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ASYMPTOTIC SPECTRAL ANALYSIS OF GROWING GRAPHS AND ORTHOGONAL MATRIX-VALUED POLYNOMIALS

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Resumo. Neste trabalho abordaremos a análise espectral de grafos por dois estudos: técnicas de probabilidade quântica e por polinômios ortogonais com valores em matrizes. No Capítulo 1, consideraremos a matriz de adjacência do grafo tal como um operador linear e sua decomposição quântica permitirá uma análise espectral que produzirá um teorema do limite central para tal grafo. No Capítulo 2, consideraremos uma medida com valores em matrizes induzida por polinômios ortogonais com valores em matrizes. Sob certas condições, é possível exibir explicitamente uma expressão de tal medida. Algumas aplicações em teoria dos grafos são dadas quando nos restringimos às matrizes estocásticas e com valores em 0-1. Do nosso conhecimento, os cálculos e exemplos obtidos nas seções 0.3.2, 0.3.3, 2.4 e 2.5 são novos.

Abstract. In this work we focus on the spectral analysis of graphs via two studies: quantum probabilistic techniques and by orthogonal matrix-valued polynomials. In Chapter 1 we consider the adjacency matrix of a graph as a linear operator, and its quantum decomposition will allow a spectral analysis that will produce a central limit theorem for such graph. In Chapter 2, we consider a matrix-valued measure induced by orthogonal matrix-valued polynomials. Under certain conditions, it is possible to display an explicit expression for such measure. Some applications to combinatorics and graph theory are given when we restrict to the stochastic and 0-1 matrices. Up to our knowledge, the calculations and examples obtained in sections 0.3.2, 0.3.3, 2.4 and 2.5 are new.

Contents

0.1 Agradecimentos

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0.2 Introduction

In this work we are interested in the spectral analysis of graphs. By this we mean that we study the spectrum of the adjacency matrix associated with the graph and here this is done by two different approaches: first, we describe the work due to A. Hora and N. Obata [7], where a sequence of finite graphs is considered, for which the number of vertices tends to infinity in some prescribed way, and then an asymptotic theorem on their spectra is obtained. In a second part, we study the description due to A. Duran [5], where spectral information is obtained via the analysis of matrix-valued polynomials. We use the results of A. Duran work's to obtain spectral information of block tridiagonal graphs in the aim to resolve counting problems on a specified class of graphs.

Unless otherwise specified, a graph is always assumed to be locally finite and connected (see definition in section 0.4.1). In Chapter 1, we will mostly work with distance-regular graphs. Such graphs possess high symmetry and then the so-called method of quantum decomposition will be most effective. A connected graph $G = (V, E)$ is called *distance-regular* if, for any choice of $x, y \in V$ with $d(x, y) = k$, the number

$$p^k_{ij} = |\{z \in V; d(x, z) = i, d(y, z) = j\}|$$

depends only on i, j, k. The numbers p^k_{ij} are called the *intersection numbers* of $G = (V, E)$.
Once we choose an initial vertex $o \in V$, we want a measure such that

$$\langle A^m \rangle_o = \int_{-\infty}^{+\infty} x^m \mu(dx), m = 1, 2, \dots,$$

where A is the *adjacency matrix* of the distance-regular graph G (defined in Section 0.4.1). Notation: $\langle A \rangle_x$ is the (x, x) entry of the matrix A, i.e, $\langle A \rangle_x = A_{xx}$.

For our analysis, we will make use of orthogonal polynomials: let μ a probability measure on $\mathbb{R}$ having finite moments of all orders (i.e. $\int_{-\infty}^{+\infty} |x|^m \mu(dx) < \infty$). Consider the Hilbert space $L^2(\mathbb{R}, \mu)$ with the inner product $< f, g >= \int_{-\infty}^{+\infty} \overline{f(x)} g(x) \mu(dx)$. We apply the Gram-Schmidt orthogonalization procedure to the monomials $\{1, x, x^2, \dots\}$ and we obtain a sequence of polynomials $P_0(x) = 1, P_1(x), P_2(x), \dots$ Each $P_n(x)$ is a polynomial of degree n, they are orthogonal to each other and we can assume them to be monic. These polynomials are called the *orthogonal polynomials* associated with μ, and they satisfy the following:

Theorem 1. [12] **(Three-term recurrence relation)** Let $\{P_n(x)\}$ be the orthogonal polynomials associated with a probability measure μ on $\mathbb{R}$, having finite moments of all orders.
Then there exists a pair of sequences $\alpha_1, \alpha_2, \dots \in \mathbb{R}$ and $\omega_1, \omega_2, \dots \in \mathbb{R}_+$ uniquely determined by

$$\begin{cases} P_0(x) = 1, \\ P_1(x) = x - \alpha_1, \\ xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n \geq 1. \end{cases}$$

The main result of Chapter 1 is given by the following theorem:

Theorem 2. (Central Limit Theorem for a Growing Distance-Regular Graph) *[[7], chapter 3 page 96]* Let $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing distance-regular graph with adjacency matrix A_ν, with degree $\kappa(\nu)$, (defined in Section 0.4.1) and intersection numbers $p^k_{i,j}(\nu)$. If the following limits exist,

$$\tilde{\omega}_n := \lim_{\nu\to\infty} \frac{p^n_{1,n-1}(\nu)p^{n-1}_{1,n}(\nu)}{\kappa(\nu)}, \qquad \tilde{\alpha}_n := \lim_{\nu\to\infty} \frac{p^{n-1}_{1,n-1}(\nu)}{\sqrt{\kappa(\nu)}}.$$

and if $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots \in \mathbb{R}$ and $\tilde{\omega}_1, \tilde{\omega}_2, \ldots \in \mathbb{R}^*_+$, then we have:

$$\lim_{\nu\to\infty} \left\langle \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m \right\rangle_o = \int_{-\infty}^{+\infty} x^m \mu(dx), m = 1, 2, \ldots,$$

where μ is a probability measure with Jacobi coefficient $(\{\tilde{\omega}_n\}, \{\tilde{\alpha}_n\})$ (defined in Section 1.2.1).

The idea of the proof is: i) Use the quantum decomposition of the adjacency matrix $A_\nu = A^+_\nu + A^o_\nu + A^-_\nu$. ii) Compute the limit of the normalized quantum components over the vacuum state as operators of an interacting Fock Space. Then the result is obtained by the following property that establishes a relation between a measure and an interacting Fock space:

Proposition 1. *[[7], chapter 1 page 25]* Let μ be a probability measure on $\mathbb{R}$, having finite moments of all orders and $(\{\omega_n\}, \{\alpha_n\})$ be its Jacobi coefficient. Consider the interacting Fock space $\Gamma_{(\{\omega_n\},\{\alpha_n\})} = (\Gamma, \{\Phi_n\}, B^+, B^-, B^o)$. Then,

$$\left\langle \Phi_0, \left(B^+ + B^- + B^o\right)^m \Phi_0 \right\rangle_\Gamma = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots$$

In the previous equation, the inner product is the one from the Hilbert Space (it will be defined at section 1.2.1). This inner product has a good behaviour when submitted to limits. This is why we will be able to work with growing graphs.

Let us give a definition to clarify the statement of Proposition 1. An *interacting Fock space associated with a Jacobi coefficient* $(\{\omega_n\}, \{\alpha_n\})$ is a quintuple $\Gamma_{(\{\omega_n\},\{\alpha_n\})} = (\Gamma, \{\Phi_n\}, B^+, B^-, B^o)$ where Γ is a pre-Hilbert space, $\{\Phi_n\}$ is an orthonormal basis of Γ and B^+, B^-, B^o are linear operators on Γ such that:

$$\begin{cases} B^+\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1}, \quad n \geq 0, \\ B^-\Phi_0 = 0, \qquad B^-\Phi_n = \sqrt{\omega_n}\Phi_{n-1}, \quad n \geq 1, \\ B^o\Phi_n = \alpha_{n+1}\Phi_n \quad n \geq 0. \end{cases}$$

The inner product of Γ is denoted by $< \cdot, \cdot >_\Gamma = < \cdot, \cdot >$.

Going back to Theorem 2, if we specify the kind of distance-regular graph we are considering, we will be able to express the measure explicitly. To illustrate this, we consider the homogeneous trees. A tree is called *homogeneous* if it is regular. These are always infinite graphs. We denote by T_κ a homogeneous tree with degree $\kappa \geq 2$ and by $A = A_\kappa$ its adjacency matrix.

Figure 1: Homogeneous Tree T_3

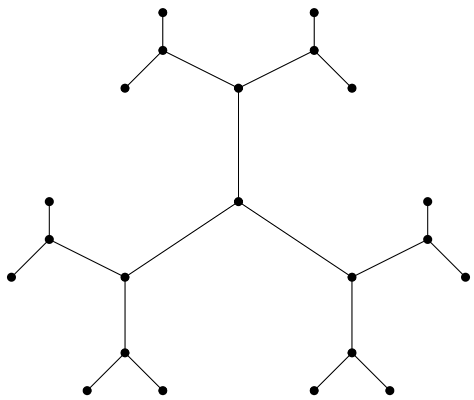
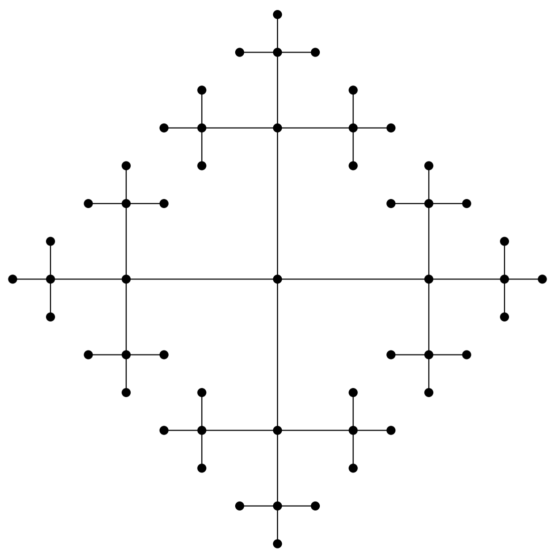


Figure 2: Homogeneous Tree T_4



Their intersection numbers are:

$$p^{n-1}_{1,n}(\kappa) = \kappa - 1, \qquad p^{n}_{1,n-1}(\kappa) = 1, \qquad p^{n}_{1,n}(\kappa) = 0.$$

Then the Jacobi coefficient $(\{\tilde{\omega}_n\}, \{\tilde{\alpha}_n\})$ of the measure given by Theorem 2 are:

$$\tilde{\omega}_n = \lim_{\kappa \to \infty} \frac{p^{n}_{1,n-1}(\kappa) p^{n-1}_{1,n}(\kappa)}{\kappa} = 1, \qquad \tilde{\alpha}_n = \lim_{\kappa \to \infty} \frac{p^{n-1}_{1,n-1}(\kappa)}{\sqrt{\kappa}} = 0.$$

The interacting Fock space associated with the Jacobi coefficient given by $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$ is called the *free Fock space* and is denoted by Γ_{free}.

Applying Proposition 1, we have that the measure with the Jacobi coefficient given by $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$ satisfies:

$$\int_{-\infty}^{+\infty} x^m \mu(dx) = \left\langle \Phi_0, \left(B^+ + B^-\right)^m \Phi_0 \right\rangle_{\Gamma_{\text{free}}} \qquad m = 1, 2, \ldots$$

where $B^+ + B^-$ are the operators of the free Fock space.

The *Wigner semicircle law* is defined as the probability measure whose density function is given by

$$\rho(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2. For the free Fock space, it holds that

$$\left\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \right\rangle_{\Gamma_{\text{free}}} = \int_{-\infty}^{+\infty} x^m \rho(dx) \quad m = 1, 2, \ldots$$

where $\rho(dx)$ is the Wigner semicircle law.

Then we have

$$\int_{-\infty}^{+\infty} x^m \mu(dx) = \int_{-\infty}^{+\infty} x^m \rho(dx) \quad m = 1, 2, \ldots$$

and as ρ is the solution of a determinate moment problem, we have that $\mu = \rho$ and then, the corresponding Theorem for homogeneous trees is:

Theorem 3. (Central Limit Theorem for homogeneous trees) *[[7], chapter 4 page 110]* Let T_κ be a homogeneous tree of degree κ with adjacency matrix A_κ. Then we have

$$\lim_{\kappa \to \infty} \left\langle \left(\frac{A_\kappa}{\sqrt{\kappa}} \right)^m \right\rangle_o = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2} dx, \quad m = 1, 2, \ldots,$$

where the probability measure on the right-side is the Wigner semicircle law.

In Chapter 2, our aim is to display an expression for a matrix-valued measure under specific conditions. Let $A_0, A_1, \ldots, A_n$ be matrices in $\mathbb{C}^{N \times N}$ with $A_n \neq 0$. We say that $P : \mathbb{R} \longrightarrow \mathbb{C}^{N \times N}$, with $P(t) = A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0$, is a *matrix-valued polynomial* of degree n.

Let $W : \mathbb{R} \longrightarrow \mathbb{C}^{N \times N}$ a bounded Hermitian matrix-valued function with $W(t) \prec W(u)$, if $t < u$ (defined in section 0.4.4).

The left-inner product is $< P, Q > := \int_{-\infty}^{+\infty} P(t) dW(t) Q(t)^*$. We have the following properties, which are proved in Section 2.1:

1. $< P, Q >= < Q, P >^*$

2. $< c_1P_1 + c_2P_2, Q >= c_1 < P_1, Q > +c_2 < P_2, Q >, \quad \forall c_1, c_2 \in \mathbb{C}^{N\times N}$

3. $< tP, Q >=< P, tQ >, \quad t \in \mathbb{R}$

Orthonormal polynomials $P_n, n = 0, 1, 2, \ldots$ are then obtained with the Gram-Schmidt procedure by normalizing $I, tI, t^2I, t^3I, \ldots$, using the left-inner product.

$$< P_n, P_m >= \int_{-\infty}^{+\infty} P_n(t)dW(t)P_m(t)^* = \delta_{n,m}I \qquad n, m \geq 0,$$

where $\delta_{n,m} = 1$ if $n = m$, and $\delta_{n,m} = 0$ otherwise.

As in the scalar case, the sequence of orthonormal matrix polynomials admits a three-term recurrence relation:

Theorem 4. *[[15], page 32]* For $n = 0, 1, 2, \ldots$ suppose P_n are left orthonormal matrix polynomials with respect to the left inner product obtained by using a matrix measure W on the real line. Then there exist invertible matrices A_n and Hermitian matrices B_n such that

$$\begin{cases} P_{-1}(t) = 0, \ P_0(t) \in \mathbb{C}^{N\times N}\backslash\{0\}, \\ tP_n(t) = A_nP_{n+1}(t) + B_nP_n(t) + A_{n-1}^*P_{n-1}(t), \qquad n \geq 0. \end{cases}$$

Let T be the following block tridiagonal matrix:

$$T = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_0^* & B_1 & A_1 & 0 & 0 & \\ 0 & A_1^* & B_2 & A_2 & 0 & \\ 0 & 0 & A_2^* & B_3 & A_3 & \\ \vdots & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Above we use the notation: if T is a block matrix, the matrix T_{ij} is the (i, j)-block of matrix T. In the above matrix, we have: $T_{00} = B_0, T_{01} = A_0, T_{10} = A_0^* \ldots$

With this, the previous recursion becomes

$$TP(t) = tP(t), \quad \text{with } P(t) := \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ \vdots \end{bmatrix}$$

This relation can be iterated:

$$T^nP(t) = t^nP(t),$$

which, together with the orthonormality of P_n, gives us an expression for the i, j entry of the block matrix T^n:

$$(T^n)_{ij} = \int_{-\infty}^{+\infty} t^n P_i(t) dW(t) P_j(t)^*.$$

The expression above is known as the Karlin-McGregor formula (see [2], p.469).

From the Karlin-McGregor formula, an interpretations for the matrix-valued measure arises in the context of random walks, and another interpretation arise in the context of number of closed walks on a block tridiagonal graph. In Section 0.3, we will see a concrete example of a matrix-valued measure over a random walk.

Aiming to display an explicit expression for the matrix-valued measure, we will assume the additional hypotheses:

- Both sequences of matrices are constant, i.e. $A_n = A$ and $B_n = B$ for $n = 0, 1, 2, \dots$
- Matrix A is positive-definite, and $P_0(t) = I$.

As all B_n are Hermitian, the same is valid for B. Since matrix A is positive-definite, it is Hermitian and then $A^* = A$.

With these assumptions, the three-term recurrence relation is reformulated as:

$$\begin{cases} P_{-1}(t) = 0, P_0(t) = I, \ A \text{ positive-definite}, \ B \text{ Hermitian}, \\ tP_n(t) = AP_{n+1}(t) + BP_n(t) + AP_{n-1}(t), \qquad n \geq 0. \end{cases}$$

The matrix measure for which the polynomials P_n are orthonormal is denoted by $W = W_{A,B}$. We also will assume that $W_{A,B}$ is a positive-definite matrix.

Observation: These polynomials are known as Chebyshev matrix polynomials of the second kind in analogy with the scalar case.

For all $z \in \mathbb{R}$, let $H_{A,B}(z) := A^{-1/2}(zI - B)A^{-1}(zI - B)A^{-1/2} - 4I$. For all $x \in \mathbb{R}$, the matrix $-H_{A,B}(x)$ admits the following orthogonal diagonalization:

$$-H_{A,B}(x) = U(x)D(x)U^*(x),$$

where $D(x)$ is a diagonal matrix with entries $d_{i,i}(x), i = 1, \dots, N$, and $U(x)$ is an unitary matrix.

With the additional hypotheses above, we will be able to display an expression for $dW_{A,B}$, and this is given by the following theorem, which is the main result of Chapter 2.

Theorem 5. *[[5], page 327]* If A is positive-definite and B Hermitian, the matrix weight $W_{A,B}$ for the Chebyshev matrix polynomials of the second kind is the matrix-valued measure given by:

$$dW_{A,B}(x) = \frac{1}{2\pi} A^{-1/2} U(x) (D^{+}(x))^{1/2} U^{*}(x) A^{-1/2} dx,$$

where $D^{+}(x)$ is the diagonal matrix with entries

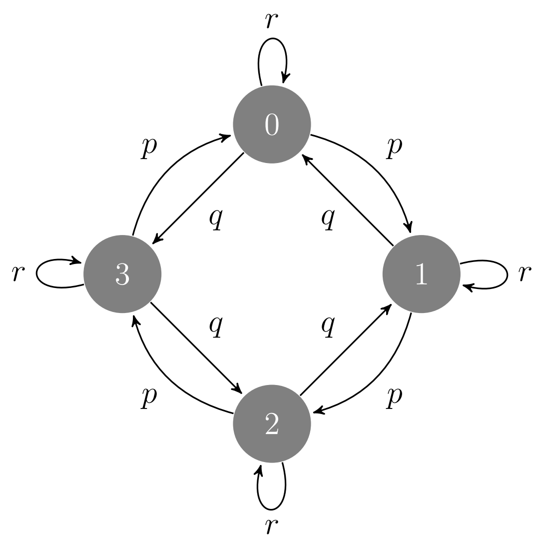
$$d^{+}_{i,i}(x) = max\{d_{i,i}(x), 0\}.$$

The strategy of the proof is first to use a known theorem about ratio polynomials in the three-term recurrence relation to produce an equation for which the unknown is the Stieltjes transform of the measure. The main result follows then by the inversion formula for the Stieltjes transform.

0.3 Applications

0.3.1 Random Walks

Imagine that you have a square with its 4 vertices named 0,1,2,3. Then a particle that starts at one of the 4 vertices can jump to one of its neighbours. Let $p, q, r \in [0, 1]$ with $p + q + r = 1$, associated with the possibilities: it can move clockwise with probability p, it can move counterclockwise with probability q, and it can rest with probability r. Let $n \in \mathbb{N}$. Starting at vertex i, what is the probability of arrival at vertex j after n jumps?



We will present two solutions. The first one uses traditional random walk theory. If we set P as the transition matrix such that P_{ij} is the probability to go from vertex i to vertex j in one step, we have:

$$P = \begin{bmatrix} r & p & 0 & q \\ q & r & p & 0 \\ 0 & q & r & p \\ p & 0 & q & r \end{bmatrix}$$

Then the answer is the probability P^n_{ij}.

The second solution is using the matrix version of the Karlin-McGregor formula. Consider a homogeneous Markov chain (see [10]) with state space

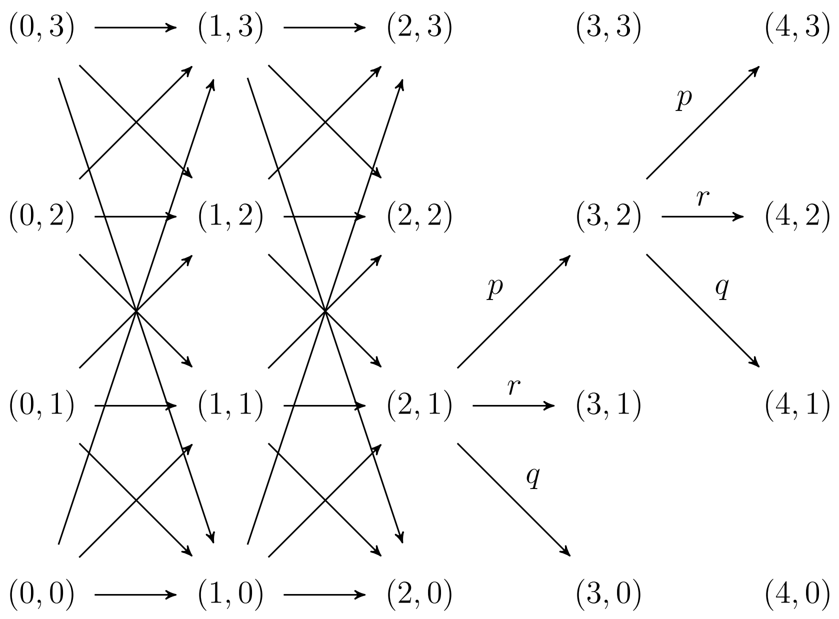
$$\mathcal{C}_N = \{(i,j) \in \mathbb{N} \times \mathbb{N} \,|\, 0 \le j \le N-1\},$$

The state (i,j) means : to be, at time i, in vertex j.

Consider the one-step transition $T = (T_{ii'})_{i,i'=0,1,\ldots}$

$$T = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \cdots & 0 \\ C_0 & B_1 & A_1 & 0 & \cdots & 0 \\ 0 & C_1 & B_2 & A_2 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & C_{d-3} & B_{d-2} & A_{d-2} \\ 0 & 0 & \cdots & 0 & C_{d-2} & B_{d-1} \end{bmatrix}.$$

The matrix T is a block tridiagonal matrix, (see definition in section 0.4.2) with blocks N by N. The probability of going in one step from state (i,j) to (i',j') is given by the element in the position (j,j') of the matrix $T_{i,i'}$. We have $T_{i,i} = B_i, T_{i,i+1} = A_i$, and $T_{i+1,i} = C_i$ The matrix T^n is the n-step transition matrix, and the element in the position (k,l) of $(T^n)_{ij}$ is the probability of going in n steps from state (i,k) to (j,l). Therefore, the Karlin-McGregor formula gives an integral representation with matrix-valued measure for the probability of going in n steps from a state to another.



In the particular case of the square, we set a random walk over the state space

$$\mathcal{C}_4 = \{(i,j) \in \mathbb{N}^* \times \mathbb{N} | 0 \leq j \leq 3\}.$$

Here, i is the number of steps and j is the vertex. The one-step transition matrix $T = (T_{ii'})_{i,i'=0,1,\ldots}$ as above is given by:

$$T_{i,i+1} = A_i = \begin{bmatrix} r & p & 0 & q \\ q & r & p & 0 \\ 0 & q & r & p \\ p & 0 & q & r \end{bmatrix},$$

$$T_{i,i} = B_i = 0,$$

$$T_{i+1,i} = C_i = 0.$$

Then the answer is the probability $(T^n)_{0,n}(i,j)$ i.e. the element (i,j) of the 4×4 matrix $(T^n)_{0,n}$ (the block in position $(0,n)$) given by Karlin-McGregor formula

$$(T^n)_{0,n} = \int_{-\infty}^{+\infty} t^n P_0(t) dW(t) P_n(t)^*.$$

Comparing the two solutions, we may conclude that:

$$P^n = (T^n)_{0,n} \qquad n = 1, 2, \ldots$$

and

$$P^n_{ij} = (T^n)_{0,n}(i,j) \qquad n = 1, 2, \ldots, \qquad 0 \leq i, j \leq 3.$$

0.3.2 Closed Walks on the Semi Infinite Ladder Graph

In this section, we will calculate the number of n-step closed walks on the semi infinite ladder graph $K^2\Box R$ (the product $\Box$ is the Cartesian product of graphs defined in section 0.4.2) starting first from a vertex $(x, x_0) \in V(G_N)$, with $x \in K^2, x_0 \in R$ and then from a vertex $(x, x_i) \in V(G_N)$, with $x \in K^2, x_i \in R$. Up to our knowledge, the following calculations are new.

The semi infinite ladder graph has adjacency matrix:

$$M = \begin{bmatrix} B & A & 0 & 0 & \cdots \\ A & B & A & 0 & \\ 0 & A & B & A & \\ \vdots & & & & \ddots \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We can define polynomials that satisfy the three-term recurrence relation:

$$P_{-1}(t) = 0, P_0(t) = I,$$

$$tP_n(t) = P_{n+1}(t) + BP_n(t) + P_{n-1}(t), \qquad n \geq 0.$$

There is a measure W for which the polynomials P_n are orthonormal, and as A is positive-definite and B is Hermitian, we can calculate dW with the result of Theorem 5. In fact, in Section 2.2, we have calculated dW for the matrix given. The result is:

$$dW(x) = \frac{1}{4\pi}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\sqrt{-x^2+2x+3}\,\chi_{[-1,3]} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\sqrt{-x^2-2x+3}\,\chi_{[-3,1]}\right)dx.$$

By the Karlin-McGregor formula, the number of n-step closed walks starting from an initial vertex (x, x_0) of the semi infinite ladder graph is the first element of the first block of the matrix (M^n) given by:

$$(M^n)_{0,0} = \int_{-\infty}^{+\infty} t^n P_0(t) dW(t) P_0(t).$$

Then we have:

$$(M^n)_{0,0} = \int_{-\infty}^{+\infty} t^n dW(t) =$$

$$\int_{-\infty}^{+\infty} t^n \frac{1}{4\pi}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\sqrt{-t^2+2t+3}\,\chi_{[-1,3]} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\sqrt{-t^2-2t+3}\,\chi_{[-3,1]}\right)dt =$$

$$\frac{1}{4\pi}\left(\int_{-1}^{3} t^n\sqrt{-t^2+2t+3}\,dt\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \int_{-3}^{1} t^n\sqrt{-t^2-2t+3}\,dt\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) =$$

$$\frac{1}{4\pi}\left(\int_{-2}^{2}(z+1)^n\sqrt{4-z^2}\,dz\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \int_{-2}^{2}(y-1)^n\sqrt{4-y^2}\,dy\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) =$$

$$\frac{1}{\pi}\left(\int_{-\pi}^{0}(2\cos\theta+1)^n\sin^2\theta\,d\theta\left[\begin{array}{cc}1&1\\1&1\end{array}\right]+\int_{-\pi}^{0}(2\cos\theta-1)^n\sin^2\theta\,d\theta\left[\begin{array}{cc}1&-1\\-1&1\end{array}\right]\right),$$

with the change of variables $z=t-1, y=t+1, \theta=\arccos(y/2), \theta=\arccos(z/2)$.

Let n be even, with $n=2m$. The first element of $(M^{2m})_{0,0}$ is given by:

$$\frac{1}{\pi}\int_{-\pi}^{0}\left((2\cos\theta+1)^{2m}+(2\cos\theta-1)^{2m}\right)\sin^2\theta\,d\theta=$$

$$\frac{1}{\pi}\int_{-\pi}^{0}\sum_{k=0}^{m}\left(\left(\begin{array}{c}2m\\2k\end{array}\right)2\cdot 2^{2k}\cos^{2k}\theta\sin^2\theta\right)\,d\theta=$$

$$\sum_{k=0}^{m}\left(\left(\begin{array}{c}2m\\2k\end{array}\right)2^{2k+1}\frac{1}{\pi}\int_{-\pi}^{0}\cos^{2k}\theta\sin^2\theta\,d\theta\right).$$

By integral formula (4) in Section 0.4.7, we have that the number of n-step closed walks, for n even, starting from an initial vertex is:

$$\sum_{k=0}^{m}\left(\begin{array}{c}2m\\2k\end{array}\right)2^{2k+1}\frac{1}{2^{2k+1}}|\mathcal{C}_k|=\sum_{k=0}^{m}\left(\begin{array}{c}2m\\2k\end{array}\right)|\mathcal{C}_k|=\sum_{k=0}^{n/2}\left(\begin{array}{c}n\\2k\end{array}\right)|\mathcal{C}_k|,$$

where $|\mathcal{C}_k|$ is the k-th Catalan number (see Lemma 1 in Section 0.4.6).

Now, we will generalize the previous result to any $i\in\mathbb{N}$, i.e., we will calculate the number of n-step closed walks on the semi infinite ladder graph, starting from a vertex (x,x_i), which is at distance i from the initial vertex (x,x_0).

By the Karlin-McGregor formula, this number is the first element of the block ii of the matrix (M^n) given by:

$$(M^n)_{i,i}=\int_{-\infty}^{+\infty}t^nP_i(t)dW(t)P_i(t).$$

Then, as P_i and $(I\pm B)$ commute, we have:

$$(M^n)_{i,i}=\frac{1}{\pi}\int_{-\pi}^{0}(2\cos\theta+1)^n\sin^2\theta P_i(2\cos\theta+1)^2\left[\begin{array}{cc}1&1\\1&1\end{array}\right]d\theta+$$
$$+\frac{1}{\pi}\int_{-\pi}^{0}(2\cos\theta-1)^n\sin^2\theta P_i(2\cos\theta-1)^2\left[\begin{array}{cc}1&-1\\-1&1\end{array}\right]d\theta.$$

For each $i\in\mathbb{N}$, let $f_i(t), g_i(t)$ polynomials in t such that:

$$P_i(t)=\left[\begin{array}{cc}f_i(t)&g_i(t)\\g_i(t)&f_i(t)\end{array}\right]=f_i(t)I+g_i(t)B.$$

The polynomials P_n satisfy:

$$P_{-1}(t)=0, P_0(t)=I,$$

$$P_{n+1}(t) = P(t)P_n(t) - P_{n-1}(t), \qquad n \geq 0,$$

where $P(t) := tI - B$.

Then, for $n \geq 0$:

$$f_{n+1}I + g_{n+1}B = (tI - B)(f_nI + g_nB) - f_{n-1}I - g_{n-1}B = (tf_n - g_n - f_{n-1})I + (tg_n - f_n - g_{n-1})B.$$

Then,

$$\begin{cases} f_0(t) = 1, g_0(t) = f_1(t) = g_1(t) = 0, \\ f_{n+1}(t) = tf_n(t) - g_n(t) - f_{n-1}(t), \\ g_{n+1}(t) = tg_n(t) - f_n(t) - g_{n-1}(t). \end{cases}$$

For any x, we want to prove by induction on n that $f_n(x-1) - g_n(x-1) = f_n(x+1) + g_n(x+1)$ for all $n = 1, 2, \ldots$

For $n = 1$, we have:

$$\begin{cases} f_1(x-1) - g_1(x-1) = (x-1)f_0 - g_0 - f_{-1} - (tg_0 - f_0 - g_{n+1}) = x - 1 + 1 = x, \\ f_1(x+1) + g_1(x+1) = (x+1)f_0 + g_0 - f_{-1} + (tg_0 - f_0 - g_{n+1}) = x + 1 - 1 = x, \end{cases}$$

i.e. $f_1(x-1) - g_1(x-1) = f_1(x+1) + g_1(x+1)$. Suppose that $f_k(x-1) - g_k(x-1) = f_k(x+1) + g_k(x+1)$ for $k = 1, 2, \ldots, n$.

We have:

$$f_{n+1}(x-1) - g_{n+1}(x-1) = x(f_n(x-1) - g_n(x-1)) - (f_{n-1}(x-1) - g_{n-1}(x-1)) =$$

$$x(f_n(x+1) + g_n(x+1)) + (f_{n-1}(x+1) - g_{n-1}(x+1)) = f_{n+1}(x+1) + g_{n+1}(x+1).$$

Let $h_n\theta := f_n(2\cos\theta + 1) + g_n(2\cos\theta + 1)$. We have that the functions $h_n\theta$ satisfy:

$$h_0\theta = 1, h_1\theta = 2\cos\theta,$$

$$h_{n+1}\theta = 2\cos\theta h_n\theta - h_{n-1}\theta, \qquad n \geq 1.$$

The closed expression for $h_n\theta$ is:

$$h_n\theta = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j . \left(\begin{array}{c} n-j \\ j \end{array} \right) (2\cos\theta)^{n-2j}.$$

Then,

$$h_i^2\theta = \sum_{l,j=0}^{\lfloor i/2 \rfloor} (-1)^{l+j} \left(\begin{array}{c} i-l \\ l \end{array} \right) \left(\begin{array}{c} i-j \\ j \end{array} \right) (2\cos\theta)^{2i-2j-2l}.$$

The first element of the block ii of the matrix (M^n) is given by:

$$(M^n)_{i,i}(0,0) = \frac{1}{\pi} \int_{-\pi}^{0} ((2\cos\theta - 1)^n + (2\cos\theta + 1)^n) h_i^2\theta \sin^2\theta \, d\theta =$$

$$\sum_k \left(\begin{array}{c} n \\ 2k \end{array} \right) 2^{2k+1} \frac{1}{\pi} \int_{-\pi}^{0} (cos\theta)^{2k} h_i^2 \theta \sin^2\theta \, d\theta =$$

$$\sum_k \left(\begin{array}{c} n \\ 2k \end{array} \right) 2^{2k+1} \sum_{l,j} \left(\begin{array}{c} i-l \\ l \end{array} \right) \left(\begin{array}{c} i-j \\ j \end{array} \right) 2^{2(i-j-l)} (-1)^{l+j} \frac{1}{\pi} \int_{-\pi}^{0} (cos\theta)^{2(k+i-j-l)} \sin^2\theta \, d\theta.$$

By integral formula (4) in Section 0.4.7, we have that the number of n-step closed walks, for n even, starting from a vertex (x, x_i) is:

$$\sum_{k,l,j} \left(\begin{array}{c} n \\ 2k \end{array} \right) \left(\begin{array}{c} i-l \\ l \end{array} \right) \left(\begin{array}{c} i-j \\ j \end{array} \right) 2^{2(k+i-j-l)+1} (-1)^{l+j} \frac{1}{2^{2(k+i-j-l)+1}} (-1)^{l+j} |\mathcal{C}_{k+i-j-l}| =$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor i/2 \rfloor} \sum_{j=0}^{\lfloor i/2 \rfloor} \left(\begin{array}{c} n \\ 2k \end{array} \right) \left(\begin{array}{c} i-l \\ l \end{array} \right) \left(\begin{array}{c} i-j \\ j \end{array} \right) (-1)^{l+j} |\mathcal{C}_{k+i-j-l}|.$$

0.3.3 Closed Walks on $K^N \square R$

Let $N \geq 2$ a fixed natural number, and $G_N := K^N \square R$. In this section we count the number of closed walks on such graph. We could not find in the literature a detailed exposition of this matter, so we provide the details in what follows. We denote by $CW_N(n,i)$ the number of n-step closed walks starting from a vertex $(x, x_i) \in V(G_N)$, with $x \in K^N, x_i \in R$. The number $CW_N(n,i)$ is well-defined because of symmetric properties of G_N. By the Karlin-McGregor formula, $CW_N(n,i)$ is any diagonal element of the block ii of the matrix (M^n) given by:

$$(M^n)_{i,i} = \int_{-\infty}^{+\infty} t^n P_i(t) dW(t) P_i(t),$$

i.e:

$$CW_N(n,i) = (M^n)_{i,i}(j,j),$$

for all $j = 1, \ldots N$.

We have:

$$(M^n)_{i,i} = \frac{1}{2N\pi} \left(\int_{-\infty}^{+\infty} t^n P_i(t)(I+B)\sqrt{d_{11}^+(t)} P_i(t) dt + \int_{-\infty}^{+\infty} t^n P_i(t)((N-1)I - B)\sqrt{d_{22}^+(t)} P_i(t) dt \right).$$

Then, as P_i and $(aI + bB)$ commute for all a, b, we have:

$$(M^n)_{i,i} = \frac{1}{2N\pi} \left(\int_{N-3}^{N+1} t^n \sqrt{d_{11}(t)} P_i^2(t)(I+B) dt + \int_{-3}^{1} t^n \sqrt{d_{22}(t)} P_i^2(t)((N-1)I - B) dt \right).$$

For each $i \in \mathbb{N}$, let $f_i(t), g_i(t)$ be polynomials in t such that:

$$P_i(t) = f_i(t)I + g_i(t)B.$$

After some matrix calculations, we obtain:

$$P_i(t)^2(I+B) = (f_i(t)+(N-1)g_i(t))^2(I+B),$$

$$P_i(t)^2((N-1)I-B) = (f_i(t)-g_i(t))^2((N-1)I-B).$$

Then,

$$(M^n)_{i,i} = \frac{1}{2N\pi}\int_{N-3}^{N+1} t^n\sqrt{d_{11}(t)}(f_i(t)+(N-1)g_i(t))^2(I+B)dt+$$

$$+\frac{1}{2N\pi}\int_{-3}^{1} t^n\sqrt{d_{22}(t)}(f_i(t)-g_i(t))^2((N-1)I-B)dt$$

$$= \left(\frac{1}{2N\pi}\int_{N-3}^{N+1} t^n\sqrt{d_{11}(t)}(f_i(t)+(N-1)g_i(t))^2dt + \frac{N-1}{2N\pi}\int_{-3}^{1} t^n\sqrt{d_{22}(t)}(f_i(t)-g_i(t))^2dt\right) I+$$

$$\left(\frac{1}{2N\pi}\int_{N-3}^{N+1} t^n\sqrt{d_{11}(t)}(f_i(t)+(N-1)g_i(t))^2dt - \frac{1}{2N\pi}\int_{-3}^{1} t^n\sqrt{d_{22}(t)}(f_i(t)-g_i(t))^2dt\right) B.$$

We just have expressed the block ii of the matrix (M^n) by a linear combination of I and B. The number $CW_N(n,i)$ is expressed by the function that multiplies I, i.e.:

$$CW_N(n,i) = \frac{1}{2N\pi}\int_{N-3}^{N+1} t^n\sqrt{d_{11}(t)}(f_i(t)+(N-1)g_i(t))^2dt + \frac{N-1}{2N\pi}\int_{-3}^{1} t^n\sqrt{d_{22}(t)}(f_i(t)-g_i(t))^2dt.$$

Let x, y such that $t+1-N = x$ and $t+1 = y$. Then $CW_N(n,i) =$

$$\frac{1}{2N\pi}\int_{-2}^{2}(x-1+N)^n\sqrt{4-x^2}(f_i(x-1+N)+(N-1)g_i(x-1+N))^2dx+$$

$$+\frac{N-1}{2N\pi}\int_{-2}^{2}(y-1)^n\sqrt{4-y^2}(f_i(y-1)-g_i(y-1))^2dy$$

$$= \frac{1}{2N\pi}\int_{-2}^{2}[(x-1+N)^n(f_i(x-1+N)+(N-1)g_i(x-1+N))^2+$$

$$+(N-1)(x-1)^n(f_i(x-1)-g_i(x-1))^2]\sqrt{4-x^2}dx.$$

It is possible to prove by induction on i, that $f_i(x-1+N)+(N-1)g_i(x-1+N) = f_i(x-1) - g_i(x-1)$, for $x \in$ and $i = 1, 2, \ldots$ Let $h_i(x) := f_i(x) - g_i(x)$. We have then:

$$CW_N(n,i) = \frac{1}{2N\pi}\int_{-2}^{2}[(x-1+N)^n + (N-1)(x-1)^n](h_i(x-1))^2\sqrt{4-x^2}dx.$$

Let θ such that $x = 2\cos\theta$. Then $dx = -2\sin\theta$ and

$$CW_N(n,i) = \frac{1}{2N\pi}\int_{-\pi}^{0}[(2\cos\theta-1+N)^n + (N-1)(2\cos\theta-1)^n](h_i(2\cos\theta-1))^2|2\sin\theta|(-2\sin\theta)d\theta$$

$$= \frac{2}{N\pi}\int_{-\pi}^{0}[(2\cos\theta-1+N)^n + (N-1)(2\cos\theta-1)^n](h_i(2\cos\theta-1))^2\sin^2\theta\, d\theta$$

We have that $h_i(2\cos\theta - 1)$ satisfies:

$$h_0(2\cos\theta-1)=1, h_1(2\cos\theta-1)=2\cos\theta,$$

$$h_{i+1}(2\cos\theta-1)=2\cos\theta h_i(2\cos\theta-1)-h_{i-1}(2\cos\theta-1), \qquad i\geq 1.$$

The closed expression for $h_i(2\cos\theta - 1)$ is:

$$h_i(2\cos\theta-1)=\sum_{j=0}^{\lfloor i/2\rfloor}(-1)^j\left(\begin{array}{c} i-j \\ j \end{array}\right)(2\cos\theta)^{i-2j}.$$

Then,

$$(h_i(2\cos\theta-1))^2=\sum_{l,j=0}^{\lfloor i/2\rfloor}(-1)^{l+j}\left(\begin{array}{c} i-l \\ l \end{array}\right)\left(\begin{array}{c} i-j \\ j \end{array}\right)(2\cos\theta)^{2i-2j-2l}. \tag{1}$$

Also, we have: $(2\cos\theta-1+N)^n+(N-1)(2\cos\theta-1)^n=$

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)2^k\cos^k\theta(N-1)^{n-k}+(N-1)\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)2^k\cos^k\theta(-1)^{n-k}$$

$$=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)2^k\cos^k\theta\left((N-1)^{n-k}+(N-1)(-1)^{n-k}\right) \tag{2}$$

With equations (1),(2), we have that $CW_N(n,i)=$

$$=\frac{2}{N\pi}\int_{-\pi}^{0}\sum_{k=0}^{n}\left(\left(\begin{array}{c} n \\ k \end{array}\right)2^k\cos^k\theta\left((N-1)^{n-k}+(N-1)(-1)^{n-k}\right)\right)\cdot$$

$$\cdot\left(\sum_{l,j=0}^{\lfloor i/2\rfloor}(-1)^{l+j}\left(\begin{array}{c} i-l \\ l \end{array}\right)\left(\begin{array}{c} i-j \\ j \end{array}\right)(2\cos\theta)^{2i-2j-2l}\right)\sin^2\,d\theta$$

$$=\frac{2}{N}\sum_{k=0}^{n}\sum_{l,j=0}^{\lfloor i/2\rfloor}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} i-l \\ l \end{array}\right)\left(\begin{array}{c} i-j \\ j \end{array}\right)(-1)^{l+j}\left((N-1)^{n-k}+(N-1)(-1)^{n-k}\right)\cdot$$

$$\cdot\,2^{k+2i-2j-2l}\frac{1}{\pi}\int_{-\pi}^{0}(\cos\theta)^{k+2i-2j-2l}\sin^2\,d\theta.$$

By integral formula, the integral vanishes when k is odd. Therefore, we can sum only the even terms. We have that $CW_N(n,i)=$

$$\frac{2}{N}\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l,j=0}^{\lfloor i/2\rfloor}\left(\begin{array}{c} n \\ 2k \end{array}\right)\left(\begin{array}{c} i-l \\ l \end{array}\right)\left(\begin{array}{c} i-j \\ j \end{array}\right)(-1)^{l+j}\left((N-1)^{n-2k}+(N-1)(-1)^{n-2k}\right)\cdot$$

$$\cdot\,2^{2k+2i-2j-2l}\frac{|\mathcal{C}_{k+i-j-l}|}{2^{2k+2i-2j-2l+1}}$$

$$=\frac{1}{N}\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l,j=0}^{\lfloor i/2\rfloor}\left(\begin{array}{c} n \\ 2k \end{array}\right)\left(\begin{array}{c} i-l \\ l \end{array}\right)\left(\begin{array}{c} i-j \\ j \end{array}\right)(-1)^{l+j}\left((N-1)^{n-2k}+(N-1)(-1)^n\right)|\mathcal{C}_{k+i-j-l}|,$$

where $|\mathcal{C}_k|$ is the k-th Catalan number.

0.4 Preliminaries

0.4.1 Graph Theory

A *graph* is a pair $G = (V, E)$ of sets such that $E \subset [V]^2$. The elements of V are the *vertices* of the graph G, the elements of E are its *edges*. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. An edge $\{x, y\}$ is usually written as xy. If $x \in X$ and $y \in Y$, then xy is an $X - Y$ *edge*. The set of all $X - Y$ edges in a set E is denoted by $E(X, Y)$. The number of vertices of a graph is its *order*, written as $|G|$.
A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\} \quad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the x_i are all distinct. The number of edges of a path is its *length*. A *walk* (of lengh k) in a graph G is a non-empty alternating sequence $v_0e_0v_1e_1...e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*. A *path* is a walk with all vertices distinct. A graph is called *connected* if any pair of distinct vertices $x, y \in V$ are connected by a walk.

The *distance* $d(x, y)$ in G of two vertices x, y is the length of a shortest $x - y$ path in G; if no such path exists, we set $d(x, y) := \infty$. The greatest distance between any two vertices in G is the *diameter* of G.
Two vertices x, y of G are *adjacent*, or *neighbours*, if xy is an edge of G. We denote that two vertices are adjacent by $x \sim y$. If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is denoted by K^n.
If $V' \subset V$ and $E' \subset E$, then G' is a *subgraph* of G, written as $G' \subset G$. If $G' \subset G$, and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G.

Let $G = (V, E)$ be a (non-empty) graph. The set of neighbours of a vertex v in G is denoted by $\Gamma(v)$. Also, we have $\Gamma_i(x) := \{y \in V : d(x, y) = i\}$. Then $\Gamma_1(x) = \Gamma(x)$. The *degree* of a vertex v is the number $|\Gamma(v)|$ of neighbours of v. The *maximum degree* of the graph G is the number $\Delta(G) = \max_{v \in V} |\Gamma(v)|$. If all the vertices of G have the same degree k, then G is *k-regular*, or simply *regular*, with degree k. A graph is called *locally finite* if $|\Gamma(v)| < \infty$ for all $v \in V$.

Let $G = (V, E)$ be a graph. An *automorphism* is a bijection $\Phi : V \to V$ with $xy \in E \iff \phi(x)\phi(y) \in E$ for all $x, y \in V$. The set of all automorphisms over G is denoted by $\mathrm{Aut}(G)$.

The *adjacency matrix* $A = (A_{xy})_{x,y\in V}$ of G is the matrix defined by

$$A_{xy} := \begin{cases} 1, & x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

According to our definition of graph, the adjacency matrix is symmetric, and all elements of its diagonal are null. As we will consider infinite graphs, this matrix can be infinite. The powers of adjacency matrix give us information about walks. We have that $(A^n)_{xy}$ is the number of $n-$step walks starting at vertex x and finishing at vertex y. In particular, $\langle A^n \rangle_x$ is the number of $n-$step closed walks starting at vertex x. Consider a fixed origin $o \in V$. The graph is stratified into a disjoint union of strata:

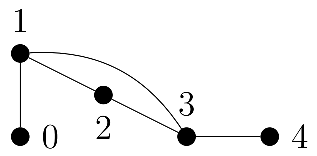
$$V = \bigcup_{n=0}^{\infty} V_n, \qquad\qquad V_n = \{x \in V : d(o,x) = n\}.$$

This is called the *stratification* associated with $o \in V$.
Given a stratification, we define three matrices $A^\epsilon, \epsilon \in \{+,-,o\}$, indexed by V as follows: For $x \in V_n$ and $n \geq 0$, we have:

$$(A^\epsilon)_{xy} := \begin{cases} A_{xy}, & y \in V_{n+\epsilon}, \\ 0, & \text{otherwise,} \end{cases}$$

where $n+\epsilon = n+1, n-1, n$ according as $\epsilon = +,-,o$. Then the adjacency matrix A is decomposed into a sum of three matrices: $A = A^+ + A^- + A^o$, which we call the *quantum decomposition* of A, and each of them is called *quantum component*.

An example:



The quantum decomposition $A = A^+ + A^- + A^o$ of the graph above is given by:

$$\begin{bmatrix} 0&1&0&0&0\\ 1&0&1&1&0\\ 0&1&0&1&0\\ 0&1&1&0&1\\ 0&0&0&1&0 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&0 \end{bmatrix}$$

Quantum decomposition will be important in Section 1.2, where the quantum components will be regarded as linear operators.

0.4.2 Block Tridiagonal Graphs

A *d-block tridiagonal graph* of dimension n is a graph whose adjacency matrix is a block tridiagonal matrix, with all blocks n by n, and $d < \infty$ blocks on the main diagonal. If d is infinity, we say that the graph is an *infinite block tridiagonal graph.*

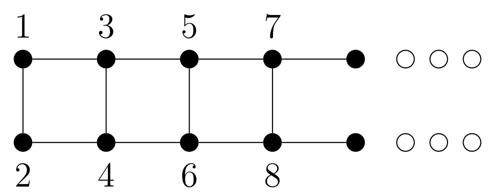
Since the adjacency matrix of a graph is symmetric, the blocks on the lower diagonal are the transpose of the blocks on the upper one. The adjacency matrix of a finite d-block tridiagonal graph $G = (V, E)$ of dimension n is represented by:

$$\begin{bmatrix} B_0 & A_0 & 0 & 0 & \cdots & 0 \\ A_0^T & B_1 & A_1 & 0 & \cdots & 0 \\ 0 & A_1^T & B_2 & A_2 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & A_{d-3}^T & B_{d-2} & A_{d-2} \\ 0 & 0 & \cdots & 0 & A_{d-2}^T & B_{d-1} \end{bmatrix}$$

The order of such graph is nd. The blocks induce a natural d-partition over the set of vertices $V = V_0, V_1, \ldots V_{d-1}$. For each $k = 0, 1, \ldots, d-1$, B_k is the adjacency matrix of the induced subgraph of V_k. Besides having a d-partition of V by n-sets, the characteristic of a block tridiagonal graph is that there is no edge between V_k and V_l if $|l - k| > 1$.

One example of a infinite block tridiagonal graph of dimension 2 is the semi infinite ladder graph. We have $A_k = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B_k = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $k = 1, 2, \ldots$

Figure 3: Semi infinite ladder



A large class of tridiagonal graphs is given by the Cartesian product of graphs. If G and H are graphs, their *Cartesian product* $G \square H$ has vertex set $V(G) \times V(H)$, where $(x_1, y_1) \sim (x_2, y_2)$ if and only if

$$\begin{cases} x_1 = y_1 \text{ and } x_2 \sim y_2, or \\ x_1 \sim y_1 \text{ and } x_2 = y_2. \end{cases}$$

Proposition 3. Let G a finite graph with $|V(G)| = d$. If H is a (finite or infinite) graph with $\Delta(H) \leq 2$, then the graph $G \square H$ is d-block tridiagonal.

The proof follows immediately from the definition. Actually, the semi infinite ladder graph is an example of cartesian product that is a block tridiagonal graph: it is the cartesian product of K^2 by the ray $R = (V, E)$ with $V = \{x_0, x_1, x_2, \ldots\}$ and $E = \{x_0x_1, x_1x_2, \ldots\}$.

Proposition 4. Let G a connected block tridiagonal graph, with blocks A_k and B_k. If both sequences of blocks are constant, i.e. $A_k = A$ and $B_k = B$ for all k, then G is not regular.

Suppose G is regular. Consider the partition $V_0, V_1, V_2, \dots$ induced by the blocks. Since the block sequence B_k is constant and the graph is regular, we have $|E(V_0, V_1)| = |E(V_0, V_1)| + |E(V_1, V_2)|$. Then $|E(V_1, V_2)| = 0$, which means $A_1 = 0$. Since the block sequence A_k is constant, we have $A_k = 0$ for all k and then G is not connected, which is a contradiction.

0.4.3 Functional Analysis

If V is a vector space over $\mathbb{F}$, (where $\mathbb{F}$ is the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$), an *inner product* on V is a function $<\cdot,\cdot>: V \times V \to \mathbb{F}$ such that, for all $\alpha, \beta \in \mathbb{F}$, and all $x, y, z \in V$, the following are satisfied:

1. $< \alpha x + \beta y, z >= \alpha < x, z > + \beta < y, z >$,
2. $< x, \alpha y + \beta z >= \overline{\alpha} < x, y > + \overline{\beta} < x, z >$,
3. $< x, y >= \overline{< y, x >}$,
4. $< x, x > \geq 0$,
5. If $< x, x >= 0$, then $x = 0$.

The quantity $\|x\| =< x, x >^{1/2}$ for an inner product $<\cdot,\cdot>$ is called the *norm* of x. A *pre-Hilbert Space* is a vector space H over $\mathbb{F}$ together with an inner product $<\cdot,\cdot>$. A *Hilbert Space* is a vector space H over $\mathbb{F}$ together with an inner product $<\cdot,\cdot>$ such that, relative to the metric $d(x, y) = \|x - y\|$ induced by the norm, H is a complete metric space.

Interacting Fock spaces are separable Hilbert Spaces with creation, annihilation and diagonal operators to be presented in Section 1.2. We will study the spectral behaviour of a growing graph via the interacting Fock spaces induced by the algebra space of the powers of its adjacency matrix.

0.4.4 Matrix Analysis

The following definitions and corresponding facts can be seen in [8]. Let $\mathbb{C}^{N\times N}$ denote the vector space of all $N \times N$ matrices over $\mathbb{C}$, i.e.

$$\mathbb{C}^{N\times N} = \{A = (A_{ij})_{0\leq i,j\leq N-1} \text{ with } A_{ij} \in \mathbb{C}\}.$$

The element ij of matrix A is denoted by A_{ij} or sometimes by $A(i, j)$. With this notation, the first element of matrix $A = (A_{ij})$ is $A_{00} = A(0, 0)$. The *conjugate transpose* of a matrix $A \in \mathbb{C}^{N\times N}$ is the matrix $\overline{A^{\mathrm{T}}}$, which is denoted by A^*. A matrix $A \in \mathbb{C}^{N\times N}$ is *Hermitian* if $A = A^*$. An *eigenvector* of a matrix $A \in \mathbb{C}^{N\times N}$ is a (non-zero) vector $z \in \mathbb{C}^N$ such that $Az = \lambda z$ for some scalar $\lambda \in \mathbb{C}$. The value λ is the corresponding *eigenvalue*. The spectral theorem guarantees that if a matrix A is *Hermitian*, there exists an orthonormal basis of $\mathbb{C}^N$ consisting of eigenvectors of A whose eigenvalues are real and whose matrix admits the *eigendecomposition* $A = P^{-1}DP$, where $P \in \mathbb{C}^{N\times N}$ is an invertible matrix and $D \in \mathbb{C}^{N\times N}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues.

A Hermitian matrix A is said to be *positive-definite* if the scalar z^*Az is real and positive for all $z \in \mathbb{C}^N\backslash\{0\}$, which is denoted by $A \succ 0$. Equivalently, a Hermitian matrix A is *positive-definite* if and only if all its eigenvalues are positive. For any matrices A, B we write $A \prec B$ if $B - A$ is positive-definite.

A matrix may have several square roots [8]. We will set the square root of any positive-definite matrix M in the following way. As M is Hermitian, it admits the eigendecomposition $M = P^{-1}D_M P$, with D_M a diagonal matrix whose diagonal elements are the corresponding eigenvalues. As A is positive-definite, all of is eigenvalues are positive. Then we can define $D_M^{1/2}$ by:

$$D_M^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_{N-1}} & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{\lambda_N} \end{bmatrix},$$

where $\lambda_1, \ldots \lambda_N$ are the eigenvalues of A. We set

$$M^{1/2} := P^{-1}D_M^{1/2}P.$$

We have

$$M^{1/2}M^{1/2} = P^{-1}D_M^{1/2}PP^{-1}D_M^{1/2}P = P^{-1}D_M^{1/2}D_M^{1/2}P = P^{-1}D_M P = M.$$

The matrix $M^{1/2}$ defined above is also positive-definite and it is the only square root of M that is positive-definite.

0.4.5 Measure Theory

The *support* of a measure W (over $\mathbb{R}$) is a subset of $\mathbb{R}$ defined by:

$$\operatorname{supp} W = \mathbb{R}\backslash A,$$

with $A = \bigcup\{U \subset \mathbb{R} \,|\, U \text{ is a open set of } \mathbb{R} \text{ such that } W(U) = 0\}$.

The *Stieltjes transform* of W is defined by:

$$G_W(z) := \int_{\mathbb{R}} \frac{dW(t)}{z-t},$$

with $z \in \mathbb{C} \setminus \operatorname{supp} W$.

We recall the classic inverse formula of Stieltjes, in a form which is suitable for our purposes:

Theorem 6. [5, 12] If a measure W is continuous over $\mathbb{R}$, we have:

$$W(x) = -\frac{1}{2i\pi} \lim_{y \to 0^+} \left(G_W(x+iy) - G_W(x-iy) \right),$$

where G_W is the Stieltjes transform of W.

0.4.6 Combinatorics

Let $m \in \mathbb{N}^*$. Given $\epsilon = (\epsilon_1 \dots, \epsilon_{2m}) \in \{+,-\}^{2m}$ we associate a path in $\mathbb{Z}^2$ defined by

$$(0,0), (1, \epsilon_1), (2, \epsilon_1 + \epsilon_2), \dots, (2m, \epsilon_1 + \epsilon_2 + \dots + \epsilon_{2m}),$$

where $+1$ is assigned to $\epsilon_i = +$ and -1, to $\epsilon_i = -$.

Definition 1. Let $\mathcal{C}_m$ denote the set of paths which end at $(2m, 0)$ and are restricted to the upper half plane, i.e.

$$\mathcal{C}_m = \left\{ (\epsilon_1 \dots, \epsilon_{2m}) \in \{+,-\}^{2m} \,\middle|\, \begin{array}{c} \epsilon_1 + \dots + \epsilon_k \geq 0, \ k = 1, \dots, 2m-1, \\ \epsilon_1 + \dots + \epsilon_{2m-1} + \epsilon_{2m} = 0 \end{array} \right\}.$$

An element in $\mathcal{C}_m$ is called a *Catalan path* of length $2m$. We call $|\mathcal{C}_m|$ the m-th *Catalan number.*

Lemma 1. *[[7], chapter 1 page 37]* For $m \geq 1$ the m-th Catalan number is given by

$$|\mathcal{C}_m| = \frac{(2m)!}{m!(m+1)!}$$

Proof.

Set

$$\tilde{\mathcal{C}}_m = \left\{ (\epsilon_1, \dots, \epsilon_{2m}) \in \{+,-\}^{2m} \,|\, \epsilon_1 + \dots + \epsilon_{2m-1} + \epsilon_{2m} = 0 \right\}.$$

We have $\mathcal{C}_m \subset \tilde{\mathcal{C}}_m$. Each $\epsilon \in \tilde{\mathcal{C}}_m$ corresponds to a path starting at $(0,0)$ and finishing at $(2m, 0)$, without any half plane restriction. As ϵ starts and finishes in the x-axis, there are as many occurrences of $\epsilon_i = +$ as of $\epsilon_i = -1$. Then,

$$|\tilde{\mathcal{C}}_m| = \begin{pmatrix} 2m \\ m \end{pmatrix}.$$

Now we will compute $|\tilde{\mathcal{C}}_m \backslash \mathcal{C}_m|$. Let $\epsilon = (\epsilon_1, \dots, \epsilon_{2m}) \in \tilde{\mathcal{C}}_m \backslash \mathcal{C}_m$. Then the path ϵ starts at $(0,0)$, pass at least once in the lower plan and finishes at $(2m, 0)$. Then, there exists $k \in 1, \dots, 2m-1$ such that $\epsilon_1 + \dots + \epsilon_l \geq 0$ for all $l = 1, \dots, k-1$ and $\epsilon_1 + \dots + \epsilon_k = -1$. Then, $\epsilon_1 + \dots + \epsilon_{k-1} = 0$ and $\epsilon_k = -1$. Let L be the horizontal line containing the point $(0, -1)$. We have that ϵ intersects L for the first time at (k, ϵ_k). Define $\bar{\epsilon}$ as the path obtained from ϵ by reflecting the first part of ϵ up to $(k, -1)$ with respect to L. Then, $\bar{\epsilon}$ is a path beginning at $(0, -2)$, intersecting L for the first time at (k, ϵ_k) and finishing at $(2m, 0)$.

We have that $\epsilon \leftrightarrow \bar{\epsilon}$ is a one-to-one correspondence between $\tilde{\mathcal{C}}_m \backslash \mathcal{C}_m$ and the set of paths connecting $(0, -2)$ and $(2m, 0)$. In fact, for each $\epsilon = (\epsilon_1 \dots, \epsilon_{2m}) \in \tilde{\mathcal{C}}_m \backslash \mathcal{C}_m$, by construction there is a path $\bar{\epsilon}$ beginning at $(0, -2)$ and finishing at $(2m, 0)$. On the other hand, each path $\bar{\epsilon}$ beginning at $(0, -2)$ and finishing at $(2m, 0)$ intersects the line L. Let $(k, -1)$ be the first intersection. Define ϵ as the path obtained from $\bar{\epsilon}$ by reflecting the first part of $\bar{\epsilon}$ up to $(k, -1)$ with respect to L. Then, ϵ is a path beginning at $(0, 0)$, intersecting L for the first time at (k, ϵ_k) and finishing at $(2m, 0)$.

As $\bar{\epsilon}$ starts at $(0, -2)$ and finishes at $(2m, 0)$, for all $2m$ elements of the path ϵ there is exactly $m+1$ of them that are positive, and $m-1$ of them that are negative. Then,

$$|\tilde{\mathcal{C}}_m \backslash \mathcal{C}_m| = \binom{2m}{m+1}.$$

Hence,

$$\begin{aligned}|\mathcal{C}_m| = |\tilde{\mathcal{C}}_m| - |\tilde{\mathcal{C}}_m \backslash \mathcal{C}_m| = \binom{2m}{m} - \binom{2m}{m+1} &= \frac{(2m)!}{m!m!} - \frac{(2m)!}{(m+1)!(m-1)!} \\ &= \frac{(2m)!(m+1)}{m!(m+1)!} - \frac{(2m)!m}{(m+1)!m!} = \frac{(2m)!}{m!(m+1)!},\end{aligned}$$

i.e.,

$$|\mathcal{C}_m| = \frac{(2m)!}{m!(m+1)!}.$$

□

0.4.7 Integrals of Trigonometric Functions

For $k \in \mathbb{N}^*$, we have:

$$\frac{1}{\pi} \int_{-\pi}^{0} \cos^{2k} \theta \, d\theta = \frac{(2k)!}{2^{2k} k! k!}, \tag{3}$$

and:

$$\frac{1}{\pi} \int_{-\pi}^{0} \cos^{2k} \theta \sin^2 \theta \, d\theta = \frac{1}{2^{2k+1}} |\mathcal{C}_k|, \tag{4}$$

where $|\mathcal{C}_k|$ is the kth Catalan number (see Lemma 1 in Section 0.4.6).

We will prove Equation (3) buy induction on k. For $k = 1$, we have:

$$\begin{cases} \frac{1}{\pi}\int_{-\pi}^{0} \cos^2\theta \, d\theta = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2} \\ \frac{(2k)!}{2^{2k}k!k!} = \frac{1}{2}. \end{cases}$$

Let $k > 1$. Suppose Equation (3) is true for k. We have:

$$\int_{-\pi}^{0} \cos^{2k}\theta \, d\theta = \int_{-\pi}^{0} \cos^{2k-1}\theta \cos\theta \, d\theta = \left[\cos^{2k-1}\theta \cos\theta\right]_{-\pi}^{0} -$$

$$-\int_{-\pi}^{0}(2k-1)\cos^{2k-2}\theta(-\sin\theta)\sin\theta \, d\theta = (2k-1)\int_{-\pi}^{0}\cos^{2k-2}\theta(1-\cos^2\theta)\, d\theta =$$

$$(2k-1)\int_{-\pi}^{0}\cos^{2(k-1)}\theta \, d\theta - (2k-1)\int_{-\pi}^{0}\cos^{2k}\theta \, d\theta.$$

From

$$\int_{-\pi}^{0}\cos^{2k}\theta \, d\theta = (2k-1)\int_{-\pi}^{0}\cos^{2(k-1)}\theta \, d\theta - (2k-1)\int_{-\pi}^{0}\cos^{2k}\theta \, d\theta,$$

we have:

$$\frac{1}{\pi}\int_{-\pi}^{0}\cos^{2k}\theta \, d\theta = \frac{2k-1}{2k}\cdot\frac{1}{\pi}\int_{-\pi}^{0}\cos^{2(k-1)}\theta \, d\theta.$$

By induction hypothesis,

$$\frac{1}{\pi}\int_{-\pi}^{0}\cos^{2k}\theta \, d\theta = \frac{2k-1}{2k}\frac{(2(k-1))!}{2^{2(k-1)}(k-1)!(k-1)!} = \frac{(2k-1)!}{2^{2k-1}k!(k-1)!} = \frac{(2k)!}{2^{2k}k!k!}.$$

For Equation (4), we have:

$$\frac{1}{\pi}\int_{-\pi}^{0}\cos^{2k}\theta\sin^2\theta \, d\theta = \frac{1}{\pi}\int_{-\pi}^{0}\cos^{2k}\theta \, d\theta - \frac{1}{\pi}\int_{-\pi}^{0}\cos^{2(k+1)}\theta \, d\theta =$$

$$\frac{(2k)!}{2^{2k}k!k!} - \frac{(2k+2)!}{2^{2k+2}(k+1)!(k+1)!} = \frac{(2k)!2^2(k+1)^2-(2k+2)!}{2^{2k+2}(k+1)!(k+1)!} = \frac{(2k)!(2k+2)}{2\cdot 2^{2k+1}(k+1)!(k+1)!}$$

$$= \frac{(2k)!(k+1)}{2^{2k+1}(k+1)!(k+1)!} = \frac{1}{2^{2k+1}}\frac{(2k)!}{k!(k+1)!} = \frac{1}{2^{2k+1}}|\mathcal{C}_k|.$$

Chapter 1

Spectral Analysis over Growing Graphs

1.1 Distance-Regular Graphs

This section follows [7]. Distance-regular graphs are important to our study because they have significant properties from the viewpoint of quantum decomposition.
In this section, some basic properties are established. They will be useful in the following section, where some spectral aspects will be analyzed. Later, in sections 1.4 and 1.6, we will study asymptotic spectral distributions of the adjacency matrix of such graphs.

Definition 2. Let $i, j, k \in \mathbb{N}$. A connected graph $G = (V, E)$ is called *distance-regular* if, for any choice of $x, y \in V$ with $d(x, y) = k$, the number

$$p_{ij}^{k} = |\{z \in V; d(x, z) = i, d(y, z) = j\}|$$

depends only on i, j, k. The numbers p_{ij}^{k} are called the *intersection numbers* of $G = (V, E)$.

Distance-regular graphs are regular with degree p_{11}^{0}. Then, by Proposition 4, distance-regular graphs and block tridiagonal graphs are disjoint classes of graphs.

A graph is called *distance-transitive* if for any $x, x', y, y' \in V$ such that $d(x, y) = d(x', y')$ there exists $\alpha \in$Aut(G) such that $\alpha(x) = x', \alpha(y) = y'$.

Proposition 5. Every distance-transitive graph is distance-regular.

Proof.
Let i, j, k positive integers. As the graph is distance-transitive, for two distinct pairs $(x, y), (x', y')$ with $d(x, y) = d(x', y') = k$, we have that $|\{z \in V; d(x, z) = i, d(y, z) = j\}|$ and $|\{z \in V; d(x', z) = i, d(y', z) = j\}|$ are the same, since there is an $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(x) = x', \alpha(y) = y'$. The number p_{ij}^{k} is then defined only depending on i, j, k.

□

Example 1. Every cyclic graph (with $|V| > 3$) is distance-regular because it is distance-transitive.

Example 2. Consider the graph $(\mathbb{Z}^2, E(\mathbb{Z}^2))$, where $\{(i,j),(k,l)\} \in E(\mathbb{Z}^2) \Longleftrightarrow |i-k|+|j-l| = 1$. This graph is not distance-regular. In fact, if we take, $O = (0,0)$, $A = (1,1)$, $B = (0,2)$, we have that $d(O,A) = d(O,B) = 2$. But we have 2 vertices that are at distance 1 of both O and A while only one vertex is at distance 1 of O and B. Therefore, p_{11}^2 does depend of the choice of the vertices.

1.2 Interacting Fock Space

This section follows [7]. We will see that the interacting Fock space of a graph together with its spectral distribution will lead to a quantum central limit theorem.

1.2.1 Definition of Interacting Fock space

In this subsection, we will define an interacting Fock space and in the following subsections, we will define the interacting Fock space of a general graph and of a distance-regular graph.

Step 1. Choose an infinite-dimensional separable Hilbert Space $\mathcal{H}$.

Step 2. Choose a complete orthonormal basis $\{\Phi_n : n \in \mathbb{N}\}$ in $\mathcal{H}$.

Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the dense subspace spanned by $\{\Phi_n : n \in \mathbb{N}\}$ i.e,

$$\mathcal{H}_0 = [\{\Phi_n\}_{n\in\mathbb{N}}] := \left\{ \sum_{k=0}^{\infty} a_k \Phi_k : a_k \in \mathbb{C} \quad k = 0, 1, \dots \right\}.$$

Definition 3. A pair of sequences $(\{\omega_n\}, \{\alpha_n\})_{n=1}^{\infty}$ is called a *Jacobi coefficient* if one of the following two conditions is satisfied:

(a) **[infinite type]** $\alpha_n \in \mathbb{R}$, $\omega_n > 0$ for all n;

(b) **[finite type]** $\alpha_n \in \mathbb{R}$, $\omega_n > 0$ for $n \in \{1, \dots, m_0\}$, and $\alpha_{n+1} = \omega_n = 0$ for $n > m_0$.

Step 3. Choose a Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})_{n=1}^{\infty}$. Then the linear operators $B^+, B^-, B^o \in L(\mathcal{H}_0)$ are defined by

$$B^+\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1}, \quad n \geq 0,$$

$$B^-\Phi_0 = 0, \qquad B^-\Phi_n = \sqrt{\omega_n}\Phi_{n-1}, \quad n \geq 1,$$

$$B^o\Phi_n = \alpha_{n+1}\Phi_n \quad n \geq 0.$$

The subspace Γ is the subspace of $\mathcal{H}_0$ spanned by $\{(B^+)^n\Phi_0 : n \in \mathbb{N}\}$, i.e,

$$\Gamma = [\{(B^+)^n\Phi_0 : n \in \mathbb{N}\}] := \left\{ \sum_{k=0}^{\infty} a_k (B^+)^k \Phi_0 : a_k \in \mathbb{C} \quad k = 0, 1, \dots \right\}.$$

Lemma 2. Γ is invariant under the actions of B^+, B^-, B^o.

Proof.

As $B^+(B^+)^n\Phi_0 = (B^+)^{n+1}\Phi_0$, Γ is invariant under the action of B^+.

Furthermore, we have:

$$(B^+)^n\Phi_0 = (B^+)^{n-1}\sqrt{\omega_1}\Phi_1 = (B^+)^{n-2}\sqrt{\omega_2\omega_1}\Phi_2 = \cdots = \sqrt{\omega_n \cdots \omega_1}\Phi_n$$

Then, Γ can be expressed by:

$$\Gamma = [\{\sqrt{\omega_n \cdots \omega_1}\Phi_n : n \in \mathbb{N}\}].$$

Hence, if $(\{\omega_n\}, \{\alpha_n\})_{n=1}^{\infty}$ is of infinite type, we have $\Gamma = \mathcal{H}_0$, and then Γ is invariant also under B^- and B^o. If $(\{\omega_n\}, \{\alpha_n\})_{n=1}^{\infty}$ is of finite type, there is some m_0 with $\alpha_n \in \mathbb{R}$, $\omega_n > 0$ for $n \in \{1, \dots, m_0\}$, and $\alpha_n = \omega_n = 0$ for $n > m_0$. As $\omega_n = 0$ for $n > m_0$, we have:

$$\Gamma = [\{\Phi_n : n = 0, 1, \dots, m_0 - 1\}].$$

Then Γ is invariant under B^- and B^o.

□

Some observations:

(a) Γ is a pre-Hilbert space.

(b) By Lemma 2, B^+, B^-, B^o are linear operators in Γ.

Definition 4. The quintuple $\Gamma_{(\{\omega_n\},\{\alpha_n\})} = (\Gamma, \{\Phi_n\}, B^+, B^-, B^o)$ is called an *interacting Fock space associated with a Jacobi coefficient* $(\{\omega_n\}, \{\alpha_n\})$.

The inner product of Γ is denoted by $< \cdot, \cdot >_\Gamma = < \cdot, \cdot >$.

1.2.2 Interacting Fock Space of a Graph

In this subsection, we will define the interacting Fock space of a graph. Let $G = (V, E)$ a graph.

Step 1. We choose $l_2(V)$, the space of square-summable functions as our Hilbert Space $\mathcal{H}$ i.e.:

$$\mathcal{H} = l_2(V) := \left\{ f : V \to \mathbb{C} \,\middle|\, \sum_{x \in V} |f(x)|^2 < \infty \right\}, \qquad \text{with } < f, g >_{\mathcal{H}} := \sum_{x \in V} \overline{f(x)} g(x).$$

For each $x \in V$, $\delta_x \in l_2(V)$ is defined by $\delta_x(y) = \begin{cases} 1, & \text{if } \quad y = x, \\ 0, & \text{otherwise.} \end{cases}$

Then, $\{\delta_x\}_{x \in V}$ is a complete orthonormal basis of $l_2(V)$, but it is still not the one that we choose for our Fock Space (see Step 2).

Now we fix a stratification on G: we fix an origin $o \in V$, and for each $n \in \mathbb{N}$, we put $V_n = \{x \in V : d(o,x) = n\}$.

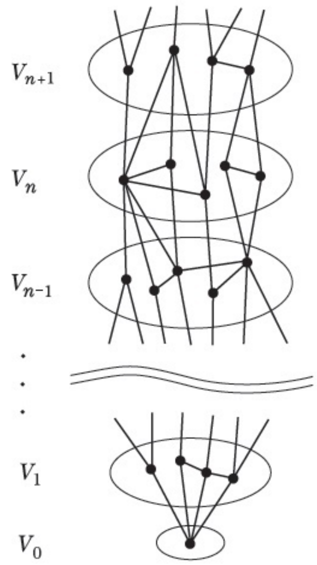


Figure 1.1: Stratification $V = \cup_{n=0}^{\infty} V_n$. Figure extracted from [7].

Now, we are ready for

Step 2. We choose $\{\Phi_n : n \in \mathbb{N}\}$ with

$$\Phi_n = \begin{cases} |V_n|^{-1/2} \sum_{x \in V_n} \delta_x, & \text{if} \quad V_n \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

We have that $\{\Phi_n : n \in \mathbb{N}\}$ is an orthonormal subset of $l_2(V)$.

We want to choose a Jacobi coefficient such that the operators of the interacting Fock space match with the quantum components of the adjacency matrix. First of all, let us see how the adjacency matrix A and the quantum components $A^{\epsilon}, \epsilon \in \{+,-,o\}$ act on δ_x:

$$A\delta_x = \sum_{y \in V} A_{xy}\delta_x = \sum_{x \sim y} \delta_x, \quad x \in V,$$

$$A^{\epsilon}\delta_x = \sum_{y \in V} A^{\epsilon}_{xy}\delta_x = \sum_{\substack{x \sim y \\ y \in V_{n+\epsilon}}} \delta_x, \quad x \in V_n.$$

The quantum components $A^{\epsilon}, \epsilon \in \{+,-,o\}$ act on Φ_n in the following way:

Proposition 6. Let $G = (V, E)$ be a graph with a fixed origin $o \in V$. Let $A = A^+ + A^- + A^o$ be the quantum decomposition of the adjacency matrix. Then we have:

$$A^+\Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n+1}} \omega_-(y)\delta_y,$$

$$A^-\Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n-1}} \omega_+(y)\delta_y,$$

$$A^o\Phi_n = |V_n|^{-1/2} \sum_{y \in V_n} \omega_o(y)\delta_y,$$

where $\omega_\epsilon(y) := |\{x \in V : x \sim y, d(o,x) = d(o,y) + \epsilon\}|$ for $\epsilon \in \{+,-,o\}$.

Proof.

We have:

$$\sum_{y \in V_{n+1}} \omega_-(y)\delta_y = \sum_{x \in V_n} \sum_{\substack{x \sim y \\ y \in V_{n+1}}} \delta_y = \sum_{x \in V_n} A^+\delta_x = |V_n|^{1/2} A^+(|V_n|^{-1/2} \sum_{x \in V_n} \delta_x) = |V_n|^{1/2} A^+\Phi_n,$$

$$\sum_{y \in V_{n-1}} \omega_+(y)\delta_y = \sum_{x \in V_n} \sum_{\substack{x \sim y \\ y \in V_{n-1}}} \delta_y = \sum_{x \in V_n} A^-\delta_x = |V_n|^{1/2} A^-(|V_n|^{-1/2} \sum_{x \in V_n} \delta_x) = |V_n|^{1/2} A^-\Phi_n,$$

$$\sum_{y \in V_n} \omega_o(y)\delta_y = \sum_{x \in V_n} \sum_{\substack{x \sim y \\ y \in V_n}} \delta_y = \sum_{x \in V_n} A^o\delta_x = |V_n|^{1/2} A^o(|V_n|^{-1/2} \sum_{x \in V_n} \delta_x) = |V_n|^{1/2} A^o\Phi_n.$$

□

Before establishing one consequence of the previous proposition, we will need the following lemma:

Lemma 3. (Matching identity). It holds that

$$\sum_{x \in V_n} \omega_+(x) = \sum_{y \in V_{n+1}} \omega_-(y), \qquad n \geq 0.$$

If $\omega_-(y), \omega_+(y)$ are constant on V_n for all $n \in \mathbb{N}$, it follows that:

$$|V_n|\omega_+(x) = |V_{n+1}|\omega_-(y), \qquad x \in V_n,\ y \in V_{n+1},\ n \geq 0.$$

Proof.

We have that both $\sum_{x \in V_n} \omega_+(x)$ and $\sum_{y \in V_{n+1}} \omega_-(y)$ are the number of edges between V_n and V_{n+1}, namely $|E(V_n, V_{n+1})|$.

In fact:

$$\sum_{x \in V_n} \omega_+(x) = \sum_{x \in V_n} |\{y \in V : y \sim x,\ d(o,y) = d(o,x) + 1\}| = \sum_{x \in V_n} |\{y \in V : y \sim x,\ d(o,y) = n+1\}|$$

$$= \sum_{x \in V_n} |\{y \in V : y \sim x,\ y \in V_{n+1}\}| = |\{xy \in E : x \in V_n,\ y \in V_{n+1}\}| = |E(V_n, V_{n+1})|.$$

On the other hand, we have:

$$\sum_{y \in V_{n+1}} \omega_-(y) = \sum_{y \in V_{n+1}} |\{x \in V : x \sim y,\ d(o,x) = d(o,y) - 1\}| = \sum_{y \in V_{n+1}} |\{x \in V : x \sim y,\ d(o,x) = n\}|$$

$$= \sum_{y \in V_{n+1}} |\{x \in V : x \sim y,\ x \in V_n\}| = |\{yx \in E : y \in V_{n+1},\ x \in V_n\}| = |E(V_n, V_{n+1})|.$$

If $\omega_-(y), \omega_+(y)$ are constant on V_n for all $n \in \mathbb{N}$, we have, for $n \geq 0$:

$$\sum_{x \in V_n} \omega_+(x) = \omega_+(x) \sum_{x \in V_n} 1 = \omega_+(x)|V_n|.$$

Similarly,

$$\sum_{y \in V_{n+1}} \omega_-(y) = \omega_-(y) \sum_{y \in V_{n+1}} 1 = \omega_-(y)|V_{n+1}|.$$

Then,

$$|V_n|\omega_+(x) = |V_{n+1}|\omega_-(y).$$

□

Corollary 1. Let $G = (V, E)$ be a graph with a fixed origin $o \in V$. Let $A = A^+ + A^- + A^o$ be the quantum decomposition of the adjacency matrix. If we assume that $\omega_-(y), \omega_+(y), \omega_o(y)$ are constant on V_n for all $n \in \mathbb{N}$, we have:

$$A^+\Phi_n = \sqrt{\frac{|V_{n+1}|}{|V_n|}\omega_-(y)^2}\ \Phi_{n+1}, \qquad y \in V_{n+1}$$

$$A^-\Phi_n = \sqrt{\frac{|V_n|}{|V_{n-1}|}\omega_-(y)^2}\ \Phi_{n-1}, \qquad y \in V_n,$$

$$A^o\Phi_n = \omega_o(y)\Phi_n, \qquad y \in V_n.$$

Proof.

By the previous proposition, we have for $y \in V_{n+1}$:

$$A^+\Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n+1}} \omega_-(y)\delta_y = |V_n|^{-1/2}|V_{n+1}|^{1/2}|V_{n+1}|^{-1/2}\omega_-(y) \sum_{y \in V_{n+1}} \delta_y$$

$$= |V_n|^{-1/2}|V_{n+1}|^{1/2}\omega_-(y)\Phi_{n+1} = \sqrt{\frac{|V_{n+1}|}{|V_n|}\omega_-(y)^2}\ \Phi_{n+1}.$$

By the previous proposition and the matching identity, we have for $x \in V_{n-1}$ and $y \in V_n$:

$$A^-\Phi_n = |V_n|^{-1/2} \sum_{x \in V_{n-1}} \omega_+(x)\delta_x = |V_n|^{-1/2}|V_{n-1}|^{-1}|V_{n-1}|\omega_+(x) \sum_{x \in V_{n-1}} \delta_x$$

$$= |V_n|^{-1/2}|V_{n-1}|^{-1}|V_n|\omega_-(y) \sum_{x \in V_{n-1}} \delta_x = |V_n|^{1/2}|V_{n-1}|^{-1/2}\omega_-(y)|V_{n-1}|^{-1/2} \sum_{x \in V_{n-1}} \delta_x$$

$$= |V_n|^{1/2}|V_{n-1}|^{-1/2}\omega_-(y)\Phi_{n-1} = \sqrt{\frac{|V_n|}{|V_{n-1}|}\omega_-(y)^2}\ \Phi_{n-1}.$$

By the previous proposition, we have for $y \in V_n$:

$$A^o\Phi_n = |V_n|^{-1/2} \sum_{y \in V_n} \omega_o(y)\delta_y = \omega_o(y)|V_n|^{-1/2} \sum_{y \in V_n} \delta_y = \omega_o(y)\Phi_n.$$

□

Step 3. We choose the following Jacobi coefficient:

$$\omega_n = \frac{|V_n|}{|V_{n-1}|}\omega_-(y)^2, \qquad y \in V_n$$

$$\alpha_n = \omega_o(y), \qquad y \in V_{n-1}, \qquad n \geq 1.$$

This coefficient defined above is well-defined only if $\omega_-(y)$ (and therefore $\omega_+(y)$ by the matching identity) and $\omega_o(y)$ are constant over V_n, for each $n \in \mathbb{N}$. So a necessary condition for the graph G to admit a Fock Space over it is that G must have a stratification such that $\omega_-(y), \omega_+(y), \omega_o(y)$ are constant over V_n for each $n \in \mathbb{N}$.

With this choice of Jacobi coefficient, the linear operators associated B^+, B^-, B^o coincide with the quantum components A^+, A^-, A^o (see Corollary 1).

We saw in Lemma 2 that Γ can be expressed by:

$$(B^+)^n\Phi_0 = \sqrt{\omega_n \cdots \omega_1}\Phi_n.$$

Then we have:

$$(B^+)^n\Phi_0 = \sqrt{\frac{|V_n|}{|V_{n-1}|}\omega_-(y_n)^2 \cdots \frac{|V_2|}{|V_1|}\omega_-(y_2)^2\frac{|V_1|}{|V_0|}\omega_-(y_1)^2}\Phi_n = \omega_-(y_n)\cdots\omega_-(y_2)\omega_-(y_1)\sqrt{|V_n|}\Phi_n$$

where $y_n \in V_n, n \geq 0$. Then

$$\Gamma(G) := \Gamma = [\{(B^+)^n\Phi_0 : n \in \mathbb{N}\}] := [\{\omega_-(y_n)\cdots\omega_-(y_2)\omega_-(y_1)\sqrt{|V_n|}\Phi_n : n \in \mathbb{N}\}].$$

If G is infinite (and connected), $|V_n| \neq 0$ for all $n \geq 0$ and then $\Gamma(G) = [\{\Phi_n : n \in \mathbb{N}\}]$. If G is finite, let m_0 such that $|V_{m_0-1}| \neq 0$ and $|V_{m_0}| = 0$. By the Corollary 1, if $\omega_-(y), \omega_+(y), \omega_o(y)$ are constant on V_n for all $n \in \mathbb{N}$, $\Gamma(G)$ is invariant under the quantum components A^+, A^-, A^o.

In that case, $(\Gamma(G), \{\Phi_n\}, A^+, A^-, A^o)$ becomes the interacting Fock Space of a graph G, and the Jacobi coefficient associated is

$$\omega_n = \frac{|V_n|}{|V_{n-1}|}\omega_-(y)^2, \qquad y \in V_n$$

$$\alpha_n = \omega_o(y), \qquad y \in V_{n-1}, \qquad n \geq 1.$$

An important class of graphs where the $\omega_-(y), \omega_+(y), \omega_o(y)$ are constant on V_n, for all $n \in \mathbb{N}$, is the class of distance-regular graphs. One observation: A graph does not need to be distance-regular in order to $\omega_-(y), \omega_+(y), \omega_o(y)$ be constant on V_n. (see Section 1.7 for such counterexample).

1.2.3 Interacting Fock Space of a Distance-Regular Graph

In this subsection, we will define the interacting Fock space of a distance-regular graph. In distance-regular graphs, $\omega_-(y)$ is constant over V_n, as the following lemma will show us. Then, such graphs admit a interacting Fock space associated with a Jacobi coefficient that can be expressed by the intersection numbers of the graph.

Lemma 4. $\omega_\epsilon(x) = p^n_{1,n+\epsilon}$, where $x \in V_n$.

Proof.
$\omega_+(x) = |\{z \in V : z \sim x, d(o,z) = d(o,x)+1\}| = |\{z \in V : d(x,z) = 1 \text{ and } d(o,z) = n+1\}| = p^n_{1,n+1}$.

□

The Jacobi coefficient depends on $\omega_-(y), \omega_o(y)$ and V_n. With the previous lemma, we can express $\omega_-(y), \omega_o(y)$ in terms of intersection numbers, and with the following one, V_n could be expressed in the same way too.

Lemma 5. $p^n_{1,n+1}|V_n| = p^{n+1}_{1,n}|V_{n+1}|$.

Proof.
By the previous lemma, and by the fact that each ω_ϵ is constant on V_i, $p^n_{1,n+1}|V_n| = \omega_+(x)|V_n| = \sum_{x\in V_n} \omega_+(x)$, with $x \in V_n$. Analogously, $p^{n+1}_{1,n}|V_{n+1}| = \omega_-(y)|V_{n+1}| = \sum_{y \in V_{n+1}} \omega_-(y)$, with $y \in V_{n+1}$. The result follows by the matching identity.

□

Let $n \in \mathbb{N}, y \in V_n$. Now, with the two previous lemmas, we can compute the Jacobi coefficient in the following way:

$$\begin{cases} \omega_n = \dfrac{|V_n|}{|V_{n-1}|}\omega_-(y)^2 = \dfrac{|V_n|}{|V_{n-1}|}(p^n_{1,n-1})^2 = \dfrac{p^{n-1}_{1,n}}{p^n_{1,n-1}}(p^n_{1,n-1})^2 = p^{n-1}_{1,n}p^n_{1,n-1}, \\ \alpha_n = \omega_o(y) = p^{n-1}_{1,n-1}. \end{cases}$$

So, for a distance-regular graph G, we have that $(\Gamma(G), \{\Phi_n\}, A^+, A^-, A^o)$ becomes the interacting Fock Space associated to the coefficient $(\{\alpha_n\}, \{\omega_n\})$, given by $\omega_n = p^n_{1,n-1}p^{n-1}_{1,n}$ and $\alpha_n = p^{n-1}_{1,n-1}$. The operators A^+, A^-, A^o are the quantum components of the adjacency matrix of G.

1.3 Proof of Proposition 1

This section follows [7]. In the Introduction we saw that Proposition 1 is important since it establishes a relation between a measure and an interacting Fock Space.

Proposition 1. Let μ be a probability measure on $\mathbb{R}$, having finite moments of all orders and $(\{\omega_n\},\{\alpha_n\})$ be its Jacobi coefficient. Consider the interacting Fock space

$$\Gamma_{(\{\omega_n\},\{\alpha_n\})} = (\Gamma, \{\Phi_n\}, B^+, B^-, B^o)$$

Then,

$$\left\langle \Phi_0, \left(B^+ + B^- + B^o\right)^m \Phi_0 \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots$$

Before proving Proposition 1, we need some preliminary results:

Lemma 6. *[[7], chapter 1 page 20]* Let $\{P_n(x)\}$ be the orthogonal polynomials associated with a probability measure μ having finite moments of all orders. Then, the Jacobi coefficient $(\{\omega_n\},\{\alpha_n\})$ is determined by

$$\omega_n \cdots \omega_2\omega_1 = \int_{-\infty}^{+\infty} P_n(x)^2 \mu(dx), \quad n = 1, 2, \ldots,$$

$$\alpha_1 = \int_{-\infty}^{+\infty} x\mu(dx),$$

$$\alpha_n\omega_{n-1} \cdots \omega_1 = \int_{-\infty}^{+\infty} xP_{n-1}(x)^2 \mu(dx), \quad n = 2, 3, \ldots$$

Proof.

Since μ is a probability measure, we have: $< P_0, P_0 >= \int_{-\infty}^{+\infty} \mu(dx) = 1$.

From the three-term recurrence relation, we have:

$$P_0(x) = 1,$$

$$P_1(x) = x - \alpha_1,$$

$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), n \geq 1.$$

Then,

$$< xP_n, P_{n-1} >=< P_{n+1}, P_{n-1} > + \alpha_{n+1} < P_n, P_{n-1} > + \omega_n < P_{n-1}, P_{n-1} >$$

$$= \omega_n < P_{n-1}, P_{n-1} >, \qquad n \geq 1,$$

where the inner product is defined by:

$$< f, g >_\mu = < f, g > = \int_{-\infty}^{+\infty} \overline{f(x)} g(x) \mu(dx).$$

Hence, for $n = 1$ we have:

$$\omega_1 < P_0, P_0 > = < xP_1, P_0 > = < P_1, xP_0 > = < P_1, x > .$$

Since

$$< P_1, P_1 > = < P_1, x - \alpha_1 > = < P_1, x - \alpha_1 P_1 > = < P_1, x > -\alpha_1 < P_1, P_0 > = < P_1, x >,$$

we have $\omega_1 < P_0, P_0 > = < P_1, P_1 >$. And for $n \geq 2$ we have:

$$\omega_n < P_{n-1}, P_{n-1} > = < xP_n, P_{n-1} > = < P_n, xP_{n-1} > =$$

$$< P_n, P_n + \alpha_n P_{n-1} + \omega_{n-1} P_{n-2} > = < P_n, P_n > +\alpha_n < P_n, P_{n-1} > +$$

$$\omega_{n-1} < P_n, P_{n-2} > = < P_n, P_n > .$$

Then for $n \geq 1$,

$$\omega_n = \frac{< P_n, P_n >}{< P_{n-1}, P_{n-1} >}.$$

Hence, for $n \geq 1$ we have:

$$\omega_n \cdots \omega_2 \omega_1 = \frac{< P_n, P_n >}{< P_{n-1}, P_{n-1} >} \frac{< P_{n-1}, P_{n-1} >}{< P_{n-2}, P_{n-2} >} \cdots \frac{< P_2, P_2 >}{< P_1, P_1 >} \frac{< P_1, P_1 >}{< P_0, P_0 >} =$$

$$\frac{< P_n, P_n >}{< P_0, P_0 >} = < P_n, P_n > = \int_{-\infty}^{+\infty} P_n(x)^2 \mu(dx).$$

For α_1 we have

$$0 = < P_1, P_0 > = < x - \alpha_1, P_0 > = < x, P_0 > - < \alpha_1 P_0, P_0 > = < x, P_0 > - \alpha_1.$$

Then,

$$\alpha_1 = < x, P_0 > = \int_{-\infty}^{+\infty} x \mu(dx).$$

Now for the α_n with $n \geq 2$, we have:

$$< xP_{n-1}, P_{n-1} > = < P_n, P_{n-1} > + \alpha_n < P_{n-1}, P_{n-1} > + \omega_n < P_{n-2}, P_{n-1} > =$$

$$\alpha_n < P_{n-1}, P_{n-1} > .$$

Then for $n \geq 2$,

$$\alpha_n = \frac{< xP_{n-1}, P_{n-1} >}{< P_{n-1}, P_{n-1} >}.$$

Hence, using the previous result for $\omega_n \cdots \omega_2\omega_1$ we have for $n \geq 1$:

$$\alpha_n\omega_{n-1}\cdots\omega_2\omega_1 = \frac{< xP_{n-1}, P_{n-1} >}{< P_{n-1}, P_{n-1} >}\frac{< P_{n-1}, P_{n-1} >}{< P_0, P_0 >} = \frac{< xP_{n-1}, P_{n-1} >}{< P_0, P_0 >} =$$

$$\int_{-\infty}^{+\infty} xP_{n-1}(x)^2\mu(dx).$$

□

Now we are ready to prove Proposition 1.

Proof.

Let $\mathcal{P}(\mathbb{R})$ the set of one-variable polynomials on $\mathbb{R}$. Let $U : \Gamma \to \mathcal{P}(\mathbb{R})$ the linear map defined by: $\Phi_0 \mapsto P_0$, $\sqrt{\omega_n \cdots \omega_1}\Phi_n \mapsto P_n$ for $n \geq 1$. Then U is a linear isometric isomorphism.
In fact, in Lemma 6, we have seen that

$$\omega_n \cdots \omega_1 =< P_n, P_n >_\mu, \quad n = 1, 2, \ldots$$

Then,

$$\left\langle U\sum_0^\infty k_n\Phi_n\, ,\, U\sum_0^\infty k_n\Phi_n\right\rangle_\mu = \left\langle \sum_0^\infty k_nU\Phi_n\, ,\, \sum_0^\infty k_nU\Phi_n\right\rangle_\mu = \langle k_0P_0\, ,\, k_0P_0\rangle_\mu$$

$$+\left\langle \sum_1^\infty \frac{k_n}{\sqrt{\omega_n\cdots\omega_1}}P_n\, ,\, \sum_1^\infty \frac{k_n}{\sqrt{\omega_n\cdots\omega_1}}P_n\right\rangle_\mu = k_0^2\,\langle P_0, P_0\rangle +$$

$$\sum_1^\infty \frac{k_n^2}{\omega_n\cdots\omega_1}\,\langle P_n, P_n\rangle_\mu = k_0^2 + \sum_1^\infty k_n^2 = \sum_0^\infty k_n^2.$$

As $\{\Phi_n\}$ is an orthonormal basis we have:

$$\left\langle \sum_0^\infty k_n\Phi_n\, ,\, \sum_0^\infty k_n\Phi_n\right\rangle_\Gamma = \sum_0^\infty k_n^2\,\langle\Phi_n, \Phi_n\rangle_\Gamma = \sum_0^\infty k_n^2.$$

Hence U is isometric. Let V be the candidate for the inverse map of U i.e.

$$V : \mathcal{P}(\mathbb{R}) \to \Gamma, \quad P_0 \mapsto \Phi_0,\;\; P_n \mapsto \sqrt{\omega_n\cdots\omega_1}\Phi_n \quad n \geq 1.$$

Also, let U^* be adjoint of U. We have:

$$< U^*P_0, \Phi_0 >_\mu = < P_0, U\Phi_0 >_\mu = < P_0, P_0 >_\mu = 1 = < \Phi_0, \Phi_0 >_\Gamma = < VP_0, \Phi_0 >_\Gamma .$$

As $\omega_n \cdots \omega_1 =< P_n, P_n >_\mu$, we have for $n, m \geq 1$:

$$< U^*P_n, \Phi_m >_\mu = < P_n, U\Phi_m >_\mu = < P_n, (\omega_m\cdots\omega_1)^{-1/2}P_m >_\mu =$$

$$(\omega_n\cdots\omega_1)^{-1/2} < P_n, P_n >_\mu \delta_{mn} = \sqrt{\omega_n\cdots\omega_1}\delta_{mn},$$

where

$$\delta_{mn} := \begin{cases} 1, & m = n \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, we have:

$$< VP_n, \Phi_m >_\Gamma = < \sqrt{\omega_n \cdots \omega_1}\Phi_n, \Phi_m >_\Gamma = \sqrt{\omega_n \cdots \omega_1} < \Phi_n, \Phi_m >_\Gamma = \sqrt{\omega_n \cdots \omega_1}\delta_{mn}.$$

As $< U^* P_n, \Phi_m > = < VP_n, \Phi_m >$ for all $n, m \geq 0$, we have that $U^* = V = U^{-1}$.

As U is a linear isometric isomorphism, we have:

$$\left\langle \Phi_0, \left(B^+ + B^- + B^o\right)^m \Phi_0 \right\rangle_\Gamma = \left\langle U^{-1} P_0, \left(B^+ + B^- + B^o\right)^m U^{-1} P_0 \right\rangle_\Gamma$$

$$= \left\langle U^* P_0, \left(B^+ + B^- + B^o\right)^m U^* P_0 \right\rangle_\Gamma = \left\langle P_0, U \left(B^+ + B^- + B^o\right)^m U^* P_0 \right\rangle_\mu$$

$$= \left\langle P_0, \left(U \left(B^+ + B^- + B^o\right) U^*\right)^m P_0 \right\rangle_\mu$$

i.e.,

$$\left\langle \Phi_0, \left(B^+ + B^- + B^o\right)^m \Phi_0 \right\rangle_\Gamma = \left\langle P_0, \left(U \left(B^+ + B^- + B^o\right) U^*\right)^m P_0 \right\rangle_\mu \tag{1.1}$$

Let $M_X : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ the multiplication operator by x defined by:

$$(M_X f)(x) = x f(x), \quad x \in \mathbb{R}$$

Rewriting the three-term recurrence relation in terms of M_X, we have:

$$P_1 = M_X P_0 - \alpha_1 P_0$$

$$M_X P_n = P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1}, \quad n = 1, 2, \ldots$$

For Φ_0, we have:

$$M_X U \Phi_0 = M_X P_0 = P_1 + \alpha_1 P_0 = U\sqrt{\omega_1}\Phi_1 + \alpha_1 U \Phi_0 = U(\sqrt{\omega_1}\Phi_1 + \alpha_1 \Phi_0) =$$

$$U(B^+\Phi_0 + B^o\Phi_0) = U(B^+\Phi_0 + B^o\Phi_0 + B^-\Phi_0) = U(B^+ + B^o + B^-)\Phi_0.$$

By the definition of U, $U\sqrt{\omega_n \cdots \omega_1}\Phi_n = P_n$, and by the definition of B^+, B^o, B^-,

$$B^+\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1}, B^-\Phi_0 = 0, B^-\Phi_n = \sqrt{\omega_n}\Phi_{n-1}$$

For Φ_n, $n \geq 1$ we have:

$$M_X U \Phi_n = M_X P_n (\omega_n \cdots \omega_1)^{-1/2} = (P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1})(\omega_n \cdots \omega_1)^{-1/2} =$$

$$(U\sqrt{\omega_{n+1}\cdots\omega_1}\Phi_{n+1}+\alpha_{n+1}U\sqrt{\omega_n\cdots\omega_1}\Phi_n+\omega_nU\sqrt{\omega_{n-1}\cdots\omega_1}\Phi_{n-1})(\omega_n\cdots\omega_1)^{-1/2}$$

$$=U(\sqrt{\omega_{n+1}}\Phi_{n+1}+\alpha_{n+1}\Phi_n+\sqrt{\omega_n}\Phi_{n-1})=U(B^+\Phi_n+B^o\Phi_n+B^-\Phi_n)$$

$$=U(B^++B^o+B^-)\Phi_n.$$

Hence, $M_XU=U(B^++B^o+B^-)$, i.e, $M_X=U(B^++B^o+B^-)U^*$.

Then for $m\geq 1$:

$$\left\langle P_0,\left(U\left(B^++B^-+B^o\right)U^*\right)^mP_0\right\rangle_\mu=\langle P_0,M_X^mP_0\rangle_\mu \tag{1.2}$$

For $m\geq 1$ we have:

$$<P_0,M_X^mP_0>_\mu=\int_{-\infty}^{+\infty}\overline{P_0(x)}(M_X^mP_0)(x)\mu(dx)=\int_{-\infty}^{+\infty}(M_X^{m-1}xP_0)(x)\mu(dx)$$

$$=\int_{-\infty}^{+\infty}(M_X^{m-2}x^2P_0)(x)\mu(dx)=\cdots=\int_{-\infty}^{+\infty}(M_Xx^{m-1}P_0)(x)\mu(dx)=$$

$$\int_{-\infty}^{+\infty}x^mP_0(x)\mu(dx)=\int_{-\infty}^{+\infty}x^m\mu(dx),$$

i.e.,

$$<P_0,M_X^mP_0>_\mu=\int_{-\infty}^{+\infty}x^m\mu(dx). \tag{1.3}$$

The result follows by combining Equations (1.1), (1.2), (1.3).

□

1.4 Asymptotic Spectral Distribution of a Distance-Regular Graph

In this section, we will demonstrate the two main theorems of this work, namely the Quantum Central Limit Theorem for a Growing Distance-Regular Graph and the Central Limit Theorem for a Growing Distance-Regular Graph.

Let $G^{(\nu)}=(V^{(\nu)},E^{(\nu)})$ be a growing distance-regular graph, where a growing parameter ν runs over a directed set. Let $p_{ij}^k(\nu)$ be the intersection numbers of $G^{(\nu)}$ and $\kappa(\nu)=p_{11}^0(\nu)$ the degree of $G^{(\nu)}$. Each graph $G^{(\nu)}$ has a fixed origin $o_\nu\in V^{(\nu)}$, associated with the corresponding stratification:

$$V^{(\nu)}=\bigcup_{n=0}^{\infty}V_n^{(\nu)},\qquad V_n^{(\nu)}=\{x\in V^{(\nu)}:d(o_\nu,x)=n\},$$

the subspace $\Gamma(G^{(\nu)})$ spanned by the unit vectors:

$$\Phi_n^{(\nu)}=|V_n^{(\nu)}|^{-1/2}\sum_{x\in V_n^{(\nu)}}\delta_x,$$

and the quantum decomposition of the adjacency matrix:

$$A_\nu = A_\nu^+ + A_\nu^- + A_\nu^o.$$

A natural normalization of adjacency matrix A_ν is given by:

$$\frac{A_\nu - \langle A_\nu \rangle_o}{\Sigma(A_\nu)}, \qquad \Sigma(A_\nu)^2 = \left\langle (A_\nu - \langle A_\nu \rangle_o)^2 \right\rangle_o.$$

The mean and the variance of are given by:

$$\langle A_\nu \rangle_o = 0 \qquad \Sigma(A_\nu) = \sqrt{\left\langle (A_\nu - 0)^2 \right\rangle_o} = \sqrt{\kappa(\nu)}.$$

Then, the normalized adjacency becomes:

$$\frac{A_\nu}{\sqrt{\kappa(\nu)}} = \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} + \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} + \frac{A_\nu^o}{\sqrt{\kappa(\nu)}}.$$

Defining

$$\bar{\omega}_n(\nu) = \frac{p_{1,n-1}^n(\nu) p_{1,n}^{n-1}(\nu)}{\kappa(\nu)},$$

$$\bar{\alpha}_n(\nu) = \frac{p_{1,n-1}^{n-1}(\nu)}{\sqrt{\kappa(\nu)}},$$

we have:

$$\frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} = \sqrt{\frac{p_{1,n}^{n+1}(\nu) p_{1,n+1}^n(\nu)}{\kappa(\nu)}} \Phi_{n+1}^{(\nu)} = \sqrt{\bar{\omega}_{n+1}(\nu)} \Phi_{n+1}^{(\nu)}, n = 0, 1, 2, \dots \tag{1.4}$$

$$\frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)} = 0, \tag{1.5}$$

$$\frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} = \sqrt{\frac{p_{1,n-1}^n(\nu) p_{1,n}^{n-1}(\nu)}{\kappa(\nu)}} \Phi_{n-1}^{(\nu)} = \sqrt{\bar{\omega}_n(\nu)} \Phi_{n-1}^{(\nu)}, n = 1, 2, \dots \tag{1.6}$$

$$\frac{A_\nu^o}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} = \frac{p_{1,n}^n(\nu)}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} = \bar{\alpha}_{n+1}(\nu) \Phi_n^{(\nu)}, n = 0, 1, 2, \dots \tag{1.7}$$

Thus we need to consider the limits of $\bar{\omega}_n(\nu), \bar{\alpha}_n(\nu)$. For $n = 1, 2, \ldots$, we define:

$$\tilde{\omega}_n := \lim_{\nu \to \infty} \bar{\omega}_n(\nu) = \lim_{\nu \to \infty} \frac{p_{1,n-1}^n(\nu) p_{1,n}^{n-1}(\nu)}{\kappa(\nu)},$$

$$\tilde{\alpha}_n = \lim_{\nu \to \infty} \bar{\alpha}_n(\nu) = \lim_{\nu \to \infty} \frac{p_{1,n-1}^{n-1}(\nu)}{\sqrt{\kappa(\nu)}}.$$

As there is no guarantee that such limits exist, we need to consider the following condition:

(DR) (i) for all $n = 1, 2, \ldots$ the limits $\tilde{\omega}_n$ and $\tilde{\alpha}_n$ exist with $\tilde{\omega}_n \in \mathbb{R}_+^*,\ \tilde{\alpha}_n \in \mathbb{R}$,
or (ii) there exists $n \in \mathbb{N}^*$ such that the limits $\tilde{\omega}_1, \ldots, \tilde{\omega}_n$ and $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ exist with

$$\begin{cases} \tilde{\omega}_1 = 1,\ \tilde{\omega}_2, \ldots, \tilde{\omega}_{n-1} \in \mathbb{R}_+^*,\ \tilde{\omega}_n = 0, \\ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \in \mathbb{R}. \end{cases}$$

If the Condition (DR) is satisfied, $(\{\tilde{\omega}_n\}, \{\tilde{\alpha}_n\})$ becomes a Jacobi coefficient.

The Condition (DR) is satisfied with the homogeneous trees (see section 1.6). The Condition (DR) is not satisfied with the complete graphs. Consider the growing graph K^N with $N \longrightarrow \infty$, where K^N is the complete graph on N vertices. We have $\kappa(N) = N - 1$ and $p_{1,1}^1(N) = N - 2$. Then,

$$\tilde{\alpha}_2 = \lim_{N\to\infty} \frac{p_{1,1}^1(N)}{\sqrt{\kappa(N)}} = \lim_{N\to\infty} \frac{N-2}{\sqrt{N-1}} = +\infty.$$

Theorem 7. (Quantum Central Limit Theorem for a Growing Distance-Regular Graph) *[[7], chapter 3 page 95]* Let $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing distance-regular graph with adjacency matrix A_ν, whose degree is $\kappa(\nu)$. Assume that the Condition (DR) is satisfied. Let $\Gamma_{(\{\tilde{\omega}_n\},\{\tilde{\alpha}_n\})} = (\Gamma, \{\psi_n\}, B^+, B^-, B^o)$ be an interacting Fock space associated with $(\{\tilde{\omega}_n\}, \{\tilde{\alpha}_n\})$. Then we have:

$$\lim_{\nu\to\infty} \frac{A_\nu^\epsilon}{\sqrt{\kappa(\nu)}} = B^\epsilon, \epsilon \in \{+, -, o\},$$

in the sense of stochastic convergence with respect to the vacuum states, i.e.,

$$\lim_{\nu\to\infty} \left\langle \Phi_0^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)} \right\rangle = \langle \psi_0, B^{\epsilon_m} \cdots B^{\epsilon_1} \psi_0 \rangle$$

for any $\epsilon_1, \ldots, \epsilon_m \in \{+, -, o\}$ and $m = 1, 2, \ldots$.

Proof.
We see from equations (1.4), (1.5), (1.6) and (1.7) that

$$\frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)}$$

is a constant multiple of $\Phi_{\epsilon_1+\cdots+\epsilon_m}^{(\nu)}$ if $\epsilon_1 + \ldots + \epsilon_k \geq 0$, for $k = 1, 2, \ldots, m$ (otherwise it vanishes) and the constant is given by a finite product of $\bar{\omega}_n(\nu)$ and $\bar{\alpha}_n(\nu), n = 1, 2, \ldots$ Hence, as Condition (DR) is satisfied, the limit $\lim_{\nu\to\infty} \left\langle \Phi_0^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)} \right\rangle$ exists. Since $\tilde{\omega}_n := \lim_{\nu\to\infty} \bar{\omega}_n(\nu)$ and $\tilde{\alpha}_n := \lim_{\nu\to\infty} \bar{\alpha}_n(\nu)$, we have that the limit coincides with $\langle \psi_0, B^{\epsilon_m} \cdots B^{\epsilon_1} \psi_0 \rangle$.

$\square$

Observation: As $\{\Phi_n\}$ is a sequence of orthonormal vectors,

$$\left\langle \Phi_0^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)} \right\rangle \neq 0 \Rightarrow \epsilon_1 + \ldots + \epsilon_m = 0$$

Now we will prove the following theorem presented in the Introduction:

Theorem 2. **(Central Limit Theorem for a Growing Distance-Regular Graph)** Let $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing distance-regular graph with adjacency matrix A_ν, with degree $\kappa(\nu)$, and intersection numbers $p_{i,j}^k(\nu)$. If Condition (DR) is satisfied, we have:

$$\lim_{\nu\to\infty} \left\langle \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m \right\rangle_o = \int_{-\infty}^{+\infty} x^m \mu(dx), m = 1, 2, \ldots,$$

where μ is a probability measure with Jacobi coefficient $(\{\tilde{\omega}_n\}, \{\tilde{\alpha}_n\})$.

Proof.

For $m = 1, 2, \ldots,$ we have

$$\lim_{\nu\to\infty} \left\langle \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m \right\rangle_o = \lim_{\nu\to\infty} \left\langle \delta_o, \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m \delta_o \right\rangle =$$

$$\lim_{\nu\to\infty} \left\langle \Phi_0^{(\nu)}, \left(\frac{A_\nu^+}{\sqrt{\kappa(\nu)}} + \frac{A_\nu^o}{\sqrt{\kappa(\nu)}} + \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \right)^m \Phi_0^{(\nu)} \right\rangle =$$

$$\lim_{\nu\to\infty} \left\langle \Phi_0^{(\nu)}, \sum_{\epsilon_1,\ldots,\epsilon_m \in \{+,-,o\}} \left(\frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \right) \Phi_0^{(\nu)} \right\rangle =$$

$$\sum_{\epsilon_1,\ldots,\epsilon_m \in \{+,-,o\}} \lim_{\nu\to\infty} \left\langle \Phi_0^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)} \right\rangle =$$

$$\sum_{\epsilon_1,\ldots,\epsilon_m \in \{+,-,o\}} \langle \psi_0, B^{\epsilon_m} \cdots B^{\epsilon_1} \psi_0 \rangle = \left\langle \psi_0, \sum_{\epsilon_1,\ldots,\epsilon_m \in \{+,-,o\}} (B^{\epsilon_m} \cdots B^{\epsilon_1}) \psi_0 \right\rangle =$$

$$\left\langle \psi_o, \left(B^+ + B^- + B^o \right)^m \psi_o \right\rangle.$$

By Proposition 1, the last line is equal to

$$\int_{-\infty}^{+\infty} x^m \mu(dx).$$

□

1.5 Proof of Proposition 2

We saw that Proposition 2 is important since it establishes a relation between the Wigner semicircle law and the free Fock Space. We first recall the statement of this proposition and then we consider some results before proving it.

Proposition 2. For the free Fock space, it holds that

$$\left\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \right\rangle_{\Gamma_{\text{free}}} = \int_{-\infty}^{+\infty} x^m \rho(dx) \quad m = 1, 2, \dots,$$

where $\rho(dx)$ is the Wigner semicircle law.

Definition 5. The interacting Fock Space associated with the Jacobi coefficient given by $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$ is called the *free Fock Space* and is denoted by Γ_{free}, i.e.,

$$\Gamma_{\text{free}} = \Gamma_{(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})} = (\Gamma, \{\Phi_n\}, B^+, B^-, 0) = (\Gamma, \{\Phi_n\}, B^+, B^-)$$

with

$$B^+ \Phi_n = \Phi_{n+1}, \quad n \geq 0,$$

$$B^- \Phi_0 = 0, \qquad B^- \Phi_n = \Phi_{n-1}, \quad n \geq 1,$$

$$B^o \Phi_n = 0, \quad n \geq 0.$$

Lemma 7. *[[7], chapter 1 page 33]* For the free Fock space, we have

$$\begin{cases} \langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \rangle = |\mathcal{C}_m|, \\ \langle \Phi_0, (B^+ + B^-)^{2m+1} \Phi_0 \rangle = 0, \end{cases}$$

for $m \geq 1$.

Proof.

For more information about Catalan numbers, see Lemma 1 in Section 0.4.6. We have that

$$\left\langle \Phi_0, (B^+ + B^-)^k \Phi_0 \right\rangle = \sum_{(\epsilon_1 \dots, \epsilon_k) \in \{+,-\}^k} \langle \Phi_0, B^{\epsilon_k} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle$$

As $B^+\Phi_n = \Phi_{n+1}, B^-\Phi_n = \Phi_{n-1}$ for the free Fock Space, a necessary condition for $B^{\epsilon_k} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0$ being a multiple of Φ_0 is $\epsilon_1 + \ldots + \epsilon_k = 0$. For this reason, $\langle \Phi_0, B^{\epsilon_{2m+1}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle = 0$ for all m, and then $\langle \Phi_0, (B^+ + B^-)^{2m+1} \Phi_0 \rangle = 0$.

In the sum $\sum_{(\epsilon_1 \dots, \epsilon_k) \in \{+,-\}^k} \langle \Phi_0, B^{\epsilon_k} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle$, since $B^-\Phi_0 = 0$, there are a lot of terms that vanish. We may state the following:

$$\left\langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \right\rangle = \sum_{(\epsilon_1 \dots, \epsilon_{2m}) \in \mathcal{C}_m} \langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle$$

In fact, let $(\epsilon_1 \ldots, \epsilon_{2m}) \in \{+,-\}^{2m} \backslash \mathcal{C}_m$. Then, $\exists k \in 1, \ldots, 2m-1$ such that $\epsilon_1 + \ldots + \epsilon_l \geq 0$ for all $l = 1, \ldots, k-1$ and $\epsilon_1 + \ldots + \epsilon_k = -1$. Then, $\epsilon_1 + \ldots + \epsilon_{k-1} = 0$ and $\epsilon_k = -1$. Hence, as $B^- \Phi_0 = 0$,

$$\left\langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \right\rangle = \left\langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_{k+1}} B^- B^{\epsilon_{k-1}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \right\rangle =$$

$$\left\langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_{k+1}} B^- \Phi_0 \right\rangle = \langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_{k+1}} 0 \rangle = 0$$

Now, we will demonstrate that $\langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \rangle = |\mathcal{C}_m|$, for $m \geq 1$, and then the result will follow by Lemma 1. In fact, we saw that

$$\left\langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \right\rangle = \sum_{\mathcal{C}_m} \langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle$$

For each $\epsilon = (\epsilon_1 \ldots, \epsilon_{2m}) \in \mathcal{C}_m$, we have

$$\left\langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \right\rangle = < \Phi_0, \Phi_0 > = 1,$$

because in the free Fock space, $B^+ \Phi_n = \Phi_{n+1}, B^- \Phi_n = \Phi_{n-1}$, and as $\epsilon \in \mathcal{C}_m$, $\epsilon_1 + \ldots + \epsilon_{2m} = 0$. Hence,

$$\left\langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \right\rangle = \sum_{\mathcal{C}_m} \langle \Phi_0, B^{\epsilon_{2m}} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle = \sum_{\mathcal{C}_m} 1 = |\mathcal{C}_m|.$$

□

Now, we are ready for the proof of Proposition 2.

Proof.

Let $\{M_m\}$ denote the moment sequence of the Wigner semicircle law, i.e.

$$M_m = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2} dx, \quad m = 1, 2, \ldots$$

As $x^{2m+1}\sqrt{4-x^2}$ is an odd function for every $m \geq 1$, , $M_{2m+1} = 0$ for every $m \geq 1$. For the even moments we have, after the substitution $x = 2cos\theta$:

$$2\pi M_{2m} = \int_{-2}^{+2} x^{2m} \sqrt{4-x^2} dx = \int_{-\pi}^{0} (2\cos\theta)^{2m} \sqrt{4 - (2\cos\theta)^2} (-2\sin\theta) d\theta =$$

$$2^{2m} \int_{-\pi}^{0} \cos^{2m}\theta \; 2\sqrt{1-\cos^2\theta} (-2\sin\theta) d\theta = -2^{2m+2} \int_{-\pi}^{0} \cos^{2m}\theta \; |sin\theta| \sin\theta d\theta =$$

$$2^{2m+2} \int_{-\pi}^{0} \cos^{2m}\theta \sin^2\theta d\theta$$

Then, for all $m \geq 1$ we have:

$$2\pi M_{2m} = 2^{2m+2} \int_{-\pi}^{0} \cos^{2m}\theta \sin^2\theta d\theta$$

By integral formula (4), we have:

$$M_{2m} = \frac{2^{2m+2}}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{0} \cos^{2m}\theta \sin^2\theta d\theta = 2^{2m+1} \frac{1}{2^{2m+1}} |\mathcal{C}_m| = |\mathcal{C}_m|.$$

In summary, we have for all $m \geq 1$:

$$\begin{cases} M_{2m} = |\mathcal{C}_m| \\ M_{2m+1} = 0, \end{cases}$$

By Lemma 7, we are able to establish for all $m \geq 1$:

$$\left\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \right\rangle = M_m.$$

□

1.6 Asymptotic Spectral Distribution of Homogeneous Trees

We will apply the theorems from the previous section for a particular type of distance-regular graph: the homogeneous trees. The main theorem of this section establishes a link between homogeneous trees and the Wigner semicircle law, and that link is the free Fock Space. A graph is called a *tree* if it has no cycle. A tree is called *homogeneous* if it is regular. A homogeneous tree is always an infinite graph. We denote by T_κ a homogeneous tree with degree $\kappa \geq 2$ and by $A = A_\kappa$ its adjacency matrix. A homogeneous tree T_k is distance-transitive and hence distance-regular. By straightforward observation we easily obtain the intersection numbers:

$$p_{11}^0 = \kappa$$

$$p_{1,n}^{n-1} = \kappa - 1, \quad n = 2, 3, \ldots$$

$$p_{1,n-1}^{n} = 1 \quad n = 1, 2, 3, \ldots$$

$$p_{1,n}^{n} = 0 \quad n = 0, 1, 2, \ldots$$

Fix an origin o of T_κ and consider the quantum decomposition:

$$A = A^+ + A^-.$$

In fact, $A^o = 0$ because if we have $A^o \neq 0$, there will be a cycle of order 3 in a tree. We are interested in a probability measure μ having finite moments of all orders such that

$$\lim_{\nu \to \infty} \left\langle \delta_o, \left(\frac{A_\kappa}{\sqrt{\kappa}} \right)^m \delta_o \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), m = 1, 2, \ldots.$$

From the previous section, the Jacobi coefficient $(\{\tilde{\omega}_n\}, \{\tilde{\alpha}_n\})$ of such measure is given by:

$$\tilde{\omega}_1 := \lim_{\kappa\to\infty} \bar{\omega}_1(\kappa) = \lim_{\kappa\to\infty} \frac{p^1_{1,0}(\kappa) p^0_{1,1}(\kappa)}{\kappa} = \lim_{\kappa\to\infty} \frac{1\cdot(\kappa)}{\kappa} = 1,$$

$$\tilde{\omega}_n := \lim_{\kappa\to\infty} \bar{\omega}_n(\kappa) = \lim_{\kappa\to\infty} \frac{p^n_{1,n-1}(\kappa) p^{n-1}_{1,n}(\kappa)}{\kappa} = \lim_{\kappa\to\infty} \frac{1\cdot(\kappa-1)}{\kappa} = 1, \text{ for } n \geq 2$$

$$\tilde{\alpha}_n = \lim_{\kappa\to\infty} \bar{\alpha}_n(\kappa) = \lim_{\kappa\to\infty} \frac{p^{n-1}_{1,n-1}(\kappa)}{\sqrt{\kappa}} = \lim_{\kappa\to\infty} \frac{0}{\sqrt{\kappa}} = 0, \text{ for } n \geq 1.$$

The Condition (DR) is then satisfied. As we have $(\{\tilde{\omega}_n\} \equiv 1, \{\tilde{\alpha}_n \equiv 0\})$, we are able to establish that it is the free Fock space that is linked to the asymptotic spectrum of homogeneous trees.
Once we adapt the Quantum Central Limit Theorem seen in the previous section to the homogeneous trees, we come to the following:

Theorem 8. (Quantum Central Limit Theorem for homogeneous trees) *[[7], chapter 4 page 110]* Let T_κ be a homogeneous tree of degree κ with adjacency matrix A_κ. Let $\Gamma_{\text{free}} = (\Gamma, \{\psi_n\}, B^+, B^-)$ be a free Fock space. Then we have:

$$\lim_{\kappa\to\infty} \frac{A^\epsilon_\kappa}{\sqrt{\kappa}} = B^\epsilon, \quad \epsilon \in \{+,-\},$$

in the sense of stochastic convergence with respect to the vacuum states.

Now we will prove the following theorem presented in the Introduction:

Theorem 3. (Central Limit Theorem for homogeneous trees) Let T_κ be a homogeneous tree of degree κ with adjacency matrix A_κ. Then we have

$$\lim_{\kappa\to\infty} \left\langle \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m \right\rangle_o = \frac{1}{2\pi}\int_{-2}^{+2} x^m \sqrt{4-x^2}\,dx, \quad m = 1, 2, \ldots,$$

where the probability measure on the right-side is the Wigner semicircle law.

Proof.
By the previous theorem, we have

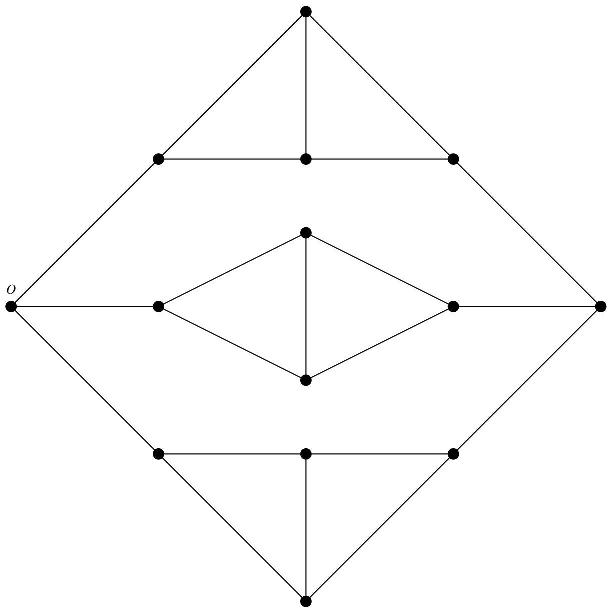
$$\lim_{\kappa\to\infty} \left\langle \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m \right\rangle_o = \lim_{\nu\to\infty} \left\langle \delta_o, \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m \delta_o \right\rangle = \lim_{\nu\to\infty} \left\langle \Phi_0, \left(\frac{A^+_\kappa + A^-_\kappa}{\sqrt{\kappa}}\right)^m \Phi_0 \right\rangle = \langle \psi_0, (B^+ + B^-)^m \psi_0 \rangle$$

and as we saw in the Proposition 2,

$$\langle \psi_0, (B^+ + B^-)^m \psi_0 \rangle = \frac{1}{2\pi}\int_{-2}^{+2} x^m \sqrt{4-x^2}\,dx.$$

1.7 Counterexample

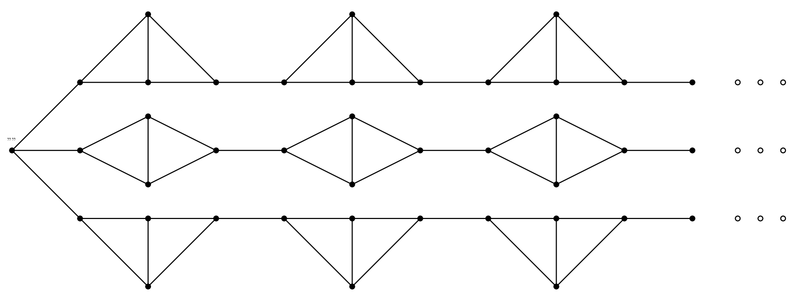
A graph does not need to be distance-regular in order to $\omega_-(y), \omega_+(y), \omega_o(y)$ be constant on V_n. We can construct a regular graph such that $\omega_-(y), \omega_+(y), \omega_o(y)$ are constant on V_n for all $n \in \mathbb{N}$, but is not a distance-regular graph:



The above graph is a 3-regular graph, and we have:

$$\omega_-(y) = \begin{cases} 0, & y \in V_0, \\ 1, & y \in V_1, \\ 1, & y \in V_2, \\ 2, & y \in V_3, \\ 3, & y \in V_4. \end{cases} \qquad \omega_o(y) = \begin{cases} 0, & y \in V_0, \\ 0, & y \in V_1, \\ 1, & y \in V_2, \\ 0, & y \in V_3, \\ 0, & y \in V_4. \end{cases} \qquad \omega_+(y) = \begin{cases} 1, & y \in V_0, \\ 2, & y \in V_1, \\ 1, & y \in V_2, \\ 1, & y \in V_3, \\ 0, & y \in V_4. \end{cases}$$

However, that graph is not distance regular: some pairs of neighbour vertices have 1 common neighbour while other pairs have 2 of them. We are also able to construct an infinite graph with same properties of the previous one:



Chapter 2

Orthogonal Matrix-Valued Polynomials and Matrix-Valued Measures

In this chapter we still make use of a spectral approach to study certain graphs. By the Gram-Schmidt orthogonalization procedure (the inner product induced by the measure is considered) applied to the sequence of monomials, we obtain a sequence of orthogonal polynomials. As previously discussed, this sequence satisfies a three-term recurrence relation from which three sequences of coefficients are extracted. With these coefficients, in certain cases, we are able to display an expression of the original measure, but now we consider a setting where the polynomials will be matrix-valued, and so will be the studied measure.

2.1 Proof of Properties of the Left-Inner Product

We prove the matrix-valued inner product properties 1,2, and 3 presented in the Introduction:

1. We have:

$$< Q, P >^* = \int_{-\infty}^{+\infty} (Q(t)dW(t)P(t)^*)^* = \int_{-\infty}^{+\infty} P(t)^{**}dW(t)^*Q(t)^*$$
$$= \int_{-\infty}^{+\infty} P(t)dW(t)Q(t)^* =< P, Q > .$$

2. By the linear properties of integral, we have $\forall c_1, c_2 \in \mathbb{C}^{N\times N}$:

$$< c_1P_1 + c_2P_2, Q >= \int_{-\infty}^{+\infty} (c_1P_1 + c_2P_2)dW(t)Q(t)^*$$

$$= c_1 \int_{-\infty}^{+\infty} P_1dW(t)Q(t)^* + c_2 \int_{-\infty}^{+\infty} P_2dW(t)Q(t)^* = c_1 < P_1, Q > +c_2 < P_2, Q > .$$

3. For $t \in \mathbb{R}$, we have:

$$< tP, Q >= \int_{-\infty}^{+\infty} tP(t)dW(t)Q(t)^*$$

$$= \int_{-\infty}^{+\infty} P(t)dW(t)tQ(t)^* = \int_{-\infty}^{+\infty} tP(t)dW(t)(tQ(t))^* =< P, tQ > .$$

2.2 Proof of Theorem 4

Theorem 9. *[[15], page 32]* For $n = 0, 1, 2, \ldots$ suppose P_n are left orthonormal matrix polynomials with respect to the left inner product obtained by using a matrix measure W on the real line. Then there exist invertible matrices A_n and Hermitian matrices B_n such that

$$tP_n(t) = A_n P_{n+1}(t) + B_n P_n(t) + A_{n-1}^* P_{n-1}(t), \qquad n \geq 0$$

$$P_{-1}(t) = 0, P_0(t) \in \mathbb{C}^{N\times N} \backslash \{0\}.$$

Proof.

Let $n \geq 1$. The degree of the polynomial $tP_n(t)$ is $n+1$, then we have:

$$tP_n(t) = \sum_{i=0}^{n+1} K_{n,i} P_i(t), \tag{2.1}$$

for some $K_{n,i} \in \mathbb{C}^{N\times N}$.

For $j \in \{0, \ldots, n-2\}$, the polynomial tP_j is a linear combination of $P_0(t), P_1(t), \ldots, P_{n-1}(t)$. Hence, $< tP_j, P_n >= 0$. On the other hand, for $k \in \{0, \ldots, n+1\}$ by property 3 we have:

$$< tP_k, P_n >=< P_k, tP_n >= \sum_{i=0}^{n+1} K_{n,i} < P_k(t), P_i(t) >= K_{n,k} < P_k(t), P_k(t) >= K_{n,k} I = K_{n,k},$$

i.e,

$$K_{n,k} =< tP_k, P_n >, \qquad k \in \{0, \ldots, n+1\}. \tag{2.2}$$

Therefore, $K_{n,j} = 0$ for $j \in \{0, \ldots, n-2\}$. Also, by Equation (2.2) and properties 1 and 3 we have $K_{n,n+1}^* = K_{n+1,n}$, and $K_{n,n}^* = K_{n,n}$. In fact,

$$K_{n,n+1}^* =< tP_{n+1}, P_n >^* =< P_n, tP_{n+1} >=< tP_n, P_{n+1} >= K_{n+1,n},$$

and

$$K_{n,n}^* =< tP_n, P_n >^* =< P_n, tP_n >=< tP_n, P_n >= K_{n,n}.$$

Then, for $n \geq 1$, we define $A_n := K_{n,n+1}$, $B_n := K_{n,n}$, on Equation (2.1). As $K_{n,n}^* = K_{n,n}$, B_n is Hermitian. The result follows, since $K_{n,n-1} = K_{n-1,n}^* = A_{n-1}^*$.

□

2.3 Proof of Theorem 5

This section follows [5]. The main objective of this section is to display an expression for a specific class of matrix-valued measure associated with matrix-valued orthonormal polynomials. Before establishing this theorem, some preliminary results are needed. The Stieltjes transform of $W_{A,B}$ is given by:

$$G_{W_{A,B}}(z) = G_{A,B}(z) := \int_{\mathbb{R}} \frac{dW_{A,B}(t)}{z-t}$$

We will assume the following theorem:

Theorem 10. Let $(P_n)_n$ be the orthonormal matrix polynomials satisfying the three-term recurrence relation with constant sequences. Then

$$\lim_{n\to\infty} P_{n-1}(z)P_n^{-1}(z) = G_{A,B}(z)A,$$

for $z \in \mathbb{C}\backslash\Lambda$, where $\Lambda \subseteq \mathbb{R}$.

We also will assume the two following lemmas:

Lemma 8. Let $(P_n)_n$ be the orthonormal matrix polynomials satisfying the three-term recurrence relation with constant sequences. Then the sequence $(P_n)_n$ has a unique matrix weight $W_{A,B}$, and there exists a positive constant $M > 0$, which does not depend on n, such that $W_{A,B}$ has compact support contained in $[-M, M]$.

Lemma 9. *[[5], page 317]* If M and N are Hermitian matrices, then the matrix $Mz + N$ is diagonalizable for $z \in \mathbb{C}\backslash\Gamma$, with $|\Gamma| < \infty$, depending of M, N, and $\Gamma \cap \mathbb{R} = \emptyset$.

As the matrix A is positive-definite, we can take the square root $A^{-1/2}$ as explained in Section 0.4.4. Then $A^{-1/2}$ is also positive-definite and, in particular, Hermitian.

Proposition 7. Let A positive-definite and B Hermitian. Then the matrix function

$$A^{-1/2}(B - zI)A^{-1/2} - \sqrt{A^{-1/2}(zI - B)A^{-1}(zI - B)A^{-1/2} - 4I}$$

is analytic in $\mathbb{C}\backslash\Delta$, where $\Delta \subset \mathbb{R}$, and $x \in \Delta \iff A^{-1/2}(B - zI)A^{-1/2}$ has at least one eigenvalue in $[-2, 2]$.

Proof.

The matrix

$$D_H(z) := A^{-1/2}(B - zI)A^{-1/2}$$

is diagonalizable for $z \in \mathbb{C}\backslash\Gamma$, with $|\Gamma| < \infty$, depending of M, N, and $\Gamma \cap \mathbb{R} = \emptyset$, i.e., there exists a invertible matrix $P(z)$ and a diagonal matrix $D_1(z)$ such that:

$$D_H(z) = P^{-1}(z)D_1(z)P(z).$$

In fact, $A^{-1/2}(B - zI)A^{-1/2} = A^{-1/2}BA^{-1/2} - A^{-1/2}zA^{-1/2} = A^{-1/2}BA^{-1/2} - A^{-1}z$. Then, it follows by applying Lemma 9 with $M = -A^{-1}$ and $N = A^{-1/2}BA^{-1/2}$ (since $A, B, A^{-1/2}$ are Hermitians, so are the matrices M, N).

The matrix

$$H_{A,B}(z) := A^{-1/2}(zI - B)A^{-1}(zI - B)A^{-1/2} - 4I$$

is diagonalizable for $z \in \mathbb{C}\backslash\Gamma$, because $H_{A,B}(z) = D_H(z)^2 - 4I$. We have then

$$H_{A,B}(z) = P^{-1}(z)(D_1^2(z) - 4I)P(z).$$

In fact,

$$H_{A,B} = D_H^2 - 4I = (P^{-1}D_1P)^2 - 4I = P^{-1}D_1^2P - 4I = P^{-1}D_1^2P - P^{-1}4IP = P^{-1}(D_1^2 - 4I)P.$$

With the notation above, we have to prove that the matrix function

$$D_H(z) - \sqrt{H_{A,B}(z)}$$

is analytic in $\mathbb{C}\backslash\Delta$. For this, we have to set how the square root of $H_{A,B}(z)$ is taken. We saw that $H_{A,B}(z)$ is diagonalizable for $z \in \mathbb{C}\backslash\Gamma$ and $\Gamma \cap \mathbb{R} = \emptyset$. The eigenvalues of $H_{A,B}(z)$ are of the form $\lambda^2 - 4$, where λ is an eigenvalue of $D_H(z)$. In order to take the square root of $H_{A,B}(z)$, we must set how we will take the square root of its eigenvalues. The function $f(z) = z - \sqrt{z^2 - 4}$ have branch points on $z = \pm 2$ and then $[-2, 2]$ is a branch line. We remark that $|f(z)| = 2$ for all $z \in [-2, 2]$. In fact, for $z \in [-2, 2]$, we have $z^2 - 4 \leq 0$, and then:

$$|f(z)| = |z - \sqrt{z^2 - 4}| = |z - i\sqrt{4 - z^2}| = \sqrt{z^2 + (4 - z^2)} = 2.$$

Hence, the two branches of the function f are defined by $|f(z)| > 2$ and $|f(z)| < 2$. We take the square root $\sqrt{z}$ such that $|z - \sqrt{z^2 - 4}| < 2$ for $z \in \mathbb{C}\backslash[-2, 2]$.

Then we have:

$$\sqrt{H_{A,B}(z)} = P^{-1}(z)\left(\sqrt{D_1^2(z) - 4I}\right)P(z).$$

where $\sqrt{D_1^2 - 4I}$ is given by taking the square root of each $\lambda^2 - 4$ such that$|\lambda - \sqrt{\lambda^2 - 4}| < 2$. For this to be possible, we cannot allow $\lambda \in [-2, 2]$, i.e. $D_H(z)$ must not have any eigenvalue in $[-2, 2]$. Hence, the matrix function $\sqrt{H_{A,B}(z)}$ is analytic in $\mathbb{C}\backslash\Delta$, where

$$\Delta = \{z \in \mathbb{C} : D_H(z) \text{ has at least one eigenvalue in } [-2, 2]\}.$$

To finish the proof, we need to show that $\Delta \subset \mathbb{R}$.
In fact, let $z \in \Delta$. Then there is $\lambda \in [-2, 2] \subset \mathbb{R}$ and a λ-eigenvector $u \in \mathbb{C}\backslash\{0\}$ such that $D_H(z)u = \lambda u$, i.e. $u^* D_H(z)u = \lambda$. By the definition of $D_H(z)$, we have $u^* A^{-1/2} B A^{-1/2} u - u^* A^{-1/2} z I A^{-1/2} u = \lambda$. As $\lambda \in \mathbb{R}$, we have that $Im(\lambda) = 0$ and then $-u^* A^{-1/2} Im(z) I A^{-1/2} u = 0$, i.e. $Im(z) u^* A^{-1} u = 0$. As the matrix A is positive-definite, we have $Im(z) = 0$, which means $z \in \mathbb{R}$.

$\square$

Proposition 8. *[[5], page 319]* Let A be positive-definite and B Hermitian. Then the Stieltjes transform of $W_{A,B}$ is given by:

$$G_{W_{A,B}}(z) = G_{A,B}(z) := \int_{\mathbb{R}} \frac{dW_{A,B}(t)}{z - t} = \frac{1}{2} A^{-1}(zI - B)A^{-1} - \frac{1}{2} A^{-1/2} \sqrt{H_{A,B}(z)} A^{-1/2},$$

with $z \in \mathbb{C}\backslash$ supp $W_{A,B}$.

Proof.
In [5], the author admits that the three-term relation can be extended to $\mathbb{C}$ as follows:

$$zP_n(z) = AP_{n+1}(z) + BP_n(z) + AP_{n-1}(z)$$

Hence, multiplying by $P_n^{-1}(z)$ we have:

$$zI = AP_{n+1}(z)P_n^{-1}(z) + B + AP_{n-1}(z)P_n^{-1}(z)$$

Taking the limit as $n \to \infty$ and using Theorem 10, we have:

$$zI = AA^{-1}G_{A,B}^{-1}(z) + B + AG_{A,B}(z)A,$$

which can be written as

$$AG_{A,B}(z)AG_{A,B}(z) + (B - zI)G_{A,B}(z) + I = 0.$$

If we write $F_{A,B}(z) = A^{1/2} G_{A,B}(z) A^{1/2}$, we have $G_{A,B}(z) = A^{-1/2} F_{A,B}(z) A^{-1/2}$, and then:

$$AA^{-1/2} F_{A,B}(z) A^{-1/2} AA^{-1/2} F_{A,B}(z) A^{-1/2} + (B - zI) A^{-1/2} F_{A,B}(z) A^{-1/2} + I = 0$$

Hence,

$$A^{1/2}F_{A,B}(z)^2A^{-1/2} + (B - zI)A^{-1/2}F_{A,B}(z)A^{-1/2} + I = 0$$

Multiplying by $A^{-1/2}$ by the left we have:

$$F_{A,B}(z)^2A^{-1/2} + A^{-1/2}(B - zI)A^{-1/2}F_{A,B}(z)A^{-1/2} + A^{-1/2} = 0$$

Multiplying by $A^{1/2}$ by the right we have:

$$F_{A,B}(z)^2 + A^{-1/2}(B - zI)A^{-1/2}F_{A,B}(z) + I = 0$$

By the definition of $D_H(z)$, we have:

$$F_{A,B}(z)^2 + D_H(z)F_{A,B}(z) + I = 0 \tag{2.3}$$

From Lemma 8, there exists $M > 0$ such that $\operatorname{supp}(W_{A,B}) \subseteq [-M, M]$. Hence, for $z \in \mathbb{R}$ with $z > M$, we have $z - t > 0$ for $t \in \operatorname{supp}(W_{A,B})$ and then $G_{A,B}(z)$ is positive-definite (by assumption $W_{A,B}$ is a positive matrix-valued measure). So is $F_{A,B}$, since $A^{1/2}$ is positive-definite. Also, $H_{A,B}$ is positive for z large enough. Looking at Equation 2.3 as a quadratic equation with unknown $F_{A,B}(z)$, we have the following expression for $F_{A,B}(z)$:

$$F_{A,B}(z) = \frac{1}{2}(-D_H(z) - T)$$

where T is a square root of the matrix $(D_H(z))^2 - 4I = H_{A,B}(z)$.

Then,

$$F_{A,B}(z) = \frac{1}{2}A^{-1/2}(zI - B)A^{-1/2} - \frac{1}{2}\sqrt{H_{A,B}(z)}.$$

And finally,

$$\begin{aligned} G_{A,B}(z) &= A^{-1/2}\left(\frac{1}{2}A^{-1/2}(zI - B)A^{-1/2} - \frac{1}{2}\sqrt{H_{A,B}(z)}\right)A^{-1/2}. \\ &= \frac{1}{2}A^{-1}(zI - B)A^{-1} - \frac{1}{2}A^{-1/2}\sqrt{H_{A,B}(z)}A^{-1/2}, \end{aligned}$$

for z positive and large enough. By definition, we have that $G_{A,B}(z)$ is analytic in $\mathbb{C}\setminus \operatorname{supp} W_{A,B}$, and by Proposition 7, the matrix-valued function $\frac{1}{2}A^{-1}(zI - B)A^{-1} - \frac{1}{2}A^{-1/2}\sqrt{H_{A,B}(z)}A^{-1/2}$ is analytic in $\mathbb{C}\setminus\Delta$. Therefore, $\operatorname{supp} W_{A,B} = \Delta$.

□

Since $H_{A,B}$ is diagonalizable, so is its opposite $-H_{A,B}$, i.e.:

$$-H_{A,B}(x) = U(x)D(x)U^*(x),$$

where $D(x)$ is a diagonal matrix with entries $d_{i,i}(x), i = 1, \ldots, N$, and $U(x)U(x)^* = I$.

We have the following theorem that establishes an expression for the matrix weight:

Theorem 11. *[[5], page 327]* 5 If A is positive-definite and B Hermitian, the matrix weight $W_{A,B}$ for the Chebyshev matrix polynomials of the second kind is the matrix-valued measure given by:

$$dW_{A,B}(x) = \frac{1}{2\pi}A^{-1/2}U(x)(D^{+}(x))^{1/2}U^{*}(x)A^{-1/2}dx,$$

where $D^{+}(x)$ is the diagonal matrix with entries

$$d^{+}_{i,i}(x) = max\{d_{i,i}(x), 0\}.$$

Proof.

From the inversion formula for the Stieltjes transform (Section 0.4.5) it is enough to prove that

$$-\frac{1}{2\pi i}\lim_{y\to 0^{+}}(G_{A,B}(x+iy) - G_{A,B}(x-iy)) = \frac{1}{2\pi}A^{-1/2}U(x)(D^{+}(x))^{1/2}U^{*}(x)A^{-1/2}, \quad x \in \mathbb{R}.$$

For a fixed real number x, we have that $-H_{A,B}(x)$ is Hermitian. Hence according to [[14], Theorem 1, pp. 33, 34], there exist analytic matrices $U(z), D(z)$, at x such that, for z in a complex neighbourhood of x, $U(z)$ is invertible, $U(x)$ is unitary, $D(z)$ is diagonal and $-H_{A,B}(z) = U(z)D(z)U^{*}(z)$ i.e.,

$$H_{A,B}(z) = U(z)(-D(z))U^{*}(z).$$

Then we have:

$$\sqrt{H_{A,B}(x)} = \sqrt{U(x)(-D(x))U^{*}(x)} = U(x)\sqrt{-D(x)}U^{*}(x),$$

where we take the square root as we did in Proposition 7. Then by Proposition 8:

$$\lim_{y\to 0^{+}}(G_{A,B}(x+iy) - G_{A,B}(x-iy)) =$$

$$-\frac{1}{2}\lim_{y\to 0^{+}}A^{-1/2}(U(x+iy)\sqrt{-D(x+iy)}U^{-1}(x+iy) - U(x-iy)\sqrt{-D(x-iy)}U^{-1}(x-iy))A^{-1/2}$$

$$= \frac{1}{2\pi}A^{-1/2}U(x)(D^{+}(x))^{1/2}U^{*}(x)A^{-1/2},$$

because

$$\lim_{y\to 0^{+}}(\sqrt{-d_{i,i}(x+iy)} - \sqrt{-d_{i,i}(x-iy)}) = \begin{cases} 2i\sqrt{d_{i,i}(x)}, & d_{i,i}(x) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

□

2.4 Example of a Matrix-Valued Measure

The examples in this section and the next are apparently new. We will compute $dW_{A,B}$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$H_{A,B}(x) = A^{-1/2}(xI - B)A^{-1}(xI - B)A^{-1/2} - 4I = (xI - B)^{2} - 4I =$$

$$\begin{bmatrix} -x & 1 \\ 1 & -x \end{bmatrix} \begin{bmatrix} -x & 1 \\ 1 & -x \end{bmatrix} - 4I = \begin{bmatrix} x^2+1 & -2x \\ -2x & x^2+1 \end{bmatrix} - 4I = \begin{bmatrix} x^2-3 & -2x \\ -2x & x^2-3 \end{bmatrix}$$

We have: $-H_{A,B}(x) = \begin{bmatrix} -x^2+3 & 2x \\ 2x & -x^2+3 \end{bmatrix}$ Setting $u_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u_2 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we have:

$$\begin{cases} -H_{A,B}(x)u_1 = (-x^2+2x+3)u_1 \\ -H_{A,B}(x)u_2 = (-x^2-2x+3)u_2 \end{cases} \tag{2.4}$$

Normalizing u_1, u_2 we get

$$v_1 := \frac{u_1}{\|u_1\|} = \frac{\sqrt{2}}{2}u_1 \qquad v_2 := \frac{u_2}{\|u_2\|} = \frac{\sqrt{2}}{2}u_2.$$

From (2.4), we see that v_1 is a normalized eigenvector for eigenvalue $-x^2+2x+3$ and v_2 is a normalized eigenvector for eigenvalue $-x^2-2x+3$. Then, $-H_{A,B}$ admits the unitary diagonalization $-H_{A,B} = U(x)D(x)U^*(x)$, where

$$U(x) = U = U^* = U^{-1} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D(x) = \begin{bmatrix} -x^2+2x+3 & 0 \\ 0 & -x^2-2x+3 \end{bmatrix}.$$

Now, we will compute $D^+(x)$: $d_{1,1}(x) = -x^2+2x+3 = -(x+1)(x-3)$. Then,

$$d_{1,1}^+(x) = \max\{0, d_{1,1}(x)\} = (-x^2+2x+3)\chi_{[-1,3]}.$$

Also $d_{2,2}(x) = -x^2-2x+3 = -(x+3)(x-1)$. Then

$$d_{2,2}^+(x) = \max\{0, d_{2,2}(x)\} = (-x^2-2x+3)\chi_{[-3,1]}.$$

We have then $D^+(x) = \begin{bmatrix} d_{1,1}^+(x) & 0 \\ 0 & d_{2,2}^+(x) \end{bmatrix} = \begin{bmatrix} d_{1,1}(x)\chi_{[-1,3]} & 0 \\ 0 & d_{2,2}(x)\chi_{[-3,1]} \end{bmatrix}$.

Finally,

$$\begin{aligned} dW_{A,B}(x) &= \frac{1}{2\pi}A^{-1/2}U(x)(D^+(x))^{1/2}U^*(x)A^{-1/2}dx \\ &= \frac{1}{2\pi}I\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{d_{1,1}^+(x)} & 0 \\ 0 & \sqrt{d_{2,2}^+(x)} \end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}dx \\ &= \frac{1}{4\pi}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\sqrt{d_{1,1}^+(x)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\sqrt{d_{2,2}^+(x)}\right)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}dx \\ &= \frac{1}{4\pi}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\sqrt{d_{1,1}^+(x)} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\sqrt{d_{2,2}^+(x)}\right)dx \\ &= \frac{1}{4\pi}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\sqrt{d_{1,1}^+(x)} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\sqrt{d_{2,2}^+(x)}\right)dx \\ &= \frac{1}{4\pi}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\sqrt{-x^2+2x+3}\,\chi_{[-1,3]} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\sqrt{-x^2-2x+3}\,\chi_{[-3,1]}\right)dx. \end{aligned}$$

◇

2.5 Generalized Example of a Matrix-Valued Measure

Let $N \geq 2$ a fixed natural number, $A_N := I_{N\times N}$ and $B := J_N - A_N$, where J_N is the $N \times N$ matrix with all entries equal to 1. For example, when $N = 4$, we have:

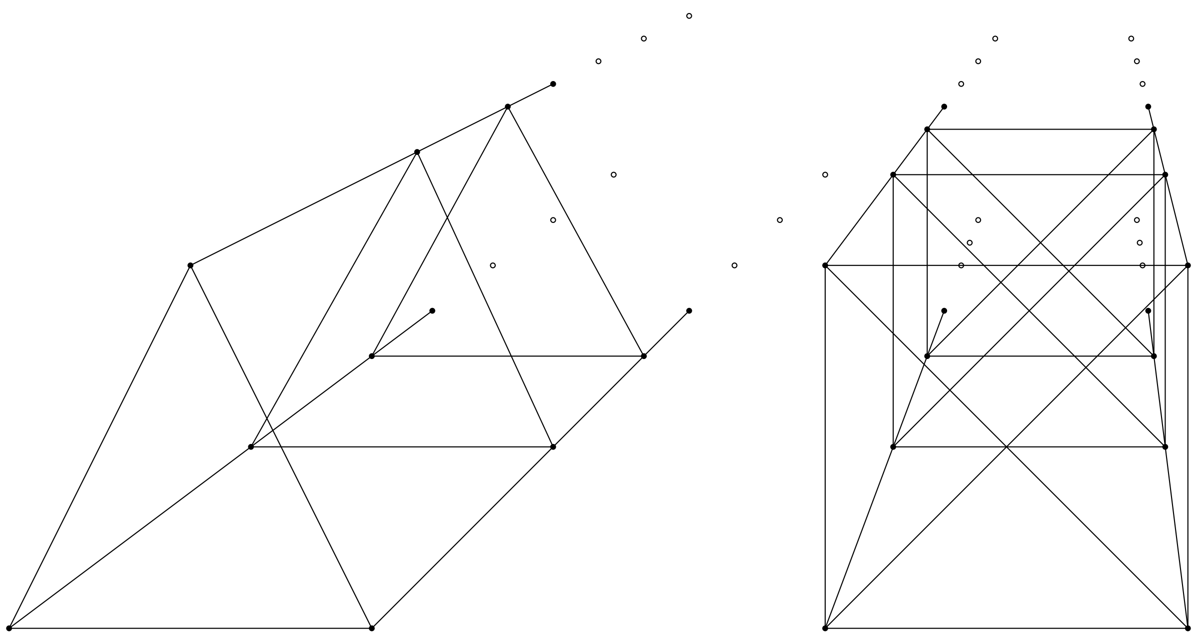
$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

We usually drop subscript N, and we usually denote A_N, B_N, I_N by A, B, I. Let $G_N := K^N \square R$ be the Cartesian product of K^N and R, where K^N is the complete graph with N vertices and $R = (V, E)$ is the ray with $V = \{x_0, x_1, x_2, \ldots\}$ and $E = \{x_0x_1, x_1x_2, \ldots\}$. With this, G_N is a block tridiagonal graph with adjacency matrix:

$$\begin{bmatrix} B & A & & & \\ A & B & A & & \\ & A & B & A & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

The graph G_2 is the semi infinite ladder already considered.

Figure 2.1: Graphs G_3 and G_4



We have:

$$H_{A,B}(z) := A^{-1/2}(zI - B)A^{-1}(zI - B)A^{-1/2} - 4I.$$

As $A = I$, we have:

$$H_{A,B}(z) = (zI - B)^2 - 4I = (z^2 - 4)I - 2zB + B^2.$$

Because of $B = J - A$, we have:

$$B^2 = (N-1)I + (N-2)B.$$

Then,

$$H_{A,B}(z) = (z^2 + N - 5)I + (N - 2 - 2z)B,$$

and

$$-H_{A,B}(z) = (5 - N - z^2)I + (2z + 2 - N)B.$$

Setting $u_1 := [1\ 1\ \cdots\ 1]^T$ and $u_k := -e_1 + e_k$, where $e_k = [0\ \cdots\ 0\ 1\ 0\ \cdots\ 0]^T$, with the number 1 in the k-th position. We have:

$$\begin{cases} -H_{A,B}(x)u_1 = d_{11}(x)u_1 \\ -H_{A,B}(x)u_k = d_{22}(x)u_k, \quad k = 2, 3, \ldots, N, \end{cases} \tag{2.5}$$

where $d_{11}(x) = -x^2 + (2N-2)x + 2N - N^2 + 3$ and $d_{22}(x) = -x^2 - 2x + 3$. The factored expressions for $d_{11}(x)$ and $d_{22}(x)$ are:

$$d_{11}(x) = -(x - N + 3)(x - N - 1), \qquad d_{22}(x) = -(x-1)(x+3).$$

Normalizing u_1 we have

$$v_1 := a_1 u_1,$$

where $a_1 = \frac{1}{\sqrt{N}}$.

Using Gram-Schimdt procedure for $u_k,\ k = 2, 3, \ldots, N$, we obtain a sequence of orthogonal vectors $w_k,\ k = 2, 3, \ldots, N$, given by:

$$w_2 = [-1\ 1\ 0\ \cdots\ 0]^T$$

$$w_3 = \left[-\frac{1}{2}\ -\frac{1}{2}\ 1\ 0\ \cdots\ 0\right]^T$$

$$w_4 = \left[-\frac{1}{3}\ -\frac{1}{3}\ -\frac{1}{3}\ 1\ 0\ \cdots\ 0\right]^T$$

$$\vdots$$

$$w_k = e_k + \frac{-1}{k-1}\sum_{l=1}^{k-1} e_l,\ k = 2, 3, \ldots, N.$$

Normalizing $w_k,\ k = 2, 3, \ldots, N$, we have:

$$v_k = (k-1)e_k + a_k \sum_{l=1}^{k-1} e_l,\ k = 2, 3, \ldots, N,$$

where $a_k = \frac{1}{\sqrt{k(k-1)}}$.

From (2.5), we see that v_1 is a normalized eigenvector for eigenvalue $d_{11}(x)$ and v_k is a normalized eigenvector for eigenvalue $d_{22}(x)$ for $k = 2, 3, \ldots, N$. Then, $-H_{A,B}$ admits the unitary diagonalization $-H_{A,B} = U(x)D(x)U^*(x)$, with $U(x) = U = U^* = U^{-1}$, and

$$U = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_N \\ a_1 & a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_N \\ a_1 & 0 & 2a_3 & -a_4 & -a_5 & \cdots & -a_N \\ a_1 & 0 & 0 & 3a_4 & -a_5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & -a_N \\ a_1 & 0 & 0 & 0 & \cdots & 0 & (N-1)a_N \end{bmatrix},$$

$$D(x) = \begin{bmatrix} d_{11}(x) & 0 & 0 & \cdots & 0 \\ 0 & d_{22}(x) & 0 & \cdots & 0 \\ 0 & 0 & d_{22}(x) & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & d_{22}(x) \end{bmatrix}.$$

Now, we will compute $D^+(x)$: $d_{11}(x) = -(x - N + 3)(x - N - 1)$. Then,

$$d^+_{1,1}(x) = \max\{0, d_{1,1}(x)\} = -(x - N + 3)(x - N - 1)\chi_{[N-3,N+1]}.$$

Also $d_{22}(x) = -(x + 3)(x - 1)$. Then

$$d^+_{22}(x) = \max\{0, d_{22}(x)\} = (-x^2 - 2x + 3)\chi_{[-3,1]}.$$

We have then

$$D^+(x) = \begin{bmatrix} d^+_{11}(x) & 0 & \cdots & 0 \\ 0 & d^+_{22}(x) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d^+_{22}(x) \end{bmatrix}$$

$$= \begin{bmatrix} d_{11}(x)\chi_{[N-3,N+1]} & 0 & \cdots & 0 \\ 0 & d_{22}(x)\chi_{[-3,1]} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_{22}(x)\chi_{[-3,1]} \end{bmatrix}.$$

Let $E_j := e_j e_j^T$. We have:

$$U(D^+(x))^{1/2}U^* = U\left(E_1\sqrt{d^+_{11}(x)} + \sum_{j=2}^{N} E_j\sqrt{d^+_{22}(x)}\right)U^* = UE_1U^*\sqrt{d^+_{11}(x)} + U\left(\sum_{j=2}^{N} E_j\right)U^*\sqrt{d^+_{22}(x)}$$

After some matrix calculations, we obtain:

$$UE_1U^* = \frac{1}{N}(I + B), \qquad U\left(\sum_{j=2}^{N} E_j\right)U^* = \frac{1}{N}((N-1)I - B).$$

And then,

$$U(D^+(x))^{1/2}U^* = \frac{1}{N}(I + B)\sqrt{d^+_{11}(x)} + \frac{1}{N}((N-1)I - B)\sqrt{d^+_{22}(x)}.$$

Finally,

$$dW_{A,B}(x) = \frac{1}{2\pi} A^{-1/2} U(x) (D^{+}(x))^{1/2} U^{*}(x) A^{-1/2} dx = \frac{1}{2\pi} U (D^{+}(x))^{1/2} U^{*} dx$$
$$= \frac{1}{2N\pi} \left((I+B)\sqrt{d_{11}^{+}(x)} + ((N-1)I - B)\sqrt{d_{22}^{+}(x)} \right) dx.$$

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