UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA
Tese de Doutorado

Teoria de turbulência fraca em plasmas empoeirados

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Porto Alegre - RS
2015

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Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal do Rio Grande do Sul como requisito parcial para a obtenção do título de Doutor em Ciências.

Orientador: Luiz Fernando Ziebell

Porto Alegre, 17 de dezembro de 2015.

*Trabalho parcialmente financiado pelo Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

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Agradecimentos

Olhando para os quatro anos que se passaram, não vejo apenas a construção de um trabalho acadêmico, mas tudo que vivi nesse período. Quanto a isto, não sei como dizer o quanto foi importante o apreço que recebi dos amigos que fiz.

Sou grato ao meu orientador, Luiz Fernando Ziebell, pelo suporte e pela oportunidade que me deu em realizar este trabalho. Sou grato ao CNPq pelo suporte financeiro, cuja ausência certamente tornaria meu doutoramento inviável.

Sumário

Resumo

Partindo das equações microscópicas exatas para uma plasma empoeirado não magnetizado, onde a carga das partículas de poeira é considerada como um novo grau de liberdade do sistema, desenvolvemos um sistema de equações básico sobre o qual aplicamos o formalismo da teoria de turbulência fraca. A partir dele, na aproximação quase linear, encontramos três relações de dispersão, em conjunto com as equações cinéticas das ondas e das partículas. O caso totalmente não linear é exemplificado em parte, para visualizarmos a aplicabilidade do formalismo a plasmas empoeirados. Entre os diversos mecanismos de carregamento que podem ocorrer sobre a superfície das partículas de poeira, consideramos, por simplicidade, somente aquele devido à absorção de partículas do plasma por colisões inelásticas.

Abstract

Starting from the exact microscopic equations for a non-magnetized dusty plasma where the dust charge particles is considered as a new degree of freedom of the plasma system, we develop a basic system of equations upon which we apply the formalism of weak turbulence theory. From this system we find three dispersion relations in the quasilinear approximation, together with the kinetic equations for the waves and for the particles. The full nonlinear case is partially exemplified, in order to provide an overview of the applicability of the formalism to dusty plasmas. Among the diverse charging mechanisms that can occur over the dust grains surface, we consider, for simplicity, only that due the absorption of plasma particles by inelastic collisions.

Capítulo 1

Introdução

De uma forma geral, turbulência corresponde a um estado dinâmico da matéria em que os graus de liberdade de um sistema, em estado sólido, estado de fluido ou em estado de plasma, são aleatórios e fortemente excitados.

Em plasmas, a dinâmica da turbulência é mais rica, dado que as partículas estão ionizadas. O movimento dos íons e elétrons é consequência da interação de longo alcance entre as partículas e do acoplamento com o campo eletromagnético do plasma. Portanto, um estado turbulento das partículas corresponde à turbulência do campo eletromagnético. A turbulência eletromagnética pode ser fraca ou forte, comparando-se a densidade de energia média dos campos em relação à densidade de energia média das partículas do plasma.

Neste trabalho, nos dispomos ao estudo da turbulência fraca em plasmas empoeirados. O que nos motiva à análise é o fato de que a presença de poeira pode modificar significativamente as propriedades eletromagnéticas de um plasma. Essa característica reside no fato de que partículas de poeira são em geral significativamente mais massivas e maiores do que íons e elétrons, e assim podem atuar como um sumidouro de partículas do plasma através de colisões inelásticas. O processo de absorção colisional de partículas carregadas pelos grãos de poeira é em grande parte influenciado pelas cargas já presentes em suas superfícies, constituindo um processo onde a interação elétrica entre as partículas do plasma e as partículas de poeira exercem um papel importante. Outros fatores também contribuem com o processo de carregamento colisional, por exemplo, as distribuições de densidade e velocidade das partículas e de tamanho dos grãos de poeira. Além disso, mecanismos de emissão de partículas também podem ser efetivos, como fotoemissão e emissão secundária de elétrons. Dessa forma, como uma consequência da presença da poeira, a propagação de ondas eletromagnéticas e eletrostáticas em plasmas empoeirados possui novas características em relação a plasmas sem poeira. Diversas investigações têm sido feitas sobre esse assunto, e citamos aqui alguns autores.

Shukla [1], em 1992, fez uma análise, utilizando o formalismo de fluidos, de propriedades dispersivas de ondas eletrostáticas e eletromagnéticas de baixa frequência em plasmas empoeirados com e sem campo magnético externo. Entre suas conclusões verificou que os espectros das ondas são modificados pela carga (estática) dos grãos de poeira, como também o surgimento de novos modos de propagação. Em 1994, Mendis e Rosenberg [2] publicaram uma extensa revisão sobre aspectos gerais de plasmas empoeirados cósmicos, incluindo, por exemplo, propriedades de carregamento e a dinâmica dos grãos sob a ação de diferentes tipos de força. De Juli e Schneider [3], em 1998, derivaram o tensor dielétrico de um plasma empoeirado magnetizado com carga variável das partículas de poeira, utilizando o formalismo cinético. Em uma de suas aplicações, eles verificaram a possibilidade de absorção pelo plasma de ondas magnetosônicas. De Angelis [4], em 2006, utilizando teoria cinética num plasma empoeirado não magnetizado, em um de seus resultados aplicando a teoria no contexto de dispositivos de fusão nuclear, verificou que grãos de poeira de carga negativa podem se atrair na interação de longo alcance. Em 2009, Gaelzer, de Juli, Schneider e Ziebell [5] utilizaram uma reformulação do tensor dielétrico de um plasma empoeirado magnetizado desenvolvido por de Juli e Schneider aplicando à análise de ondas de Alfvén com propagação oblíqua em relação à direção do campo magnético externo. A análise é rica em desenvolvimento e em resultados, e citamos um dos mais importantes, que é a verificação de que a frequência das colisões inelásticas afeta diretamente as propriedades de dispersão e absorção de ondas de Aflvén. Em 2012, Galvão e Ziebell [6] estenderam o formalismo de de Juli e Schneider introduzindo o efeito de fotoemissão de elétrons pelos grãos de poeira. Além da carga, a massa da poeira é particularmente importante em dois modos de propagação de baixa freqüência em plasmas não magnetizados de baixa densidade, o modo poeira-acústico e o modo poeira íon-acústico [7], que foram teoricamente previstos por Rao, Shukla e Yu [8], e Shukla e Silin [9].

Os modos de propagação que surgem das interações coletivas em plasmas podem se tornar instáveis, e podem também interagir uns com os outros. Se as amplitudes dos modos forem suficientemente pequenas, suas interações mútuas serão fracas, mas não necessariamente desprezáveis. Essas interações podem ser classificadas em três tipos. A primeira é a interação onda-partícula quase linear, que ocorre mais fortemente quando a condição de ressonância $\omega = \mathbf{k} \cdot \mathbf{v}$ é satisfeita. A segunda, a interação não linear de três ondas, satisfaz a condição $\omega_1 + \omega_2 = \omega_3$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. A terceira é a interação onda-partícula não linear, que satisfaz $\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{v}$, e corresponde ao espalhamento de uma onda por uma partícula [11].

A teoria cujo formalismo incorpora a descrição de tais modos não lineares é a denominada teoria de turbulência fraca, cujos fundamentos teóricos foram em grande parte desenvolvidos nos anos sessenta e setenta. Existem na literatura excelentes livros para uma introdução ao tema da turbulência fraca, como por exemplo os livros de Kadomtsev, Sagdeev e Galeev, Davidson, Tsytovitch, e Sitenko, [10, 11, 12, 13, 14]. O formalismo da turbulência fraca foi revisitado em anos mais recentes por Yoon. Em 2000, [15] ele generalizou a teoria em relação à análise de interações não lineares com a incorporação de modos não lineares, focando apenas ondas eletrostáticas, e posteriormente estendeu o formalismo para incluir ondas eletromagnéticas [16], incorporando ainda posteriormente a presença de efeitos de emissão espontânea [17].

O formalismo da teoria de turbulência fraca tem sido aplicado em problemas fundamentais em física de plasmas, visto que provê ferramentas para o estudo e entendimento de fenômenos não lineares. Dentro do grande número de artigos que aplicam a teoria, citamos somente alguns como exemplo. Muschietti e Dum [18] fizeram uma análise da influência do espalhamento não linear de ondas de Langmuir durante a relaxação quase linear de um feixe de elétrons. Ziebell, Gaelzer e Yoon [19] fizeram uma análise da instabilidade bump-in-tail e da subsequente evolução não linear em ondas de Langmuir e íon-acústicas, considerando uma distribuição de velocidades unidimensional. A análise é rica em detalhes e, entre os principais resultados, destacamos a saturação de ondas de Langmuir seguida de processos não lineares de decaimento e espalhamento de ondas. Em [20] foram obtidos diversos resultados em que, pela primeira vez, a interação feixe-plasma foi tomada numa solução matemática totalmente autoconsistente das equações cinéticas, considerando a interação quase linear e o decaimento de três ondas, em um sistema bidimensional. Numa aplicação à astrofísica solar, Gaelzer *et al.* [21] demonstrou que as funções de distribuição de velocidades do vento solar tem a forma de distribuições do tipo kappa, confirmando que a aceleração turbulenta é uma explicação plausível para o ganho de energia dos elétrons supertérmicos. Em 2012, Ziebell *et al.* [22] verificou que a emissão espontânea por elétrons desempenha um papel chave no desenvolvimento da turbulência de Langmuir no limite de longo comprimento de onda. Em 2014, [23], foi verificada a transição de um estado de equilíbrio térmico para outro caracterizado por um espectro finito de radiação eletromagnética.

A teoria de turbulência fraca também já foi utilizada para a investigação de efeitos não lineares de modos em plasmas empoeirados. Por exemplo, Yi, Ryu e Yoon investigaram o desvio de frequência não linear em ondas poeira-acústicas e ondas poeira íon-acústicas em plasmas empoeirados não magnetizados, causado por turbulência poeira-acústica e poeira íon-acústica, respectivamente [24, 25].

O presente trabalho se desenvolve considerando a carga da poeira como uma quantidade variável [26], incorporando tal característica no contexto da turbulência fraca [27]. Partimos das distribuições microscópicas exatas das partículas de poeira e dos íons e elétrons, considerando a carga da poeira como um novo grau de liberdade do sistema, e considerando a equação de primeira ordem na hierarquia BBGKY (Bogolyubov-Born-Green-Kirkwood-Yvon) para um plasma empoeirado [28]. Entre os diversos mecanismos de carregamento possíveis, consideramos por simplicidade somente o mecanismo de absorção de partículas por colisões inelásticas. A interação poeira-poeira também foi desprezada por simplicidade. Esta aproximação é válida em investigações em plasmas empoeirados onde a distância média entre grãos de poeira é maior que o comprimento de Debye, $r_d > \lambda_D$, condição tipicamente encontrada em plasmas de sistemas experimentais e em plasmas de diversos sistemas espaciais [29].

No que diz respeito ao formalismo utilizado, em analogia com a teoria de turbulência fraca para plasmas sem poeira, expandimos as flutuações das distribuições das partículas do plasma e da poeira em potências do campo elétrico do plasma, como também em potências das oscilações associadas ao processo de carregamento dos grãos de poeira. Assim pudemos encontrar uma equação não linear para a densidade de energia espectral que leva em conta o carregamento das partículas de poeira, que está em consistência com duas outras novas equações que correspondem às densidades de energia espectrais associadas às flutuações das distribuições da poeira e às flutuações das correntes de carregamento, formando um conjunto autoconsistente de equações. Como uma aplicação, a partir desse conjunto, pudemos obter as equações cinéticas para as ondas em conexão com a evolução temporal das funções de distribuição das partículas do plasma na aproximação quase linear. Ilustramos o caso não linear parcialmente, de modo a exemplificar a aplicabilidade do formalismo a plasmas empoeirados.

A tese está estruturada da seguinte forma. No Capítulo 2, apresentamos um modelo para a descrição de um plasma empoeirado no formalismo cinético. Ao longo deste capítulo desenvolvemos um sistema de equações que forma a base para o que queremos encontrar, que são as equações cinéticas para ondas eletrostáticas, em correspondência com as equações cinéticas das partículas do plasma e da poeira. No Capítulo 3, escrevemos o sistema de equações obtido em variáveis espectrais, separando-o em duas escalas de tempo e fazendo uma expansão das flutuações das distribuições de partículas em potências dos campos. No Capítulo 4, desenvolvemos as equações das densidades espectrais de energia e definimos diversas funções resposta. No Capítulo 5, obtemos um sistema de equações na aproximação quase linear, do qual esperamos ser possível extrair diversas propriedades dispersivas e dinâmicas do plasma, como consequência da interação quase linear onda-partícula. No Capítulo 6, desenvolvemos a equação cinética para a evolução temporal das ondas do plasma incluindo interações não lineares onda-onda e onda-partícula.

Capítulo 2

Descrição cinética clássica

Um conjunto de equações para descrever fenômenos turbulentos em plasmas pode ser obtido pelo formalismo cinético, começando pela distribuição e equação microscópica para as partículas do plasma e então supondo que as grandezas microscópicas diferem de seus valores médios por pequenas frações que oscilam no tempo e no espaço. Seguindo o formalismo de Klimontovich, adotamos o modelo para um plasma empoeirado de Schram *et al.* [28].

Começamos a descrição do sistema assumindo que todo elétron e íon que colide com um grão de poeira é absorvido. Matematicamente isto equivale a introduzir uma função de distribuição microscópica modificada para as partículas do plasma, na forma [28]

$$N_{i\sigma}(X,t) = \delta(X - X_{i\sigma}(t))\theta(t_{i\sigma} - t) \ , \tag{2.1}$$

onde $t_{i\sigma}$ identifica o instante de colisão da i-ésima partícula do tipo σ com um grão de poeira. As variáveis $X \equiv (\mathbf{r}, \mathbf{v})$ e $X_{i\sigma}(t) \equiv (\mathbf{r}_{i\sigma}(t), \mathbf{v}_{i\sigma}(t))$ são vetores de seis dimensões, e

$$\theta(x) = \left(\begin{array}{ll} 0, & x < 0 \\ 1, & x \geq 0 \end{array} \right.$$

é a função degrau de Heaviside.

A distribuição total de partículas do tipo σ é dada pela soma

$$N_{\sigma} = \sum_{i=1}^{N_\sigma} N_{i\sigma}(X,t) = \sum_{i=1}^{N_\sigma} \delta(X - X_{i\sigma}(t))\theta(t_{i\sigma} - t) \ , \tag{2.2}$$

onde N_σ no limite superior indica o número total de partículas.

Tomando a derivada parcial em relação tempo de (2.1) e considerando as equações de movimento

$$\frac{d\mathbf{r}_{i\sigma}(t)}{dt} = \mathbf{v}_{i\sigma}(t) \quad ; \quad \frac{d\mathbf{v}_{i\sigma}(t)}{dt} = \frac{1}{m_\sigma}\mathbf{F}^m_{i\sigma}(\mathbf{r}_{i\sigma}(t), t) \ , \tag{2.3}$$

obtemos

$$\frac{\partial N_{i\sigma}(X,t)}{\partial t} = \theta(t_{i\sigma} - t)\frac{\partial}{\partial t}\delta(X - X_{i\sigma}(t)) - \delta(X - X_{i\sigma}(t))\delta(t_{i\sigma} - t)$$

$$= -\theta(t_{i\sigma} - t)\frac{\partial}{\partial \mathbf{r}_{i\sigma}}\delta(X - X_{i\sigma}(t)) \cdot \frac{\partial \mathbf{r}_{i\sigma}}{\partial t} - \theta(t_{i\sigma} - t)\frac{\partial}{\partial \mathbf{v}_{i\sigma}}\delta(X - X_{i\sigma}(t)) \cdot \frac{\partial \mathbf{v}_{i\sigma}}{\partial t}$$

$$-\delta(X - X_{i\sigma}(t))\delta(t_{i\sigma} - t)$$

$$= -\theta(t_{i\sigma}-t)\frac{\partial}{\partial \mathbf{r}}\delta(X-X_{i\sigma}(t))\cdot\frac{\partial \mathbf{r}}{\partial t}-\theta(t_{i\sigma}-t)\frac{\partial}{\partial \mathbf{v}}\delta(X-X_{i\sigma}(t))\cdot\frac{\partial \mathbf{v}}{\partial t}-\delta(X-X_{i\sigma}(t))\delta(t_{i\sigma}-t) \ .$$

Assim, a evolução temporal da função de distribuição microscópica $dN_{i\sigma}/dt$ é dada de acordo com

$$\frac{dN_{i\sigma}}{dt} = \frac{\partial N_{i\sigma}}{\partial t} + \frac{\partial N_{i\sigma}}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{\partial N_{i\sigma}}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}^m_{i\sigma}}{m_\sigma} = -\delta(X - X_{i\sigma}(t))\delta(t - t_{i\sigma}) \ . \tag{2.4}$$

Como $t_{i\sigma}$ representa o instante de colisão de uma partícula com um grão de poeira, e considerando que a função delta de Dirac de uma função $f(x)$ satisfaz a relação

$$\delta(f(x)) = \sum_i \delta(x - x_i)\left|\frac{df}{dx}\right|^{-1}_{x=x_i} \ ,$$

onde x_i são raízes de $f(x)$, podemos escrever

$$\delta(|\mathbf{r}_{i\sigma}(t) - \mathbf{r}_d(t)| - a)\left|\frac{\partial|\mathbf{r}_{i\sigma}(t) - \mathbf{r}_d(t)|}{\partial t}\right|_{t=t_{i\sigma}} = \delta(t - t_{i\sigma}) \ ,$$

onde $\mathbf{r}_d(t)$ descreve a trajetória de uma partícula de poeira de forma esférica, e o parâmetro a representa seu raio.

Dado que $|\mathbf{r}_{i\sigma}(t) - \mathbf{r}_d(t)| = (\mathbf{r}_{i\sigma}(t) - \mathbf{r}_d(t)) \cdot \mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}$, onde $\mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}$ é o vetor unitário ao longo da direção do vetor diferença $\mathbf{r}_{i\sigma}(t) - \mathbf{r}_d(t)$, tomamos a derivada em relação ao tempo de seu módulo,

$$\frac{\partial}{\partial t}|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)| = \frac{\partial}{\partial t}\left((\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t))\cdot\frac{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}{|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)|}\right)$$

$$= (\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t))\cdot\mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)} + (\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t))$$

$$\cdot\left(\frac{\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t)}{|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)|} - \frac{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}{|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)|^2}\frac{\partial}{\partial t}|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)|\right)$$

$$= 2(\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t))\cdot\mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)} - \frac{\partial}{\partial t}|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)|\ ,$$

tal que

$$\frac{\partial}{\partial t}|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)| = (\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t))\cdot\mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}\ . \tag{2.5}$$

Então podemos escrever que

$$\delta(t-t_{i\sigma}) = \delta(|\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)|-a)\left|(\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t))\cdot\mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}\right|_{t=t_{i\sigma}}\ . \tag{2.6}$$

Assim (2.4) se torna

$$\frac{\partial N_{i\sigma}}{\partial t} + \frac{\partial N_{i\sigma}}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{\partial N_{i\sigma}}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}^m_{i\sigma}}{m_\sigma}$$
$$= -\int d\mathcal{X}'\delta(\mathcal{X}'-\mathcal{X}_d(t))\,|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}-\mathbf{r}'}|\,\delta(|\mathbf{r}-\mathbf{r}'|-a)N_{i\sigma}\ , \tag{2.7}$$

onde $\mathcal{X}\equiv(\mathbf{r},\mathbf{v},q)$ são as coordenadas no espaço de fase de sete dimensões das partículas de poeira [26].

Para a trajetória de uma partícula do plasma que se aproxima de um grão de poeira, vale a relação

$$(\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t))\cdot\mathbf{e}_{\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)}\big|_{t=t_{i\sigma}} < 0\ ,$$

que significa que o ângulo entre os vetores $\mathbf{r}_{i\sigma}(t)-\mathbf{r}_d(t)$ e $\mathbf{v}_{i\sigma}(t)-\mathbf{v}_d(t)$ deve satisfazer a condição $\vartheta > \pi/2$, de tal modo que

$$\frac{\partial N_{i\sigma}}{\partial t} + \frac{\partial N_{i\sigma}}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{\partial N_{i\sigma}}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}^m_{i\sigma}}{m_\sigma} = -\int d\mathcal{X}'\delta(\mathcal{X}'-\mathcal{X}_d(t))\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a)N_{i\sigma}\ . \tag{2.8}$$

onde a função

$$\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a) = \delta(|\mathbf{r}-\mathbf{r}'|-a)\theta(-(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}-\mathbf{r}'})|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}-\mathbf{r}'}| \quad (2.9)$$

tem dimensão de frequência.

Considerando a distribuição total dos grãos de poeira

$$N_d(\mathcal{X},t) = \sum_{k=1}^{N_d} \delta(\mathcal{X}-\mathcal{X}_k(t)) \;, \quad (2.10)$$

e (2.2), a equação (2.8) pode ser escrita como

$$\frac{\partial N_\sigma}{\partial t} + \frac{\partial N_\sigma}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{\partial N_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_\sigma^m}{m_\sigma} = -\int d\mathcal{X}'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a)N_d(\mathcal{X}',t)N_\sigma(X,t) \;. \quad (2.11)$$

A equação microscópica das partículas de poeira, desprezando a interação entre grãos de poeira, é dada por [28]

$$\frac{\partial N_d}{\partial t} + \frac{\partial N_d}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{\partial N_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_d^m}{m_d} = -\sum_{\sigma=e,i}\int dX'\big(\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a)N_d(\mathcal{X},t)$$
$$-\,\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\delta\mathbf{v}_\sigma-\mathbf{v}',a)N_d(\mathbf{r},\mathbf{v}-\delta\mathbf{v}_\sigma,q-q_\sigma,t)\big)N_\sigma(X',t) \;. \quad (2.12)$$

O símbolo q_σ se refere à carga elétrica das espécies do plasma, e $\delta\mathbf{v}_\sigma$ é a variação no vetor velocidade de um grão de poeira devido a uma colisão inelástica com uma partícula do plasma. Considerando que a variação de velocidade num grão de poeira pela colisão é pequena, e que a carga da poeira em geral é muito maior que a carga de uma partícula absorvida, esses parâmetros, $\delta\mathbf{v}_\sigma$ e q_σ, são pequenos na interação, de modo que o lado direito da equação (2.12) pode ser expandido sobre eles. Esta aproximação em que o caráter discreto da carga da poeira pode ser desprezado é válida para grãos suficientemente grandes, com raio da ordem de $a \sim 10^{-5}$cm [30].

Levando em conta a conservação de *momenta* na colisão

$$\delta\mathbf{v}_\sigma = \frac{m_d\mathbf{v}+m_\sigma\mathbf{v}_\sigma}{m_\sigma+m_d} - \mathbf{v} = -\frac{m_\sigma}{m_\sigma+m_d}(\mathbf{v}-\mathbf{v}_\sigma) \simeq -\frac{m_\sigma}{m_d}(\mathbf{v}-\mathbf{v}') \;,$$

consideramos uma expansão de Taylor em torno de valores arbitrários $\mathbf{v}_1$ e q_1, de modo que

$$N_d(\mathbf{r},\mathbf{v},q,t) \simeq N_d(\mathbf{r},\mathbf{v}_1,q_1,t) + (\mathbf{v}-\mathbf{v}_1)\frac{\partial N_d}{\partial \mathbf{v}}\bigg|_{\substack{\mathbf{v}=\mathbf{v}_1\\ q=q_1}} + (q-q_1)\frac{\partial N_d}{\partial q}\bigg|_{\substack{\mathbf{v}=\mathbf{v}_1\\ q=q_1}}$$

$$+\frac{1}{2!}\left(\sum_{ij}(v_i-v_{i1})(v_j-v_{j1})\frac{\partial^2 N_d}{\partial v_i \partial v_j}\bigg|_{\substack{\mathbf{v}=\mathbf{v}_1\\ q=q_1}} +2(q-q_1)(\mathbf{v}-\mathbf{v}_1)\cdot\frac{\partial^2 N_d}{\partial \mathbf{v}\partial q}\bigg|_{\substack{\mathbf{v}=\mathbf{v}_1\\ q=q_1}} +(q-q_1)^2\frac{\partial^2 N_d}{\partial q^2}\bigg|_{\substack{\mathbf{v}=\mathbf{v}_1\\ q=q_1}}\right) .$$

Com $\delta\mathbf{v}_\sigma = \mathbf{v}-\mathbf{v}_1$ e $q-q_1 = q_\sigma$, obtemos

$$N_d(\mathbf{r},\mathbf{v}-\delta\mathbf{v}_\sigma,q-q_\sigma,t) \simeq N_d(\mathbf{r},\mathbf{v},q,t) + \frac{m_\sigma}{m_d}(\mathbf{v}-\mathbf{v}')\frac{\partial N_d}{\partial \mathbf{v}} - q_\sigma\frac{\partial N_d}{\partial q}$$
$$+\frac{1}{2!}\left(\left(\frac{m_\sigma}{m_d}\right)^2\sum_{ij}(v_i-v_i')(v_j-v_j')\frac{\partial^2 N_d}{\partial v_i \partial v_j} - 2q_\sigma\frac{m_\sigma}{m_d}(\mathbf{v}-\mathbf{v}')\cdot\frac{\partial^2 N_d}{\partial \mathbf{v}\partial q} + q_\sigma^2\frac{\partial^2 N_d}{\partial q^2}\right) . \quad (2.13)$$

Substituindo (2.13) em (2.12),

$$\frac{\partial N_d}{\partial t} + \frac{\partial N_d}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{\partial N_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_d^m}{m_d} = -\frac{\partial}{\partial q}\big(I^m N_d(\mathcal{X},t)\big) , \quad (2.14)$$

onde definimos a corrente microscópica

$$I^m = \sum_\sigma q_\sigma \int dX' N_\sigma(X',t)\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a) . \quad (2.15)$$

Na equação (2.14) mantivemos somente o termo devido às colisões inelásticas, que correspondem ao carregamento elétrico.

Tomando a média estatística das equações (2.11) e (2.14) obtemos,

$$\frac{\partial f_\sigma}{\partial t} + \frac{\partial f_\sigma}{\partial \mathbf{r}}\cdot\mathbf{v} + \left\langle \frac{\partial N_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_\sigma^m}{m_\sigma}\right\rangle = -\int d\mathcal{X}'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a)\big\langle N_d(\mathcal{X}',t)N_\sigma(X,t)\big\rangle , \quad (2.16)$$

$$\frac{\partial f_d}{\partial t} + \frac{\partial f_d}{\partial \mathbf{r}}\cdot\mathbf{v} + \left\langle \frac{\partial N_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_d^m}{m_d}\right\rangle = -\left\langle \frac{\partial}{\partial q}\big(I^m N_d(\mathcal{X},t)\big)\right\rangle , \quad (2.17)$$

onde

$$\langle N_\sigma(X,t)\rangle = \sum_{i=1}^{N_\sigma}\int dX_{1\sigma}\ldots X_{N_\sigma}\delta(X-X_{i\sigma}(t))\theta(t_{i\sigma}-t)G(X_{1\sigma}\ldots X_{N_\sigma}) = f_\sigma(X,t) ,$$

e G é a função densidade de probabilidade de que no instante t as partículas do plasma estejam no estado $(X_{1\sigma}(t)\ldots X_{N_\sigma}(t);t)$.

Introduzindo a flutuação

$$\delta N = N - \langle N \rangle \ , \tag{2.18}$$

de onde seguem as relações

$$\delta N_1 \delta N_2 + \delta N_1 \langle N_2 \rangle + \langle N_1 \rangle \delta N_2 + \langle N_1 \rangle \langle N_2 \rangle = N_1 N_2 \ ,$$

$$\langle \delta N_1 \delta N_2 \rangle + \langle N_1 \rangle \langle N_2 \rangle = \langle N_1 N_2 \rangle \ ,$$

e considerando a expressão de I^m em (2.15), temos

$$-\Big\langle \frac{\partial}{\partial q}\big(I^m N_d(\mathcal{X},t)\big)\Big\rangle = -\frac{\partial}{\partial q}\sum_\sigma q_\sigma \int dX'\Omega\big\langle N_\sigma(X',t) N_d(\mathcal{X},t)\big\rangle$$

$$= -\frac{\partial}{\partial q}\sum_\sigma q_\sigma \int dX'\Omega\big(\big\langle N_\sigma(X',t)\big\rangle\big\langle N_d(\mathcal{X},t)\big\rangle$$

$$+\big\langle \delta N_\sigma(X',t)\delta N_d(\mathcal{X},t)\big\rangle\big) = -\frac{\partial}{\partial q}\big(I f_d(\mathcal{X},t)\big) - \frac{\partial}{\partial q}\big\langle \delta I^m \delta N_d(\mathcal{X},t)\big\rangle \ ,$$

onde

$$I(X,t) = \sum_\sigma q_\sigma \int dX'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a) f_\sigma(X',t) \ , \tag{2.19}$$

$$\delta I^m(X,t) = \sum_\sigma q_\sigma \int dX'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a)\ \delta N_\sigma(X',t) \ . \tag{2.20}$$

Portanto

$$\frac{\partial f_d}{\partial t} + \frac{\partial f_d}{\partial \mathbf{r}}\cdot\mathbf{v} + \left\langle \frac{\partial N_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_d^m}{m_d}\right\rangle = -I(X,t)\frac{\partial}{\partial q} f_d(\mathcal{X},t) - \frac{\partial}{\partial q}\big\langle \delta I^m(X,t)\delta N_d(\mathcal{X},t)\big\rangle \ . \tag{2.21}$$

De forma similar,

$$\frac{\partial f_\sigma}{\partial t} + \frac{\partial f_\sigma}{\partial \mathbf{r}}\cdot\mathbf{v} + \left\langle \frac{\partial N_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{F}_\sigma^m}{m_\sigma}\right\rangle$$
$$= -\int d\mathcal{X}'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a)\big(f_d(\mathcal{X}',t) f_\sigma(X,t) + \big\langle \delta N_d(\mathcal{X}',t)\delta N_\sigma(X,t)\big\rangle\big) \ . \tag{2.22}$$

Considerando a aproximação eletrostática, a natureza da força indicada no terceiro termo das equações cinéticas depende apenas do campo elétrico. A equação para o campo deve conter também a contribuição da componente da poeira, de tal modo que a lei de

Gauss em sua forma diferencial é dada, no sistema de unidades Gaussiano, por

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{E}^m(\mathbf{r}, t) = 4\pi\bigg(\sum_\sigma q_\sigma \int d\mathbf{v} N_\sigma(X, t) + \int dq\; q \int d\mathbf{v} N_d(\mathcal{X}, t)\bigg) \; . \tag{2.23}$$

Notemos que o caráter discreto da carga da poeira é perdido na segunda integral do lado direito de (2.23).

Para a flutuação da força temos

$$\delta \mathbf{F}_\sigma^m = \mathbf{F}_\sigma^m - \langle \mathbf{F}_\sigma^m \rangle \; , \tag{2.24}$$

de modo que

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{E}^m(\mathbf{r}, t) = \frac{\partial}{\partial \mathbf{r}} \cdot \big(\delta\mathbf{E}^m(\mathbf{r}, t) + \big\langle \mathbf{E}^m(\mathbf{r}, t) \big\rangle\big) = \frac{\partial}{\partial \mathbf{r}} \cdot \big(\delta\mathbf{E}^m(\mathbf{r}, t) + \mathbf{E}(\mathbf{r}, t)\big) \; , \tag{2.25}$$

onde $\mathbf{E}(\mathbf{r}, t)$ é o campo elétrico (macroscópico) médio. Portanto,

$$\frac{\partial}{\partial \mathbf{r}} \cdot \delta\mathbf{E}^m(\mathbf{r}, t) = 4\pi\bigg(\sum_\sigma q_\sigma \int d\mathbf{v} \delta N_\sigma(X, t) + \int dq\; q \int d\mathbf{v} \delta N_d(\mathcal{X}, t)\bigg) \; , \tag{2.26}$$

Realizando a média sobre (2.23), obtemos

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{E}(\mathbf{r}, t) = 4\pi\bigg(\sum_\sigma q_\sigma \int d\mathbf{v} f_\sigma(X, t) + \int dq\; q \int d\mathbf{v} f_d(\mathcal{X}, t)\bigg) \; , \tag{2.27}$$

onde $\langle \delta\mathbf{E}^m(\mathbf{r}, t)\rangle = 0$.

A média do produto no terceiro termo de (2.22) pode ser escrita como

$$\begin{aligned} \left\langle \frac{\partial N_\sigma}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}_\sigma^m}{m_\sigma} \right\rangle &= \left\langle \frac{\partial}{\partial \mathbf{v}} \left(\delta N_\sigma + \langle N_\sigma \rangle\right) \cdot \frac{1}{m_\sigma} \left(\delta \mathbf{F}_\sigma^m + \langle \mathbf{F}_\sigma^m \rangle\right) \right\rangle \\ &= \left\langle \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \frac{\delta \mathbf{F}_\sigma^m}{m_\sigma} \right\rangle + \frac{\partial \langle N_\sigma \rangle}{\partial \mathbf{v}} \cdot \frac{\langle \mathbf{F}_\sigma^m \rangle}{m_\sigma} \; . \end{aligned}$$

Com $\mathbf{F}_\sigma^m = q_\sigma \mathbf{E}^m$ a equação (2.22) é dada por

$$\begin{aligned} \frac{\partial f_\sigma}{\partial t} + \frac{\partial f_\sigma}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q_\sigma}{m_\sigma} \mathbf{E} \cdot \frac{\partial f_\sigma}{\partial \mathbf{v}} &= -\frac{q_\sigma}{m_\sigma} \left\langle \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle \\ &- \int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a)\big(f_d(\mathcal{X}', t) f_\sigma(X, t) + \big\langle \delta N_d(\mathcal{X}', t) \delta N_\sigma(X, t) \big\rangle\big) \; , \end{aligned} \tag{2.28}$$

de tal modo que, subtraindo a equação (2.28) de (2.11),

$$\frac{\partial \delta N_\sigma}{\partial t} + \frac{\partial \delta N_\sigma}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q_\sigma}{m_\sigma} \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \mathbf{E} + \frac{q_\sigma}{m_\sigma} \frac{\partial f_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m = -\frac{q_\sigma}{m_\sigma} \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m + \frac{q_\sigma}{m_\sigma} \left\langle \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle$$
$$- \int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) \big(\delta N_d(\mathcal{X}', t) \delta N_\sigma(X, t) + \delta N_d(\mathcal{X}', t) f_\sigma(X, t)$$
$$+ f_d(\mathcal{X}', t) \delta N_\sigma(X, t) - \big\langle \delta N_d(\mathcal{X}', t) \delta N_\sigma(X, t) \big\rangle \big) \ . \quad (2.29)$$

Podemos reduzir a notação na equação (2.29) notando que a integral no lado direito pode ser reescrita identificando algumas grandezas com dimensões de campo elétrico e frequência,

$$\int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) \delta N_d(\mathcal{X}', t) \delta N_\sigma(X, t)$$
$$= \frac{q_\sigma}{m_\sigma v_*} \frac{m_\sigma v_*}{q_\sigma} \int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) \delta N_d(\mathcal{X}', t) \delta N_\sigma(X, t)$$
$$= \frac{q_\sigma}{m_\sigma v_*} \delta N_\sigma(X, t) \delta E_\sigma(X, t) \ ,$$
$$\int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) \delta N_d(\mathcal{X}', t) f_\sigma(X, t) = \frac{q_\sigma}{m_\sigma v_*} f_\sigma(X, t) \delta E_\sigma(X, t) \ ,$$
$$\int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) f_d(\mathcal{X}', t) \delta N_\sigma(X, t) = \omega_\sigma(X, t) \delta N_\sigma(X, t) \ ,$$
$$\int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) \big\langle \delta N_d(\mathcal{X}', t) \delta N_\sigma(X, t) \big\rangle = \frac{q_\sigma}{m_\sigma v_*} \big\langle \delta N_\sigma(X, t) \delta E_\sigma(X, t) \big\rangle \ ,$$

onde

$$\delta E_\sigma(X, t) = \frac{m_\sigma v_*}{q_\sigma} \int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) \delta N_d(\mathcal{X}', t) \ , \quad (2.30)$$

$$\omega_\sigma(X, t) = \int d\mathcal{X}' \Omega(\mathbf{r} - \mathbf{r}', \mathbf{v} - \mathbf{v}', a) f_d(\mathcal{X}', t) \ . \quad (2.31)$$

A velocidade v_* foi introduzida como uma velocidade característica do problema a ser analisado. Por exemplo, no caso de ondas eletrostáticas, pode ser conveniente escolher a velocidade térmica dos elétrons.

Então

$$\frac{\partial \delta N_\sigma}{\partial t} + \frac{\partial \delta N_\sigma}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q_\sigma}{m_\sigma}\frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \mathbf{E} + \frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m$$
$$= -\frac{q_\sigma}{m_\sigma}\frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m + \frac{q_\sigma}{m_\sigma}\left\langle \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle - \frac{q_\sigma}{m_\sigma v_*}\big(\delta N_\sigma(X,t)\delta E_\sigma(X,t)$$
$$- \big\langle \delta N_\sigma(X,t)\delta E_\sigma(X,t)\big\rangle + f_\sigma(X,t)\delta E_\sigma(X,t)\big) - \omega_\sigma(X,t)\delta N_\sigma(X,t) \ . \quad (2.32)$$

Para a componente da poeira temos que $\mathbf{F}_d^m = q\mathbf{E}^m$,

$$\left\langle \frac{\partial N_d}{\partial \mathbf{v}} \cdot \frac{q\mathbf{E}^m}{m_d} \right\rangle = \left\langle \frac{\partial}{\partial \mathbf{v}} \left(\delta N_d + \langle N_d \rangle\right) \cdot \frac{q}{m_d} \left(\delta \mathbf{E}^m + \langle \mathbf{E}^m \rangle\right) \right\rangle$$

$$= \frac{q}{m_d}\left\langle \frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle + \frac{q}{m_d}\mathbf{E} \cdot \frac{\partial f_d}{\partial \mathbf{v}} \ ,$$

e então

$$\frac{\partial f_d}{\partial t} + \frac{\partial f_d}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q}{m_d}\mathbf{E} \cdot \frac{\partial f_d}{\partial \mathbf{v}} = -\frac{q}{m_d}\left\langle \frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle$$
$$- I(X,t)\frac{\partial}{\partial q} f_d(\mathcal{X},t) - \frac{\partial}{\partial q}\big\langle \delta I^m(X,t)\delta N_d(\mathcal{X},t)\big\rangle \ . \quad (2.33)$$

Subtraindo a equação (2.33) de (2.14),

$$\frac{\partial \delta N_d}{\partial t} + \frac{\partial \delta N_d}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q}{m_d}\frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \mathbf{E} + \frac{q}{m_d}\frac{\partial f_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m = -\frac{q}{m_d}\frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m + \left\langle \frac{q}{m_d}\frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle$$
$$- \delta I^m \frac{\partial}{\partial q}\delta N_d - I\frac{\partial}{\partial q}\delta N_d - \delta I^m \frac{\partial}{\partial q} f_d + \frac{\partial}{\partial q}\big\langle \delta I^m(X,t)\delta N_d(\mathcal{X},t)\big\rangle \ . \quad (2.34)$$

Assim, o sistema de equações que determina a evolução temporal do plasma, considerando que o campo elétrico médio é nulo, $\mathbf{E} = 0$, pode ser descrito pelo conjunto

$$\frac{\partial f_\sigma(X,t)}{\partial t} + \frac{\partial f_\sigma}{\partial \mathbf{r}} \cdot \mathbf{v} = -\frac{q_\sigma}{m_\sigma}\left\langle \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle - \frac{q_\sigma}{m_\sigma v_*}\big\langle \delta N_\sigma \delta E_\sigma \big\rangle - \omega_\sigma f_\sigma \ , \quad (2.35)$$

$$\frac{\partial f_d(\mathcal{X},t)}{\partial t} + \frac{\partial f_d}{\partial \mathbf{r}} \cdot \mathbf{v} = -\frac{q}{m_d}\left\langle \frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle - \frac{\partial}{\partial q}\langle \delta I^m \delta N_d \rangle - I\frac{\partial f_d}{\partial q} \ , \quad (2.36)$$

$$\frac{\partial \delta N_\sigma(X,t)}{\partial t} + \frac{\partial \delta N_\sigma}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m = -\frac{q_\sigma}{m_\sigma}\frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m + \frac{q_\sigma}{m_\sigma}\left\langle \frac{\partial \delta N_\sigma}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle$$
$$- \frac{q_\sigma}{m_\sigma v_*}\delta N_\sigma \delta E_\sigma + \frac{q_\sigma}{m_\sigma v_*}\left\langle \delta N_\sigma \delta E_\sigma \right\rangle - \frac{q_\sigma}{m_\sigma v_*} f_\sigma \delta E_\sigma - \omega_\sigma \delta N_\sigma \ , \quad (2.37)$$

$$\frac{\partial \delta N_d(\mathcal{X},t)}{\partial t} + \frac{\partial \delta N_d}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{q}{m_d}\frac{\partial f_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m = -\frac{q}{m_d}\frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m + \left\langle \frac{q}{m_d}\frac{\partial \delta N_d}{\partial \mathbf{v}} \cdot \delta \mathbf{E}^m \right\rangle$$
$$- \delta I^m \frac{\partial}{\partial q}\delta N_d - I\frac{\partial}{\partial q}\delta N_d - \delta I^m \frac{\partial}{\partial q} f_d + \frac{\partial}{\partial q}\langle \delta I^m \delta N_d \rangle \ , \quad (2.38)$$

$$\delta I^m(X,t) = \sum_\sigma q_\sigma \int dX' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) \ \delta N_\sigma(X',t) \ , \quad (2.39)$$

$$I(X,t) = \sum_\sigma q_\sigma \int dX' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) f_\sigma(X',t) \ , \quad (2.40)$$

$$\omega_\sigma(X,t) = \int d\mathcal{X}' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) f_d(\mathcal{X}',t) \ , \quad (2.41)$$

$$\delta E_\sigma(X,t) = \frac{m_\sigma v_*}{q_\sigma} \int d\mathcal{X}' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) \delta N_d(\mathcal{X}',t) \ , \quad (2.42)$$

$$\frac{\partial}{\partial \mathbf{r}} \cdot \delta \mathbf{E}^m(\mathbf{r},t) = 4\pi \left(\sum_\sigma q_\sigma \int d\mathbf{v} \ \delta N_\sigma + \int dq \ q \int d\mathbf{v} \ \delta N_d \right) \ . \quad (2.43)$$

A dependência sobre a velocidade das equações (2.39) e (2.42) traz uma complicação no cálculo das correlações que apresentamos no Capítulo (4). Ela pode ser contornada tomando valores médios sobre a velocidade das equações (2.39) e (2.42). Com estas considerações obtemos

$$\delta I^m(X,t) \rightarrow \delta I^m(\mathbf{r},t) = \frac{\int d\mathbf{v}dq \ \delta I^m(X,t) f_d}{\int d\mathbf{v}dq \ f_d}$$

$$= \frac{1}{n_d} \int d\mathbf{v}dq \ f_d \sum_\sigma q_\sigma \int dX' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) \ \delta N_\sigma(X',t) \ ,$$

$$\delta E_\sigma(X,t) \rightarrow \delta E_\sigma(\mathbf{r},t) = \frac{\int d\mathbf{v} \ \delta E_\sigma(X,t) f_\sigma}{\int d\mathbf{v} \ f_\sigma}$$

$$= \frac{m_\sigma v_*}{n_\sigma q_\sigma} \int d\mathbf{v} \ f_\sigma \int d\mathcal{X}' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) \delta N_d(\mathcal{X}',t) \ ,$$

onde $n_d(\mathbf{r},t) = \int d\mathbf{v}dq \ f_d$ e $n_\sigma(\mathbf{r},t) = \int d\mathbf{v} \ f_\sigma$ são o número de partículas de poeira e de partículas do plasma por unidade de volume respectivamente. Também procedemos da mesma forma com I e ω_σ, equações (2.40) e (2.41),

$$I(X,t) \approx I(\mathbf{r},t) = \frac{1}{n_d}\int d\mathbf{v}dq\ f_d \sum_\sigma q_\sigma \int dX'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a) f_\sigma(X',t)\ ,$$

$$\omega_\sigma(X,t) \approx \omega_\sigma(\mathbf{r},t) = \frac{1}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathcal{X}'\Omega(\mathbf{r}-\mathbf{r}',\mathbf{v}-\mathbf{v}',a) f_d(\mathcal{X}',t)\ .$$

A rigor, o sistema de equações (2.35)-(2.43) não está fechado, visto que os termos de correlações binárias do tipo $\langle \delta N_1 \delta N_2 \rangle$ também requerem uma equação cinética própria. Mesmo se as escrevêssemos, estas dependeriam de correlações de ordem superior, e assim em diante, formando uma cadeia de equações que caracteriza a hierarquia BBGKY. Em vez disto, adotaremos nos próximos capítulos uma abordagem perturbativa expandindo as flutuações das distribuições das partículas em potências dos campos. Isto nos permitirá dar fechamento à cadeia de equações em termos das correlações quadráticas dos campos, supondo que as quantidades flutuantes, por exemplo, δI^m, são suficientemente pequenas de modo a validar a aproximação.

Capítulo 3

Solução das equações cinéticas

Dando prosseguimento ao desenvolvimento, faremos neste capítulo uma análise de Fourier do sistema de equações obtido, transformando as esquações pertinentes para o espaço de vetores de onda e frequência. Neste procedimento, preservaremos os termos lineares e não lineares das equações, aqueles que contêm produtos das quantidades flutuantes até segunda ordem. Isto nos permitirá, então, considerar efeitos de interações não lineares.

Para simplificar a notação removemos o símbolo δ e o índice m das equações.

Começamos considerando a equação (2.42) que, após realizada a média sobre a velocidade, é dada por

$$E_\sigma(\mathbf{r},t) = \frac{m_\sigma v_*}{n_\sigma q_\sigma} \int d\mathbf{v}\ f_\sigma \int d\mathcal{X}' \Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) N_d(\mathcal{X}', t)\ . \tag{3.1}$$

Multiplicando (3.1) por $(2\pi)^{-3}\exp(-i\mathbf{k}.\mathbf{r})$ e integrando em $\mathbf{r}$,

$$\begin{aligned}\frac{1}{(2\pi)^3}\int & d\mathbf{r} e^{-i\mathbf{k}.\mathbf{r}} E_\sigma(\mathbf{r},t) \\ &= \frac{m_\sigma v_*}{n_\sigma q_\sigma}\frac{1}{(2\pi)^3}\int d\mathbf{v}\ f_\sigma(\mathbf{v},t)\int d\mathbf{r} e^{-i\mathbf{k}.\mathbf{r}}\int d\mathcal{X}'\Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) N_d(\mathcal{X}', t)\ .\end{aligned}$$

Do lado direito desta equação temos a convolução

$$\int d\mathbf{r} e^{-i\mathbf{k}.\mathbf{r}}\int d\mathbf{r}'\Omega(\mathbf{r}-\mathbf{r}', \mathbf{v}-\mathbf{v}', a) N_d(\mathcal{X}', t) = (2\pi)^6 \Omega_\mathbf{k}(\mathbf{v}-\mathbf{v}', a) N^d_\mathbf{k}(\mathbf{v}', q', t)\ ,$$

onde

$$\Omega_\mathbf{k}(\mathbf{v}-\mathbf{v}', a) = \frac{1}{(2\pi)^3}\int d\mathbf{r}\Omega(\mathbf{r}, \mathbf{v}-\mathbf{v}', a) e^{-i\mathbf{k}.\mathbf{r}}\ , \tag{3.2}$$

$$N^d_{\mathbf{k}}(\mathbf{v}', q', t) = \frac{1}{(2\pi)^3}\int d\mathbf{r} N_d(\mathbf{r}, \mathbf{v}', q', t) e^{-i\mathbf{k}.\mathbf{r}} \ .$$

Então

$$\begin{aligned} E^\sigma_{\mathbf{k}}(t) &= \frac{m_\sigma v_*}{n_\sigma q_\sigma} 8\pi^3 \int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) N^d_{\mathbf{k}}(\mathbf{v}', q', t) \\ &= -\frac{4\pi i}{k}\frac{q_\sigma}{n_\sigma} \int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) N^d_{\mathbf{k}}(\mathbf{v}', q', t) \ . \end{aligned} \tag{3.3}$$

onde definimos a função adimensional

$$\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) = 2\pi^2 i \frac{m_\sigma v_* k}{q^2_\sigma}\Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) = 8\pi^3 i \frac{n_\sigma v_* k}{\omega^2_{p\sigma}}\Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) \ . \tag{3.4}$$

O procedimento para a equação (2.39) é o mesmo, de modo que

$$\begin{aligned} I_{\mathbf{k}}(t) &= \frac{8\pi^3}{n_d}\int d\mathbf{v} dq\ f_d \sum_\sigma q_\sigma \int d\mathbf{v}' \Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) N^\sigma_{\mathbf{k}}(\mathbf{v}', t) \\ &= \sum_\sigma q_\sigma \frac{4\pi q^2_\sigma}{m_\sigma v_* k} 2\pi^2 \frac{m_\sigma v_* k}{q^2_\sigma}\frac{1}{n_d}\int d\mathbf{v} dq\ f_d \int d\mathbf{v}' \Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) N^\sigma_{\mathbf{k}}(\mathbf{v}', t) \\ &= -\frac{4\pi i}{k}\sum_\sigma q_\sigma \frac{q^2_\sigma}{m_\sigma v_*}\frac{1}{n_d}\int d\mathbf{v} dq\ f_d \int d\mathbf{v}' \Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) N^\sigma_{\mathbf{k}}(\mathbf{v}', t) \ . \end{aligned} \tag{3.5}$$

Supondo que as distribuições f_σ and f_d são uniformes no espaço, consideramos que também o são as quantidades que dependem delas de forma integral. Assim, as equações (2.40) e (2.41) tornam-se

$$I(t) = \frac{1}{n_d}\int d\mathbf{v} dq\ f_d \sum_\sigma q_\sigma \int d\mathbf{r}' d\mathbf{v}' \delta(r'-a) |(\mathbf{v}-\mathbf{v}')\cdot \mathbf{e}_{\mathbf{r}'}| f_\sigma(\mathbf{v}', t) \ , \tag{3.6}$$

$$\omega_\sigma(t) = \frac{1}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathbf{r}' d\mathbf{v}' dq' |(\mathbf{v}-\mathbf{v}')\cdot \mathbf{e}_{\mathbf{r}'}| \delta(r'-a) f_d(\mathbf{v}', q', t) \ . \tag{3.7}$$

As expressões de (3.6) e (3.7) integradas, assim como de $\Omega_{\mathbf{k}}$, equação (3.2), estão no Apêndice A.

Escrevendo o lado direito das equações cinéticas de f_σ e f_d, (2.35) e (2.36), em termos de suas transformadas de Fourier,

$$\begin{aligned} \frac{\partial f_\sigma(\mathbf{v}, t)}{\partial t} &= -\frac{q_\sigma}{m_\sigma}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial \mathbf{v}}\langle N^\sigma_{\mathbf{k}} E_{\mathbf{k}'}\rangle \\ &\quad - \frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\langle N^\sigma_{\mathbf{k}} E^\sigma_{\mathbf{k}'}\rangle - \omega_\sigma f_\sigma \ , \end{aligned} \tag{3.8}$$

$$\frac{\partial f_d(\mathbf{v},q,t)}{\partial t} = -\frac{q}{m_d}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial \mathbf{v}}\big\langle N^d_{\mathbf{k}}E_{\mathbf{k}'}\big\rangle - \frac{\partial}{\partial q}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\big\langle N^d_{\mathbf{k}}I_{\mathbf{k}'}\big\rangle - I\frac{\partial f_d}{\partial q}\ . \quad (3.9)$$

Da equação para a flutuação N_σ, (2.37),

$$\int d\mathbf{k}\bigg(\frac{\partial N^\sigma_{\mathbf{k}}}{\partial t} + \frac{\partial N^\sigma_{\mathbf{k}}}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}\bigg)e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= \int d\mathbf{k}'\int d\mathbf{k}'' e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}}$$

$$\times\bigg(- \frac{q_\sigma}{m_\sigma}\frac{\partial N^\sigma_{\mathbf{k}''}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'} + \frac{q_\sigma}{m_\sigma}\bigg\langle\frac{\partial N^\sigma_{\mathbf{k}''}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\bigg\rangle\bigg)$$

$$- \int d\mathbf{k}\frac{q_\sigma}{m_\sigma v_*}f_\sigma E^\sigma_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} - \int d\mathbf{k}\ \omega_\sigma N^\sigma_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\int d\mathbf{k}''\left(E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}''} - \big\langle E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}''}\big\rangle\right)e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}}\ .$$

Fazendo $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, obtemos

$$\int d\mathbf{k}\bigg(\frac{\partial N^\sigma_{\mathbf{k}}}{\partial t} + i(\mathbf{k}\cdot\mathbf{v})N^\sigma_{\mathbf{k}} + \omega_\sigma N^\sigma_{\mathbf{k}} + \frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}\bigg)e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= \int d\mathbf{k}'\int d\mathbf{k}\bigg(- \frac{q_\sigma}{m_\sigma}\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'} + \frac{q_\sigma}{m_\sigma}\bigg\langle\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\bigg\rangle\bigg)e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$- \int d\mathbf{k}\frac{q_\sigma}{m_\sigma v_*}f_\sigma(\mathbf{v},t)E^\sigma_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\int d\mathbf{k}\left(E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'} - \big\langle E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'}\big\rangle\right)e^{i\mathbf{k}\cdot\mathbf{r}}\ ,$$

$$\frac{\partial N^\sigma_{\mathbf{k}}}{\partial t} + i(\mathbf{k}\cdot\mathbf{v})N^\sigma_{\mathbf{k}} + \omega_\sigma N^\sigma_{\mathbf{k}}$$

$$= -\frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}} - \frac{q_\sigma}{m_\sigma v_*}f_\sigma E^\sigma_{\mathbf{k}}$$

$$- \int d\mathbf{k}'\bigg(\frac{q_\sigma}{m_\sigma}\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'} - \frac{q_\sigma}{m_\sigma}\bigg\langle\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\bigg\rangle\bigg)$$

$$-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\left(E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'} - \big\langle E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'}\big\rangle\right)\ ,$$

$$
\begin{aligned}
-i\Big(i\frac{\partial}{\partial t}-\mathbf{k}\cdot\mathbf{v}+i\omega_\sigma\Big)N^\sigma_{\mathbf{k}} &= -\frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}-\frac{q_\sigma}{m_\sigma v_*}f_\sigma E^\sigma_{\mathbf{k}} \\
&\quad -\int d\mathbf{k}'\Big(\frac{q_\sigma}{m_\sigma}\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}-\frac{q_\sigma}{m_\sigma}\Big\langle\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\Big\rangle\Big) \\
&\quad -\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\left(E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'}-\langle E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'}\rangle\right) \ . \qquad (3.10)
\end{aligned}
$$

Para a equação (2.38) das partículas de poeira,

$$
\begin{aligned}
&\int d\mathbf{k}\Big(\frac{\partial N^d_{\mathbf{k}}}{\partial t}+\frac{\partial N^d_{\mathbf{k}}}{\partial \mathbf{r}}\cdot\mathbf{v}+I\frac{\partial N^d_{\mathbf{k}}}{\partial q}+\frac{q}{m_d}\frac{\partial f_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}+I_{\mathbf{k}}\frac{\partial f_d}{\partial q}\Big)e^{i\mathbf{k}\cdot\mathbf{r}} \\
&= \int d\mathbf{k}'\int d\mathbf{k}''\Big(-\frac{q}{m_d}\frac{\partial N^d_{\mathbf{k}''}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}+\frac{q}{m_d}\Big\langle\frac{\partial N^d_{\mathbf{k}''}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\Big\rangle\Big)e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}} \\
&\quad -\frac{\partial}{\partial q}\int d\mathbf{k}'\int d\mathbf{k}''\big(I_{\mathbf{k}'}N^d_{\mathbf{k}''}-\langle I_{\mathbf{k}'}N^d_{\mathbf{k}''}\rangle\big)e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}} \ ,
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{v}+I\frac{\partial}{\partial q}\Big)N^d_{\mathbf{k}} = -\frac{q}{m_d}\frac{\partial f_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}-I_{\mathbf{k}}\frac{\partial f_d}{\partial q} \\
&-\frac{\partial}{\partial q}\int d\mathbf{k}'\big(I_{\mathbf{k}'}N^d_{\mathbf{k}-\mathbf{k}'}-\langle I_{\mathbf{k}'}N^d_{\mathbf{k}-\mathbf{k}'}\rangle\big)-\int d\mathbf{k}'\Big(\frac{q}{m_d}\frac{\partial N^d_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}-\frac{q}{m_d}\Big\langle\frac{\partial N^d_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\Big\rangle\Big) \ .
\end{aligned}
$$

Para o campo elétrico efetivo do plasma, segue da lei de Gauss, equação (2.43), que

$$
E_{\mathbf{k}} = -\frac{4\pi i}{k}\Big(\sum_\sigma q_\sigma\int d\mathbf{v}\ N^\sigma_{\mathbf{k}}+\int dq\ q\int d\mathbf{v}\ N^d_{\mathbf{k}}\Big) \ , \qquad (3.11)
$$

onde $k=|\mathbf{k}|$.

Resumindo até este ponto, escrevemos o conjunto de equações transformado,

$$
\begin{aligned}
\frac{\partial f_\sigma(\mathbf{v},t)}{\partial t} &= -\frac{q_\sigma}{m_\sigma}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial \mathbf{v}}\langle N^\sigma_{\mathbf{k}}E_{\mathbf{k}'}\rangle \\
&\quad -\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\langle N^\sigma_{\mathbf{k}}E^\sigma_{\mathbf{k}'}\rangle-\omega_\sigma f_\sigma \ ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_d(\mathbf{v},q,t)}{\partial t} &= -\frac{q}{m_d}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial \mathbf{v}}\langle N^d_{\mathbf{k}}E_{\mathbf{k}'}\rangle \\
&\quad -\frac{\partial}{\partial q}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\langle N^d_{\mathbf{k}}I_{\mathbf{k}'}\rangle-I\frac{\partial f_d}{\partial q} \ ,
\end{aligned}
$$

$$
\begin{aligned}
-i\Big(i\frac{\partial}{\partial t}-\mathbf{k}\cdot\mathbf{v}+i\omega_\sigma\Big)N^\sigma_{\mathbf{k}} &= -\frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}-\frac{q_\sigma}{m_\sigma v_*}f_\sigma E^\sigma_{\mathbf{k}} \\
&\quad -\int d\mathbf{k}'\Big(\frac{q_\sigma}{m_\sigma}\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}-\frac{q_\sigma}{m_\sigma}\Big\langle\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\Big\rangle\Big) \\
&\quad -\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\left(E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'}-\left\langle E^\sigma_{\mathbf{k}'}N^\sigma_{\mathbf{k}-\mathbf{k}'}\right\rangle\right) \ ,
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{v}+I\frac{\partial}{\partial q}\Big)N^d_{\mathbf{k}} = -\frac{q}{m_d}\frac{\partial f_d}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}-I_{\mathbf{k}}\frac{\partial f_d}{\partial q} \\
&-\frac{\partial}{\partial q}\int d\mathbf{k}'\big(I_{\mathbf{k}'}N^d_{\mathbf{k}-\mathbf{k}'}-\langle I_{\mathbf{k}'}N^d_{\mathbf{k}-\mathbf{k}'}\rangle\big)-\int d\mathbf{k}'\Big(\frac{q}{m_d}\frac{\partial N^d_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}-\frac{q}{m_d}\Big\langle\frac{\partial N^d_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\Big\rangle\Big) \ ,
\end{aligned}
$$

$$
I_{\mathbf{k}}(t) = -\frac{4\pi i}{k}\sum_\sigma q_\sigma\frac{q_\sigma^2}{m_\sigma v_*}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d\int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)N^\sigma_{\mathbf{k}}(\mathbf{v}',t) \ ,
$$

$$
I(t) = \frac{1}{n_d}\int d\mathbf{v}dq\ f_d\sum_\sigma q_\sigma\int d\mathbf{r}'d\mathbf{v}'\delta(r'-a)|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}|f_\sigma(\mathbf{v}',t) \ ,
$$

$$
\omega_\sigma(t) = \frac{1}{n_\sigma}\int d\mathbf{v}\ f_\sigma\int d\mathbf{r}'d\mathbf{v}'dq'|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}|\delta(r'-a)f_d(\mathbf{v}',q',t) \ ,
$$

$$
E^\sigma_{\mathbf{k}}(t) = -\frac{4\pi i}{k}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma\int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)N^d_{\mathbf{k}}(\mathbf{v}',q',t) \ ,
$$

$$
E_{\mathbf{k}} = -\frac{4\pi i}{k}\Big(\sum_\sigma q_\sigma\int d\mathbf{v}\ N^\sigma_{\mathbf{k}}+\int dq\ q\int d\mathbf{v}\ N^d_{\mathbf{k}}\Big) \ .
$$

Podemos considerar que este sistema de equações é composto por quantidades que evoluem de acordo com escalas de tempo diferentes. É possível realizar uma expansão considerando múltiplas escalas de tempo sobre o parâmetro de interação não linear, $\varepsilon < 1$, estendendo o número de variáveis temporais tais que [12, 14]

$$
t=\varepsilon^{-1}t' \ , \quad t'=\varepsilon^{-1}t'' \ , \quad t''=\varepsilon^{-1}t'''\dots
$$

e

$$
N_{\mathbf{k}}(t) = \varepsilon N^{(1)}_{\mathbf{k}}(t,t',t'',\dots)+\varepsilon^2 N^{(2)}_{\mathbf{k}}(t,t',t'',\dots)+\varepsilon^3 N^{(3)}_{\mathbf{k}}(t,t',t'',\dots)+\dots \ ,
$$

$$
E_{\mathbf{k}}(t) = \varepsilon E^{(1)}_{\mathbf{k}}(t,t',t'',\dots)+\varepsilon^2 E^{(2)}_{\mathbf{k}}(t,t',t'',\dots)+\varepsilon^3 E^{(3)}_{\mathbf{k}}(t,t',t'',\dots)+\dots \ ,
$$

$$\langle N_{\mathbf{k}}(t)E_{\mathbf{k}}(t)\rangle = \varepsilon\langle N_{\mathbf{k}}(t,t',t'',\dots)E_{\mathbf{k}}(t,t',t'',\dots)\rangle^{(1)} + \varepsilon^2\langle N_{\mathbf{k}}(t,t',t'',\dots)E_{\mathbf{k}}(t,t',t'',\dots)\rangle^{(2)} + \dots \ ,$$

$$f(t) = f^{(0)}(t,t',t'',\dots) + \varepsilon f^{(1)}(t,t',t'',\dots) + \varepsilon^2 f^{(2)}(t,t',t'',\dots) + \dots \ ,$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t'} + \varepsilon^2\frac{\partial}{\partial t''} + \dots \ .$$

A escala de tempo de oscilações lentas, ou longa, governa as distribuições médias de partículas e de poeira, enquanto a escala de tempo de oscilações rápidas, ou curta, é relevante para as flutuações.

Substituindo estas formas expandidas nas equações para f_σ e $N^\sigma_{\mathbf{k}}$, temos, em múltiplas escalas,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t'} + \varepsilon^2\frac{\partial}{\partial t''} + \dots\right)\left(f^{(0)}_\sigma + \varepsilon f^{(1)}_\sigma + \varepsilon^2 f^{(2)}_\sigma + \dots\right) \\
&= -\frac{q_\sigma}{m_\sigma}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\left(\varepsilon\left\langle N^\sigma_{\mathbf{k}}E_{\mathbf{k}'}\right\rangle^{(1)} + \varepsilon^2\left\langle N^\sigma_{\mathbf{k}}E_{\mathbf{k}'}\right\rangle^{(2)} + \dots\right)\right. \\
&\quad -\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\left(\varepsilon\left\langle N^\sigma_{\mathbf{k}}E^\sigma_{\mathbf{k}'}\right\rangle^{(1)} + \varepsilon^2\left\langle N^\sigma_{\mathbf{k}}E^\sigma_{\mathbf{k}'}\right\rangle^{(2)} + \dots\right) \\
&\quad -\left(\omega^{(0)}_\sigma + \varepsilon\omega^{(1)}_\sigma + \varepsilon^2\omega^{(2)}_\sigma + \dots\right)\left(f^{(0)}_\sigma + \varepsilon f^{(1)}_\sigma + \varepsilon^2 f^{(2)}_\sigma + \dots\right) \ ,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&-i\left(i\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t'} + \varepsilon^2\frac{\partial}{\partial t''} + \dots\right) - \mathbf{k}\cdot\mathbf{v} + i\left(\omega^{(0)}_\sigma + \varepsilon\omega^{(1)}_\sigma + \varepsilon^2\omega^{(2)}_\sigma + \dots\right)\right) \qquad (3.13) \\
&\times\left(\varepsilon N^{\sigma(1)}_{\mathbf{k}} + \varepsilon^2 N^{\sigma(2)}_{\mathbf{k}} + \dots\right) \\
&= -\frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial\mathbf{v}}\left(f^{(0)}_\sigma(\mathbf{v},t,t',\dots) + \varepsilon f^{(1)}_\sigma + \varepsilon^2 f^{(2)}_\sigma + \dots\right)\cdot\frac{\mathbf{k}}{k}\left(\varepsilon E^{(1)}_{\mathbf{k}'} + \varepsilon^2 E^{(2)}_{\mathbf{k}'} + \dots\right) \\
&\quad -\frac{q_\sigma}{m_\sigma v_*}\left(f^{(0)}_\sigma + \varepsilon f^{(1)}_\sigma + \varepsilon^2 f^{(2)}_\sigma + \dots\right)\left(\varepsilon E^{\sigma(1)}_{\mathbf{k}'} + \varepsilon^2 E^{\sigma(2)}_{\mathbf{k}'} + \dots\right) \\
&\quad -\int d\mathbf{k}'\left(\frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial\mathbf{v}}\left(\varepsilon N^{\sigma(1)}_{\mathbf{k}-\mathbf{k}'} + \varepsilon^2 N^{\sigma(2)}_{\mathbf{k}-\mathbf{k}'} + \dots\right)\cdot\frac{\mathbf{k}'}{k'}\left(\varepsilon E^{(1)}_{\mathbf{k}'} + \varepsilon^2 E^{(2)}_{\mathbf{k}'} + \dots\right)\right. \\
&\quad \left. -\epsilon\frac{q_\sigma}{m_\sigma}\left\langle\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\right\rangle^{(1)} - \epsilon^2\frac{q_\sigma}{m_\sigma}\left\langle\frac{\partial N^\sigma_{\mathbf{k}-\mathbf{k}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\right\rangle^{(2)} + \dots\right) \\
&\quad -\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\left(\left(\varepsilon E^{\sigma(1)}_{\mathbf{k}'} + \varepsilon^2 E^{\sigma(2)}_{\mathbf{k}'} + \dots\right)\left(\varepsilon N^{\sigma(1)}_{\mathbf{k}-\mathbf{k}'} + \varepsilon^2 N^{\sigma(2)}_{\mathbf{k}-\mathbf{k}'} + \dots\right)\right.
\end{aligned}$$

$$-\varepsilon\big\langle E^{\sigma}_{\mathbf{k}'}N^{\sigma}_{\mathbf{k}-\mathbf{k}'}\big\rangle^{(1)}+\varepsilon^2\big\langle E^{\sigma}_{\mathbf{k}'}N^{\sigma}_{\mathbf{k}-\mathbf{k}'}\big\rangle^{(2)}+\dots\Big)\ .$$

Aqui cada termo da expansão é uma função de $(t,t',t'',\dots)$.

Para as equações cinéticas de f_d e $N^d_{\mathbf{k}}$,

$$\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial t'}+\varepsilon^2\frac{\partial}{\partial t''}+\dots\right)\big(f_d^{(0)}+\varepsilon f_d^{(1)}+\varepsilon^2 f_d^{(2)}+\dots\big)$$

$$=-\frac{q}{m_d}\int d\mathbf{k}\int d\mathbf{k}'e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\big(\varepsilon\big\langle N^d_{\mathbf{k}}E_{\mathbf{k}'}\big\rangle^{(1)}+\varepsilon^2\big\langle N^d_{\mathbf{k}}E_{\mathbf{k}'}\big\rangle^{(2)}+\dots\big)$$
$$-\frac{\partial}{\partial q}\int d\mathbf{k}\int d\mathbf{k}'e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\big(\varepsilon\big\langle N^d_{\mathbf{k}}I_{\mathbf{k}'}\big\rangle^{(1)}+\varepsilon^2\big\langle N^d_{\mathbf{k}}I_{\mathbf{k}'}\big\rangle^{(2)}+\dots\big)$$
$$-\big(I^{(0)}+\varepsilon I^{(1)}+\varepsilon^2 I^{(2)}+\dots\big)\frac{\partial}{\partial q}\big(f_d^{(0)}+\varepsilon f_d^{(1)}+\varepsilon^2 f_d^{(2)}+\dots\big)\ , \tag{3.14}$$

$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial t'}+\varepsilon^2\frac{\partial}{\partial t''}+\dots\right)+i\mathbf{k}\cdot\mathbf{v}+\big(I^{(0)}+\varepsilon I^{(1)}+\varepsilon^2 I^{(2)}+\dots\big)\frac{\partial}{\partial q}\right)$$
$$\times\big(\varepsilon N^{d(1)}_{\mathbf{k}}+\varepsilon^2 N^{d(2)}_{\mathbf{k}}+\dots\big)$$
$$=-\frac{q}{m_d}\frac{\partial}{\partial\mathbf{v}}\big(f_d^{(0)}+\varepsilon f_d^{(1)}+\varepsilon^2 f_d^{(2)}+\dots\big)\cdot\frac{\mathbf{k}}{k}\big(\varepsilon E^{(1)}_{\mathbf{k}}+\varepsilon^2E^{(2)}_{\mathbf{k}}+\dots\big)$$
$$-\big(\varepsilon I^{(1)}_{\mathbf{k}'}+\varepsilon^2 I^{(2)}_{\mathbf{k}'}+\dots\big)\frac{\partial}{\partial q}\big(f_d^{(0)}+\varepsilon f_d^{(1)}+\varepsilon^2 f_d^{(2)}+\dots\big)$$
$$-\frac{\partial}{\partial q}\int d\mathbf{k}'\big(\big(\varepsilon I^{(1)}_{\mathbf{k}'}+\varepsilon^2 I^{(2)}_{\mathbf{k}'}+\dots\big)\big(\varepsilon N^{d(1)}_{\mathbf{k}-\mathbf{k}'}+\varepsilon^2 N^{d(2)}_{\mathbf{k}-\mathbf{k}'}+\dots\big)$$
$$-\big(\varepsilon\big\langle I_{\mathbf{k}'}N^d_{\mathbf{k}-\mathbf{k}'}\big\rangle^{(1)}+\varepsilon^2\big\langle I_{\mathbf{k}'}N^d_{\mathbf{k}-\mathbf{k}'}\big\rangle^{(2)}+\dots\big)\big)$$
$$-\int d\mathbf{k}'\bigg(\frac{q}{m_d}\frac{\partial}{\partial\mathbf{v}}\big(\varepsilon N^{d(1)}_{\mathbf{k}-\mathbf{k}'}+\varepsilon^2 N^{d(2)}_{\mathbf{k}-\mathbf{k}'}+\dots\big)\cdot\frac{\mathbf{k}'}{k'}\big(\varepsilon E^{(1)}_{\mathbf{k}'}+\varepsilon^2E^{(2)}_{\mathbf{k}'}+\dots\big)$$
$$-\frac{q}{m_d}\varepsilon\left\langle\frac{\partial}{\partial\mathbf{v}}N^d_{\mathbf{k}-\mathbf{k}'}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\right\rangle^{(1)}-\frac{q}{m_d}\varepsilon^2\left\langle\frac{\partial}{\partial\mathbf{v}}N^d_{\mathbf{k}-\mathbf{k}'}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}\right\rangle^{(2)}\bigg)\ . \tag{3.15}$$

Consideremos agora a hierarquia em potências de ϵ da equação (3.12). Para cada ordem em ϵ temos

$$\frac{\partial}{\partial t}f^{(0)}_{\sigma}=-\omega^{(0)}_{\sigma}f^{(0)}_{\sigma}\ , \tag{3.16}$$

$$\frac{\partial}{\partial t}f^{(1)}_{\sigma}+\frac{\partial}{\partial t'}f^{(0)}_{\sigma}=-\omega^{(0)}_{\sigma}f^{(1)}_{\sigma}-\omega^{(1)}_{\sigma}f^{(0)}_{\sigma}\ , \tag{3.17}$$

$$-\frac{q_\sigma}{m_\sigma}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\big\langle N^\sigma_{\mathbf{k}}E_{\mathbf{k}'}\big\rangle^{(1)}-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}\int d\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\big\langle N^\sigma_{\mathbf{k}}E^\sigma_{\mathbf{k}'}\big\rangle^{(1)}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Em primeira ordem na hierarquia para $N^\sigma_{\mathbf{k}}$, temos

$$-i\Big(i\frac{\partial}{\partial t}-\mathbf{k}\cdot\mathbf{v}+i\omega^{(0)}_\sigma(t,t',\dots)\Big)N^{\sigma(1)}_{\mathbf{k}}(t,t',\dots)$$

$$\begin{aligned}=&-\frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial\mathbf{v}}f^{(0)}_\sigma(\mathbf{v},t,t',\dots)\cdot\frac{\mathbf{k}}{k}E^{(1)}_{\mathbf{k}'}(t,t',\dots)-\frac{q_\sigma}{m_\sigma v_*}f^{(0)}_\sigma(\mathbf{v},t,t',\dots)E^{\sigma(1)}_{\mathbf{k}'}(t,t',\dots)\\&-\int d\mathbf{k}'\Big(-\frac{q_\sigma}{m_\sigma}\Big\langle\frac{\partial}{\partial\mathbf{v}}N^\sigma_{\mathbf{k}-\mathbf{k}'}(t,t',\dots)\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}(t,t',\dots)\Big\rangle^{(1)}\Big)\\&-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\Big(-\big\langle E^\sigma_{\mathbf{k}'}(t,t',\dots)N^\sigma_{\mathbf{k}-\mathbf{k}'}(t,t',\dots)\big\rangle^{(1)}\Big)\ ,\end{aligned}\tag{3.18}$$

em segunda ordem,

$$-i\Big(i\frac{\partial}{\partial t}-\mathbf{k}\cdot\mathbf{v}+i\omega^{(0)}_\sigma(t,t',\dots)\Big)N^{\sigma(2)}_{\mathbf{k}}(t,t',\dots))-i\Big(i\frac{\partial}{\partial t'}+i\omega^{(1)}_\sigma(t,t',\dots)\Big)N^{\sigma(1)}_{\mathbf{k}}(t,t',\dots)$$

$$\begin{aligned}=&-\frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial\mathbf{v}}f^{(0)}_\sigma(\mathbf{v},t,t',\dots)\cdot\frac{\mathbf{k}}{k}E^{(2)}_{\mathbf{k}'}(t,t',\dots)-\frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial\mathbf{v}}f^{(1)}_\sigma(\mathbf{v},t,t',\dots)\cdot\frac{\mathbf{k}}{k}E^{(1)}_{\mathbf{k}'}(t,t',\dots)\\&-\frac{q_\sigma}{m_\sigma v_*}f^{(0)}_\sigma(\mathbf{v},t,t',\dots)E^{\sigma(2)}_{\mathbf{k}'}(t,t',\dots)-\frac{q_\sigma}{m_\sigma v_*}f^{(1)}_\sigma(\mathbf{v},t,t',\dots)E^{\sigma(1)}_{\mathbf{k}'}(t,t',\dots)\\&-\int d\mathbf{k}'\Big(\frac{q_\sigma}{m_\sigma}\frac{\partial}{\partial\mathbf{v}}N^{\sigma(1)}_{\mathbf{k}-\mathbf{k}'}(t,t',\dots)\cdot\frac{\mathbf{k}'}{k'}E^{(1)}_{\mathbf{k}'}(t,t',\dots)\\&-\frac{q_\sigma}{m_\sigma}\Big\langle\frac{\partial}{\partial\mathbf{v}}N^\sigma_{\mathbf{k}-\mathbf{k}'}(t,t',\dots)\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}(t,t',\dots)\Big\rangle^{(2)}\Big)\\&-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\Big(E^{\sigma(1)}_{\mathbf{k}'}(t,t',\dots)N^{\sigma(1)}_{\mathbf{k}-\mathbf{k}'}(t,t',\dots)-\big\langle E^\sigma_{\mathbf{k}'}(t,t',\dots)N^\sigma_{\mathbf{k}-\mathbf{k}'}(t,t',\dots)\big\rangle^{(2)}\Big)\ .\end{aligned}\tag{3.19}$$

Integrando (3.16) em t, obtemos

$$f^{(0)}_\sigma(t,t')=f^{(0)}_\sigma(t=0,t')e^{-\omega^{(0)}_\sigma t}\simeq f_\sigma(t')e^{-\omega_\sigma t}\ .\tag{3.20}$$

Esta solução deve então ser substituída em (3.17), resultando numa equação cinética em escala lenta para f_σ.

Em ordem zero em ε na equação para a poeira (3.14), temos

$$\frac{\partial f_d^{(0)}}{\partial t} = I^{(0)} \frac{\partial}{\partial q} f_d^{(0)} \ , \tag{3.21}$$

cuja solução é dada por $f_d^{(0)}(t,t') = F(t', q + I^{(0)}t)$, onde F é uma função arbitrária de $q + I^{(0)}t$. Podemos escrever, por exemplo, que

$$f_d^{(0)}(t,t') = f_d(t') \exp\big(-(q + It)/e \big) \ . \tag{3.22}$$

Integrando a equação (3.17) em t, obtemos a solução

$$f_\sigma^{(1)}(t,t') = -t \frac{\partial f_\sigma^{(0)}(t')}{\partial t'} - \int_0^t dt\ \omega_\sigma^{(1)}(t,t') f_\sigma^{(0)}(t')$$

$$-\frac{q_\sigma}{m_\sigma} \int_0^t dt \int d\mathbf{k} d\omega \int d\mathbf{k}' d\omega' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} e^{-i(\omega+\omega')t} \frac{\mathbf{k}'}{k'} \cdot \frac{\partial}{\partial \mathbf{v}} \big\langle N_{\mathbf{k}\omega}^\sigma E_{\mathbf{k}'\omega'} \big\rangle^{(1)}$$

$$-\frac{q_\sigma}{m_\sigma v_*} \int_0^t dt \int d\mathbf{k} d\omega \int d\mathbf{k}' d\omega' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} e^{-i(\omega+\omega')t} \big\langle N_{\mathbf{k}\omega}^\sigma E_{\mathbf{k}'\omega'}^\sigma \big\rangle^{(1)} \ , \tag{3.23}$$

onde escrevemos os cumulantes em termos de suas transformadas de Laplace. As duas últimas integrações em t só são não-nulas para $\omega = -\omega'$, de modo que, neste caso, as integrais resultantes são proporcionais a t. Fazendo a aproximação $\omega_\sigma^{(1)}(t,t') \simeq \omega_\sigma^{(0)}(t')$ e requerendo $f_\sigma^{(1)}(t,t') = 0$ no limite $t \to \infty$, obtemos do lado direito de (3.23) a solução

$$\begin{aligned}\frac{\partial f_\sigma(\mathbf{v},t')}{\partial t'} = &-\frac{q_\sigma}{m_\sigma} \int d\mathbf{k} d\omega \int d\mathbf{k}' d\omega' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} e^{-i(\omega+\omega')t} \frac{\mathbf{k}'}{k'} \cdot \frac{\partial}{\partial \mathbf{v}} \big\langle N_{\mathbf{k}\omega}^\sigma(t') E_{\mathbf{k}'\omega'}(t') \big\rangle \\ &- \frac{q_\sigma}{m_\sigma v_*} \int d\mathbf{k} d\omega \int d\mathbf{k}' d\omega' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} e^{-i(\omega+\omega')t} \big\langle N_{\mathbf{k}\omega}^\sigma(t') E_{\mathbf{k}'\omega'}^\sigma(t') \big\rangle - \omega_\sigma(t') f_\sigma(t') \ .\end{aligned}$$

Procedendo da mesma forma para a componente da poeira, com a aproximação $|I|t \ll |q|$, obtemos

$$\begin{aligned}\frac{\partial f_d(\mathbf{v},q,t')}{\partial t'} = &-\frac{q}{m_d} \int d\mathbf{k} d\omega \int d\mathbf{k}' d\omega' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} e^{-i(\omega+\omega')t} \frac{\mathbf{k}'}{k'} \cdot \frac{\partial}{\partial \mathbf{v}} \big\langle N_{\mathbf{k}\omega}^d(t') E_{\mathbf{k}'\omega'}(t') \big\rangle \\ &- \frac{\partial}{\partial q} \int d\mathbf{k} d\omega \int d\mathbf{k}' d\omega' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} e^{-i(\omega+\omega')t} \big\langle N_{\mathbf{k}\omega}^d(t') I_{\mathbf{k}'\omega'}(t') \big\rangle - I(t') \frac{\partial f_d(t')}{\partial q} \ ,\end{aligned}$$

Para $N_{\mathbf{k}}^\sigma$, somamos (3.18) com (3.19) em termos de $N_{\mathbf{k}}^{\sigma(1)}$, desprezando termos proporcionais a $N_{\mathbf{k}}^{\sigma(2)}$, obtendo

$$
\begin{aligned}
-i\Big(i\frac{\partial}{\partial t}+i\frac{\partial}{\partial t'}-\mathbf{k}\cdot\mathbf{v}+i\omega_\sigma\Big)N_{\mathbf{k}}^{\sigma}(t,t') &= -\frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma(t')}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}(t,t')-\frac{q_\sigma}{m_\sigma v_*}f_\sigma(t')E_{\mathbf{k}}^{\sigma}(t,t')\\
&-\int d\mathbf{k}'\Big(\frac{q_\sigma}{m_\sigma}\frac{\partial N_{\mathbf{k}-\mathbf{k}'}^{\sigma}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}(t,t')-\frac{q_\sigma}{m_\sigma}\Big\langle\frac{\partial N_{\mathbf{k}-\mathbf{k}'}^{\sigma}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}(t,t')\Big\rangle\Big)\\
&-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{k}'\left(E_{\mathbf{k}'}^{\sigma}(t,t')N_{\mathbf{k}-\mathbf{k}'}^{\sigma}-\left\langle E_{\mathbf{k}'}^{\sigma}(t,t')N_{\mathbf{k}-\mathbf{k}'}^{\sigma}\right\rangle\right)\ ,
\end{aligned}
$$

Para $N_{\mathbf{k}}^{d}(t,t')$ obtemos

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}+i\mathbf{k}\cdot\mathbf{v}+I\frac{\partial}{\partial q}\Big)N_{\mathbf{k}}^{d}(t,t') &= -\frac{q}{m_d}\frac{\partial f_d(t')}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{k}}(t,t')-I_{\mathbf{k}}(t,t')\frac{\partial f_d(t')}{\partial q}\\
&-\frac{\partial}{\partial q}\int d\mathbf{k}'\big(I_{\mathbf{k}'}(t,t')N_{\mathbf{k}-\mathbf{k}'}^{d}-\langle I_{\mathbf{k}'}(t,t')N_{\mathbf{k}-\mathbf{k}'}^{d}\rangle\big)\\
&-\int d\mathbf{k}'\Big(\frac{q}{m_d}\frac{\partial N_{\mathbf{k}-\mathbf{k}'}^{d}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}(t,t')-\frac{q}{m_d}\Big\langle\frac{\partial N_{\mathbf{k}-\mathbf{k}'}^{d}}{\partial \mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{k}'}(t,t')\Big\rangle\Big)\ ,
\end{aligned}
$$

onde, como consequência, teremos

$$
I_{\mathbf{k}}(t,t') = -\frac{4\pi i}{k}\sum_\sigma q_\sigma\frac{q_\sigma^2}{m_\sigma v_*}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d\int d\mathbf{v}'\Omega_{\mathbf{k}}^{\sigma}(\mathbf{v}-\mathbf{v}',a)N_{\mathbf{k}}^{\sigma}(\mathbf{v}',t,t')\ ,
$$

$$
I(t') = \frac{1}{n_d}\int d\mathbf{v}dq\ f_d\sum_\sigma q_\sigma\int d\mathbf{r}'d\mathbf{v}'\delta(r'-a)|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}|f_\sigma(\mathbf{v}',t')\ ,
$$

$$
\omega_\sigma(t') = \frac{1}{n_\sigma}\int d\mathbf{v}\ f_\sigma\int d\mathbf{r}'d\mathbf{v}'dq'|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}|\delta(r'-a)f_d(\mathbf{v}',q',t')\ ,
$$

$$
E_{\mathbf{k}}^{\sigma}(t,t') = -\frac{4\pi i}{k}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma\int d\mathbf{v}'dq'\Omega_{\mathbf{k}}^{\sigma}(\mathbf{v}-\mathbf{v}',a)N_{\mathbf{k}}^{d}(\mathbf{v}',q',t,t')\ ,
$$

$$
E_{\mathbf{k}}(t,t') = -\frac{4\pi i}{k}\Big(\sum_\sigma q_\sigma\int d\mathbf{v}\ N_{\mathbf{k}}^{\sigma}(t,t')+\int dq\ q\int d\mathbf{v}\ N_{\mathbf{k}}^{d}(t,t')\Big)\ .
$$

Agora tomamos a transformada de Laplace sobre a escala de tempo curta de todas as equações, com exceção das equações cinéticas de f_σ e f_d porque já o fizemos e reproduzimos aqui novamente,

$$
\begin{aligned}
\frac{\partial f_\sigma(\mathbf{v},t')}{\partial t'} &= -\frac{q_\sigma}{m_\sigma}\int d\mathbf{n}\int d\mathbf{n}'\ \frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial \mathbf{v}}\left\langle N_{\mathbf{n}}^{\sigma}(t')E_{\mathbf{n}'}(t')\right\rangle e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}-i(\omega+\omega')t}\\
&-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{n}\int d\mathbf{n}'\ \left\langle N_{\mathbf{n}}^{\sigma}(t')E_{\mathbf{n}'}^{\sigma}(t')\right\rangle e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}-i(\omega+\omega')t}-\omega_\sigma(t')f_\sigma(t')\ , \quad (3.24)
\end{aligned}
$$

$$\frac{\partial f_d(\mathbf{v},q,t')}{\partial t'} = -\frac{q}{m_d}\int d\mathbf{n}\int d\mathbf{n}'\ \frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}\big\langle N^d_{\mathbf{n}}(t')E_{\mathbf{n}'}(t')\big\rangle e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}-i(\omega+\omega')t}$$
$$-\frac{\partial}{\partial q}\int d\mathbf{n}\int d\mathbf{n}'\ \big\langle N^d_{\mathbf{n}}(t')I_{\mathbf{n}'}(t')\big\rangle e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}-i(\omega+\omega')t} - I(t')\frac{\partial f_d(t')}{\partial q}\ , \quad (3.25)$$

$$-i\Big(\omega+i\frac{\partial}{\partial t'}-\mathbf{k}\cdot\mathbf{v}+i\omega_\sigma\Big)N^\sigma_{\mathbf{n}}(t') = -\frac{q_\sigma}{m_\sigma}\frac{\partial f_\sigma(t')}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{n}}(t') - \frac{q_\sigma}{m_\sigma v_*}f_\sigma(t')E^\sigma_{\mathbf{n}}(t')$$
$$-\int d\mathbf{k}'\Big(\frac{q_\sigma}{m_\sigma}\frac{\partial N^\sigma_{\mathbf{n}-\mathbf{n}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}(t') - \frac{q_\sigma}{m_\sigma}\Big\langle\frac{\partial N^\sigma_{\mathbf{n}-\mathbf{n}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}(t')\Big\rangle\Big)$$
$$-\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{n}'\ \big(E^\sigma_{\mathbf{n}'}(t')N^\sigma_{\mathbf{n}-\mathbf{n}'} - \big\langle E^\sigma_{\mathbf{n}'}(t')N^\sigma_{\mathbf{n}-\mathbf{n}'}\big\rangle\big)\ , \quad (3.26)$$

$$\Big(-i\omega+\frac{\partial}{\partial t'}+i\mathbf{k}\cdot\mathbf{v}+I\frac{\partial}{\partial q}\Big)N^d_{\mathbf{n}}(t') = -\frac{q}{m_d}\frac{\partial f_d(t')}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{n}}(t') - I_{\mathbf{n}}(t')\frac{\partial f_d(t')}{\partial q}$$
$$-\frac{\partial}{\partial q}\int d\mathbf{n}'\big(I_{\mathbf{n}'}(t')N^d_{\mathbf{n}-\mathbf{n}'} - \langle I_{\mathbf{n}'}(t')N^d_{\mathbf{n}-\mathbf{n}'}\rangle\big)$$
$$-\int d\mathbf{n}'\Big(\frac{q}{m_d}\frac{\partial N^d_{\mathbf{n}-\mathbf{n}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}(t') - \frac{q}{m_d}\Big\langle\frac{\partial N^d_{\mathbf{n}-\mathbf{n}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}(t')\Big\rangle\Big)\ , \quad (3.27)$$

$$I_{\mathbf{n}}(t') = -\frac{4\pi i}{k}\sum_\sigma q_\sigma\frac{q_\sigma^2}{m_\sigma v_*}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d\int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)N^\sigma_{\mathbf{n}}(\mathbf{v}',t')\ , \quad (3.28)$$

$$I(t') = \frac{1}{n_d}\int d\mathbf{v}dq\ f_d\sum_\sigma q_\sigma\int d\mathbf{r}'d\mathbf{v}'\delta(r'-a)|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}|f_\sigma(\mathbf{v}',t')\ ,$$

$$\omega_\sigma(t') = \frac{1}{n_\sigma}\int d\mathbf{v}\ f_\sigma\int d\mathbf{r}'d\mathbf{v}'dq'|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}|\delta(r'-a)f_d(\mathbf{v}',q',t')\ ,$$

$$E^\sigma_{\mathbf{n}}(t') = -\frac{4\pi i}{k}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma\int d\mathbf{v}'dq'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)N^d_{\mathbf{n}}(\mathbf{v}',q',t')\ , \quad (3.29)$$

$$E_{\mathbf{n}}(t') = -\frac{4\pi i}{k}\Big(\sum_\sigma q_\sigma\int d\mathbf{v}\ N^\sigma_{\mathbf{n}}(t') + \int dq\ q\int d\mathbf{v}\ N^d_{\mathbf{n}}(t')\Big)\ , \quad (3.30)$$

onde

$$A_{\mathbf{k}}(t,t') = \int_L d\omega\ A_{\mathbf{n}}(t')\exp(-i\omega t)\quad ,\quad A_{\mathbf{n}}(t') = \frac{1}{2\pi}\int_0^\infty dt\ A_{\mathbf{n}}(t,t')\exp(i\omega t)\ ,$$

e $A_{\mathbf{k}}(t,t')$ representa as quantidades $N^\sigma_{\mathbf{k}}(t,t')$, $N^d_{\mathbf{k}}(t,t')$, $E_{\mathbf{k}}(t,t')$, $E^\sigma_{\mathbf{k}}(t,t')$ e $I_{\mathbf{k}}(t,t')$. O vetor $\mathbf{n}$ representa o par $\mathbf{k}\omega$. O símbolo L indica integração sobre contorno de Landau na frequência angular. Consideraremos a solução para tempo assintótico, $t\to\infty$, de modo que as contribuições não-nulas da integral nesse contorno vem das raízes da relação de dispersão.

Introduzindo os operadores

$$\mathbf{g}^{\sigma}_{\mathbf{n}}(\mathbf{v}) = -\frac{q_\sigma}{m_\sigma}\frac{\partial/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v} + i\omega_\sigma}\ , \tag{3.31}$$

e

$$h^{\sigma}_{\mathbf{n}}(\mathbf{v}) = -\frac{q_\sigma}{m_\sigma}\frac{v_*^{-1}}{\omega - \mathbf{k}\cdot\mathbf{v} + i\omega_\sigma}\ , \tag{3.32}$$

ressaltando o fato de que $\mathbf{g}^{\sigma}_{\mathbf{n}}$ e $h^{\sigma}_{\mathbf{n}}$ estão associados com a frequência de carregamento ω_σ, a equação (3.10) pode ser escrita como

$$\begin{aligned} N^{\sigma}_{\mathbf{n}} &= i\left(\mathbf{g}^{\sigma}_{\mathbf{n}}\cdot\frac{\mathbf{k}}{k}\right)E_{\mathbf{n}}f_\sigma + ih^{\sigma}_{\mathbf{n}}E^{\sigma}_{\mathbf{n}}f_\sigma \\ &+ i\mathbf{g}^{\sigma}_{\mathbf{k}}\int d\mathbf{n}'\left(N^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'} - \left\langle N^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\right\rangle\right) \\ &+ ih^{\sigma}_{\mathbf{n}}\int d\mathbf{n}'\left(E^{\sigma}_{\mathbf{n}'}N^{\sigma}_{\mathbf{n}-\mathbf{n}'} - \left\langle E^{\sigma}_{\mathbf{n}'}N^{\sigma}_{\mathbf{n}-\mathbf{n}'}\right\rangle\right)\ . \end{aligned} \tag{3.33}$$

Para a equação de $N^{d}_{\mathbf{n}}$ temos que lidar com o operador $\partial/\partial q$ primeiro. Tentamos encontrar uma solução aproximada de sua ação considerando $N^{d}_{\mathbf{n}}$ na aproximação linear, supondo que existe uma frequência ω_d tal que

$$I\frac{\partial N^{d(1)}_{\mathbf{n}}}{\partial q} = \omega_d N^{d(1)}_{\mathbf{n}}\ . \tag{3.34}$$

Então escrevemos

$$\begin{aligned} N^{d}_{\mathbf{n}} = -i\Big(\omega - \mathbf{k}\cdot\mathbf{v} + i\omega_d\Big)^{-1}\Bigg(& I_{\mathbf{n}}\frac{\partial f_d}{\partial q} + \frac{q}{m_d}\frac{\partial f_d}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}}{k}E_{\mathbf{n}} \\ &+ \frac{\partial}{\partial q}\int d\mathbf{n}'\big(I_{\mathbf{n}'}N^{d}_{\mathbf{n}-\mathbf{n}'} - \langle I_{\mathbf{n}'}N^{d}_{\mathbf{n}-\mathbf{n}'}\rangle\big) \\ &+ \int d\mathbf{n}'\left(\frac{q}{m_d}\frac{\partial N^{d}_{\mathbf{n}-\mathbf{n}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'} - \frac{q}{m_d}\left\langle\frac{\partial N^{d}_{\mathbf{n}-\mathbf{n}'}}{\partial\mathbf{v}}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\right\rangle\right)\Bigg)\ , \end{aligned}$$

$$N_{\mathbf{n}}^{d} = i\left(\mathbf{g}_{\mathbf{n}}^{d} \cdot \frac{\mathbf{k}}{k}\right) E_{\mathbf{n}} f_d + i h_{\mathbf{n}}^{d} \hat{E}_{\mathbf{n}} f_d$$
$$+ i h_{\mathbf{n}}^{d} \int d\mathbf{n}' \big(\hat{E}_{\mathbf{n}'} N_{\mathbf{n}-\mathbf{n}'}^{d} - \big\langle \hat{E}_{\mathbf{n}'} N_{\mathbf{n}-\mathbf{n}'}^{d} \big\rangle\big)$$
$$+ i \mathbf{g}_{\mathbf{n}}^{d} \int d\mathbf{n}' \left(N_{\mathbf{n}-\mathbf{n}'}^{d} \cdot \frac{\mathbf{k}'}{k'} E_{\mathbf{n}'} - \left\langle N_{\mathbf{n}-\mathbf{n}'}^{d} \cdot \frac{\mathbf{k}'}{k'} E_{\mathbf{n}'} \right\rangle \right) , \tag{3.35}$$

onde

$$\mathbf{g}_{\mathbf{n}}^{d}(\mathbf{v}, q) = -\frac{q}{m_d} \frac{\partial/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v} + i\omega_d} , \tag{3.36}$$

$$h_{\mathbf{n}}^{d}(\mathbf{v}) = -\frac{e^2}{m_d} \frac{v_*^{-1} \partial/\partial q}{\omega - \mathbf{k}\cdot\mathbf{v} + i\omega_d} . \tag{3.37}$$

Aqui também definimos

$$\hat{E}_{\mathbf{n}} = \frac{m_d v_*}{e^2} I_{\mathbf{n}} , \tag{3.38}$$

onde $I_{\mathbf{n}}$ é dada pela equação (3.28).

3.1 Expansão das flutuações

A equação (3.33) pode ser expandida numa série,

$$N_{\mathbf{n}}^{\sigma} = \sum_{j=1}^{\infty} N_{\mathbf{n}}^{\sigma(j)} , \tag{3.39}$$

em potências das amplitudes dos campos elétricos. Temos em primeira ordem

$$N_{\mathbf{n}}^{\sigma(1)} = i\left(\mathbf{g}_{\mathbf{n}}^{\sigma} \cdot \frac{\mathbf{k}}{k}\right) E_{\mathbf{n}} f_\sigma + i h_{\mathbf{n}}^{\sigma} E_{\mathbf{n}}^{\sigma} f_\sigma ,$$

e em j-ésima ordem

$$N_{\mathbf{n}}^{\sigma(j)} = i \int d\mathbf{n}' \Bigg(\mathbf{g}_{\mathbf{n}}^{\sigma} \left(N_{\mathbf{n}-\mathbf{n}'}^{\sigma(j-1)} \cdot \frac{\mathbf{k}'}{k'} E_{\mathbf{n}'} - \left\langle N_{\mathbf{n}-\mathbf{n}'}^{\sigma(j-1)} \cdot \frac{\mathbf{k}'}{k'} E_{\mathbf{n}'} \right\rangle \right)$$
$$+ h_{\mathbf{n}}^{\sigma} \big(E_{\mathbf{n}'}^{\sigma} N_{\mathbf{n}-\mathbf{n}'}^{\sigma(j-1)} - \big\langle E_{\mathbf{n}'}^{\sigma} N_{\mathbf{n}-\mathbf{n}'}^{\sigma(j-1)} \big\rangle \big) \Bigg) . \tag{3.40}$$

Para nossos propósitos é suficiente tomar a expansão (3.39) até terceira ordem para ter em conta a interação entre as ondas e a interação não linear onda-partícula [31], de modo que, até terceira ordem, temos

$$
\begin{aligned}
N^{\sigma}_{\mathbf{n}} &= i\left(\mathbf{g}^{\sigma}_{\mathbf{n}} \cdot \frac{\mathbf{k}}{k}\right) E_{\mathbf{n}} f_{\sigma} + i h^{\sigma}_{\mathbf{n}} E^{\sigma}_{\mathbf{n}} f_{\sigma} - \int d\mathbf{n}_1 \int d\mathbf{n}_2\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2) \\
&\times \Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big) + \left(\mathbf{g}^{\sigma}_{\mathbf{n}} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^{\sigma}_{\mathbf{n}_2}\big(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\big) \\
&\quad + h^{\sigma}_{\mathbf{n}}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2} - \langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big) + h^{\sigma}_{\mathbf{n}} h^{\sigma}_{\mathbf{n}_2}\big(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2} - \langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\big)\Bigg) f_{\sigma} \\
&\quad - i \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3) \\
&\times \Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right)\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\big) \\
&\quad + \left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^{\sigma}_{\mathbf{n}_2}\big(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3} + \langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\big)\Bigg) \\
&\quad + h^{\sigma}_{\mathbf{n}}\Bigg(\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E^{\sigma}_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}\rangle\big)\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) \\
&\quad + \left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^{\sigma}_{\mathbf{n}_2}\big(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle E^{\sigma}_{\mathbf{n}_3} + \langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}\rangle\big)\Bigg) \\
&\quad + \left(\mathbf{g}^{\sigma}_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right)\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\big) \\
&\quad + h^{\sigma}_{\mathbf{n}-\mathbf{n}_3} h^{\sigma}_{\mathbf{n}_2}\big(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\big)\Bigg) \\
&\quad + h^{\sigma}_{\mathbf{n}}\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E^{\sigma}_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}\rangle\big) \\
&\quad + h^{\sigma}_{\mathbf{n}-\mathbf{n}_3} h^{\sigma}_{\mathbf{n}_2}\big(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle E^{\sigma}_{\mathbf{n}_3} - \langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}\rangle\big)\Bigg)\Bigg) f_{\sigma}\ . \qquad (3.41)
\end{aligned}
$$

A expansão da equação (3.35) desenvolve-se da mesma forma,

$$
N^{d}_{\mathbf{n}} = \sum_{j=1}^{\infty} N^{d(j)}_{\mathbf{n}}\ , \qquad (3.42)
$$

resultando em

$$
N^{d(1)}_{\mathbf{n}} = i h^{d}_{\mathbf{n}} \hat{E}_{\mathbf{n}} f_d + i\left(\mathbf{g}^{d}_{\mathbf{n}} \cdot \frac{\mathbf{k}}{k}\right) E_{\mathbf{n}} f_d,
$$

e

$$
\begin{aligned}
N^{d(j)}_{\mathbf{n}} &= i \int d\mathbf{n}' \Bigg(h^{d}_{\mathbf{n}}\big(\hat{E}_{\mathbf{n}'} N^{d(j-1)}_{\mathbf{n}-\mathbf{n}'} - \langle \hat{E}_{\mathbf{n}'} N^{d(j-1)}_{\mathbf{n}-\mathbf{n}'}\rangle\big) \\
&+ \mathbf{g}^{d}_{\mathbf{n}}\left(N^{d(j-1)}_{\mathbf{n}-\mathbf{n}'} \cdot \frac{\mathbf{k}'}{k'} E_{\mathbf{n}'} - \left\langle N^{d(j-1)}_{\mathbf{n}-\mathbf{n}'} \cdot \frac{\mathbf{k}'}{k'} E_{\mathbf{n}'}\right\rangle \right)\Bigg)\ . \qquad (3.43)
\end{aligned}
$$

Assim,

$$
\begin{aligned}
N^d_{\mathbf{n}} = {} & i h^d_{\mathbf{n}} \hat{E}_{\mathbf{n}} f_d + i \mathbf{g}^d_{\mathbf{n}} f_d \cdot \frac{\mathbf{k}}{k} E_{\mathbf{n}} - \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \\
& \times \Bigg(h^d_{\mathbf{n}} h^d_{\mathbf{n}_2} \big(\hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle\big) + \left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2} \big(\hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_1} - \big\langle \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_1} \big\rangle\big) \\
& + \big(\hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle\big) h^d_{\mathbf{n}} \left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) + \left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_1}{k_1}\right) \left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) \big(E_{\mathbf{n}_2} E_{\mathbf{n}_1} - \big\langle E_{\mathbf{n}_2} E_{\mathbf{n}_1} \big\rangle\big) \Bigg) f_d \\
& - i \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) \\
& \times \Bigg(h^d_{\mathbf{n}} \Bigg(h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2} \big(\hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle \hat{E}_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} \big\rangle\big) \\
& + \left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2} \big(E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle \hat{E}_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} \big\rangle\big) \Bigg) \\
& + \left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right) \Bigg(h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2} \big(\hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle E_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} \big\rangle\big) \\
& + \left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2} \big(E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle E_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} \big\rangle\big) \Bigg) \\
& + \Bigg(\big(\hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle \hat{E}_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} \big\rangle\big) h^d_{\mathbf{n}} h^d_{\mathbf{n}-\mathbf{n}_3} \\
& + \big(E_{\mathbf{n}_1} E_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle \hat{E}_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} \big\rangle\big) h^d_{\mathbf{n}} \left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) \\
& + \big(\hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle E_{\mathbf{n}_3} - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{\mathbf{n}_3} \big\rangle\big) \left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right) h^d_{\mathbf{n}-\mathbf{n}_3} \\
& + \big(E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle E_{\mathbf{n}_3} - \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{\mathbf{n}_3} \big\rangle\big) \left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right) \left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) \Bigg) \left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) \Bigg) f_d \, .
\end{aligned}
\tag{3.44}
$$

Os termos recursivos utilizados na obtenção de (3.41) e (3.44) estão desenvolvidos no Apêndice B.

Os diversos produtos dos campos até terceira ordem, presentes em (3.41) e (3.44), refletem a não linearidade das interações. Junto a cada termo, há um produto de operadores $\mathbf{g}$ e/ou h, que estão associados à resposta do plasma a essas interações. Dessa forma, eles representam susceptibilidades não lineares, como função dos vetores de onda e frequência dessas ondas. Essas funções-resposta serão definidas no próximo capítulo.

Capítulo 4

Densidades espectrais de energia

No Capítulo (2) introduzimos as flutuações dos campos, cujos valores médios são nulos, e no Capítulo (3) escrevemos esses campos em termos de coordenadas espectrais. Para descrevermos esses campos em conexão com as funções de distribuição médias, consideramos a densidade espectral de energia do campo elétrico, $\langle E_{\mathbf{n}}^2 \rangle$, que também pode ser definida como a correlação quadrática ou binária. Neste capítulo, lidamos com os detalhes na obtenção de $\langle E_{\mathbf{n}}^2 \rangle$.

Começamos multiplicando a equação (3.30) por $E_{-\mathbf{n}}$

$$E_{\mathbf{n}}^2 = -\frac{4\pi i}{k}\left(\sum_{\sigma} q_\sigma \int d\mathbf{v}\ N_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} + \int dq\ q \int d\mathbf{v}\ N_{\mathbf{n}}^{d} E_{-\mathbf{n}} \right), \tag{4.1}$$

onde utilizamos a propriedade $E_{-\mathbf{n}} = E_{\mathbf{n}}^*$. Tomando a média da equação (4.1),

$$\langle E_{\mathbf{n}}^2 \rangle = -\frac{4\pi i}{k}\left(\sum_{\sigma} q_\sigma \int d\mathbf{v}\ \langle N_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} \rangle + \int dq\ q \int d\mathbf{v}\ \langle N_{\mathbf{n}}^{d} E_{-\mathbf{n}} \rangle \right). \tag{4.2}$$

As médias do lado direito da equação (4.2), podem ser obtidas multiplicando (3.41) e (3.44) pela direita por $E_{-\mathbf{n}}$ e realizando a média. O resultado são as expressões

$$\begin{aligned} \langle N_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} \rangle = {} & i\mathbf{g}_{\mathbf{n}}^{\sigma} f_\sigma \cdot \frac{\mathbf{k}}{k} \langle E_{\mathbf{n}}^2 \rangle + i h_{\mathbf{n}}^{\sigma} \langle E_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} \rangle f_\sigma \\ & - \int d\mathbf{n}_1 \int d\mathbf{n}_2\, \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \\ & \times \left(\left(\mathbf{g}_{\mathbf{n}}^{\sigma} \cdot \frac{\mathbf{k}_1}{k_1} \right) \left(\mathbf{g}_{\mathbf{n}_2}^{\sigma} \cdot \frac{\mathbf{k}_2}{k_2} \right) \langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{-\mathbf{n}} \rangle + \left(\mathbf{g}_{\mathbf{n}}^{\sigma} \cdot \frac{\mathbf{k}_1}{k_1} \right) h_{\mathbf{n}_2}^{\sigma} \langle E_{\mathbf{n}_1} E_{\mathbf{n}_2}^{\sigma} E_{-\mathbf{n}} \rangle \right) f_\sigma \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&-\int d\mathbf{n}_1 \int d\mathbf{n}_2\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\Bigg(h^{\sigma}_{\mathbf{n}}\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle + h^{\sigma}_{\mathbf{n}}h^{\sigma}_{\mathbf{n}_2}\big\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle\Bigg) f_\sigma \\
&-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3) \\
&\times\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\bigg)\Bigg(\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big) \\
&+\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)h^{\sigma}_{\mathbf{n}_2}\big(\big\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)\Bigg) f_\sigma \\
&-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3) \\
&\times h^{\sigma}_{\mathbf{n}}\Bigg(\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E^{\sigma}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg) \\
&+\bigg\langle E^{\sigma}_{\mathbf{n}_3}\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)h^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{-\mathbf{n}}\bigg\rangle - \big\langle E^{\sigma}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)h^{\sigma}_{\mathbf{n}_2}\big\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\big\rangle\Bigg) f_\sigma \\
&-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3) \\
&\times\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\bigg)\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(\big\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big) \\
&+h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}h^{\sigma}_{\mathbf{n}_2}\big(\big\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\Bigg) f_\sigma \\
&-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3) \\
&\times h^{\sigma}_{\mathbf{n}}\Bigg(\Bigg(\bigg\langle E^{\sigma}_{\mathbf{n}_3}h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-\mathbf{n}}\bigg\rangle - \big\langle E^{\sigma}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\bigg(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\Bigg) \\
&+h^{\sigma}_{\mathbf{n}-\mathbf{n}_3}h^{\sigma}_{\mathbf{n}_2}\big(\big\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\big\rangle\big\langle E^{\sigma}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)\Bigg) f_\sigma \ ,
\end{aligned}
$$

e

$$
\begin{aligned}
\big\langle N^{d}_{\mathbf{n}}E_{-\mathbf{n}}\big\rangle &= ih^{d}_{\mathbf{n}}\big\langle \hat{E}_{\mathbf{n}}E_{-\mathbf{n}}\big\rangle f_d + i\mathbf{g}^{d}_{\mathbf{n}} f_d\cdot\frac{\mathbf{k}}{k}\big\langle E^2_{\mathbf{n}}\big\rangle \qquad (4.4)\\
&-\int d\mathbf{n}_1 \int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)h^{d}_{\mathbf{n}}h^{d}_{\mathbf{n}_2}\big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle f_d \\
&-\int d\mathbf{n}_1 \int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(\mathbf{g}^{d}_{\mathbf{n}}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)h^{d}_{\mathbf{n}_2}\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle f_d \\
&-\int d\mathbf{n}_1 \int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\big\langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle h^{d}_{\mathbf{n}}\bigg(\mathbf{g}^{d}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\bigg) f_d
\end{aligned}
$$

$$-\int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{-\mathbf{n}}\big\rangle f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\Bigg(h^d_{\mathbf{n}} h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2}\big(\big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times h^d_{\mathbf{n}}\left(\left\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2} E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\right\rangle - \left\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2}\right\rangle\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\right) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right) h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2}\big(\big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right)\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2}\big(\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\big(\big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2}\big\rangle\big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) h^d_{\mathbf{n}} h^d_{\mathbf{n}-\mathbf{n}_3}\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\big(\big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2}\big\rangle\big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) h^d_{\mathbf{n}}\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right)\big(\big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) h^d_{\mathbf{n}-\mathbf{n}_3}\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) f_d$$

$$-i\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)$$

$$\times\left(\mathbf{g}^d_{\mathbf{n}} \cdot \frac{\mathbf{k}_3}{k_3}\right)\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\big(\big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3} E_{-\mathbf{n}}\big\rangle\big) f_d\ .$$

Substituindo estas duas médias, (4.3) e (4.4), em (4.2), obtemos

$$\big\langle E_{\mathbf{n}}^2\big\rangle - \frac{4\pi}{k^2}\sum_\sigma q_\sigma\int d\mathbf{v}\ \big(\mathbf{g}_{\mathbf{n}}^\sigma\cdot\mathbf{k}\big)f_\sigma\big\langle E_{\mathbf{n}}^2\big\rangle = \frac{4\pi}{k^2}\sum_\sigma q_\sigma\int d\mathbf{v}\ h_{\mathbf{n}}^\sigma k\big\langle E_{\mathbf{n}}^\sigma E_{-\mathbf{n}}\big\rangle f_\sigma \tag{4.5}$$

$$+i\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\frac{4\pi}{kk_1k_2}\sum_\sigma q_\sigma\int d\mathbf{v}$$

$$\times\Big(\big(\mathbf{g}_{\mathbf{n}}^\sigma\cdot\mathbf{k}_1\big)\big(\mathbf{g}_{\mathbf{n}_2}^\sigma\cdot\mathbf{k}_2\big)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle + \big(\mathbf{g}_{\mathbf{n}}^\sigma\cdot\mathbf{k}_1\big)h_{\mathbf{n}_2}^\sigma k_2\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^\sigma E_{-\mathbf{n}}\big\rangle\Big)f_\sigma$$

$$+i\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\frac{4\pi}{kk_1k_2}\sum_\sigma q_\sigma\int d\mathbf{v}$$

$$\times\Big(h_{\mathbf{n}}^\sigma k_1\big(\mathbf{g}_{\mathbf{n}_2}^\sigma\cdot\mathbf{k}_2\big)\big\langle E_{\mathbf{n}_1}^\sigma E_{\mathbf{n}_2}E_{-\mathbf{n}}\big\rangle + h_{\mathbf{n}}^\sigma k_1 h_{\mathbf{n}_2}^\sigma k_2\big\langle E_{\mathbf{n}_1}^\sigma E_{\mathbf{n}_2}^\sigma E_{-\mathbf{n}}\big\rangle\Big)f_\sigma$$

$$-\int d\mathbf{n}_1\int d\mathbf{n}_2\int d\mathbf{n}_3\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\sum_\sigma q_\sigma\int d\mathbf{v}$$

$$\times\big(\mathbf{g}_{\mathbf{n}}^\sigma\cdot\mathbf{k}_3\big)\Big(\big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^\sigma\cdot\mathbf{k}_1\big)\big(\mathbf{g}_{\mathbf{n}_2}^\sigma\cdot\mathbf{k}_2\big)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)$$

$$+\big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^\sigma\cdot\mathbf{k}_1\big)h_{\mathbf{n}_2}^\sigma k_2\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^\sigma E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^\sigma\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)\Big)f_\sigma$$

$$-\int d\mathbf{n}_1\int d\mathbf{n}_2\int d\mathbf{n}_3\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\sum_\sigma q_\sigma\int d\mathbf{v}$$

$$\times h_{\mathbf{n}}^\sigma k_3\Big(\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}^\sigma E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}^\sigma E_{-\mathbf{n}}\big\rangle\big)\big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^\sigma\cdot\mathbf{k}_1\big)\big(\mathbf{g}_{\mathbf{n}_2}^\sigma\cdot\mathbf{k}_2\big)$$

$$+\Big\langle E_{\mathbf{n}_3}^\sigma E_{-\mathbf{n}}\big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^\sigma\cdot\mathbf{k}_1\big)h_{\mathbf{n}_2}^\sigma k_2 E_{\mathbf{n}_1}E_{\mathbf{n}_2}^\sigma\Big\rangle$$

$$-\big\langle E_{\mathbf{n}_3}^\sigma E_{-\mathbf{n}}\big\rangle\big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^\sigma\cdot\mathbf{k}_1\big)h_{\mathbf{n}_2}^\sigma k_2\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^\sigma\big\rangle\Big)f_\sigma$$

$$-\int d\mathbf{n}_1\int d\mathbf{n}_2\int d\mathbf{n}_3\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\sum_\sigma q_\sigma\int d\mathbf{v}$$

$$\times\big(\mathbf{g}_{\mathbf{n}}^\sigma\cdot\mathbf{k}_3\big)\Big(h_{\mathbf{n}-\mathbf{n}_3}^\sigma k_1\big(\mathbf{g}_{\mathbf{n}_2}^\sigma\cdot\mathbf{k}_2\big)\big(\big\langle E_{\mathbf{n}_1}^\sigma E_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}^\sigma E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)$$

$$+h_{\mathbf{n}-\mathbf{n}_3}^\sigma k_1 h_{\mathbf{n}_2}^\sigma k_2\big(\big\langle E_{\mathbf{n}_1}^\sigma E_{\mathbf{n}_2}^\sigma E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}^\sigma E_{\mathbf{n}_2}^\sigma\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\Big)f_\sigma$$

$$-\int d\mathbf{n}_1\int d\mathbf{n}_2\int d\mathbf{n}_3\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\sum_\sigma q_\sigma\int d\mathbf{v}$$

$$\times h^{\sigma}_{\mathbf{n}} k_3 \Bigg(\Bigg(\Bigg\langle E^{\sigma}_{\mathbf{n}_3} E_{-\mathbf{n}} h^{\sigma}_{\mathbf{n}-\mathbf{n}_3} k_1 \big(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \mathbf{k}_2 \big) E^{\sigma}_{\mathbf{n}_1} E_{\mathbf{n}_2} \Bigg\rangle$$

$$- \big\langle E^{\sigma}_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle h^{\sigma}_{\mathbf{n}-\mathbf{n}_3} k_1 \big(\mathbf{g}^{\sigma}_{\mathbf{n}_2} \cdot \mathbf{k}_2 \big) \big\langle E^{\sigma}_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle \Bigg)$$

$$+ h^{\sigma}_{\mathbf{n}-\mathbf{n}_3} k_1 h^{\sigma}_{\mathbf{n}_2} k_2 \big(\big\langle E^{\sigma}_{\mathbf{n}_1} E^{\sigma}_{\mathbf{n}_2} E^{\sigma}_{\mathbf{n}_3} E_{-\mathbf{k}} \big\rangle - \big\langle E^{\sigma}_{\mathbf{n}_1} E^{\sigma}_{\mathbf{n}_2} \big\rangle \big\langle E^{\sigma}_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle \big) \Bigg) f_{\sigma}$$

$$+ \frac{4\pi}{k^2} \int dq\; q \int d\mathbf{v}\;\, h^{d}_{\mathbf{n}} k \big\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \big\rangle f_d + \frac{4\pi}{k^2} \int dq\; q \int d\mathbf{v}\;\, \big(\mathbf{g}^{d}_{\mathbf{n}} \cdot \mathbf{k} \big) \big\langle E^2_{\mathbf{n}} \big\rangle f_d$$

$$+ i \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \frac{4\pi}{k k_1 k_2} \int dq\; q \int d\mathbf{v} h^{d}_{\mathbf{n}} k_1 h^{d}_{\mathbf{n}_2} k_2 \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{-\mathbf{n}} \big\rangle f_d$$

$$+ i \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \frac{4\pi}{k k_1 k_2} \int dq\; q \int d\mathbf{v} \big(\mathbf{g}^{d}_{\mathbf{n}} \cdot \mathbf{k}_1 \big) h^{d}_{\mathbf{n}_2} k_2 \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{-\mathbf{n}} \big\rangle f_d$$

$$+ i \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \frac{4\pi}{k k_1 k_2} \int dq\; q \int d\mathbf{v} \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{-\mathbf{n}} \big\rangle h^{d}_{\mathbf{n}} k_1 \big(\mathbf{g}^{d}_{\mathbf{n}_2} \cdot \mathbf{k}_2 \big) f_d$$

$$+ i \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \frac{4\pi}{k k_1 k_2} \int dq\; q \int d\mathbf{v} \big(\mathbf{g}^{d}_{\mathbf{n}} \cdot \mathbf{k}_1 \big) \big(\mathbf{g}^{d}_{\mathbf{n}_2} \cdot \mathbf{k}_2 \big) \big\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{-\mathbf{n}} \big\rangle f_d$$

$$- \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) \frac{4\pi}{k k_1 k_2 k_3} \int dq\; q \int d\mathbf{v}$$

$$\times h^{d}_{\mathbf{n}} k_3 h^{d}_{\mathbf{n}-\mathbf{n}_3} k_1 h^{d}_{\mathbf{n}_2} k_2 \Big(\big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}} \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle - \big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle \Big) f_d$$

$$- \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) \frac{4\pi}{k k_1 k_2 k_3} \int dq\; q \int d\mathbf{v}$$

$$\times h^{d}_{\mathbf{n}} k_3 \Bigg(\Bigg\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}} \big(\mathbf{g}^{d}_{\mathbf{n}-\mathbf{n}_3} \cdot \mathbf{k}_1 \big) h^{d}_{\mathbf{n}_2} k_2 E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \Bigg\rangle$$

$$- \big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle \big(\mathbf{g}^{d}_{\mathbf{n}-\mathbf{n}_3} \cdot \mathbf{k}_1 \big) h^{d}_{\mathbf{n}_2} k_2 \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle \Bigg) f_d$$

$$- \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) \frac{4\pi}{k k_1 k_2 k_3} \int dq\; q \int d\mathbf{v}$$

$$\times \big(\mathbf{g}^{d}_{\mathbf{n}} \cdot \mathbf{k}_3 \big) h^{d}_{\mathbf{n}-\mathbf{n}_3} k_1 h^{d}_{\mathbf{n}_2} k_2 \Big(\big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle - \big\langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle \big\langle E_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle \Big) f_d$$

$$- \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) \frac{4\pi}{k k_1 k_2 k_3} \int dq\; q \int d\mathbf{v}$$

$$\times \big(\mathbf{g}^{d}_{\mathbf{n}} \cdot \mathbf{k}_3 \big) \big(\mathbf{g}^{d}_{\mathbf{n}-\mathbf{n}_3} \cdot \mathbf{k}_1 \big) h^{d}_{\mathbf{n}_2} k_2 \Big(\big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle - \big\langle E_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \big\rangle \big\langle E_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle \Big) f_d$$

$$- \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) \frac{4\pi}{k k_1 k_2 k_3} \int dq\; q \int d\mathbf{v}$$

$$\times \big(\big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle - \big\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \big\rangle \big\langle \hat{E}_{\mathbf{n}_3} E_{-\mathbf{n}} \big\rangle \big) h^{d}_{\mathbf{n}} k_3 h^{d}_{\mathbf{n}-\mathbf{n}_3} k_1 \big(\mathbf{g}^{d}_{\mathbf{n}_2} \cdot \mathbf{k}_2 \big) f_d$$

$$-\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\int dq\, q \int d\mathbf{v}$$

$$\times\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle \hat{E}_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)h^d_{\mathbf{n}}k_3\big(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)f_d$$

$$-\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\int dq\, q \int d\mathbf{v}$$

$$\times\left(\mathbf{g}^d_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\right)\big(\big\langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)h^d_{\mathbf{n}-\mathbf{n}_3}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)f_d$$

$$-\int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{kk_1k_2k_3}\int dq\, q \int d\mathbf{v}$$

$$\times\big(\mathbf{g}^d_{\mathbf{n}}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big\langle E_{\mathbf{n}_3}E_{-\mathbf{n}}\big\rangle\big)f_d\ .$$

Vemos que a equação para a correlação quadrática depende de correlações de ordem superior, de terceira e quarta ordens. Elas podem ser eliminadas reduzindo-as a funções da correção quadrática. Podemos lidar mais facilmente com as de quarta ordem, pois para oscilações aleatórias com fases completamente não correlacionadas, a função de correlação de quarta ordem pode ser escrita na forma do produto [31],

$$\big\langle C^a_{\mathbf{n}_1}C^b_{\mathbf{n}_2}C^c_{\mathbf{n}_3}C^d_{\mathbf{n}_4}\big\rangle$$
$$= \big\langle C^a_{\mathbf{n}_1}C^b_{\mathbf{n}_2}\big\rangle\big\langle C^c_{\mathbf{n}_3}C^d_{\mathbf{n}_4}\big\rangle + \big\langle C^a_{\mathbf{n}_1}C^c_{\mathbf{n}_3}\big\rangle\big\langle C^b_{\mathbf{n}_2}C^d_{\mathbf{n}_4}\big\rangle + \big\langle C^a_{\mathbf{n}_1}C^d_{\mathbf{n}_4}\big\rangle\big\langle C^b_{\mathbf{n}_2}C^c_{\mathbf{n}_3}\big\rangle\ . \quad (4.6)$$

As correlações triplas possuem um desenvolvimento mais longo e trataremos delas mais adiante. Assim, eliminando as correlações quádruplas de (4.5) aplicando a propriedade (4.6) e fazendo a equivalência $k = |\mathbf{k}_1 + \cdots + \mathbf{k}_N|$, $\omega = \omega_1 + \cdots + \omega_N$ no integrando das coordenadas espectrais, (4.5) torna-se

$$\big\langle E^2_{\mathbf{n}}\big\rangle - \frac{4\pi}{k^2}\sum_\sigma q_\sigma \int d\mathbf{v}\ \big(\mathbf{g}^\sigma_{\mathbf{n}}\cdot\mathbf{k}\big)f_\sigma\big\langle E^2_{\mathbf{n}}\big\rangle = \frac{4\pi}{k^2}\sum_\sigma q_\sigma \int d\mathbf{v}\ h^\sigma_{\mathbf{n}}k\big\langle E^\sigma_{\mathbf{n}}E_{-\mathbf{n}}\big\rangle f_\sigma \quad (4.7)$$

$$+i\int d\mathbf{n}_1 \int d\mathbf{n}_2\, \delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2|k_1k_2}\sum_\sigma q_\sigma \int d\mathbf{v}$$

$$\times\Bigg(\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\Big(\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle + h^\sigma_{\mathbf{n}_2}k_2\big\langle E_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle\Big)$$

$$+h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1\Big(\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle + h^\sigma_{\mathbf{n}_2}k_2\big\langle E^\sigma_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle\Big)\Bigg)f_\sigma$$

$$
\begin{aligned}
&-\int d\mathbf{n}_1\int d\mathbf{n}_2\int d\mathbf{n}_3\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|k_1k_2k_3}\sum_\sigma q_\sigma\int d\mathbf{v}\\
&\times\Bigg(\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\Bigg(\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\\
&+\big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big)+h^\sigma_{\mathbf{n}_2}k_2\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\\
&+\big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big)\Bigg)\\
&+h^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\Bigg(\big(\big\langle E_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\\
&+\big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E^\sigma_{\mathbf{n}_3}\big\rangle\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\\
&+\big\langle E^\sigma_{\mathbf{n}_3}E_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E^\sigma_{\mathbf{n}_2}\big\rangle\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^\sigma_{\mathbf{n}_2}k_2\\
&+\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}\big\rangle\big\langle E^\sigma_{\mathbf{n}_3}E^\sigma_{\mathbf{n}_2}\big\rangle\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^\sigma_{\mathbf{n}_2}k_2\Bigg)\\
&+\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1\Bigg(\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big(\big\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\\
&+\big\langle E^\sigma_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big)+h^\sigma_{\mathbf{n}_2}k_2\big(\big\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\\
&+\big\langle E^\sigma_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big)\Bigg)\\
&+h^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1\Bigg(\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big\langle E^\sigma_{\mathbf{n}_3}E^\sigma_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_2}\big\rangle\\
&+\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big\langle E^\sigma_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E^\sigma_{\mathbf{n}_1}\big\rangle+h^\sigma_{\mathbf{n}_2}k_2\big(\big\langle E^\sigma_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\\
&+\big\langle E^\sigma_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E^\sigma_{\mathbf{n}_3}\big\rangle\big)\Bigg)\Bigg)f_\sigma\\
&+\frac{4\pi}{k^2}\int dq\;q\int d\mathbf{v}\;h^d_{\mathbf{n}}k\big\langle\hat{E}_{\mathbf{n}}E_{-\mathbf{n}}\big\rangle f_d+\frac{4\pi}{k^2}\int dq\;q\int d\mathbf{v}\;\big(\mathbf{g}^d_{\mathbf{n}}\cdot\mathbf{k}\big)\big\langle E^2_{\mathbf{n}}\big\rangle f_d\\
&+i\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2|k_1k_2}\int dq\;q\int d\mathbf{v}\\
&\times\Bigg(h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1h^d_{\mathbf{n}_2}k_2\big\langle\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle\\
&+\big\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle\Bigg)f_d\\
&-\int d\mathbf{n}_1\int d\mathbf{n}_2\int d\mathbf{n}_3\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3)\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|k_1k_2k_3}\int dq\;q\int d\mathbf{v}
\end{aligned}
$$

$$\times\Bigg(h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\Bigg(h^d_{\mathbf{n}_1+\mathbf{k}_2}k_1h^d_{\mathbf{n}_2}k_2\big\langle\hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_2}\big\rangle$$

$$+h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1h^d_{\mathbf{n}_2}k_2\big\langle\hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_1}\big\rangle$$

$$+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2\big\langle\hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_2}\big\rangle$$

$$+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2\big\langle\hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}\big\rangle\Bigg)$$

$$+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\Bigg(h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1h^d_{\mathbf{n}_2}k_2\big(\big\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle$$

$$+\big\langle\hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big)\Bigg)$$

$$+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\Bigg(\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle$$

$$+\big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big)\Bigg)$$

$$+\Bigg(\big\langle\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle+\big\langle\hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle\hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\Bigg)$$

$$\times h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$$

$$+\Bigg(\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle+\big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle\hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\Bigg)$$

$$\times h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$$

$$+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\Bigg(\big\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle$$

$$+\big\langle\hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\Bigg)h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$$

$$+\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$$

$$\times\Bigg(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle+\big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\Bigg)\Bigg)f_d\ .$$

Da equação (4.7) podemos identificar o tensor dielétrico

$$\epsilon(\mathbf{n})=1+\chi(\mathbf{n}) \tag{4.8}$$

que é definido em termos da susceptibilidade elétrica linear

$$\chi(\mathbf{n}) = -\frac{4\pi}{k^2}\sum_\sigma q_\sigma \int d\mathbf{v}\ \big(\mathbf{g}_\mathbf{n}^\sigma \cdot \mathbf{k}\big) f_\sigma\ . \tag{4.9}$$

A equação (4.9) é modificada em relação ao caso sem poeira pela frequência de colisões no operador $\mathbf{g}_\mathbf{n}^\sigma$, em concordância com outros trabalhos da literatura, [3, 32, 33].

Identificamos também os seguintes termos de primeira ordem devido à presença da poeira, e que contribuem com a susceptibilidade linear efetiva,

$$\psi(\mathbf{n}) = -\frac{4\pi}{k^2}\int dq\ q \int d\mathbf{v}\ \big(\mathbf{g}_\mathbf{n}^d \cdot \mathbf{k}\big) f_d\ , \tag{4.10}$$

$$\chi_{1\sigma}^{(1)}(\mathbf{n}) = \frac{4\pi}{k^2} q_\sigma \int d\mathbf{v}\ h_\mathbf{n}^\sigma k f_\sigma\ , \tag{4.11}$$

$$\psi_1^{(1)}(\mathbf{n}) = \frac{4\pi}{k^2}\int dq\ q \int d\mathbf{v}\ h_\mathbf{n}^d k f_d\ . \tag{4.12}$$

Termos não lineares de segunda e terceira ordem ocorrem na forma

$$\chi_{j\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2) = \frac{4\pi i}{|\mathbf{k}_1+\mathbf{k}_2| k_1 k_2} q_\sigma \int d\mathbf{v} A_j^\sigma f_\sigma\ , \tag{4.13}$$

$$\chi_{j\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3) = -\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3| k_1 k_2 k_3} q_\sigma \int d\mathbf{v} B_j^\sigma f_\sigma\ , \tag{4.14}$$

$$\psi_j^{(2)}(\mathbf{n}_1;\mathbf{n}_2) = \frac{4\pi i}{|\mathbf{k}_1+\mathbf{k}_2| k_1 k_2}\int dq\ q \int d\mathbf{v}\ C_j f_d\ , \tag{4.15}$$

$$\psi_j^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3) = -\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3| k_1 k_2 k_3}\int dq\ q \int d\mathbf{v}\ D_j f_d\ , \tag{4.16}$$

onde A_j^σ, B_j^σ, C_j e D_j são indexados de acordo com

j	A_j^σ
1	$\frac{1}{2}\Big(\big(\mathbf{g}_{\mathbf{n}_1+\mathbf{n}_2}^\sigma \cdot \mathbf{k}_1\big)\big(\mathbf{g}_{\mathbf{n}_2}^\sigma \cdot \mathbf{k}_2\big) + (\mathbf{n}_1 \leftrightarrows \mathbf{n}_2)\Big)$
2	$\big(\mathbf{g}_{\mathbf{n}_1+\mathbf{n}_2}^\sigma \cdot \mathbf{k}_1\big) h_{\mathbf{n}_2}^\sigma k_2$
3	$h_{\mathbf{n}_1+\mathbf{n}_2}^\sigma k_1 \big(\mathbf{g}_{\mathbf{n}_2}^\sigma \cdot \mathbf{k}_2\big)$
4	$\frac{1}{2}\Big(h_{\mathbf{n}_1+\mathbf{n}_2}^\sigma k_1 h_{\mathbf{n}_2}^\sigma k_2 + (\mathbf{n}_1 \leftrightarrows \mathbf{n}_2)\Big)$

j	B_j^σ
1	$\frac{1}{3}\big(\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_3)\big)$
2	$\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^\sigma_{\mathbf{n}_2}k_2$
3	$\frac{1}{2}\big(h^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
4	$h^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^\sigma_{\mathbf{n}_2}k_2$
5	$\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$
6	$\frac{1}{2}\big(\big(\mathbf{g}^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1h^\sigma_{\mathbf{n}_2}k_2+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
7	$h^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^\sigma_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$
8	$\frac{1}{3}\big(h^\sigma_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3h^\sigma_{\mathbf{n}_1+\mathbf{n}_2}k_1h^\sigma_{\mathbf{n}_2}k_2+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_3)\big)$

j	C_j
1	$\frac{1}{2}\big(h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1h^d_{\mathbf{n}_2}k_2+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
2	$\frac{1}{2}\big(\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
3	$\frac{1}{2}\big(h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
4	$\frac{1}{2}\big(\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$

j	D_j
1	$\frac{1}{3}\big(h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1h^d_{\mathbf{n}_2}k_2+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_3)\big)$
2	$h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2$
3	$\frac{1}{2}\big(\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1h^d_{\mathbf{n}_2}k_2+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
4	$\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)h^d_{\mathbf{n}_2}k_2$
5	$h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$
6	$\frac{1}{2}\big(h^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}k_3\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)\big)$
7	$\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)h^d_{\mathbf{n}_1+\mathbf{n}_2}k_1\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)$
8	$\frac{1}{3}\big(\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\cdot\mathbf{k}_3\big)\big(\mathbf{g}^d_{\mathbf{n}_1+\mathbf{n}_2}\cdot\mathbf{k}_1\big)\big(\mathbf{g}^d_{\mathbf{n}_2}\cdot\mathbf{k}_2\big)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_2)+(\mathbf{n}_1\leftrightarrows\mathbf{n}_3)\big)$

Introduzindo a notação

$$\sum_{\mathbf{n}_1+\cdots+\mathbf{n}_N=\mathbf{n}} = \int d\mathbf{n}_1\ldots d\mathbf{n}_N\,\delta(\mathbf{n}-\mathbf{n}_1-\cdots-\mathbf{n}_N)\ ,$$

e usando as definições das susceptibilidades acima, obtemos uma forma relativamente compacta de representar a densidade espectral de energia,

$$
\begin{aligned}
&\big\langle E_{\mathbf{n}}^2\big\rangle\big(\epsilon(\mathbf{n})+\psi(\mathbf{n})\big) - \sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}} \chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle \\
&- \sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}} \chi_1^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\Big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\Big) \\
&= \sum_{\sigma}\chi_{1\sigma}^{(1)}(\mathbf{n})\big\langle E_{\mathbf{n}}^{\sigma}E_{-\mathbf{n}}\big\rangle + \psi_1^{(1)}(\mathbf{n})\big\langle \hat{E}_{\mathbf{n}}E_{-\mathbf{n}}\big\rangle \\
&+ \sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\sum_{\sigma}\Big(\chi_{2\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle + \chi_{3\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle \\
&\quad +\chi_{4\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle\Big) \\
&+ \sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\sum_{\sigma}\Big(\chi_{2\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_3}\big\rangle\big) \\
&\quad +\chi_{3\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}^{\sigma}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}^{\sigma}\big\rangle\big) \\
&\quad +\chi_{4\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_2}^{\sigma}\big\rangle + \big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}\big\rangle\big\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\big\rangle\big) \\
&\quad +\chi_{5\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&\quad +\chi_{6\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_1}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big) \\
&\quad +\chi_{7\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_1}^{\sigma}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_2}\big\rangle + \big\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_2}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}^{\sigma}\big\rangle\big) \\
&\quad +\chi_{8\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_3}^{\sigma}\big\rangle\big\langle E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_3}^{\sigma}\big\rangle\big)\Big) \\
&+ \sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\Big(\psi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2}\big\rangle + \psi_2^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle \\
&\quad +\psi_3^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle + \psi_4^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle\Big) \\
&+ \sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\Big(\psi_1^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle \hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_2}\big\rangle + \big\langle \hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_1}\big\rangle\big) \\
&\quad +\psi_2^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle \hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_1}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_2}\big\rangle + \big\langle \hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_2}\big\rangle\big\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}\big\rangle\big) \\
&\quad +\psi_3^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle \hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle \hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle \hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&\quad +\psi_4^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle \hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle \hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&\quad +\psi_5^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle \hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle \hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\big) \\
&\quad +\psi_6^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle \hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\big) \\
&\quad +\psi_7^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle \hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\big) \\
&\quad +\psi_8^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_3}E_{\mathbf{n}_2}\big\rangle\big)\Big)\ .
\end{aligned}
\tag{4.17}
$$

Podemos ver de uma forma mais clara em (4.17) a dependência das flutuações do campo elétrico do plasma com a presença da poeira. No escopo do sistema que definimos, o campo elétrico do plasma sofre o efeito dos campos relacionados ao processo de carregamento das partículas de poeira. Estes campos estão definidos pelas equações (3.29) e (3.38), e ocorrem na equação (4.17) combinados com o campo elétrico efetivo do plasma. Assim, cada combinação está associada a diferentes correlações.

Veremos mais adiante que as equações cinéticas das partículas do plasma e da poeira são funcionais de $\langle \hat{E}_{\mathbf{n}}^2 \rangle$ e $\langle E_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}}^{\sigma'} \rangle$, cujas expressões podem ser obtidas de forma análoga a (4.17), o que faremos agora.

A fim de obter uma equação para $\langle \hat{E}_{\mathbf{n}}^2 \rangle$, multiplicamos a equação (3.38) por $\hat{E}_{-\mathbf{n}}$ e fazemos o procedimento da média,

$$\langle \hat{E}_{\mathbf{n}}^2 \rangle = -\frac{4\pi i}{k} \sum_{\sigma} \frac{q_{\sigma}^3}{e^2} \frac{m_d}{m_{\sigma}} \frac{1}{n_d} \int d\mathbf{v} dq \; f_d \int d\mathbf{v}' \Omega_{\mathbf{k}}^{\sigma}(\mathbf{v} - \mathbf{v}', a) \langle N_{\mathbf{n}}^{\sigma} \hat{E}_{-\mathbf{n}} \rangle \; . \tag{4.18}$$

Em seguida, a equação (3.41) é inserida em (4.18). O resultado é a expressão

$$
\begin{aligned}
\big\langle \hat{E}^2_{\mathbf{n}} \big\rangle &= \rho_1^{(1)}(\mathbf{n})\big\langle E_{\mathbf{n}}\hat{E}_{-\mathbf{n}}\big\rangle + \rho_{2\sigma}^{(1)}(\mathbf{n})\big\langle E^\sigma_{\mathbf{n}}\hat{E}_{-\mathbf{n}}\big\rangle \\
&+ \sum_{\mathbf{n}=\mathbf{n}_1+\mathbf{n}_2} \Bigg(\rho_1^{(2)}(\mathbf{n}_1,\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle + \sum_\sigma \rho_{2\sigma}^{(2)}(\mathbf{n}_1,\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle \Bigg) \\
&+ \sum_\sigma \rho_{3\sigma}^{(2)}(\mathbf{n}_1,\mathbf{n}_2)\big\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle + \sum_\sigma \rho_{4\sigma}^{(2)}(\mathbf{n}_1,\mathbf{n}_2)\big\langle E^\sigma_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle \Bigg) \\
&+ \sum_{\mathbf{n}=\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3} \Bigg(\rho_1^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{2\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{3\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E^\sigma_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{4\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E^\sigma_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{5\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E^\sigma_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{6\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E^\sigma_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{7\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E^\sigma_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_3}\big\rangle\big\langle E_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E^\sigma_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E_{\mathbf{n}_2}E^\sigma_{\mathbf{n}_3}\big\rangle\big) \\
&+ \sum_\sigma \rho_{8\sigma}^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)\big(\big\langle E^\sigma_{\mathbf{n}_1}E^\sigma_{\mathbf{n}_3}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle + \big\langle E^\sigma_{\mathbf{n}_1}\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\big\rangle\big\langle E^\sigma_{\mathbf{n}_2}E^\sigma_{\mathbf{n}_3}\big\rangle\big)\Bigg) f_\sigma \ ,
\end{aligned}
\tag{4.19}
$$

onde

$$
\rho_1^{(1)}(\mathbf{n}) = \frac{4\pi}{k^2}\sum_\sigma \frac{q_\sigma^3}{e^2}\frac{m_d}{m_\sigma}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d \int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)\big(\mathbf{g}^\sigma_{\mathbf{n}}\cdot\mathbf{k}\big)f_\sigma \ , \tag{4.20}
$$

$$
\rho_{2\sigma}^{(1)}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma^3}{e^2}\frac{m_d}{m_\sigma}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d \int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)h^\sigma_{\mathbf{n}}kf_\sigma \ , \tag{4.21}
$$

$$
\begin{aligned}
\rho_j^{(2)}(\mathbf{n}_1,\mathbf{n}_2) &= -\frac{4\pi i}{|\mathbf{k}_1+\mathbf{k}_2|k_1k_2}\sum_\sigma \frac{q_\sigma^3}{e^2}\frac{m_d}{m_\sigma}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d \\
&\times \int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}_1+\mathbf{k}_2}(\mathbf{v}-\mathbf{v}',a)A^\sigma_j f_\sigma \ ,
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
\rho_j^{(3)}(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3) &= -\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|k_1k_2k_3}\sum_\sigma \frac{q_\sigma^3}{e^2}\frac{m_d}{m_\sigma}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d \\
&\times \int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}(\mathbf{v}-\mathbf{v}',a)B^\sigma_j f_\sigma \ .
\end{aligned}
\tag{4.23}
$$

Procedemos da mesma forma para obter uma equação para $\langle E^{\sigma}_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle$, multiplicando a equação (3.29) por $E^{\sigma}_{-\mathbf{n}}$ e fazendo a média,

$$\langle E^{\sigma}_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle = -\frac{4\pi i}{k}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega^{\sigma}_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a)\langle N^{d}_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle\ . \tag{4.24}$$

Inserindo a equação (3.44) no lado direito de (4.24),

$$\begin{aligned}
\langle E^{\sigma}_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle &= \zeta^{(1)}_{1\sigma}(\mathbf{n})\langle \hat{E}_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle + \zeta^{(1)}_{2\sigma}(\mathbf{n})\langle E_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle \\
&+ \sum_{\mathbf{n}=\mathbf{n}_1+\mathbf{n}_2} \Big(\zeta^{(2)}_{1\sigma}(\mathbf{n}_1;\mathbf{n}_2)\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2)} \rangle + \zeta^{(2)}_{2\sigma}(\mathbf{n}_1;\mathbf{n}_2)\langle \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2)} \rangle \\
&\quad + \zeta^{(2)}_{3\sigma}(\mathbf{n}_1;\mathbf{n}_2)\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2)} \rangle + \zeta^{(2)}_{4\sigma}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_2} E_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2)} \rangle \Big) \\
&+ \sum_{\mathbf{n}=\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3} \Big(\zeta^{(3)}_{1\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\rangle\langle \hat{E}_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle \hat{E}_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle \hat{E}_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{2\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\rangle\langle \hat{E}_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle E_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle \hat{E}_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{3\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_3}\rangle\langle \hat{E}_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle \hat{E}_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{4\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle E_{\mathbf{n}_1} E_{\mathbf{n}_3}\rangle\langle \hat{E}_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle E_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{5\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle \hat{E}_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{6\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle E_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{7\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle \hat{E}_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2} E_{\mathbf{n}_3}\rangle\big) \\
&+ \zeta^{(3)}_{8\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\langle E_{\mathbf{n}_1} E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle + \langle E_{\mathbf{n}_1} E^{\sigma'}_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2} E_{\mathbf{n}_3}\rangle\big) \Big)\ ,
\end{aligned} \tag{4.25}$$

onde

$$\zeta^{(1)}_{1\sigma}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega^{\sigma}_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) h^{d}_{\mathbf{n}} k f_d\ , \tag{4.26}$$

$$\zeta^{(1)}_{2\sigma}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega^{\sigma}_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a)\big(\mathbf{g}^{d}_{\mathbf{n}}\cdot\mathbf{k}\big) f_d\ , \tag{4.27}$$

$$\zeta^{(2)}_{j\sigma}(\mathbf{n}_1;\mathbf{n}_2) = \frac{4\pi i}{|\mathbf{k}_1+\mathbf{k}_2| k_1 k_2}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega^{\sigma}_{\mathbf{k}_1+\mathbf{k}_2}(\mathbf{v}-\mathbf{v}', a) C_j f_d\ , \tag{4.28}$$

$$\zeta^{(3)}_{j\sigma}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3) = -\frac{4\pi}{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3| k_1 k_2 k_3}\frac{q_\sigma}{n_\sigma}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega^{\sigma}_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}(\mathbf{v}-\mathbf{v}', a) D_j f_d\ . \tag{4.29}$$

As três equações de densidades espectrais, (4.17), (4.19) e (4.25) são autoconsistentes em conexão com as equações cinéticas das partículas, na forma explícita das equações

(5.30) e (5.31), por exemplo. A partir delas podemos tomar as aproximações que são relevantes para uma análise quase linear ou não linear.

Como dissemos, as correlações de terceira ordem podem ser aproximadas e escritas em termos das correlações quadráticas, e daremos um exemplo ao final do Capítulo (6), visto que faremos uso de expressões que são desenvolvidas na aproximação quase linear, que é tema do próximo capítulo.

Capítulo 5

Aproximação quase linear

Como aplicação do que já obtivemos, consideramos o limite quase linear da interação onda-partícula. Considerando que efeitos não lineares podem ser desprezados, somente termos proporcionais a f_σ e ω_σ restarão do lado direito da equação (2.32), e termos proporcionais à corrente I e a $\partial f_d/\partial q$ restarão em (2.34). Nessa aproximação, as equações (3.41) e (3.44) reduzem-se a [27]

$$N_{\mathbf{n}}^{\sigma} = i\left(\mathbf{g}_{\mathbf{n}}^{\sigma} \cdot \frac{\mathbf{k}}{k}\right) E_{\mathbf{n}} f_\sigma + i h_{\mathbf{n}}^{\sigma} E_{\mathbf{n}}^{\sigma} f_\sigma \ , \tag{5.1}$$

$$N_{\mathbf{n}}^{d} = i h_{\mathbf{n}}^{d} \hat{E}_{\mathbf{n}} f_d + i\left(\mathbf{g}_{\mathbf{n}}^{d} \cdot \frac{\mathbf{k}}{k}\right) E_{\mathbf{n}} f_d \ . \tag{5.2}$$

Inserindo (5.1) e (5.2) na equação para o campo $E_{\mathbf{n}}$, multiplicando pela direita por $E_{-\mathbf{n}}$ e realizando a média, obtemos a equação para a densidade $\langle E_{\mathbf{n}}^2 \rangle$ na aproximação quase linear, dada por

$$\langle E_{\mathbf{n}}^2 \rangle (\epsilon + \psi) = \sum\nolimits_\sigma \chi_{1\sigma} \langle E_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} \rangle + \psi_1 \langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \rangle \ , \tag{5.3}$$

que também pode ser obtida desprezando-se todos os termos não lineares em (4.17). As correlações no lado direito de (5.3) são dadas por

$$\langle E_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} \rangle = \zeta_{1\sigma} \langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \rangle + \zeta_{2\sigma} \langle E_{\mathbf{n}}^2 \rangle \ , \tag{5.4}$$

$$\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \rangle = \rho_1 \langle E_{\mathbf{n}}^2 \rangle + \sum_\sigma \rho_{2\sigma} \langle E_{\mathbf{n}}^{\sigma} E_{-\mathbf{n}} \rangle \ , \tag{5.5}$$

que obtivemos da mesma forma mas considerando os campos (3.29) e (3.38). As equações (5.3), (5.4) e (5.5) formam um sistema fechado de três equações em $\langle E_{\mathbf{n}}^2 \rangle$, $\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \rangle$ e

$\langle E^{\sigma}_{\mathbf{n}} E_{-\mathbf{n}} \rangle$, e podemos resolvê-lo em termos de $\langle E^2_{\mathbf{n}} \rangle$.

Com substituição de (5.4) em (5.3) e (5.5), e com substituição em seguida da equação resultante na equação (5.3), chegamos a

$$\begin{aligned} \langle E^2_{\mathbf{n}} \rangle \Bigg(\epsilon + \psi - \sum_{\sigma} \chi_{1\sigma} \Bigg(\zeta_{1\sigma} \frac{\rho_1 + \sum_{\sigma'} \rho_{2\sigma'} \zeta_{2\sigma'}}{1 - \sum_{\sigma'} \rho_{2\sigma'} \zeta_{1\sigma'}} + \zeta_{2\sigma} \Bigg) \\ - \psi_1 \frac{\rho_1 + \sum_{\sigma'} \rho_{2\sigma'} \zeta_{2\sigma'}}{1 - \sum_{\sigma'} \rho_{2\sigma'} \zeta_{1\sigma'}} \Bigg) = 0 \ . \end{aligned} \quad (5.6)$$

Definindo por D_1 a quantidade que multiplica a densidade espectral, a identidade

$$D_1 = \epsilon + \psi - \sum_{\sigma} \chi_{1\sigma} \Bigg(\zeta_{1\sigma} \frac{\rho_1 + \sum_{\sigma'} \rho_{2\sigma'} \zeta_{2\sigma'}}{1 - \sum_{\sigma'} \rho_{2\sigma'} \zeta_{1\sigma'}} + \zeta_{2\sigma} \Bigg) - \psi_1 \frac{\rho_1 + \sum_{\sigma'} \rho_{2\sigma'} \zeta_{2\sigma'}}{1 - \sum_{\sigma'} \rho_{2\sigma'} \zeta_{1\sigma'}} = 0 \quad (5.7)$$

define a relação de dispersão relacionada à energia espectral $\langle E^2_{\mathbf{n}} \rangle$.

Consideremos que a solução para (5.6) é dada por [14]

$$E^{\alpha}_{\mathbf{k}}(t) = E^{\alpha}_{\mathbf{k}} \cos(\omega^{\alpha}_{1\mathbf{k}} t + \phi^{\alpha}_{\mathbf{k}}) \quad , \quad E^{\alpha}_{\mathbf{k}} = E^{\alpha}_{\mathbf{k}0} \exp\big(- \mathrm{Im}\,(\omega^{\alpha}_{1\mathbf{k}}) t \big) \quad (5.8)$$

onde $E^{\alpha}_{\mathbf{k}0}$ e $\phi^{\alpha}_{\mathbf{k}}$ são a amplitude e fase iniciais, e $\omega^{\alpha}_{1\mathbf{k}}$ são as frequências angulares que são as raízes da relação de dispersão (5.7). O índice α foi introduzido para distinguir entre os diferentes modos de oscilação.

Tomando a transformada de Fourier de (5.8) em relação ao tempo obtemos,

$$\frac{1}{2\pi} \int dt \ E^{\alpha}_{\mathbf{k}}(t) e^{i\omega t} = E_{\mathbf{k}} \frac{1}{2\pi} \int dt \ \frac{e^{i(\omega^{\alpha}_{1\mathbf{k}} t + \phi^{\alpha}_{\mathbf{k}})} + e^{-i(\omega^{\alpha}_{1\mathbf{k}} t + \phi^{\alpha}_{\mathbf{k}})}}{2} e^{i\omega t} \ ,$$

$$E^{\alpha}_{\mathbf{n}} = \frac{1}{2} E^{\alpha}_{\mathbf{k}} \big(e^{-i\phi^{\alpha}_{\mathbf{k}}} \delta(\omega - \omega^{\alpha}_{1\mathbf{k}}) + e^{i\phi^{\alpha}_{\mathbf{k}}} \delta(\omega + \omega^{\alpha}_{1\mathbf{k}}) \big) \ . \quad (5.9)$$

O valor médio do produto das flutuações do campo elétrico é dado por

$$\langle E_{\mathbf{k}\omega} E^{*}_{\mathbf{k}\omega'} \rangle = V \delta(\omega - \omega') \langle E^2_{\mathbf{k}\omega} \rangle \quad (5.10)$$

onde $\langle E^2_{\mathbf{k}\omega} \rangle$ é a função de correlação espectral e V o volume do plasma. Inserindo (5.9) no lado esquerdo de (5.10), somando sobre os modos α, integrando em ω' e fazendo a média sobre as fases iniciais no intervalo $[0, 2\pi]$, temos

$$\sum_\alpha \frac{1}{2\pi}\int_0^{2\pi} d\phi_{\mathbf{k}}^\alpha \int d\omega' \, \frac{|E_{\mathbf{k}}^\alpha|^2}{4}\bigg(\delta(\omega-\omega_{1\mathbf{k}}^\alpha)\delta(\omega'-\omega_{1\mathbf{k}}^\alpha)+e^{-2i\phi_{\mathbf{k}}^\alpha}\delta(\omega-\omega_{1\mathbf{k}}^\alpha)\delta(\omega'+\omega_{1\mathbf{k}}^\alpha)$$

$$+e^{2i\phi_{\mathbf{k}}^\alpha}\delta(\omega+\omega_{1\mathbf{k}}^\alpha)\delta(\omega'-\omega_{1\mathbf{k}}^\alpha)+\delta(\omega+\omega_{1\mathbf{k}}^\alpha)\delta(\omega'+\omega_{1\mathbf{k}}^\alpha)\bigg)$$

$$= V\int d\omega'\delta\big(\omega-\omega'\big)\big\langle E_{\mathbf{k}\omega}^2\big\rangle \ ,$$

$$\sum_\alpha \frac{|E_{\mathbf{k}}^\alpha|^2}{4}\big(\delta(\omega-\omega_{1\mathbf{k}}^\alpha)+\delta(\omega+\omega_{1\mathbf{k}}^\alpha)\big) = V\big\langle E_{\mathbf{k}\omega}^2\big\rangle \ ,$$

$$\big\langle E_{\mathbf{k}\omega}^2\big\rangle = \big\langle E_{\mathbf{n}}^2\big\rangle = \sum_\alpha\sum_{j=\pm}\mathcal{I}_{\mathbf{k}}^{\alpha j}\delta(\omega+j\omega_{1\mathbf{k}}^\alpha) \ , \qquad (5.11)$$

onde

$$\mathcal{I}_{\mathbf{k}}^{\alpha j} = \frac{1}{4}\frac{|E_{\mathbf{k}}^{\alpha j}|^2}{V}$$

é a intensidade das flutuações.

Nós expandiremos as funções dielétricas ϵ e ψ supondo que $|\mathrm{Im}\,(\omega)| \ll |\omega|$, levando em conta que a parte imaginária da freqüência incorpora a variação lenta das amplitudes das ondas. Portanto [14],

$$\big(\epsilon(\mathbf{n})+\psi(\mathbf{n})\big)\big\langle E_{\mathbf{n}}^2\big\rangle \simeq \bigg(\epsilon(\mathbf{n})+\psi(\mathbf{n})+\frac{i}{2}\bigg(\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}+\frac{\partial \mathrm{Re}\,\psi}{\partial\omega}\bigg)\frac{\partial}{\partial t}\bigg)\big\langle E_{\mathbf{n}}^2\big\rangle \ . \qquad (5.12)$$

Da equação (5.6) obtemos

$$\frac{i}{2}\bigg(\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}+\frac{\partial \mathrm{Re}\,\psi}{\partial\omega}\bigg)\frac{\partial}{\partial t}\big\langle E_{\mathbf{n}}^2\big\rangle = -\big(\epsilon(\mathbf{n})+\psi(\mathbf{n})\big)\big\langle E_{\mathbf{n}}^2\big\rangle$$
$$+\big\langle E_{\mathbf{n}}^2\big\rangle\bigg(\sum_\sigma \chi_{1\sigma}\bigg(\zeta_{1\sigma}\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+\zeta_{2\sigma}\bigg)+\psi_1\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}\bigg) \ . \qquad (5.13)$$

A evolução temporal de $\big\langle E_{\mathbf{n}}^2\big\rangle$ é obtida igualando a parte imaginária de (5.13) a zero, de modo que

$$\frac{1}{2}\left(\frac{\partial \mathrm{Re}\,\epsilon}{\partial\omega}+\frac{\partial \mathrm{Re}\,\psi}{\partial\omega}\right)\frac{\partial}{\partial t}\big\langle E_{\mathbf{n}}^2\big\rangle = -\mathrm{Im}\,\big(\epsilon(\mathbf{n})+\psi(\mathbf{n})\big)\big\langle E_{\mathbf{n}}^2\big\rangle$$
$$+\,\mathrm{Im}\left(\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+\zeta_{2\sigma}\right)+\psi_1\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}\right)\big\langle E_{\mathbf{n}}^2\big\rangle\ . \quad (5.14)$$

Inserindo (5.11) em (5.14) e integrando em ω, encontramos a equação cinética para as ondas

$$\frac{\mathrm{Re}}{2}\left(\frac{\partial\epsilon(j_1\omega_{1\mathbf{k}}^\alpha)}{\partial\omega}+\frac{\partial\psi(j_1\omega_{1\mathbf{k}}^\alpha)}{\partial\omega}\right)\frac{\partial\mathcal{I}_{\mathbf{k}}^{\alpha j_1}}{\partial t} = -\mathrm{Im}\,\big(\epsilon(j_1\omega_{1\mathbf{k}}^\alpha)+\psi(j_1\omega_{1\mathbf{k}}^\alpha)\big)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}$$
$$+\,\mathrm{Im}\left(\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+\zeta_{2\sigma}\right)+\psi_1\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}\right)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\ . \quad (5.15)$$

No caso de $\big\langle \hat{E}_{\mathbf{n}}^2\big\rangle$ procedemos com abordagem similar, mas multiplicando por $\hat{E}_{-\mathbf{n}}$ as equações (3.29), (3.30) e (3.38). Outro conjunto de três equações é então obtido,

$$\big\langle E_{\mathbf{n}}\hat{E}_{-\mathbf{n}}\big\rangle(\epsilon+\psi) = \sum\nolimits_\sigma \chi_{1\sigma}\big\langle E_{\mathbf{n}}^\sigma\hat{E}_{-\mathbf{n}}\big\rangle+\psi_1\big\langle\hat{E}_{\mathbf{n}}^2\big\rangle\ , \quad (5.16)$$

$$\big\langle E_{\mathbf{n}}^\sigma\hat{E}_{-\mathbf{n}}\big\rangle = \zeta_{1\sigma}\big\langle\hat{E}_{\mathbf{n}}^2\big\rangle+\zeta_{2\sigma}\big\langle E_{\mathbf{n}}\hat{E}_{-\mathbf{n}}\big\rangle\ , \quad (5.17)$$

$$\big\langle\hat{E}_{\mathbf{n}}^2\big\rangle = \rho_1\big\langle E_{\mathbf{n}}\hat{E}_{-\mathbf{n}}\big\rangle+\sum\nolimits_\sigma \rho_{2\sigma}\big\langle E_{\mathbf{n}}^\sigma\hat{E}_{-\mathbf{n}}\big\rangle\ . \quad (5.18)$$

Inserindo a equação (5.17) em (5.16) e (5.18), e combinando as expressões resultantes, obtemos

$$\big\langle\hat{E}_{\mathbf{n}}^2\big\rangle\Bigg(1-\rho_1\frac{\sum_\sigma\chi_{1\sigma}\zeta_{1\sigma}+\psi_1}{\epsilon+\psi-\sum_\sigma\chi_{1\sigma}\zeta_{2\sigma}}$$
$$-\sum_\sigma\rho_{2\sigma}\left(\zeta_{1\sigma}+\zeta_{2\sigma}\frac{\sum_{\sigma'}\chi_{1\sigma'}\zeta_{1\sigma'}+\psi_1}{\epsilon+\psi-\sum_{\sigma'}\chi_{1\sigma'}\zeta_{2\sigma'}}\right)\Bigg)=0\ , \quad (5.19)$$

que conduz a uma relação de dispersão associada a $\langle\hat{E}_{\mathbf{n}}^2\rangle$,

$$D_2 = 1-\rho_1\frac{\sum_\sigma\chi_{1\sigma}\zeta_{1\sigma}+\psi_1}{\epsilon+\psi-\sum_\sigma\chi_{1\sigma}\zeta_{2\sigma}}-\sum_\sigma\rho_{2\sigma}\left(\zeta_{1\sigma}+\zeta_{2\sigma}\frac{\sum_{\sigma'}\chi_{1\sigma'}\zeta_{1\sigma'}+\psi_1}{\epsilon+\psi-\sum_{\sigma'}\chi_{1\sigma'}\zeta_{2\sigma'}}\right)=0\ . \quad (5.20)$$

Por analogia com $\langle E_{\mathbf{n}}^2\rangle$, tomamos

$$\big\langle\hat{E}_{\mathbf{n}}^2\big\rangle = \sum_\beta\sum_{j_2=\pm}\hat{\mathcal{I}}_{\mathbf{k}}^{\beta j_2}\delta(\omega+j_2\omega_{2\mathbf{k}}^\beta)\ , \quad (5.21)$$

onde $\omega_{2\mathbf{k}}^{\beta}$ são as raízes de D_2, com a equação para a evolução temporal da intensidade das flutuações dada por

$$\frac{\mathrm{Re}}{2}\left(\frac{\partial D_2(j_2\omega_{2\mathbf{k}}^{\beta})}{\partial\omega}\right)\frac{\partial\hat{\mathcal{I}}_{\mathbf{k}}^{\beta j_2}}{\partial t} = -\mathrm{Im}\left(D_2(j_2\omega_{2\mathbf{k}}^{\beta})\right)\hat{\mathcal{I}}_{\mathbf{k}}^{\beta j_2} \ . \tag{5.22}$$

Para $\langle E_{\mathbf{n}}^{\sigma'} E_{-\mathbf{n}}^{\sigma}\rangle$, considerando as mesmas equações (3.29), (3.30) e (3.38), mas multiplicadas por $E_{-\mathbf{n}}^{\sigma}$,

$$\langle E_{\mathbf{n}} E_{-\mathbf{n}}^{\sigma}\rangle(\epsilon+\psi) = \sum\nolimits_{\sigma''}\chi_{1\sigma''}\langle E_{\mathbf{n}}^{\sigma''}E_{-\mathbf{n}}^{\sigma}\rangle + \psi_1\langle \hat{E}_{\mathbf{n}}E_{-\mathbf{n}}^{\sigma}\rangle \ , \tag{5.23}$$

$$\langle E_{\mathbf{n}}^{\sigma'} E_{-\mathbf{n}}^{\sigma}\rangle = \zeta_{1\sigma'}\langle \hat{E}_{\mathbf{n}}E_{-\mathbf{n}}^{\sigma}\rangle + \zeta_{2\sigma'}\langle E_{\mathbf{n}}E_{-\mathbf{n}}^{\sigma}\rangle \ , \tag{5.24}$$

$$\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}}^{\sigma}\rangle = \rho_1\langle E_{\mathbf{n}}E_{-\mathbf{n}}^{\sigma}\rangle + \sum_{\sigma''}\rho_{2\sigma''}\langle E_{\mathbf{n}}^{\sigma''}E_{-\mathbf{n}}^{\sigma}\rangle \ . \tag{5.25}$$

Combinando essas equações obtemos

$$\begin{aligned}\langle E_{\mathbf{n}}^{\sigma'} E_{-\mathbf{n}}^{\sigma}\rangle &= \zeta_{1\sigma'}\sum_{\sigma''}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1\sigma''}}{\epsilon+\psi}+\rho_{2\sigma''}\big)\langle E_{\mathbf{n}}^{\sigma''}E_{-\mathbf{n}}^{\sigma}\rangle \\ &+ \frac{\zeta_{2\sigma'}}{\epsilon+\psi}\sum_{\sigma''}\left(\chi_{1\sigma''}\langle E_{\mathbf{n}}^{\sigma''}E_{-\mathbf{n}}^{\sigma}\rangle + \psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1\sigma''}}{\epsilon+\psi}+\rho_{2\sigma''}\big)\langle E_{\mathbf{n}}^{\sigma''}E_{-\mathbf{n}}^{\sigma}\rangle\right) \ . \end{aligned}\tag{5.26}$$

Uma relação de dispersão

$$D_3 = 0 \tag{5.27}$$

pode ser obtida do sistema de equações que surge de (5.26) quando as espécies do plasma, elétrons e íons, são definidas. Um exemplo é dado na próxima seção. Se $\omega_{3\mathbf{k}}^{\gamma}$ são as raízes de D_3, então podemos escrever para cada par $\sigma'\sigma$

$$\frac{\mathrm{Re}}{2}\left(\frac{\partial D_3(j_3\omega_{3\mathbf{k}}^{\gamma})}{\partial\omega}\right)\frac{\partial\mathcal{I}_{\mathbf{k},\sigma'\sigma}^{\gamma j_3}}{\partial t} = -\mathrm{Im}\,(D_3(j_3\omega_{3\mathbf{k}}^{\gamma}))\mathcal{I}_{\mathbf{k},\sigma'\sigma}^{\gamma j_3} \ , \tag{5.28}$$

onde, em analogia com os dois casos anteriores,

$$\langle E_{\mathbf{n}}^{\sigma'} E_{-\mathbf{n}}^{\sigma}\rangle = \sum_{\gamma}\sum_{j_3=\pm}\mathcal{I}_{\mathbf{k},\sigma'\sigma}^{\gamma j_3}\delta(\omega+j_3\omega_{3\mathbf{k}}^{\gamma}) \ . \tag{5.29}$$

É importante notar que existem tantas equações (5.28) quanto existem combinações entre tipos iguais e diferentes de partículas do plasma.

A evolução temporal das funções de distribuição das partículas do plasma e das partículas de poeira são dadas por (3.24) e (3.25). Por substituição de (5.1) e (5.2),

$$\frac{\partial f_\sigma(\mathbf{v},t')}{\partial t'} = -\omega_\sigma f_\sigma - i\frac{q_\sigma}{m_\sigma}\int d\mathbf{n}\ \frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}$$
$$\times\Big(\mathbf{g}^\sigma_\mathbf{n}\cdot\frac{\mathbf{k}}{k}\big\langle E^2_\mathbf{n}\big\rangle + h^\sigma_\mathbf{n}\big\langle E^\sigma_\mathbf{n} E_{-\mathbf{n}}\big\rangle\Big) f_\sigma$$
$$-i\frac{q_\sigma}{m_\sigma v_*}\int d\mathbf{n}\ \Big(\mathbf{g}^\sigma_\mathbf{n}\cdot\frac{\mathbf{k}}{k}\big\langle E_\mathbf{n} E^\sigma_{-\mathbf{n}}\big\rangle + h^\sigma_\mathbf{n}\big\langle E^{\sigma 2}_\mathbf{n}\big\rangle\Big) f_\sigma, \tag{5.30}$$

$$\frac{\partial f_d(\mathbf{v},q,t')}{\partial t'} = -I\frac{\partial f_d}{\partial q} - i\frac{q}{m_d}\int d\mathbf{n}\ \frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}$$
$$\times\Big(h^d_\mathbf{n}\big\langle \hat{E}_\mathbf{n} E_{-\mathbf{n}}\big\rangle + \mathbf{g}^d_\mathbf{n}\cdot\frac{\mathbf{k}}{k}\big\langle E^2_\mathbf{n}\big\rangle\Big) f_d$$
$$-i\frac{e^2}{m_d v_*}\frac{\partial}{\partial q}\int d\mathbf{n}\ \Big(h^d_\mathbf{n}\big\langle \hat{E}^2_\mathbf{n}\big\rangle + \mathbf{g}^d_\mathbf{n}\cdot\frac{\mathbf{k}}{k}\big\langle E_\mathbf{n}\hat{E}_{-\mathbf{n}}\big\rangle\Big) f_d\ . \tag{5.31}$$

Até aqui somente correlações entre campos iguais foram dadas. As expressões para as correlações entre campos diferentes podem ser obtidas da combinação das equações (5.4), (5.5), (5.16), (5.23), (5.11), (5.21) e (5.29),

$$\big\langle E^\sigma_\mathbf{n} E_{-\mathbf{n}}\big\rangle = \zeta_{1\sigma}\big(\rho_1\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_\mathbf{k}\delta(\omega + j_1\omega^\alpha_{1\mathbf{k}})$$
$$+\sum_{\sigma'}\rho_{2\sigma'}\big\langle E^{\sigma'}_\mathbf{n} E_{-\mathbf{n}}\big\rangle\big) + \zeta_{2\sigma}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_\mathbf{k}\delta(\omega + j_1\omega^\alpha_{1\mathbf{k}})\ , \tag{5.32}$$

$$\big\langle \hat{E}_\mathbf{n} E_{-\mathbf{n}}\big\rangle = \frac{\rho_1 + \sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1 - \sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_\mathbf{k}\delta(\omega + j_1\omega^\alpha_{1\mathbf{k}})\ , \tag{5.33}$$

$$\big\langle E_\mathbf{n}\hat{E}_{-\mathbf{n}}\big\rangle = \frac{\sum_\sigma \chi_{1\sigma}\zeta_{1\sigma} + \psi_1}{\epsilon + \psi - \sum_\sigma \chi_{1\sigma}\zeta_{2\sigma}}\sum_\beta\sum_{j_2=\pm}\hat{\mathcal{I}}^{\beta j_2}_\mathbf{k}\delta(\omega + j_2\omega^\beta_{2\mathbf{k}})\ , \tag{5.34}$$

$$\big\langle E_\mathbf{n} E^\sigma_{-\mathbf{n}}\big\rangle = \frac{1}{\epsilon+\psi}\sum_{\sigma''}\Big(\chi_{1\sigma''} + \psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}$$
$$\times\big(\frac{\rho_1\chi_{1\sigma''}}{\epsilon+\psi} + \rho_{2\sigma''}\big)\Big)\sum_\gamma\sum_{j_3=\pm}\mathcal{I}^{\gamma j_3}_{\mathbf{k},\sigma''\sigma}\delta(\omega + j_3\omega^\gamma_{3\mathbf{k}})\ . \tag{5.35}$$

As expressões (5.33), (5.34) e (5.35) podem ser inseridas em (5.30) e (5.31), mas (5.32) precisa, como já mencionado, uma definição prévia das espécies do plasma. Por exemplo, num plasma composto por elétrons, íons e poeira devemos encontrar

$$\left\langle E^e_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle = \frac{(\zeta_{1e}\rho_1 + \zeta_{2e})(1 - \zeta_{1i}\rho_{2i}) + (\zeta_{1i}\rho_1 + \zeta_{2i})\zeta_{1e}\rho_{2i}}{(1 - \zeta_{1e}\rho_{2e})(1 - \zeta_{1i}\rho_{2i}) - \zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}} \left\langle E^2_{\mathbf{n}} \right\rangle$$
$$, \quad \left\langle E^i_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle = \left\langle E^e_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle (e \leftrightarrows i) \ . \quad (5.36)$$

Com substituição nas equações cinéticas, o resultado é um sistema de equações fechado dado pelas equações (5.7), (5.15), (5.20), (5.22), (5.27), (5.28), (5.30), (5.31) e (5.32), que descreve a evolução temporal do sistema na aproximação quasilinear.

A dependência explícita sobre a carga da poeira q na equação cinética para a distribuição da poeira é complementada com a equação para a corrente de carregamento, $I = dq/dt$, que é dada pela equação (3.6), e pela condição de neutralidade de carga,

$$\sum_\sigma n_\sigma q_\sigma + n_d Z e = 0 \ . \quad (5.37)$$

5.1 O caso de um plasma empoeirado de duas componentes

Consideremos o caso de um plasma empoeirado composto de elétrons e somente uma espécie de íon. As equações cinéticas explícitas do plasma são dadas por

$$\frac{\mathrm{Re}}{2}\left(\frac{\partial \epsilon(j\omega^\alpha_{1\mathbf{k}})}{\partial \omega} + \frac{\partial \psi(j\omega^\alpha_{1\mathbf{k}})}{\partial \omega} \right) \frac{\partial \mathcal{I}^{\alpha j}_{\mathbf{k}}}{\partial t} = -\mathrm{Im}\left(D_1(j\omega^\alpha_{1\mathbf{k}}) \right) \mathcal{I}^{\alpha j}_{\mathbf{k}} \ , \quad (5.38)$$

$$\frac{\mathrm{Re}}{2}\left(\frac{\partial D_2(j\omega^\beta_{2\mathbf{k}})}{\partial \omega} \right) \frac{\partial \hat{\mathcal{I}}^{\beta j}_{\mathbf{k}}}{\partial t} = -\mathrm{Im}\left(D_2(j\omega^\beta_{2\mathbf{k}}) \right) \hat{\mathcal{I}}^{\beta j}_{\mathbf{k}} \ , \quad (5.39)$$

$$\frac{\mathrm{Re}}{2}\left(\frac{\partial D_3(j\omega^\gamma_{3\mathbf{k}})}{\partial \omega} \right) \frac{\partial \mathcal{I}^{\gamma j}_{\mathbf{k},\sigma'\sigma}}{\partial t} = -\mathrm{Im}\left(D_3(j\omega^\gamma_{3\mathbf{k}}) \right) \mathcal{I}^{\gamma j}_{\mathbf{k},\sigma'\sigma} \ , \quad (5.40)$$

onde em cada caso o índice $\alpha, \beta, \gamma = 0, 1, \ldots, n$, onde n é o número de raízes da relação de dispersão correspondente. Na equação (5.40), os índices $\sigma'\sigma$ indicam as combinações

ee, ei, ie, e ii. Além disso, D_1, D_2 e D_3 são dadas por,

$$D_1 = \epsilon + \psi - \sum_\sigma \chi_{1\sigma}\zeta_{2\sigma} - \left(\psi_1 + \sum_\sigma \chi_{1\sigma}\zeta_{1\sigma}\right)\frac{\rho_1 + \sum_{\sigma'} \rho_{2\sigma'}\zeta_{2\sigma'}}{1 - \sum_{\sigma'} \rho_{2\sigma'}\zeta_{1\sigma'}} \ , \tag{5.41}$$

$$D_2 = 1 - \left(\rho_1 + \sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}\right)\frac{\sum_{\sigma'} \chi_{1\sigma'}\zeta_{1\sigma'} + \psi_1}{\epsilon + \psi - \sum_{\sigma'} \chi_{1\sigma'}\zeta_{2\sigma'}} - \sum_\sigma \rho_{2\sigma}\zeta_{1\sigma} \ , \tag{5.42}$$

$$\begin{aligned} D_3 = &\left(1 - \zeta_{1e}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big) - \frac{\zeta_{2e}}{\epsilon+\psi}\left(\chi_{1e}+\psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big)\right)\right) \\ &\times\left(1 - \zeta_{1i}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big) - \frac{\zeta_{2i}}{\epsilon+\psi}\left(\chi_{1i}+\psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\right)\right) \\ &-\left(\zeta_{1e}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big) + \frac{\zeta_{2e}}{\epsilon+\psi}\left(\chi_{1i}+\psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\right)\right) \\ &\times\left(\zeta_{1i}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big) + \frac{\zeta_{2i}}{\epsilon+\psi}\left(\chi_{1e}+\psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big)\right)\right) \ . \end{aligned} \tag{5.43}$$

Demonstrando a equação (5.43) obtida, para o caso de um plasma empoeirado elétron-íon, a equação (5.26) leva a um sistema acoplado de quatro equações,

$$\begin{aligned} \langle E_{\mathbf{n}}^{e2}\rangle = &\zeta_{1e}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\left(\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big)\langle E_{\mathbf{n}}^{e2}\rangle + \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\langle E_{\mathbf{n}}^{i}E_{-\mathbf{n}}^{e}\rangle\right) \\ &+\frac{\zeta_{2e}}{\epsilon+\psi}\left(\chi_{1e}\langle E_{\mathbf{n}}^{e2}\rangle + \psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big)\langle E_{\mathbf{n}}^{e2}\rangle\right) \\ &+\frac{\zeta_{2e}}{\epsilon+\psi}\left(\chi_{1i}\langle E_{\mathbf{n}}^{i}E_{-\mathbf{n}}^{e}\rangle + \psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\langle E_{\mathbf{n}}^{i}E_{-\mathbf{n}}^{e}\rangle\right) \ , \end{aligned}$$

$$\begin{aligned} \langle E_{\mathbf{n}}^{e}E_{-\mathbf{n}}^{i}\rangle = &\zeta_{1e}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\left(\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big)\langle E_{\mathbf{n}}^{e}E_{-\mathbf{n}}^{i}\rangle + \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\langle E_{\mathbf{n}}^{i2}\rangle\right) \\ &+\frac{\zeta_{2e}}{\epsilon+\psi}\left(\chi_{1e}\langle E_{\mathbf{n}}^{e}E_{-\mathbf{n}}^{i}\rangle + \psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi}+\rho_{2e}\big)\langle E_{\mathbf{n}}^{e}E_{-\mathbf{n}}^{i}\rangle\right) \\ &+\frac{\zeta_{2e}}{\epsilon+\psi}\left(\chi_{1i}\langle E_{\mathbf{n}}^{i2}\rangle + \psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\langle E_{\mathbf{n}}^{i2}\rangle\right) \ , \end{aligned}$$

$$\big\langle E^i_{\mathbf{n}} E^e_{-\mathbf{n}} \big\rangle = \zeta_{1i} \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \Bigg(\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) \big\langle E^{e2}_{\mathbf{n}} \big\rangle \Bigg) + \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi} + \rho_{2i}\big) \big\langle E^i_{\mathbf{n}} E^e_{-\mathbf{n}} \big\rangle \Bigg)$$
$$+ \frac{\zeta_{2i}}{\epsilon+\psi} \Bigg(\chi_{1e} \big\langle E^{e2}_{\mathbf{n}} \big\rangle + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) \big\langle E^{e2}_{\mathbf{n}} \big\rangle \Bigg)$$
$$+ \frac{\zeta_{2i}}{\epsilon+\psi} \Bigg(\chi_{1i} \big\langle E^i_{\mathbf{n}} E^e_{-\mathbf{n}} \big\rangle + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi} + \rho_{2i}\big) \big\langle E^i_{\mathbf{n}} E^e_{-\mathbf{n}} \big\rangle \Bigg) ,$$

$$\big\langle E^{i2}_{\mathbf{n}} \big\rangle = \zeta_{1i} \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \Bigg(\big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) \big\langle E^e_{\mathbf{n}} E^i_{-\mathbf{n}} \big\rangle + \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi} + \rho_{2i}\big) \big\langle E^{i2}_{\mathbf{n}} \big\rangle \Bigg)$$
$$+ \frac{\zeta_{2i}}{\epsilon+\psi} \Bigg(\chi_{1e} \big\langle E^e_{\mathbf{n}} E^i_{-\mathbf{n}} \big\rangle + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) \big\langle E^e_{\mathbf{n}} E^i_{-\mathbf{n}} \big\rangle \Bigg)$$
$$+ \frac{\zeta_{2i}}{\epsilon+\psi} \Bigg(\chi_{1i} \big\langle E^{i2}_{\mathbf{n}} \big\rangle + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi} + \rho_{2i}\big) \big\langle E^{i2}_{\mathbf{n}} \big\rangle \Bigg) .$$

Escrevendo o sistema acima em forma matricial, um determinante 4×4 é obtido dos coeficientes,

$$D_3 = \begin{vmatrix} D_{11} & 0 & D_{13} & 0 \\ 0 & D_{11} & 0 & D_{13} \\ D_{31} & 0 & D_{33} & 0 \\ 0 & D_{31} & 0 & D_{33} \end{vmatrix} = D_{11}D_{33} - D_{13}D_{31}, \tag{5.44}$$

onde

$$D_{11} = 1 - \zeta_{1e} \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) - \frac{\zeta_{2e}}{\epsilon+\psi} \Bigg(\chi_{1e} + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) \Bigg) ,$$

$$D_{13} = -\zeta_{1e} \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi} + \rho_{2i}\big) - \frac{\zeta_{2e}}{\epsilon+\psi} \Bigg(\chi_{1i} + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi} + \rho_{2i}\big) \Bigg) ,$$

$$D_{31} = -\zeta_{1i} \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) - \frac{\zeta_{2i}}{\epsilon+\psi} \Bigg(\chi_{1e} + \psi_1 \frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1} \big(\frac{\rho_1\chi_{1e}}{\epsilon+\psi} + \rho_{2e}\big) \Bigg) ,$$

$$D_{33} = 1 - \zeta_{1i}\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big) - \frac{\zeta_{2i}}{\epsilon+\psi}\left(\chi_{1i}+\psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1i}}{\epsilon+\psi}+\rho_{2i}\big)\right) .$$

A relação de dispersão dada pela equação (5.43) corresponde à equação (5.44) igual a zero.

A conexão das equações cinéticas para f_σ e f_d com as propriedades dispersivas do plasma torna-se mais explícita com integração em termos de (5.11), (5.21), (5.33), (5.34), (5.35) e (5.36),

$$\begin{aligned}
\frac{\partial f_\sigma(\mathbf{v},t')}{\partial t'} &= -i\frac{q_\sigma}{m_\sigma}\sum_\alpha\sum_{j=\pm}\int d\mathbf{k}\,\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\left(\mathbf{g}^\sigma_{\mathbf{k},j\omega^\alpha_{1\mathbf{k}}}\cdot\frac{\mathbf{k}}{k}\right)\mathcal{I}^{\alpha j}_{\mathbf{k}} f_\sigma - i\frac{q_\sigma}{m_\sigma}\sum_\alpha\sum_{j=\pm}\int d\mathbf{k}\,\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}} \\
&\times\left(h^e_{\mathbf{k},j\omega^\alpha_{1\mathbf{k}}}\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\delta_{\sigma e}+\delta_{\sigma i}(e\leftrightarrows i)\right)\mathcal{I}^{\alpha j}_{\mathbf{k}} f_\sigma \\
&\quad -i\frac{q_\sigma}{m_\sigma v_*}\sum_\gamma\sum_{j=\pm}\int d\mathbf{k}\,\left(\left(\mathbf{g}^\sigma_{\mathbf{k},j\omega^\gamma_{3\mathbf{k}}}\cdot\frac{\mathbf{k}}{k}\right)\right. \\
&\times\frac{1}{\epsilon+\psi}\sum_{\sigma''}\left(\chi_{1\sigma''}+\psi_1\frac{\epsilon+\psi}{\epsilon+\psi-\rho_1\psi_1}\big(\frac{\rho_1\chi_{1\sigma''}}{\epsilon+\psi}+\rho_{2\sigma''}\big)\right)\mathcal{I}^{\gamma j}_{\mathbf{k},\sigma''\sigma} \\
&\quad\left.+h^\sigma_{\mathbf{k},j\omega^\gamma_{3\mathbf{k}}}\mathcal{I}^{\gamma j}_{\mathbf{k},\sigma\sigma}\right)f_\sigma-\omega_\sigma f_\sigma\ ,
\end{aligned}\tag{5.45}$$

$$\begin{aligned}
\frac{\partial f_d(\mathbf{v},q,t')}{\partial t'} &= -i\frac{q}{m_d}\sum_\alpha\sum_{j=\pm}\int d\mathbf{k}\,\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}\left(h^d_{\mathbf{k},j\omega^\alpha_{1\mathbf{k}}}\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}+g^d_{\mathbf{k},j\omega^\alpha_{1\mathbf{k}}}\right)\mathcal{I}^{\alpha j}_{\mathbf{k}} f_d \\
-i\frac{e^2}{m_d v_*}\sum_\beta\sum_{j=\pm}\frac{\partial}{\partial q}\int d\mathbf{k}\,&\left(h^d_{\mathbf{k},j\omega^\beta_{2\mathbf{k}}}+g^d_{\mathbf{k},j\omega^\beta_{2\mathbf{k}}}\frac{\sum_\sigma\chi_{1\sigma}\zeta_{1\sigma}+\psi_1}{\epsilon+\psi-\sum_\sigma\chi_{1\sigma}\zeta_{2\sigma}}\right)\hat{\mathcal{I}}^{\beta j}_{\mathbf{k}} f_d - I\frac{\partial f_d}{\partial q}\ .
\end{aligned}\tag{5.46}$$

Exemplificando um caso concreto, no caso de uma distribuição de velocidades do tipo Maxwell-Boltzmann e uma distribuição de Gibbs da carga da poeira em torno da carga de quilíbrio adimensional Z_0, temos que a relação de dispersão (5.41) é dada por

$$
\begin{aligned}
1 &+ \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{2}{\ell^2 u_\sigma^2}\big(1+\zeta_\sigma Z(\zeta_\sigma)\big) + \frac{\omega_{pd}^2}{\omega_*^2}\frac{(Z_d^2/Z_0^2+2)}{\ell^2 u_d^2}\big(1+\zeta_d Z(\zeta_d)\big) \\
&+ i\sqrt{\pi}\sum_\sigma \frac{\omega_{pd}^2}{\omega_*^2}\frac{a^2 v_* n_\sigma}{\ell^3 u_d^2 \omega_*}\frac{q_\sigma}{eZ_0}\left(1-\frac{\ell^2 g^2}{8}\right)Z(\zeta_\sigma)(1+\zeta_d Z(\zeta_d)) \\
-i\frac{1}{Z_0^2}\frac{\omega_{pd}^2}{\omega_*^2}\frac{1}{\ell^2 u_d}Z(\zeta_d)&\Bigg(\sum_\sigma n_\sigma \frac{g^2}{\ell u_\sigma^2}\frac{v_*^3}{\omega_*^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\Bigg(\sqrt{\pi}\big(1+\zeta_\sigma Z(\zeta_\sigma)\big)-\ell^2 g^2\frac{\sqrt{\pi}}{8}-\ell^2 g^2\zeta_\sigma^2\frac{\sqrt{\pi}}{2}+i\ell g\zeta_\sigma\frac{\pi}{2} \\
&+ 4i\ell g\zeta_\sigma^3\big(\sqrt{\eta}-\arctan\sqrt{\eta}\big) + 2i\ell g\zeta_\sigma^4\int_0^\infty ds\,\frac{s^{1/2}}{(1+s)^{1/2}}Z(\zeta_\sigma^s) \\
&- \ell^2 g^2\zeta_\sigma \lim_{\nu\to 0}\int_0^\infty ds\,\frac{s^{1/2}}{1+s}Z(\nu,-\zeta_\sigma^s) - \ell^2 g^2\zeta_\sigma\int_0^\infty ds\,\frac{s^{1/2}}{1+s}Z(\zeta_\sigma^s)\Bigg) \\
&+ i\sqrt{\pi}\sum_\sigma \frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\Bigg(\frac{\sqrt{\pi}}{2}Z(\zeta_\sigma)+i\ell g\frac{\pi}{4}+2i\ell g\zeta_\sigma^2\big(\sqrt{\eta}-\arctan\sqrt{\eta}\big) \\
&+ i\ell g\zeta_\sigma^3\int_0^\infty ds\,\frac{s^{1/2}}{(1+s)^{1/2}}Z(\zeta_\sigma^s) - \ell^2 g^2\zeta_\sigma\frac{\sqrt{\pi}}{4} - \frac{\ell^2 g^2}{2}\lim_{\nu\to 0}\int_0^\infty ds\,\frac{s^{1/2}}{1+s}Z(\nu,-\zeta_\sigma^s) \\
&- \frac{\ell^2 g^2}{2}\int_0^\infty ds\,\frac{s^{1/2}}{1+s}Z(\zeta_\sigma^s)\Bigg)\Bigg(\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{u_\sigma}{u_d^2}\left(1-\frac{\ell^2 g^2}{8}\right)(1+\zeta_d Z(\zeta_d))\Bigg)\Bigg) = 0\,.
\end{aligned}
\tag{5.47}
$$

As integrações aplicadas em (5.47) estão no Apêndice A.

As equações (5.38), (5.39), (5.40), (5.41), (5.42), (5.44), (5.45) e (5.46) formam um conjunto fechado para uma análise quase linear. A partir delas poderemos analisar quais os efeitos da presença da poeira em propriedades como conservação de partículas, momentum, energia e carga. Presumimos que haja uma perda das três primeiras propriedades, visto que as colisões são inelásticas e termos de recuo dos grãos de poeira devido às colisões foram desprezados inicialmente no Capítulo 2.

No que diz respeito às propriedades dispersivas, a relação de dispersão (5.41) é linear em ϵ e ψ, e as relações de dispersão (5.42) e (5.44) são lineares em 1, mas todas também são não lineares por produtos de susceptibilidades até terceira ordem. Portanto, as raízes $\omega_{\mathbf{1k}}^\alpha$, por exemplo, correspondem a modos lineares, como Langmuir e íon-acústicos, com correções não lineares devido à absorção das partiículas pelos grãos de poeira.

Capítulo 6

Evolução temporal não linear de $\left\langle E_{\mathbf{n}}^2 \right\rangle$

A evolução temporal não linear de $\left\langle E_{\mathbf{n}}^2 \right\rangle$ pode ser obtida procedendo de forma similar ao caso quase linear.

Por simplicidade, escrevemos por enquanto todos os termos não lineares de (4.17) na variável $NL1$. Então consideramos a substituição das flutuações expandidas (3.41) e (3.44) em (3.29) e (3.38), incluindo todos termos não lineares até terceira ordem nos campos. O resultado é, em forma condensada, o sistema

$$\begin{aligned}
\left\langle E_{\mathbf{n}}^2 \right\rangle (\epsilon + \psi) &= \textstyle\sum_\sigma \chi_{1\sigma} \left\langle E_{\mathbf{n}}^\sigma E_{-\mathbf{n}} \right\rangle + \psi_1 \left\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle + NL1 \;, && (6.1)\\
\left\langle E_{\mathbf{n}}^\sigma E_{-\mathbf{n}} \right\rangle &= \zeta_{1\sigma} \left\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle + \zeta_{2\sigma} \left\langle E_{\mathbf{n}}^2 \right\rangle + NL2 \;, && (6.2)\\
\left\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle &= \rho_1 \left\langle E_{\mathbf{n}}^2 \right\rangle + \sum_\sigma \rho_{2\sigma} \left\langle E_{\mathbf{n}}^\sigma E_{-\mathbf{n}} \right\rangle + NL3 \;, && (6.3)
\end{aligned}$$

onde $NL2$ e $NL3$ são também outros diversos termos não lineares, que manteremos implícitos. Combinando essas equações de modo a obter uma expressão para $\left\langle E_{\mathbf{n}}^2 \right\rangle$, fazemos a substituição de (6.2) em (6.1) e (6.3),

$$\left\langle E_{\mathbf{n}}^2 \right\rangle (\epsilon + \psi) = \sum_\sigma \chi_{1\sigma} \big(\zeta_{1\sigma} \langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \rangle + \zeta_{2\sigma} \langle E_{\mathbf{n}}^2 \rangle + NL2 \big) + \psi_1 \left\langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \right\rangle + NL1 \;, \quad (6.4)$$

$$(1 - \sum_\sigma \rho_{2\sigma} \zeta_{1\sigma}) \langle \hat{E}_{\mathbf{n}} E_{-\mathbf{n}} \rangle = \rho_1 \langle E_{\mathbf{n}}^2 \rangle + \sum_\sigma \rho_{2\sigma} \zeta_{2\sigma} \langle E_{\mathbf{n}}^2 \rangle + \sum_\sigma \rho_{2\sigma} NL2 + NL3 \;. \quad (6.5)$$

Com a substituição de (6.5) em (6.4)

$$\langle E_{\mathbf{n}}^2\rangle(\epsilon+\psi)=\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+\zeta_{2\sigma}\right)\langle E_{\mathbf{n}}^2\rangle+\psi_1\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}\langle E_{\mathbf{n}}^2\rangle$$

$$+\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\sum_{\sigma'}\rho_{2\sigma'}NL2+NL3}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+NL2\right)+\psi_1\frac{\sum_{\sigma'}\rho_{2\sigma'}NL2+NL3}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+NL1$$

$$\langle E_{\mathbf{n}}^2\rangle\left(\epsilon+\psi-\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+\zeta_{2\sigma}\right)-\psi_1\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}\right)$$

$$=\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\sum_{\sigma'}\rho_{2\sigma'}NL2+NL3}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+NL2\right)+\psi_1\frac{\sum_{\sigma'}\rho_{2\sigma'}NL2+NL3}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+NL1\ . \quad (6.6)$$

A equação (6.6) difere de (5.6) pelo lado direito não-nulo. No contexto de (5.12) obtemos

$$\left(\epsilon(\mathbf{n})+\psi(\mathbf{n})+\frac{i}{2}\left(\frac{\partial\mathrm{Re}\,\epsilon}{\partial\omega}+\frac{\partial\mathrm{Re}\,\psi}{\partial\omega}\right)\frac{\partial}{\partial t}\right)\langle E_{\mathbf{n}}^2\rangle$$
$$=\langle E_{\mathbf{n}}^2\rangle\left(\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+\zeta_{2\sigma}\right)+\psi_1\frac{\rho_1+\sum_{\sigma'}\rho_{2\sigma'}\zeta_{2\sigma'}}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}\right)$$
$$+\sum_\sigma \chi_{1\sigma}\left(\zeta_{1\sigma}\frac{\sum_{\sigma'}\rho_{2\sigma'}NL2+NL3}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+NL2\right)+\psi_1\frac{\sum_{\sigma'}\rho_{2\sigma'}NL2+NL3}{1-\sum_{\sigma'}\rho_{2\sigma'}\zeta_{1\sigma'}}+NL1\ , \quad (6.7)$$

que corresponde à equação cinética não linear para as ondas. O limite quase linear corresponde a fazer $NL1\to 0$, $NL2\to 0$, $NL3\to 0$, de modo que (6.7) se reduz a (5.13).

Podemos perceber, tomando $NL1$ em (4.17) como ilustração, que no caso não linear há um acoplamento de forma integral com as outras duas correlações, $\langle \hat{E}_{\mathbf{a}}\hat{E}_{\mathbf{b}}\rangle$ e $\langle E_{\mathbf{c}}^\sigma E_{\mathbf{d}}^\sigma\rangle$, entre diversas outras. De forma explícita, baseado nas integrações que fizemos no Apêndice C, em termos das intensidades das ondas, $NL1$ é dado por

$$NL1=\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle \quad (6.8)$$

$$+\sum_\alpha\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_3\left(\chi_1^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\mathcal{I}_{-\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3-j_1\omega_{1,-\mathbf{k}_3}^{\alpha})\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})\right.$$

$$+\chi_1^{(3)}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})\mathcal{I}_{-\mathbf{k}_3}^{\alpha' j_1'}\delta(\omega_3-j_1'\omega_{1,-\mathbf{k}_3}^{\alpha'})\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\Bigg(\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_3\big(\chi_{2e}^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\mathcal{I}_{-\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3-j_1\omega_{1,-\mathbf{k}_3}^{\alpha})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\chi_{2e}^{(3)}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_3}\mathcal{I}_{-\mathbf{k}_3}^{\alpha' j_1'}\delta(\omega_3-j_1'\omega_{1,-\mathbf{k}_3}^{\alpha'})\big)$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}$$

$$\times\Big(\int d\mathbf{n}_1\chi_{3e}^{(3)}(\mathbf{n}_1;\mathbf{n};-\mathbf{n}_1)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_1}\mathcal{I}_{-\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1-j_1\omega_{1,-\mathbf{k}_1}^{\alpha})$$

$$\times\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\int d\mathbf{n}_2\chi_{3e}^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\Big)$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_1\chi_{4e}^{(3)}(\mathbf{n}_1;\mathbf{n};-\mathbf{n}_1)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_1}\mathcal{I}_{-\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1-j_1\omega_{1,-\mathbf{k}_1}^{\alpha})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\gamma}\sum_{j_3=\pm}\int d\mathbf{n}_2\chi_{4e}^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})\mathcal{I}_{-\mathbf{k}_2,ee}^{\gamma j_3}\delta(\omega_2-j_3\omega_{3,-\mathbf{k}_2}^{\gamma})$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_3\big(\chi_{5e}^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_3}\mathcal{I}_{-\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3-j_1\omega_{1,-\mathbf{k}_3}^{\alpha})\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\chi_{5e}^{(3)}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})$$

$$\times\mathcal{I}_{-\mathbf{k}_3}^{\alpha' j_1'}\delta(\omega_3-j_1'\omega_{1,-\mathbf{k}_3}^{\alpha'})\big)$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_3\big(\chi^{(3)}_{6e}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_3}\mathcal{I}^{\alpha j_1}_{-\mathbf{k}_3}\delta(\omega_3-j_1\omega^{\alpha}_{1,-\mathbf{k}_3})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}}\delta(\omega+j_1'\omega^{\alpha'}_{1\mathbf{k}})$$

$$+\chi^{(3)}_{6e}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_3}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_2}\delta(\omega_3-j_1\omega^{\alpha}_{1,-\mathbf{k}_3})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}}\delta(\omega+j_1'\omega^{\alpha'}_{1\mathbf{k}})\big)$$

$$+\sum_{\gamma}\sum_{j_3=\pm}\sum_{\alpha}\sum_{j_1=\pm}\int d\mathbf{n}_1\chi^{(3)}_{7e}(\mathbf{n}_1;\mathbf{n};-\mathbf{n}_1)\mathcal{I}^{\gamma j_3}_{-\mathbf{k}_1,ee}\delta(\omega_1-j_3\omega^{\gamma}_{3,-\mathbf{k}_1})\mathcal{I}^{\alpha j_1}_{\mathbf{k}}\delta(\omega+j_1\omega^{\alpha}_{1\mathbf{k}})$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_2\chi^{(3)}_{7e}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha j_1}_{-\mathbf{k}_2}\delta(\omega_2-j_1\omega^{\alpha}_{1,-\mathbf{k}_2})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}}\delta(\omega+j_1'\omega^{\alpha'}_{1\mathbf{k}})$$

$$+\sum_{\gamma}\sum_{j_3=\pm}\sum_{\alpha}\sum_{j_1=\pm}\int d\mathbf{n}_3\big(\chi^{(3)}_{8e}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\mathcal{I}^{\gamma j_3}_{-\mathbf{k}_3,ee}\delta(\omega_3-j_3\omega^{\gamma}_{3,-\mathbf{k}_3})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha j_1}_{\mathbf{k}}\delta(\omega+j_1\omega^{\alpha}_{1\mathbf{k}})$$

$$+\chi^{(3)}_{8e}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha j_1}_{\mathbf{k}}\delta(\omega+j_1\omega^{\alpha}_{1\mathbf{k}})$$

$$\times\mathcal{I}^{\gamma j_3}_{-\mathbf{k}_3,ee}\delta(\omega_3-j_3\omega^{\gamma}_{3,-\mathbf{k}_3})\big)$$

$$+\big(\quad e\leftrightarrows i\quad\big)\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\psi^{(2)}_4(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle$$

$$+\Big(\sum_{\beta}\sum_{j_2=\pm}\sum_{\alpha}\sum_{j_1=\pm}\big(\int d\mathbf{n}_1\psi^{(3)}_1(\mathbf{n}_1;\mathbf{n};-\mathbf{n}_1)\hat{\mathcal{I}}^{\beta j_2}_{-\mathbf{k}_1}\delta(\omega_1-j_2\omega^{\beta}_{2,-\mathbf{k}_1})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha j_1}_{\mathbf{k}}\delta(\omega+j_1\omega^{\alpha}_{1\mathbf{k}})$$

$$+\int d\mathbf{n}_2\psi_1^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\hat{\mathcal{I}}_{-\mathbf{k}_2}^{\beta j_2}\delta(\omega_2-j_2\omega_{2,-\mathbf{k}_2}^{\beta})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})\big)$$

$$+\sum_\alpha\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_1\psi_2^{(3)}(\mathbf{n}_1;\mathbf{n};-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_1}\mathcal{I}_{-\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1-j_1\omega_{1,-\mathbf{k}_1}^{\alpha})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\sum_\beta\sum_{j_2=\pm}\sum_\alpha\sum_{j_1=\pm}\int d\mathbf{n}_2\psi_2^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\hat{\mathcal{I}}_{-\mathbf{k}_2}^{\beta j_2}\delta(\omega_2-j_2\omega_{2,-\mathbf{k}_2}^{\beta})\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})$$

$$+\sum_\alpha\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_3\big(\psi_3^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_3}\mathcal{I}_{-\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3-j_1\omega_{1,-\mathbf{k}_3}^{\alpha})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\psi_3^{(3)}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_3}\mathcal{I}_{-\mathbf{k}_3}^{\alpha' j_1'}\delta(\omega_3-j_1'\omega_{1,-\mathbf{k}_3}^{\alpha'})\big)$$

$$+\sum_\alpha\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_3\big(\psi_4^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\mathcal{I}_{-\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3-j_1\omega_{1,-\mathbf{k}_3}^{\alpha})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\psi_4^{(3)}(\mathbf{n};-\mathbf{n}_3;\mathbf{n}_3)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_3}\mathcal{I}_{-\mathbf{k}_3}^{\alpha' j_1'}\delta(\omega_3-j_1'\omega_{1,-\mathbf{k}_3}^{\alpha'})\big)$$

$$+\sum_\beta\sum_{j_2=\pm}\sum_\alpha\sum_{j_1=\pm}\int d\mathbf{n}_3\psi_5^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\hat{\mathcal{I}}_{-\mathbf{k}_3}^{\beta j_2}\delta(\omega_3-j_2\omega_{2,-\mathbf{k}_3}^{\beta})\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})$$

$$+\sum_\alpha\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\int d\mathbf{n}_2\psi_5^{(3)}(\mathbf{n}_1;\mathbf{n}_2;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$+\sum_\alpha\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\Big(\int d\mathbf{n}_1\psi_6^{(3)}(\mathbf{n}_1;\mathbf{n};-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_1}\mathcal{I}_{-\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1-j_1\omega_{1,-\mathbf{k}_1}^{\alpha})$$

$$\times\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\int d\mathbf{n}_2\psi_6^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}}^{\alpha j_1}\delta(\omega+j_1\omega_{1\mathbf{k}}^{\alpha})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\Big)$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\Big(\int d\mathbf{n}_3\psi_7^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_3}\mathcal{I}^{\alpha j_1}_{-\mathbf{k}_3}\delta(\omega_3-j_1\omega^{\alpha}_{1,-\mathbf{k}_3})$$

$$\times\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}}\delta(\omega+j_1'\omega^{\alpha'}_{1\mathbf{k}})$$

$$+\int d\mathbf{n}_2\psi_7^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha j_1}_{\mathbf{k}}\delta(\omega+j_1\omega^{\alpha}_{1\mathbf{k}})$$

$$\times\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\Big)$$

$$+\sum_{\alpha}\sum_{j_1=\pm}\sum_{\alpha'}\sum_{j_1'=\pm}\Big(\int d\mathbf{n}_3\psi_8^{(3)}(-\mathbf{n}_3;\mathbf{n};\mathbf{n}_3)\mathcal{I}^{\alpha j_1}_{-\mathbf{k}_3}\delta(\omega_3-j_1\omega^{\alpha}_{1,-\mathbf{k}_3})\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}}\delta(\omega+j_1'\omega^{\alpha'}_{1\mathbf{k}})$$

$$+\int d\mathbf{n}_2\psi_8^{(3)}(\mathbf{n};\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}^{\alpha j_1}_{\mathbf{k}}\delta(\omega+j_1\omega^{\alpha}_{1\mathbf{k}})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\Big)\Bigg)\,.$$

As funções de correlação tripla, dadas nas expressões $\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle$ e $\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\psi_4^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle$, podem ser escritas, como dissemos anteriormente, em termos das correlações quadráticas, de modo que podemos dar fechamento para a cadeia de equações em segunda ordem nas intensidades das ondas. No Apêndice C desenvolvemos uma expressão para $\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle$ dada por (C.16).

Escreveremos somente a expressão para o termo proporcional a $\chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)$, visto que o segundo, que é proporcional a $\psi_4^{(2)}(\mathbf{n}_1;\mathbf{n}_2)$, é dado de forma análoga.

Multiplicamos então a equação (C.16) por $\delta\big(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2\big)\chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)$. Integrando em $\mathbf{n}_1$ e $\mathbf{n}_2$ e dividindo por 3, obtemos

$$\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle \tag{6.9}$$

$$=\frac{1}{3}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}\int d\mathbf{n}_2\frac{\chi_1^{(2)}(\mathbf{n}-\mathbf{n}_2;\mathbf{n}_2)}{\epsilon(\mathbf{n}-\mathbf{n}_2)+\psi(\mathbf{n}-\mathbf{n}_2)}$$

$$\times\Bigg(\Bigg(\chi_1^{(2)}(-\mathbf{n}_2;\mathbf{n})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}}\delta(\omega+j_1''\omega^{\alpha''}_{1,\mathbf{k}})$$

$$+\chi_1^{(2)}(\mathbf{n};-\mathbf{n}_2)\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}'}\delta(\omega+j_1'\omega^{\alpha'}_{1,\mathbf{k}})\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})\Bigg)$$

$$+\Bigg(\chi_{2e}^{(2)}(-\mathbf{n}_2;\mathbf{n})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}}\delta(\omega+j_1''\omega^{\alpha''}_{1\mathbf{k}})$$

$$+\chi_{2e}^{(2)}(\mathbf{n};-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{3e}^{(2)}(-\mathbf{n}_2;\mathbf{n})\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\chi_{3e}^{(2)}(\mathbf{n};-\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{4e}^{(2)}(-\mathbf{n}_2;\mathbf{n})\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\chi_{4e}^{(2)}(\mathbf{n};-\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)+\Big(\quad e\leftrightarrows i\quad\Big)\Bigg)$$

$$+\frac{1}{3}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}\int d\mathbf{n}_2\frac{\chi_1^{(2)}(\mathbf{n}-\mathbf{n}_2;\mathbf{n}_2)}{\epsilon(\mathbf{n}-\mathbf{n}_2)+\psi(\mathbf{n}-\mathbf{n}_2)}$$

$$\times\Bigg(\Bigg(\psi_1^{(2)}(-\mathbf{n}_2;\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\psi_1^{(2)}(\mathbf{n};-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\psi_2^{(2)}(-\mathbf{n}_2;\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\psi_2^{(2)}(\mathbf{n};-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\Bigg)$$

$$+\Bigg(\psi_3^{(2)}(-\mathbf{n}_2;\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\psi_3^{(2)}(\mathbf{n};-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Big)$$

$$+\Big(\psi_4^{(2)}(-\mathbf{n}_2;\mathbf{n})\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\psi_4^{(2)}(\mathbf{n};-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\Big)\Big)$$

$$+\frac{1}{3}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}\int d\mathbf{n}_2\frac{\chi_1^{(2)}(\mathbf{n}-\mathbf{n}_2;\mathbf{n}_2)}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}$$

$$\times\Big(\Big(\chi_1^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha' j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\chi_1^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'' j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\Big)$$

$$+\Big(\chi_{2e}^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha' j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\chi_{2e}^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'' j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\Big)$$

$$+\Big(\chi_{3e}^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha' j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})$$

$$\times\mathcal{I}_{\mathbf{k}}^{\alpha'' j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\chi_{3e}^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha' j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'' j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\Big)$$

$$+\Big(\chi_{4e}^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha' j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha''j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\chi_{4e}^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha'j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha''j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\bigg)$$

$$+\Big(\quad e\leftrightarrows i\quad\Big)\bigg)$$

$$+\frac{1}{3}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}\int d\mathbf{n}_2\frac{\chi_1^{(2)}(\mathbf{n}-\mathbf{n}_2;\mathbf{n}_2)}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}$$

$$\times\Bigg(\bigg(\psi_1^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha''j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\psi_1^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha'j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha''j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\bigg)$$

$$+\bigg(\psi_2^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha''j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\mathcal{I}_{\mathbf{k}}^{\alpha'j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\psi_2^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha''j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})\bigg)$$

$$+\bigg(\psi_3^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})\mathcal{I}_{\mathbf{k}}^{\alpha''j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})$$

$$+\psi_3^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}}\mathcal{I}_{\mathbf{k}}^{\alpha'j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha''j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\bigg)$$

$$+\bigg(\psi_4^{(2)}(\mathbf{n}_2-\mathbf{n};\mathbf{n})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha''j_1''}\delta(\omega-\omega_2-j_1''\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha''})\mathcal{I}_{\mathbf{k}}^{\alpha'j_1'}\delta(\omega+j_1'\omega_{1\mathbf{k}}^{\alpha'})$$

$$+\psi_4^{(2)}(\mathbf{n};\mathbf{n}_2-\mathbf{n})\mathcal{I}_{\mathbf{k}}^{\alpha''j_1''}\delta(\omega+j_1''\omega_{1\mathbf{k}}^{\alpha''})\mathcal{I}_{\mathbf{k}_2-\mathbf{k}}^{\alpha'j_1'}\delta(\omega-\omega_2-j_1'\omega_{1,\mathbf{k}_2-\mathbf{k}}^{\alpha'})\bigg)\Bigg)$$

$$+\frac{1}{3}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}\int d\mathbf{n}_2\frac{\chi_1^{(2)}(\mathbf{n}-\mathbf{n}_2;\mathbf{n}_2)}{\epsilon(-\mathbf{n})+\psi(-\mathbf{n})}$$

$$
\times\Bigg(\bigg(\chi_1^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})
$$

$$
+\chi_1^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\bigg)
$$

$$
+\bigg(\chi_{2e}^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})
$$

$$
+\chi_{2e}^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\bigg)
$$

$$
+\bigg(\chi_{3e}^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})
$$

$$
\times\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})
$$

$$
+\chi_{3e}^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})
$$

$$
\times\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\bigg)
$$

$$
+\bigg(\chi_{4e}^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})
$$

$$
+\chi_{4e}^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\bigg)
$$

$$
+\left(\quad e\leftrightarrows i\quad\right)\Bigg)
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}\int d\mathbf{n}_2\frac{\chi_1^{(2)}(\mathbf{n}-\mathbf{n}_2;\mathbf{n}_2)}{\epsilon(-\mathbf{n})+\psi(-\mathbf{n})}\\
&\times\Bigg(\Bigg(\psi_1^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})\\
&\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})\\
&+\psi_1^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\\
&\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\Bigg)\\
&+\Bigg(\psi_2^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\\
&+\psi_2^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})\Bigg)\\
&+\Bigg(\psi_3^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_2-\mathbf{n}}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})\\
&+\psi_3^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\Bigg)\\
&+\Bigg(\psi_4^{(2)}(\mathbf{n}_2-\mathbf{n};-\mathbf{n}_2)\mathcal{I}^{\alpha'' j_1''}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1''\omega^{\alpha''}_{1,\mathbf{k}_2-\mathbf{k}})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\\
&+\psi_4^{(2)}(-\mathbf{n}_2;\mathbf{n}_2-\mathbf{n})\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2-\mathbf{k}}\delta(\omega-\omega_2-j_1'\omega^{\alpha'}_{1,\mathbf{k}_2-\mathbf{k}})\Bigg)\Bigg)\ .
\end{aligned}
$$

Devido à grande quantidade de termos, as expressões (6.9) e (6.8), apesar de consistentes do ponto de vista formal, parecem requerer alguma aproximação adicional quando de sua aplicação prática, onde se poderá desprezar termos que não sejam relevantes para uma determinada situação. Por exemplo, onde a interação onda-onda seja mais relevante que a interação não linear onda-partícula, sem que isso comprometa o caráter da interação não linear.

Os três tipos de interações que citamos no início do texto estão presentes em (6.7). A interação quase linear é representada pelo primeiro parêntese no lado direito de (6.7). Se a velocidade de uma partícula do plasma satisfizer a condição de ressonância $\omega = \mathbf{k}\cdot\mathbf{v}$, energia das ondas é transferida para as partículas por amortecimento de Landau. Neste aspecto, as partículas de poeira devem contribuir pouco uma vez que suas velocidades térmicas são em geral muitos menores do que a velocidade de grupo de qualquer onda eletrostática que propague no plasma. No entanto, já foi demonstrado que a flutuação da carga da poeira leva a um amortecimento em ondas de Alfvén em plasmas empoeirados magnetizados [21, 34].

A interação não linear onda-onda ocorre apenas entre três ondas. Nas equações isto pode ser visto nos diversos termos de (6.9) que contêm uma função delta como função de três frequências. O processo de decaimento ou confluência das três ondas satisfaz então as leis de conservação de energia e momentum,

$$\omega_1 = \omega_2 + \omega_3 \quad , \quad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 \ .$$

Os termos que correspondem à interação não linear onda-partícula não satisfazem a estas duas condições, e ocorrem na forma de espalhamento onda-partícula. Nas equações, todos os termos não lineares restantes são responsáveis por esse processo, que também pode ser interpretado como um amortecimento de Landau não linear, caracterizado pela transferência de energia das ondas para as partículas numa certa região de ressonância, $\mathbf{k}_1 \cdot \mathbf{v}$ com posterior emissão pela partícula de uma nova onda $\pm\mathbf{k}_2 \cdot \mathbf{v}$, em oposição ao amortecimento de Landau linear, onde só ocorre absorção da energia da onda. Também esperamos, devido à presença da poeira, que efeitos não lineares devido às flutuações no carregamento também desempenhem um papel no espalhamento onda-partícula.

Capítulo 7

Conclusões

Utilizando o formalismo básico da teoria de turbulência fraca para um plasma não-magnetizado, e um modelo de carregamento dos grãos de poeira por colisões inelásticas, desenvolvemos um conjunto de equações autoconsistente a partir do qual podem ser obtidas as equações relevantes para investigações sobre a influência dos grãos de poeira em fenômenos turbulentos em ondas eletrostáticas em plasmas.

A exemplo da interação quase linear onda-partícula, obtivemos um sistema de equações fechado definido pelas equações (5.38), (5.39), (5.40), (5.41), (5.42), (5.43), (5.45) e (5.46). Elas formam um conjunto autoconsistente entre as equações cinéticas das ondas e das partículas, e incorporam o aspecto não linear do carregamento da poeira. As três relações de dispersão associadas a esse sistema são consequência do acoplamento dos campos quando do carregamento da poeira. Elas podem fornecer propriedades relevantes sobre o efeito da presença da poeira em ondas eletrostáticas já conhecidas, como Langmuir, íon-acústicas, poeira íon-acústicas e poeira-acústicas, como também novos modos de oscilação que podem surgir como raízes de D_2 e D_3. Junto com as equações cinéticas para ondas, (5.38), (5.39), (5.40), e as equações cinéticas para as partículas, (5.45) e (5.46), é possível analisar a evolução temporal das distribuições de velocidade das partículas e de carga dos grãos de poeira, como consequência da interação onda-partícula e da absorção de partículas do plasma pelos grãos de poeira.

No caso não linear, como explanamos no Capítulo 6, deduzimos uma equação cinética associada a $\langle E_{\mathbf{n}}^2 \rangle$, equação (6.7), utilizando o formalismo padrão da teoria de turbulência fraca em termos de potências dos campos, com base nas expansões (3.41) e (3.44). Então, deduzimos uma expressão para as correlações triplas relevantes em termos de correlações binárias e uma expressão para o termo não linear $NL1$, cujas deduções encontram-se no Apêndice (C). Em princípio, o sistema não linear completo requer uma equação cinética associada a $\langle \hat{E}^2_{\mathbf{n}} \rangle$ e outra a $\langle E^{\sigma}_{\mathbf{n}} E^{\sigma'}_{-\mathbf{n}} \rangle$, e todas as correlações triplas relevantes. Ressaltamos que o procedimento é o mesmo do apresentado no Capítulo 6, porém julgamos conveniente não deduzi-las para não sermos repetitivos em procedimentos muito longos.

Concluímos com a perspectiva inicial de aplicar o que fizemos a sistemas físicos de interesse. Em princípio, o que fizemos se aplica a plasmas empoeirados com baixa densidade de poeira, tipicamente encontrados em plasmas espaciais e experimentais. A generalização para plasmas com densidade de poeira arbitrária é possível. No entanto, isto requer, a rigor, a inclusão da interação poeira-poeira nas equações microscópicas básicas, o que implica em mudanças relativas a esta interação em todas equações cinéticas. Generalizações para o estudo de ondas eletromagnéticas e a inclusão de um campo magnético externo também são possíveis, utilizando o mesmo formalismo.

Apêndice A

Integração de $\Omega_{\mathbf{k}}$ e ω_σ, ω_d, I, ϵ, $\chi_{1\sigma}$, ψ_1, ρ_1, $\rho_{2\sigma}$, $\zeta_{1\sigma}$ e $\zeta_{2\sigma}$

Considerando que as partículas de poeira têm velocidades muito menores do que as velocidades dos íons e elétrons, $|\mathbf{v}| \ll |\mathbf{v}'|$, procedemos como segue.

Escrevemos

$$|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}| = |(\mathbf{v}'-\mathbf{v})\cdot\mathbf{e}_{\mathbf{r}'}| = ||\mathbf{v}'-\mathbf{v}|(\mathbf{e}_{\mathbf{v}'-\mathbf{v}}\cdot\mathbf{e}_{\mathbf{r}'})| = |\mathbf{v}'-\mathbf{v}||\mathbf{e}_{\mathbf{v}'-\mathbf{v}}\cdot\mathbf{e}_{\mathbf{r}'}| \ ,$$

Expandindo $|\mathbf{v}'-\mathbf{v}|$ obtemos até primeira ordem

$$|\mathbf{v}'-\mathbf{v}| \approx v' - \mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'} \ .$$

O vetor unitário $\mathbf{e}_{\mathbf{v}'-\mathbf{v}}$ pode ser escrito como

$$\mathbf{e}_{\mathbf{v}'-\mathbf{v}} = \frac{\partial}{\partial\mathbf{v}'}|\mathbf{v}'-\mathbf{v}| \approx \frac{\partial}{\partial\mathbf{v}'}(v' - \mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'})$$

$$= \mathbf{e}_{\mathbf{v}'} - \frac{\partial}{\partial\mathbf{v}'}(\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'}) \ .$$

Mas

$$\frac{\partial}{\partial\mathbf{v}'}(\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'}) = \left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}'}\right)\mathbf{e}_{\mathbf{v}'} + \left(\mathbf{e}_{\mathbf{v}'}\cdot\frac{\partial}{\partial\mathbf{v}'}\right)\mathbf{v} + \mathbf{v}\times\left(\frac{\partial}{\partial\mathbf{v}'}\times\mathbf{e}_{\mathbf{v}'}\right) + \mathbf{e}_{\mathbf{v}'}\times\left(\frac{\partial}{\partial\mathbf{v}'}\times\mathbf{v}\right)$$

$$= \left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}'}\right)\mathbf{e}_{\mathbf{v}'} \ .$$

Então

$$\mathbf{e}_{\mathbf{v}'-\mathbf{v}} = \left(1 - \left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}'}\right)\right)\mathbf{e}_{\mathbf{v}'} \ .$$

Desprezando o segundo termo obtemos

$$|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}'}| = (v' - \mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'})\left|\left(1-\left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}'}\right)\right)\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}'}\right| \approx (v' - \mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'})|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}'}| \ . \quad \text{(A.1)}$$

Das equações (2.9) e (3.2)

$$\Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) = \frac{1}{(2\pi)^3}\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\ \theta\big(-(\mathbf{v}-\mathbf{v}')\cdot\mathbf{e}_{\mathbf{r}}\big)$$

$$= \frac{1}{(2\pi)^3}(v' - \mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'})\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\ \theta\big(-\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}\big) \ . \quad \text{(A.2)}$$

Em componentes esféricas, o produto escalar $\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}$ é dado por

$$\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}} = \sin\theta_2'\sin\theta_1\cos(\phi_2'-\phi_1) + \cos\theta_2'\cos\theta_1 \ .$$

Para oscilações longitudinais, tomamos o vetor de onda $\mathbf{k}$ na direção z, tal que

$$\mathbf{k}\cdot\mathbf{r} = k_{\|}r\cos\theta_1 = kr\cos\theta_1 \ .$$

Além disso, selecionamos partículas do plasma cujos vetores velocidade apontam para o centro do grão de poeira, tal que

$$\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}} = \sin\theta_2'\sin\theta_1\cos(\phi_2'-\phi_1) + \cos\theta_2'\cos\theta_1 = -1 \ .$$

Assim, nessa aproximação, a integral (A.2) é equivalente a

$$\Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) = \frac{1}{(2\pi)^3}(v' - \mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'}) \quad \text{(A.3)}$$

$$\times\Bigg(\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\theta_1)\delta(\theta_2'-\pi) + \int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\theta_1-\pi)\delta(\theta_2')$$

$$+\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\phi_2'-\phi_1)\delta(\theta_1-\pi-\theta_2')\big(1-\delta(\theta_1)-\delta(\theta_1-\pi)\big)$$

$$+\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\phi_2'-\phi_1-2\pi)\delta(\theta_1-\pi-\theta_2')\big(1-\delta(\theta_1)-\delta(\theta_1-\pi)\big)$$

$$+\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\phi_2'-\phi_1+2\pi)\delta(\theta_1-\pi-\theta_2')\big(1-\delta(\theta_1)-\delta(\theta_1-\pi)\big)$$

$$+\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\phi_2'-\phi_1+\pi)\delta(\theta_1-\pi+\theta_2')\big(1-\delta(\theta_1)-\delta(\theta_1-\pi)\big)$$

$$+\int d\mathbf{r}e^{-i\mathbf{k}.\mathbf{r}}|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}}|\delta(r-a)\delta(\phi_2'-\phi_1-\pi)\delta(\theta_1-\pi+\theta_2')\big(1-\delta(\theta_1)-\delta(\theta_1-\pi)\big)\Bigg)$$

$$=\frac{1}{(2\pi)^3}(v'-\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'})$$

$$\times\Bigg(a^2\sin\theta_2' e^{ika\cos\theta_2'}\theta\big(\pi-\phi_2'\big)+a^2\sin\theta_2' e^{ika\cos\theta_2'}\theta\big(\phi_2'-\pi\big)\Bigg)$$

$$=\frac{a^2}{(2\pi)^3}(1-\frac{\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'}}{v'})v_\perp'\exp\left(ika\frac{v_\parallel'}{v'}\right)\,.$$

Aqui utilizamos a aproximação (A.1) para calcular (3.6) e (3.7).

$$I(t)=\frac{1}{n_d}\int d\mathbf{v}dq\; f_d\sum_\sigma q_\sigma\int d\mathbf{r}'d\mathbf{v}'\delta(r'-a)(v'-\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'})|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}'}|\theta\big(-\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}'}\big)f_\sigma(\mathbf{v}',t)\,. \quad (A.4)$$

Como o vetor **v** dá contribuição ímpar à integração,

$$I(t)=\sum_\sigma q_\sigma\int d\mathbf{r}'d\mathbf{v}'v'\delta(r'-a)|\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}'}|\theta\big(-\mathbf{e}_{\mathbf{v}'}\cdot\mathbf{e}_{\mathbf{r}'}\big)f_\sigma(\mathbf{v}',t)$$

$$=a^2\sum_\sigma q_\sigma\int d\mathbf{v}'v_\perp' f_\sigma(\mathbf{v}',t)=a^2v_*\sum_\sigma q_\sigma\int d\mathbf{u}\; u_\perp f_\sigma(\mathbf{u},t)$$

$$=\sum_\sigma q_\sigma\frac{n_\sigma}{n_d}\omega_\sigma\,. \quad (A.5)$$

A frequência média de colisões ω_σ é dada por

$$\omega_\sigma(t)=\frac{1}{n_\sigma}\int d\mathbf{v}\; f_\sigma\int d\mathbf{r}'d\mathbf{v}'dq'(v-\mathbf{v}'\cdot\mathbf{e}_{\mathbf{v}})|\mathbf{e}_{\mathbf{v}}\cdot\mathbf{e}_{\mathbf{r}'}|\delta(r'-a)f_d(\mathbf{v}',q',t)$$

$$=a^2v_*\frac{n_d}{n_\sigma}\int d\mathbf{u}\; u_\perp f_\sigma\,. \quad (A.6)$$

Para ω_d, utilizamos a equação (3.34), cuja solução é

$$N_{\mathbf{n}}^{d(1)}=N_{\mathbf{n}}^{d(1)}|_{q=0}\exp\big(\omega_d q/I\big)\,. \quad (A.7)$$

Invertendo a equação para ω_d,

$$\omega_d = \frac{I}{q} \ln \left(N_{\mathbf{n}}^{d(1)} / N_{\mathbf{n}}^{d(1)}|_{q=0} \right) \ . \tag{A.8}$$

Substituindo explicitamente as expressões de $N_{\mathbf{n}}^{d(1)}$, temos

$$\begin{aligned}
\omega_d &= \frac{I}{q} \ln \left(\frac{i h_{\mathbf{n}}^d \hat{E}_{\mathbf{n}} f_d + i \left(\mathbf{g}_{\mathbf{n}}^d \cdot \frac{\mathbf{k}}{k} \right) E_{\mathbf{n}} f_d}{i h_{\mathbf{n}}^d \hat{E}_{\mathbf{n}} f_d \big|_{q=0}} \right) \\
&= \frac{I}{q} \ln \left(\frac{e^2 v_*^{-1} \hat{E}_{\mathbf{n}} \partial f_d / \partial q + q \left(\frac{\mathbf{k}}{k} \cdot \partial f_d / \partial \mathbf{v} \right) E_{\mathbf{n}}}{e^2 v_*^{-1} \hat{E}_{\mathbf{n}} \partial f_d / \partial q \big|_{q=0}} \right) \\
&= \frac{I}{q} \left(\ln \left(\frac{\partial f_d}{\partial q} + \frac{q v_*}{e^2} \left(\frac{\mathbf{k}}{k} \cdot \frac{\partial f_d}{\partial \mathbf{v}} \right) \frac{E_{\mathbf{n}}}{\hat{E}_{\mathbf{n}}} \right) - \ln \left(\frac{\partial f_d}{\partial q} \bigg|_{q=0} \right) \right) \ .
\end{aligned} \tag{A.9}$$

É conveniente introduzirmos as quantidades adimensionais

$$v = u v_* \quad , \quad q = Ze \quad , \quad \omega = \omega_* z \quad , \quad k = \ell \frac{\omega_*}{v_*} \quad , \quad a = g \frac{v_*}{\omega_*} \quad , \quad t = \frac{\tau}{\omega_*} \ ,$$

e considerarmos que as funções de distribuição de velocidade e de carga da poeira são normalizadas de tal modo que

$$\int d\mathbf{v} f_\sigma(\mathbf{v}) = \int d\mathbf{u} f_\sigma(\mathbf{u}) \quad , \quad \int d\mathbf{v} dq f_d(\mathbf{v}, q) = \int d\mathbf{u} dZ f_d(\mathbf{u}, Z) \ ,$$

$$f_\sigma(\mathbf{v}) = \frac{1}{v_*^3} f_\sigma(\mathbf{u}) \quad , \quad f_d(\mathbf{v}, q) = \frac{1}{v_*^3 e} f_d(\mathbf{u}, Z) \ .$$

As distribuições iniciais são dadas por [28]

$$f_\sigma = n_{\sigma 0} \left(\frac{m_\sigma}{2\pi k_B T_\sigma} \right)^{3/2} \exp \left(- \frac{m_\sigma v^2}{2 k_B T_\sigma} \right) \ ,$$

$$f_d = \frac{n_{d0}}{\sqrt{2\pi a k_B \tilde{T}_{eff}}} \exp \left(- \frac{(q - q_0)^2}{2 a k_B \tilde{T}_{eff}} \right) \left(\frac{m_d}{2\pi k_B T_{eff}} \right)^{3/2} \exp \left(- \frac{m_d v^2}{2 k_B T_{eff}} \right) \ .$$

As temperaturas $\tilde{T}_{eff}$ e T_{eff} na distribuição de carga e velocidade dos grãos de poeira são temperaturas efetivas que se correlacionam pelo efeito mútuo entre o carregamento dos grãos de poeira e as variações em suas velocidades pelas colisões inelásticas. Para um plasma empoeirado elétron-íon, elas são da ordem aproximada de $\tilde{T}_{eff} \simeq T_e$ e $T_{eff} \simeq 2T_i$.

Definindo

$$v_\sigma = \sqrt{\frac{2k_B T_\sigma}{m_\sigma}} \quad , \quad v_d = \sqrt{\frac{2k_B T_{eff}}{m_d}} \quad , \quad q_d = \sqrt{2ak_B \tilde{T}_{eff}} \ ,$$

obtemos

$$f_\sigma(\mathbf{u}) = \frac{n_{\sigma 0}}{\pi^{3/2} u_\sigma^3} \exp\left(-\frac{u^2}{u_\sigma^2} \right) \ , \ f_d(Z, \mathbf{u}) = \frac{n_{d0}}{\sqrt{\pi} Z_d} \exp\left(-\frac{(Z - Z_0)^2}{Z_d^2} \right) \frac{1}{\pi^{3/2} u_d^3} \exp\left(-\frac{u^2}{u_d^2} \right)$$

onde

$$u_\sigma = \frac{v_\sigma}{v_*} \quad , \quad u_d = \frac{v_d}{v_*} \quad , \quad Z_d = \frac{q_d}{e}$$

e e é a carga elementar, $e = 4.8032 \times 10^{-10}$ statcoul.

A frequência de colisões ω_σ é dada por

$$\omega_\sigma = \frac{\sqrt{\pi}}{2} a^2 v_* n_d u_\sigma \ , \tag{A.10}$$

de modo que a corrente (A.5) é dada por

$$I = \frac{\sqrt{\pi}}{2} a^2 v_* \sum_\sigma q_\sigma n_\sigma u_\sigma \ , \tag{A.11}$$

A frequência ω_d é dada como uma função da velocidade e da carga da poeira. De (A.9), temos

$$\begin{aligned} \omega_d &= \frac{I}{q} \left(\ln\left(\frac{-2(q - q_0)}{q_d^2} f_d - \frac{q v_*}{e^2} \left(\frac{\mathbf{k}}{k} \cdot \frac{2\mathbf{v}}{v_d^2} \right) \frac{E_\mathbf{n}}{\hat{E}_\mathbf{n}} f_d \right) - \ln\left(\frac{2q_0}{q_d^2} f_d \Big|_{q=0} \right) \right) \\ &= \frac{I}{q} \ln\left(\left(1 - \frac{q}{q_0} \right) e^{-(q-q_0)^2/q_d^2} e^{q_0^2/q_d^2} - \frac{q}{q_0} \frac{q_d^2}{e^2} \left(\frac{\mathbf{k}}{k} \cdot \frac{v_* \mathbf{v}}{v_d^2} \right) \frac{E_\mathbf{n}}{\hat{E}_\mathbf{n}} e^{-(q-q_0)^2/q_d^2} e^{q_0^2/q_d^2} \right) \\ &= \frac{I}{q} \ln\left(1 - \frac{q}{q_0} \left(1 + \frac{\tilde{T}_{eff}}{T_{eff}} \left(\frac{\mathbf{k}}{k} \cdot \mathbf{v} \right) a \frac{E_\mathbf{n}}{I_\mathbf{n}} \right) \right) + I \frac{2q_0 - q}{q_d^2} \ . \end{aligned} \tag{A.12}$$

Se o termo proporcional à velocidade for muito menor que um,

$$\omega_d = \frac{I}{q} \ln\left(1 - \frac{q}{q_0} \right) + I \frac{2q_0 - q}{q_d^2} \ . \tag{A.13}$$

Para $q \neq q_0$, se q tem o mesmo sinal de q_0, devemos requerer em princípio, para que ω_d seja real, que $|q_0| > |q|$. No limite em que $|q| \to |q_0|$, teremos $I = 0$ e então $\omega_d = 0$.

Nos dois casos de sinais iguais ou opostos de q e q_0, devemos requerer que o sinal de I e $\ln(1 - q/q_0)/q + (2q_0 - q)/q_d^2$ sejam os mesmos.

Uma aproximação simples para esta equação seria considerar ω_d no limite $q \to 0$, de modo que

$$\omega_d = I\left(\frac{2q_0}{q_d^2} - \frac{1}{q_0}\right) \ . \tag{A.14}$$

Para que a frequência ω_d seja positiva, I e q_0 devem satisfazer um dos quatro casos abaixo,

$$\begin{array}{ccccc} I > 0 & , & q_0 > 0 & , & q_0 > q_d/\sqrt{2}\ , \\ I > 0 & , & q_0 < 0 & , & -q_d/\sqrt{2} < q_0\ , \\ I < 0 & , & q_0 > 0 & , & q_0 < q_d/\sqrt{2}\ , \\ I < 0 & , & q_0 < 0 & , & q_0 < -q_d/\sqrt{2}\ . \end{array}$$

A carga de equilíbrio $q_0 = eZ_0$ é a carga média dos grãos de poeira quando $I = 0$. Da condição de neutralidade (5.37) e (A.11),

$$Z_0 = Z_i \frac{n_i}{n_d}\left(\frac{u_i}{u_e} - 1\right) \ . \tag{A.15}$$

No que segue, procedemos com os cálculos das susceptibilidades lineares. Com o vetor de onda na direção paralela à direção z, como no Apêndice A, temos que $|\mathbf{k}| = k_z = k_\parallel = k$.

$$\begin{aligned} \epsilon(\mathbf{n}) &= 1 + \chi(\mathbf{n}) \\ &= 1 + \frac{4\pi}{k}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\int d\mathbf{v}\ \frac{\partial f_\sigma/\partial v_\parallel}{\omega - kv_\parallel + i\omega_\sigma} = 1 + \frac{4\pi}{k^2}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\frac{2}{v_\sigma^2}\int d\mathbf{v}\ \frac{v_\parallel f_\sigma}{v_\parallel - (\omega + i\omega_\sigma)/k} \\ &= 1 + \frac{4\pi}{\ell^2}\frac{v_*^2}{\omega_*^2}\sum_\sigma \frac{q_\sigma^2}{m_\sigma}\frac{2}{u_\sigma^2 v_*^2}\int d\mathbf{u}\ \frac{u_\parallel f_\sigma}{u_\parallel - (z + iz_\sigma)/\ell} \\ &= 1 + \frac{1}{\ell^2}\sum_\sigma \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{2}{\sqrt{\pi}u_\sigma^3}\int du_\parallel\ \frac{u_\parallel e^{-u_\parallel^2/u_\sigma^2}}{u_\parallel - (z + iz_\sigma)/\ell} = 1 + \frac{1}{\ell^2}\sum_\sigma \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{2}{\sqrt{\pi}u_\sigma^2}\int du\ \frac{ue^{-u^2}}{u - \zeta_\sigma} \\ &= 1 + \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{2}{\ell^2 u_\sigma^2}\big(1 + \zeta_\sigma Z(\zeta_\sigma)\big)\ , \end{aligned} \tag{A.16}$$

onde

$$\zeta_\sigma = \frac{z + iz_\sigma}{\ell u_\sigma} \quad , \quad Z(\zeta_\sigma) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} du\ \frac{e^{-u^2}}{u - \zeta_\sigma}\ .$$

Também

$$\psi(\mathbf{n}) = \frac{4\pi}{k^2}\int dq\ \frac{q^2}{m_d}\frac{2}{v_d^2}\int d\mathbf{v}\ \frac{v_\parallel f_d}{v_\parallel - (\omega + i\omega_d)/k}$$

$$= \frac{4\pi}{\ell^2}\frac{v_*^2}{\omega_*^2}e^2\int dZ\ \frac{Z^2}{m_d}\frac{2}{u_d^2 v_*^2}\int d\mathbf{u}\ \frac{u_\parallel f_d(Z,u)}{u_\parallel - (z + iz_d)/\ell}$$

$$= \frac{1}{\ell^2}\frac{4\pi e^2 n_{d0}}{\omega_*^2 m_d}\frac{2}{\sqrt{\pi}Z_d u_d^3}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dZ\ Z^2 e^{-\frac{(Z-Z_0)^2}{Z_d^2}}\int_{-\infty}^{\infty} du_\parallel\ \frac{u_\parallel e^{-u_\parallel^2/u_d^2}}{u_\parallel - (z + iz_d)/\ell}$$

$$= \frac{\omega_{pd}^2}{\omega_*^2}\frac{2}{\ell^2 u_d^2}\frac{1}{\sqrt{\pi}Z_d}\left(\frac{\sqrt{\pi}}{2}\frac{Z_d^3}{Z_0^2} + \sqrt{\pi}Z_d\right)\big(1 + \zeta_d Z(\zeta_d)\big) = \frac{\omega_{pd}^2}{\omega_*^2}\frac{(Z_d^2/Z_0^2 + 2)}{\ell^2 u_d^2}\big(1 + \zeta_d Z(\zeta_d)\big)\ , \tag{A.17}$$

onde

$$\omega_{pd}^2 = \frac{4\pi Z_0^2 e^2 n_{d0}}{m_d} \quad , \quad \zeta_d = \frac{z + iz_d}{\ell u_d}\ .$$

Continuando,

$$\chi_{1\sigma}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma^2}{m_\sigma}v_*^{-1}\int d\mathbf{v}\ \frac{f_\sigma}{v_\parallel - (\omega + i\omega_\sigma)/k}$$

$$= \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{1}{\ell^2 n_{\sigma 0}}\int d\mathbf{u}\ \frac{f_\sigma}{u_\parallel - (z + iz_\sigma)/\ell} = \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{1}{\ell^2 u_\sigma}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} du_\parallel\ \frac{e^{-u_\parallel^2/u_\sigma^2}}{u_\parallel - (z + iz_\sigma)/\ell}$$

$$= \frac{\omega_{p\sigma}^2}{\omega_*^2}\frac{1}{\ell^2 u_\sigma}Z(\zeta_\sigma)\ , \tag{A.18}$$

$$\psi_1(\mathbf{n}) = -\frac{4\pi}{k^2}\frac{e^2}{m_d}v_*^{-1}\frac{2}{q_d^2}\int dq\ q\int d\mathbf{v}\ \frac{(q - q_0)f_d}{v_\parallel - (\omega + i\omega_d)/k}$$

$$= -\frac{4\pi v_*^2}{\ell^2\omega_*^2}\frac{e^2}{m_d}v_*^{-1}\frac{2}{Z_d^2 e^2}\frac{1}{v_*}e^2\int dZ\ Z(Z - Z_0)\int d\mathbf{u}\ \frac{f_d(Z,u)}{u_\parallel - (z + iz_d)/\ell}$$

$$= -\frac{1}{\ell^2}\frac{\omega_{pd}^2}{\omega_*^2}\frac{2}{Z_d^2 Z_0^2}\frac{1}{\sqrt{\pi}Z_d}\frac{1}{\sqrt{\pi}u_d}\int_{-\infty}^{\infty} dZ\ Z(Z - Z_0)e^{-\frac{(Z-Z_0)^2}{Z_d^2}}\int_{-\infty}^{\infty} du_\parallel\ \frac{e^{-u^2/u_d^2}}{u_\parallel - (z + iz_d)/\ell}$$

$$= -\frac{1}{\ell^2}\frac{\omega_{pd}^2}{\omega_*^2}\frac{2}{Z_d^2 Z_0^2}\frac{1}{\sqrt{\pi}Z_d}\frac{1}{u_d}\left(\frac{\sqrt{\pi}}{2}Z_d^3 + \sqrt{\pi}Z_d Z_0^2 - \sqrt{\pi}Z_d Z_0^2\right)Z(\zeta_d)$$

$$= -\frac{1}{Z_0^2}\frac{\omega_{pd}^2}{\omega_*^2}\frac{1}{\ell^2 u_d}Z(\zeta_d)\ . \tag{A.19}$$

Para o cálculo de ρ_1, $\rho_{2\sigma}$, $\zeta_{1\sigma}$ e $\zeta_{2\sigma}$ utilizamos $\Omega_{\mathbf{k}}$ na aproximação (A.3). De (4.20) e (A.3), temos

$$\rho_1^{(1)}(\mathbf{n}) = \frac{4\pi}{k^2}\sum_\sigma \frac{q_\sigma^3}{e^2}\frac{m_d}{m_\sigma}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d\int d\mathbf{v}'\Omega_{\mathbf{k}}^\sigma(\mathbf{v} - \mathbf{v}', a)\big(\mathbf{g}_{\mathbf{n}}^\sigma \cdot \mathbf{k}\big)f_\sigma \tag{A.20}$$

$$= \frac{4\pi}{k^2}\sum_\sigma \frac{q_\sigma^4}{e^2}\frac{m_d}{m_\sigma^2}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\frac{1}{n_d}\int d\mathbf{v}dq\ f_d(\mathbf{v},q)\int d\mathbf{v}'$$

$$\times\frac{a^2}{(2\pi)^3}(1-\frac{\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'}}{v'})v'_\perp \exp\left(ika\frac{v'_\parallel}{v'}\right)\frac{\partial f_\sigma(\mathbf{v}')/\partial v'_\parallel}{v'_\parallel-(\omega+i\omega_\sigma)/k}$$

$$= i\frac{a^2}{k^2}v_* k\sum_\sigma \frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\int d\mathbf{v}'v'_\perp \exp\left(ika\frac{v'_\parallel}{v'}\right)\frac{\partial f_\sigma(\mathbf{v}')/\partial v'_\parallel}{v'_\parallel-(\omega+i\omega_\sigma)/k}$$

$$= -i\frac{v_*}{\omega_*}\frac{g^2}{\ell}\frac{v_*^2}{\omega_*^2}v_*\sum_\sigma \frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2 v_*^2}v_*\int d\mathbf{u}'u'_\perp \exp\left(i\ell g\frac{u'_\parallel}{u'}\right)\frac{u_\parallel f_\sigma(\mathbf{u}')}{u'_\parallel-(z+iz_\sigma)/\ell}$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}\int d\mathbf{u}'u'_\perp \exp\left(i\ell g\frac{u'_\parallel}{u'}\right)\frac{u_\parallel f_\sigma(\mathbf{u}')}{u'_\parallel-(z+iz_\sigma)/\ell}$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi\int_0^\infty du_\perp u_\perp\int_{-\infty}^\infty du_\parallel$$

$$\times u_\perp\ e^{-\frac{u_\perp^2}{u_\sigma^2}}\exp\left(i\ell g\frac{u_\parallel}{u}\right)\frac{u_\parallel\ e^{-\frac{u_\parallel^2}{u_\sigma^2}}}{u_\parallel-(z+iz_\sigma)/\ell}$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi\int_0^\infty du_\perp u_\perp\int_{-\infty}^\infty du_\parallel$$

$$\times u_\perp\ e^{-\frac{u_\perp^2}{u_\sigma^2}}\exp\left(i\ell g\frac{u_\parallel}{u}\right)\left(1+\frac{(z+iz_\sigma)/\ell}{u_\parallel-(z+iz_\sigma)/\ell}\right)e^{-\frac{u_\parallel^2}{u_\sigma^2}}\ .$$

Se o produto $ka=\ell g$ é suficientemente pequeno, podemos considerar a aproximação

$$\exp\left(i\ell g\frac{u_\parallel}{u}\right)\simeq 1+i\ell g\frac{u_\parallel}{u}-\frac{\ell^2g^2}{2}\frac{u_\parallel^2}{u^2}=1+i\ell g\frac{u_\parallel}{(u_\perp^2+u_\parallel^2)^{1/2}}-\frac{\ell^2g^2}{2}\frac{u_\parallel^2}{u_\perp^2+u_\parallel^2}\ , \qquad (A.21)$$

de modo que

$$\rho_1^{(1)}(\mathbf{n}) = -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi u_\sigma^3\int_0^\infty du_\perp u_\perp\int_{-\infty}^\infty du_\parallel \qquad (A.22)$$

$$\times u_\perp\ e^{-u_\perp^2}\left(1+i\ell g\frac{u_\parallel}{(u_\perp^2+u_\parallel^2)^{1/2}}-\frac{\ell^2g^2}{2}\frac{u_\parallel^2}{u_\perp^2+u_\parallel^2}\right)\left(1+\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)e^{-u_\parallel^2}$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi u_\sigma^3$$

$$\times\Bigg(\int_0^\infty du_\perp u_\perp \int_{-\infty}^\infty du_\parallel u_\perp\ e^{-u_\perp^2}\left(1+\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)e^{-u_\parallel^2}$$

$$+i\ell g\int_0^\infty du_\perp \int_{-\infty}^\infty du_\parallel\ \frac{u_\perp^2 e^{-u_\perp^2}}{(1+u_\perp^2/u_\parallel^2)^{1/2}}\frac{u_\parallel}{|u_\parallel|}\left(\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)e^{-u_\parallel^2}$$

$$-\frac{\ell^2 g^2}{2}\int_0^\infty du_\perp \int_{-\infty}^\infty du_\parallel\ \frac{u_\perp^2 e^{-u_\perp^2}}{1+u_\perp^2/u_\parallel^2}\left(1+\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)e^{-u_\parallel^2}\Bigg)$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi u_\sigma^3$$

$$\times\Bigg(\frac{\sqrt{\pi}}{4}\sqrt{\pi}+\frac{\sqrt{\pi}}{4}\sqrt{\pi}\zeta_\sigma Z(\zeta_\sigma)$$

$$+i\ell g\frac{1}{2}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\ \frac{s^{1/2}e^{-su_\parallel^2}}{(1+s)^{1/2}}u_\parallel^3\left(\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)e^{-u_\parallel^2}$$

$$-\frac{\ell^2 g^2}{4}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\ \frac{s^{1/2}e^{-su_\parallel^2}}{1+s}|u_\parallel|^3\left(1+\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)e^{-u_\parallel^2}\Bigg)$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi u_\sigma^3$$

$$\times\Bigg(\frac{\pi}{4}\big(1+\zeta_\sigma Z(\zeta_\sigma)\big)+i\ell g\frac{\zeta_\sigma}{2}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\ \frac{s^{1/2}e^{-su_\parallel^2}}{(1+s)^{1/2}}\big(u_\parallel^2+\zeta_\sigma u_\parallel+\zeta_\sigma^2\big)e^{-u_\parallel^2}$$

$$+i\ell g\frac{\zeta_\sigma^4}{2}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\ \frac{s^{1/2}e^{-su_\parallel^2}}{(1+s)^{1/2}}\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}$$

$$-\frac{\ell^2 g^2}{4}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\ \frac{s^{1/2}e^{-su_\parallel^2}}{1+s}|u_\parallel|^3 e^{-u_\parallel^2}$$

$$-\frac{\ell^2 g^2}{4}\zeta_\sigma\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\ \frac{s^{1/2}e^{-su_\parallel^2}}{1+s}\left(|u_\parallel|(u_\parallel+\zeta_\sigma)+\mathrm{sgn}(u_\parallel)\zeta_\sigma^2\left(1+\frac{\zeta_\sigma}{u_\parallel-\zeta_\sigma}\right)\right)e^{-u_\parallel^2}\Bigg)$$

$$= -i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi u_\sigma^3$$

$$\times\Bigg(\frac{\pi}{4}\big(1+\zeta_\sigma Z(\zeta_\sigma)\big)+i\ell g\frac{\zeta_\sigma}{2}\int_0^\infty ds\ \frac{s^{1/2}}{(1+s)^{1/2}}\left(\frac{\sqrt{\pi}}{2(s+1)^{3/2}}+\zeta_\sigma^2\frac{\sqrt{\pi}}{(s+1)^{1/2}}\right)$$

$$+i\ell g\frac{\zeta_\sigma^4}{2}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\;\frac{s^{1/2}e^{-su_\parallel^2}}{(1+s)^{1/2}}\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}$$

$$-\frac{\ell^2g^2}{4}\int_0^\infty ds\;\frac{s^{1/2}}{1+s}\frac{1}{(1+s)^2}$$

$$-\frac{\ell^2g^2}{4}\zeta_\sigma\int_0^\infty ds\;\frac{s^{1/2}}{1+s}\left(\frac{\zeta_\sigma}{1+s}\right)$$

$$\left.-\frac{\ell^2g^2}{4}\zeta_\sigma^4\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\;\frac{s^{1/2}e^{-su_\parallel^2}}{1+s}\mathrm{sgn}(u_\parallel)\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}\right)$$

$$=-i\frac{g^2}{\ell}\frac{v_*^3}{\omega_*^3}\sum_\sigma\frac{n_\sigma}{\sqrt{\pi}^3u_\sigma^3}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{u_\sigma^2}2\pi u_\sigma^3$$

$$\times\left(\frac{\pi}{4}\big(1+\zeta_\sigma Z(\zeta_\sigma)\big)-\ell^2g^2\frac{\pi}{32}-\ell^2g^2\zeta_\sigma^2\frac{\pi}{8}+i\ell g\zeta_\sigma\frac{\pi}{4}\frac{\sqrt{\pi}}{2}+i\ell g\zeta_\sigma^3\frac{\sqrt{\pi}}{2}\int_0^\infty ds\;\frac{s^{1/2}}{1+s}\right.$$

$$+i\ell g\frac{\zeta_\sigma^4}{2}\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\;\frac{s^{1/2}}{(1+s)^{1/2}}\frac{e^{-(1+s)u_\parallel^2}}{u_\parallel-\zeta_\sigma}$$

$$\left.-\frac{\ell^2g^2}{4}\zeta_\sigma\int_0^\infty ds\int_{-\infty}^\infty du_\parallel\;\frac{s^{1/2}}{1+s}\mathrm{sgn}(u_\parallel)\frac{e^{-(1+s)u_\parallel^2}}{u_\parallel-\zeta_\sigma}\right)\;.$$

Com relação às três últimas integrais em (A.22) devemos fazer algumas considerações.

A primeira integral é dada por $2(\sqrt{s}-\arctan\sqrt{s})$ e diverge no infinito, que é o limite superior. Mas podemos tentar regularizá-la com um limite superior finito. Se fizermos o limite superior $s_2=\eta$, onde $\eta>0$, estamos considerando, dado que a variável s é definida por $s=u_\perp^2/u_\parallel^2$, que $u_\perp$ como função de $u_\parallel$ varre a região entre a reta $u_\perp=0$ e $u_\perp=\eta|u_\parallel|$, excluindo o cone em que $u_\perp>\eta|u_\parallel|$. Nessa aproximação, obtemos

$$\int_0^\eta ds\;\frac{s^{1/2}}{1+s}=2\big(\sqrt{\eta}-\arctan\sqrt{\eta}\big)\;.$$

É possível que o crescimento da integral com η esteja relacionado com alguma instabilidade associada à direção do vetor velocidade das partículas incidentes num grão de poeira em relação ao eixo de propagação das ondas. Matematicamente vemos isto como consequência da equação (A.3).

A integral em s para as duas outras integrais, também dependerá da integração em $u_\parallel$, e depois podemos utilizar o mesmo procedimento de regularização para as integrais em s que resultarem divergentes.

Para a integração em $u_\parallel$ dos últimos dois termos em (A.22), fazemos a substituição $u_\parallel \to u_\parallel/(1+s)^{1/2}$, de modo que

$$\int_{-\infty}^{\infty} du_\parallel \, \frac{e^{-(1+s)u_\parallel^2}}{u_\parallel - \zeta_\sigma} = \frac{1}{(1+s)^{1/2}} \int_{-\infty}^{\infty} du_\parallel \, \frac{e^{-u_\parallel^2}}{u_\parallel/(1+s)^{1/2} - \zeta_\sigma} = \int_{-\infty}^{\infty} du_\parallel \, \frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_\sigma^s}$$

$$= \sqrt{\pi} Z(\zeta_\sigma^s) \ ,$$

onde $Z(\zeta)$ é a função de dispersão e

$$\zeta_\sigma^s = (1+s)^{1/2} \zeta_\sigma \ .$$

Para a última integral em (A.22) temos

$$\int_{-\infty}^{\infty} du_\parallel \, \mathrm{sgn}(u_\parallel) \frac{e^{-(1+s)u_\parallel^2}}{u_\parallel - \zeta_\sigma} = \frac{1}{(1+s)^{1/2}} \int_{-\infty}^{\infty} du_\parallel \, \mathrm{sgn}\left(\frac{u_\parallel}{(1+s)^{1/2}} \right) \frac{e^{-u_\parallel^2}}{u_\parallel/(1+s)^{1/2} - \zeta_\sigma}$$

$$= \int_{-\infty}^{\infty} du_\parallel \, \mathrm{sgn}(u_\parallel) \frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_\sigma^s}$$

$$= -\int_{-\infty}^{0} du_\parallel \, \frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_\sigma^s} + \int_{0}^{\infty} du_\parallel \, \frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_\sigma^s}$$

$$= -2 \int_{-\infty}^{0} du_\parallel \, \frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_\sigma^s} + \sqrt{\pi} Z(\zeta_\sigma^s) = 2 \int_{0}^{\infty} du_\parallel \, \frac{e^{-u_\parallel^2}}{u_\parallel + \zeta_\sigma^s} + \sqrt{\pi} Z(\zeta_\sigma^s)$$

$$= 2 \lim_{\nu \to 0} \sqrt{\pi} Z(\nu, -\zeta_\sigma^s) + \sqrt{\pi} Z(\zeta_\sigma^s) \ ,$$

onde $Z(\nu, \zeta)$ é a função de dispersão incompleta [35], definida por

$$Z(\nu, \zeta) = \frac{1}{\sqrt{\pi}} \int_{\nu}^{\infty} dt \, \frac{e^{-t^2}}{t - \zeta} \ . \tag{A.23}$$

A função de dispersão $Z(\zeta)$ corresponde ao limite $\lim_{\nu \to -\infty} Z(\nu, \zeta)$.

Substituindo as integrais em (A.22),

$$\rho_1^{(1)}(\mathbf{n}) = -i \frac{g^2}{\ell} \frac{v_*^3}{\omega_*^3} \sum_\sigma \frac{n_\sigma}{\sqrt{\pi}^3 u_\sigma^3} \frac{q_\sigma^2}{e^2} \frac{m_d}{m_\sigma} \frac{2}{u_\sigma^2} 2\pi u_\sigma^3 \tag{A.24}$$

$$\times \left(\frac{\pi}{4} \big(1 + \zeta_\sigma Z(\zeta_\sigma)\big) - \ell^2 g^2 \frac{\pi}{32} - \ell^2 g^2 \zeta_\sigma^2 \frac{\pi}{8} + i \ell g \zeta_\sigma \frac{\pi}{4} \frac{\sqrt{\pi}}{2} + i \ell g \zeta_\sigma^3 \sqrt{\pi} \big(\sqrt{\eta} - \arctan \sqrt{\eta} \big) \right.$$

$$+i\ell g\frac{\zeta_\sigma^4}{2}\sqrt{\pi}\int_0^\infty ds\ \frac{s^{1/2}}{(1+s)^{1/2}}Z(\zeta_\sigma^s) - \frac{\ell^2 g^2}{4}\zeta_\sigma\sqrt{\pi}\lim_{\nu\to 0}\int_0^\infty ds\ \frac{s^{1/2}}{1+s}Z(\nu,-\zeta_\sigma^s)$$

$$-\frac{\ell^2 g^2}{4}\zeta_\sigma\sqrt{\pi}\int_0^\infty ds\ \frac{s^{1/2}}{1+s}Z(\zeta_\sigma^s)\Bigg)\ .$$

A integração em velocidade de $\rho_{2\sigma}$ é similar à de ρ_1, diferindo (A.20) pelo operador $h^\sigma_{\mathbf{n}}$,

$$\rho^{(1)}_{2\sigma}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma^3}{e^2}\frac{m_d}{m_\sigma}\frac{1}{n_d}\int d\mathbf{v}dq\ f_d\int d\mathbf{v}'\Omega^\sigma_{\mathbf{k}}(\mathbf{v}-\mathbf{v}',a)h^\sigma_{\mathbf{n}}kf_\sigma$$

$$= \frac{4\pi}{k^2}\frac{q_\sigma^4}{e^2}\frac{m_d}{m_\sigma^2}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\frac{1}{n_d}v_*^{-1}\int d\mathbf{v}dq\ f_d(\mathbf{v},q)\int d\mathbf{v}'$$

$$\times\frac{a^2}{(2\pi)^3}(1-\frac{\mathbf{v}\cdot\mathbf{e}_{\mathbf{v}'}}{v'})v'_\perp\exp\left(ika\frac{v'_\parallel}{v'}\right)\frac{f_\sigma(\mathbf{v}')}{v'_\parallel-(\omega+i\omega_\sigma)/k}$$

$$= i\frac{a^2}{k^2}k\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\int d\mathbf{v}'v'_\perp\exp\left(ika\frac{v'_\parallel}{v'}\right)\frac{f_\sigma(\mathbf{v}')}{v'_\parallel-(\omega+i\omega_\sigma)/k}$$

$$= i\frac{a^2}{\ell}\frac{v_*}{\omega_*}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{n_\sigma}{\pi^{3/2}u_\sigma^3}2\pi\int_0^\infty du_\perp u_\perp\int_{-\infty}^\infty du_\parallel u_\perp\ e^{-\frac{u_\perp^2}{u_\sigma^2}}\exp\left(i\ell g\frac{u_\parallel}{u}\right)\frac{e^{-\frac{u_\parallel^2}{u_\sigma^2}}}{u_\parallel-(z+iz_\sigma)/\ell}$$

$$= i\frac{a^2}{\ell}\frac{v_*}{\omega_*}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{n_\sigma}{\pi^{3/2}u_\sigma^3}2\pi u_\sigma^3\int_0^\infty du_\perp u_\perp\int_{-\infty}^\infty du_\parallel u_\perp\ e^{-u_\perp^2}$$

$$\times\left(1+i\ell g\frac{u_\parallel}{(u_\perp^2+u_\parallel^2)^{1/2}}-\frac{\ell^2 g^2}{2}\frac{u_\parallel^2}{u_\perp^2+u_\parallel^2}\right)\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}$$

$$= i\frac{a^2}{\ell}\frac{v_*}{\omega_*}\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{n_\sigma}{\sqrt{\pi}}2\Bigg(\int_0^\infty du_\perp u_\perp\int_{-\infty}^\infty du_\parallel u_\perp\ e^{-u_\perp^2}\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}$$

$$+i\ell g\int_0^\infty du_\perp\int_{-\infty}^\infty du_\parallel\ \frac{u_\perp^2 e^{-u_\perp^2}}{(1+u_\perp^2/u_\parallel^2)^{1/2}}\frac{u_\parallel}{|u_\parallel|}\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}$$

$$-\frac{\ell^2 g^2}{2}\int_0^\infty du_\perp\int_{-\infty}^\infty du_\parallel\ \frac{u_\perp^2 e^{-u_\perp^2}}{1+u_\perp^2/u_\parallel^2}\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_\sigma}\Bigg)$$

$$= i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma^2}{e^2}\frac{m_d}{m_\sigma}\frac{2}{\sqrt{\pi}}\Bigg(\frac{\pi}{4}Z(\zeta_\sigma)+i\ell g\frac{\pi}{4}\frac{\sqrt{\pi}}{2}$$

$$+i\ell g\zeta_\sigma^2\sqrt{\pi}\big(\sqrt{\eta}-\arctan\sqrt{\eta}\big)+i\ell g\frac{\zeta_\sigma^3}{2}\sqrt{\pi}\int_0^\infty ds\ \frac{s^{1/2}}{(1+s)^{1/2}}Z(\zeta_\sigma^s)$$

$$-\ell^2 g^2\zeta_\sigma\frac{\pi}{8}-\frac{\ell^2 g^2}{4}\sqrt{\pi}\lim_{\nu\to 0}\int_0^\infty ds\ \frac{s^{1/2}}{1+s}Z(\nu,-\zeta_\sigma^s)-\frac{\ell^2 g^2}{4}\sqrt{\pi}\int_0^\infty ds\ \frac{s^{1/2}}{1+s}Z(\zeta_\sigma^s)\Bigg)\ . \quad \text{(A.25)}$$

De (4.26),

$$\zeta_{1\sigma}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a) h_{\mathbf{n}}^d k f_d$$

$$= \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\frac{e^2}{m_d}v_*^{-1}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq'$$

$$\times\frac{a^2}{(2\pi)^3}(1-\frac{\mathbf{v}'\cdot\mathbf{e_v}}{v})v_\perp \exp\left(ika\frac{v_\parallel}{v}\right)\frac{\partial f_d/\partial q}{v_\parallel' - (\omega+i\omega_d)/k}$$

$$= -i\frac{a^2}{k}\frac{4\pi e^2}{\omega_{p\sigma}^2}\frac{q_\sigma}{m_d}\frac{2}{q_d^2}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' (1-\frac{v_\parallel'}{v})v_\perp \exp\left(ika\frac{v_\parallel}{v}\right)\frac{(q'-q_0')f_d}{v_\parallel' - (\omega+i\omega_d)/k}$$

$$= 0\ ,$$

que é zero pela integração sobre a carga da poeira.

De (4.27),

$$\zeta_{2\sigma}^{(1)}(\mathbf{n}) = \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq' \Omega_{\mathbf{k}}(\mathbf{v}-\mathbf{v}', a)\big(\mathbf{g}_{\mathbf{n}}^d\cdot\mathbf{k}\big) f_d$$

$$= \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\frac{1}{m_d}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq'$$

$$\times\frac{a^2}{(2\pi)^3}(1-\frac{\mathbf{v}'\cdot\mathbf{e_v}}{v})v_\perp \exp\left(ika\frac{v_\parallel}{v}\right)\frac{q'\ \partial f_d/\partial v_\parallel'}{v_\parallel' - (\omega+i\omega_d)/k}$$

$$= \frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\frac{1}{m_d}\int d\mathbf{v}\ f_\sigma \int d\mathbf{v}' dq'$$

$$\times\frac{a^2}{(2\pi)^3}(1-\frac{v_\parallel'}{v})v_\perp \cos\left(ka\frac{v_\parallel}{v}\right)\frac{q'\ \partial f_d/\partial v_\parallel'}{v_\parallel' - (\omega+i\omega_d)/k}$$

$$= -\frac{4\pi}{k^2}\frac{q_\sigma}{n_\sigma}\left(8\pi^3 i\frac{n_\sigma v_* k}{\omega_{p\sigma}^2}\right)\frac{a^2}{(2\pi)^3}\frac{1}{m_d}\frac{2}{u_d^2 v_*^2}e v_* \int d\mathbf{u}\ f_\sigma \int d\mathbf{u}' dZ'$$

$$\times(1-\frac{u_\parallel'}{u})u_\perp \cos\left(\ell g\frac{u_\parallel}{u}\right)\frac{Z'\ u_\parallel' f_d}{u_\parallel' - (z+iz_d)/\ell}$$

$$= -ia^2\frac{4\pi}{k}\frac{q_\sigma}{n_\sigma}\frac{n_\sigma}{\omega_{p\sigma}^2}\frac{1}{m_d}\frac{2}{u_d^2}e\frac{1}{\sqrt{\pi}Z_d}\frac{n_d}{(\pi)^{3/2}u_d^3}\sqrt{\pi}Z_0 Z_d\frac{n_\sigma}{(\pi)^{3/2}u_\sigma^3}(2\pi)^2$$

$$\times\int du_\perp\ u_\perp \int du_\parallel\ e^{-u^2/u_\sigma^2}\int du_\perp'\ u_\perp' \int du_\parallel' (1-\frac{u_\parallel'}{u})u_\perp(1-\frac{1}{2}\frac{\ell^2 g^2 u_\parallel^2}{u_\perp^2+u_\parallel^2})\frac{u_\parallel' e^{-u'^2/u_d^2}}{u_\parallel' - (z+iz_d)/\ell}$$

$$= -i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{8}{u_d^5 u_\sigma^3\pi}$$

$$\times\int du_\perp\ u_\perp\int du_\parallel\ e^{-u^2/u_\sigma^2}\int du'_\perp\ u'_\perp\int du'_\parallel(1-\frac{u'_\parallel}{u})u_\perp(1-\frac{1}{2}\frac{\ell^2g^2u_\parallel^2}{u_\perp^2+u_\parallel^2})\frac{u'_\parallel e^{-u'^2/u_d^2}}{u'_\parallel-(z+iz_d)/\ell}$$

$$= -i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{8}{u_d^2\pi}u_\sigma$$

$$\times\int du_\perp\ u_\perp\int du_\parallel\ e^{-u_\perp^2}e^{-u_\parallel^2}\int du'_\perp\ u'_\perp\int du'_\parallel(1-\frac{u_d}{u_\sigma}\frac{u'_\parallel}{u})u_\perp(1-\frac{1}{2}\frac{\ell^2g^2u_\parallel^2}{u_\perp^2+u_\parallel^2})\frac{u'_\parallel e^{-u'^2_\perp}e^{-u'^2_\parallel}}{u'_\parallel-\zeta_d}$$

$$= -i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{8}{u_d^2\pi}u_\sigma$$

$$\times\int du_\perp\ u_\perp\int du_\parallel\ e^{-u_\perp^2}e^{-u_\parallel^2}\int du'_\perp\ u'_\perp\int du'_\parallel u_\perp(1-\frac{1}{2}\frac{\ell^2g^2u_\parallel^2}{u_\perp^2+u_\parallel^2})\frac{u'_\parallel e^{-u'^2_\perp}e^{-u'^2_\parallel}}{u'_\parallel-\zeta_d}$$

$$= -i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{8}{u_d^2\pi}u_\sigma\left(\frac{\pi}{4}\frac{1}{2}\int du'_\parallel(e^{-u'^2_\parallel}+\zeta_d\frac{e^{-u'^2_\parallel}}{u'_\parallel-\zeta_d})\right.$$

$$\left.-\frac{1}{2}\ell^2g^2\frac{1}{2}\int_0^\infty ds\frac{\sqrt{s}}{1+s}\int du_\parallel\ |u_\parallel^3|e^{-(s+1)u_\parallel^2}\frac{1}{2}\int du'_\parallel(e^{-u'^2_\parallel}+\zeta_d\frac{e^{-u'^2_\parallel}}{u'_\parallel-\zeta_d})\right)$$

$$= -i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{8}{u_d^2\pi}u_\sigma\left(\frac{\pi}{4}\frac{1}{2}\sqrt{\pi}(1+\zeta_dZ(\zeta_d))\right.$$

$$\left.-\frac{1}{2}\ell^2g^2\frac{1}{2}\int_0^\infty ds\frac{\sqrt{s}}{(1+s)^3}\frac{1}{2}\sqrt{\pi}(1+\zeta_dZ(\zeta_d))\right)$$

$$= -i\frac{a^2}{\ell}\frac{v_*}{\omega_*}n_\sigma\frac{q_\sigma}{eZ_0}\frac{\omega_{pd}^2}{\omega_{p\sigma}^2}\frac{u_\sigma}{u_d^2}\left(1-\frac{\ell^2g^2}{8}\right)\sqrt{\pi}(1+\zeta_dZ(\zeta_d))\ .$$

Apêndice B

Termos das expansões de $N_{\mathbf{n}}^{\sigma}$ e $N_{\mathbf{n}}^{d}$

Os três primeiros termos da série (3.39) são

$$N_{\mathbf{n}}^{\sigma(1)} = i\bigg(\Big(\mathbf{g}_{\mathbf{n}}^{\sigma}\cdot\frac{\mathbf{k}}{k}\Big)E_{\mathbf{n}} + h_{\mathbf{n}}^{\sigma}E_{\mathbf{n}}^{\sigma}\bigg)f_{\sigma}\ ;$$

$$N_{\mathbf{n}}^{\sigma(2)} = i\mathbf{g}_{\mathbf{n}}^{\sigma}\int d\mathbf{k}_1\bigg(N_{\mathbf{n}-\mathbf{n}_1}^{\sigma(1)}\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1} - \Big\langle N_{\mathbf{n}-\mathbf{n}_1}^{\sigma(1)}\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}\Big\rangle\bigg)$$
$$+ih_{\mathbf{n}}^{\sigma}\int d\mathbf{k}_1\left(E_{\mathbf{n}_1}^{\sigma}N_{\mathbf{n}-\mathbf{n}_1}^{\sigma(1)} - \langle E_{\mathbf{n}_1}^{\sigma}N_{\mathbf{n}-\mathbf{n}_1}^{\sigma(1)}\rangle\right)$$

$$= i\mathbf{g}_{\mathbf{n}}^{\sigma}\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(N_{\mathbf{n}_2}^{\sigma(1)}\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1} - \Big\langle N_{\mathbf{n}_2}^{\sigma(1)}\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}\Big\rangle\bigg)$$
$$+ih_{\mathbf{n}}^{\sigma}\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\left(E_{\mathbf{n}_1}^{\sigma}N_{\mathbf{n}_2}^{\sigma(1)} - \langle E_{\mathbf{n}_1}^{\sigma}N_{\mathbf{n}_2}^{\sigma(1)}\rangle\right)$$

$$= -\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(\Big(\mathbf{g}_{\mathbf{n}}^{\sigma}\cdot\frac{\mathbf{k}_1}{k_1}\Big)\Big(\mathbf{g}_{\mathbf{n}_2}^{\sigma}\cdot\frac{\mathbf{k}_2}{k_2}\Big)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$
$$+\Big(\mathbf{g}_{\mathbf{k}\omega}^{\sigma}\cdot\frac{\mathbf{k}_1}{k_1}\Big)h_{\mathbf{n}_2}^{\sigma}\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2}^{\sigma} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^{\sigma}\rangle\big)\bigg)f_{\sigma}$$
$$-\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(h_{\mathbf{n}}^{\sigma}\Big(\mathbf{g}_{\mathbf{n}_2}^{\sigma}\cdot\frac{\mathbf{k}_2}{k_2}\Big)\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}\rangle\big)$$
$$+h_{\mathbf{n}}^{\sigma}h_{\mathbf{n}_2}^{\sigma}\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma} - \langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\rangle\big)\bigg)f_{\sigma}\ ;$$

$$N_{\mathbf{n}}^{\sigma(3)} = i\mathbf{g}_{\mathbf{n}}^{\sigma}\int d\mathbf{n}'\bigg(N_{\mathbf{n}-\mathbf{n}'}^{\sigma(2)}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'} - \Big\langle N_{\mathbf{n}-\mathbf{n}'}^{\sigma(2)}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\Big\rangle\bigg) + ih_{\mathbf{n}}^{\sigma}\int d\mathbf{n}'\left(E_{\mathbf{n}'}^{\sigma}N_{\mathbf{n}-\mathbf{n}'}^{\sigma(2)} - \langle E_{\mathbf{n}'}^{\sigma}N_{\mathbf{n}-\mathbf{n}'}^{\sigma(2)}\rangle\right)$$

$$= -i\mathbf{g}_{\mathbf{n}}^{\sigma}\int d\mathbf{n}'\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}'-\mathbf{n}_1-\mathbf{n}_2)$$

$$\times\Bigg(\Bigg(\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}$$

$$+\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}\left(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\Bigg)$$

$$-\Bigg\langle\Bigg(\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}$$

$$+\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}\left(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\Bigg)\Bigg\rangle\Bigg)$$

$$-ih^{\sigma}_{\mathbf{n}}\int d\mathbf{n}'\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}'-\mathbf{n}_1-\mathbf{n}_2)$$

$$\times\Bigg(E^{\sigma}_{\mathbf{n}'}\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)$$

$$+E^{\sigma}_{\mathbf{n}'}\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}\left(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)\Bigg)$$

$$-\Bigg\langle E^{\sigma}_{\mathbf{n}'}\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)$$

$$+E^{\sigma}_{\mathbf{n}'}\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}'}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}\left(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)f_\sigma\Bigg)\Bigg)\Bigg\rangle\Bigg)$$

$$-i\mathbf{g}^{\sigma}_{\mathbf{n}}\int d\mathbf{n}'\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}'-\mathbf{n}_1-\mathbf{n}_2)$$

$$\times\Bigg(\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}'}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)+h^{\sigma}_{\mathbf{n}-\mathbf{n}'}h^{\sigma}_{\mathbf{n}_2}\left(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{b}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)\Bigg)f_\sigma\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}$$

$$-\Bigg\langle\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}'}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)+h^{\sigma}_{\mathbf{n}-\mathbf{n}'}h^{\sigma}_{\mathbf{n}_2}\left(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)\Bigg)f_\sigma\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\Bigg\rangle\Bigg)$$

$$-ih^{\sigma}_{\mathbf{n}}\int d\mathbf{n}'\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}'-\mathbf{n}_1-\mathbf{n}_2)$$

$$\times\Bigg(E^{\sigma}_{\mathbf{n}'}\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}'}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)+h^{\sigma}_{\mathbf{n}-\mathbf{n}'}h^{\sigma}_{\mathbf{n}_2}\left(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)\Bigg)f_\sigma$$

$$-\Bigg\langle E^{\sigma}_{\mathbf{n}'}\Bigg(h^{\sigma}_{\mathbf{n}-\mathbf{n}'}\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\right)+h^{\sigma}_{\mathbf{n}-\mathbf{n}'}h^{\sigma}_{\mathbf{n}_2}\left(E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E^{\sigma}_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)\Bigg)f_\sigma\Bigg\rangle\Bigg)$$

$$=-i\mathbf{g}^{\sigma}_{\mathbf{n}}\int d\mathbf{n}_3\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2)$$

$$\times\Bigg(\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\right)$$

$$+\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}\left(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3}-\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3}+\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\right)\Bigg)\cdot\frac{\mathbf{k}_3}{k_3}f_\sigma$$

$$-ih^{\sigma}_{\mathbf{n}}\int d\mathbf{n}_3\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2)$$

$$\times\Bigg(\left(E_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E^{\sigma}_{\mathbf{n}_3}-\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E^{\sigma}_{\mathbf{n}_3}\rangle\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^{\sigma}_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)$$

$$+E^{\sigma}_{\mathbf{n}_3}\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}\left(E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}-\langle E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\rangle\right)+\left\langle E^{\sigma}_{\mathbf{n}_3}\left(\mathbf{g}^{\sigma}_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)h^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_1}E^{\sigma}_{\mathbf{n}_2}\right\rangle\Bigg)f_\sigma$$

$$
\begin{aligned}
&-i\mathbf{g}_{\mathbf{n}}^{\sigma}\int d\mathbf{n}_3\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2)\\
&\times\Bigg(\bigg(h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}\bigg(\mathbf{g}_{\mathbf{n}_2}^{\sigma}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}-\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}\big\rangle\big)+h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}h_{\mathbf{n}_2}^{\sigma}\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}-\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\big\rangle\big)\bigg)f_{\sigma}\cdot\frac{\mathbf{k}_3}{k_3}E_{\mathbf{n}_3}\\
&-\bigg\langle\bigg(h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}\bigg(\mathbf{g}_{\mathbf{n}_2}^{\sigma}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}\big)+h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}h_{\mathbf{n}_2}^{\sigma}\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\big)\bigg)f_{\sigma}\cdot\frac{\mathbf{k}_3}{k_3}E_{\mathbf{n}_3}\bigg\rangle\Bigg)\\
&-ih_{\mathbf{n}}^{\sigma}\int d\mathbf{n}_3\int d\mathbf{n}_1\int d\mathbf{n}_2\,\delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2)\\
&\times\Bigg(E_{\mathbf{n}_3}^{\sigma}\bigg(h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}\bigg(\mathbf{g}_{\mathbf{n}_2}^{\sigma}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}-\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}\big\rangle\big)+h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}h_{\mathbf{n}_2}^{\sigma}\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}-\big\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\big\rangle\big)\bigg)f_{\sigma}\\
&-\bigg\langle E_{\mathbf{n}_3}^{\sigma}\bigg(h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}\bigg(\mathbf{g}_{\mathbf{n}_2}^{\sigma}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}\big)+h_{\mathbf{n}-\mathbf{n}_3}^{\sigma}h_{\mathbf{n}_2}^{\sigma}\big(E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\big)\bigg)f_{\sigma}\bigg\rangle\Bigg)\;.
\end{aligned}
$$

Para $N_{\mathbf{n}}^{d}$ temos

$$
N_{\mathbf{n}}^{d(1)}=ih_{\mathbf{n}}^{d}\hat{E}_{\mathbf{n}}f_d+i\bigg(\mathbf{g}_{\mathbf{n}}^{d}\cdot\frac{\mathbf{k}}{k}\bigg)E_{\mathbf{n}}f_d,
$$

$$
\begin{aligned}
N_{\mathbf{n}}^{d(2)}&=ih_{\mathbf{n}}^{d}\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\big(\hat{E}_{\mathbf{n}_1}N_{\mathbf{n}_2}^{d(1)}-\langle\hat{E}_{\mathbf{n}_1}N_{\mathbf{n}_2}^{d(1)}\rangle\big)\\
&+i\mathbf{g}_{\mathbf{n}}^{d}\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(N_{\mathbf{n}_2}^{d(1)}\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}-\bigg\langle N_{\mathbf{n}_2}^{d(1)}\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}\bigg\rangle\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&=-h_{\mathbf{n}}^{d}\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\big(\hat{E}_{\mathbf{n}_1}h_{\mathbf{n}_2}^{d}\hat{E}_{\mathbf{n}_2}f_d-\langle\hat{E}_{\mathbf{n}_1}h_{\mathbf{n}_2}^{d}\hat{E}_{\mathbf{n}_2}f_d\rangle\big)\\
&-\mathbf{g}_{\mathbf{n}}^{d}\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(h_{\mathbf{n}_2}^{d}\hat{E}_{\mathbf{n}_2}f_d\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}-\bigg\langle h_{\mathbf{n}_2}^{d}\hat{E}_{\mathbf{n}_2}f_d\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}\bigg\rangle\bigg)\\
&-h_{\mathbf{n}}^{d}\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(\hat{E}_{\mathbf{n}_1}\bigg(\mathbf{g}_{\mathbf{n}_2}^{d}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)E_{\mathbf{n}_2}f_d-\langle\hat{E}_{\mathbf{n}_1}\bigg(\mathbf{g}_{\mathbf{n}_2}^{d}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)E_{\mathbf{n}_2}f_d\rangle\bigg)\\
&-\mathbf{g}_{\mathbf{n}}^{d}\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(\bigg(\mathbf{g}_{\mathbf{n}_2}^{d}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)E_{\mathbf{n}_2}f_d\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}-\bigg\langle\bigg(\mathbf{g}_{\mathbf{n}_2}^{d}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)E_{\mathbf{n}_2}f_d\cdot\frac{\mathbf{k}_1}{k_1}E_{\mathbf{n}_1}\bigg\rangle\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&=-\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)h_{\mathbf{n}}^{d}h_{\mathbf{n}_2}^{d}\big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}-\langle\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\rangle\big)f_d\\
&-\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(\mathbf{g}_{\mathbf{n}}^{d}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)h_{\mathbf{n}_2}^{d}\big(\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_1}-\big\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_1}\big\rangle\big)f_d\\
&-\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\big(\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}-\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)h_{\mathbf{n}}^{d}\bigg(\mathbf{g}_{\mathbf{n}_2}^{d}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)f_d\\
&-\int d\mathbf{n}_1\int d\mathbf{n}_2\delta(\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2)\bigg(\mathbf{g}_{\mathbf{n}}^{d}\cdot\frac{\mathbf{k}_1}{k_1}\bigg)\bigg(\mathbf{g}_{\mathbf{n}_2}^{d}\cdot\frac{\mathbf{k}_2}{k_2}\bigg)\big(E_{\mathbf{n}_2}E_{\mathbf{n}_1}-\big\langle E_{\mathbf{n}_2}E_{\mathbf{n}_1}\big\rangle\big)f_d\;;
\end{aligned}
$$

$$
\begin{aligned}
N_{\mathbf{n}}^{d(3)}&=ih_{\mathbf{n}}^{d}\int d\mathbf{n}'\big(\hat{E}_{\mathbf{n}'}N_{\mathbf{n}-\mathbf{n}'}^{d(2)}-\big\langle\hat{E}_{\mathbf{n}'}N_{\mathbf{n}-\mathbf{n}'}^{d(2)}\big\rangle\big)\\
&+i\mathbf{g}_{\mathbf{n}}^{d}\int d\mathbf{n}'\bigg(N_{\mathbf{n}-\mathbf{n}'}^{d(2)}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}-\bigg\langle N_{\mathbf{n}-\mathbf{n}'}^{d(2)}\cdot\frac{\mathbf{k}'}{k'}E_{\mathbf{n}'}\bigg\rangle\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -ih_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\big(\hat{E}_{\mathbf{n}_3} h_{\mathbf{n}-\mathbf{n}_3}^d h_{\mathbf{n}_2}^d \big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big) - \Big\langle \hat{E}_{\mathbf{n}_3} h_{\mathbf{n}-\mathbf{n}_3}^d h_{\mathbf{n}_2}^d \big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big)\Big\rangle\big) f_d \\
&-ih_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\big(\hat{E}_{\mathbf{n}_3}\Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big) h_{\mathbf{n}_2}^d \big(E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big) - \Big\langle \hat{E}_{\mathbf{n}_3}\Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big) h_{\mathbf{n}_2}^d \big(E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big)\Big\rangle\big) f_d \\
&-i\mathbf{g}_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\Big(h_{\mathbf{n}-\mathbf{n}_3}^d h_{\mathbf{n}_2}^d \big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3} - \Big\langle h_{\mathbf{n}-\mathbf{n}_3}^d h_{\mathbf{n}_2}^d \big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3}\Big\rangle\Big) f_d \\
&-i\mathbf{g}_{\mathbf{k}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\Big(\Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big) h_{\mathbf{n}_2}^d \big(E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3} - \Big\langle \Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big) h_{\mathbf{n}_2}^d \big(E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3}\Big\rangle\Big) f_d \\
&-ih_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\big(\hat{E}_{\mathbf{n}_3} h_{\mathbf{n}-\mathbf{n}_3}^d \big(\hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2}\big\rangle\big) \\
&-\Big\langle \hat{E}_{\mathbf{n}_3} h_{\mathbf{n}-\mathbf{n}_3}^d \big(\hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2}\big\rangle\big)\Big\rangle\big) f_d \\
&-ih_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\big(\hat{E}_{\mathbf{n}_3}\Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big)\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big) \\
&-\Big\langle \hat{E}_{\mathbf{n}_3}\Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big)\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big)\Big\rangle\big) f_d \\
&-i\mathbf{g}_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\Big(h_{\mathbf{n}-\mathbf{n}_3}^d \big(\hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3} \\
&-\Big\langle h_{\mathbf{n}-\mathbf{n}_3}^d \big(\hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2} - \big\langle \hat{E}_{\mathbf{n}_1}\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big) E_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3}\Big\rangle\Big) f_d \\
&-i\mathbf{g}_{\mathbf{n}}^d \int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\Big(\Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big)\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3} \\
&-\Big\langle \Big(\mathbf{g}_{\mathbf{n}-\mathbf{n}_3}^d \cdot \frac{\mathbf{k}_1}{k_1}\Big)\Big(\mathbf{g}_{\mathbf{n}_2}^d \cdot \frac{\mathbf{k}_2}{k_2}\Big)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2} - \big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\big\rangle\big) \cdot \frac{\mathbf{k}_3}{k_3} E_{\mathbf{n}_3}\Big\rangle\Big) f_d
\end{aligned}
$$

$$
\begin{aligned}
&= -i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\left(h^d_{\mathbf{n}}\hat{E}_{\mathbf{n}_3} h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2}\big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\rangle\big) - \left\langle h^d_{\mathbf{n}}\hat{E}_{\mathbf{n}_3} h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\right\rangle\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\left(h^d_{\mathbf{n}}\hat{E}_{\mathbf{n}_3}\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2}\big(E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\rangle\big) - \left\langle h^d_{\mathbf{n}}\hat{E}_{\mathbf{n}_3}\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2} E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\right\rangle\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\left(\mathbf{g}^d_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\right)\left(h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2}\big(\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle \hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3}\big) - \left\langle h^d_{\mathbf{n}-\mathbf{n}_3} h^d_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\right\rangle\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\left(\mathbf{g}^d_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\right)\left(\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2}\big(E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3}\big) - \left\langle \left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2} E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\right\rangle\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\big(\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3} - \langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\hat{E}_{\mathbf{n}_3} - \langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}\rangle\big) h^d_{\mathbf{n}} h^d_{\mathbf{n}-\mathbf{n}_3}\left(\mathbf{g}^d_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\hat{E}_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\hat{E}_{\mathbf{n}_3}\rangle\big) h^d_{\mathbf{n}}\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^d_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\left(\mathbf{g}^d_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\right)\big(\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3} - \langle \hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\big) h^d_{\mathbf{n}-\mathbf{n}_3}\left(\mathbf{g}^d_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right) f_d \\
&- i\int d\mathbf{n}_3 \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n}-\mathbf{n}_3-\mathbf{n}_1-\mathbf{n}_2) \\
&\times\left(\mathbf{g}^d_{\mathbf{n}}\cdot\frac{\mathbf{k}_3}{k_3}\right)\left(\mathbf{g}^d_{\mathbf{n}-\mathbf{n}_3}\cdot\frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^d_{\mathbf{n}_2}\cdot\frac{\mathbf{k}_2}{k_2}\right)\big(E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle E_{\mathbf{n}_3} - \langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\big) f_d \ .
\end{aligned}
$$

Apêndice C

Cálculo da correlação $\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{-(\mathbf{n}_1+\mathbf{n}_2)} \rangle$ e do termo não linear $NL1$

Primeiramente consideramos as expansões de $N^\sigma_\mathbf{n}$ e $N^d_\mathbf{n}$ até segunda ordem,

$$
\begin{aligned}
N^\sigma_\mathbf{n} = i\left(\mathbf{g}^\sigma_\mathbf{n} \cdot \frac{\mathbf{k}}{k}\right) E_\mathbf{n} f_\sigma + i h^\sigma_\mathbf{n} E^d_\mathbf{n} f_\sigma - \int d\mathbf{n}_1 \int d\mathbf{n}_2\, \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \\
\times \left(\left(\mathbf{g}^\sigma_\mathbf{n} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^\sigma_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_1} E_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} \rangle\right) + \left(\mathbf{g}^\sigma_\mathbf{n} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^\sigma_{\mathbf{n}_2}\left(E_{\mathbf{n}_1} E^d_{\mathbf{n}_2} - \langle E_{\mathbf{n}_1} E^d_{\mathbf{n}_2} \rangle\right) \right. \\
\left. + h^\sigma_\mathbf{n}\left(\mathbf{g}^\sigma_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\left(E^d_{\mathbf{n}_1} E_{\mathbf{n}_2} - \langle E^d_{\mathbf{n}_1} E_{\mathbf{n}_2} \rangle\right) + h^\sigma_\mathbf{n} h^\sigma_{\mathbf{n}_2}\left(E^d_{\mathbf{n}_1} E^d_{\mathbf{n}_2} - \langle E^d_{\mathbf{n}_1} E^d_{\mathbf{n}_2} \rangle\right)\right) f_\sigma
\end{aligned}
\tag{C.1}
$$

e

$$
\begin{aligned}
N^d_\mathbf{n} = i h^d_\mathbf{n} \hat{E}_\mathbf{n} f_d + i \mathbf{g}^d_\mathbf{n} f_d \cdot \frac{\mathbf{k}}{k} E_\mathbf{n} - \int d\mathbf{n}_1 \int d\mathbf{n}_2 \delta(\mathbf{n} - \mathbf{n}_1 - \mathbf{n}_2) \\
\times \left(h^d_\mathbf{n} h^d_{\mathbf{n}_2}\left(\hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} - \langle \hat{E}_{\mathbf{n}_1} \hat{E}_{\mathbf{n}_2} \rangle\right) + \left(\mathbf{g}^d_\mathbf{n} \cdot \frac{\mathbf{k}_1}{k_1}\right) h^d_{\mathbf{n}_2}\left(\hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_1} - \langle \hat{E}_{\mathbf{n}_2} E_{\mathbf{n}_1} \rangle\right) \right. \\
+ \left(\hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} - \langle \hat{E}_{\mathbf{n}_1} E_{\mathbf{n}_2} \rangle\right) h^d_\mathbf{n}\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right) \\
\left. + \left(\mathbf{g}^d_\mathbf{n} \cdot \frac{\mathbf{k}_1}{k_1}\right)\left(\mathbf{g}^d_{\mathbf{n}_2} \cdot \frac{\mathbf{k}_2}{k_2}\right)\left(E_{\mathbf{n}_2} E_{\mathbf{n}_1} - \langle E_{\mathbf{n}_2} E_{\mathbf{n}_1} \rangle\right)\right) f_d \,,
\end{aligned}
\tag{C.2}
$$

Inserindo estas soluções de segunda ordem (C.1) e (C.2) em (3.30), obtemos

$$
E_{\mathbf{n}'''}\left(\epsilon(\mathbf{n}''') + \psi(\mathbf{n}''')\right) = \sum_\sigma \chi^{(1)}_{1\sigma}(\mathbf{n}''') E^\sigma_{\mathbf{n}'''} + \psi^{(1)}_1(\mathbf{n}''') \hat{E}_{\mathbf{n}'''} \tag{C.3}
$$

$$
+ \sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}'''} \left(\chi^{(2)}_1 \left(E_{\mathbf{n}'} E_{\mathbf{n}''} - \langle E_{\mathbf{n}'} E_{\mathbf{n}''} \rangle\right) + \sum_\sigma \chi^{(2)}_{2\sigma}\left(E_{\mathbf{n}'} E^\sigma_{\mathbf{n}''} - \langle E_{\mathbf{n}'} E^\sigma_{\mathbf{n}''} \rangle\right) \right.
$$

$$+\sum_{\sigma}\chi_{3\sigma}^{(2)}\big(E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}-\big\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}\big\rangle\big)+\sum_{\sigma}\chi_{4\sigma}^{(2)}\big(E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}^{\sigma}-\big\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}^{\sigma}\big\rangle\big)\Big)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}'''}\Big(\psi_1^{(2)}\big(\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\big)+\psi_2^{(2)}\big(\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}-\big\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\big)$$

$$+\psi_3^{(2)}\big(\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\big)+\psi_4^{(2)}\big(E_{\mathbf{n}''}E_{\mathbf{n}'}-\big\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\big)\Big)\ .$$

Fazendo sucessivamente $\mathbf{n}'''=\mathbf{n}_1$, $\mathbf{n}_2$ e $-(\mathbf{n}_1+\mathbf{n}_2)$, e multiplicando o resultado pela direita pelos outros campos e fazendo a média, obtemos os termos apropriados para construir a correlação tripla,

$$\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big(\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)\big)$$

$$=\sum_{\sigma}\chi_{1\sigma}^{(1)}(\mathbf{n}_1)\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\psi_1^{(1)}(\mathbf{n}_1)\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\Big(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}^{\sigma}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}^{\sigma}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{3}^{(2)}\big(\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{4\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}^{\sigma}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}^{\sigma}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Big)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\Big(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Big)\ ; \qquad \text{(C.4)}$$

$$\langle E_{\mathbf{n}_2}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big(\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)\big) \qquad \text{(C.5)}$$

$$=\sum_{\sigma}\chi_{1\sigma}^{(1)}(\mathbf{n}_2)\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\psi_1^{(1)}(\mathbf{n}_2)\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\Big(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}^{\sigma}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}^{\sigma}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{3\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{4\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}^{\sigma}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}^{\sigma}E_{\mathbf{n}''}^{\sigma}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Big)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\Bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Bigg)\ ;$$

$$\langle E_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big(\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))\big)$$

$$=\sum_\sigma\chi_{1\sigma}^{(1)}(-(\mathbf{n}_1+\mathbf{n}_2))\langle E^\sigma_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle+\psi_1^{(1)}(\mathbf{n})\langle\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\Bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\sum_\sigma\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\sum_\sigma\chi_{3\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\sum_\sigma\chi_{4\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E^\sigma_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)\Bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\Bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)\Bigg)\ . \tag{C.6}$$

De (C.4), (C.5) e (C.6),

$$3\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle \tag{C.7}$$

$$=\sum_\sigma\frac{\chi_{1\sigma}^{(1)}(\mathbf{n}_1)}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\frac{\psi_1^{(1)}(\mathbf{n}_1)}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\Bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_\sigma\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_\sigma\chi_{3\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_\sigma\chi_{4\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E^\sigma_{\mathbf{n}'}E^\sigma_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\Bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Bigg)$$

$$+\sum_{\sigma}\frac{\chi_{1\sigma}^{(1)}(\mathbf{n}_2)}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\langle E^{\sigma}_{\mathbf{n}_2}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\frac{\psi_1^{(1)}(\mathbf{n}_2)}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\Bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{3\sigma}^{(2)}\big(\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\sum_{\sigma}\chi_{4\sigma}^{(2)}\big(\langle E^{\sigma}_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E^{\sigma}_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\Bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle-\big\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\big\rangle\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\big)\Bigg)$$

$$+\sum_{\sigma}\frac{\chi_{1\sigma}^{(1)}(-(\mathbf{n}_1+\mathbf{n}_2))}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}\langle E^{\sigma}_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle$$

$$+\frac{\psi_1^{(1)}(\mathbf{n})}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}\langle\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}$$

$$\times\Bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\sum_{\sigma}\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\sum_{\sigma}\chi_{3\sigma}^{(2)}\big(\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)$$

$$+\sum_{\sigma}\chi_{4\sigma}^{(2)}\big(\langle E^{\sigma}_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\big\langle E^{\sigma}_{\mathbf{n}'}E^{\sigma}_{\mathbf{n}''}\big\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)\Bigg)$$

$$
+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)} \frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}
$$

$$
\times\Bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\langle\hat{E}_{\mathbf{n}'}\hat{E}_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}'}\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}''}\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}'}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle-\langle E_{\mathbf{n}''}E_{\mathbf{n}'}\rangle\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle\big)\Bigg)\,.
$$

Eliminando as correlações quádruplas em termos de (4.6),

$$
3\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle \tag{C.8}
$$

$$
=\sum_\sigma \frac{\chi_{1\sigma}^{(1)}(\mathbf{n}_1)}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\langle E^\sigma_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\frac{\psi_1^{(1)}(\mathbf{n}_1)}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle
$$

$$
+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\Bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\sum_\sigma\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E^\sigma_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\sum_\sigma\chi_{3\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E^\sigma_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\sum_\sigma\chi_{4\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E^\sigma_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E^\sigma_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\big)\Bigg)
$$

$$
+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\Bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle\hat{E}_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle\hat{E}_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle\hat{E}_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle\hat{E}_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\big)
$$

$$
+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\big)\Bigg)
$$

$$
+\sum_\sigma \frac{\chi_{1\sigma}^{(1)}(\mathbf{n}_2)}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\langle E^\sigma_{\mathbf{n}_2}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\frac{\psi_1^{(1)}(\mathbf{n}_2)}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle
$$

$$
+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\Bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)
$$

$$
+\sum_\sigma\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E^\sigma_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)
$$

$$
+\sum_\sigma\chi_{3\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E^\sigma_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)
$$

$$
+\sum_\sigma\chi_{4\sigma}^{(2)}\big(\langle E^\sigma_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E^\sigma_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E^\sigma_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E^\sigma_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)\Bigg)
$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle\hat{E}_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle\hat{E}_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle\hat{E}_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle\hat{E}_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}'}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\langle E_{\mathbf{n}''}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\big)\bigg)$$

$$+\sum_{\sigma}\frac{\chi_{1\sigma}^{(1)}(-(\mathbf{n}_1+\mathbf{n}_2))}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}\langle E^{\sigma}_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle$$

$$+\frac{\psi_1^{(1)}(\mathbf{n})}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}\langle\hat{E}_{-(\mathbf{n}_1+\mathbf{n}_2)}E_{\mathbf{n}_1}E_{\mathbf{n}_2}\rangle$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}$$

$$\times\bigg(\chi_1^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle+\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\sum_{\sigma}\chi_{2\sigma}^{(2)}\big(\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle+\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\sum_{\sigma}\chi_{3\sigma}^{(2)}\big(\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle+\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\sum_{\sigma}\chi_{4\sigma}^{(2)}\big(\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle+\langle E^{\sigma}_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E^{\sigma}_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)\bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}$$

$$\times\bigg(\psi_1^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle+\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\psi_2^{(2)}\big(\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle+\langle\hat{E}_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\big)$$

$$+\psi_3^{(2)}\big(\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle+\langle\hat{E}_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\big)$$

$$+\psi_4^{(2)}\big(\langle E_{\mathbf{n}''}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_2}\rangle+\langle E_{\mathbf{n}''}E_{\mathbf{n}_2}\rangle\langle E_{\mathbf{n}'}E_{\mathbf{n}_1}\rangle\big)\bigg)\ .$$

O lado esquerdo de (C.8) é o que queremos. Se considerarmos que as correlações triplas de diferentes campos, como por exemplo $\langle E^{\sigma}_{\mathbf{n}}E_{\mathbf{a}}E_{\mathbf{b}}\rangle$ e $\langle\hat{E}_{\mathbf{n}}E_{\mathbf{a}}E_{\mathbf{b}}\rangle$, são devidas somente a efeitos de absorção de partículas, e que a interação não linear onda-onda devida a esses termos é desprezável em relação a correlações triplas de campos iguais, como $\langle E_{\mathbf{a}}E_{\mathbf{b}}E_{\mathbf{c}}\rangle$, $\langle\hat{E}_{\mathbf{a}}\hat{E}_{\mathbf{b}}\hat{E}_{\mathbf{c}}\rangle$ e $\langle E^{\sigma}_{\mathbf{a}}E^{\sigma'}_{\mathbf{b}}E^{\sigma''}_{\mathbf{c}}\rangle$, podemos desprezar aquelas no lado direito de (C.8), visto que elas não contribuem para a intensidade das ondas. Mesmo com esta aproximação, a equação (C.8) ainda contém o efeito de absorção de partículas do plasma pelas partículas de poeira.

Da discussão feita no Capítulo 5,

$$\langle E_{\mathbf{a}} E_{\mathbf{b}} \rangle = \langle E_{\mathbf{a}}^2 \rangle \delta\big(\mathbf{a}+\mathbf{b}\big) = \sum_{\alpha} \sum_{j_1=\pm} \mathcal{I}_{\mathbf{k_a}}^{\alpha j_1} \delta(\omega_{\mathbf{a}} + j_1 \omega_{1\mathbf{k_a}}^{\alpha}) \delta\big(\mathbf{a}+\mathbf{b}\big) \tag{C.9}$$

$$\langle \hat{E}_{\mathbf{a}} \hat{E}_{\mathbf{b}} \rangle = \langle \hat{E}_{\mathbf{a}}^2 \rangle \delta\big(\mathbf{a}+\mathbf{b}\big) = \sum_{\beta} \sum_{j_2=\pm} \hat{\mathcal{I}}_{\mathbf{k_a}}^{\beta j_2} \delta(\omega_{\mathbf{a}} + j_2 \omega_{2\mathbf{k_a}}^{\beta}) \delta\big(\mathbf{a}+\mathbf{b}\big) \tag{C.10}$$

$$\langle E_{\mathbf{a}}^{\sigma} E_{\mathbf{b}}^{\sigma'} \rangle = \langle E_{\mathbf{a}}^{\sigma} E_{-\mathbf{a}}^{\sigma'} \rangle \delta\big(\mathbf{a}+\mathbf{b}\big) = \sum_{\gamma} \sum_{j_3=\pm} \mathcal{I}_{\mathbf{k_a},\sigma'\sigma}^{\gamma j_3} \delta(\omega_{\mathbf{a}} + j_3 \omega_{3\mathbf{k_a}}^{\gamma}) \delta\big(\mathbf{a}+\mathbf{b}\big) \tag{C.11}$$

$$\langle \hat{E}_{\mathbf{a}} E_{\mathbf{b}} \rangle = \big\langle \hat{E}_{\mathbf{a}} E_{-\mathbf{a}} \rangle \delta\big(\mathbf{a}+\mathbf{b}\big) = \frac{\rho_1 + \sum_{\sigma} \rho_{2\sigma} \zeta_{2\sigma}}{1 - \sum_{\sigma} \rho_{2\sigma} \zeta_{1\sigma}} \bigg|_{\mathbf{a}} \times \sum_{\alpha} \sum_{j_1=\pm} \mathcal{I}_{\mathbf{k_a}}^{\alpha j_1} \delta(\omega_{\mathbf{a}} + j_1 \omega_{1\mathbf{k_a}}^{\alpha}) \delta\big(\mathbf{a}+\mathbf{b}\big) \tag{C.12}$$

$$\langle E_{\mathbf{a}}^{e} E_{\mathbf{b}} \rangle = \langle E_{\mathbf{a}}^{e} E_{-\mathbf{a}} \rangle \delta\big(\mathbf{a}+\mathbf{b}\big) = \frac{(\zeta_{1e}\rho_1 + \zeta_{2e})(1 - \zeta_{1i}\rho_{2i}) + (\zeta_{1i}\rho_1 + \zeta_{2i})\zeta_{1e}\rho_{2i}}{(1 - \zeta_{1e}\rho_{2e})(1 - \zeta_{1i}\rho_{2i}) - \zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}} \bigg|_{\mathbf{a}} \times \sum_{\alpha} \sum_{j_1=\pm} \mathcal{I}_{\mathbf{k_a}}^{\alpha j_1} \delta(\omega_{\mathbf{a}} + j_1 \omega_{1\mathbf{k_a}}^{\alpha}) \delta\big(\mathbf{a}+\mathbf{b}\big) \tag{C.13}$$

$$\langle E_{\mathbf{a}}^{i} E_{\mathbf{b}} \rangle = \langle E_{\mathbf{a}}^{i} E_{-\mathbf{a}} \rangle \delta\big(\mathbf{a}+\mathbf{b}\big) = \frac{(\zeta_{1i}\rho_1 + \zeta_{2i})(1 - \zeta_{1e}\rho_{2e}) + (\zeta_{1e}\rho_1 + \zeta_{2e})\zeta_{1i}\rho_{2e}}{(1 - \zeta_{1i}\rho_{2i})(1 - \zeta_{1e}\rho_{2e}) - \zeta_{1e}\rho_{2i}\zeta_{1i}\rho_{2e}} \bigg|_{\mathbf{a}} \times \sum_{\alpha} \sum_{j_1=\pm} \mathcal{I}_{\mathbf{k_a}}^{\alpha j_1} \delta(\omega_{\mathbf{a}} + j_1 \omega_{1\mathbf{k_a}}^{\alpha}) \delta\big(\mathbf{a}+\mathbf{b}\big) \tag{C.14}$$

Então, considerando um plasma empoeirado elétron-íon, (C.8) se torna

$$3\langle E_{\mathbf{n}_1} E_{\mathbf{n}_2} E_{-(\mathbf{n}_1+\mathbf{n}_2)} \rangle = \sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1} \frac{1}{\epsilon(\mathbf{n}_1) + \psi(\mathbf{n}_1)} \tag{C.15}$$

$$\times\Bigg(\chi_1^{(2)}\Bigg(\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega' + j_1' \omega_{1\mathbf{k}'}^{\alpha'}) \delta\big(\mathbf{n}'+\mathbf{n}_2\big) \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega'' + j_1'' \omega_{1\mathbf{k}''}^{\alpha''}) \delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega' + j_1' \omega_{1\mathbf{k}'}^{\alpha'}) \delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big) \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega'' + j_1'' \omega_{1\mathbf{k}''}^{\alpha''}) \delta\big(\mathbf{n}''+\mathbf{n}_2\big)\Bigg)$$

$$+\chi_{2e}^{(2)}\Bigg(\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega' + j_1' \omega_{1\mathbf{k}'}^{\alpha'}) \delta\big(\mathbf{n}'+\mathbf{n}_2\big) \frac{(\zeta_{1e}\rho_1 + \zeta_{2e})(1 - \zeta_{1i}\rho_{2i}) + (\zeta_{1i}\rho_1 + \zeta_{2i})\zeta_{1e}\rho_{2i}}{(1 - \zeta_{1e}\rho_{2e})(1 - \zeta_{1i}\rho_{2i}) - \zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}} \bigg|_{\mathbf{n}''}$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega'' + j_1'' \omega_{1\mathbf{k}''}^{\alpha''}) \delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega' + j_1' \omega_{1\mathbf{k}'}^{\alpha'}) \delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times \frac{(\zeta_{1e}\rho_1 + \zeta_{2e})(1 - \zeta_{1i}\rho_{2i}) + (\zeta_{1i}\rho_1 + \zeta_{2i})\zeta_{1e}\rho_{2i}}{(1 - \zeta_{1e}\rho_{2e})(1 - \zeta_{1i}\rho_{2i}) - \zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}} \bigg|_{\mathbf{n}''} \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega'' + j_1'' \omega_{1\mathbf{k}''}^{\alpha''}) \delta\big(\mathbf{n}''+\mathbf{n}_2\big)\Bigg)$$

$$+\chi_{3e}^{(2)}\Bigg(\frac{(\zeta_{1e}\rho_1 + \zeta_{2e})(1 - \zeta_{1i}\rho_{2i}) + (\zeta_{1i}\rho_1 + \zeta_{2i})\zeta_{1e}\rho_{2i}}{(1 - \zeta_{1e}\rho_{2e})(1 - \zeta_{1i}\rho_{2i}) - \zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}} \bigg|_{\mathbf{n}'} \sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega' + j_1' \omega_{1\mathbf{k}'}^{\alpha'}) \delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega'' + j_1'' \omega_{1\mathbf{k}''}^{\alpha''}) \delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)\bigg)$$

$$+\chi_{4e}^{(2)}\bigg(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}''}$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)\bigg)$$

$$+\big(\quad e\leftrightarrows i\quad\big)\bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_1}\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\bigg(\psi_1^{(2)}\bigg(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)\bigg)$$

$$+\psi_2^{(2)}\bigg(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)\bigg)$$

$$+\psi_3^{(2)}\bigg(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}'} \sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)\bigg)$$

$$+\psi_4^{(2)}\bigg(\sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)\bigg)\bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}$$

$$\times\bigg(\chi_1^{(2)}\bigg(\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)\sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\bigg)$$

$$+\chi_{2e}^{(2)}\bigg(\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}''}$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\bigg)$$

$$+\chi_{3e}^{(2)}\bigg(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\bigg)$$

$$+\chi_{4e}^{(2)}\bigg(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm} \mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'} \delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}''}$$

$$\times \sum_{\alpha''}\sum_{j_1''=\pm} \mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''} \delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)$$

$$
+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Big)
$$

$$
+\big(\quad e\leftrightarrows i\quad\big)\Big)
$$

$$
+\sum_{\mathbf{n}'+\mathbf{n}''=\mathbf{n}_2}\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\Bigg(\psi_1^{(2)}\Bigg(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)
$$

$$
\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)
$$

$$
+\psi_2^{(2)}\Bigg(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)
$$

$$
\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\Bigg)
$$

$$
+\psi_3^{(2)}\Bigg(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)
$$

$$
\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)
$$

$$
+\psi_4^{(2)}\Bigg(\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'-(\mathbf{n}_1+\mathbf{n}_2)\big)
$$

$$
+\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''-(\mathbf{n}_1+\mathbf{n}_2)\big)\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\Bigg)\Bigg)
$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}$$

$$\times\Bigg(\chi_1^{(2)}\Bigg(\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$+\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)$$

$$+\chi_{2e}^{(2)}\Bigg(\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$+\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)$$

$$+\chi_{3e}^{(2)}\Bigg(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)$$

$$+\chi_{4e}^{(2)}\Bigg(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)$$

$$+\big(\quad e\leftrightarrows i\quad\big)\Bigg)$$

$$+\sum_{\mathbf{n}'+\mathbf{n}''=-(\mathbf{n}_1+\mathbf{n}_2)}\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}$$

$$\times\Bigg(\psi_1^{(2)}\Bigg(\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$+\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)$$

$$+\psi_2^{(2)}\Bigg(\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$+\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}''}\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\Bigg)$$

$$+\psi_3^{(2)}\Bigg(\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)$$

$$+\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}'}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$\times\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\Bigg)$$

$$+\psi_4^{(2)}\Bigg(\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_1\big)\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_2\big)$$

$$+\sum_{\alpha''}\sum_{j_1''=\pm}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega''+j_1''\omega_{1\mathbf{k}''}^{\alpha''})\delta\big(\mathbf{n}''+\mathbf{n}_2\big)\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega'+j_1'\omega_{1\mathbf{k}'}^{\alpha'})\delta\big(\mathbf{n}'+\mathbf{n}_1\big)\Bigg)\Bigg)\ .$$

Integrando em $\mathbf{n}'$ e $\mathbf{n}''$,

$$3\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle=\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm} \tag{C.16}$$

$$\times\Bigg(\Bigg(\chi_1^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_1^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{2e}^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_{2e}^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{3e}^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_{3e}^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}$$

$$\times\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{4e}^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_{4e}^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)+\Big(\quad e\leftrightarrows i\quad\Big)\Bigg)$$

$$+\frac{1}{\epsilon(\mathbf{n}_1)+\psi(\mathbf{n}_1)}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}$$

$$\times\Bigg(\Bigg(\psi_1^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\psi_1^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}'}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\psi_2^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$+\psi_2^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\Bigg)$$

$$+\Bigg(\psi_3^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{-\mathbf{n}_2}\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\psi_3^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\Bigg)$$

$$+\Bigg(\psi_4^{(2)}(-\mathbf{n}_2;\mathbf{n}_1+\mathbf{n}_2)\mathcal{I}_{-\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$+\psi_4^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_2)\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})\mathcal{I}_{-\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})\Bigg)\Bigg)$$

$$+\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}$$

$$\times\Bigg(\Bigg(\chi_1^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\mathcal{I}_{-\mathbf{k}_1}^{\alpha' j_1'}\delta(\omega_1-j_1'\omega_{1,-\mathbf{k}_1}^{\alpha'})\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_1^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})\mathcal{I}_{\mathbf{k}''}^{\alpha'' j_1''}\delta(\omega_1-j_1''\omega_{1,-\mathbf{k}_1}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{2e}^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\mathcal{I}_{-\mathbf{k}_1}^{\alpha' j_1'}\delta(\omega_1-j_1'\omega_{1,-\mathbf{k}_1}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_{2e}^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_1}\mathcal{I}_{-1\mathbf{k}_1}^{\alpha'' j_1''}\delta(\omega_1-j_1''\omega_{1,-\mathbf{k}_1}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{3e}^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_1}\mathcal{I}_{-\mathbf{k}_1}^{\alpha' j_1'}\delta(\omega_1-j_1'\omega_{1,-\mathbf{k}_1}^{\alpha'})$$

$$\times\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_{3e}^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\mathcal{I}_{-\mathbf{k}_1}^{\alpha'' j_1''}\delta(\omega_1-j_1''\omega_{1,-\mathbf{k}_1}^{\alpha''})\Bigg)$$

$$+\Bigg(\chi_{4e}^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{-\mathbf{n}_1}\mathcal{I}_{-\mathbf{k}_1}^{\alpha' j_1'}\delta(\omega_1-j_1'\omega_{1,-\mathbf{k}_1}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha'' j_1''}\delta(\omega_1+\omega_2+j_1''\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha''})$$

$$+\chi_{4e}^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\bigg|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}_{\mathbf{k}_1+\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_1+\omega_2+j_1'\omega_{1,\mathbf{k}_1+\mathbf{k}_2}^{\alpha'})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{-\mathbf{n}_1}\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_1}\delta(\omega_1-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Big)$$

$$+\Big(\quad e\leftrightarrows i\quad\Big)\Big)$$

$$+\frac{1}{\epsilon(\mathbf{n}_2)+\psi(\mathbf{n}_2)}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}$$

$$\times\Big(\Big(\psi_1^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_1}\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_1}\delta(\omega_1-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}^{\alpha''j_1''}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1''\omega^{\alpha''}_{1,\mathbf{k}_1+\mathbf{k}_2})$$

$$+\psi_1^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}^{\alpha'j_1'}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1'\omega^{\alpha'}_{1,\mathbf{k}_1+\mathbf{k}_2})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_1}\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_1}\delta(\omega_1-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Big)$$

$$+\Big(\psi_2^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_1}\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_1}\delta(\omega_1-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\mathcal{I}^{\alpha'j_1'}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1'\omega^{\alpha'}_{1,\mathbf{k}_1+\mathbf{k}_2})$$

$$+\psi_2^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}^{\alpha''j_1''}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1''\omega^{\alpha''}_{1,\mathbf{k}_1+\mathbf{k}_2})\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_1}\delta(\omega_1-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})\Big)$$

$$+\Big(\psi_3^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{-\mathbf{n}_1}\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_1}\delta(\omega_1-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})\mathcal{I}^{\alpha''j_1''}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1''\omega^{\alpha''}_{1,\mathbf{k}_1+\mathbf{k}_2})$$

$$+\psi_3^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1+\mathbf{n}_2}\mathcal{I}^{\alpha'j_1'}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1'\omega^{\alpha'}_{1,\mathbf{k}_1+\mathbf{k}_2})$$

$$\times\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_1}\delta(\omega_1-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Big)$$

$$+\Big(\psi_4^{(2)}(-\mathbf{n}_1;\mathbf{n}_1+\mathbf{n}_2)\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_1}\delta(\omega_1-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\mathcal{I}^{\alpha'j_1'}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1'\omega^{\alpha'}_{1,\mathbf{k}_1+\mathbf{k}_2})$$

$$+\psi_4^{(2)}(\mathbf{n}_1+\mathbf{n}_2;-\mathbf{n}_1)\mathcal{I}^{\alpha''j_1''}_{\mathbf{k}_1+\mathbf{k}_2}\delta(\omega_1+\omega_2+j_1''\omega^{\alpha''}_{1,\mathbf{k}_1+\mathbf{k}_2})\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_1}\delta(\omega_1-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})\Big)\Big)$$

$$+\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}$$

$$\times\Big(\Big(\chi_1^{(2)}(-\mathbf{n}_1;-\mathbf{n}_2)\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_1}\delta(\omega_1-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})$$

$$+\chi_1^{(2)}(-\mathbf{n}_2;-\mathbf{n}_1)\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_2}\delta(\omega_2-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_1}\delta(\omega_1-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Big)$$

$$+\Big(\chi_{2e}^{(2)}(-\mathbf{n}_1;-\mathbf{n}_2)\mathcal{I}^{\alpha'j_1'}_{-\mathbf{k}_1}\delta(\omega_1-j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{-\mathbf{n}_2}\mathcal{I}^{\alpha''j_1''}_{-\mathbf{k}_2}\delta(\omega_2-j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})$$

$$+\chi^{(2)}_{2e}(-\mathbf{n}_2;-\mathbf{n}_1)\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_1}\delta(\omega_1 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Bigg)$$

$$+\Bigg(\chi^{(2)}_{3e}(-\mathbf{n}_1;-\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_1}\delta(\omega_1 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})$$

$$\times\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})$$

$$+\chi^{(2)}_{3e}(-\mathbf{n}_2;-\mathbf{n}_1)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})$$

$$\times\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_1}\delta(\omega_1 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Bigg)$$

$$+\Bigg(\chi^{(2)}_{4e}(-\mathbf{n}_1;-\mathbf{n}_2)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_1}\delta(\omega_1 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})$$

$$+\chi^{(2)}_{4e}(-\mathbf{n}_2;-\mathbf{n}_1)\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_1}\delta(\omega_1 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Bigg)$$

$$+\left(\quad e\leftrightarrows i\quad\right)\Bigg)$$

$$+\frac{1}{\epsilon(-(\mathbf{n}_1+\mathbf{n}_2))+\psi(-(\mathbf{n}_1+\mathbf{n}_2))}\sum_{\alpha'}\sum_{j_1'=\pm}\sum_{\alpha''}\sum_{j_1''=\pm}$$

$$\times\Bigg(\Bigg(\psi^{(2)}_1(-\mathbf{n}_1;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_1}\delta(\omega_1 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})$$

$$+\psi^{(2)}_1(-\mathbf{n}_2;-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_1}\delta(\omega_1 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Bigg)$$

$$+\Bigg(\psi^{(2)}_2(-\mathbf{n}_1;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_1}\delta(\omega_1 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})$$

$$+\psi^{(2)}_2(-\mathbf{n}_2;-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_1}\delta(\omega_1 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})\Bigg)$$

$$+\Bigg(\psi^{(2)}_3(-\mathbf{n}_1;-\mathbf{n}_2)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_1}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_1}\delta(\omega_1 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_1})\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_2}\delta(\omega_2 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_2})$$

$$+\psi^{(2)}_3(-\mathbf{n}_2;-\mathbf{n}_1)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{-\mathbf{n}_2}\mathcal{I}^{\alpha' j_1'}_{-\mathbf{k}_2}\delta(\omega_2 - j_1'\omega^{\alpha'}_{1,-\mathbf{k}_2})\mathcal{I}^{\alpha'' j_1''}_{-\mathbf{k}_1}\delta(\omega_1 - j_1''\omega^{\alpha''}_{1,-\mathbf{k}_1})\Bigg)$$

$$+\Big(\psi_4^{(2)}(-\mathbf{n}_1;-\mathbf{n}_2)\mathcal{I}_{-\mathbf{k}_1}^{\alpha''j_1''}\delta(\omega_1-j_1''\omega_{1,-\mathbf{k}_1}^{\alpha''})\mathcal{I}_{-\mathbf{k}_2}^{\alpha'j_1'}\delta(\omega_2-j_1'\omega_{1,-\mathbf{k}_2}^{\alpha'})$$

$$+\psi_4^{(2)}(-\mathbf{n}_2;-\mathbf{n}_1)\mathcal{I}_{-\mathbf{k}_2}^{\alpha''j_1''}\delta(\omega_2-j_1''\omega_{1,-\mathbf{k}_2}^{\alpha''})\mathcal{I}_{-\mathbf{k}_1}^{\alpha'j_1'}\delta(\omega_1-j_1'\omega_{1,-\mathbf{k}_1}^{\alpha'})\Big)\Big)\ .$$

De (4.17) segue que

$$NL1=\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\chi_1^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\Big(\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\sum_{\sigma}\Big(\chi_{2\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\chi_{3\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\chi_{4\sigma}^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\sum_{\sigma}\Big(\chi_{2\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_3}\rangle)$$

$$+\chi_{3\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}^{\sigma}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}^{\sigma}\rangle)$$

$$+\chi_{4\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_1}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_2}^{\sigma}\rangle+\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}\rangle\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_2}^{\sigma}\rangle)$$

$$+\chi_{5\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle)$$

$$+\chi_{6\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_1}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle)$$

$$+\chi_{7\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_1}^{\sigma}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_2}\rangle+\langle E_{\mathbf{n}_3}^{\sigma}E_{\mathbf{n}_2}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}^{\sigma}\rangle)$$

$$+\chi_{8\sigma}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}^{\sigma}E_{\mathbf{n}_3}^{\sigma}\rangle\langle E_{\mathbf{n}_2}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}^{\sigma}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_2}^{\sigma}E_{\mathbf{n}_3}^{\sigma}\rangle)\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\Big(\psi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2}\rangle+\psi_2^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle$$

$$+\psi_3^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle+\psi_4^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\rangle\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\Big(\psi_1^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle\hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_1}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_2}\rangle+\langle\hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_2}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_1}\rangle)$$

$$+\psi_2^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle\hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_1}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\hat{E}_{\mathbf{n}_2}\rangle+\langle\hat{E}_{\mathbf{n}_3}\hat{E}_{\mathbf{n}_2}\rangle\langle E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}E_{\mathbf{n}_1}\rangle)$$

$$+\psi_3^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_3}\rangle\langle\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle\hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle)$$

$$+\psi_4^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\rangle\langle\hat{E}_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle\hat{E}_{\mathbf{n}_2}E_{\mathbf{n}_3}\rangle)$$

$$+\psi_5^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle\hat{E}_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle\hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle\hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_2}\rangle)$$

$$+\psi_6^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}\hat{E}_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle\hat{E}_{\mathbf{n}_3}E_{\mathbf{n}_2}\rangle)$$

$$+\psi_7^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle\hat{E}_{\mathbf{n}_1}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle\hat{E}_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_3}E_{\mathbf{n}_2}\rangle)$$

$$+\psi_8^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)(\langle E_{\mathbf{n}_1}E_{\mathbf{n}_3}\rangle\langle E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle+\langle E_{\mathbf{n}_1}E_{-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)}\rangle\langle E_{\mathbf{n}_3}E_{\mathbf{n}_2}\rangle)\Big)\ . \qquad \text{(C.17)}$$

Com os mesmos argumentos que utilizamos na derivação de (C.16), desprezaremos correlações triplas de campos diferentes, de modo que, por substituição direta das equações (C.9)-(C.14), obtemos

$$
\begin{aligned}
NL1 = & \sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}} \chi_1^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle \qquad \text{(C.18)}\\
& + \sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}} \chi_1^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\\
& \times\Bigg(\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& +\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)\Bigg)\\
& + \sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\Bigg(\chi_{2e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\Big(\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)\\
& \times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& +\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& \times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)\Big)\\
& +\chi_{3e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\Big(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_3}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3+j_1\omega_{1\mathbf{k}_3}^{\alpha})\delta\big(\mathbf{n}_3+\mathbf{n}_1\big)\\
& \times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& +\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& \times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_3}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_3}^{\alpha' j_1'}\delta(\omega_3+j_1'\omega_{1\mathbf{k}_3}^{\alpha'})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\Big)\\
& +\chi_{4e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\Big(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_3}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3+j_1\omega_{1\mathbf{k}_3}^{\alpha})\delta\big(\mathbf{n}_3+\mathbf{n}_1\big)\\
& \times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& +\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\sum_{\gamma}\sum_{j_3=\pm}\mathcal{I}_{\mathbf{k}_3,ee}^{\gamma j_3}\delta(\omega_3+j_3\omega_{3\mathbf{k}_3}^{\gamma})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\Big)\\
& +\chi_{5e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\Big(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_1}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)\\
& \times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\\
& +\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Bigg|_{\mathbf{n}_1}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)
\end{aligned}
$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)\big)$$

$$+\chi_{6e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_1}$$

$$\times\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha' j_1'}\delta(\omega_2+j_1'\omega_{1\mathbf{k}_2}^{\alpha'})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_2}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha j_1}\delta(\omega_2+j_1\omega_{1\mathbf{k}_2}^{\alpha})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_1}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha' j_1'}\delta(\omega_1+j_1'\omega_{1\mathbf{k}_1}^{\alpha'})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\big)$$

$$+\chi_{7e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\sum_{\gamma}\sum_{j_3=\pm}\mathcal{I}_{\mathbf{k}_3,ee}^{\gamma j_3}\delta(\omega_3+j_3\omega_{3\mathbf{k}_3}^{\gamma})\delta\big(\mathbf{n}_3+\mathbf{n}_1\big)\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha j_1}\delta(\omega_2+j_1\omega_{1\mathbf{k}_2}^{\alpha})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_3}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_3}^{\alpha j_1}\delta(\omega_3+j_1\omega_{1\mathbf{k}_3}^{\alpha})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_1}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha' j_1'}\delta(\omega_1+j_1'\omega_{1\mathbf{k}_1}^{\alpha'})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\big)$$

$$+\chi_{8e}^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\sum_{\gamma}\sum_{j_3=\pm}\mathcal{I}_{\mathbf{k}_1,ee}^{\gamma j_3}\delta(\omega_1+j_3\omega_{3\mathbf{k}_1}^{\gamma})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)$$

$$\times\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_2}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha j_1}\delta(\omega_2+j_1\omega_{1\mathbf{k}_2}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\frac{(\zeta_{1e}\rho_1+\zeta_{2e})(1-\zeta_{1i}\rho_{2i})+(\zeta_{1i}\rho_1+\zeta_{2i})\zeta_{1e}\rho_{2i}}{(1-\zeta_{1e}\rho_{2e})(1-\zeta_{1i}\rho_{2i})-\zeta_{1i}\rho_{2e}\zeta_{1e}\rho_{2i}}\Big|_{\mathbf{n}_1}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$\times\sum_{\gamma}\sum_{j_3=\pm}\mathcal{I}_{\mathbf{k}_2,ee}^{\gamma j_3}\delta(\omega_2+j_3\omega_{3\mathbf{k}_2}^{\gamma})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)\big)$$

$$+\big(\quad e\leftrightarrows i\quad\big)\Big)$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}\psi_4^{(2)}(\mathbf{n}_1;\mathbf{n}_2)\big\langle E_{\mathbf{n}_1}E_{\mathbf{n}_2}E_{-(\mathbf{n}_1+\mathbf{n}_2)}\big\rangle$$

$$+\sum_{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=\mathbf{n}}\Big(\psi_1^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\sum_{\beta}\sum_{j_2=\pm}\hat{\mathcal{I}}_{\mathbf{k}_3}^{\beta j_2}\delta(\omega_3+j_2\omega_{2\mathbf{k}_3}^{\beta})\delta\big(\mathbf{n}_3+\mathbf{n}_1\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_2}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_2}^{\alpha j_1}\delta(\omega_2+j_1\omega_{1\mathbf{k}_2}^{\alpha})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\sum_{\beta}\sum_{j_2=\pm}\hat{\mathcal{I}}_{\mathbf{k}_3}^{\beta j_2}\delta(\omega_3+j_2\omega_{2\mathbf{k}_3}^{\beta})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1}\sum_{\alpha}\sum_{j_1=\pm}\mathcal{I}_{\mathbf{k}_1}^{\alpha j_1}\delta(\omega_1+j_1\omega_{1\mathbf{k}_1}^{\alpha})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\big)$$

$$+\psi_2^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_3}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_3}\delta(\omega_3+j_1\omega^\alpha_{1\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_1\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\sum_\beta\sum_{j_2=\pm}\hat{\mathcal{I}}^{\beta j_2}_{\mathbf{k}_3}\delta(\omega_3+j_2\omega^\beta_{2\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\big)$$

$$+\psi_3^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)\big)$$

$$+\psi_4^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_2}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2+\mathbf{n}_3\big)\big)$$

$$+\psi_5^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\sum_\beta\sum_{j_2=\pm}\hat{\mathcal{I}}^{\beta j_2}_{\mathbf{k}_1}\delta(\omega_1+j_2\omega^\beta_{2\mathbf{k}_1})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_2}\delta(\omega_2+j_1\omega^\alpha_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$\times\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_3}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_3}\delta(\omega_3+j_1'\omega^{\alpha'}_{1\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\big)$$

$$+\psi_6^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_3}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_3}\delta(\omega_3+j_1\omega^\alpha_{1\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_1\big)$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_3}\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_3}\delta(\omega_3+j_1'\omega^{\alpha'}_{1\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\big)$$

$$+\psi_7^{(3)}(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\frac{\rho_1+\sum_\sigma\rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma\rho_{2\sigma}\zeta_{1\sigma}}\Big|_{\mathbf{n}_1}\sum_\alpha\sum_{j_1=\pm}\mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^\alpha_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)$$

$$\times\sum_{\alpha'}\sum_{j_1'=\pm}\mathcal{I}^{\alpha' j_1'}_{\mathbf{k}_2}\delta(\omega_2+j_1'\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\frac{\rho_1+\sum_\sigma \rho_{2\sigma}\zeta_{2\sigma}}{1-\sum_\sigma \rho_{2\sigma}\zeta_{1\sigma}}\Bigg|_{\mathbf{n}_1} \sum_{\alpha}\sum_{j_1=\pm} \mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^{\alpha}_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$\times\sum_{\alpha'}\sum_{j'_1=\pm} \mathcal{I}^{\alpha' j'_1}_{\mathbf{k}_3}\delta(\omega_3+j'_1\omega^{\alpha'}_{1\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\big)$$

$$+\psi^{(3)}_8(\mathbf{n}_1;\mathbf{n}_2;\mathbf{n}_3)\big(\sum_{\alpha}\sum_{j_1=\pm} \mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^{\alpha}_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1+\mathbf{n}_3\big)\sum_{\alpha'}\sum_{j'_1=\pm} \mathcal{I}^{\alpha' j'_1}_{\mathbf{k}_2}\delta(\omega_2+j'_1\omega^{\alpha'}_{1\mathbf{k}_2})\delta\big(\mathbf{n}_2-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)$$

$$+\sum_{\alpha}\sum_{j_1=\pm} \mathcal{I}^{\alpha j_1}_{\mathbf{k}_1}\delta(\omega_1+j_1\omega^{\alpha}_{1\mathbf{k}_1})\delta\big(\mathbf{n}_1-(\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3)\big)\sum_{\alpha'}\sum_{j'_1=\pm} \mathcal{I}^{\alpha' j'_1}_{\mathbf{k}_3}\delta(\omega_3+j'_1\omega^{\alpha'}_{1\mathbf{k}_3})\delta\big(\mathbf{n}_3+\mathbf{n}_2\big)\big)\bigg)\ .$$

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