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Generation of harmonic Langmuir mode by beam-plasma instability

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In this article, numerical solutions of the generalized weak turbulence equation [P. H. Yoon, Phys. Plasmas **7**, 4858 (2000)] are carried out. In the generalized weak turbulence theory, the generation of the $2\omega_{pe}$-harmonic Langmuir mode is treated as a fundamental process in turbulent beam-plasma interaction process, in addition to, and concomitant to, the well-known nonlinear processes such as Langmuir and ion-sound mode coupling and wave-particle interactions. The present numerical analysis shows that the harmonic mode, which is a solution to a nonlinear dispersion equation, hence a ''nonlinear'' eigenmode, grows primarily due to an induced emission process, which is a ''linear'' wave-particle interaction process. The harmonic Langmuir mode generation has been observed since the late 1960s in laboratory experiments, simulations, and in space. However, adequate and quantitative theoretical explanation has not been forthcoming. The present work represents a step toward an understanding of such a phenomenon.
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I. INTRODUCTION

The beam-plasma (or bump-on-tail) instability has played a crucial role in the development of nonlinear plasma turbulence theories, the simplest of which is the well-known quasilinear theory.[1,2] Although turbulence theories based upon random-phase approximation cannot describe coherent nonlinear processes such as particle trapping,[3–5] which was observed in early numerical simulations[6–8] and experiments[9–12] and thought to be one of the primary nonlinear saturation mechanisms for the beam-plasma instability, it is now known largely in part due to more carefully designed simulations[8,13–23] and experiments,[24–28] that for a weak or sufficiently warm beam, the incoherent turbulence theories provide a rather good description of the beam-plasma instability development.[17,29,30] Among these theories, a straightforward perturbative theory called the weak turbulence theory,[31–44] which uses the wave-field amplitude as an expansion parameter, and which generalizes the quasilinear theory, is most widely used.

An interesting phenomenon, first observed in 1967 by Apel in his beam-plasma interaction laboratory experiment,[45] and yet not adequately investigated in detail to this date, is the generation of harmonic Langmuir modes. Subsequent laboratory experiments[10–12,46,47] and particle-in-cell and Vlasov simulations[13,14,16,18,21,23] confirmed this finding, and space observations show that such a phenomenon may occur in a natural environment.[48] Most of the early theoretical attempts to explain such a phenomenon were based upon particle trapping dynamics and/or the so-called ''ballistic'' quasi-beam mode generation.[4,5,13,49]

In the theory by Manheimer,[49] a beam-mode dispersion relation,

$$\omega \approx k v_b, \tag{1}$$

is assumed, where ω, k, and v_b represent the wave angular frequency, wave number, and the average beam speed, respectively. According to linear theory, a cold beam cannot excite electrostatic mode with frequency above the plasma frequency, $\omega > \omega_{pe}$, where $\omega_{pe} = (4\pi \hat{n} e^2/m_e)^{1/2}$ is the electron plasma frequency, e, $\hat{n}$, and m_e being the unit electric charge, the ambient plasma density, and electron mass, respectively. However, Manheimer assumes that such a solution is valid for all frequencies, and finds that an approximate renormalized plasma dispersion relation[50] computed on the basis of trapped particle equilibrium supports enhanced fluctuations at the multiple harmonics of the plasma frequency along the beam-mode dispersion line Eq. (1),

$$\omega_n \approx n\omega_{pe}, \quad k_n \approx n\omega_{pe}/v_b, \tag{2}$$

where $n = 1,2,3,\cdots$. However, as Dupree indicates later,[51] such modes are not genuine eigenmodes of a system, but are quasi-ballistic modes which can be related to the clump formation.

According to the theory proposed by O'Neil and his colleagues,[4,5] harmonic generation is attributed to electrons trapped in a monochromatic large-amplitude wave. These authors consider the perturbed electric potential field computed on the basis of complicated electron orbits in the field of a large-amplitude wave. Then they show that the wave electric field potential possesses the harmonic structure in frequency space with the plasma frequency as the fundamental frequency unit. This theory assumes a cold beam interacting in a large-amplitude coherent wave, and as such, the theory is not fully self-consistent. Also, the harmonics predicted by such a theory are, again, not eigenmodes of the system.

On the other hand, Joyce *et al.*'s explanation[13] and a similar theory by Klimas[14] are based upon a concept similar to the recently proposed theory by Yoon[52] in that they involve an eigenmode of a nonlinear dispersion relation. In the theories by Joyce *et al.*[13] and Klimas,[14] the harmonic modes are treated as legitimate eigenmodes of a plasma in which large-amplitude Langmuir waves are excited. However, these theories are strictly coherent versions, which do not take the finite wave spectrum into account. (Later interpretation by Klimas,[16] however, relies on the electron-beam phase-space vortex formation, or particle trapping dynamics, which is similar in spirit with the earlier theoretical ideas.[4,5,49])

Detailed Vlasov simulation analyses of electrostatic Langmuir harmonic modes were first performed by Klimas,[14,16] although Joyce *et al.*'s earlier simulation study[13] also discusses the excitation of such modes. Klimas' works show that harmonic Langmuir modes possess the characteristics specified by Eq. (2), namely, $\omega_n \sim n\omega_{pe}$ and $k_n \sim nk_L$, where (ω_n, k_n) represents the angular frequency-wave number pair of the the nth harmonic. Moreover, it was shown that the harmonic modes possess the beam-mode characteristics, $\omega_n \sim k_n v_b$. On such a basis, it could be argued that the results of Klimas' Vlasov simulations are in overall agreement with ballistic beam-mode or single-wave theories.[4,5,49] The simulation by Nishikawa and Cairns[18] shows that the harmonic Langmuir modes can also be excited in the particle-in-cell simulation, which confirms the Vlasov simulation results. Their work shows that the excitations of these modes are not likely to be numerical artifacts peculiar to Vlasov simulation somehow, but rather, are genuine dynamical results of a nonlinear plasma system.

Klimas' observation that the harmonic modes begin to grow even in the linear stage where the full phase-space vortex characteristics of particle trapping is presumed to be not fully developed, is extremely interesting. He also noted that the higher the harmonic mode number, the faster the initial (linear) growth rate, although the harmonic modes saturate at low amplitudes. As we will discuss, these findings are highly relevant to the present theory.

It should be emphasized that the simulation studies by Joyce *et al.*,[13] Klimas,[14,16] and by Nishikawa and Cairns[18] are limited in the sense that full dispersive characteristics of the harmonic modes are not revealed. The characteristics of a mode is best described in terms of an instantaneous frequency-wave number dispersion relation. In a simulation study, this can be obtained by plotting the intensity of a mode in ω-k space, averaged over a period of time interval. The above cited works did not present such an analysis, and failed to observe that the harmonic modes are not excited along nondispersive beam-mode line, $\omega \approx kv_b$ [Eq. (1)], but rather, they appear as genuine eigenmodes, much like the higher-order cyclotron harmonic modes in a warm magnetized plasma, $\omega \sim n\Omega_j$ (e.g., Bernstein modes), where $\Omega_j = e_jB/m_jc$ is the cyclotron frequency of particle species j, B being the ambient magnetic field, are thermal eigenmodes of the fundamental cyclotron frequency, $\omega \sim \Omega_j$.

Such a characteristics associated with the harmonic modes were revealed in more recent simulations by Kasaba and his colleagues[22,23] and by Schriver *et al.*[21] These authors show that when plotted in ω-k diagram, the Langmuir mode (denoted by L) and the harmonic Langmuir mode (denoted by N in accordance with Ref. 52, and which corresponds to harmonic mode number $n=2$) appear to satisfy approximate dispersion relations given, respectively, by

$$\omega_{\mathbf{k}}^{L1} \approx \omega_{pe}(1+3k^2\lambda_{De}^2/2) \equiv \omega_{\mathbf{k}}^{L}, \qquad \omega_{\mathbf{k}}^{L2} \approx \omega_{pe}(2+3k^2\lambda_{De}^2/2+\eta) \equiv \omega_{\mathbf{k}}^{N}, \tag{3}$$

with a broad spectrum for each mode centered around $k \approx \omega_{pe}/v_b$ for the case of $L1$ and $k \approx 2\omega_{pe}/v_b$ for $L2$, spectral widths of both modes being roughly comparable. Here $\lambda_{De}^2 = T_e/(4\pi\hat{n}e^2)$ is the square of the Debye length, T_e is the electron temperature, and η represents a small but finite nonlinear frequency shift the magnitude of which is on the order of the fundamental Langmuir wave intensity.

Note that the simulation by Schriver *et al.*[21] deals with a situation with a relatively dense beam. As a consequence, the simulated $L1$ and $L2$ mode dispersion relations do not exactly follow Eq. (3), but rather the simulation exhibits signatures of both the beam-acoustic and Langmuir modes. In particular, the most intense simulated $L1$ mode lies just below ω_{pe}, which is typical of a strong-beam instability, and likewise, $L2$ mode is also observed to be exicted slightly below $2\omega_{pe}$. In contrast, the present representation of $L1$ and $L2$ mode dispersion relations (3), is applicable to a classical bump-on-tail instability situation, i.e., a tenuous beam and sufficiently broad beam velocity spread, as will be specifically considered under the subsequent choice of physical parameters. In this respect, results obtained by Kasaba and his colleagues[22,23] are more closely related to the present theory.

References 21–23 only analyzed the fundamental and first harmonic, but presumably higher harmonics are expected to possess similar characteristics, namely,

$$\omega_{\mathbf{k}}^{Ln} \approx \omega_{pe}(n+3k^2\lambda_{De}^2/2+\eta_n), \tag{4}$$

with a spectrum of k values centered around $k \approx n\omega_{pe}/v_b$. In the above, we have denoted the nonlinear correction factor η_n to indicate the possibility that η_n may be different for each higher harmonics. In light of these developments, it is imperative that we re-examine the theory of harmonic Langmuir mode generation. In particular, it is desirable to explain the harmonic Langmuir modes in terms of turbulence theories which imply broad wave spectrum by definition. It is also desirable to formulate a theory in which the harmonic modes can be described as eigenmodes of a turbulent plasma. Note that random-phase averaged turbulence theories exclude the trapping dynamics by default. Hence, the present approach is an alternative explanation to those theories which rely on phase-space vortex dynamics.[4,5,13,16,49]

With these aims in mind, Yoon[52] obtained an eigenmode solution for the (first) harmonic Langmuir mode, by formulating and solving a nonlinear dispersion equation which includes broad spectrum of waves with random phases, i.e., the turbulent generalization of the coherent nonlinear dispersion equations discussed by Joyce *et al.*[13] and by Klimas.[14] Yoon also formulated a generalized weak turbulence theory which incorporates the harmonic Langmuir wave as part of the system of eigenmodes in a turbulent plasma.

The purpose of the present paper is to present the numerical solution of the generalized weak turbulence kinetic equations, and to examine the characteristics and time-development of the self-consistent system of electrons, Langmuir and ion-sound modes as well as the harmonic mode. The structure of the paper is the following: In Sec. II we formulate the theoretical equations to be numerically analyzed in detail. In Sec. III we conduct the numerical computation of the equations. Finally, some comments on the results obtained and on the perspectives for future work appear in Sec. IV.

II. THEORETICAL FRAMEWORK

A. Nonlinear dispersion relation for the harmonic Langmuir mode

For the sake of completeness, let us briefly review the nonlinear eigenmode analysis within the framework of the generalized weak turbulence theory.[52] By way of doing so, we also address some ambiguities associated with the approximate form of the nonlinear dispersion relation, as it appears in Ref. 52. The eigenmode of the nonlinear dispersion equation of interest to us corresponds to the first harmonic ($n=2$) Langmuir mode, with wave frequency in the vicinity of $2\omega_{pe}$ (some authors refer to $2\omega_{pe}$-mode as ''second'' harmonic, $3\omega_{pe}$-mode as ''third'' harmonic, etc., but our convention here is that ω_{pe} Langmuir mode is the fundamental, $2\omega_{pe}$-mode being its first harmonic, $3\omega_{pe}$-mode being the ''second'' harmonic, and so on). In order for this mode to exist, a finite but not necessarily substantially high level of Langmuir wave turbulence is required. If we retain the nonlinear wave coupling term which arises from the presence of a broaden spectrum of incoherent Langmuir waves, then the nonlinear dispersion equation can be shown to be given by a generic form,[52]

$$\mathrm{Re}\left(\epsilon(\mathbf{k},\omega)-4\sum_{\sigma'=\pm1}\times\int d\mathbf{k}'\frac{|\chi^{(2)}(\mathbf{k}',\sigma'\omega^L_{\mathbf{k}'}|\mathbf{k}-\mathbf{k}',\omega-\sigma'\omega^L_{\mathbf{k}'})|^2}{\epsilon(\mathbf{k}-\mathbf{k}',\omega-\sigma'\omega^L_{\mathbf{k}'})}I_L^{\sigma'}(\mathbf{k}')\right)=0, \tag{5}$$

where ω is the dispersion relation for the nonlinear eigenmode (harmonic Langmuir mode), which is expected to possess frequency near $2\omega_{pe}$, $I_L^{\sigma'}(\mathbf{k}')$ is the spectral wave intensity associated with the primary (fundamental) Langmuir wave. In this notation, $\sigma=1$ and $\sigma=-1$ represents, respectively, forward and backward propagating components of the primary Langmuir waves. The quantity $\epsilon(\mathbf{k},\omega)$ and $\chi^{(2)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)$ are the linear dielectric response function and the second-order nonlinear susceptibility, respectively, defined in Eq. (4) of Ref. 52. From Eq. (5) one can easily see that if we ignore the wave coupling term, the dispersion relation reduces to the usual linear limit,

$$\mathrm{Re}\,\epsilon(\mathbf{k},\omega)=0.$$

In the present formalism we restrict ourselves to the so-called kinetic instability limit, in which the angular frequency is considered to be real with an implicit infinitesimally small but positive imaginary part, $\omega=\omega+i0$, in Eq. (5). Essentially, the harmonic Langmuir mode solution, $\omega=\omega^N_{\mathbf{k}}\sim2\omega_{pe}$ and $\mathbf{k}=\mathbf{k}_N\sim2\mathbf{k}_L$ is possible because $(\omega,\mathbf{k})=(2\omega_{pe},2\mathbf{k}_L)$ is a quasi-root of the denominator on the right-hand side of Eq. (5), $\epsilon(\mathbf{k}-\mathbf{k}',\omega-\omega^L_{\mathbf{k}'})\sim\epsilon(\mathbf{k}_N-\mathbf{k}_L,\omega^N_{\mathbf{k}}-\omega_{pe})\sim\epsilon(\mathbf{k}_L,\omega_{pe})\approx0$. However, such a solution cannot be an exact root, otherwise it will lead to a singularity, but rather, the denominator must be of the same order of magnitude as the numerator, which balances the linear response term $\epsilon(\mathbf{k},\omega)$. The simplified expression for the nonlinear dispersion equation is derived in Ref. 52, and is given by

$$\begin{aligned}0&=1-\frac{\omega_{pe}^2}{\omega^2}\left(1+3k^2\lambda_{De}^2\frac{\omega_{pe}^2}{\omega^2}\right)\\&\quad-\frac{1}{\omega_{pe}^4}\frac{e^2}{(4m_e)^2}\int d\mathbf{k}'\frac{a^2_{\mathbf{k},\mathbf{k}'}(\omega-\omega^L_{\mathbf{k}'})^2I_L(\mathbf{k}')}{(\omega-\omega^L_{\mathbf{k}'})^2-\omega^{L2}_{\mathbf{k}-\mathbf{k}'}},\\a_{\mathbf{k},\mathbf{k}'}&=\frac{(\mathbf{k}\times\mathbf{k}')^2+3(\mathbf{k}\cdot\mathbf{k}')[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]}{k\,k'\,|\mathbf{k}-\mathbf{k}'|},\end{aligned} \tag{6}$$

where $\omega^L_{\mathbf{k}}=\omega_{pe}(1+3k^2\lambda^2_{De}/2)$ is the familiar fundamental Langmuir mode dispersion relation. If we assume that the solution ω of interest lies near ω_{pe} then one can see that the denominator $(\omega-\omega^L_{\mathbf{k}'})^2-\omega^{L2}_{\mathbf{k}-\mathbf{k}'}\sim-\omega^2_{pe}\neq0$. As such, the wave coupling term can be ignored. This situation corresponds to the fundamental Langmuir mode solution. On the other hand, if we are interested in the regime, $\omega\sim2\omega_{pe}$, then we can see that the denominator can be very small, as explained above. Thus, the balance of the linear response (the first two terms on the right-hand side) and the nonlinear wave coupling term (the last term on the right-hand side) leads to the desired solution with characteristic frequency near $2\omega_{pe}$, i.e., the eigenmode of a nonlinear dispersion relation.

In Ref. 52, an approximate analytical solution to Eq. (5) is given, which follows from a number of simplifying assumptions. The specific form of the approximate solution suggested in Ref. 52 is given by

$$\begin{aligned}\omega&=\omega^N_{\mathbf{k}}=\omega_{pe}(2+3k^2\lambda^2_{De}/4+\eta_{\mathbf{k}}),\\\eta_{\mathbf{k}}&=\frac{2}{3\omega_{pe}^4}\frac{e^2}{(4m_e)^2}\int d\mathbf{k}'\,a^2_{\mathbf{k},\mathbf{k}'}\,I_L(\mathbf{k}').\end{aligned} \tag{7}$$

However, the thermal correction factor to the dispersion relation, namely, $3\omega_{pe}k^2\lambda^2_{De}/4$ does not seem to be in agreement with the simulated nonlinear mode dispersion relation,[21–23] which closely resembles $3\omega_{pe}k^2\lambda^2_{De}/2$. The original solution Eq. (7) is based upon a number of approximations, and therefore, it is possible that the detailed numerical factor of 3/4 may not necessarily represent the best choice. For this reason, it is better to devise an alternative approximation scheme. In the present approach, let us first approximate the resonant denominator by

$$(\omega-\omega_{\mathbf{k}'}^{L})^2-\omega_{\mathbf{k}-\mathbf{k}'}^{L2}=(\omega-\omega_{\mathbf{k}'}^{L}+\omega_{\mathbf{k}-\mathbf{k}'}^{L})(\omega-\omega_{\mathbf{k}'}^{L}-\omega_{\mathbf{k}-\mathbf{k}'}^{L})$$
$$\approx 2\omega_{pe}[\omega-2\omega_{pe}-3\omega_{pe}k^2\lambda_{De}^2 -3\omega_{pe}(k'^2-\mathbf{k}\cdot\mathbf{k}')].$$

We then approximate Eq. (6) by assuming that $\omega\approx 2\omega_{pe}$ everywhere except in the denominator of the $\mathbf{k}'$ integral. Neglecting the small thermal correction associated with the linear response and ignoring terms which contain the integration variable $\mathbf{k}'$ in the denominator, we obtain

$$0\approx 1-\frac{2}{3\omega_{pe}^3}\frac{e^2}{(4m_e)^2}\frac{\int d\mathbf{k}'\, a_{\mathbf{k},\mathbf{k}'}^2\, I_L(\mathbf{k}')}{\omega-2\omega_{pe}-3\omega_{pe}k^2\lambda_{De}^2/2}.$$

From this, we now obtain an alternative approximate expression for the harmonic Langmuir mode,

$$\omega=\omega_{\mathbf{k}}^{N}=\omega_{pe}(2+3k^2\lambda_{De}^2/2+\eta_{\mathbf{k}}). \tag{8}$$

In the subsequent analysis, we shall resort to this expression for the harmonic Langmuir mode, which is an eigenvalue of the nonlinear dispersion equation, hence the superscript "N." The above dispersion relation is to be considered together with the linear eigenmode dispersion relations, the well-known (fundamental) Langmuir mode dispersion relation,

$$\omega_{\mathbf{k}}^{L}=\omega_{pe}(1+3k^2\lambda_{De}^2/2),$$

and the ion-sound (or ion-acoustic) mode dispersion relation,

$$\omega_{\mathbf{k}}^{S}=\omega_{pe}(m_e/m_i)^{1/2}(1+3T_i/T_e)^{1/2}k\lambda_{De},$$

and these three modes form the basic excitations of turbulent unmagnetized plasma in the electrostatic approximation.

Note that we are concerned with the first harmonic (i.e., $\omega_{\mathbf{k}}^{N}=\omega_{\mathbf{k}}^{L2}$) mode only. However, the analysis by Yoon[52] can be generalized to all higher harmonics by making use of the first-harmonic ($\omega_{\mathbf{k}}^{L2}$) mode as the source of nonlinear wave-coupling for the second-harmonic ($\omega_{\mathbf{k}}^{L3}$) mode, and so on. The resulting nonlinear dispersion relation is expected to be of the form

$$0=\mathrm{Re}\Bigg(\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{Ln})-4\sum_{\sigma'=\pm1}\int d\mathbf{k}' \times\frac{|\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{L(n-1)}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{Ln}-\sigma'\omega_{\mathbf{k}'}^{L(n-1)})|^2}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{Ln}-\sigma'\omega_{\mathbf{k}'}^{L(n-1)})} \times I_{L(n-1)}^{\sigma'}(\mathbf{k}')\Bigg),$$

where $n=2,3,4,\cdots$, the superscripts $n-1$ and n designate the $(n-1)$th and nth harmonics of the Langmuir mode. For $n=1$ (the fundamental), we simply ignore the nonlinear wave-coupling term. The detailed analysis of the above dispersion relation is beyond the scope of the present article, however.

Finally, note that although we consider only the real part of Eq. (5) to determine the wave dispersion relation, the imaginary part of Eq. (5) is not discarded, but rather, it is incorporated in the wave kinetic equation for the N mode, where, together with other nonlinear responses, the imaginary part of Eq. (5) balances the time-rate of change in the wave intensity, $\partial I_N(\mathbf{k},t)/\partial t$,[52]

$$\frac{\partial I_N(\mathbf{k},t)}{\partial t}=-\frac{2\,\mathrm{Im}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{N})}{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{N})/\partial\sigma\omega_{\mathbf{k}}^{N}}I_N(\mathbf{k},t)-\frac{4}{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{N})/\partial\sigma\omega_{\mathbf{k}}^{N}}$$
$$\times\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,\mathrm{Im}\Bigg(2\{\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{L}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{N}-\sigma'\omega_{\mathbf{k}'}^{L})\}^2$$
$$\times\mathcal{P}\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{N}-\sigma'\omega_{\mathbf{k}'}^{L})}-\overline{\chi}^{(3)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{L}|-\mathbf{k}',-\sigma'\omega_{\mathbf{k}'}^{L}|\mathbf{k},\sigma\omega_{\mathbf{k}}^{N})\Bigg)I_L(\mathbf{k}',t)\,I_N(\mathbf{k},t)$$
$$-\frac{4\pi}{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{N})/\partial\sigma\omega_{\mathbf{k}}^{N}}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'|\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{L}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})|^2$$
$$\times\left(\frac{I_L(\mathbf{k}',t)\,I_N(\mathbf{k},t)}{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})/\partial\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}}+\frac{I_L(\mathbf{k}-\mathbf{k}',t)\,I_N(\mathbf{k},t)}{\partial\,\mathrm{Re}\,\epsilon(\mathbf{k}',\sigma'\omega_{\mathbf{k}}^{L})/\partial\sigma'\omega_{\mathbf{k}}^{L}}\right)\delta(\sigma\omega_{\mathbf{k}}^{N}-\sigma'\omega_{\mathbf{k}}^{L}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L}). \tag{9}$$

In the above, $\overline{\chi}^{(3)}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3)$ is the third-order nonlinear susceptibility [see Eq. (4) of Ref. 52].

B. Particle and wave kinetic equations

In Ref. 52, particle kinetic equations for the ions and electrons which generalize the conventional quasilinear diffusion equation were obtained. In the present approach, however, we take a simpler view in that the ions are considered as quasi-stationary and the electrons are assumed to be governed by the simple quasilinear diffusion equation, except that the wave intensity which enters the diffusion coefficient is the combined Langmuir and the nonlinear eigenmode intensities,

$$\frac{\partial F_e(\mathbf{v},t)}{\partial t}=\frac{\pi e^2}{m_e^2}\sum_{\sigma=\pm 1}\int\frac{d\mathbf{k}}{k^2}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\times\left\{[\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\,I_L^\sigma(\mathbf{k},t)+\delta(\sigma\omega_{\mathbf{k}}^N-\mathbf{k}\cdot\mathbf{v})\times I_N^\sigma(\mathbf{k},t)]\left(\mathbf{k}\cdot\frac{\partial F_e(\mathbf{v},t)}{\partial\mathbf{v}}\right)\right\}. \tag{10}$$

The Langmuir wave kinetic equation which takes into account the nonlinear coupling with the harmonic nonlinear mode is derived in Ref. 52, which is given by

$$\frac{\partial I_L^\sigma(\mathbf{k},t)}{\partial t}=\left(\frac{\partial}{\partial t}\bigg|_{\text{ind. emiss.}}+\frac{\partial}{\partial t}\bigg|_{\text{decay } LS}+\frac{\partial}{\partial t}\bigg|_{\text{decay } LN}+\frac{\partial}{\partial t}\bigg|_{\text{ind. scatt. } LL}+\frac{\partial}{\partial t}\bigg|_{\text{ind. scatt. } LN}\right)I_L^\sigma(\mathbf{k},t), \tag{11}$$

where the first term on the right-hand side represents the induced emission/absorption process, which is the only term considered in the traditional quasilinear theory, the second term represents the three-wave decay/coalescence term considered in the literature on conventional weak turbulence theory, the third term corresponds to the modification to the traditional Langmuir wave kinetic equation which comes from the three-wave decay/coalescence process involving the Langmuir wave and the nonlinear harmonic mode, the fourth term represents the conventional induced-scattering term which involves two Lagmuir waves interacting with the particles via nonlinear wave-particle interaction process, and the final term corresponds to the induced-scattering process involving a Langmuir mode and the nonlinear eigenmode, mediated by the particles (electrons). Specific expressions for the various processes are given by

$$\frac{\partial I_L^\sigma(\mathbf{k},t)}{\partial t}\bigg|_{\text{ind. emiss.}}=\sigma\omega_{\mathbf{k}}^L\int d\mathbf{v}\,\Gamma_L^\sigma(\mathbf{k},\mathbf{v})\,\mathbf{k}\cdot\frac{\partial F_e(\mathbf{v},t)}{\partial\mathbf{v}}\,I_L^\sigma(\mathbf{k},t),$$

$$\Gamma_L^\sigma(\mathbf{k},\mathbf{v})=\pi\frac{\omega_{pe}^2}{k^2}\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v}),$$

$$\frac{\partial I_L^\sigma(\mathbf{k},t)}{\partial t}\bigg|_{\text{decay } LS}=\sum_{\sigma',\sigma''=\pm 1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,V_{L,S}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')\{\sigma\omega_{\mathbf{k}}^L I_L^{\sigma'}(\mathbf{k}',t)\,I_S^{\sigma''}(\mathbf{k}-\mathbf{k}',t)-[\sigma'\omega_{\mathbf{k}'}^L I_S^{\sigma''}(\mathbf{k}-\mathbf{k}',t)+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_L^{\sigma'}(\mathbf{k}',t)]\,I_L^\sigma(\mathbf{k},t)\},$$

$$V_{L,S}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')=\frac{\pi}{2}\frac{e^2}{T_e^2}\frac{\mu_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S),$$

$$\mu_{\mathbf{k}}=|k|^3\lambda_{De}^3(m_e/m_i)^{1/2}(1+3T_i/T_e)^{1/2},$$

$$\frac{\partial I_L^\sigma(\mathbf{k},t)}{\partial t}\bigg|_{\text{decay } LN}=\sum_{\sigma',\sigma''=\pm 1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,V_{L,N}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')[\sigma\omega_{\mathbf{k}}^L I_L^{\sigma''}(\mathbf{k}-\mathbf{k}',t)-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_L^\sigma(\mathbf{k},t)]\,I_N^{\sigma'}(\mathbf{k}',t),$$

$$V_{L,N}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')=\frac{\pi}{32\omega_{pe}^4}\frac{e^2}{m_e^2}a_{\mathbf{k}',\mathbf{k}}^2\,\delta(\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^N),$$

$$\frac{\partial I_L^\sigma(\mathbf{k},t)}{\partial t}\bigg|_{\text{ind. scatt. } LL}=-\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v}\,U_{L,L}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial}{\partial\mathbf{v}}\left[(\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L)\,F_e(\mathbf{v},t)-\frac{m_e}{m_i}\sigma\omega_{\mathbf{k}}^L F_i(\mathbf{v})\right]I_L^{\sigma'}(\mathbf{k}',t)\,I_L^\sigma(\mathbf{k},t),$$

$$U_{L,L}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})=\frac{\pi}{\omega_{pe}^2}\frac{e^2}{m_e^2}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}],$$

$$\frac{\partial I_L^\sigma(\mathbf{k},t)}{\partial t}\bigg|_{\text{ind. scatt. } LN}=\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v}\,U_{L,N}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial F_e(\mathbf{v},t)}{\partial\mathbf{v}}\,I_N^{\sigma'}(\mathbf{k}',t)\,I_L^\sigma(\mathbf{k},t),$$

$$U_{L,N}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})=\frac{\pi}{16\omega_{pe}^2}\frac{e^2}{m_e^2}\frac{a_{\mathbf{k}',\mathbf{k}}^2\,\delta_{\sigma,\sigma'}}{|\mathbf{k}-\mathbf{k}'|^2\,|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^N)|^2}\,\delta[\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^N-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}],$$

$$|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^N)|^2=|\,3\,[2\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')+k'^2]\,\lambda_{De}^2-2\,\eta_{\mathbf{k}'}\,|^2. \tag{12}$$

In the above, the coefficient $a_{\mathbf{k}',\mathbf{k}}$ is defined exactly as in Eq. (6), except that $\mathbf{k}$ and $\mathbf{k}'$ are interchanged. Similarly, the quantity $\eta_{\mathbf{k}'}$ is the same as that defined in Eq. (7), except for the argument, $\mathbf{k}'$. In the definition for $U_{L,N}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})$, we have explicitly used the fact that in its derivation, the assumption that σ and σ' be of the same sign has been used. Such an assumption was implicitly used in Ref. 52, but not explicitly stated there.

In Eq. (11), an induced scattering term involving an ion-sound wave, Langmuir wave and the ions, which appears in Ref. 52 is ignored. Reference 52 shows that such a term is dictated by the resonance condition, $\delta(\sigma'\omega_{\mathbf{k}'}^S-\mathbf{k}\cdot\mathbf{v})$, with an appropriate scattering coefficient $U_{L,S}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})$. This type of process, which appears as a nonlinear correction term in the wave kinetic equation, yet is dictated by linear wave-particle resonant interaction condition, is not considered in the standard literature, and although such a term naturally appears in the generalized weak turbulence theory of Ref. 52, its consequence or interpretation is not clear at this point. A preliminary numerical analysis of weak beam-plasma interaction which includes such a term on the right-hand side of Langmuir wave kinetic equation shows that it only has a marginal effect. Therefore, we have decided to ignore such an interaction term at the outset in the present numerical analysis.

The wave kinetic equation for the harmonic nonlinear mode, the generic form of which is given by Eq. (9), can be made explicit as

$$\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}=\left.\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{ind. emiss.}}+\left.\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{decay }NL}+\left.\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{ind. scatt. }NL}, \tag{13}$$

where the first term on the right-hand side represents the induced emission of harmonic nonlinear eigenmode, the second term depicts the decay/coalescence of nonlinear mode into two Langmuir waves and vice versa, and the third term represents the induced scattering of nonlinear mode off electrons mediated by the enhanced Langmuir turbulence. These terms are given by

$$\left.\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{ind. emiss.}}=\sigma\omega_{\mathbf{k}}^N\int d\mathbf{v}\,\Gamma_N^\sigma(\mathbf{k},\mathbf{v})\,\mathbf{k}\cdot\frac{\partial F_e(\mathbf{v},t)}{\partial\mathbf{v}}I_N^\sigma(\mathbf{k},t),$$

$$\Gamma_N^\sigma(\mathbf{k},\mathbf{v})=4\pi\frac{\omega_{pe}^2}{k^2}\delta(\sigma\omega_{\mathbf{k}}^N-\mathbf{k}\cdot\mathbf{v}),$$

$$\left.\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{decay }NL}=-\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^N\int d\mathbf{k}'\times V_{N,L}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')\times[\sigma'\omega_{\mathbf{k}'}^L I_L^{\sigma''}(\mathbf{k}-\mathbf{k}',t)+\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_L^{\sigma'}(\mathbf{k}',t)]\,I_N^\sigma(\mathbf{k},t),$$

$$V_{N,L}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')=\frac{\pi}{16\omega_{pe}^4}\frac{e^2}{m_e^2}a_{\mathbf{k},\mathbf{k}'}^2\times\delta(\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L),$$

$$\left.\frac{\partial I_N^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{ind. scatt. }NL}=\sum_{\sigma'=\pm1}\sigma\omega_{\mathbf{k}}^N\int d\mathbf{k}'\int d\mathbf{v}\times U_{N,L}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial F_e(\mathbf{v},t)}{\partial\mathbf{v}}I_L^{\sigma'}(\mathbf{k}',t)\,I_N^\sigma(\mathbf{k},t),$$

$$U_{N,L}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}',\mathbf{v})=\frac{\pi}{4\omega_{pe}^2}\frac{e^2}{m_e^2}\times\frac{a_{\mathbf{k},\mathbf{k}'}^2\,\delta_{\sigma,\sigma'}}{|\mathbf{k}-\mathbf{k}'|^2\,|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L)|^2}\times\delta[\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}],$$

$$|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L)|^2=|\,3\,[2(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'-k^2]\lambda_{De}^2+2\,\eta_{\mathbf{k}}|^2. \tag{14}$$

The reader may notice that the detailed expression for $|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^N)|^2$ and $|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L)|^2$ in Eqs. (12) and (14) differ from those found in Ref. 52. The primary reason is because of the different thermal correction factor associated with the nonlinear mode dispersion relation. In Ref. 52, the expression $\omega_{\mathbf{k}}^N=\omega_{pe}(2+3k^2\lambda_{De}^2/4+\eta_{\mathbf{k}})$ was used. In contrast, we now adopt the expression $\omega_{\mathbf{k}}^N=\omega_{pe}(2+3k^2\lambda_{De}^2/2+\eta_{\mathbf{k}})$, as already explained. However, in view of the fact that the detailed expression for thermal correction factor in the wave dispersion relation is uncertain, and that $\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^N)$ and $\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L)$ are both quite small in magnitude such that any small modification to the thermal factor may lead to potentially large differences, we have decided to adopt a much simpler approach in computing the detailed expression for $|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^N)|^2$ and $|\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^N-\sigma'\omega_{\mathbf{k}'}^L)|^2$. Specifically, we have employed the simplifying approximation, $\omega_{\mathbf{k}}^N\simeq\omega_{pe}(2+\eta_{\mathbf{k}})$.

Finally, the wave kinetic equation which governs the development of ion-sound wave intensity is given by

$$\frac{\partial I_S(\mathbf{k},t)}{\partial t}=\left.\frac{\partial I_S(\mathbf{k},t)}{\partial t}\right|_{\text{ind. emiss.}}+\left.\frac{\partial I_S(\mathbf{k},t)}{\partial t}\right|_{\text{decay }SL}, \tag{15}$$

where the first term represents the induced emission/absorption of ion-sound waves and the second term represents the decay process. These are given by

$$\left.\frac{\partial I_S(\mathbf{k},t)}{\partial t}\right|_{\text{ind. emiss.}} = \sigma\omega_{\mathbf{k}}^L \int d\mathbf{v}\, \Gamma_S^\sigma(\mathbf{k},\mathbf{v}) \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\left[F_e(\mathbf{v},t) + \frac{m_e}{m_i}F_i(\mathbf{v})\right] I_S^\sigma(\mathbf{k},t),$$

$$\Gamma_S^\sigma(\mathbf{k},\mathbf{v}) = \pi\,\mu_{\mathbf{k}}\frac{\omega_{pe}^2}{k^2}\,\delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v}),$$

$$\left.\frac{\partial I_S^\sigma(\mathbf{k},t)}{\partial t}\right|_{\text{decay } SL} = \sum_{\sigma',\sigma''=\pm1} \sigma\mu_{\mathbf{k}}\,\omega_{\mathbf{k}}^L \int d\mathbf{k}' \times V_{S,L}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}')\,\{\sigma\omega_{\mathbf{k}}^L I_L^{\sigma'}(\mathbf{k}',t) \times I_L^{\sigma''}(\mathbf{k}-\mathbf{k}',t) - [\sigma'\omega_{\mathbf{k}'}^L I_L^{\sigma''}(\mathbf{k}-\mathbf{k}',t) + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_L^{\sigma'}(\mathbf{k}',t)]\, I_S^\sigma(\mathbf{k},t)\},$$

$$V_{S,L}^{\sigma,\sigma',\sigma''}(\mathbf{k},\mathbf{k}') = \frac{\pi}{4}\frac{e^2}{T_e^2}\frac{\mu_{\mathbf{k}}[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2 k'^2|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L). \tag{16}$$

In Eq. (15), we have ignored the induced scattering term which involves two ion-sound waves and the ions. This process is much slower than any of the processes considered hitherto.

C. One-dimensional formulation

In what follows, we simplify the analysis by considering a one-dimensional limit where both the particles and the excited waves propagate along the same or opposite directions. Defining the following nondimensional quantities:

$$z = \omega/\omega_{pe}, \quad q = kv_e/\omega_{pe}, \quad \tau = \omega_{pe}t, \quad u = v/v_e, \tag{17}$$

where $v_e^2 = 2T_e/m_e$ is the thermal velocity of bulk electrons, and normalizing the distribution functions and the wave spectral intensities,

$$\int dv\, F_a(v) = \int du\, F_a(u), \qquad \frac{1}{8\pi\hat{n}T_e}\int dk\, I_\alpha^\sigma(k) = \int dq\, I_\alpha^\sigma(q), \tag{18}$$

where $\alpha = L,S,N$ denotes the wave modes, the kinetic equations for waves and particles can be written as follows:

$$\begin{aligned}\frac{\partial I_L^\sigma(q)}{\partial\tau} =& \frac{\pi\,\sigma z_q^L}{q}\int_{-\infty}^{\infty} du\,\frac{\partial F_e(u)}{\partial u}\,\delta(\sigma z_q^L - qu)\, I_L^\sigma(q) + \pi\,\sigma z_q^L \sum_{\sigma',\sigma''=\pm1}\int_{-\infty}^{\infty} dq' \Big(\xi|q-q'|\{\sigma z_q^L I_L^{\sigma'}(q')\, I_S^{\sigma''}(q-q') \\ &- [\sigma' z_{q'}^L I_S^{\sigma''}(q-q') + \sigma'' z_{q-q'}^L I_L^{\sigma'}(q')]\, I_L^\sigma(q)\}\,\delta(\sigma z_q^L - \sigma' z_{q'}^L - \sigma'' z_{q-q'}^S) \\ &+ \frac{9\,q'^2}{8}[\sigma z_q^L I_L^{\sigma''}(q-q') - \sigma'' z_{q-q'}^L I_L^\sigma(q)]\, I_N^{\sigma'}(q')\,\delta(\sigma z_q^L - \sigma' z_{q'}^N - \sigma'' z_{q-q'}^L)\Big) - \pi\sum_{\sigma'=\pm1}\int_{-\infty}^{\infty} dq'\int_{-\infty}^{\infty} du \\ &\times\Big[(q-q')\, I_L^{\sigma'}(q')\left((\sigma z_q^L - \sigma' z_{q'}^L)\frac{\partial F_e(u)}{\partial u} - \frac{m_e}{m_i}\sigma z_q^L\frac{\partial F_i(u)}{\partial u}\right)\delta[\sigma z_q^L - \sigma' z_{q'}^L - (q-q')\,u] \\ &- \frac{\sigma z_q^L\, q'^2\,\delta_{\sigma,\sigma'}\, I_N^{\sigma'}(q')}{4(q-q')\,[2q(q-q') + q'^2 - 4\eta_{q'}/3]^2}\frac{\partial F_e(u)}{\partial u}\,\delta[\sigma z_q^L - \sigma' z_{q'}^N - (q-q')\,u]\Big]\, I_L^\sigma(q),\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial I_S^\sigma(q)}{\partial\tau} =& \frac{\pi\,\xi}{2}q\,|q|\,\sigma z_q^L\int_{-\infty}^{\infty} du\left(\frac{\partial F_e(u)}{\partial u} + \frac{m_e}{m_i}\frac{\partial F_i(u)}{\partial u}\right)\delta(\sigma z_q^S - qu)\, I_S^\sigma(q) \\ &+ \frac{\pi\,\xi}{2}|q|\,\sigma z_q^L\sum_{\sigma',\sigma''=\pm1}\int_{-\infty}^{\infty} dq'\,\{\sigma z_q^L I_L^{\sigma'}(q')\, I_L^{\sigma''}(q-q') \\ &- [\sigma' z_{q'}^L I_L^{\sigma''}(q-q') + \sigma'' z_{q-q'}^L I_L^{\sigma'}(q')]\, I_S^\sigma(q)\}\,\delta(\sigma z_q^S - \sigma' z_{q'}^L - \sigma'' z_{q-q'}^L),\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial I_N^\sigma(q)}{\partial\tau} =& 4\pi\frac{\sigma z_q^N}{q}\int_{-\infty}^{\infty} du\,\frac{\partial F_e(u)}{\partial u}\,\delta(\sigma z_q^N - qu)\, I_N^\sigma(q) \\ &- \frac{9\pi}{8}\sigma z_q^N\sum_{\sigma',\sigma''=\pm1}\int_{-\infty}^{\infty} dq'\, q^2[\sigma' z_{q'}^L I_L^{\sigma''}(q-q') + \sigma'' z_{q-q'}^L I_L^{\sigma'}(q')]\, I_N^\sigma(q)\,\delta(\sigma z_q^N - \sigma' z_{q'}^L - \sigma'' z_{q-q'}^L) \\ &+ \pi\sum_{\sigma'=\pm1}\int_{-\infty}^{\infty} dq'\int_{-\infty}^{\infty} du\,\frac{\sigma z_q^N\, q^2\,\delta_{\sigma,\sigma'}\, I_L^{\sigma'}(q')\, I_N^\sigma(q)}{(q-q')\,[2(q-q')q' - q^2 + 4\eta_q/3]^2}\frac{\partial F_e(u)}{\partial u}\,\delta[\sigma z_q^N - \sigma' z_{q'}^L - (q-q')u],\end{aligned} \tag{21}$$

$$\frac{\partial F_e(u)}{\partial\tau} = 2\pi\sum_{\sigma=\pm1}\frac{\partial}{\partial u}\left(\int_0^\infty dq\, I_L^\sigma(q)\,\delta(\sigma z_q^L - qu) + I_N^\sigma(q)\,\delta(\sigma z_q^N - qu)\right)\frac{\partial F_e(u)}{\partial u}, \tag{22}$$

where we have omitted the time variable τ from various dynamic quantities for the sake of simplicity. In the above

$$z_q^L = 1 + 3q^2/4,$$
$$z_q^S = \xi q (1+q^2/2)^{-1/2} \approx \xi q,$$
$$\xi = (m_e/2m_i)^{1/2}(1+3T_i/T_e)^{1/2}, \tag{23}$$
$$z_q^N = 2 + 3q^2/4 + 3\gamma q^2/4, \quad \gamma = \sum_{\sigma=\pm 1} \int_0^\infty dq\, I_L^\sigma(q,\tau),$$

are the dispersion relations for Langmuir, ion-sound and nonlinear waves, respectively, and where we have used the fact that $\lambda_{De}^2 = T_e/(4\pi \hat{n} e^2) = v_e^2/(2\omega_{pe}^2)$.

Equations (19)–(20) have to be solved simultaneously in order to obtain a self-consistent solution for the evolution of the spectral intensities of the waves and particle distributions. Before we present the numerical solution, we note that the spectral intensity for waves propagating in the forward direction ($\sigma=1$) is different from the intensity for waves propagating backwards ($\sigma=-1$), that is, $I_\alpha^\pm(q) \neq I_\alpha^\mp(q)$. The difference shows itself when one considers the solutions for the resonance conditions contained in the delta functions in Eqs. (19)–(20). Thus, first of all, one has to perform a careful investigation of all existing physical solutions of this set of delta distributions. In order to accomplish that, one has to consider all possible combinations of the dummy variables $(\sigma,\sigma',\sigma'')$. For those terms of Eqs. (19)–(20) that correspond to the standard weak-turbulence theory, this analysis was already performed in Ref. 53 and will not be repeated here. We will only mention that, thanks to the symmetry relations

$$z_{-q}^\alpha = -z_q^\alpha, \quad I_\alpha^\sigma(-q) = I_\alpha^\sigma(q), \tag{24}$$

for $\alpha = L,S,N$, we can restrict the analysis to $q>0$ only. We warn the readers that $I_\alpha^\pm(q) \neq I_\alpha^\mp(-q)$, but that the above symmetry is applicable only for the same sign of σ.

Here, a word of caution on the application of the above symmetry condition is called for. From Eq. (23), it is easy to see that the ion-sound wave dispersion relation, $z_q^S = \xi q$, automatically satisfies the symmetry condition, $z_{-q}^S = -z_q^S$. However, at first sight, the L and N mode dispersion relations, $z_q^L = 1+3q^2/4$ and $z_q^N = 2+3q^2/4+3\gamma q^2/4$, respectively, do not appear to satisfy the required symmetry. This apparent contradiction can be easily resolved if we interpret that the expressions $z_q^L = 1+3q^2/4$, $z_q^S = \xi q$, and $z_q^N = 2 + 3q^2/4 + 3\gamma q^2/4$, are applicable only for positive q, while for negative q, we simply impose symmetry rule, $z_{-q}^\alpha = -z_q^\alpha$, $(\alpha = L,S,N)$. In short, we apply the following rule for the dispersion relations:

$$z_q^L = 1 + 3q^2/4, \quad z_{-q}^L = -z_q^L, \quad (q>0),$$
$$z_q^S = \xi q, \quad z_{-q}^S = -z_q^S, \quad (q>0), \tag{25}$$
$$z_q^N = 2 + 3q^2/4 + 3\gamma q^2/4, \quad z_{-q}^N = -z_q^N, \quad (q>0).$$

In the following, we are only interested in positive q space. In Eqs. (19)–(21), we also convert q'-integrals, $\int_{-\infty}^\infty dq'$, to integrals over positive q' range, $\int_0^\infty dq'$, by invoking the symmetry condition Eq. (24). Here, we note that although we compute the wave intensities over positive-q space only, in the subsequent plots of numerical results, however, we will plot the intensity of the backward-propagating mode, $I_\alpha^-(q)$ in the negative-q space, again invoking the symmetry relation Eq. (24), in order to provide the view of complete q-space. Moreover, since we will pay our attention to beam velocity in the range of 5 to 10 times the thermal velocity of the bulk electrons, it is evident that the significant wave-particle and wave–wave interactions take place only for $q<1$.

In what follows, let us pay attention to the terms which pertain to the harmonic nonlinear eigenmode, which appear in the three-wave decay and induced-scattering resonance conditions in Eqs. (19) and (21). These are also the terms labeled with LN and NL in Eqs. (12) and (14). Consider the three-wave resonant interaction condition,

$$\sigma z_q^L - \sigma' z_{q'}^N - \sigma'' z_{q\pm q'}^L = 0. \tag{26}$$

Suppose that the ratio of beam to thermal speeds is ≈ 5. Then, according to the linear growth rate prediction, the primary Langmuir wave should peak around $q \sim 0.2$. A simple analysis of the above relation shows that the only possible solutions occur for $\sigma = -\sigma' = \sigma''$ and for $q \sim 1.6$, which is high above the q values where the interactions are bound to occur. Therefore, all waves in this q regime are completely damped. Thus, we can conclude that

$$\left.\frac{\partial I_L^\sigma(q,\tau)}{\partial \tau}\right|_{\text{decay } LN} \approx 0. \tag{27}$$

The induced-scattering processes involving the nonlinear mode are dictated by the nonlinear wave-particle resonance condition,

$$\sigma z_q^L - \sigma' z_{q'}^N - (q \pm q')\, u = 0, \tag{28}$$

where the $\pm$ signal takes into account the fact that we are considering only the $q>0$ region. This means that the electrons must have a resonant velocity of

$$u_{\text{res}} = \frac{\sigma z_q^L - \sigma' z_{q'}^N}{q \pm q'}. \tag{29}$$

Again, assuming a normalized beam velocity for the electrons around five times the thermal speed, the maximum linear growth should occur for Langmuir waves with $q \sim q_L \approx 0.2$. Since the nonlinear mode has a frequency about twice that of fundamental Langmuir wave, the main wave-particle interaction is expected to occur for a phase velocity similar to the fundamental, which means that $q' \sim q_N \approx 2q_L = 0.4$. For forward-propagating Langmuir waves ($\sigma=1$) we have the following possibilities: $u_{\text{res}} = (z_q^L - z_{q'}^N)/(q-q') \approx 1/q_L = 5$, in this case, the resonant velocity is near the beam velocity and this term can have an important effect on the induced LN scattering. The only other possibility is $u_{\text{res}} = (z_q^L + z_{q'}^N)/(q+q') \approx 5$, which has to be kept. For backward-propagating Langmuir waves ($\sigma=-1$) we have the possibilities: $u_{\text{res}} = (-z_q^L + z_{q'}^N)/(q-q') \approx -1/q_L = -5$ and $u_{\text{res}} = -(z_q^L + z_{q'}^N)/(q+q') \approx -1/q_L = -5$, which will affect the particles in the tail of the distribution with negative velocity.

For the *NL* term, one can easily show that the three-wave decay processes will also not contribute to the dynamics of the system. Thus,

$$\left.\frac{\partial I_N^\sigma(q,\tau)}{\partial\tau}\right|_{\text{decay } NL}\approx 0. \tag{30}$$

On the other hand, for the induced-scattering process to be effective, the electron resonant velocity must be

$$u_{\text{res}}=\frac{\sigma z_q^N-\sigma' z_{q'}^L}{q\pm q'}. \tag{31}$$

Again, using the case of beam to thermal speed ratio of 5, we see that in this case, the main contribution to the resonance will come from $q\sim q_N\approx 0.4$, and $q'\sim q_L\approx 0.2$. For $\sigma=1$ we have $u_{\text{res}}=(z_q^N-z_{q'}^L)/(q-q')\approx 1/q_L=5$ and $u_{\text{res}}=(z_q^N+z_{q'}^L)/(q+q')\approx 1/q_L=5$. Again, both terms are equally important, and thus, will be kept. For $\sigma=-1$ we have $u_{\text{res}}=(-z_q^N+z_{q'}^L)/(q-q')\approx -5$ and $u_{\text{res}}=-(z_q^N+z_{q'}^L)/(q+q')\approx -5$, which can also be of equal significance, and may affect the electrons in the tail with negative speeds.

As mentioned, the ions will be considered stationary, with their distribution function in normalized form given by

$$F_i(u)=\frac{1}{\sqrt{\pi}}\left(\frac{m_i}{m_e}\frac{T_e}{T_i}\right)^{1/2}\exp\left(-\frac{m_i}{m_e}\frac{T_e}{T_i}u^2\right).$$

After performing all possible integrations, the resulting equations are

$$\begin{aligned}
\frac{\partial I_L^\pm(q)}{\partial\tau}=&\pm\frac{\pi z_q^L}{q^2}\left.\frac{\partial F_e(u)}{\partial u}\right|_{u=\pm z_q^L/q} I_L^\pm(q)+\pi\, p\, z_q^L\,(z_q^L I_L^\mp(q+p)\,I_S^\mp(2q+p)-[z_{q+p}^L I_S^\mp(2q+p)-z_{2q+p}^L I_L^\mp(q+p)]\,I_L^\pm(q)\\
&+\{z_q^L I_L^\pm(p-q)\,I_S^\pm(2q-p)-[z_{p-q}^L I_S^\pm(2p-q)+z_{2q-p}^L I_L^\pm(p-q)]\,I_L^\pm(q)\}\Theta(p-q)\Theta(2q-p)\\
&+\{z_q^L I_L^\pm(p-q)\,I_S^\pm(2q-p)-[z_{p-q}^L I_S^\pm(p-2q)-z_{p-2q}^L I_L^\pm(p-q)]\,I_L^\pm(q)\}\Theta(p-q)\Theta(p-2q)\\
&+\{z_q^L I_L^\mp(q-p)\,I_S^\pm(2q-p)-[z_{q-p}^L I_S^\pm(2q-p)+z_{2q-p}^L I_L^\mp(q-p)]\,I_L^\pm(q)\}\Theta(q-p))\\
&-\pi\int_0^\infty dq'\left\{\;|q-q'|\,u\left(\frac{\partial F_e(u)}{\partial u}\pm\frac{2T_e}{T_i}\frac{z_q^L}{q-q'}F_i(u)\right)_{u=\pm 3(q+q')/4} I_L^\pm(q')\right.\\
&+(q+q')\,u\left(\frac{\partial F_e(u)}{\partial u}\pm\frac{2T_e}{T_i}\frac{z_q^L}{q+q'}F_i(u)\right)_{u=\pm 3(q-q')/4} I_L^\mp(q')\\
&\mp\frac{z_q^L}{4}\frac{q'^2(q-q')^{-1}|q'-q|^{-1}}{[2q(q-q')+q'^2-4\eta_{q'}/3]^2}\left.\frac{\partial F_e(u)}{\partial u}\right|_{u=\pm(z_{q'}^N-z_q^L)/(q'-q)} I_N^\pm(q')\\
&\left.\mp\frac{z_q^L}{4}\frac{q'^2(q+q')^{-2}}{[2q(q+q')+q'^2-4\eta_{q'}/3]^2}\left.\frac{\partial F_e(u)}{\partial u}\right|_{u=\pm(z_q^L+z_{q'}^N)/(q+q')} I_N^\pm(q')\right\} I_L^\pm(q),
\end{aligned}\tag{32}$$

$$\begin{aligned}
\frac{\partial I_S^\pm(q)}{\partial\tau}=&\pm\frac{3\pi}{8}p\,q\,z_q^L\left(\frac{\partial F_e(u)}{\partial u}-\frac{2T_e}{T_i}u\,F_i(u)\right)_{u=\pm z_q^S/q} I_S^\pm(q)+\frac{\pi\, p\, z_q^L}{2}\left\{z_q^L I_L^\pm\left(\frac{q+p}{2}\right)I_L^\pm\left(\frac{p-q}{2}\right)\right.\\
&\left.\mp\left[z_{(p+q)/2}^L I_L^\pm\left(\frac{p-q}{2}\right)-z_{(p-q)/2}^L I_L^\pm\left(\frac{p+q}{2}\right)\right]I_S^\pm(q)\right\}\Theta(p-q)\\
&+\frac{\pi\, p\, z_q^L}{2}\left\{z_q^L I_L^\pm\left(\frac{p+q}{2}\right)I_L^\mp\left(\frac{q-p}{2}\right)\mp\left[z_{(p+q)/2}^L I_L^\mp\left(\frac{q-p}{2}\right)-z_{(q-p)/2}^L I_L^\pm\left(\frac{p+q}{2}\right)\right]I_S^\pm(q)\right\}\Theta(q-p),
\end{aligned}\tag{33}$$

$$\begin{aligned}
\frac{\partial I_N^\pm(q)}{\partial\tau}=&\pm\frac{4\pi z_q^N}{q^2}\left.\frac{\partial F_e(u)}{\partial u}\right|_{u=\pm z_q^N/q} I_N^\pm(q)\pm\,\pi q^2 z_q^N\int_0^\infty dq'\left(\frac{(q-q')^{-1}|q-q'|^{-1}}{[2(q-q')q'-q^2+4\eta_q/3]^2}\left.\frac{\partial F_e(u)}{\partial u}\right|_{u=\pm(z_q^N-z_{q'}^L)/(q-q')}\right.\\
&\left.\times I_L^\pm(q')+\frac{(q+q')^{-2}}{[2(q+q')q'+q^2-4\eta_q/3]^2}\left.\frac{\partial F_e(u)}{\partial u}\right|_{u=\pm(z_q^N+z_{q'}^L)/(q+q')} I_L^\mp(q')\right)I_N^\pm(q),
\end{aligned}\tag{34}$$

$$\frac{\partial F_e(u)}{\partial\tau}=2\pi\sum_{+,-}\frac{\partial}{\partial u}\left(\frac{1}{|u|}\{[I_L^\pm(q)]_{q=1/|u|}\,\Theta(\pm u)+[I_N^\pm(q)]_{q=2/|u|}\,\Theta(\pm u)\}\frac{\partial F_e(u)}{\partial u}\right). \tag{35}$$

In the above, p is defined by

$$p=4\xi/3,$$

ξ being defined in Eq. (23). Equations (32)–(35) must be solved simultaneously in order to describe correctly the dynamics of the wave-particle system. In the next section, we present numerical solutions of this set of equations. We reiterate that Eqs. (32)–(34) are applicable for $q>0$ only. However, in plotting the numerical results, we invoke the symmetries, $I_L^-(q)=I_L^-(-q)$, $I_S^-(q)=I_S^-(-q)$, and $I_N^-(q)=I_N^-(-q)$, to plot backward-propagating wave intensities in the negative q-range.

III. NUMERICAL SOLUTIONS

In this section, we present some numerical solutions of Eqs. (32)–(35). Let us further define normalized quantities,

$$\delta=\frac{n_e}{n_0},\quad \rho_b=\frac{T_b}{T_e},\quad \rho_i=\frac{T_i}{T_e},\quad u_b=\frac{V_b}{v_b},\quad \mu=\frac{m_e}{m_i}, \tag{36}$$

where n_e and n_0 are number densities for the beam electrons and the background thermal electrons, respectively; T_b represents the thermal spread (kinetic temperature) associated with the beam; V_b and $v_b=(2T_b/m_e)^{1/2}$ are the average beam speed and thermal speed of the beam electrons defined in the reference frame of the beam. In terms of these quantities, the initial electron distribution function and the quasi-stationary ion distribution are given, respectively, by

$$F_e(u,0)=\frac{1-\delta}{\pi^{1/2}}\exp(-u^2)+\frac{\delta}{\pi^{1/2}\rho_b^{1/2}}\exp\left(-\frac{(u-u_b)^2}{\rho_b}\right),$$
$$F_i(u)=\frac{1}{\pi^{1/2}(\mu\rho_i)^{1/2}}\exp\left(-\frac{u^2}{\mu\rho_i}\right). \tag{37}$$

In the numerical computation, we have chosen $\delta=n_e/n_0=2\times10^{-4}$, $\rho_b=T_b/T_e=1$, $\rho_i=T_i/T_e=1/7$, $u_b=V_b/v_b=5$, and of course, $\mu=1/1836$. Sets of parameters which include the present choice were adopted in our recent study on numerical analysis of conventional weak turbulence kinetic equation in which nonlinear eigenmode is not considered as part of the eigenmode system.[53] The purpose of choosing the same set of parameters as in Ref. 53 is so that we can make direct comparisons, and assess the dynamical role of the nonlinear eigenmode (the harmonic Langmuir wave).

Figure 1 shows the self-consistent time-development of the total (thermal plus beam) electron distribution, $F_e(u)$. The vertical axis represents the logarithmic of the distribution, $\log_{10}F_e(u)$, while the two horizontal axes represent the normalized speed, $u=v/v_e$, and normalized time, $\tau=\omega_{pe}t$, respectively. We have chosen to plot the numerical results at time intervals corresponding to $\tau=0$, 250, 500, 1000, 2000, 3000, 4000, 6000, 8000, and 1×10^4. The development of $F_e(u)$ during relatively early times is as expected according to the usual quasilinear diffusion theory in that the bump-on-tail feature rapidly flattens out to form a plateau in velocity space. The long-time behavior of the electrons shows that the

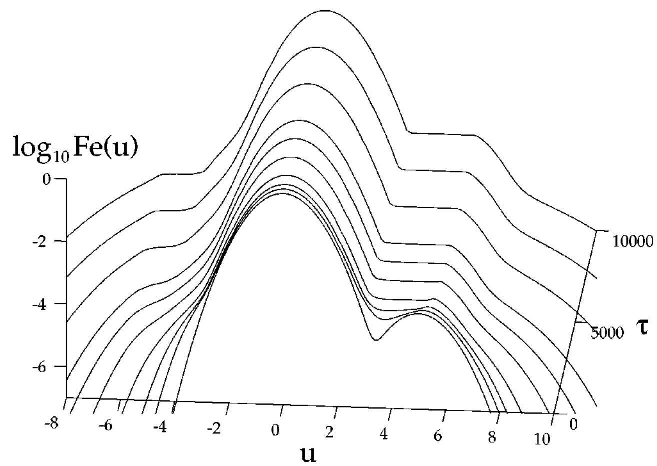

FIG. 1. Plot of electron distribution function, $F_e(u,\tau)$, vs u and τ, for the case of $n_b/n_0=2\times10^{-4}$, $u_b=5$, $T_e/T_i=7$.

energetic tail develops for both the forward- and backward-propagating electrons, and a plateaulike feature is seen to form for the electrons possessing negative speed. These features are more or less identical to that already discussed in Ref. 53, where it is explained that the heating of electrons and negative-u plateau are owing to the combined effects of continuous absorption of the initial level of turbulence and the feedback effects of wave-coupling processes. From this, it may be concluded that the excitation of additional eigenmode (i.e., the harmonic Langmuir mode) does not affect the particles in any significant manner.

Let us now move on to the discussion of wave dynamics. In Fig. 2, we plot the (fundamental) Langmuir-wave intensity spectrum, $I_L(q)$, in logarithmic vertical scale versus normalized wave number, $q=kv_e/\omega_{pe}$, and normalized time, τ (left-hand panel). Here, the time-dependence of $I_L(q,\tau)$ is implicitly assumed. To aid the visualization, we have added a small constant, $\epsilon=1\times10^{-6}$ to the spectrum $I_L(q)$. As a result, Fig. 2 actually displays $\log_{10}[I_L(q)+\epsilon]$. In order to help the readers interpret the numerical results in a more quantifiable way, we have also plotted the same results in two-dimensional format (right-hand panel), superposing the curves at different time steps, although in this format it becomes difficult to distinguish results at different time steps. The best way to look at our result is to compare the left- and right-hand panels.

Note that the portion of the wave spectrum corresponding to $q>0$ is the forward-propagating mode, $I_L^+(q)$, while the negative q represents the backward-propagating mode, $I_L^-(q)$ (originally computed over $q>0$), which is replotted over $-q$ using the symmetry property, $I_L^-(-q)=I_L^-(q)$. The initial level of Langmuir mode is taken to be a flat spectrum over the entire range of q values considered in our computation ($-0.5<q<0.5$). That is, we assumed that initially, the spectrum is give by $I_L^+(q,0)=I_L^-(q,0)=2\times10^{-4}$. Again, this choice is the same as that of Ref. 53. We could have adopted a more realistic profile such as a certain power-law spectrum, or that computed on the basis of thermal spontaneous fluctuation theory. However, as the present formalism

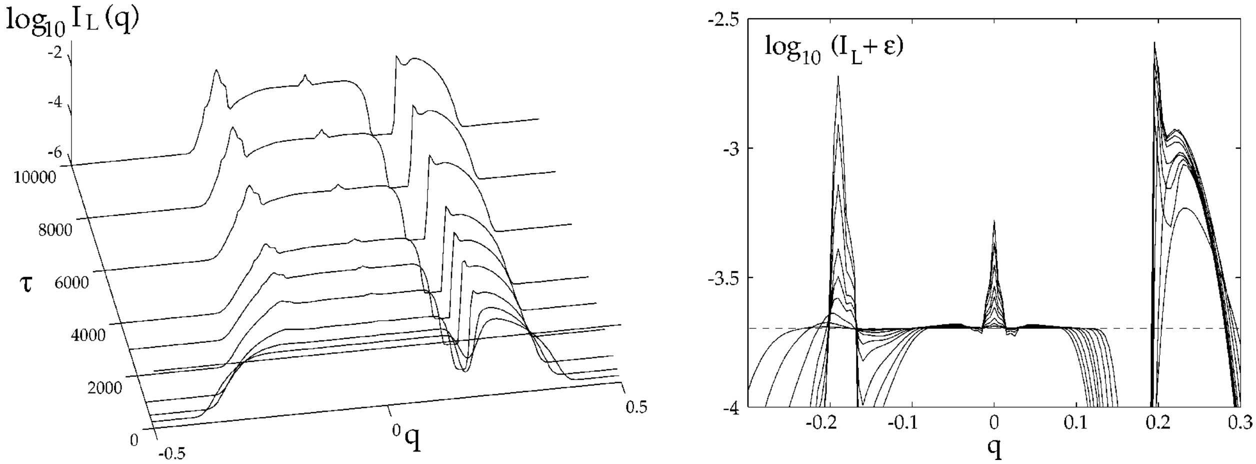

FIG. 2. Plot of Langmuir wave intensity, $I_q^L(\tau)$, vs q and τ (left-hand panel), for the case of $n_b/n_0=2\times10^{-4}$, $u_b=5$, $T_e/T_i=7$, and with initial wave level of $I_q^L(0)=2\times10^{-4}$. To aid the visualization, we have added a small constant $\epsilon=1\times10^{-6}$, so that the actual quantity plotted above is $\log_{10}[I_L(q)+\epsilon]$. The right-hand panel is the same result shown in two-dimensional, $\log_{10}[I_L(q)+\epsilon]$ versus q, format.

does not have the single-particle fluctuation effects, the specific choice of initial wave spectrum becomes arbitrary. For this reason, we chose a simple white-noise flat spectrum. The initial wave level is indicated by the straight line.

As Fig. 2 shows, the modes in the linearly unstable regime (around $q\approx1/5=0.2$) begin to grow exponentially at early times, but rapidly saturate as a result of the flattening out of the beam electrons. Those modes in the damped portions of the q range undergo Landau damping. The damping of the initial waves lead to the bulk electron heating in the tail, as already indicated. Relatively small-q range ($|q|<0.2$) does not suffer heavy Landau damping, and as a result, the initial level of turbulence spectrum remains more or less unchanged over time. Beyond the initial quasilinear stage, the combined effects of induced-scattering off ions and three-wave decay process lead to the formation of backscattered Langmuir modes (negative q in the range roughly corresponding to $q\approx-0.2$) and long-wavelength ($q\sim0$) Langmuir modes, as discussed in Ref. 53.

Overall feature associated with $I_L(q)$, when compared against that of Ref. 53, shows that the Langmuir mode dynamics is largely unaffected by the presence of nonlinear eigenmode, N. This means that the additional nonlinear coupling terms on the right-hand side of Eqs. (32) and (34), which involve corrections due to the presence of N-mode, have relatively insignificant role in the Langmuir wave kinetic equation. As is well-known, the induced-scattering term involving the electrons is largely unimportant. It turns out that the nonlinear mechanisms responsible for the generation of backward-propagating Langmuir mode are the induced-scattering off ions and three-wave decay process, although the scattering off ions is the more important of the two, which is also well-known in the literature, and confirmed in our recent work.[53]

Figure 3 plots the ion-sound wave intensity spectrum in the same format as that of Fig. 2. The evolution of $I_S(q)$ in time is very similar to that shown in Ref. 53 in that the S-mode continuously damp out over time, except near q

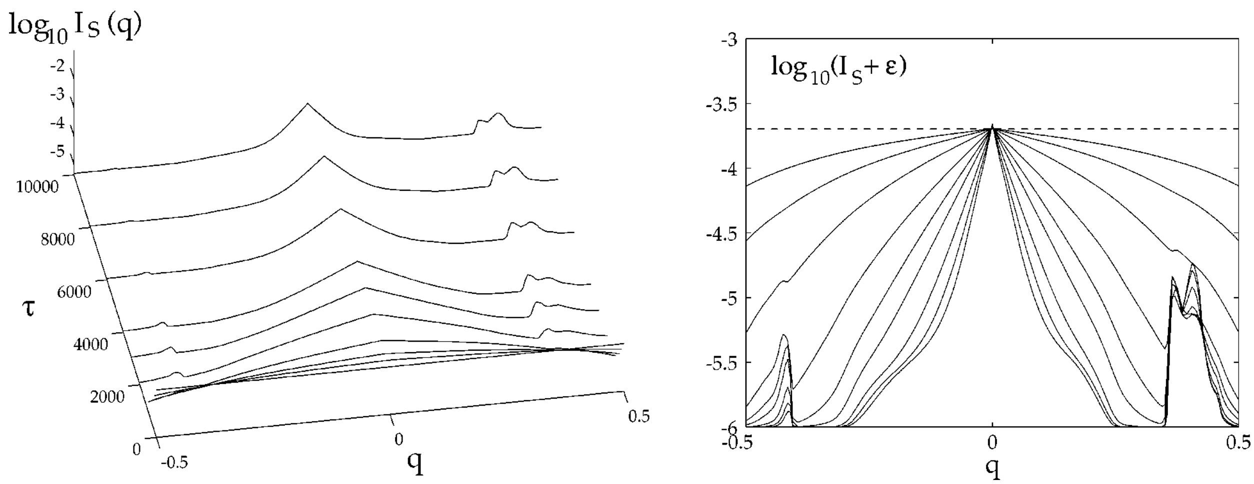

FIG. 3. Plot of ion-sound wave intensity, $I_q^S(\tau)$, in the same format as Fig. 2. Note that the peak near $q=0$ is simply the remnant wave intensity initially imposed which is undamped.

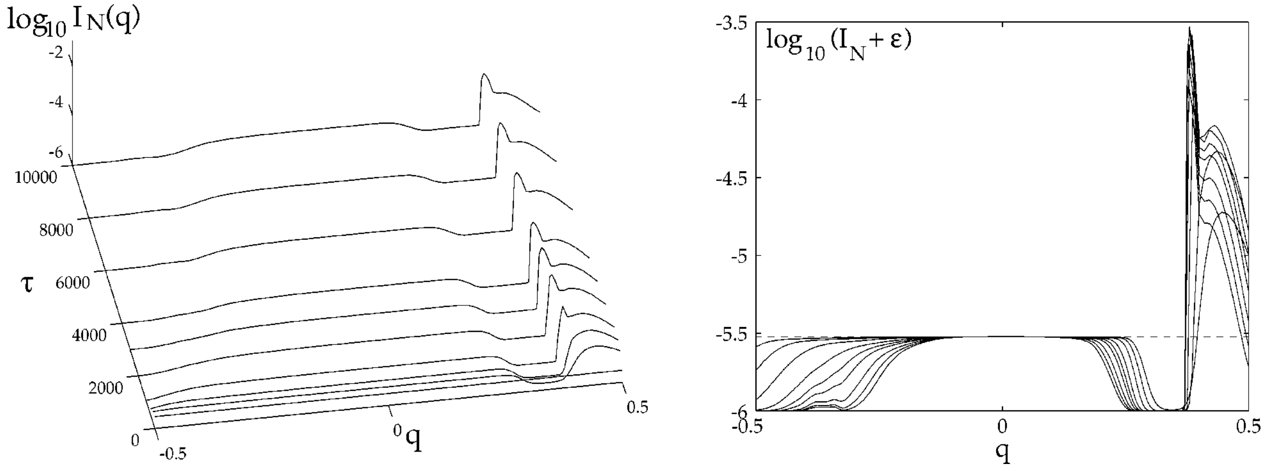

FIG. 4. Plot of nonlinear eigenmode (harmonic Langmuir mode) intensity, $I_q^N(\tau)$, in the same format as Fig. 2, for the case of $n_b/n_0 = 2\times10^{-4}$, $u_b = 5$, $T_e/T_i = 7$, and with initial wave level of $I_q^N(0) = 2\times10^{-6}$.

$=0$, where the initially imposed wave intensity is largely unchanged (note that the intensity does not grow higher than the initial level), and near $q=0.4$, where the peak in the q spectrum is prominent. This is the result of weak decay instability.

The dependence of the beam-plasma interaction process on the various input parameters has been investigated in a fairly extensive manner in Ref. 53, and thus we shall not repeat such a study here. Among the findings of Ref. 53 are that the increase in the electron-to-ion temperature ratio, T_e/T_i increases the efficacy of both the induced scattering off ions and the decay instability, and that the choice of initial wave level has a significant effect on the later evolution of the wave intensity.

We now focus our attention on the generation of harmonic Langmuir mode, or equivalently, the nonlinear eigenmode, N. Figure 4 is the plot of wave intensity for N mode in the same format as before. For the harmonic Langmuir mode, we expect that the initial wave level should be very low, since in the quiescent plasma, such a mode should be at an extremely low level. The exact level of initial N mode cannot be determined within the context of the present collisionless theory. We thus made an arbitrary choice of $I_N^+(q,0) = I_N^-(q,0) = 2\times10^{-6}$. The time evolution of N-mode intensity shows that the initial exponential growth is followed by quasilinearlike saturation, with an overall temporal rate of intensity amplification comparable to that of L-mode. Note that the wave frequency of N mode is near $2\omega_{pe}$, and the characteristic wave number is also about twice that of L mode. Note also that the bandwidth associated with N mode in q-space is comparable to L mode. These characteristics are in a reasonably good and qualitative agreement with recent simulation results of harmonic Langmuir mode.[21–23] It is interesting to note that the backward-propagating N mode, i.e., $I_N^-(q)$, corresponding to the negative q portion of the total $I_N(q)$, is not excited.

To understand the dynamical role of each term on the right-hand side of wave kinetic equation for N mode, we have first ignored the nonlinear wave-coupling term on the right-hand of Eq. (34). The resulting equation is equivalent to the usual quasilinear wave kinetic in which only the induced emission/absorption effect is taken into account. We then have ignored the induced emission term, but only retained the second and third terms on the right-hand side of Eq. (34), which represent the scattering of N mode off electrons induced by the presence of L mode. By comparing the two results, we may then determine the primary effect which is responsible for the excitation of the harmonic Langmuir mode, N. In principle, both the induced emission and induced scattering can be of equal significance since the linear resonant velocity,

$$u_{\rm res} = \pm z_q^N/q \sim \pm 5,$$

and nonlinear resonant velocity,

$$u_{\rm res} = \pm (z_q^N - z_{q'}^L)/(q-q') \sim \pm 5,$$

are comparable. In the above estimates, we have used the relations, $z_{q'}^L \sim 1$, $z_q^N \sim 2$, $q \sim 0.4$, and $q' \sim 0.2$. However, the result of our analysis shows that the induced emission/absorption (quasilinear process) is the dominant term for the excitation of N mode, although the induced scattering effect leads to gradual damping of the mode in later times.

The result of the comparative numerical studies is presented in Fig. 5. The two panels show the respective numerical result in which one of the two competing effects are ignored. Figure 5 clearly shows that the induced emission is the primary cause of the wave growth. As is well-known, induced emission (or quasilinear process) is largely driven by the positive gradient in the velocity distribution, such that the quasi-saturation of the waves should be concomitant with the particle plateau formation. Comparison of Fig. 1 and Figs. 4 or 5 (left-hand panel) shows that, indeed, the saturation of N mode largely follows the plateau formation associated with the electron beam distribution.

This finding is highly relevant to the earlier findings by Klimas,[14] who reports on the basis of his Vlasov simulations that the harmonic modes begin to grow even in the linear stage, and that the initial growth rates of the harmonic modes are higher than the linear growth rate of the primary Lang-

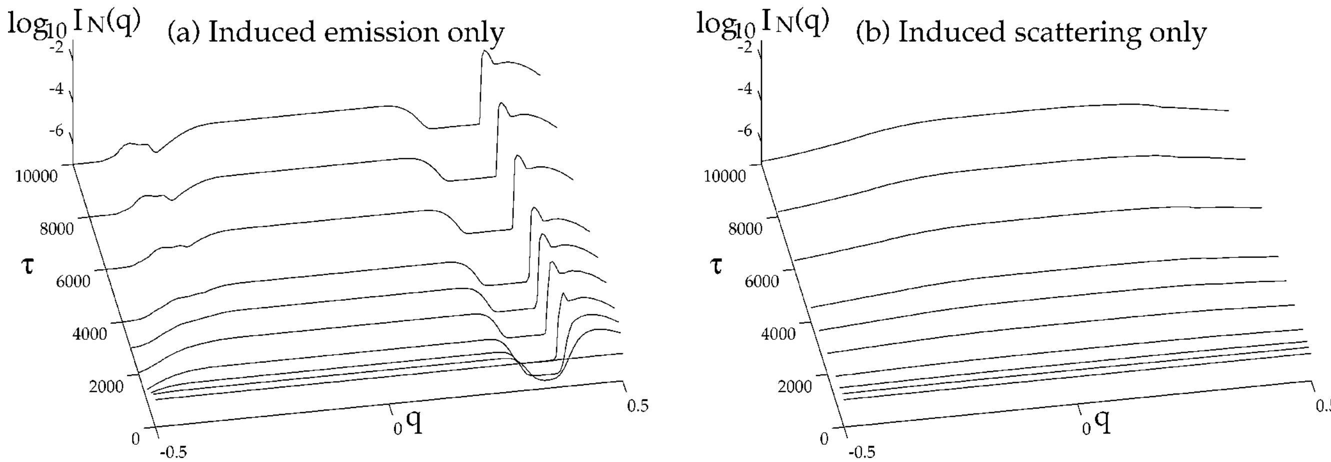

FIG. 5. Comparative study of the effects due to induced emission and induced scattering. The result clearly shows that the induced emission process is the primary effect responsible for the generation of the harmonic Langmuir mode [panel (a)], although panel (b) shows that the scattering leads to damping of the harmonic mode at later times.

muir mode roughly by the mode number ($\gamma_{Ln}^{\max} \sim n\gamma_{L1}^{\max}$). The growth of the harmonic mode during the linear stage is easily explained by the fact that the quasilinear process is the dominant wave generation mechanism for N mode, which is equivalent to $L2$ mode in the general scheme of harmonic Langmuir modes of all order, Ln. The induced emission process, which is essentially a linear instability in nature, causes the harmonic mode to amplify during the same time period over which the primary Langmuir waves amplify.

The observation by Klimas that the harmonic modes possess higher initial growth rates than the fundamental Langmuir mode (although harmonic modes saturate at much lower levels) can also be explained. From Eq. (32), the linear growth rate of the primary Langmuir mode L (or, $L1$) is given by

$$\gamma_L(q) = \gamma_{L1}(q) = \frac{\pi z_q^L}{q^2} \frac{\partial F_e(u)}{\partial u}\bigg|_{u=z_q^L/q}. \tag{38}$$

In contrast, the "linear" growth rate of the "nonlinear" eigenmode, N (or $L2$), can be defined by [see Eq. (34)]

$$\gamma_N(q) = \gamma_{L2}(q) = \frac{4\pi z_q^N}{q^2} \frac{\partial F_e(u)}{\partial u}\bigg|_{u=z_q^N/q}. \tag{39}$$

Noting the fact that γ_L should maximizes around $q \sim 0.2$, while γ_N should possess the peak around $q \sim 0.4$, it can be seen clearly that

$$\gamma_N^{\max} \sim 2\gamma_L^{\max}, \quad \text{or} \quad \gamma_{L2}^{\max} \sim 2\gamma_{L1}^{\max}. \tag{40}$$

In general, we expect that

$$\gamma_{Ln}^{\max} \sim n\gamma_{L1}^{\max}, \tag{41}$$

although the present analysis is restricted to $n=2$ only. Despite the fact that the N mode grows twice as fast as the fundamental L mode, we again note that it derives its free energy from the beam. As the beam flattens out forming a plateau, the free energy source of the harmonic mode (and the fundamental mode, for that matter) is exhausted. Consequently, the saturation of the harmonic mode occurs at the same time period during which the primary Langmuir mode undergoes the quasilinear saturation. Since the harmonic mode starts to grow from a much lower initial level when compared with the L mode, the saturation amplitude of N mode is also comparatively lower than that of the L mode. This explains why the harmonic mode saturates at a much lower level although it grow twice as fast initially. The basic reason is the initially low level of harmonic mode which leads to low subsequent saturation level.

Within the framework of the present collisionless theory, the choice of initial wave level is arbitrary, although from physical grounds we know that the harmonic Langmuir mode should possess much lower initial wave level than that of the fundamental Langmuir mode, since the notion of harmonic mode is ill-defined when the plasma is quiescent. The level of harmonic mode in a thermal plasma must be determined from the theory of spontaneous fluctuation in which the effects of nonlinear mode coupling is incorporated. This is beyond the scope of the present analysis. However, we may discuss the effects of initial choice of N mode on the later time development of the same mode, by simply choosing a different number for the wave intensity at $t=0$. Therefore, we have considered a case of initial N-mode with one-tenth the wave level as considered before, namely, $I_N(q,0) = 2\times 10^{-7}$. The result is plotted in Fig. 6, where the earlier case of $I_N(q,0) = 2\times 10^{-6}$ and the new case are plotted side by side, in a format where the fundamental Langmuir intensity and the harmonic mode intensity are plotted in a combined logarithmic plot, $\log_{10}[I_L(q,\tau) + I_N(q,\tau) + \epsilon]$, where $\epsilon = 1\times 10^{-6}$ is added, as before, to aid visual presentation. The same results are plotted in a 2D format in Fig. 7, in order to aid the readers read the vertical scales more or less accurately, although superposition of curves makes it difficult to distinguish each curve. The dashed lines are the initial levels of L and N modes, respectively. The best way to look at the present result is compare both Figs. 6 and 7. Note that for case (b), which corresponds to lower initial N-mode level, the saturated N-mode level is much lower than that of case (a). Note also that the primary and backscattered Langmuir

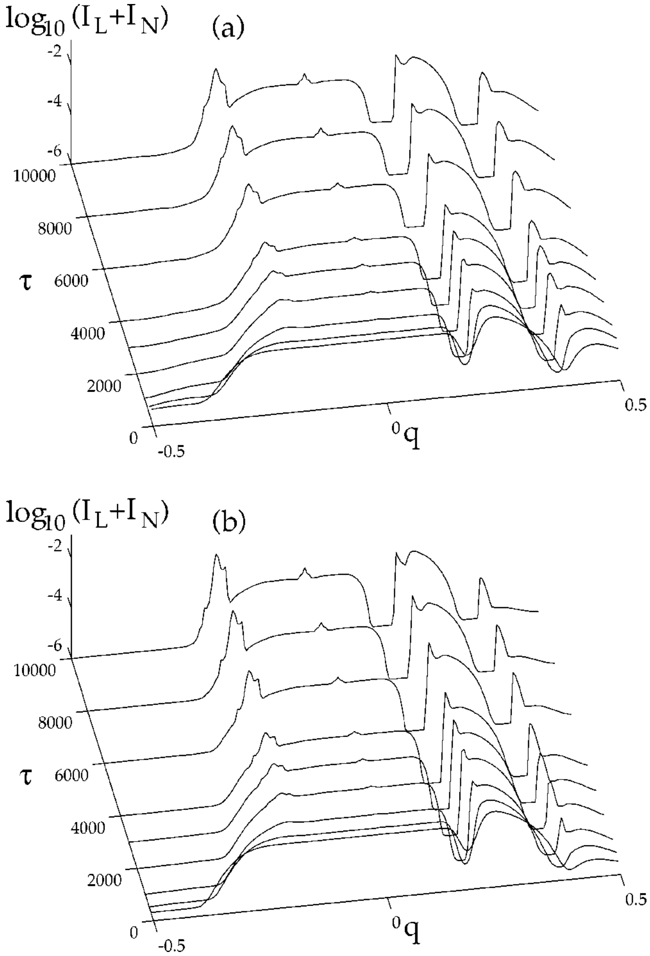

FIG. 6. Plot of combined wave intensity, $\log_{10}[I_L(q,\tau)+I_N(q,\tau)+1\times10^{-6}]$ vs q and τ for the case when the initial N-mode wave level is (a) $I_N(q,0)=2\times10^{-6}$, and when it is (b) $I_N(q,0)=2\times10^{-7}$. For both cases, the initial Langmuir and ion-sound (not shown) mode intensities are the same, $I_L(q,0)=I_S(q,0)=2\times10^{-4}$.

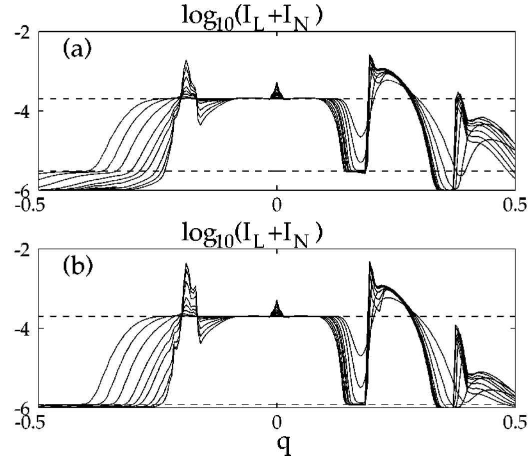

FIG. 7. The same plot as in the previous figure, except that combined wave intensity, $\log_{10}[I_L(q,\tau)+I_N(q,\tau)+1\times10^{-6}]$, is plotted versus q for cases (a) and (b).

waves possess slightly higher peaks in the case of lower initial N-mode level (b), than case (a). This can be attributed to the fact that, even though the harmonic mode does not greatly affect the evolution of L mode, it nevertheless affects the L mode, since the harmonic mode extracts wave energy from L mode.

IV. CONCLUSIONS AND DISCUSSION

In this article, we have numerically solved a one-dimensional version of the generalized weak turbulence equation for the first time. The generalized weak turbulence theory[52] incorporates the harmonic Langmuir mode as part of the eigenmode system in a turbulent plasma. Textbook plasma theory only considers Langmuir and ion-sound modes as the eigenmodes of an unmagnetized plasma interacting through electrostatic field in a uniform medium. In the conventional view, plasma turbulence is described in terms of mode coupling among Langmuir and ion-sound modes. The generalized weak turbulence theory considers the harmonic mode generation as part of the basic turbulent beam-plasma interaction process. The present numerical analysis shows that the harmonic mode grows from a low initial level to a finite level due to an induced emission process, which is essentially a linear instability process. That is, the generation of "nonlinear" eigenmode is dictated by a "linear" wave-particle interaction process. The initial growth rate of the $2\omega_{pe}$ harmonic mode is shown to be twice as high as the bump-on-tail instability growth rate, although the harmonic mode saturates at a much lower level than that of the Langmuir mode. The basic reason is because the primary excitation and saturation mechanisms for both the Langmuir and harmonic modes are the same linear growth and quasilinear saturation by plateau formation as in the classic bump-on-tail instability and quasilinear saturation theory. The initially much lower level of the harmonic mode when compared with the fundamental Langmuir wave level is thus, directly reflected in the saturated levels of the two modes.

The excitation of the $2\omega_{pe}$ harmonic Langmuir mode (and even higher-order harmonics as well, $3\omega_{pe}$, $4\omega_{pe}$, etc.) is known since the late 1960s, first discovered through laboratory experiments.[45–47,10–12] An early simulation study by Joyce *et al.*[13] also revealed the existence of such a mode. Early theoretical attempts to explain such a phenomenon were largely based upon the trapped particle dynamics.[4,5,49] The harmonic Langmuir mode phenomemon was independently rediscovered in the 1980s by Klimas through his Vlasov simulations[14,16] and confirmed by Nishikawa and Cairns in their particle simulation.[18] More recent carefully designed particle-in-cell and Vlasov simulations[21–23] revealed that the ($2\omega_{pe}$) harmonic Langmuir mode possesses broad spectrum comparable to the fundamental Langmuir mode, and that such a mode is better characterized as a legitimate eigenoscillation of a plasma. This means that it is better to describe the generation of broad-spectrum harmonic mode in terms of random-phase turbulence theory, rather than in terms of coherent theories such as trapping theory. Generalized weak turbulence theory[52] does just that, as it treats the harmonic Langmuir mode as a solution of a nonlinear dispersion equation, hence, a nonlinear eigenmode.

The present numerical solution confirms a number of features associated with the harmonic mode as revealed through simulation studies. These include the excitation of the harmonic mode during the linear stage, and the comparable spectral characteristics of the fundamental and harmonic Langmuir modes.

At this point, it is appropriate remark on the following relevant fact: In an earlier theoretical discussion by Yoon[54] on the so-called nonlinear beam instability, he predicts that the nonlinear beam mode, which is essentially the same as the present harmonic Langmuir mode in that it is a solution to a nonlinear dispersion equation with frequency in the vicinity of $2\omega_{pe}$, should grow in a linear fashion. This is in qualitative agreement with the present finding. However, the major difference is that the nonlinear beam instability predicts long-wavelength mode ($k\approx 0$ range of the harmonic mode) to grow, whereas the present numerical results show that short-wavelength harmonic mode ($k\approx\omega_{pe}/v_b$) is excited with no evidence for the excitation of long-wavelength harmonic Langmuir mode.

The limitations of the present study are, first, that the single-particle fluctuations are ignored. As such, the choice of the initial wave spectrum for both linear modes, i.e., Langmuir and ion-sound modes, as well as the harmonic mode becomes arbitrary. From physical grounds, the harmonic mode at the initial stage cannot be very high. However, the present theory cannot dictate the relative level of harmonic to fundamental Langmuir wave level. The second limitation is the fact that the present study is limited to first harmonic ($2\omega_{pe}$) mode only, although this deficiency can be easily overcome by a straightforward generalization of the present theory to include higher-order modes. These are the subject of future studies.

ACKNOWLEDGMENTS

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