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Hypothesis Test in Long-Range Dependence Processes with Stable Innovations

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I do believe you think what now you speak,
But what we do determine oft we break.
Purpose is but the slave to memory,
Of violent birth, but poor validity,
Which now, like fruit unripe, sticks on the tree,
But fall, unshaken, when they mellow be.
Most necessary 'tis that the forget
To pay ourselves what to ourselves is debt.

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Hypothesis Test in Long-Range Dependence Processes with Stable Innovations

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The main objetive of this paper is to analyze hypothesis tests for the stability parameter α of the stable innovations, which determines the heaviness of the distribution's tails. We will consider both a simple z-test and also the likelihood ratio test. Power analysis for both situations will be studied as well. Four estimators for the long-range dependence parameter d will be used, including parametric and semi-parametric procedures with robust versions. Three different estimators for the tail index α will be considered in this work. A Monte Carlo study will be presented, as well as an application to real data.

Keywords: hypothesis test; long-range dependence; stable distributions.

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1. Introduction

In time series analysis, it is common to find studies where long-range dependence is present. In fact, significant correlation between distant time periods can be found in time series in different areas such as economics, hydrology and meteorology. Characteristics of *long memory* time series are such that its autocorrelation function ρ_k decays hyperbolically to zero and its spectral density function $f_X(\cdot)$ is unbounded in the neighbourhood of the zero frequency. Models for *long-range dependence* were first introduced by the seminal work of Hurst (1951). One of the models capable of representing this phenomena is the autoregressive fractionally integrated moving average, also known as ARFIMA(p,d,q) process, introduced by Granger and Joyeux (1980) and Hosking (1981). This model can represent the hyperbolic decay rate of the autocorrelation function in a *long memory* time series, while an ARMA process can only represent a geometric decay rate. In this model, $d \in (-0.5, 0.5)$ is called the long-range dependence parameter, since its value is closely related to the autocorrelation function decay rate.

Usually, time series models assume a Gaussian process as innovations. Several estimation methods for the long-range dependence parameter d in the ARFIMA(p, d, q) process consider this assumption to analyze the limiting distribution properties of the estimators. However, this work will consider α-stable innovations for the time series. This class of distributions can model several characteristics, and allows heavy tails to be introduced on the innovation process. Therefore, it is rather interesting to analyze the performance of the estimation methods under these circumstances. We refer the reader to Kokoszka and Taqqu (1995, 1996) for a detailed study on α-stable innovations on long-range dependence processes.

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This paper consists in analyzing the estimation procedure in an ARFIMA(p,d,q) model with α-stable innovations, specifically in estimating the stability parameter α. To achieve this goal, hypothesis tests for the stability parameter α, which determines the heaviness of the distribution's tails, will be performed. We will analyze in which circumstances a hypothesis test can detect non-Gaussianity in the innovations. For this, we will consider both a simple Z-test and also the likelihood ratio test. Power analysis for both situations will be studied as well. Four estimators of the long-range dependence parameter d will be considered, including parametric and semi-parametric procedures with robust versions. Three different estimators for the tail index α will be considered in this work. A Monte Carlo study will be presented, as well as an application to a tree-ring measurements time series.

2. Preliminary Concepts

2.1 ARFIMA*(p,d,q) Model*

An ARFIMA(p,d,q) process is the stationary solution to the equation

$$\Phi(\mathcal{B})(1-\mathcal{B})^d(X_t-\mu)=\Theta(\mathcal{B})\varepsilon_t,\ t\in\mathbb{Z}, \tag{2.1}$$

where $\mathcal{B}$ is the backward shift operator, $\Phi(\cdot)$ and $\Theta(\cdot)$ are the p and q order polynomials, respectively given by $\Phi(\mathcal{B})=\sum_{i=0}^{p}(-\phi_i)\,\mathcal{B}^i$ and $\Theta(\mathcal{B})=\sum_{j=0}^{q}(-\theta_j)\,\mathcal{B}^j$, where ϕ_i and θ_j are real constants with $\phi_0=-1=\theta_0$, with roots outside the unit circle. The innovation $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance equal to σ_ε^2. When $d\in(-0.5,0.5)$, the process is both stationary and invertible. Moreover, when $d\in(0.0,0.5)$, the process exhibits long-range dependence, namely the autocorrelation function has an hyperbolic decay, slowly approaching zero.

2.2 *Stable Distributions*

A random variable X is stable if its characteristic function $\varphi_X(\cdot)$ is given by

$$\varphi_X(t)=\begin{cases}\exp\{-\lambda^\alpha|t|^\alpha\left[1-i\beta(\tan\frac{\pi\alpha}{2})(\text{sign } t)\right]+i\mu t\}, & \alpha\neq 1\\ \exp\{-\lambda|t|\left[1-i\beta\frac{2}{\pi}(\text{sign } t)\log(t)\right]+i\mu t\}, & \alpha=1.\end{cases} \tag{2.2}$$

The parameter α is the stability index parameter ($0<\alpha\leq 2$), β is a symmetry parameter ($|\beta|\leq 1$), λ is the scale parameter ($\lambda>0$) and μ is the location parameter ($\mu\in\mathbb{R}$). The word *stable* is used in this case because its shape remains unchanged - or stable - under summation. Stable distributions also arise as a limit distribution of sums of i.i.d. random variables, as shown by the Generalized Central Limit Theorem (see Lévy, 1924 for a complete study of these distributions).

The stability index α is the most important parameter to be estimated, since it is related to the probability of ocurrence of extreme values, that is, $P(|X|>x)\sim Cx^{-\alpha}$. Therefore, its value may represent the existence of heavy-tailed data. The lack of closed formulas for all but a few distributions in this class has been a drawback to models using this kind of random variable. Fortunately, the advance of numerical computing has allowed the generation of densities, quantiles and random numbers from these distributions. Those computational programs made it possible to use stable models in several practical problems.

3. Long Memory Parameter Estimation

The literature on stochastic long-memory processes has presented several estimation procedures for the fractional differencing parameter d. In this section we summarize four of such methods that shall be used in this paper. Two of them are in the parametric class, and the other two are in the semi-parametric class. For the latest, we also consider their robust versions. They are:

- maximum likelihood estimator (MLE), proposed by Sowell (1992);
- parametric approximated maximum likelihood method (W), proposed by Whittle (1953) and Fox and Taqqu (1986);
- semi-parametric regression method based on the periodogram function (GPH), proposed by Geweke and Porter-Hudak (1983);
- semi-parametric regression method based on the smoothed periodogram which considers the Bartlett lag window (BA), proposed by Lopes and Mendes (2006).

3.1 *Maximum Likelihood Estimator*

Consider a stationary normally distributed fractionally integrated process $\{X_t\}_{t\in\mathbb{Z}}$, generated by the model in (2.1). Let $\{x_t\}_{t=1}^{T}$ be a sample of T observations. The normal probability density function of this sample is given by

$$f(z_t, \Sigma) = (2\pi)^{-T/2}|\Sigma|^{-1/2} \exp\left\{\frac{-1}{2} z_t' \Sigma z_t\right\}, \tag{3.1}$$

where Σ is the covariance matrix of the observations. Sowell (1992) noted that stationarity implies that this covariance matrix has a Toeplitz form as in $\Sigma = [\gamma(i-j)]$, for $i, j = 1, 2, \cdots, T$, where $\gamma(\cdot)$ is the autocovariance function of the process $\{X_t\}_{t\in\mathbb{Z}}$. The joint estimation of the model parameters requires the specification and the estimation of the above covariance matrix. For that, the author wrote the autocorrelation function in terms of the model parameters and derived the autocovariance function

$$\gamma(s) = \frac{1}{2\pi}\int_0^{2\pi} f_z(\lambda) e^{i\lambda s} d\lambda. \tag{3.2}$$

The maximization procedure involves the evaluation of the likelihood function for a global maximum for a given set of parameter values. The spectral density function of $\{X_t\}_{t\in\mathbb{Z}}$ is used in the specification of the autocovariance function and can be derived using the spectral density of the diminished ARMA process. Details on the derivation of the maximum likelihood function and the recursive algorithm implementation are also available in Sowell (1992). We also refer the reader to Andrews et al. (2009) for an account of the maximum likelihood estimation for α-stable autoregressive models, that is, only short-range dependence is present on the process.

3.2 *Whittle estimator*

The Whittle (W) estimator, proposed by Fox and Taqqu (1986) is based on an approximation of the likelihood function suggested by Whittle (1953). The authors considered the function

$$\mathcal{Q}(\boldsymbol{\eta}) = \int_{-\pi}^{\pi} \frac{I_n(w)}{f_X(w, \boldsymbol{\eta})} dw, \tag{3.3}$$

where $\boldsymbol{\eta}$ denotes the vector of unknown parameters, $f_X(\cdot, \boldsymbol{\eta})$ is the spectral density function and $I(\cdot)$ is the periodogram function. The W estimator is the value of $\boldsymbol{\eta}$ which minimizes the function $\mathcal{Q}(\boldsymbol{\eta})$. Here, $\boldsymbol{\eta}$ is the vector $(\phi_1, \cdots, \phi_p, d, \sigma_\varepsilon, \theta_1, \cdots, \theta_q)$.

3.3 *Regression methods using the periodogram function*

Let $\{X_t\}_{t\in\mathbb{Z}}$ be an ARFIMA(p, d, q) process with $d \in (-0.5, 0.5)$ given by equation (2.1). Its spectral density function is given by

$$f_X(w) = f_U(w)\left[2\sin\left(\frac{w}{2}\right)\right]^{-2d}, \quad 0 < w \le \pi, \tag{3.4}$$

where $f_U(w)$ is the spectral density function of the ARMA process. We can rewrite (3.4) as

$$\log(I(w_j)) = \log(f_U(0)) - d\log\left[2\sin\left(\frac{w_j}{2}\right)\right]^2 + \log\left\{\frac{f_U(w_j)}{f_U(0)}\right\} + \log\left\{\frac{I(w_j)}{f_X(w_j)}\right\}, \tag{3.5}$$

where $I(\cdot)$ is the periodogram function of X_t, given by

$$I_n(w) = \frac{1}{2\pi}\left(\hat{\gamma}_X(0) + 2\sum_{k=1}^{n-1}\hat{\gamma}_X(k)\cos(wk)\right), w \in [-\pi, \pi], \tag{3.6}$$

where $\hat{\gamma}(\cdot)$ is the sample autocovariance function of the process and w_j is the j-th Fourier frequency, for $j = 1, \cdots, g(n)$, with $g(n)$ as the number of regressors. A semi-parametric estimator may be obtained by applying a regression model to equation (3.5). In this work, we will consider three different methods based on this regression:

- *Ordinary Least Squares* (OLS) method
- *Least Trimmed Squares* (LTS) method, proposed by Rousseeuw (1984).
- MM method, proposed by Yohai (1987).

3.4 *Classical and Robust* GPH *estimators*

Geweke and Porter-Hudak (1983) introduced the first estimation method based on the periodogram function. The classical GPH estimator is given by

$$\hat{d}_{GPH} = \frac{\sum_{j=1}^{g(n)}(x_j - \overline{x})(y_j - \overline{y})}{\sum_{j=1}^{g(n)}(x_j - \overline{x})^2}, \tag{3.7}$$

where the trimming value $g(n)$ is set to $g(n) = \lfloor n^{0.85}\rfloor$ in this work. To obtain the robust versions of the GPH estimator, we apply the LTS and the MM methodologies to the used regression model in (3.5) with $m = g(n)$.

3.5 *Classical and Robust* BA *estimators*

Another estimator is obtained by considering the smoothed version of the periodogram function, since the periodogram function is not a consistent estimator for the spectral density function. By considering the Bartlett lag window, a consistent estimator for the spectral density function may be obtained, as seen in Lopes and Mendes (2006). The smoothed version of the periodogram function is defined by

$$I_s(w) = \frac{1}{2\pi} \sum_{j=-\nu}^{\nu} \kappa\left(\frac{j}{\nu}\right) \hat{\gamma}_X(j) \cos(jw), \tag{3.8}$$

where $\kappa(\cdot)$ is the Bartlett lag window given by

$$\kappa(x) = \begin{cases} 1-|x|, & \text{if } |x| \leq 1 \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

with ν being the truncation point of the weighted function. To obtain the robust versions of the BA estimator we apply the LTS and the MM methodologies to the regression model in (3.5) using the smoothed periodogram given by (3.8).

4. Stability Index Estimation

After estimating the fractional differencing parameter d the stability index α of the α-stable distributions can be estimated from the residuals of the proposed model. This paper will use three estimators:

- Hill-Hall estimator, proposed by Hill (1975) and denoted by $\hat{\alpha}_{hh}$;
- empirical characteristic function (ecf) estimator, proposed by Press (1972) and denoted by $\hat{\alpha}_{ecf}$;
- maximum likelihood estimator (mle), proposed by Nolan (2001) and denoted by $\hat{\alpha}_{mle}$.

4.1 *Hill-Hall estimator*

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics. Let $m = \lfloor n^r \rfloor$ be the truncation point that considers only the extreme observations. The Hill's estimator is given by

$$\hat{\alpha}_{\rm hh} = \left(m^{-1} \sum_{j=1}^{m} \ln X_{(n-j+1)} - \ln X_{(n-m)} \right)^{-1}. \tag{4.1}$$

Hall (1982) established the asymptotic normality of this estimator and determined the optimal choice of the truncation parameter m. However, it is given by a function of the unknown parameters of the distribution. Therefore, it is common to use a different set of values. In this work we shall consider $r \in (0.5, 0.8)$. The asymptotic variance for this estimator is given by

$$AVar(\hat{\alpha}_{\rm hh}) = \frac{\alpha^2}{m}. \tag{4.2}$$

4.2 *Empirical Characteristic Function estimator*

The empirical characteristic function (ecf) estimator is given by the equation

$$\hat{\alpha}_{\rm ecf} = \frac{\ln\left(\left|\frac{\ln \hat{\varphi}_X(t_1)}{\ln \hat{\varphi}_X(t_2)}\right|\right)}{\ln\left(\frac{t_1}{t_2}\right)}, \tag{4.3}$$

where $\widehat{\varphi}_X(t) = \sum_{j=1}^n e^{itX_j}$ is the empirical characteristic function and $t_1, t_2 \in (0,1)$. Press (1972) determines the asymptotic normality of the estimator, as well as its asymptotic variance,

$$AVar(\hat{\alpha}_{\text{ecf}}) = \frac{\sigma^2_{\hat{\alpha}_{\text{ecf}}}}{n} \ , \tag{4.4}$$

where

$$\sigma^2_{\hat{\alpha}_{\text{ecf}}} = \frac{1 + |\hat{\varphi}_X(2t_1)| - 2|\hat{\varphi}_X(t_1)|^2}{2[|\hat{\varphi}_X(t_1)| \ln |\hat{\varphi}_X(t_1)| \ln \frac{t_1}{t_2}]^2} + \frac{1 + |\hat{\varphi}_X(2t_2)| - 2|\hat{\varphi}_X(t_2)|^2}{2[|\hat{\varphi}_X(t_2)| \ln |\hat{\varphi}_X(t_2)| \ln(\frac{t_1}{t_2})]^2}$$
$$- \frac{|\hat{\varphi}_X(t_1+t_2)| + |\hat{\varphi}_X(t_1-t_2)| - 2|\hat{\varphi}_X(t_1)||\hat{\varphi}_X(t_2)|}{[|\hat{\varphi}_X(t_1)||\hat{\varphi}_X(t_2)| \ln |\hat{\varphi}_X(t_1)| \ln |\hat{\varphi}_X(t_2)| \ln(\frac{t_1}{t_2})]^2}.$$

Besbeas and Morgan (2008) consider $t_1 = 0.2$ and $t_2 = 0.8$ as the optimal selection values for t_1 and t_2 in (4.3).

4.3 *Maximum Likelihood Estimator*

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a vector of i.i.d. stable random variables. The maximum likelihood estimator (mle) is the value $\hat{\alpha}_{mle}$ which maximizes the likelihood function, that is,

$$\hat{\alpha}_{mle} = \arg\max_{\alpha\in\Theta} L(\alpha|\mathbf{x}), \tag{4.5}$$

where $L(\alpha|\mathbf{x}) = \prod_{i=1}^n f(\alpha|x_i)$ and Θ is the parameter space. Since there are no closed expression for the stable density function, numerical algorithms must be used. The asymptotic variance for the $\hat{\alpha}_{mle}$ estimator is given by

$$Var(\hat{\alpha}_{mle}) = \frac{\sigma^2_{\hat{\alpha}_{mle}}}{n}, \tag{4.6}$$

where $\sigma^2_{\hat{\alpha}_{mle}}$ is the first diagonal entrance of the Fisher information matrix.

5. Hypothesis Tests for the Stability Index

The main objetive of this paper is to test the hyphotesis that the residuals of the ARFIMA(p,d,q) process follow an α-stable distribution with heavy tails, that is, $\alpha < 2$. The case $\alpha = 2$ in expression (2.2) is equivalent to the Gaussian distribution. Therefore, by using the stability parameter α of the stable distributions, the null and alternative hypothesis may be presented as

$$H_0 : \alpha = 2 \quad \text{vs} \quad H_1 : \alpha < 2, \tag{5.1}$$

that is, under the null hypothesis, the residuals follow a Gaussian distribution, and under the alternative hypothesis, an α-stable distribution with large tails.

5.1 *Statistical Test Power*

The power of a statistical test can be defined as the probability that it will correctly lead to the rejection of a false null hypothesis. Formally, if $\varphi(\cdot)$ is a test function for the problem $(\alpha_s, \Theta_0, \Theta_1)$ - as the one established in (5.1) - , the power function $\beta_\varphi(\cdot)$ for the test is given by

$$\beta_\varphi(\theta) = P_\theta\{\text{Reject } H_0 | H_0 \text{ is false}\}, \tag{5.2}$$

where $\theta \in \Theta$ and Θ_0 and Θ_1 are subsets of the parameter space Θ such that $\Theta_0 \cup \Theta_1 = \Theta$. One can notice that $\beta_\varphi(\cdot)$ is a function of the parameter value θ, which means that the power of the statistical test depends on how close the evaluated value in Θ_1 is close to the value in Θ_0. In our context, the power of the statistical test to evaluate the hypothesis given in (5.1) is the probability that the test will detect non-Gaussianity when the residuals are, in fact, non-Gaussian, namely α-stable with $\alpha < 2$.

5.2 *Z-test*

For the $\hat{\alpha}_{\text{hh}}$ and $\hat{\alpha}_{\text{ecf}}$ estimators, we will consider the asymptotic normal distribution, as shown in Hall (1982) and Press (1972), respectively. Thereby, the proper test statistic for the problem is given by

$$Z = \frac{\hat{\alpha} - 2}{\sqrt{Var(\hat{\alpha})}}. \tag{5.3}$$

Under H_0, $Z \sim \mathcal{N}(0, 1)$. For a 5% significance level, H_0 will be rejected if $Z < -1.64$.

5.3 *Likelihood Ratio Test*

The $\hat{\alpha}_{mle}$ estimator allow us to consider the likelihood ratio test, which is the uniformly most powerful test among all tests of a given size. The likelihood ratio is given by

$$\Lambda(\boldsymbol{x}) = \frac{\sup\{L(\boldsymbol{\theta}; \boldsymbol{x}) : \boldsymbol{\theta} \in \Theta_0\}}{\sup\{L(\boldsymbol{\theta}; \boldsymbol{x}) : \boldsymbol{\theta} \in \Theta_1\}}, \tag{5.4}$$

where Θ_0 and Θ_1 are subsets of the parameter space Θ such that $\Theta_0 \cup \Theta_1 = \Theta$. The test statistic D, defined as $-2\log(\Lambda)$, is asymptotically χ^2_k distributed, where k is the number of degrees of freedom equal to the difference of parameters in Θ_1 and Θ_0.

In this paper, we shall estimate the parameter λ^2_ε for the model under H_0, where the residuals are normally distributed with zero mean and σ^2_ε variance, and α and λ parameters of the stable distribution for the model under H_1. Under this hypothesis, the residuals follow an α-stable distribution with stability index $\alpha < 2$, scale parameter $\lambda > 0$, symmetry parameter $\beta = 0$ and location parameter $\mu = 0$. Informally, this means that we will test the hypothesis that the residuals are normally distributed with a *large* variance or are, in fact, *stable* distributed, that is, they have heavy tails.

6. Monte Carlo Simulations

In this simulation study we have considered Monte Carlo simulations for ARFIMA(p,d,q) processes with α-stable white noise. The results were obtained by time series of size $n = 500$, over $re = 5000$ replications. For the ARFIMA(p,d,q) model, we considered $d \in \{0.2, 0.3, 0.45\}$ and $p = 0 = q$. For the α-stable white noise process we considered $\alpha \in \{1.25, 1.5, 1.75\}$, $\beta = 0 = \mu$ and $\sigma = 1$. The truncation point used for the GPH and BA estimators for the long-range dependence parameter d is $g(n) = n^{0.85}$.

For the estimation of the stability parameter α, we considered $r \in \{0.7, 0.75, 0.8\}$ for the hh estimator, and $t_1 = 0.2$ and $t_2 = 0.8$ for the ecf estimator. The maximization of the likelihood function for the mle estimator was performed using the Nelder-Mead algorithm. The estimates of the stability index α, denoted by $\hat{\alpha}$, and the variance of this estimates, denoted by $\hat{\sigma}^2_{\hat{\alpha}}$ are respectively given by

$$\hat{\alpha} = \frac{1}{re} \sum_{j=1}^{re} \alpha_j \tag{6.1}$$

and

$$\hat{\sigma}^2_{\hat{\alpha}} = \frac{1}{re} \sum_{j=1}^{re} (\alpha_j - \hat{\alpha})^2, \tag{6.2}$$

where α_j is the estimation of α in the j-th replication for all three estimation procedures presented in Section 4. For the mle estimator, the scale parameter λ was also estimated: the estimates $\hat{\lambda}$ and $\hat{\sigma}^2_{\hat{\lambda}}$ are given by $\hat{\lambda} = \frac{1}{re}\sum_{j=1}^{re} \lambda_j$ and $\hat{\sigma}^2_{\hat{\lambda}} = \frac{1}{re}\sum_{j=1}^{re}(\lambda_j - \hat{\lambda})^2$, respectively. For the power analysis, we have computed the value

$$\hat{p} = \frac{1}{re} \sum_{j=1}^{re} \pi_j, \tag{6.3}$$

where π_j is given by

$$\pi_j = \begin{cases} 1, & \text{if } H_0 \text{ is rejected} \\ 0, & \text{otherwise.} \end{cases} \tag{6.4}$$

Since H_0 should be rejected in every situation, since we have $\alpha < 2$ in every simulation scenario, $\hat{p}$ gives us an estimate of the probability given in equation (5.2) - the percentage of times H_0 was correctly rejected. The hypothesis tests were performed for a significance level $\gamma = 0.05$.

Table 1: Estimation Results when $\hat{\alpha}$ is the hh estimator and $d = 0.2$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
				$r = 0.7$				
$\hat{\alpha}$	1.2891	1.2900	1.2939	1.2931	1.2943	1.2975	1.2994	1.3321
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0218	0.0229	0.0238	0.0239	0.0258	0.0253	0.0306	1.7396
$\hat{p}$	0.9990	0.9982	0.9984	0.9984	0.9982	0.9980	0.9974	0.9924
				$r = 0.75$				
$\hat{\alpha}$	1.2075	1.2098	1.2119	1.2115	1.2121	1.2114	1.2141	1.2344
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0157	0.0165	0.0181	0.0177	0.0197	0.0216	0.0235	0.7519
$\hat{p}$	0.9992	0.9986	0.9986	0.9982	0.9980	0.9974	0.9970	0.9928
				$r = 0.8$				
$\hat{\alpha}$	1.0414	1.0422	1.0424	1.0424	1.0414	1.0412	1.0425	1.0564
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0134	0.0140	0.0158	0.0157	0.0183	0.0221	0.0247	0.3795
$\hat{p}$	0.9968	0.9972	0.9950	0.9962	0.9958	0.9906	0.9930	0.9884
				$\alpha = 1.5$				
				$r = 0.7$				
$\hat{\alpha}$	1.8028	1.8048	1.8058	1.8050	1.8039	1.7979	1.8001	1.8036
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0458	0.0468	0.0479	0.0477	0.0482	0.0490	0.0488	0.0603
$\hat{p}$	0.7708	0.7670	0.7636	0.7636	0.7686	0.7730	0.7698	0.7638
				$r = 0.75$				
$\hat{\alpha}$	1.6670	1.6689	1.6658	1.6663	1.6652	1.6550	1.6569	1.6579
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0281	0.0283	0.0294	0.0287	0.0292	0.0344	0.0343	0.0454
$\hat{p}$	0.9534	0.9508	0.9490	0.9526	0.9536	0.9470	0.9474	0.9418
				$r = 0.8$				
$\hat{\alpha}$	1.3632	1.3651	1.3613	1.3628	1.3607	1.3490	1.3508	1.3529
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0195	0.0187	0.0199	0.0195	0.0212	0.0281	0.0279	0.0391
$\hat{p}$	0.9996	0.9994	0.9990	0.9992	0.9992	0.9976	0.9976	0.9924
				$\alpha = 1.75$				
				$r = 0.7$				
$\hat{\alpha}$	2.9249	2.9260	2.9085	2.9127	2.9033	2.8724	2.8725	2.8556
$\hat{\sigma}^2_{\hat{\alpha}}$	0.1512	0.1516	0.1537	0.1530	0.1529	0.1565	0.1548	0.1649
$\hat{p}$	0.0006	0.0008	0.0010	0.0004	0.0014	0.0046	0.0046	0.0068
				$r = 0.75$				
$\hat{\alpha}$	2.5714	2.5740	2.5559	2.5586	2.5509	2.5195	2.5194	2.4987
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0868	0.0870	0.0900	0.0889	0.0900	0.0974	0.0974	0.1056
$\hat{p}$	0.0116	0.0096	0.0152	0.0136	0.0176	0.0306	0.0286	0.0412
				$r = 0.8$				
$\hat{\alpha}$	1.8863	1.8867	1.8758	1.8789	1.8747	1.8547	1.8549	1.8414
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0492	0.0480	0.0510	0.0503	0.0522	0.0596	0.0588	0.0646
$\hat{p}$	0.6640	0.6666	0.6788	0.6762	0.6882	0.7002	0.7026	0.7182

Table 2: Estimation Results when $\hat{\alpha}$ is the ecf estimator and $d = 0.2$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
$\hat{\alpha}$	1.2783	1.2785	1.2818	1.2814	1.2815	1.2851	1.2852	1.2833
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0068	0.0067	0.0067	0.0067	0.0072	0.0071	0.0074	0.0114
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
				$\alpha = 1.5$				
$\hat{\alpha}$	1.5726	1.5728	1.5740	1.5738	1.5741	1.5742	1.5747	1.5735
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0067	0.0066	0.0066	0.0066	0.0066	0.0068	0.0067	0.0082
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9996
				$\alpha = 1.75$				
$\hat{\alpha}$	1.8661	1.8662	1.8665	1.8665	1.8665	1.8659	1.8661	1.8655
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0055
$\hat{p}$	0.5454	0.5458	0.5462	0.5472	0.5462	0.5518	0.5506	0.5530

Table 3: Estimation Results when $\hat{\alpha}$ is the mle estimator and $d = 0.2$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
$\hat{\alpha}$	1.2916	1.2912	1.2951	1.2939	1.2942	1.3012	1.3008	1.3031
$\sigma^2_{\hat{\alpha}}$	0.0048	0.0050	0.0050	0.0052	0.0054	0.0051	0.0052	0.0068
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	0.9996	0.9998	1.0000	0.9996
$\hat{\lambda}$	1.1396	1.1329	1.1525	1.1468	1.1565	1.1959	1.1908	1.2238
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0106	0.0083	0.0106	0.0100	0.0190	0.0236	0.0232	0.0789
				$\alpha = 1.5$				
$\hat{\alpha}$	1.6017	1.6022	1.6019	1.6019	1.6019	1.6007	1.6011	1.6019
$\sigma^2_{\hat{\alpha}}$	0.0041	0.0041	0.0042	0.0042	0.0042	0.0045	0.0045	0.0051
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{\lambda}$	1.1640	1.1628	1.1693	1.1677	1.1712	1.1844	1.1849	1.2006
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0032	0.0030	0.0033	0.0032	0.0037	0.0052	0.0059	0.0198
				$\alpha = 1.75$				
$\hat{\alpha}$	1.8654	1.8655	1.8652	1.8653	1.8652	1.8640	1.8640	1.8634
$\sigma^2_{\hat{\alpha}}$	0.0021	0.0021	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024
$\hat{p}$	0.9912	0.9908	0.9910	0.9912	0.9912	0.9906	0.9910	0.9902
$\hat{\lambda}$	1.1206	1.1204	1.1237	1.1229	1.1244	1.1310	1.1314	1.1397
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0009	0.0009	0.0011	0.0010	0.0011	0.0014	0.0015	0.0052

Table 4: Estimation Results when $\hat{\alpha}$ is the hh estimator and $d = 0.3$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
				$r = 0.7$				
$\hat{\alpha}$	1.2923	1.2924	1.2974	1.2969	1.2985	1.2996	1.3028	1.3187
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0262	0.0280	0.0288	0.0294	0.0297	0.0338	0.0350	0.0946
$\hat{p}$	0.9960	0.9954	0.9952	0.9950	0.9946	0.9934	0.9936	0.9876
				$r = 0.75$				
$\hat{\alpha}$	1.2115	1.2126	1.2158	1.2154	1.2170	1.2160	1.2182	1.2294
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0226	0.0239	0.0253	0.0259	0.0262	0.0317	0.0334	0.0631
$\hat{p}$	0.9934	0.9934	0.9930	0.9924	0.9926	0.9884	0.9898	0.9838
				$r = 0.8$				
$\hat{\alpha}$	1.0453	1.0465	1.0495	1.0480	1.0486	1.0504	1.0512	1.0598
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0236	0.0232	0.0252	0.0262	0.0261	0.0354	0.0369	0.0645
$\hat{p}$	0.9862	0.9882	0.9864	0.9876	0.9862	0.9762	0.9802	0.9742
				$\alpha = 1.5$				
				$r = 0.7$				
$\hat{\alpha}$	1.7983	1.8004	1.7995	1.8012	1.7996	1.7937	1.7962	1.8017
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0452	0.0466	0.0475	0.0486	0.0470	0.0526	0.0543	0.1109
$\hat{p}$	0.7740	0.7718	0.7694	0.7692	0.7710	0.7702	0.7630	0.7636
				$r = 0.75$				
$\hat{\alpha}$	1.6636	1.6662	1.6625	1.6642	1.6612	1.6516	1.6544	1.6545
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0311	0.0318	0.0330	0.0352	0.0326	0.0410	0.0417	0.0739
$\hat{p}$	0.9486	0.9474	0.9490	0.9460	0.9454	0.9412	0.9404	0.9370
				$r = 0.8$				
$\hat{\alpha}$	1.3634	1.3651	1.3606	1.3622	1.3606	1.3511	1.3524	1.3515
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0242	0.0242	0.0263	0.0269	0.0257	0.0364	0.0382	0.0704
$\hat{p}$	0.9982	0.9982	0.9978	0.9982	0.9974	0.9952	0.9948	0.9902
				$\alpha = 1.75$				
				$r = 0.7$				
$\hat{\alpha}$	2.9101	2.9135	2.8976	2.9014	2.8933	2.8517	2.8533	2.8342
$\hat{\sigma}^2_{\hat{\alpha}}$	0.1470	0.1463	0.1503	0.1507	0.1511	0.1612	0.1613	0.1719
$\hat{p}$	0.0020	0.0014	0.0020	0.0014	0.0020	0.0078	0.0080	0.0110
				$r = 0.75$				
$\hat{\alpha}$	2.5699	2.5724	2.5519	2.5549	2.5477	2.5066	2.5065	2.4838
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0897	0.0869	0.0875	0.0874	0.0897	0.1062	0.1057	0.1170
$\hat{p}$	0.0120	0.0118	0.0170	0.0160	0.0214	0.0386	0.0384	0.0534
				$r = 0.8$				
$\hat{\alpha}$	1.8850	1.8877	1.8752	1.8787	1.8746	1.8479	1.8472	1.8349
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0567	0.0539	0.0555	0.0558	0.0569	0.0717	0.0700	0.0788
$\hat{p}$	0.6616	0.6584	0.6794	0.6726	0.6752	0.6912	0.6918	0.7098

Table 5: Estimation Results when $\hat{\alpha}$ is the ecf estimator and$d = 0.3$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
$\hat{\alpha}$	1.2820	1.2818	1.2850	1.2843	1.2843	1.2894	1.2887	1.2866
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0082	0.0074	0.0077	0.0075	0.0083	0.0083	0.0078	0.0115
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996
				$\alpha = 1.5$				
$\hat{\alpha}$	1.5723	1.5724	1.5737	1.5737	1.5741	1.5739	1.5740	1.5729
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0066	0.0066	0.0066	0.0065	0.0066	0.0070	0.0068	0.0085
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
				$\alpha = 1.75$				
$\hat{\alpha}$	1.8661	1.8662	1.8666	1.8664	1.8664	1.8657	1.8657	1.8646
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0056
$\hat{p}$	0.5464	0.5470	0.5470	0.5474	0.5498	0.5606	0.5582	0.5654

Table 6: Estimation Results when $\hat{\alpha}$ is the mle estimator and $d = 0.3$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
$\hat{\alpha}$	1.3017	1.2992	1.3011	1.3024	1.3033	1.3132	1.3112	1.3145
$\sigma^2_{\hat{\alpha}}$	0.0055	0.0060	0.0061	0.0061	0.0061	0.0062	0.0064	0.0082
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996
$\hat{\lambda}$	1.1613	1.1502	1.1719	1.1661	1.1751	1.2222	1.2134	1.2444
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0224	0.0178	0.0207	0.0202	0.0237	0.0391	0.0326	0.0870
				$\alpha = 1.5$				
$\hat{\alpha}$	1.6006	1.6009	1.6009	1.6009	1.6010	1.6002	1.6002	1.6009
$\sigma^2_{\hat{\alpha}}$	0.0041	0.0042	0.0043	0.0043	0.0043	0.0045	0.0045	0.0050
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{\lambda}$	1.1681	1.1663	1.1730	1.1714	1.1750	1.1912	1.1904	1.2065
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0040	0.0034	0.0037	0.0034	0.0040	0.0071	0.0071	0.0200
				$\alpha = 1.75$				
$\hat{\alpha}$	1.8651	1.8652	1.8649	1.8650	1.8646	1.8635	1.8636	1.8629
$\sigma^2_{\hat{\alpha}}$	0.0021	0.0021	0.0022	0.0022	0.0022	0.0023	0.0023	0.0025
$\hat{p}$	0.9926	0.9928	0.9924	0.9924	0.9922	0.9912	0.9914	0.9906
$\hat{\lambda}$	1.1235	1.1230	1.1263	1.1255	1.1273	1.1357	1.1360	1.1466
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0012	0.0012	0.0013	0.0013	0.0015	0.0020	0.0025	0.0086

Table 7: Estimation Results when $\hat{\alpha}$ is the hh estimator and $d = 0.45$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
				$r = 0.7$				
$\hat{\alpha}$	1.3073	1.3079	1.3137	1.3135	1.3123	1.3243	1.3315	1.3414
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0530	0.0577	0.0603	0.0679	0.0605	0.0779	0.3829	0.2831
$\hat{p}$	0.9726	0.9738	0.9720	0.9744	0.9746	0.9598	0.9642	0.9630
				$r = 0.75$				
$\hat{\alpha}$	1.2262	1.2286	1.2323	1.2315	1.2308	1.2480	1.2498	1.2584
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0639	0.0670	0.0720	0.0738	0.0685	0.0924	0.2431	0.1681
$\hat{p}$	0.9640	0.9672	0.9652	0.9670	0.9668	0.9450	0.9546	0.9518
				$r = 0.8$				
$\hat{\alpha}$	1.0737	1.0721	1.0770	1.0759	1.0738	1.0984	1.0944	1.0969
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0782	0.0802	0.0860	0.0865	0.0777	0.1096	0.1815	0.1543
$\hat{p}$	0.9404	0.9498	0.9458	0.9476	0.9490	0.9190	0.9290	0.9308
				$\alpha = 1.5$				
				$r = 0.7$				
$\hat{\alpha}$	1.7916	1.7933	1.7923	1.7937	1.7919	1.7929	1.7911	1.7942
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0672	0.0648	0.0797	0.0839	0.0717	0.0860	0.0819	0.1126
$\hat{p}$	0.7476	0.7604	0.7546	0.7532	0.7518	0.7320	0.7390	0.7368
				$r = 0.75$				
$\hat{\alpha}$	1.6540	1.6584	1.6552	1.6577	1.6534	1.6549	1.6513	1.6497
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0618	0.0567	0.0750	0.0773	0.0652	0.0827	0.0788	0.1051
$\hat{p}$	0.9014	0.9094	0.9076	0.9110	0.9120	0.8782	0.8876	0.8882
				$r = 0.8$				
$\hat{\alpha}$	1.3558	1.3598	1.3598	1.3603	1.3580	1.3653	1.3602	1.3623
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0653	0.0555	0.0704	0.0770	0.0630	0.0835	0.0788	0.0992
$\hat{p}$	0.9782	0.9824	0.9764	0.9776	0.9776	0.9474	0.9584	0.9526
				$\alpha = 1.75$				
				$r = 0.7$				
$\hat{\alpha}$	2.8857	2.8946	2.8729	2.8758	2.8702	2.8336	2.8383	2.8210
$\hat{\sigma}^2_{\hat{\alpha}}$	0.1685	0.1657	0.1704	0.1682	0.1709	0.2008	0.1958	0.2179
$\hat{p}$	0.0064	0.0050	0.0068	0.0072	0.0088	0.0230	0.0178	0.0204
				$r = 0.75$				
$\hat{\alpha}$	2.5407	2.5470	2.5252	2.5304	2.5221	2.4847	2.4893	2.4724
$\hat{\sigma}^2_{\hat{\alpha}}$	0.1247	0.1156	0.1215	0.1196	0.1247	0.1621	0.1518	0.1680
$\hat{p}$	0.0422	0.0314	0.0438	0.0396	0.0442	0.0786	0.0688	0.0808
				$r = 0.8$				
$\hat{\alpha}$	1.8726	1.8763	1.8610	1.8627	1.8608	1.8441	1.8450	1.8357
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0967	0.0845	0.0903	0.0890	0.0922	0.1244	0.1207	0.1325
$\hat{p}$	0.6354	0.6404	0.6572	0.6576	0.6558	0.6594	0.6692	0.6796

Table 8: Estimation Results when $\hat{\alpha}$ is the ecf estimator and $d = 0.45$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
$\hat{\alpha}$	1.2824	1.2809	1.2841	1.2838	1.2829	1.2862	1.2855	1.2838
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0102	0.0099	0.0102	0.0101	0.0107	0.0125	0.0120	0.0145
$\hat{p}$	1.0000	0.9988	0.9990	0.9990	0.9996	0.9992	0.9994	0.9988
				$\alpha = 1.5$				
$\hat{\alpha}$	1.5735	1.5734	1.5746	1.5742	1.5746	1.5748	1.5753	1.5743
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0071	0.0068	0.0069	0.0070	0.0068	0.0074	0.0071	0.0084
$\hat{p}$	1.0000	1.0000	0.9998	1.0000	1.0000	0.9998	1.0000	0.9990
				$\alpha = 1.75$				
$\hat{\alpha}$	1.8650	1.8650	1.8656	1.8654	1.8651	1.8649	1.8647	1.8632
$\hat{\sigma}^2_{\hat{\alpha}}$	0.0050	0.0049	0.0048	0.0048	0.0049	0.0049	0.0050	0.0060
$\hat{p}$	0.5436	0.5440	0.5456	0.5460	0.5482	0.5570	0.5554	0.5624

Table 9: Estimation Results when $\hat{\alpha}$ is the mle estimator and $d = 0.45$.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
				$\alpha = 1.25$				
$\hat{\alpha}$	1.3294	1.3218	1.3262	1.3245	1.3240	1.3434	1.3358	1.3367
$\sigma^2_{\hat{\alpha}}$	0.0118	0.0114	0.0123	0.0119	0.0115	0.0156	0.0142	0.0153
$\hat{p}$	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996
$\hat{\lambda}$	1.2454	1.2191	1.2420	1.2330	1.2427	1.3139	1.2949	1.3167
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0897	0.0733	0.0856	0.0753	0.0798	0.1492	0.1334	0.1603
				$\alpha = 1.5$				
$\hat{\alpha}$	1.6070	1.6059	1.6064	1.6061	1.6057	1.6105	1.6085	1.6090
$\sigma^2_{\hat{\alpha}}$	0.0045	0.0046	0.0048	0.0047	0.0048	0.0056	0.0055	0.0063
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{\lambda}$	1.1950	1.1878	1.1968	1.1934	1.1969	1.2285	1.2210	1.2372
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0130	0.0105	0.0129	0.0120	0.0113	0.0290	0.0237	0.0335
				$\alpha = 1.75$				
$\hat{\alpha}$	1.8635	1.8635	1.8636	1.8635	1.8632	1.8629	1.8627	1.8624
$\sigma^2_{\hat{\alpha}}$	0.0022	0.0022	0.0022	0.0022	0.0023	0.0023	0.0024	0.0025
$\hat{p}$	0.9916	0.9920	0.9920	0.9922	0.9926	0.9916	0.9918	0.9910
$\hat{\lambda}$	1.1332	1.1307	1.1342	1.1331	1.1355	1.1484	1.1471	1.1587
$\hat{\sigma}^2_{\hat{\lambda}}$	0.0023	0.0020	0.0020	0.0019	0.0024	0.0045	0.0050	0.0160

7. Application

In this section we present an application of the methodology developed in the previous sections. The data consists of a 4500 observation time series of tree-ring measurements, at Campito Mountain, in California. The tree-ring widths were observed from 3435BC to 1064AD. Telesca and Lovallo (2010) present a study on long-range dependence on tree-ring measurements as well as its connection to weather behaviour. The graphics of the time series and its periodogram, autocorrelation and partial autocorrelation functions are shown in Figures 1 and 2.

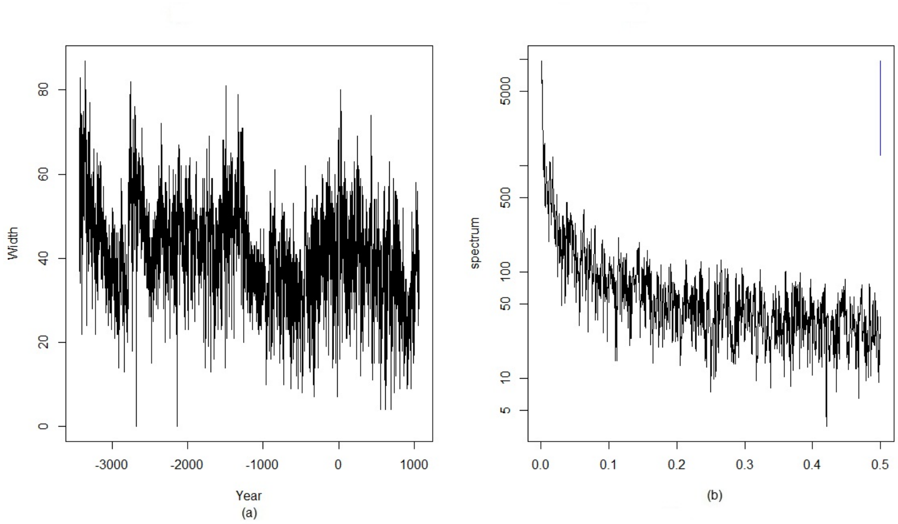

Figure 1: Time series: (a) tree-ring width at Campito Mountain; (b) periodogram for the tree-ring measurements time series.

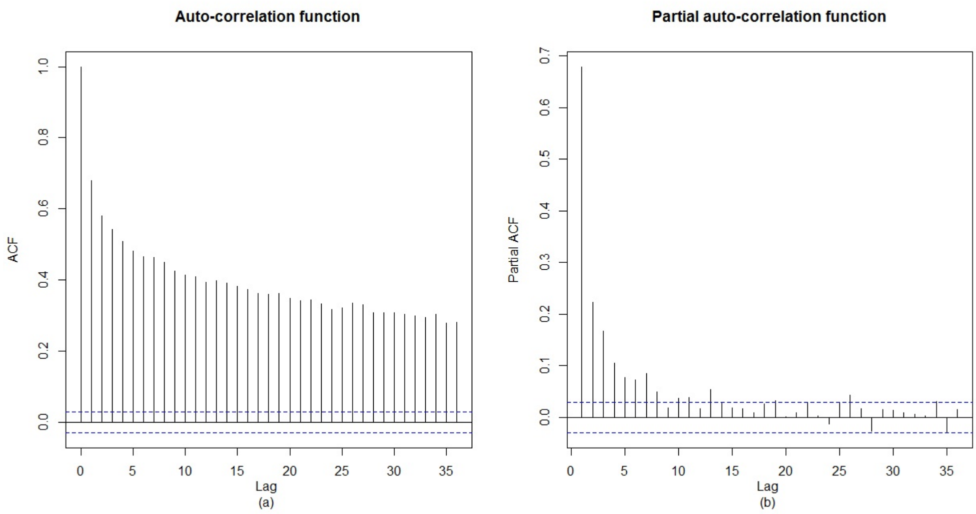

Figure 2: (a) Auto-correlation; (b) Partial auto-correlation functions for the tree-ring width at Campito Mountain.

The results for the parameter estimates are shown in Table 10.

Table 10: Parameter estimates for the tree-ring measurements time series.

Estimate	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
$\hat{d}$	0.4305	0.4361	0.4333	0.4346	0.4323	0.4835	0.4822	0.4762
$\hat{\alpha}_{hh;r=0.8}$	> 2	> 2	> 2	> 2	> 2	> 2	> 2	> 2
$\hat{\alpha}_{hh;r=0.85}$	1.6379	1.6318	1.6336	1.6346	1.6370	1.6849	1.6773	1.6788
$\hat{\alpha}_{hh;r=0.875}$	1.2508	1.2531	1.2564	1.2555	1.2574	1.2429	1.2397	1.2440
$\hat{\alpha}_{ecf}$	1.2452	1.2452	1.2452	1.2452	1.2452	1.2305	1.2312	1.2338
$\hat{\alpha}_{mle}$	1.8134	1.8134	1.8134	1.8134	1.8134	1.8136	1.8136	1.8135
$\hat{\lambda}_{mle}$	4.9953	4.9949	4.9950	4.9950	4.9951	5.0028	5.0023	5.0004

For all the long-range dependence parameter d estimators, we have performed the Wald−Wolfowitz runs test for randomness. We refer the reader to Wald and Wolfowitz (1943) for details on the procedure. Table 11 shows the p-values for the tests: the randomness hypothesis was not rejected for neither of the procedures used, which indicates that the residuals are, in fact, a white noise process.

Table 11: p-values for the Wald−Wolfowitz runs test for randomness.

Statistic	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
p-value	0.3165	0.4565	0.3846	0.4168	0.3603	0.9976	0.9970	0.9916

We have performed hypothesis tests for normality as well. The Z-tests were performed for the hh and ecf estimates, and the likelihood ratio test was performed for the mle estimates. The null hypothesis, given in expression (5.1), was rejected in most part of the cases. The p-values for the tests results are shown in Table 12.

Table 12: p-values for hypothesis tests.

Estimate	MLE	W	GPH	GPHMM	GPHLTS	BA	BAMM	BALTS
$\hat{\alpha}_{hh;r=0.8}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{\alpha}_{hh;r=0.85}$	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001
$\hat{\alpha}_{hh;r=0.875}$	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001
$\hat{\alpha}_{ecf}$	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001
$\hat{\alpha}_{mle}$	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001

In Figure 3 we present the residuals for the ARFIMA$(0, 0.43, 0)$ model applied to the tree-ring time series. In Figure 4 we present the auto-correlation and partial autocorrelation functions for the residuals of the fitted model, and in Figure 5 we present both a comparison between the α-stable density function with $\alpha = 1.8$ and $\lambda = 5$ - values close to the mle estimates for all different estimators for the long-range dependence parameter d - and a non- parametric estimate for the *pdf* of the residuals using a simple kernel function and the confidence interval at 95% confidence level for the 5-step ahead forecasting in the tree-ring time series based on the fitted model.

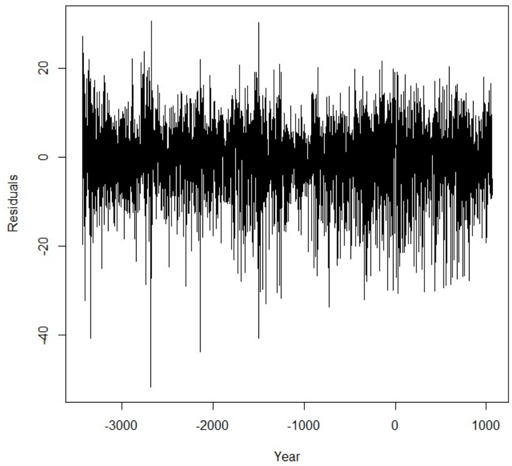

Figure 3: Residuals for the ARFIMA$(0, 0.43, 0)$ model.

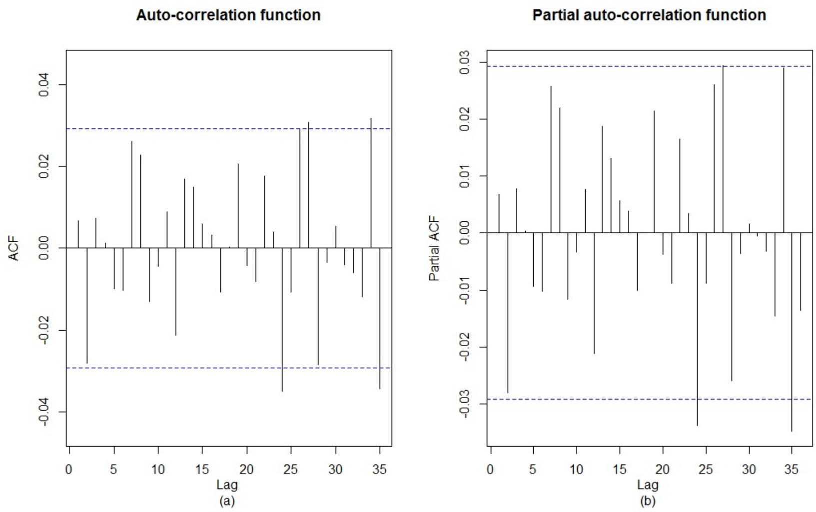

Figure 4: Autocorrelation function for the residuals of the fitted model.

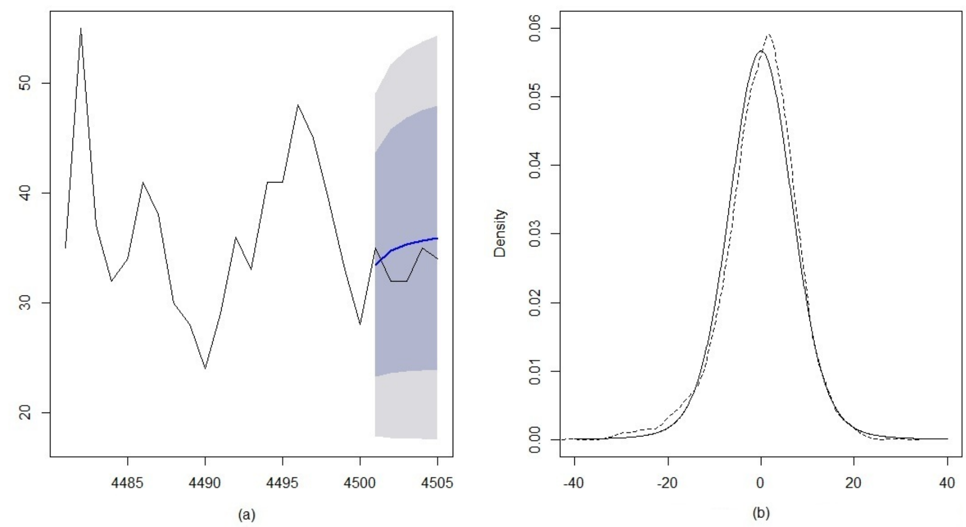

Figure 5: Fitted model: (a) Confidence interval at 95% confidence level for the 5-step ahead forecasting in the tree-ring time series based on the fitted model; (b) Density comparison: α-stable distribution with $\alpha = 1.8$ and $\sigma = 5$ (solid line) and non-parametric estimate from the residuals (dashed line).

8. Conclusions

In this paper, we have presented hypothesis tests for the stability parameter α of the α-stable distributions. The objetive was to analyze in which cases the tests on the three studied estimators were able to distinguish between an α-stable distribution and a Gaussian distribution. We have performed both regular Z-tests and likelihood ratio test. In this context, it is clear that the hh estimator is the most sensitive of the three selected methods: the choice of the truncation point r is crucial to determine the precision of the estimates and, therefore, the results of the hypothesis tests. Unfortunately, the proper truncation point depends on prior knowledge of the data itself - fact that makes it difficult to application in real data sets. The ecf estimator performed well in all of the studied cases. A single choice of parameters is enough to provide precise test results. The mle estimator performed better than the other three for the largest values of stability parameter α and long-range dependence parameter d, in fact the most difficult cases to study. In these cases, this method provided better test results than the other two methods. Power analysis for the three procedures was also presented. The likelihood ratio test proved to be the better option for the test problem, as we notice the largest power estimates.

An application to real data was also presented. We have adjusted the proposed model in a tree-ring measurements time series with 4500 observations and performed hypothesis tests for normality. The estimates for the stability parameter α, as well as the tests results, oscilated depending on the estimation method. Most of the hypothesis tests rejected normality: only a few of the hh estimates could not reject H_0. A comparison between densities was also presented as a visual indication of the fitted model, showing that the mle estimator provided the closest estimate for the residuals distribution.

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