}$ and $c_{14|23}$ - DGP dynamic D-vine.

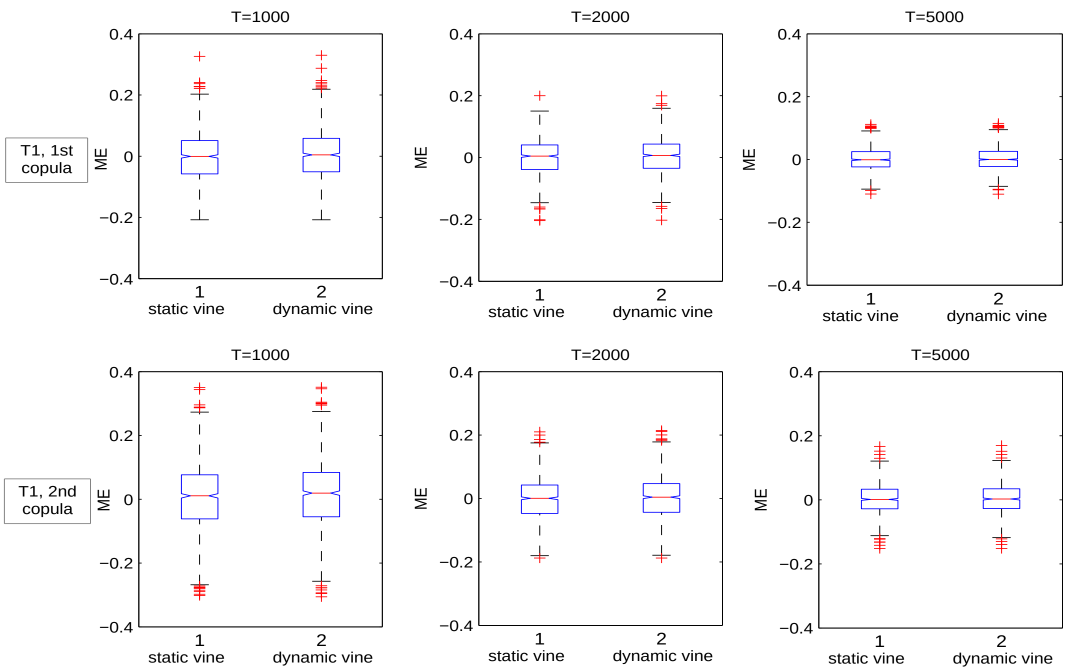

Figure 4.5: Mean errors for the parameters of c_{12} and c_{23} - DGP static D-vine.

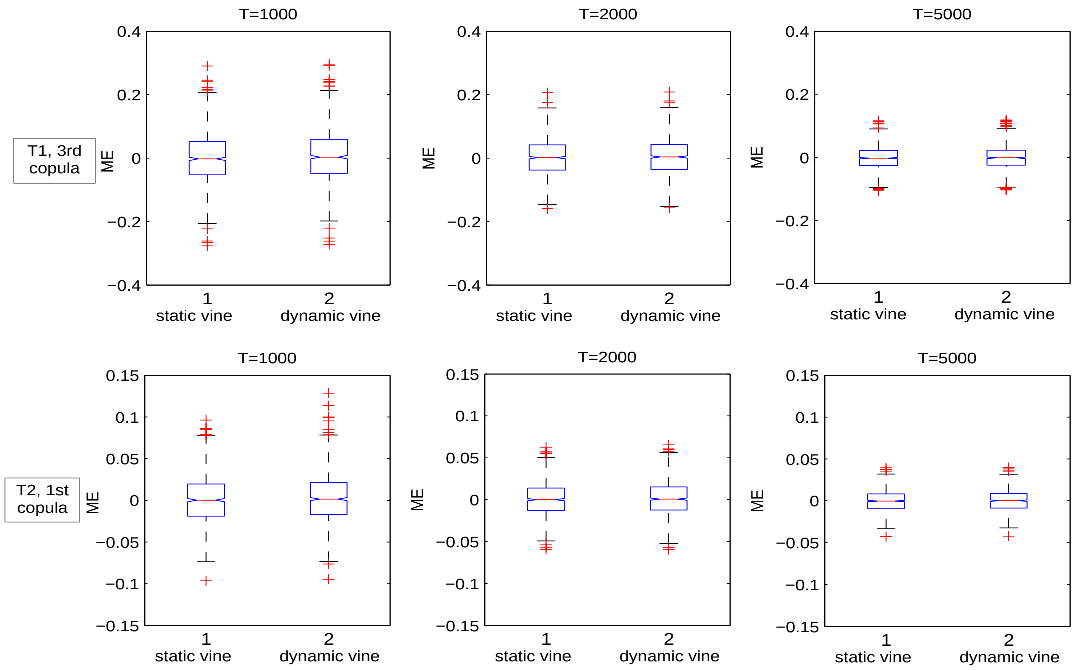

Figure 4.6: Mean errors for the parameters of c_{34} and $c_{13|2}$ - DGP static D-vine.

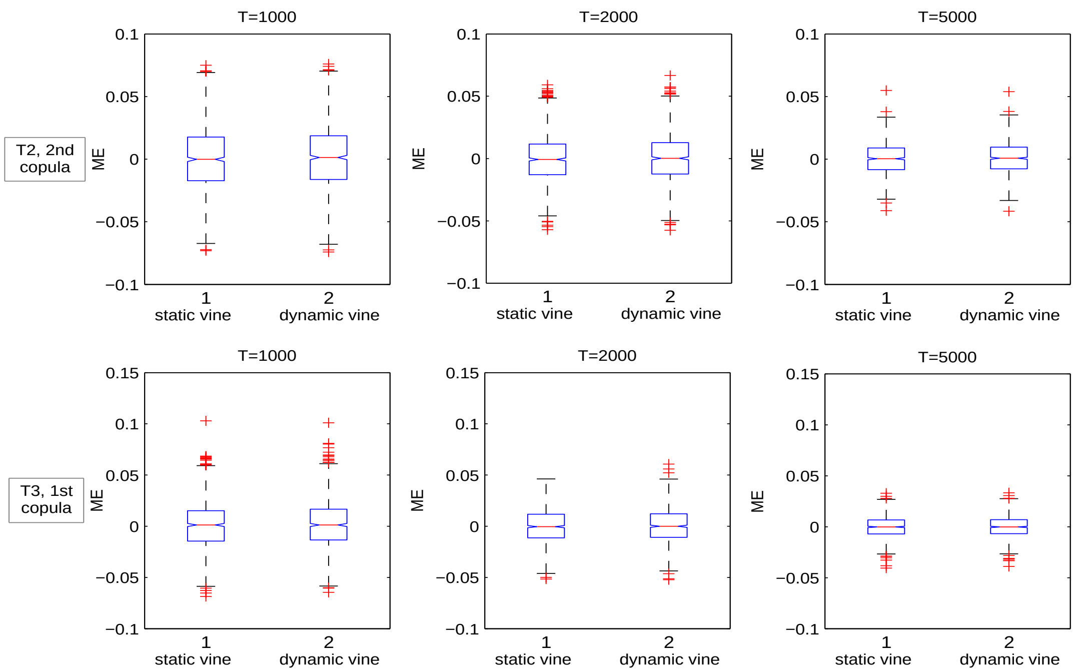

Figure 4.7: Mean errors for the parameters of $c_{24|3}$ and $c_{14|23}$ - DGP static D-vine.

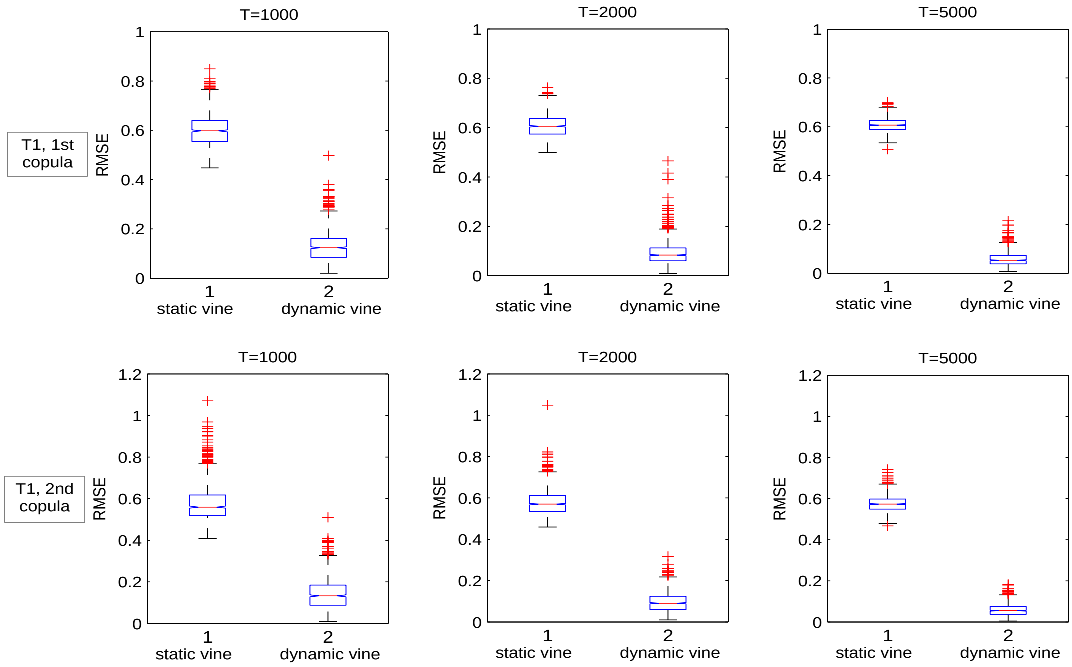

Figure 4.8: Root mean squared errors for the parameters of c_{12} and c_{23} - DGP dynamic D-vine.

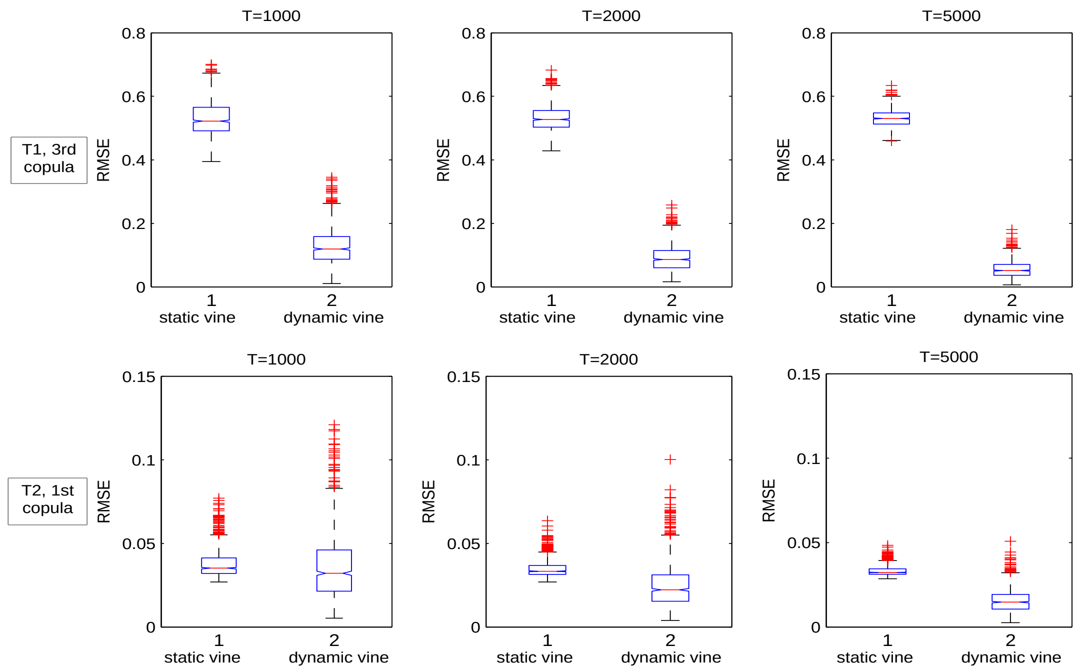

Figure 4.9: Root mean squared errors for the parameters of c_{34} and $c_{13|2}$ - DGP dynamic D-vine.

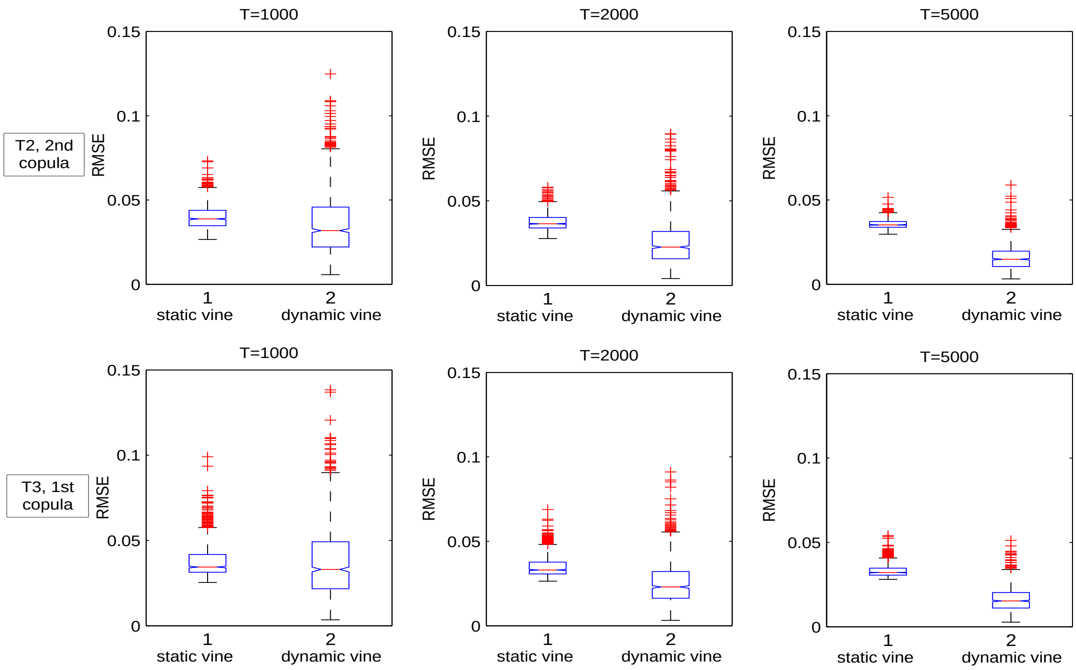

Figure 4.10: Root mean squared errors for the parameters of $c_{24|3}$ and $c_{14|23}$ - DGP dynamic D-vine.

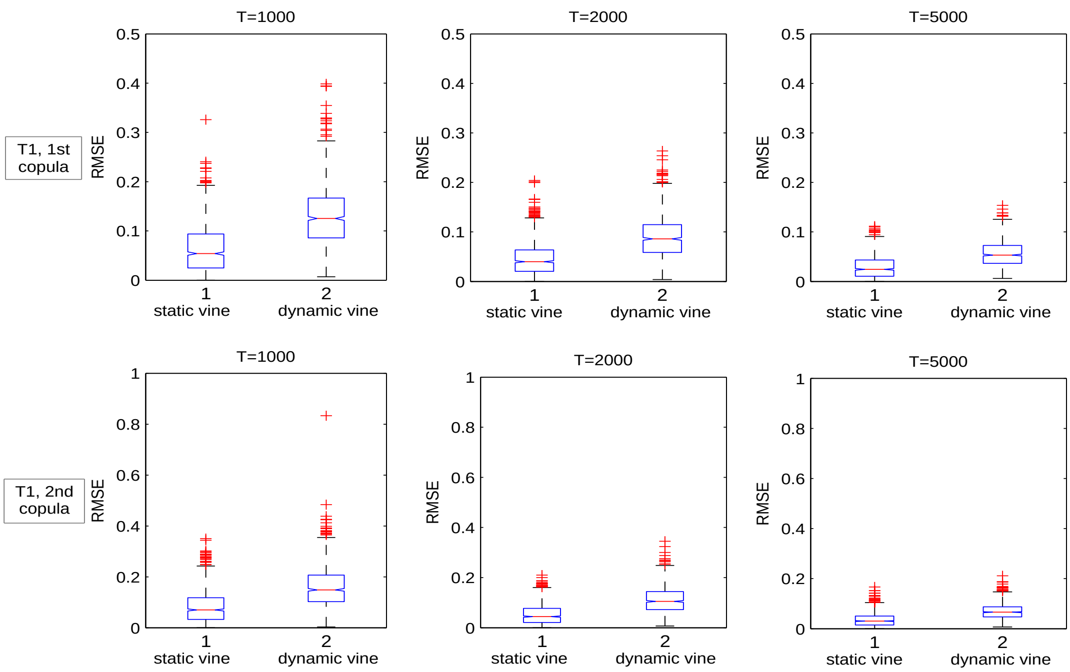

Figure 4.11: Root mean squared errors for the parameters of c_{12} and c_{23} - DGP static D-vine.

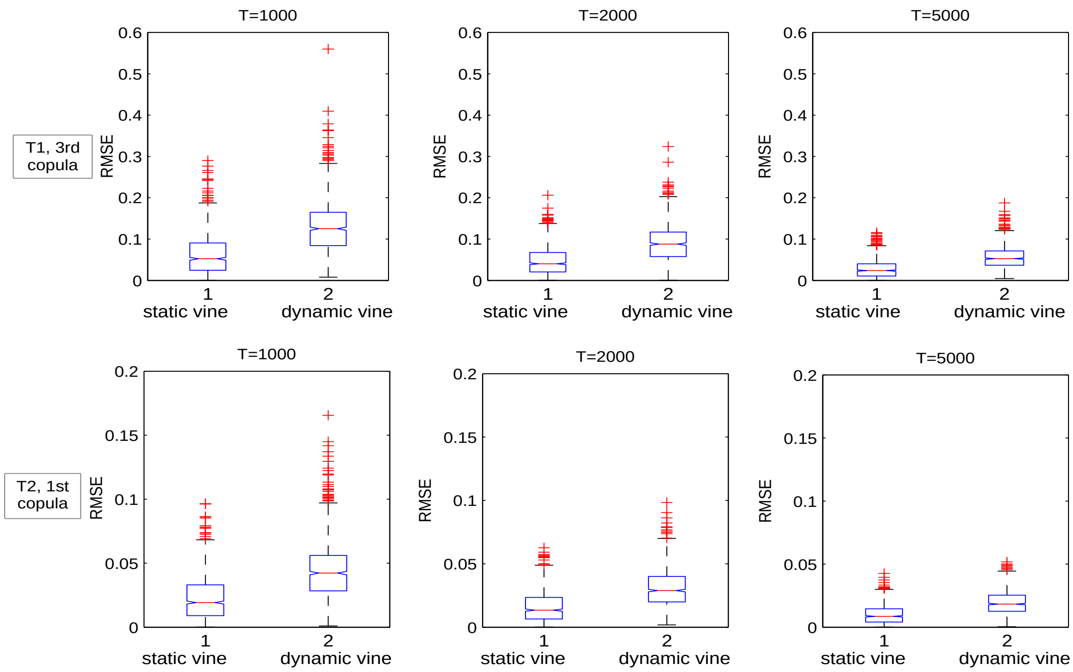

Figure 4.12: Root mean squared errors for the parameters of c_{34} and $c_{13|2}$ - DGP static D-vine.

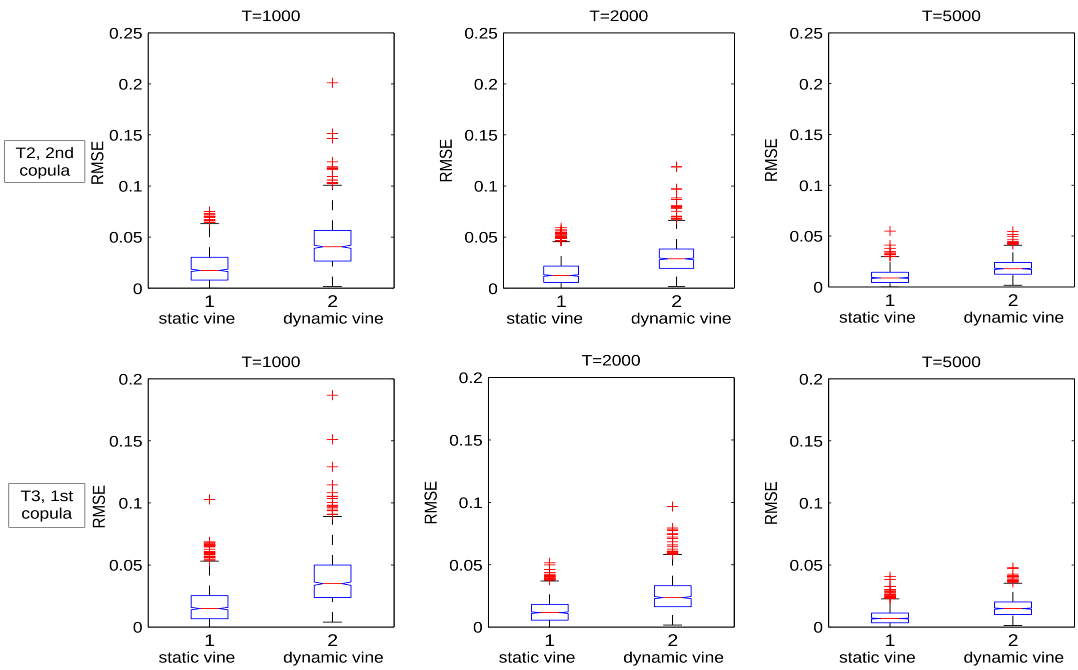

Figure 4.13: Root mean squared errors for the parameters of $c_{24|3}$ and $c_{14|23}$ - DGP static D-vine.

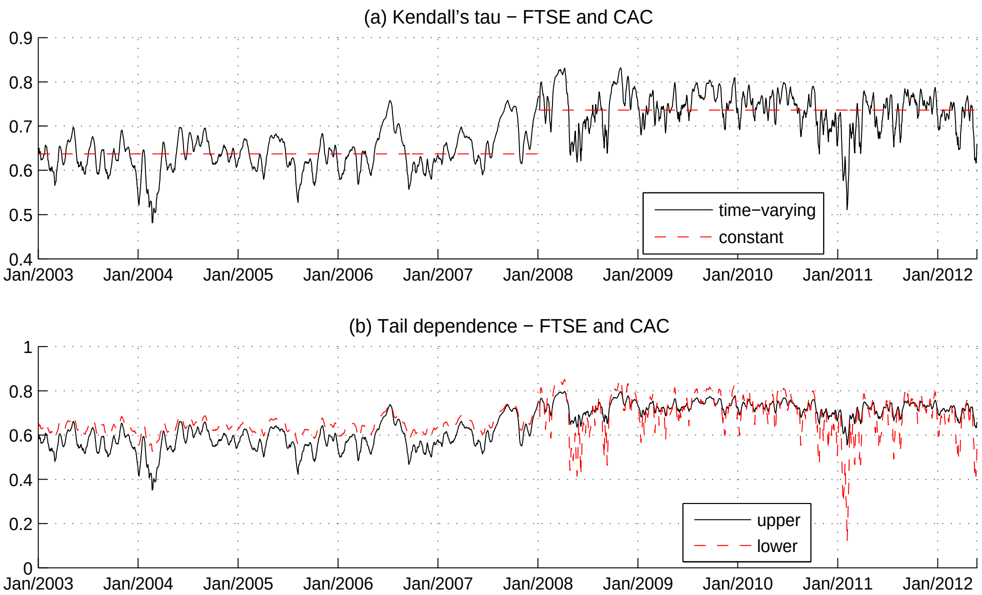

Figure 4.14: The dynamics of the Kendall's tau and the tail dependence parameters for the pair FTSE-CAC.

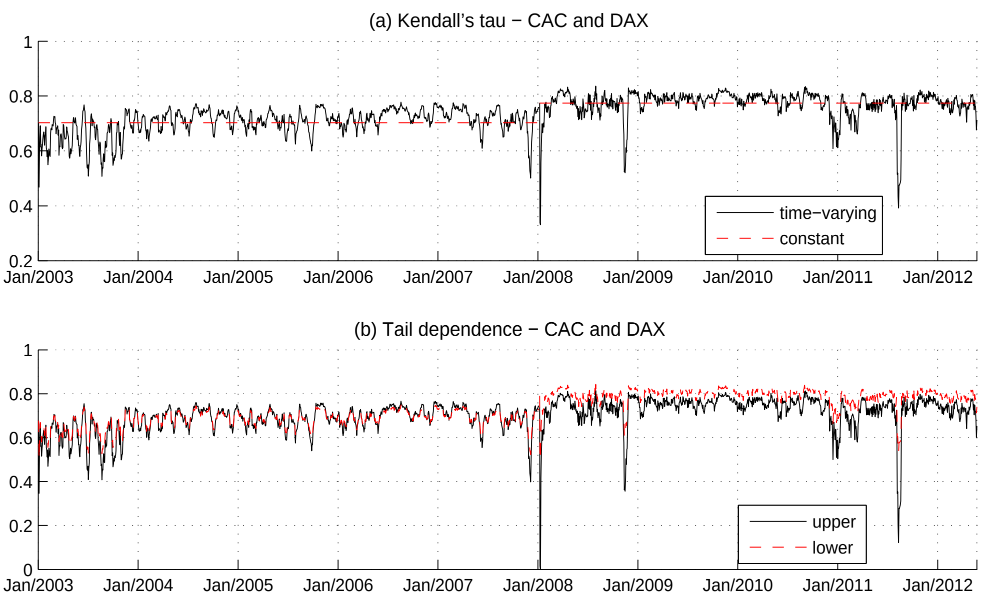

Figure 4.15: The dynamics of the Kendall's tau and the tail dependence parameters for the pair CAC-DAX.

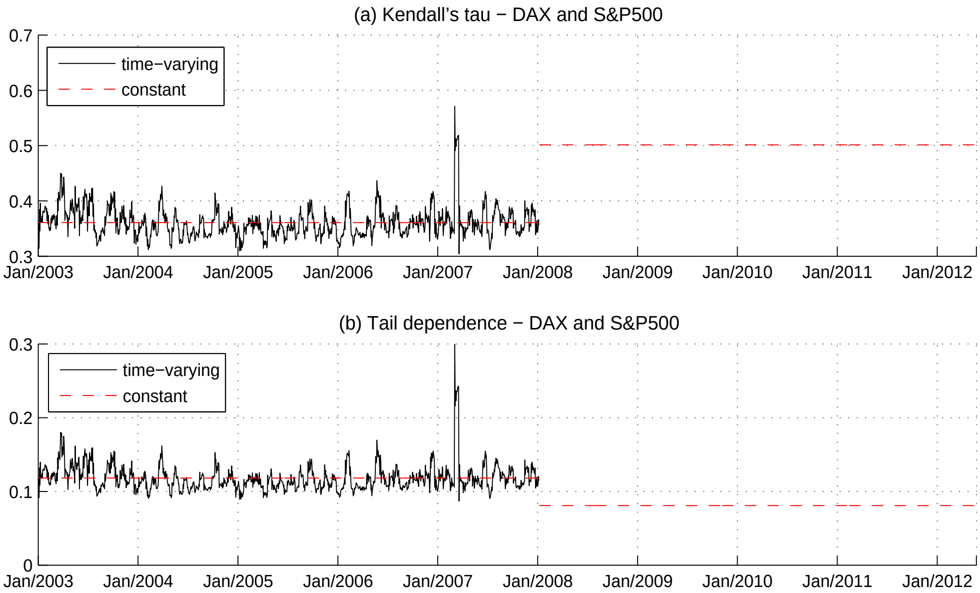

Figure 4.16: The dynamics of the Kendall's tau and the tail dependence parameters for the pair DAX-S&P500.

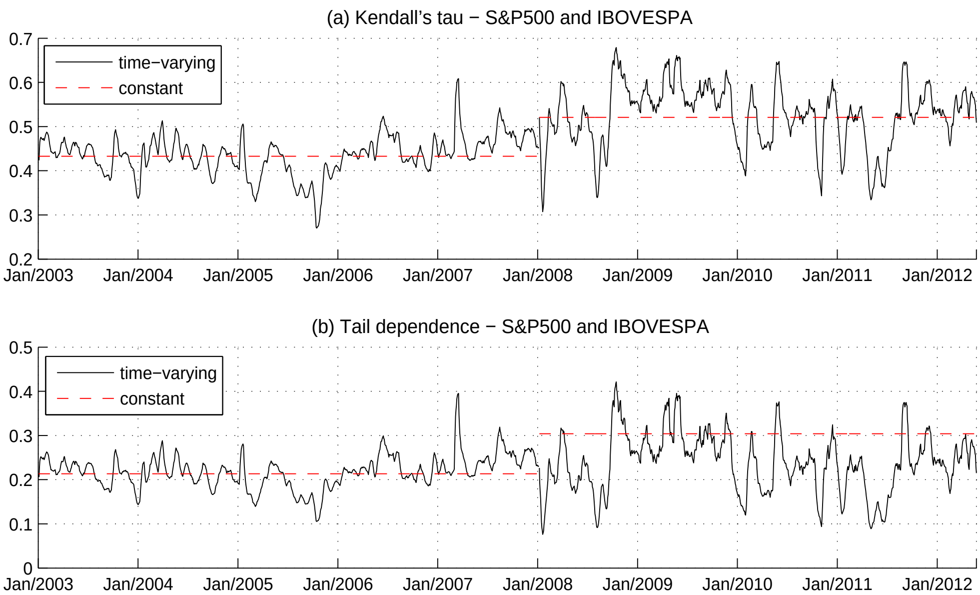

Figure 4.17: The dynamics of the Kendall's tau and the tail dependence parameters for the pair S&P500-IBOVESPA.

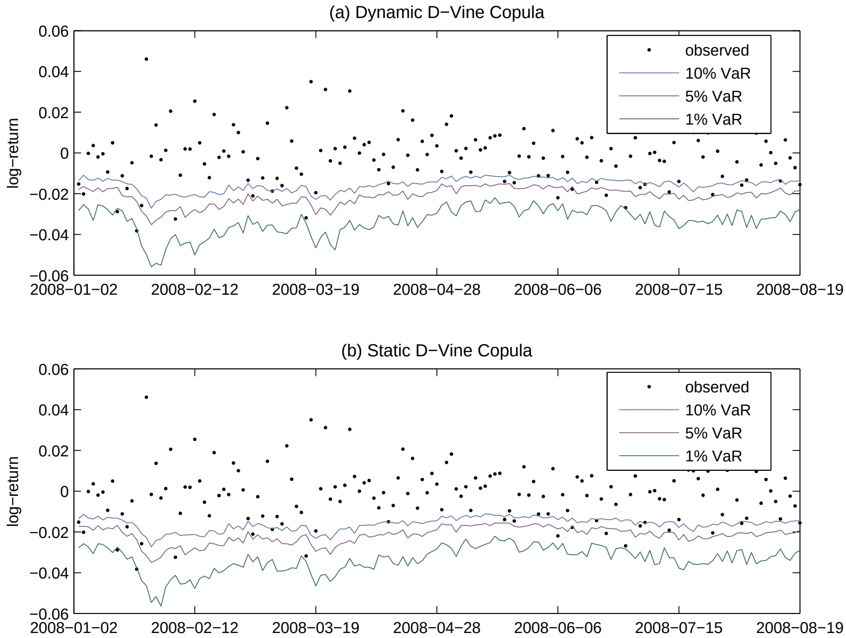

Figure 4.18: Log-returns of the portfolio for the period from January 02, 2008 to August 19, 2008 along with 10%, 5% and 1% VaR forecasts from the dynamic and static D-vine copula models.

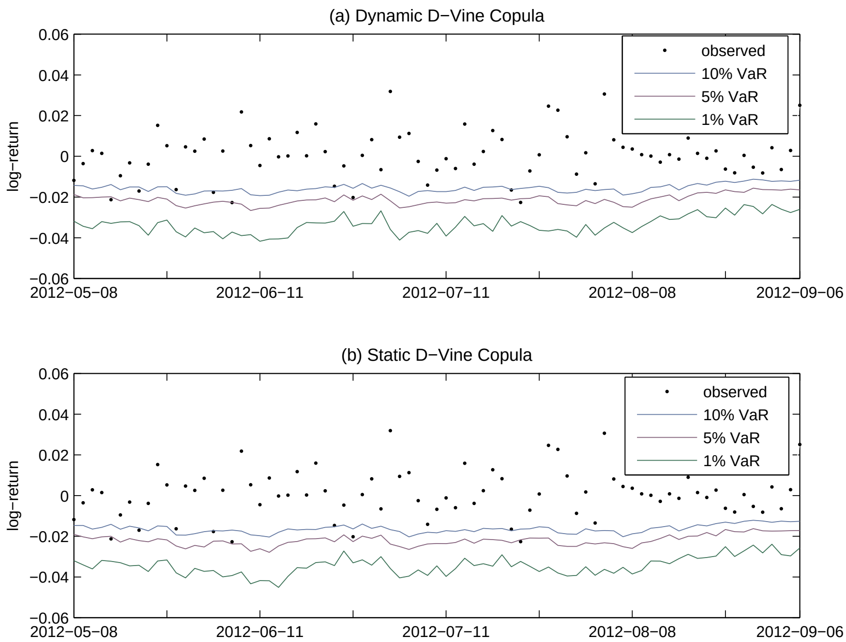

Figure 4.19: Log-returns of the portfolio for the period from May 08, 2012 to September 06, 2012 along with 10%, 5% and 1% VaR forecasts from the dynamic and static D-vine copula models.

5 Considerações Finais

Na presente tese, desenvolvemos três ensaios em que exploramos a teoria de cópulas para obter modelos de dependência suficientemente flexíveis, capazes de capturar os fatos estilizados presentes na estrutura de dependência de retornos financeiros multivariados, tais como dependência caudal e dependência assimétrica.

Sklar (1959) mostra que existe uma função cópula que liga a função de distribuição conjunta às suas distribuições marginais, de modo que a função cópula contém toda a informação sobre a dependência entre as variáveis. O conjunto de famílias de cópulas bivariadas é extenso e cada uma descreve um diferente padrão de dependência. O número de cópulas com dimensão mais elevada, por sua vez, era bastante limitado até recentemente. Apesar das tentativas de se estender as cópulas Arquimedianas para o caso de $n > 2$ variáveis e da popularidade das cópulas elípticas multivariadas, essas classes de cópulas ainda envolvem muitas restrições. A construção de cópulas multivariadas com base em *pair-copulas*, proposta por Joe (1996), reverteu o problema.

O teorema de Sklar foi estendido por Patton (2006) para o caso de distribuições condicionais, o que abriu a possibilidade de se incorporar dinâmica à estrutura de dependência. No primeiro ensaio, Capítulo 2 desta tese, propusemos um novo enfoque para modelar a dependência entre retornos financeiros internacionais ao longo do tempo, combinando a metodologia de Patton (2006), que permite que o parâmetro de dependência da cópula evolua de forma determinística ao longo do tempo, e o modelo de mudança Markoviana, que permite incorporar mais assimetrias ao modelo de cópula de uma forma bastante natural. Realizamos um estudo comparativo do modelo proposto e dos modelos de cópulas tempo-variantes propostos por Patton (2006), Jondeau e Rockinger (2006) e Silva Filho et al. (2012a), em termos das dinâmicas de dependência capturadas pelos modelos e das suas habilidades em prever Valor em Risco (VaR). Todos os modelos identificaram um longo período de alta dependência entre os retornos começando em 2007, quando a crise do *subprime* teve início oficialmente. A alta dependência no período de crise era esperada, porém os modelos com mudança de regime também indicaram períodos de alta dependência entre os anos de 2003 e 2006, quando as condições nos mercados financeiros eram favoráveis. Como trabalho futuro, sugerimos estimar o número de regimes, para verificar de que modo os resultados anteriores seriam alterados. O melhor ajuste dos modelos de cópulas elípticas às caudas das distribuições conjuntas sugere que parte da assimetria pode ser gerada por parâmetros de dependência tempo-variantes. Adicionalmente, a falta de dependência caudal pode ser parcialmente compensada pela possibilidade de uma forte dependência mais geral.

No segundo ensaio, Capítulo 3 desta tese, estendemos nosso estudo para o caso de $n > 2$ variáveis, usando o modelo de *vine* cópula para investigar a estrutura de dependência dos índices DAX, CAC 40, FTSE 100, S&P 500 e IBOVESPA, e, particularmente, checar a hipótese de dependência assimétrica nas caudas conjuntas da distribuição dos retornos dos índices financeiros. Com base em nossas descobertas empíricas, entretanto, não pudemos verificar a hipótese de dependência assimétrica, uma vez que uma decomposição *vine* canônica com todas as *pair-copulas* t-Student teve o melhor ajuste aos dados, segundo o critério de verossimilhança. Segundo Nikoloulopoulos et al. (2010), pode ser que a dependência assimétrica com cauda inferior mais forte ocorra apenas temporariamente, o que motivaria a incorporação de dinâmica ao modelo de *vine* cópula regular, como foi feito no terceiro ensaio.

A pesquisa em modelagem de dependência de séries financeiras multivariadas usando *vine* cópulas está concentrada principalmente no caso de estruturas de dependência homogêneas no tempo, porém, enfoques promissores para incorporação de dinâmica aos modelos de cópulas têm sido apresentados. No terceiro ensaio, Capítulo 4 desta tese, nós incorporamos dinâmica ao modelo de *vine* cópula, permitindo que os parâmetros de dependência das *pair-copulas* em uma decomposição *D-vine* variassem ao longo do tempo, seguindo um processo ARMA(1,m) restrito como em Patton (2006). Essa extensão do modelo nos permitiu ampliar a flexibilidade da modelagem da estrutura de dependência de dados financeiros multivariados sem, entretanto, incorrer em perda em termos de tratabilidade do modelo, que pode ser estimado utilizando-se um procedimento sequencial muito rápido e ainda assintoticamente eficiente. O modelo proposto foi avaliado, primeiramente, em um estudo de Monte Carlo, cujos resultados foram bastante favoráveis à cópula *D-vine* dinâmica em comparação a uma cópula *D-vine* estática. Também avaliamos o modelo com respeito à acurácia das previsões de VaR em períodos de crise: a cópula *D-vine* dinâmica superou a cópula *D-vine* estática em termos de acurácia preditiva para os nossos conjuntos de dados. O modelo de cópula *D-vine* dinâmica ainda pode ser melhorado se estendido para o caso geral de cópulas *vine* regulares e investigado em aplicações de dimensão mais elevada. Adicionalmente, no futuro, podemos assumir que os parâmetros de dependência das *pair-copulas* não apenas seguem um processo ARMA(1,m), mas também podem ser influenciados por uma cadeia de Markov, uma vez que encontramos evidência de quebra no grau de dependência do período de 2003-2007 para o período de 2008-2012.

Ainda há muito que ser explorado a respeito de estruturas de dependência multivariadas ($n > 2$) tempo-variantes. Também seria interessante, como tópico de pesquisa futura, investigar a possibilidade de se combinar cópulas e medidas realizadas, como variâncias, calculadas com base em dados em alta frequência, para criar distribuições multivariadas flexíveis. As medidas realizadas fornecem melhores previsões do que as tradicionais, porém o conhecimento dos segundos momentos apenas não determina a função de distribuição conjunta.

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