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Axiomatic systemic risk measures forecasting

Porto Alegre

Maio de 2018

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Dissertação apresentada à Escola de Administração da Universidade Federal do Rio Grande do Sul como requisito parcial para obtenção do grau de mestre em Administração.

Supervisor: Prof. Dr. Marcelo Brutti Righi

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RESUMO

Neste trabalho, aprofundamos o estudo sobre risco sistêmico via funções de agregação. Consideramos três carteiras diferentes como proxy para um sistema econômico, estas carteiras são consistidas por duas funções de agregação, baseadas em todos as ações do E.U.A, e um índice de mercado. As medidas de risco aplicadas são Value at Risk (VaR), Expected Shortfall (ES) and Expectile Value at Risk (EVaR), elas são previstas através do modelo GARCH clássico unido com nove funções de distribuição de probabilidade diferentes e mais por um método não paramétrico. As previsões são avaliadas por funções de perda e backtests de violação. Os resultados indicam que nossa abordagem pode gerar uma função de agregação adequada para processar o risco de um sistema previamente selecionado.

Palavras-chaves: medidas de risco, risco sistêmico, função de agregação, mensuração axiomática, GARCH.

ABSTRACT

In this work, we deepen the study of systemic risk measurement via aggregation functions. We consider three different portfolios as a proxy for an economic system, these portfolios are consisted in two aggregation functions, based on all U.S. stocks and a market index. The risk measures applied are Value at Risk (VaR), Expected Shortfall (ES) and Expectile Value at Risk (EVaR), they are forecasted via the classical GARCH model along with nine distribution probability functions and also by a nonparametric approach. The forecasts are evaluated by loss functions and violation backtests. Results indicate that our approach can generate an adequate aggregation function to process the risk of a system previously selected.

Key-words: risk measures, systemic risk, aggregation function, axiomatic measurement, GARCH.

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1 INTRODUCTION

Risk measurement is a core component of the operation of financial institutions such as banks, insurance companies, investment funds and others including financial regulators and even any diversified firm. The objective of risk measurement is to aid decision-making and predictions about the impact of risk on many financial applications, such as portfolio management, asset allocation, derivatives pricing, economic capital and financial stability. Since Markowitz (1952), much research has been devoted to economics, finance, and mathematics, trying to continually answer questions as to what properties should be expected from a risk measure, what should be considered as a good risk measure, and even if there is a best of all.

The concept of risk was initiated as a deviation, and focused in quantifying market risk. Volatility is commonly expressed by the standard deviation, and it is the main measure in most financial analyses. However, the standard deviation and variance do not consider tail risk, which inspired the development of another type of measures, for instance Value-at-Risk (VaR)[1] and Expected Shortfall (ES), both bothering with the far left tail of the distribution. Recently, the literature has given attention to the measure called Expectile Value at Risk (EvaR), for being elicitable[2]. Over the years, the increase of instabilities in the financial market worldwide - especially the collapse on September 2008 - led to criticisms about the then-current risk management systems and motivated the search for more appropriate methodologies. Following recent crises, the necessity of perceiving and comprehending financial network dynamics and characteristics has increased, and the desire to ensure financial stability over the globe highlighted systemic risk in the risk discussion. Systemic risk emerges when the possibility of the breakdown of an entire complex system due to the aftermath of actions taken by any individual or agents with the comprised of the system. In other words, it arises when generalized malfunctioning menaces the overall economy and welfare. Even though there is vast research on its theoretical properties and its main mechanisms of propagation; it is

1 VaR is the most common measure in use, due to the fact that the Basel Committee had required that each financial institutions use this measure (SUPERVISION, 1996), although they had recommended ES as a complementary tool (SUPERVISION, 2013) and posteriorly they have implemented the replacement of VaR to ES for capital adequacy internal models as a rule of Basel 4 (SUPERVISION, 2016).

2 A measure is defined as elicitable, according to Ziegel (2016), it is possible the verification and comparison of competing for estimation procedures, which allows its forecasts to be evaluated.

continuously being conceptualized as hard to define but you nonetheless know it when you see it, as in the case of Benoit et al. (2017).

The study on systemic risk has begun focusing on the balance sheet or accounting information (non-performing loan ratios, earnings and profitability, liquidity and capital adequacy ratio, as cited by Huang et al. (2009) - and measuring correlation coefficients. These traditional procedures were criticized because of the low frequency of the data available that did not include the tail behavior. This lack generated a wide new range of methods to correct it. Benoit et al. (2017) developed an extensive survey on systemic risk strands, defining the current two main ones as the source-specific approach and the global approach. [3] As cited by the authors, both methods have their own virtues, however, it is necessary to have a better understanding of the integration between them and not study them purely in isolation.

Independent of the research line, the literature had been exceedingly concerned with proposing new measures - as can be seen in Billio et al. (2016), Giglio et al. (2016), Bernardi et al. (2017) - and on extracting sources of risk - as in Cherubini & Mulinacci (2014), Acemoglu et al. (2015). Despite the arguments like those of Danielsson et al. (2012), who criticized the leading systemic risk measurement tools and had shown that they performed worse than VaR; simpler measures are not habitual in systemic risk studies. Pursuing the theoretical foundations of single-firm risk measures developed by Artzner et al. (1999)[4], Kusuoka (2001), Föllmer & Schied (2002)[5], and also Rockafellar et al. (2006)[6,7], Chen et al. (2013) proposed an axiomatic approach for the measurement and the management of systemic risk, which regards a system consisted of multiple elements. Based on a set of definitions, the authors demonstrated that any measure

3 The first strand relates to qualitative models, which analyzes specific courses of risk, examples are contagion (CHEN, 1999; ACEMOGLU et al., 2015), bank runs (CHEN et al., 2010; MARTIN et al., 2014), and liquidity crises (BRUNNERMEIER et al., 2013). The other strand focus is to capture the mechanisms, previously quoted, all in a global measure, as viewed in the measurement of connectedness (BILLIO et al., 2012), conditional value at risk (CoVaR) (ADRIAN; BRUNNERMEIER, 2016), capital shortfall (SRISK) (ACHARYA et al., 2012; BROWNLEES; ENGLE, 2012) and marginal expected shortfall (MES), also named systemic expected shortfall (SES), (ACHARYA et al., 2017).

4 In the sense proposed by Artzner et al. (1999), a functional is defined as a coherent risk measure if it respects the following axioms: translation invariance, subadditivity, positive homogeneity, monotonicity.

5 They relaxed the axioms of positive homogeneity and of subadditivity and required instead a new axiom denominated convexity, which is weaker than the originals

6 A deviation measure, according to Rockafellar et al. (2006), will be meant any functional that satisfies the following axioms: Translation insensitivity, positive homogeneity, subadditivity and non-negativity

7 Among the development made by other authors, a well-known class is the spectral risk measure developed by Acerbi (2002), Acerbi (2003).

that follows their definition can be characterized by an aggregation function and a base risk measure. Later on, Kromer et al. (2016) extended this axiomatic framework, allowing a general measurable space in contrast with the finite probability space that was considered before.

Thus, in contrast to the majority of preceding studies, we develop a confrontation between broadly consolidated measures that are acquainted with market risk - VaR, ES, and EVaR - in a systemic risk view. Risk market framework - many studies had ensured the acceptance and effectiveness of a group of consolidated measures. On the other hand, though the property defined amount of measures, there is no consensus about a definitive or superior risk measure, resulting in the defense of the use of distinct measures for distinguished situations, however almost no author argues about a measure procedure as the dominant overall.

This procedure is corroborated by Benoit et al. (2013), who followed the work of Brownlees & Engle (2012) and added CoVaR to the risk comparison, stating that the most popular systemic risk measures can be represented as linear transformations of single-firm market risk. As found out by them, both measures miscarried catching the multifaceted essence of systemic risk and, from theoretical and empirical sight, one market risk measure can capture most of the variability of it. Yun & Moon (2014) applied the same method among Korean banks and had similar results, suggesting that the measures were qualitatively quite related in explaining the cross-sectional distinct contributions across firms, and it was closely associated with specific variables (e.g., VaR). Brownlees et al. (2017) developed an deep comparison between CoVaR and SRISK during eight financial panics[8] placed in the era before Federal Deposit Insurance Corporation (FDIC) insurance. The authors showed that the measures performed similarly, but it seemed that relative progress was not made. Finally, they argued that VaR was a suitable instrument for systemic risk monitoring. Additionally, Righi & Borenstein (2017) compared eleven risk measures and concluded there is no clear dominance of any measure. Previous works have commonly analyzed systemic risk based on isolated groups of individual firms, usually financial institutions[9] - as can be

[8] the panics of 1873 and 1884, the Barings Crisis of 1890, the subsequent panics of 1893 and 1896, the panic of 1907, the monetary and fiscal consolidation of 1921, and the panic of 1933.

[9] In addition, studies are Banulescu & Dumitrescu (2012), Castro & Ferrari (2014) focus primally on identifying systemically important financial institutions (SIFIs).

seen in Benoit (2014) and Kreis & Leisen (2017) - or even employing index prices as a proxy for systemic risk - see Bernal et al. (2014), Mensi et al. (2017) - going against the fundamental concept that systemic risk enfolds the simultaneous outcomes across all agents in an economic system. In order to contribute to the literature, we extend the analysis of systemic risk to an entire economy, regarding the outlook of a regulator as a portfolio manager who desires to manifest preference across all feasible outcomes of the economy.

The correct estimation of risk is necessary in order to ensure that the investor has an adequate position in the market without being overexposed and underexposed[10]. As pointed out by Boucher et al. (2014), the uncertainty inherent in the model's specification and estimation generated the forecasting failure during the global financial crisis. The uncertainty around the data statistical properties can cause errors in estimation (which generates parameter risk) in misspecification and in identification. Even though the required necessity of model risk quantification is well-known, there is no broadly-acknowledged approach for it. One of the major differences of modeling approaches is how they deal with the uncertainty around the distribution of the risk factors and what assumptions are made about its shape. A risk estimative is tightly connected to the assumption of data's distribution. As reviewed, manifold measures are attached to the tail behavior of the data - as VaR and ES - therefore it is feasible to assume the relevance of the distribution of data to risk definition and the requirement of an accurate supposition of it. Therefore, the analysis structured in this study is defined under dissimilar assumptions of distribution functions, including the leading procedures, jointly with an econometric GARCH model. The GARCH models are deeply consecrated over the literature, owing to their capacity to generate more accurate predictions of the variance and covariance of financial series over time. Arguments on behalf of the use of GARCH models to forecast volatility compared to parsimonious models can be seen in Angelidis et al. (2004), Zhou (2012). A large number of extensions of the GARCH have already been proposed, however, the standard GARCH model can be widely improved by simply replacing the normal conditional distribution assumed to the random shock z_t by another that considers characteristics as fat tails. This approach can generate more parsimonious models. In place of it, the classical model is applied in this work due to

[10] Overexposed is caused by risk underestimation, what could be considered as uncovered losses, otherwise underexposed is caused by risk and is an overestimation, generating opportunity costs.

the desire to obtain the best trade-off between parsimony in the parameters, as well as better estimations. The majority of works usually focus on a small set of distributions, as in Marinelli et al. (2007), and Tolikas (2014) on estimating with EVT, and Diamandis et al. (2011) which assumed only the student distribution and its variants, due to empirical literature who had shown that distributions which allow for fat-tail are more suitable to financial data, see Kuester et al. (2006), Degiannakis & Potamia (2017), we contrast by widening the set and applying it to the systemic overview.

In order to set up a portfolio compounded by the joint distribution of outcomes over entities; the data set is based on all individual stock of the U.S. market that has been negotiated in the period from January 2000 to December 2016. Based on the axiomatic definition of Chen et al. (2013), we apply two distinct types of aggregation functions to enable the evaluation of systemic risk according to a single-firm risk measure. To the best of our knowledge, our work is the first to consider a data set consisted of all U.S. Stocks, and to additionally use it to model systemic risk. As a way to ensure that our approach is distinct; we analyze an index price of the economy in question as an alternative portfolio selection. The choice is made based on the widespread use of both theoretical and empirical fields and recognition by market agents. So, the selected index is the $S\&P500$. We also build an $1/N$ portfolio, also known as the naive diversification, compounded by all U.S. Stocks and a five-factor portfolio (FF) based on Fama & French (2015) approach.

Since the main objective of this work is to analyze systemic risk under aggregation function, not risk measures themselves, and considering that the literature agrees that there is no dominant risk measure, suggesting the use of distinct procedures for different situations and emphasizing the role of the manager to decide what to consider in every situation, and additionally, from the best of our knowledge, this work is the first of applying this approach, we limit it to develop a comparison among the three main risk measures currently in use: VaR, ES, EVaR. The empirical evaluation of this study is conducted as follows: The first step consisted in assembling the data of all American stocks by the aggregation functions stipulated, resulting in two portfolio sets and building the $S\&P500$, $1/N$ and the five-factor (FF) portfolios. Each one of them is modeled, according to its log-returns, to engender estimation of VaR, ES, and EVaR for every joint of GARCH. Finally, the performances of the risk measures are compared. The outcomes in this study also contribute to the financial industry due to the positive result acquired for its innovative approach in dealing with systemic risk. The methodology based on aggregating all stocks of an economy shows solid theoretical properties and proper empirical performance, it also outperforms the $S\$P500$ index concerning ES forecasting. Our findings can be used as a foundation of major and new lines of study.

This work will is divided into 5 sections, being this the first one, which ought to present the theme, reporting in a comprehensive way the issues to be discussed later on. The remainder of this paper is structured as follows: section 2 reports the background of risk measures theory. Section 3 discloses the methodological procedures of the techniques, which is used to select and analyze the data chosen for this research, and demonstrates how empirical results are obtained.ă Section 4 expounds the empirical analysis and its results. Finally, 5 concludes with the final considerations.

2 BACKGROUND

Throughout the study, we work with an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω represents the sample space, $\mathcal{F}$ is a set of possible events in Ω and $\mathbb{P}$ refers the probability that is defined in Ω of the events in $\mathcal{F}$. Considering n the number of a finite set of firms, assets or portfolio, the random result of any firm in an economy is expressed by X_i, where $i = 1 \ldots n$, and so an economy is denoted by $X = (X_1, \ldots, X_n)$ as the juncture of all firms belonging to it. $X \geq 0$ is represented as a gain and $X < 0$ as a loss, and the expected value of X under $\mathbb{P}$ is defined by $E[X]$. In addition, F_X is the probability function of a given X, and F_X^{-1} is its inverse. Let $\mathcal{X} = \mathcal{X}(\Omega, \mathcal{F}, \mathbb{P})$ be a space of random variables, of which X is an element. 1_n is a vector denoting a cross-sectional loss profile in a scenario where a unit loss occurs by each component, 1_Ω is a vector denoting a unit loss of an individual agent in all scenarios and $1'_\Omega$ is its transposed. Following this context, expressing the risk of X with a real number is equivalent to establishing a function $\rho_0 : \mathcal{X} \to \mathbb{R}$. Analogously, measuring an economy X risk is equivalent to establishing a function $\rho : \mathcal{X}^n \to \mathbb{R}$.

The search for an answer for the typical question about the quality of risk measures (and if it is possible to find the best of all) inspired the research to define the characteristics that a measure should possess. The first to determine the properties any good risk measure shall have were Artzner et al. (1999), they listed four axioms and called any measure which satisfied them a coherent risk measure.ă In the sense proposed by Artzner et al. (1999),ă a functională $\rho_0 : \mathcal{X} \to \mathbb{R}$ is defined as a coherent risk measure if it respects the following axioms:

Translation Invariance: $\rho_0(X + C) = \rho_0(X) - C, \forall\, X \in \mathcal{X}, C \in \mathbb{R}$.

Convexity[11]: $\rho_0(\lambda X + (1-\lambda)Y) \leq \lambda\rho_0(X) + (1-\lambda)\rho_0(Y), \forall\, X, Y \in \mathcal{X}$, $0 \leq \lambda \leq 1$.

Positive Homogeneity: $\rho_0(\lambda X) = \lambda\rho_0(X), \forall\, X \in \mathcal{X}, \lambda \geq 0$.

Monotonicity: if $X \leq Y$, then $\rho_0(X) \geq \rho_0(Y), \forall\, X, Y \in \mathcal{X}$.

The axioms were designed to correspond with the characteristics required for a consistent measure. The first axiom, translation invariance, ensures that if adding a certain gain C to a position X, the risk of this position should decrease by the

[11] In this case, convexity coincides with subadditivity.

same amount. Convexity is a relaxed version of subadditivity, which corresponds to the principle of diversification, in other words, the risk of two or more combined assets, a portfolio, is less than, or equal to, the sum of individual risks of the portfolio assets. Positive homogeneity demonstrates the risk of a position that increases proportionally with its size. The last axiom guarantees that if the loss of a financial position X is always greater than another position Y, so the value of X is smaller than Y, the expected risk of X is greater than the risk of Y. Artzner et al. (1999) defined an acceptance set to any coherent risk measure of positions which do not causes losses, expressed by $A_{\rho_0} = \{X \in \mathcal{X} : \rho_0(X) \leq 0\}$. In addition, they demonstrated the dual representation of a coherent measure. This representation, in the example of a discrete case, could be perceived as the worst expected loss of a position X, among the scenarios $\mathbb{Q} \in \mathcal{P}_{\rho'} \subseteq \mathcal{P}$, where $\mathcal{P}$ is a probability space, and is formally expressed by $\rho_0(X) = \sup_{\mathbb{Q} \in \mathcal{P}_{\rho'}} E_{\mathbb{Q}}[-X]$.

In the sense of Chen et al. (2013), a measure of systemic risk is defined as a functional $\rho : \mathcal{X}^n \to \mathbb{R}$ on the juncture of all possible scenarios of distribution outcomes across any agent belonging to all economies X, $Y \in \mathcal{X}^n$ if it satisfies the following set of axioms:

Monotonicity: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$, $\forall\, X, Y \in \mathcal{X}^n$.

Positive Homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, with $\lambda \geq 0$, $\forall\, X \in \mathcal{X}^n$.

Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, with $0 \leq \lambda \leq 1$, $\forall\, X, Y \in \mathcal{X}^n$.

Preference consistency: given cross-sectional profiles $X_i, Y_i \in \mathcal{X}$, its said $X_i \preceq_p Y_i$ iff $\rho(X_i 1'_\Omega) \geq \rho(Y_i 1'_\Omega)$. So, if for every scenario $\omega \in \Omega$, $X(\omega) \preceq_p Y(\omega)$, then $\rho(X) \geq \rho(Y)$, $\forall\, X, Y \in \mathcal{X}^n$.

Normalization: $\rho(1_n) = n$.

The axioms proposed were designed to correspond to the characteristics required for a consistent measure. The first three axioms are similar to the ones for a single-firm risk measure, monotonicity guarantees that if the loss of a financial position X is always greater than another position Y, and the value of X is smaller than Y: the expected risk of X is greater than the risk of Y, positive homogeneity demonstrates the risk of a position increases proportionally with its size, and convexity corresponds to the principle of diversification. In other words, the risk of two or more combined assets (a portfolio) is less than, or equal to, the weighted sum ofă individual risks of the portfolio assets.

The fourth axiom determines a preference relationship on cross-sectional profiles by comparing the systemic risk of the constant economies. If an economy if compounded only by one firm, it follows from monotonicity. This axiom claims that for every scenario ω, if $Y(\omega)$ is always preferred to $X(\omega)$, and the systemic risk of the economy Y must be consistent with this preference and so it cannot be greater than the one of the economy X. The last axiom, normalization, ensures that the risk of a unit loss by all firms with certainty to be the total loss.

To determine a systemic risk measure, it is necessary to capture both the cross-sectional loss profiles per firms and the distribution of joint outcomes per scenario of the economy in question. According to Chen et al. (2013) any convex systemic risk measure $\rho : \mathcal{X}^n \to \mathbb{R}$ can be composed by a convex single-firm risk measure $\rho_0 : \mathcal{X} \to \mathbb{R}$ and a convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$. A convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is a function that sets up an encapsulated statistic of the cross-sectional loss profile into a real number, if, for all cross-sectional loss profiles $x, y \in \mathbb{R}^n$, it satisfies the following axioms:

Monotonicity: if $x \leq y$, then $\Lambda(x) \geq \Lambda(y)$.

Positive Homogeneity: $\Lambda(\lambda x) = \lambda\Lambda(x)$, with $\lambda \geq 0$.

Convexity: $\Lambda(\lambda x + (1-\lambda)y) \leq \lambda\Lambda(x) + (1-\lambda)\Lambda(y)$, with $0 \leq \lambda \leq 1$.

Normalization: $\Lambda(1_n) = n$.

So the relation of a systemic risk measure ρ and Λ is defined as:

$$\rho(X) = (\rho_0 \circ \Lambda)(X) = \rho_0(\Lambda(X_1, X_2, ..., X_n)), \forall X \in \mathcal{X} \tag{2.1}$$

where ρ_0 is a base risk measure and Λ is an aggregation function.

Since the cross-sectional outcomes become aggregated, the estimation is simplified to single outcomes across scenarios. The evaluation of systemic risk can be acquired through base risk measure- a single-firm risk measure, for instance. Diverse aggregation functions were propounded by the literature, ones that treated other particularities of the data (as optimization, resource allocation, flow network and contagion). See Chen et al. (2013) and Kromer et al. (2016) for more details.

3 METHOD

The empirical study is based on available adjusted closing prices of the U.S. stocks since it is the most important and expressive market around the world. In order to make the work broader, and to guarantee the definition of a portfolio composed by the joint distribution of outcomes across entities in an economy; the data set is composed of daily log-returns of all U.S. stocks that have been negotiated in the selected period, generating a survival bias. The time interval chosen is from January 2000 to December 2016, resulting in 17 years of observations. It is defined in order to capture turbulent and calm observations, and also to avoid any distortion generated in small samples. The computational implementations are performed in the R programming language[12] and the empirical data is acquired through Quandl[13] and the Database of Kenneth R. French[14].

The selection of an aggregation function and a base risk measure is defined according to specific preferences of the regulator. A base risk measure is selected by an appropriate single-firm risk measure - measures such as VaR, conditional VaR, and even contagium models - otherwise the definition of the aggregation function is not so simple due to the small amount of research performed in the area and the consolidation of the characteristics of the market into a single function. An aggregation function can be derived in many forms. The individual stocks data are aggregated by two distinct procedures. The first function, symbolized as Λ_S, is simply compounded by the sum of all adjusted closing prices of the stocks. Similarly, the Λ_W also sums the adjusted closing prices. However, every stock is weighted by its own volume of transactions in the sum. Once aggregated, each set has its daily log-returns extracted. From the sum up of all losses and profits of each individual firm i, a linear aggregation function is defined:

$$\Lambda_{S_t}(x_i) = \sum_{i=1}^{n} x_{i,t} \tag{3.1}$$

where $x_{i,t}$ corresponds to the position of each individual firm i in the time period t.

[12] R Core Team, 2017.

[13] Quandl is a platform for financial, economic, and alternative data that serves investment professionals. All stocks and S&P500 data were acquired through the WIKI Prices Database developed and offered by Quandl, Inc. More info can be seen at https://www.quandl.com/databases/WIKIP

[14] All the data to build the Fama/French five-factor (FF) Portfolio was acquired through http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/index.html website.

This function covers the definition of a portfolio composed by the joint distribution of outcomes across firms in an economy. Meantime Λ_S might subsidize losses of some firms from gains of others and it considers firms as equals on the aggregation, not regarding their liquidity, size, volume of transactions and other features. A reasonable procedure to weigh entities is through its own volume of traded stocks.

$$\Lambda_{W_t}(x_i) = \frac{\sum_{i=1}^{n} x_{i,t} w_{i,t}}{W_t} \tag{3.2}$$

where $W = \sum_{i=1}^{n} w_{i,t}$ is the total volume of all stocks in the economy.

We also analyze the $S\&P500$ index price as an alternative portfolio selection, as well the $1/N$ portfolio, also known as the naive diversification, and the approach presented by Fama & French (2015) to build a five-factor portfolio. The naive diversification is defined to be one in which a fraction $1/N$ of wealth is allocated to each of the N assets available for investment at each rebalancing date.

The Value at Risk (VaR) is the most widely-disseminated measure, despite the existence of a large number of others. VaR quantifies the maximum loss that is expected to occur over a given time period and a given confidence level. The popularity of this measure, according to Daníelsson et al. (2013), is due to its simplicity and practical advantages that outweigh its theoretical deficiencies. VaR does not possess the subadditivity axiom, however, since it is the most used measure, we include it in this study. Since Basel I, VaR has been required for all financial institutions as an integral part of risk management operations. VaR is formally defined by the expression:

$$\text{VaR}_{\alpha}(X) = -\text{inf}\{x : F_X(x) \geq \alpha\} = -F_X^{-1}(\alpha) \tag{3.3}$$

where $\alpha \in (0, 1)$ corresponds to the significance level, usually is close to zero.

Acerbi & Tasche (2002) suggested the use of the Expected Shortfall instead of the VaR to alleviate the problems inherent in VaR and, in addition, they proved that ES is a coherent risk measure. This proposition was due to the fact that VaR is not subaddivity (that is not a coherent measure) and it does not consider the severity of losses occurred beyond the quantile of interest. ES measures the expected value of losses occurred when VaR is violated given an α-quantile of interest, in other words, it represents the expected losses that exceed VaR losses. ES is defined to be:

$$\text{ES}_\alpha(X) = -E[X|X \leq F_X^{-1}(\alpha)] = -\frac{1}{\alpha}\int_{-\infty}^{F_X^{-1}(\alpha)} xF_X(x)dx = -\frac{1}{\alpha}\int_0^{\alpha} F_X^{-1}(s)ds \quad (3.4)$$

where $\alpha \in (0,1)$, as in VaR, corresponds to the significance level, usually is close to zero.

A recently used measure that gained attention is the Expectile Value at Risk (EVaR). This method is connected to the concept of an expectile[15], which is a generalized quantile function. Bellini et al. (2014) showed that EVaR is a coherent risk measure for $\alpha \leq 0.5$. EVaR is expressed by:

$$\text{EVaR}_\alpha(X) = -\text{arg min}_\gamma E[|\alpha - 1_{X\leq\gamma}|(X-\gamma)^2] \quad (3.5)$$

where $\alpha \in (0,1)$.

A statistical model can be represented in parallel with risk measures and a conditional heteroskedastic model. Assuming X as a variable with a fully parametric location-scale specification based on the expectation, dispersion and random component; represented as: $X_t = \mu_t + \sigma_t z_t$, where t is a period of time, X_t is the return of a portfolio or asset in t, μ_t is the conditional mean (usually equal to zero), σ_t is the conditional standard deviation and z_t represents the innovations, which could admit different probability distribution functions F_z. Following this definition, any risk measures are expressed as:

$$\rho(X_t) = -\mu_t + \sigma_t\rho(z_t) \quad (3.6)$$

where σ_t is the conditional standard deviation and it is defined according to the GARCH model applied to the estimation.

Bellini & Bernardino (2017) confirmed the competitiveness of the measure against others, and also connected EVaR, as the financial risk measure to expectiles in the same way that VaR is to quantiles. According to the author, EVaR is defined

[15] The concept of expectile was introduced by Aigner et al. (1976) as the τ-level partial moment, called expectile with $\tau \in (0,1)$, for a continuous random variable Y where the value of μ_τ minimizes the expectation $E[|\tau - I[Y < \mu_\tau]|(Y-\mu_\tau)^2]$, where $I(.)$ is an indicator function which is one when $Y < \mu_\tau$ and zero otherwise. According to Gerlach (2016) from a sample $y_1, ..., y_T$ on Y , and a known-fixed τ, the constant τ-level expectile of Y (μ_τ) could be evaluated by minimizing the asymmetric sum of a squares function defined as: $\sum_{t=1}^{T} |\tau - I[y_T < \mu_\tau]|(y_t - \mu_\tau)^2$

as the amount of money that should be added to a position to guarantee a sufficiently high gain-loss ratio.

The competing models are obtained by Historic Simulation[16] (HS) and by the classical GARCH model in combination with nine distributions. The conditional variance (σ^2) of the GARCH(L_1, L_2) model is determined by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{L_1} \alpha_i X_{t-i}^2 + \sum_{j=1}^{L_2} \beta_j \sigma_{t-j}^2 \tag{3.7}$$

where L_1 and L_2 are the orders of the dependencies of the variance with the past returns and the past variances themselves, respectively.

The parameters are acquired by the maximum log-likelihood technique, which has desirable asymptotic properties as shown in Bollerslev & Wooldridge (1992). Each model is computed under different assumptions of distribution functions represented by F_z, these being the normal distribution, the skew-normal distribution, the student-t distribution, the skew-student-t distribution, the generalized error distribution, the skew-generalized error distribution, the logistic distribution, and also the empirical distribution and EVT as semiparametric approaches.

The most well-known parametric distribution is the normal one, which is entirely described by its first and second moments, mean (μ) and variance (σ^2), respectively. Thus, a random variable X is normally distributed with a probability density function (PDF) [17] of:

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \tag{3.8}$$

As an alternative to fit fatter tails of data, the student distribution was introduced by Bollerslev (1987) into GARCH modeling. This distribution is described by a location parameter(α), a scale parameter(β) and a shape parameter (υ) and its density is given

[16] HS uses the empirical distribution of the data to model the time series and no statistical model and parameters are assumed. This estimation method just considers the risk measure models with F_z as the empirical distribution of past returns, and makes no assumption about the distribution of the data, except that the returns are iid.

[17] A density, or PDF, of a continuous random variable is a function that provides a relative likelihood of a random value, in a sample space, which would equal the sample.

by:

$$f(z) = \frac{\Gamma(\frac{\upsilon+1}{2})}{\sqrt{\beta\upsilon\pi}\Gamma(\frac{\upsilon}{2})}[1 + \frac{(z-\alpha)^2}{\beta\upsilon}]^{-(\frac{\upsilon+1}{2})} \tag{3.9}$$

where Γ is the Gamma function.

The generalized error distribution (GED) is a distribution that belongs to the exponential family and, like the student distribution, it is symmetric, unimodal and its location parameter is the mean, mode, and median of the distribution (i.e μ). Alongside the location parameter (α), it is described by the scale parameter (β) and the shape parameter (κ) with density expressed by:

$$f(z) = \frac{\kappa e^{-0.5|\frac{z-\alpha}{\beta}|^{\kappa}}}{2^{1+\kappa^{-1}}\beta\Gamma(\kappa^{-1})} \tag{3.10}$$

where Γ is the Gamma function.

As a form to add asymmetry for probability distributions, Fernández & Steel (1998) inserted skewness into unimodal and symmetric distributions - Normal, Student and GED - by adding inverse scale factors to the positive and negative real half lines. According to a skew parameter (ξ), the density of a random variable X is described as:

$$f(z|\xi) = \frac{2}{\xi + \xi^{-1}}[f(\xi z)H(-z) + f(\xi^{-1}z)H(z)] \tag{3.11}$$

where $\xi \in \mathbb{R}^{+}$ and $H(.)$ is the Heaviside function.

The Logistic distribution possesses a similar shape of the normal, only with higher kurtosis (fatter tails). The logistic function is commonly used in logistic regression and feedforward neural networks, and its PDF is expressed as:

$$f(z) = \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma(1 + e^{-\frac{x-\mu}{\sigma}})^2} \tag{3.12}$$

where μ and σ are respectively the location and scale parameters.

As an alternative method to deal with distribution assumptions; the data can be modeled by nonparametric methods. Nonparametric models, in contrast to parametric ones, do not specify any assumption a priority but rather leave to the own data determined it. The empirical distributions are entirely defined based on the historical data of a sample, and are obtained by extracting the observed first two moments of it. It is most commonly applied to the Historical Simulation (HS)[18] estimation approach, Daníelsson (2011) affirms that it is a simple approach for forecasting risk, which relies on the assumption that history repeats itself. However, it is often criticized for resulting in bad estimations, as expressed by Christoffersen & Gonçalves (2005). Meanwhile, the empirical distribution can be used together with a parametric model, generating the Filtered Historical Simulation (FHS), in an attempt to generate a better estimation. The main motivation of this method is that it allows us to capture conditional heteroskedasticity through a parametric model, while still being somewhat unrestrictive about the shape of the distribution of the data, introducing the non-normality of the risk factors. The empirical distribution function is defined as:

$$\hat{F}_n(z) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x) \tag{3.13}$$

where $I(.)$ is the indicator function, assuming 1 if the variable has a probability and 0 otherwise.

Another well-known nonparametric process in use is the Extreme Value Theory (EVT), which uses its own type of distribution, only of the extreme events. Singh et al. (2011) suggested the use of EVT as a better option to model risk, because it has a advantage of dynamically reacting to the changeable market conditions, resulting in better forecasts.

$$f(z) = \frac{1}{\sigma} t(x)^{\xi+1} e^{-t(x)} \tag{3.14}$$

where $t(x) = (1 + \xi(x-\mu)/\sigma)^{-1/\xi}$ if $\xi \neq 0$ and $t(x) = e^{-(x-\mu)/\sigma}$ otherwise.

The risk measures are estimated in accordance with the definitions already exposed, except for EVaR, which is drawn following the approach stipulated by Taylor (2008), considering it as a proxy for expectiles. Each measure was forecasted by a one-step-ahead procedure for each portfolio set.

To ensure robustness, we considered different levels of significance and window sizes. The significance levels of 0.01, 0.025 and 0.05 are the most commonly used for

18 HS uses the empirical distribution of the data and no statistical model and parameters are assumed.

practical purposes and are more widespread in the risk literature, thus they are defined for this analysis too. The Basel committee requires an estimated window of at least 1 year, approximately 250 business days, so this is the value chosen for this purpose. As a form of robustness, the windows of 500 and 1,000 are also applied. All models are allowed to change their parameters over time and may be updated for each day of the moving window.

As a form to evaluate the risk forecasts, backtests techniques and loss functions for each measure are applied. Otherwise, there are more traditional procedures utilized instead as goodness-of-fit measures, some examples are the mean squared error, root mean squared error and mean absolute error, as can be seen in Telmoudi et al. (2016), however, these procedures do not address correctly the risk-forecasting property of the models under consideration. As a way to ensure the robustness of the study, descriptive statistics and individual characteristics of each model will be analyzed, such as mean and variance.

In this work, VaR was backtested according to the violation rate (VR), which is determined as the rate of violations and also could be interpreted as the proportion of observations for which the observed returns are more extreme than those forecasted. The most desirable measure is the one with VR close to 1, which means that the measure in question is doing its job, otherwise when $VR < 1$ the forecasting is conservative, resulting in underutilized capital, and when $VR > 1$ the forecasting is generating overexposure, which means that the institution may not have sufficient resources to cover a likely future loss. A risk measure violation rate (VR) can be expressed as:

$$\text{VR} = \frac{\text{realized violations of } VaR(X)_t}{\text{expected violations of } VaR(X)_t} \tag{3.15}$$

where the realized violations of $VaR(X)_t$ = $\sum_{t=1}^{T} I(x_t < VaR(X)_t)$.

Backtesting ES is more difficult, owing to it being defined by an expectation and not by a single quantile, such as VaR. McNeil & Frey (2000) determined an analogous methodology to test ES as the violations for VaR, which consists of a hypothesis test. The test is based on a null hypothesis (H_0) of the residuals with a mean of zero that considers the measure as corrected estimated, and an alternative hypothesis (H_A) of a mean greater than zero, considering ES as systematically underestimated since this is the likely direction of failure.

$$\begin{cases} H_0 : \mu = 0 \\ H_A : \mu > 0 \end{cases} \tag{3.16}$$

As EVaR has a definition related to a gain-loss ration, Bellini & Bernardino (2017) argued that the notion of violation is not meaningful to it, and according to it this measure was only qualified by its loss function.

The intention of risk measures is to support decision-making, so the best manner to rate such measures is to detect how well they perform in an intended task. In the context of this work, the ending goal is to ensure the minimum loss over the aggregate system, which can be directly reached by loss functions. It also can be argued that computing the number of violations occurred does not provide the magnitude of the violation, but only a perception of it. Loss function is conceptually preferable to violation backtest, thus it is used as the principal appliance for interpreting results and inferring conclusions, although VR is considered a support tool. A loss function is a function that maps an event, or even values of variables, onto a real number; it represents a "cost" or "loss" associated with the event. The concept is that the closer the mean score of a model is to zero, the better. The loss functions are calculated following the methodologies of Acerbi & Szekely (2017), Gerlach et al. (2017) and Bellini et al. (2014), for VaR, ES and EVaR respectively and are expressed as:

$$Z_{VaR_\alpha} = \frac{1}{T}\sum_{t=1}^{T}[\alpha(x_t + VaR_\alpha(X)_t)_+ + (1-\alpha)(x_t + VaR_\alpha(X)_t)_-] \tag{3.17}$$

Where T is the estimation length, $\alpha \in (0,1)$ is the significance level, x_t is a realized value of the serie and $VaR_\alpha(X)_t$ is a risk estimative.

$$Z_{ES_\alpha} = \frac{1}{T}\sum_{t=1}^{T}[\alpha(ES_\alpha(X)_t - VaR_\alpha(X)_t) + (x_t + VaR_\alpha(X)_t)_-] \tag{3.18}$$

Where T is the estimation length, $\alpha \in (0,1)$ is the significance level, x_t is a realized value of the serie and $VaR_\alpha(X)_t$ and $ES_\alpha(X)_t$ are risk estimatives.

$$Z_{EVaR_\alpha} = \frac{1}{T}\sum_{t=1}^{T}[\alpha((x_t + EVaR_\alpha(X)_t)_+)^2 + (1-\alpha)((x_t + EVaR_\alpha(X)_t)_-)^2] \tag{3.19}$$

Where T is the estimation length, $\alpha \in (0, 1)$ is the significance level, x_t is a realized value of the serie and $EVaR_\alpha(X)_t$ is a risk estimative.

4 RESULTS

As defined earlier, the empirical study is based on available daily data of the U.S. stocks aggregated by two distinct aggregation procedures, Λ_S and Λ_W, and by the $S\&P500$ index, the $1/N$ and the fife-factor (FF) portfolios. These five portfolios are used as a proxy for economies in a systemic context for modeling and forecasting risk. They are detailed in Figures 4.1, 4.2, 4.3, 4.4 and 4.5. It can be easily seen that Λ_S resembling behaves as the $S\&P500$, disregarding the very beginning of it which is expected since $S\&P500$ is compounded by the 500 most representative stocks of the ones considered in Λ_S. The $1/N$ and the FF portfolios also behave close to the $S\&P500$ index. However, the similarity between these four, Λ_S, and additionally Λ_W, present more turbulent behavior through time and greater deviations than the others. All portfolios exhibited excess kurtosis and skewness, generating fatter tails, are more accurately seen in Table 4.1. The weighted portfolio has shown a distinguished behavior wherein a huge crisis, such as the one from 2008, cannot be identified with ease, in contrast with the sum portfolio and the index portfolio that evidenced the effect and subsequent recovery from a crisis. It is also observed the greater standard deviation of Λ_W than the four others, which is seen either in 4.2.

Table 4.1 – Descriptive Statistics of the Portfolios

	$\Lambda_S(X)$	$\Lambda_W(X)$	$S\&P500$	$1/N$	FF
Mean	-0.0001	0.0003	0.0001	0.0002	0.0010
Standart Deviation	0.0198	0.1833	0.0124	0.0134	0.0161
Kurtosis	7.5463	8.0047	8.1605	7.2481	5.1121
Skewness	0.0402	-0.2408	-0.01937	-0.4579	-0.1574

Figure 4.1 – Time series and histogram of $\Lambda_S(X)$

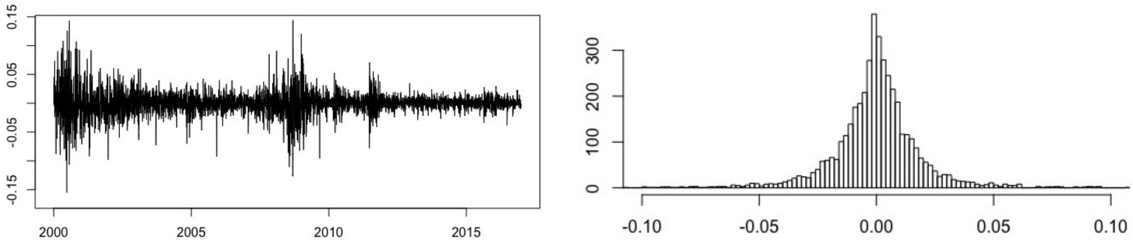


Figure 4.2 – Time series and histogram of $\Lambda_W(X)$

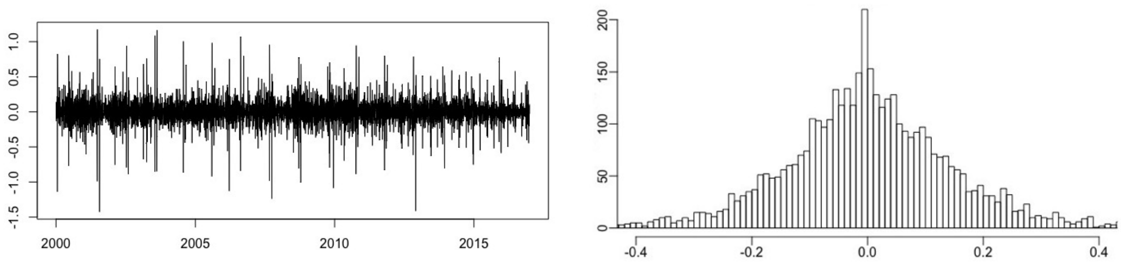


Figure 4.3 – Time series and histogram of $S\&P500$

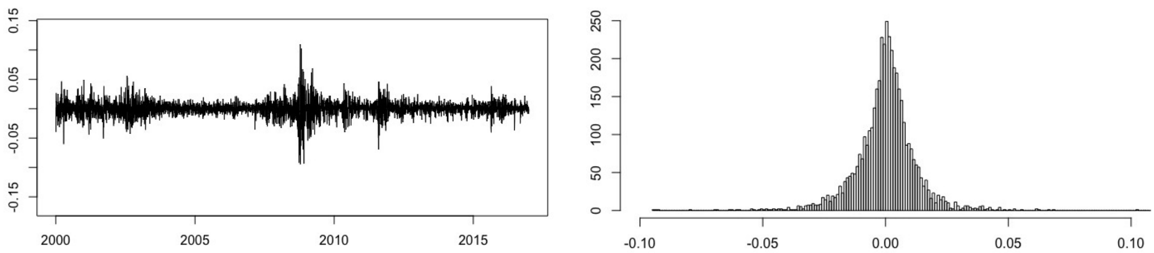


Figure 4.4 – Time series and histogram of $1/N$ portfolio

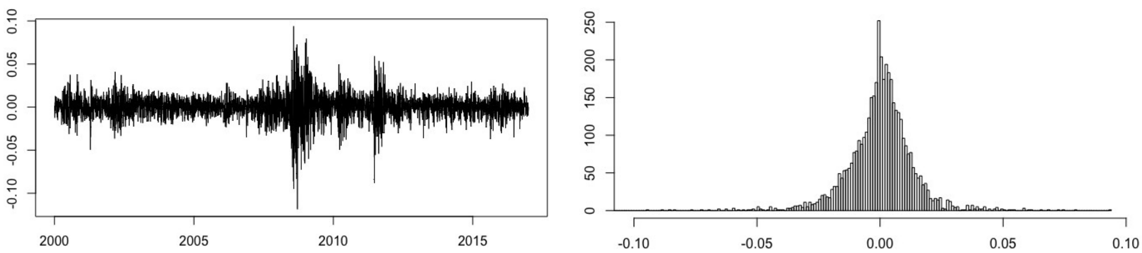
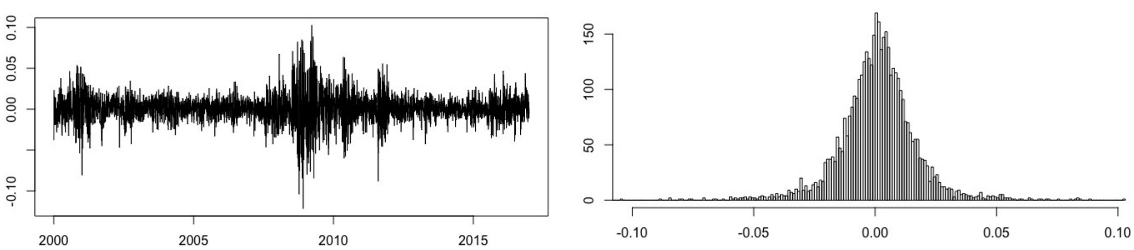


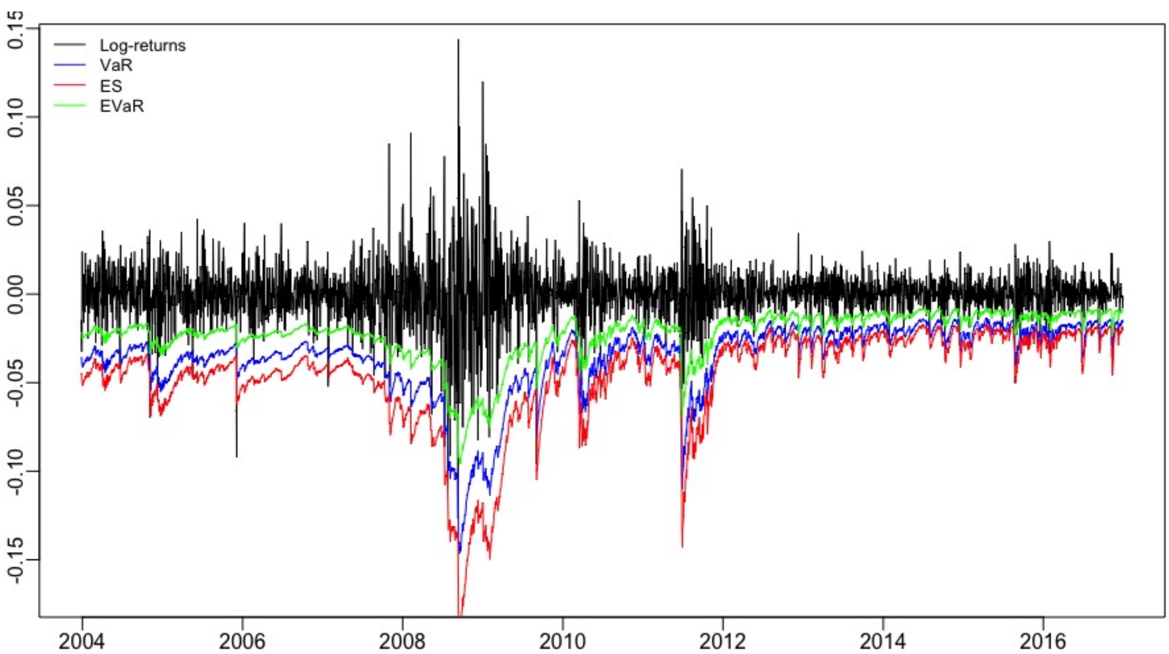
Figure 4.5 – Time series and histogram of FF portfolio



After the statistical analysis of the portfolio series, we turn the attention to the forecasts of risk. Proceeding from the introduced models and estimation procedures, each portfolio generates a one-step-ahead forecast series for VaR, ES, and EVaR for 10 models with different distributions in different scenarios concerning three significance levels and three estimation windows. The results acquired are segregated by measure and by the significance level (α) - which is linked to the quantile of interest - to better expound the portfolio behavior and collate its performances.

Summarizing the descriptive statistics obtained, the econometric models of Λ_S and of $S\&P500$ show similar behaviors for all competing models and measures, slightly changing their means and deviations for all analyses.The models of Λ_W presented relatively greater descriptive statistics for all models, although it was quite consistent with the criteria of estimation chosen, as was window estimation size and α. Additionally we can see that the $1/N$ and the FF portfolios presented almost the same behaviors, concerning mean and standard deviation, of one another and very similar to the $S\&P500$ portfolio.

Figure 4.6 – Risk forecasting for $\Lambda_S(X)$ at 1% significance level with estimation window of 1000 days under conditional GARCH with student distribution.



To emphasize the distinction of the estimated models, we selected the more suitable characteristics to set up a visual analysis. Figures 4.6, 4.7, 4.8,4.9 and 4.10 display the forecast of the five portfolios studied under the student distribution at a 1% significance level with the estimation window of 1,000 business days. Both α and the window are selected for being the most conservative and in line to secure minimum loss over the system.

Figure 4.7 – Risk forecasting for $\Lambda_W(X)$ at 1% significance level with estimation window of 1000 days under conditional GARCH with student distribution.

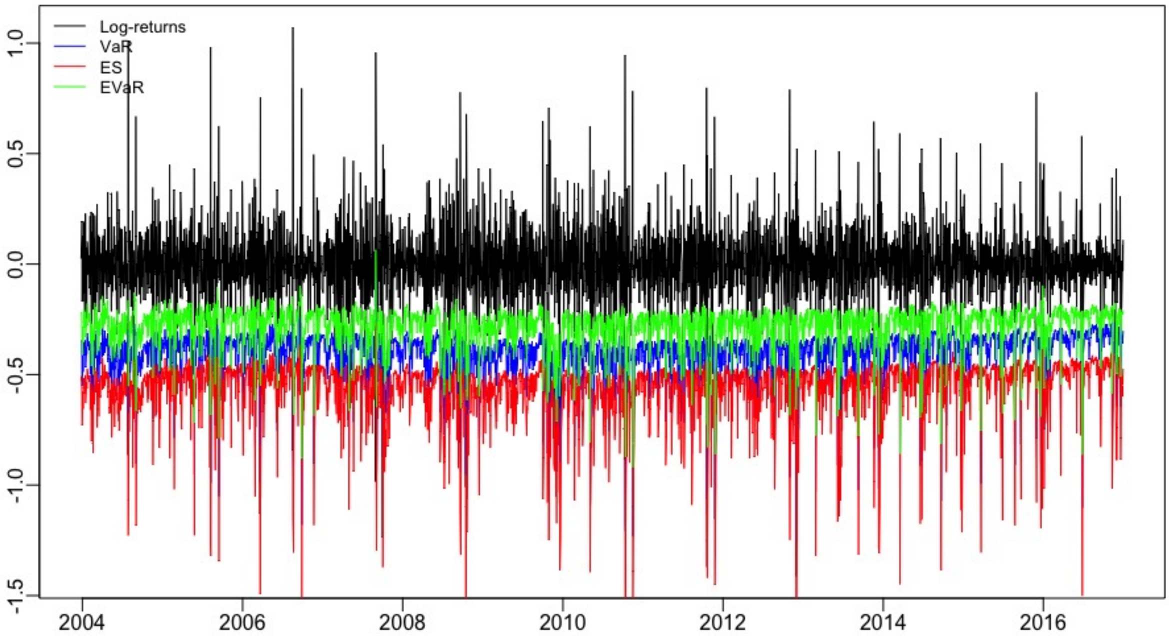


Figure 4.8 – Risk forecasting for $S\&P500$ at 1% significance level with estimation window of 1000 days under conditional GARCH with student distribution.

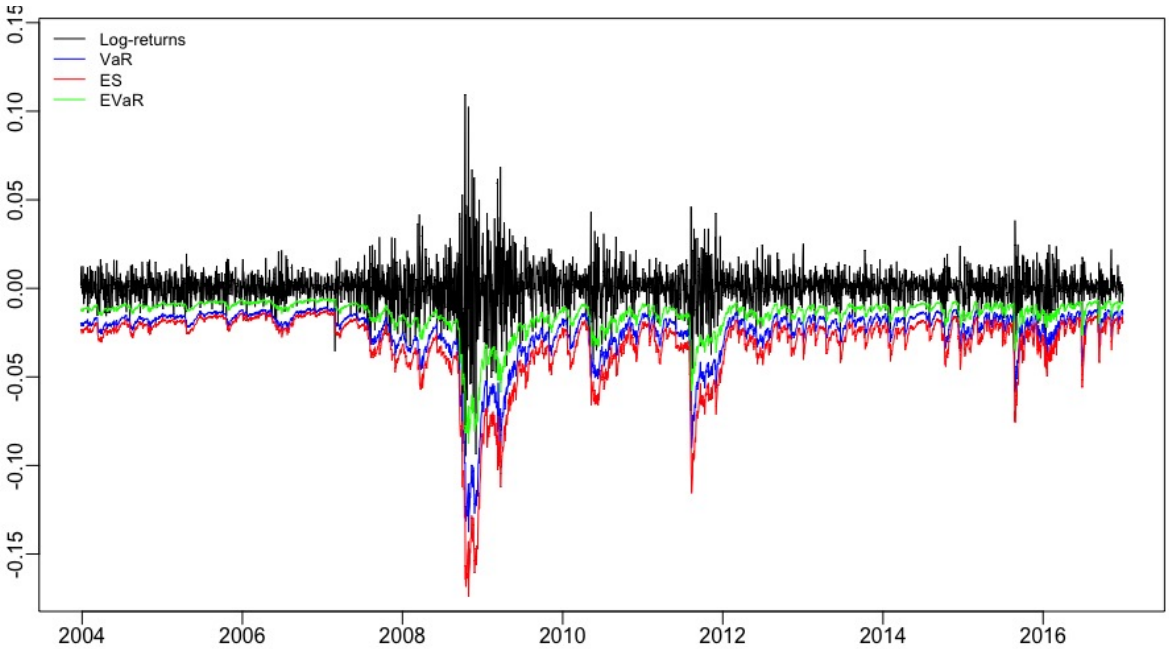


Figure 4.9 – Risk forecasting for $1/N$ at 1% significance level with estimation window of 1000 days under conditional GARCH with student distribution.

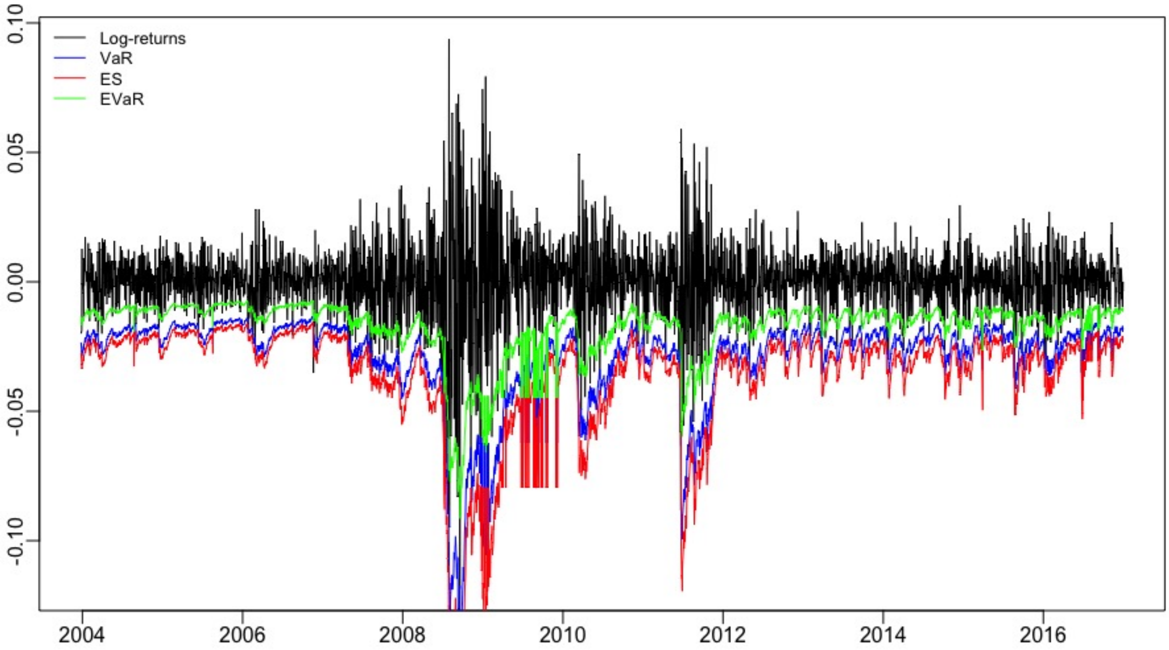
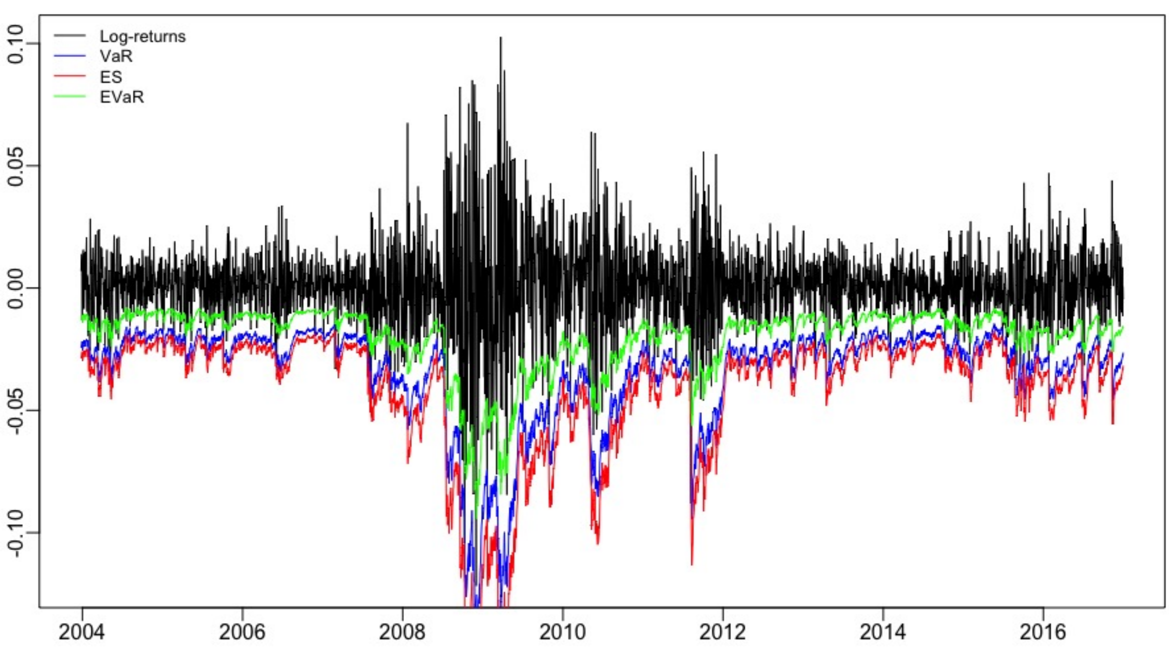


Figure 4.10 – Risk forecasting for FF at 1% significance level with estimation window of 1000 days under conditional GARCH with student distribution.



Visually analyzing the graphics, the magnitude of the violations and how each measure deals with them can be viewed. The effect of the 2008 crisis is marked in $S\&P500$, Λ_S, $1/N$ and FF, in the meantime, all forecasts look quite well-fitted for the five portfolios. There are not any discrepant failures or violation clusters among forecasts, although it is necessary to analyze the numerical results in order to formulate an adequate understanding of the data. Regarding the ES forecasting, it can be seen that for the sum the measure is more conservative, distancing more from VaR forecasts than what is observed in the others.

The forecasts generate some patterns among portfolios and models. Focusing the attention on VaR, which is shown in Tables 4.2, 4.3 and 4.4, the aggregation Λ_S presents a good fit in relation to the violations and it also has fewer violations than $S\&P500$ and $1/N$ for all competing models. The $S\&P500$ index exceeded the violation expected for all series forecasted, even though just exceeding the violation in a backtest does not always suffice to accept the bad quality of the forecast. It is also exceeded by an expressive ratio, usually greater than 1.5 and in extreme cases greater than 2. In contrast to Λ_S, Λ_W and $S\&P500$ present an increase of violations when the significance level decreases, and Λ_W also show an increase when the window size decreases. The sum portfolio was far superior in VR performance than $S\&P500$, $N/1$ and FF. It is important to highlight that $S\&P500$, $1/N$, FF shown very similar results for all models, with FF presenting a slightly better behavior.

Comparing the portfolios among distributions, Λ_W performs better with the historical simulation and empirical distribution in general, followed by the skew-student. The same is observed with $S\&P500$, even though the skew-student performed far better considering $\alpha = 1\%$. The sum portfolio does not possess a global dominant distribution, but three could be cited as the most significant, these being the generalized error distribution and its skewed variant, as well as the logistic distribution. The $1/N$ and the FF also did not present a dominant distribution over all. The generalized extreme value distribution is among the worst results overall - with two extremes exceptions for FF with $\alpha = 5\%$ and $1/N$ with $\alpha = 1\%$ - opposing studies as Kuester et al. (2006), it is also relevant to cite that GEV generated $VR \geq 2$ for $S\&P500$ with $\alpha = 1\%$.

Table 4.2 – Results for VaR forecasting with $\alpha = 5\%$

	$\Lambda_S(X)$				$\Lambda_W(X)$				$S\&P500$				$1/N$				FF			
	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}
Estimation window: 250																				
HS	0.03	0.01	0.95	1.08E-07	0.25	0.03	**1.02**	3.71E-06	0.02	0.01	**1.06**	9.79E-08	0.02	0.01	**1.10**	2.03E-07	0.02	0.01	1.07	**1.71E-07**
Normal	0.03	0.01	1.06	9.92E-08	0.26	0.11	0.81	2.45E-06	0.02	0.01	1.31	5.39E-08	0.02	0.01	1.43	**1.45E-07**	0.02	0.01	1.15	1.78E-07
Skew-Normal	0.03	0.01	**1.03**	1.02E-07	0.27	0.12	0.72	2.97E-06	0.02	0.01	1.23	6.61E-08	0.02	0.01	1.29	1.61E-07	0.02	0.01	**1.02**	**1.71E-07**
Student	0.02	0.01	1.15	**7.95E-08**	0.24	0.10	0.91	2.17E-06	0.02	0.01	1.49	**2.88E-08**	0.02	0.01	1.45	**1.40E-07**	0.02	0.01	1.21	1.77E-07
Skew-Student	0.02	0.01	1.15	**8.56E-08**	0.23	0.10	**1.00**	2.59E-06	0.02	0.01	1.33	5.08E-08	0.02	0.01	1.29	2.03E-07	0.02	0.01	1.09	**1.69E-07**
GED	0.03	0.01	1.06	9.04E-08	0.25	0.11	0.91	2.39E-06	0.02	0.01	1.35	5.89E-08	0.02	0.01	1.46	1.45E-07	0.02	0.01	1.19	1.80E-07
Skew-GED	0.03	0.01	**1.03**	9.59E-08	0.25	0.12	**0.98**	**2.12E-06**	0.02	0.01	1.24	6.00E-08	0.02	0.01	**1.24**	1.62E-07	0.02	0.01	**1.02**	**1.71E-07**
Empirical	0.03	0.01	**1.03**	**8.99E-08**	0.22	0.11	1.16	**1.90E-06**	0.02	0.01	**1.10**	**4.78E-08**	0.02	0.01	**1.13**	1.46E-07	0.02	0.01	**1.04**	**1.57E-07**
GEV	0.03	0.01	0.95	1.16E-07	0.28	0.12	0.70	2.33E-06	0.02	0.01	**1.14**	7.00E-08	0.02	0.01	1.44	1.58E-07	0.02	0.02	1.22	1.75E-07
Logistic	0.03	0.01	1.05	1.00E-07	0.24	0.11	0.97	**2.15E-06**	0.02	0.01	1.32	**5.04E-08**	0.02	0.01	1.42	**1.43E-07**	0.02	0.01	1.13	1.95E-07
Estimation window: 500																				
HS	0.03	0.01	0.93	1.42E-07	0.25	0.03	**1.03**	4.15E-06	0.02	0.01	**1.06**	1.30E-07	0.02	0.01	**1.05**	2.21E-07	0.02	0.01	1.03	1.84E-07
Normal	0.02	0.01	**0.99**	1.08E-07	0.26	0.10	0.65	2.79E-06	0.02	0.01	1.32	6.60E-08	0.02	0.01	1.35	1.64E-07	0.02	0.01	1.05	1.68E-07
Skew-Normal	0.02	0.01	0.95	1.09E-07	0.27	0.11	0.63	2.62E-06	0.02	0.01	1.23	6.63E-08	0.02	0.01	1.19	1.62E-07	0.02	0.01	**0.98**	1.63E-07
Student	0.02	0.01	1.09	**9.94E-08**	0.24	0.10	0.78	2.51E-06	0.02	0.01	1.40	**5.08E-08**	0.02	0.01	1.40	**1.50E-07**	0.02	0.01	1.13	1.68E-07
Skew-Student	0.02	0.01	1.05	**1.00E-07**	0.23	0.10	**0.86**	**2.32E-06**	0.02	0.01	1.29	**5.38E-08**	0.02	0.01	1.19	2.21E-07	0.02	0.01	**0.98**	**1.56E-07**
GED	0.02	0.01	**0.99**	1.07E-07	0.25	0.10	0.73	2.70E-06	0.02	0.01	1.34	6.39E-08	0.02	0.01	1.34	**1.55E-07**	0.02	0.01	1.08	1.75E-07
Skew-GED	0.02	0.01	0.95	1.08E-07	0.25	0.11	0.81	**2.46E-06**	0.02	0.01	1.19	**6.35E-08**	0.02	0.01	**1.14**	1.61E-07	0.02	0.01	0.97	**1.59E-07**
Empirical	0.02	0.01	0.97	**9.67E-08**	0.22	0.10	**1.01**	**2.44E-06**	0.02	0.01	**1.09**	7.09E-08	0.02	0.01	**1.07**	1.81E-07	0.02	0.01	0.94	**1.51E-07**
GEV	0.03	0.01	0.90	1.12E-07	0.27	0.11	0.64	2.91E-06	0.02	0.01	**1.12**	7.77E-08	0.02	0.01	1.20	**1.34E-07**	0.02	0.01	**0.99**	1.66E-07
Logistic	0.02	0.01	**1.02**	1.06E-07	0.24	0.10	0.81	2.60E-06	0.02	0.01	1.35	6.45E-08	0.02	0.01	1.32	1.65E-07	0.02	0.01	**1.02**	1.69E-07
Estimation window: 1000																				
HS	0.03	0.01	0.85	1.58E-07	0.26	0.02	**1.07**	4.95E-06	0.02	0.01	**1.02**	1.37E-07	0.02	0.01	0.89	2.47E-07	0.02	0.01	**1.00**	**1.92E-07**
Normal	0.02	0.01	**0.96**	1.23E-07	0.26	0.10	0.60	3.43E-06	0.02	0.01	1.25	7.26E-08	0.02	0.01	1.14	1.85E-07	0.02	0.01	**0.95**	1.93E-07
Skew-Normal	0.02	0.01	0.92	1.23E-07	0.27	0.10	0.55	**3.27E-06**	0.02	0.01	1.15	8.19E-08	0.02	0.01	**1.03**	1.93E-07	0.02	0.01	0.87	2.01E-07
Student	0.02	0.01	1.10	**1.17E-07**	0.24	0.09	0.79	3.78E-06	0.02	0.01	1.37	**6.12E-08**	0.02	0.01	1.19	**1.77E-07**	0.02	0.01	**0.97**	1.98E-07
Skew-Student	0.02	0.01	1.06	**1.22E-07**	0.23	0.09	**0.83**	3.46E-06	0.02	0.01	1.22	7.37E-08	0.02	0.01	**1.02**	2.47E-07	0.02	0.01	0.89	2.08E-07
GED	0.02	0.01	**0.97**	1.23E-07	0.26	0.09	0.64	3.88E-06	0.02	0.01	1.27	**6.87E-08**	0.02	0.01	1.13	**1.81E-07**	0.02	0.01	**0.95**	1.97E-07
Skew-GED	0.02	0.01	0.95	1.28E-07	0.25	0.10	0.67	3.61E-06	0.02	0.01	**1.13**	8.02E-08	0.02	0.01	**0.98**	1.98E-07	0.02	0.01	0.86	2.08E-07
Empirical	0.02	0.01	0.94	1.23E-07	0.22	0.09	**0.97**	**2.99E-06**	0.02	0.01	**1.03**	9.00E-08	0.02	0.01	0.86	2.15E-07	0.02	0.01	0.85	**1.80E-07**
GEV	0.02	0.01	1.11	1.30E-07	0.28	0.09	0.59	3.35E-06	0.02	0.01	1.23	8.73E-08	0.02	0.01	0.95	2.10E-07	0.02	0.01	0.90	2.08E-07
Logistic	0.02	0.01	**1.01**	**1.21E-07**	0.24	0.09	0.78	**3.27E-06**	0.02	0.01	1.29	**7.08E-08**	0.02	0.01	1.12	**1.82E-07**	0.02	0.01	0.91	**1.93E-07**

The expected violations values for the windows of 250, 500 and 1000 are, respectively, 201, 188 and 163.
The values in bold correspond to the best results for each criteria.

Table 4.3 – Results for VaR forecasting with $\alpha = 2.5\%$

	$\Lambda_S(X)$				$\Lambda_W(X)$				$S\&P500$				$1/N$				FF			
	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}
Estimation window: 250																				
HS	0.03	0.02	1.11	7.54E-08	0.34	0.04	1.15	2.15E-06	0.02	0.01	**1.32**	8.64E-08	0.03	0.01	1.28	1.44E-07	0.03	0.01	1.28	1.23E-07
Normal	0.03	0.02	1.24	**6.30E-08**	0.31	0.12	**1.02**	1.44E-06	0.02	0.01	1.73	**3.79E-08**	0.02	0.01	1.63	**8.67E-08**	0.03	0.02	1.38	1.14E-07
Skew-Normal	0.03	0.02	1.22	6.48E-08	0.32	0.13	**0.96**	1.67E-06	0.02	0.01	1.53	4.59E-08	0.02	0.01	1.39	9.70E-08	0.03	0.02	1.19	**1.08E-07**
Student	0.03	0.02	1.19	**5.74E-08**	0.31	0.12	**0.97**	1.41E-06	0.02	0.01	1.87	**2.54E-08**	0.02	0.01	1.54	**8.58E-08**	0.03	0.02	1.40	1.16E-07
Skew-Student	0.03	0.02	1.21	**6.24E-08**	0.30	0.12	1.08	1.49E-06	0.02	0.01	1.46	4.31E-08	0.02	0.01	**1.25**	1.44E-07	0.03	0.02	**1.17**	**1.09E-07**
GED	0.03	0.02	**1.10**	**6.30E-08**	0.32	0.14	1.33	1.50E-06	0.02	0.01	1.51	4.68E-08	0.02	0.01	1.52	8.91E-08	0.03	0.02	1.28	1.16E-07
Skew-GED	0.03	0.02	**1.09**	6.70E-08	0.32	0.14	1.41	**1.30E-06**	0.02	0.01	**1.40**	4.76E-08	0.02	0.01	**1.21**	9.85E-08	0.03	0.02	**1.12**	1.10E-07
Empirical	0.03	0.02	1.15	7.43E-08	0.29	0.12	1.31	**1.12E-06**	0.02	0.01	**1.29**	4.22E-08	0.02	0.02	**1.18**	**8.62E-08**	0.03	0.02	**1.17**	1.17E-07
GEV	0.03	0.02	1.16	7.16E-08	0.33	0.13	0.91	1.33E-06	0.02	0.01	1.52	4.67E-08	0.02	0.01	1.75	9.52E-08	0.03	0.02	1.40	**1.08E-07**
Logistic	0.03	0.02	**1.10**	6.74E-08	0.30	0.12	1.13	**1.31E-06**	0.02	0.01	1.56	**3.89E-08**	0.02	0.01	1.39	8.94E-08	0.03	0.02	1.19	1.30E-07
Estimation window: 500																				
HS	0.03	0.02	**1.04**	1.01E-07	0.34	0.03	1.09	2.47E-06	0.02	0.01	**1.19**	9.36E-08	0.03	0.01	**1.21**	1.47E-07	0.03	0.01	**1.10**	1.34E-07
Normal	0.03	0.02	1.16	**6.81E-08**	0.31	0.11	0.89	1.64E-06	0.02	0.01	1.68	**4.63E-08**	0.02	0.01	1.72	9.70E-08	0.03	0.02	1.31	1.06E-07
Skew-Normal	0.03	0.02	1.13	6.91E-08	0.32	0.12	0.83	**1.53E-06**	0.02	0.01	1.46	4.78E-08	0.02	0.01	1.43	9.79E-08	0.03	0.02	1.15	**1.02E-07**
Student	0.03	0.02	1.14	**6.73E-08**	0.31	0.11	0.90	1.66E-06	0.02	0.01	1.67	**4.16E-08**	0.02	0.01	1.64	**9.21E-08**	0.03	0.02	1.24	1.10E-07
Skew-Student	0.03	0.02	1.12	**6.79E-08**	0.30	0.11	**0.99**	**1.51E-06**	0.02	0.01	1.35	**4.39E-08**	0.02	0.01	1.35	1.47E-07	0.03	0.02	**1.06**	**1.02E-07**
GED	0.03	0.02	0.95	7.14E-08	0.32	0.12	**0.99**	1.73E-06	0.02	0.01	1.52	4.89E-08	0.02	0.01	1.61	**9.46E-08**	0.03	0.02	1.16	1.12E-07
Skew-GED	0.03	0.02	**0.96**	7.21E-08	0.32	0.13	1.11	1.57E-06	0.02	0.01	**1.24**	4.90E-08	0.02	0.02	**1.26**	9.90E-08	0.03	0.02	**1.03**	1.03E-07
Empirical	0.03	0.02	1.11	7.14E-08	0.29	0.11	1.09	**1.47E-06**	0.02	0.01	**0.17**	5.81E-08	0.03	0.02	**1.20**	1.05E-07	0.03	0.02	1.12	9.79E-08
GEV	0.03	0.02	1.12	6.92E-08	0.32	0.13	0.91	1.68E-06	0.02	0.01	1.47	5.22E-08	0.02	0.01	1.50	**7.94E-08**	0.03	0.02	1.28	**1.02E-07**
Logistic	0.03	0.02	**1.03**	7.08E-08	0.30	0.11	**0.94**	1.59E-06	0.02	0.01	1.53	4.88E-08	0.02	0.01	1.43	1.01E-07	0.03	0.02	1.12	1.13E-07
Estimation window: 1000																				
HS	0.04	0.01	**0.99**	1.10E-07	0.35	0.02	**1.07**	3.44E-06	0.03	0.01	**1.15**	9.52E-08	0.03	0.01	1.28	1.59E-07	0.03	0.01	1.21	1.30E-07
Normal	0.03	0.02	1.09	**7.80E-08**	0.31	0.10	0.83	2.03E-06	0.02	0.01	1.70	**5.08E-08**	0.02	0.01	1.69	**1.11E-07**	0.03	0.02	1.38	**1.22E-07**
Skew-Normal	0.03	0.02	**1.02**	**7.96E-08**	0.32	0.10	0.69	**1.91E-06**	0.02	0.01	1.56	5.77E-08	0.02	0.01	1.33	1.17E-07	0.03	0.02	**1.15**	1.28E-07
Student	0.03	0.02	1.04	**7.85E-08**	0.31	0.10	0.78	2.37E-06	0.02	0.01	1.60	**4.90E-08**	0.02	0.01	1.60	**1.10E-07**	0.03	0.02	1.35	1.29E-07
Skew-Student	0.03	0.02	1.06	8.25E-08	0.30	0.10	0.78	2.15E-06	0.02	0.01	1.41	5.71E-08	0.03	0.02	**1.23**	1.59E-07	0.03	0.02	**1.15**	1.36E-07
GED	0.03	0.02	0.95	8.16E-08	0.33	0.10	0.74	2.40E-06	0.02	0.01	1.48	**5.26E-08**	0.02	0.01	1.48	1.12E-07	0.03	0.02	1.27	1.27E-07
Skew-GED	0.03	0.02	0.93	8.56E-08	0.32	0.11	0.77	2.21E-06	0.02	0.01	**1.30**	5.98E-08	0.03	0.02	**1.14**	1.23E-07	0.03	0.02	**1.07**	1.35E-07
Empirical	0.03	0.01	**1.02**	8.46E-08	0.29	0.10	**0.98**	**1.88E-06**	0.02	0.01	**1.14**	6.76E-08	0.03	0.02	**1.09**	1.26E-07	0.03	0.02	1.16	**1.18E-07**
GEV	0.03	0.01	1.33	8.06E-08	0.34	0.10	0.80	**1.92E-06**	0.02	0.01	1.73	5.88E-08	0.02	0.01	1.46	1.25E-07	0.03	0.02	1.37	**1.27E-07**
Logistic	0.03	0.02	**1.02**	8.12E-08	0.30	0.10	**0.88**	2.01E-06	0.02	0.01	1.52	5.35E-08	0.02	0.01	1.42	**1.40E-08**	0.03	0.02	**1.00**	1.29E-07

The expected violations values for the windows of 250, 500 and 1000 are, respectively, 100, 94 and 81.
The values in bold correspond to the best results for each criteria.

Table 4.4 – Results for VaR forecasting with $\alpha = 1\%$

	$\Lambda_S(X)$				$\Lambda_W(X)$				$S\&P500$				$1/N$				FF			
	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}	Mean	SD	VR	Z_{VaR_α}
Estimation window: 250																				
HS	0.04	0.02	0.65	4.22E-08	0.50	0.08	**1.48**	1.21E-06	0.03	0.02	**1.63**	4.91E-08	0.03	0.02	1.75	7.19E-08	0.04	0.02	1.70	6.52E-08
Normal	0.04	0.02	1.18	**2.62E-08**	0.37	0.14	1.68	6.79E-07	0.02	0.01	2.60	**2.03E-08**	0.03	0.02	2.20	**4.12E-08**	0.03	0.02	1.75	5.68E-08
Skew-Normal	0.04	0.02	**1.03**	**2.71E-08**	0.38	0.15	1.58	7.48E-07	0.03	0.02	2.10	2.44E-08	0.03	0.02	1.65	4.65E-08	0.03	0.02	1.65	**5.40E-08**
Student	0.04	0.02	0.85	**2.80E-08**	0.42	0.14	**1.15**	7.68E-07	0.03	0.02	2.23	**1.61E-08**	0.03	0.02	1.83	**4.24E-08**	0.03	0.02	1.70	5.94E-08
Skew-Student	0.04	0.02	0.48	3.13E-08	0.41	0.15	**1.48**	7.09E-07	0.03	0.02	**1.63**	2.79E-08	0.03	0.02	**1.30**	7.19E-08	0.03	0.02	**1.30**	5.57E-08
GED	0.04	0.02	**0.88**	2.90E-08	0.41	0.16	2.15	7.69E-07	0.03	0.02	1.98	2.78E-08	0.03	0.02	1.75	**4.36E-08**	0.03	0.02	1.70	5.87E-08
Skew-GED	0.04	0.02	0.73	3.12E-08	0.41	0.18	2.83	**6.46E-07**	0.03	0.02	**1.43**	2.82E+00	0.03	0.02	**1.30**	4.76E-08	0.03	0.02	**1.35**	**5.53E-08**
Empirical	0.04	0.02	0.80	3.42E-08	0.43	0.18	1.55	7.40E-07	0.03	0.02	1.93	3.03E-08	0.03	0.02	1.65	4.41E-08	0.03	0.02	1.63	6.27E-08
GEV	0.04	0.02	0.73	2.95E-08	0.39	0.15	1.53	**6.08E-07**	0.02	0.02	2.20	2.40E-08	0.03	0.02	2.75	4.54E-08	0.03	0.02	2.13	**5.21E-08**
Logistic	0.04	0.02	**0.93**	3.07E-08	0.38	0.14	1.60	**6.46E-07**	0.03	0.02	1.88	**2.27E-08**	0.03	0.02	**1.38**	4.51E-08	0.04	0.02	**1.33**	6.92E-08
Estimation window: 500																				
HS	0.04	0.02	1.38	5.31E-08	0.51	0.08	**1.22**	1.47E-06	0.03	0.01	1.62	5.47E-08	0.03	0.02	1.57	7.31E-08	0.04	0.02	1.27	7.18E-08
Normal	0.03	0.02	1.70	**3.39E-08**	0.37	0.13	1.54	7.75E-07	0.02	0.01	2.43	**2.47E-08**	0.03	0.02	2.08	**4.57E-08**	0.03	0.02	1.76	5.28E-08
Skew-Normal	0.04	0.02	1.57	**3.45E-08**	0.38	0.14	1.46	**7.13E-07**	0.03	0.02	1.92	**2.60E-08**	0.03	0.02	1.49	4.71E-08	0.03	0.02	1.43	**5.08E-08**
Student	0.04	0.02	**1.24**	3.65E-08	0.42	0.13	**1.08**	9.19E-07	0.03	0.02	1.76	**2.56E-08**	0.03	0.02	1.62	**4.54E-08**	0.03	0.02	1.43	5.67E-08
Skew-Student	0.04	0.02	**1.11**	3.69E-08	0.40	0.13	**1.18**	8.23E-07	0.03	0.02	**1.46**	2.69E-08	0.03	0.02	**1.16**	7.31E-08	0.03	0.02	**1.16**	5.29E-08
GED	0.04	0.02	1.19	3.75E-08	0.42	0.14	1.35	8.94E-07	0.03	0.02	1.76	2.81E-08	0.03	0.02	1.54	4.61E-08	0.03	0.02	1.43	5.68E-08
Skew-GED	0.04	0.02	**1.11**	3.78E-08	0.41	0.15	1.62	8.09E-07	0.03	0.02	**1.43**	2.82E-08	0.03	0.02	**1.19**	4.84E-08	0.03	0.02	**1.16**	5.24E-08
Empirical	0.04	0.02	1.35	3.73E-08	0.44	0.19	1.30	**6.86E-07**	0.03	0.02	**1.46**	3.44E-08	0.03	0.02	1.35	5.15E-08	0.03	0.02	**1.16**	**4.86E-08**
GEV	0.03	0.02	1.68	**3.35E-08**	0.38	0.15	1.78	**7.71E-07**	0.02	0.02	2.00	2.69E-08	0.03	0.02	2.03	**3.75E-08**	0.03	0.02	1.70	**4.89E-08**
Logistic	0.04	0.02	1.30	3.76E-08	0.38	0.13	1.46	7.82E-07	0.03	0.02	1.73	2.81E-08	0.03	0.02	**1.32**	5.04E-08	0.04	0.02	**1.14**	6.01E-08
Estimation window: 1000																				
HS	0.05	0.02	**1.06**	5.99E-08	0.50	0.05	**1.09**	6.71E-07	0.03	0.01	1.56	5.63E-08	0.04	0.02	1.66	7.74E-08	0.04	0.02	1.56	7.75E-08
Normal	0.03	0.02	1.53	**3.89E-08**	0.37	0.12	1.34	**1.00E-07**	0.02	0.01	2.53	**2.71E-08**	0.03	0.02	2.22	**5.26E-08**	0.03	0.02	1.94	**6.05E-08**
Skew-Normal	0.03	0.02	1.44	**4.02E-08**	0.38	0.12	**1.09**	**8.33E-08**	0.02	0.02	2.19	3.09E-08	0.03	0.02	1.59	5.65E-08	0.03	0.02	1.47	**6.35E-08**
Student	0.04	0.02	1.16	4.19E-08	0.42	0.12	0.88	3.08E-07	0.03	0.02	1.81	**2.97E-08**	0.03	0.02	1.81	**5.47E-08**	0.03	0.02	1.63	6.67E-08
Skew-Student	0.04	0.02	1.19	4.44E-08	0.40	0.12	**1.03**	2.30E-07	0.03	0.02	**1.38**	3.36E-08	0.03	0.02	**1.25**	7.74E-08	0.04	0.02	**1.16**	7.03E-08
GED	0.04	0.02	**1.13**	4.27E-08	0.42	0.12	**1.00**	2.10E-07	0.03	0.02	1.69	**3.01E-08**	0.03	0.02	1.59	**5.52E-08**	0.03	0.02	1.53	6.46E-08
Skew-GED	0.04	0.02	**1.13**	4.49E-08	0.41	0.13	1.16	1.68E-07	0.03	0.02	**1.28**	3.36E-08	0.03	0.02	**1.16**	6.07E-08	0.04	0.02	**1.16**	6.88E-08
Empirical	0.04	0.02	1.25	4.56E-08	0.42	0.14	1.13	1.43E-07	0.03	0.02	**1.44**	3.51E-08	0.03	0.02	1.31	6.07E-08	0.04	0.02	1.19	6.69E-08
GEV	0.03	0.02	1.75	**3.92E-08**	0.42	0.13	**1.09**	**7.78E-08**	0.02	0.01	2.59	3.04E-08	0.03	0.01	**1.00**	5.89E-08	0.03	0.02	1.88	**6.08E-08**
Logistic	0.04	0.02	1.34	4.33E-08	0.38	0.12	1.31	1.26E-07	0.03	0.02	1.78	3.07E-08	0.03	0.02	1.47	5.75E-08	0.04	0.02	**1.06**	6.83E-08

The expected violations values for the windows of 250, 500 and 1000 are, respectively, 40, 37 and 32.
The values in bold correspond to the best results for each criteria.

Concerning the VaR loss function analysis, there are generally no relevant changes in patterns among models. The five portfolios had their loss function reduced as the significance level decreased. What was not observed with window size, what can be explained owing to as the estimation window gets larger, risk predictions tend to be more conservative, contradicting Degiannakis et al. (2013), who argues that as the forecasting horizon increases, VaR underestimation turns less prevalent. For all forecasts, Λ_S presents results slightly bigger than $S\&P500$. $1/N$ and FF exhibits greater differences from both Λ_S and $S\&P500$, although not highly significant ones. Λ_W presents the worst results over all models and scenarios. Contrasting what is observed in the VaR violation analysis, the HS produces the worst outcome for all models in all scenarios - considering the exception of FF -, corroborating Pritsker (2006) as being a poor choice. This bad result is justifiable due to the fact that this nonparametric approach delays the change of the volatility in the forecasting. Another relevant point is that the GEV distributions show quite good results in the aggregate analysis. The sum portfolio performs better with the student distribution, as well as the $S\&P500$ and the $1/N$, as was expected. The weighted and the FF portfolios present a superior fit with the empirical distribution.

Artzner et al. (1999) proposed the use of Expected Shortfall after identifying VaR as not being coherent due its lack of subaddivity[19]. Although other authors defend the use of VaR and mention methods to correct it, as seen in Daníelsson et al. (2013) and McNeil & Frey (2000), ES is widely known as being more conservative than VaR and has thus been gaining reputability since the 2008 crisis.

The Expected Shortfall is defined by an expectation, so its forecast cannot be simply backtested by the number of violations occurred, although it can be done by a hypothesis test. The test considered a null hypothesis (H_0) of the residuals with a mean of zero versus an alternative hypothesis (H_A) of a mean greater than zero. As the ideal scenario is the measure possessing a mean as close to zero as possible, the objective here is not to reject H_0, otherwise, ES is considered systematically underestimated. The data shown in Tables 4.5, 4.6 and 4.7 expose the results for ES forecasting and where the H_0 is rejected with 1%, 5% and 10% of significance. All portfolios behave

[19] Subadditivity is an important property for a risk measure, and due to its relation with the diversification principle of modern portfolio theory, its means of a measure is subadditive it shall generate lower measured risk for a diversified portfolio than for a non-diversified portfolio.

differently among significance levels, Λ_S lowered with the decrease of α, while Λ_W, $1/N$ and FF were practically constant and $S\&P500$ has mildly risen. With $\alpha = 5\%$, Λ_S and $1/N$presents the best performance - especially with the Student distribution for Λ_S - , however $S\&P500$ improves its conduct with $\alpha = 2.5\%$ and $\alpha = 1\%$ and became more suitable. The augmentation of the estimation window size has a positive effect on the significance of the backtest, which can be understood because when days are added to the window, it brings about additional information to improve the forecast.

As hoped, the U.S. market index performed expressively well, modeled by the skew-student distribution, since it had been established by the literature from a long time and has already has been shown in the previous test with VaR. Λ_W had the worst results in general among models, even though it performs far better than others with the nonparametric distribution and with the semi-parametric GARCH with the empirical distribution which also is detected in the VaR backtest. The $1/N$ and FF portfolios present similar behavior in general, performing better with logistic and skew-student distributions. Considering the first three portfolios, there is no distribution that generated a significance greater than 10% for all the scenarios and models. On the other hand, as already said, the $1/N$ and FF present their best results constant for every analysis. On the other hand, the normal distribution and the GEV distribution exposed a significance of 1% for every model in all scenarios.

Alternating the analysis of the outcomes of ES loss function, there are also some patterns found previously. Overall, there is a tendency of the losses to grow as the windows rise. The same result is also perceived as the significance level grew. Both portfolios present constant behavior through window size and significance levels. As seen before, Λ_S produces outcomes quite similar to $S\&P500$, whereby it is observed that there is an increase of it when significance level became smaller. Additionally, Λ_S performs better than both $1/N$ and FF. The results of Λ_W present the worst outcomes for ES forecasting, while the $S\&P500$ present the best, even though Λ_S is extremely similar to the index.

Table 4.5 – Results for ES forecasting with $\alpha = 5\%$

	$\Lambda_S(X)$				$\Lambda_W(X)$				$S\&P500$				$1/N$				FF			
	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}
Estimation window: 250																				
HS	0.04	0.02	**0.06**	8.61E-08	0.42	0.05	**0.17**	1.23E-06	0.03	0.01	0.01	9.18E-08	0.03	0.01	0.02	9.88E-08	0.03	0.02	0.01	8.96E-08
Normal	0.03	0.02	0.00	**3.56E-08**	0.33	0.13	0.00	**5.79E-07**	0.02	0.01	0.00	**2.92E-08**	0.02	0.01	0.00	**3.70E-08**	0.03	0.02	0.00	**6.48E-08**
Skew-Normal	0.03	0.02	0.00	**3.70E-08**	0.34	0.14	0.00	**4.69E-07**	0.02	0.01	0.00	3.42E-08	0.02	0.01	0.01	**4.37E-08**	0.03	0.02	0.00	**6.04E-08**
Student	0.03	0.02	**0.12**	5.69E-08	0.36	0.13	0.00	1.14E-06	0.02	0.02	0.00	**3.29E-08**	0.02	0.01	0.13	4.52E-08	0.03	0.02	**0.03**	7.41E-08
Skew-Student	0.03	0.02	**0.10**	6.40E-08	0.35	0.13	0.00	6.38E-07	0.02	0.01	**0.73**	5.88E-08	0.03	0.02	**0.52**	9.88E-08	0.03	0.02	**0.10**	6.81E-08
GED	0.03	0.02	0.02	5.02E-08	0.35	0.14	0.00	9.07E-07	0.02	0.01	0.02	4.98E-08	0.02	0.01	0.10	4.46E-08	0.03	0.02	0.02	7.00E-08
Skew-GED	0.03	0.02	0.01	5.36E-08	0.35	0.15	0.00	6.91E-07	0.02	0.02	**0.35**	5.03E-08	0.03	0.02	**0.23**	4.66E-08	0.03	0.02	0.02	6.49E-08
Empirical	0.03	0.02	0.00	7.00E-08	0.35	0.14	0.00	8.16E-07	0.02	0.01	0.00	6.42E-08	0.03	0.02	0.01	5.09E-08	0.03	0.02	0.01	8.23E-08
GEV	0.03	0.02	0.00	**3.57E-08**	0.35	0.14	0.00	**4.34E-07**	0.02	0.01	0.00	**3.06E-08**	0.02	0.01	0.00	**4.22E-08**	0.03	0.02	0.00	**5.23E-08**
Logistic	0.03	0.02	0.03	4.93E-08	0.33	0.13	0.00	6.71E-07	0.02	0.01	**0.03**	3.91E-08	0.03	0.02	**0.72**	5.13E-08	0.03	0.02	**0.52**	9.39E-08
Estimation window: 500																				
HS	0.04	0.02	0.19	8.52E-08	0.43	0.03	**0.58**	1.71E-06	0.03	0.01	0.03	9.34E-08	0.03	0.01	0.05	8.53E-08	0.03	0.02	0.24	1.01E-07
Normal	0.03	0.02	0.00	**3.79E-08**	0.33	0.12	0.00	**6.68E-07**	0.02	0.01	0.00	**3.53E-08**	0.02	0.01	0.00	**3.91E-08**	0.03	0.02	0.00	**5.91E-08**
Skew-Normal	0.03	0.02	0.00	**3.90E-08**	0.34	0.12	0.00	**5.80E-07**	0.02	0.01	0.00	**3.89E-08**	0.03	0.02	0.06	4.52E-08	0.03	0.02	0.01	**5.59E-08**
Student	0.03	0.02	**0.45**	5.29E-08	0.36	0.12	0.00	1.43E-06	0.02	0.01	**0.36**	4.94E-08	0.03	0.02	0.57	**7.72E-09**	0.03	0.02	0.40	7.21E-08
Skew-Student	0.03	0.02	**0.36**	5.35E-08	0.35	0.12	0.00	1.22E-06	0.02	0.02	**0.96**	5.13E-08	0.03	0.02	**0.94**	8.53E-08	0.03	0.02	**0.48**	6.72E-08
GED	0.03	0.02	**0.23**	4.96E-08	0.35	0.12	0.00	1.10E-06	0.02	0.01	0.24	4.75E-08	0.03	0.02	0.30	4.64E-08	0.03	0.02	0.23	6.70E-08
Skew-GED	0.03	0.02	0.20	5.01E-08	0.35	0.13	0.00	9.87E-07	0.02	0.02	**0.82**	4.78E-08	0.03	0.02	**0.88**	4.95E-08	0.03	0.02	**0.41**	6.31E-08
Empirical	0.03	0.02	0.04	6.34E-08	0.36	0.13	**0.10**	8.22E-07	0.02	0.02	0.18	5.89E-08	0.03	0.02	0.31	4.90E-08	0.03	0.02	0.07	6.86E-08
GEV	0.03	0.02	0.00	**3.39E-08**	0.34	0.13	0.00	**5.80E-07**	0.02	0.01	0.00	**3.49E-08**	0.02	0.01	0.00	**3.28E-08**	0.03	0.02	0.00	**4.81E-08**
Logistic	0.03	0.02	0.12	5.10E-08	0.33	0.12	0.00	8.15E-07	0.02	0.01	0.07	4.73E-08	0.03	0.02	**0.78**	5.40E-08	0.03	0.02	**0.66**	8.17E-08
Estimation window: 1000																				
HS	0.04	0.01	0.10	8.86E-08	0.43	0.02	**0.85**	2.50E-06	0.03	0.01	0.03	8.79E-08	0.03	0.01	0.04	8.85E-08	0.04	0.01	0.18	1.03E-07
Normal	0.03	0.02	0.00	**4.40E-08**	0.33	0.11	0.00	**7.66E-07**	0.02	0.01	0.00	**3.84E-08**	0.02	0.01	0.00	**4.80E-08**	0.03	0.02	0.00	**6.72E-08**
Skew-Normal	0.03	0.02	0.00	**4.77E-08**	0.34	0.11	0.00	**6.63E-07**	0.02	0.01	0.00	**4.47E-08**	0.03	0.02	0.07	5.49E-08	0.03	0.02	0.03	**7.13E-08**
Student	0.03	0.02	**0.45**	5.84E-08	0.36	0.11	0.00	2.03E-06	0.02	0.01	**0.19**	5.57E-08	0.03	0.02	0.40	6.04E-08	0.03	0.02	0.29	8.44E-08
Skew-Student	0.03	0.02	**0.40**	6.29E-08	0.34	0.11	0.00	1.70E-06	0.02	0.02	**0.64**	5.97E-08	0.03	0.02	**0.89**	8.85E-08	0.03	0.02	**0.69**	8.96E-08
GED	0.03	0.02	0.19	5.59E-08	0.36	0.11	0.00	1.32E-06	0.02	0.01	0.07	5.06E-08	0.03	0.02	0.29	5.84E-08	0.03	0.02	0.21	7.76E-08
Skew-GED	0.03	0.02	**0.26**	5.95E-08	0.35	0.11	0.00	1.14E-06	0.02	0.02	**0.51**	5.41E-08	0.03	0.02	**0.72**	6.51E-08	0.03	0.02	**0.45**	8.34E-08
Empirical	0.03	0.02	0.13	6.75E-08	0.37	0.12	**0.32**	1.38E-06	0.02	0.02	0.09	6.07E-08	0.03	0.02	0.13	**5.23E-08**	0.03	0.02	0.21	9.30E-08
GEV	0.03	0.01	0.00	**4.05E-08**	0.37	0.11	0.00	**5.46E-07**	0.02	0.01	0.00	**3.97E-08**	0.02	0.01	0.00	**5.14E-08**	0.03	0.02	0.00	**5.85E-08**
Logistic	0.03	0.02	0.04	5.98E-08	0.33	0.11	0.00	1.05E-06	0.02	0.01	0.02	5.15E-08	0.03	0.02	**0.58**	6.57E-08	0.03	0.02	**0.64**	9.24E-08

The values in bold correspond to the best results for each criteria.

Table 4.6 – Results for ES forecasting with $\alpha = 2.5\%$

	$\Lambda_S(X)$				$\Lambda_W(X)$				$S\&P500$				$1/N$				FF			
	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}
Estimation window: 250																				
HS	0.04	0.02	**0.06**	4.85E-08	0.54	0.07	**0.15**	7.19E-07	0.03	0.02	0.09	3.56E-08	0.03	0.02	0.03	3.03E-08	0.04	0.02	0.03	3.71E-08
Normal	0.04	0.02	0.00	**1.61E-08**	0.37	0.14	0.00	**2.62E-07**	0.02	0.02	0.00	**1.32E-08**	0.03	0.02	0.00	**1.67E-08**	0.03	0.02	0.00	**2.93E-08**
Skew-Normal	0.04	0.02	0.00	**1.67E-08**	0.38	0.15	0.00	**2.10E-07**	0.03	0.02	0.00	**1.55E-08**	0.03	0.02	0.00	**1.99E-08**	0.03	0.02	0.00	**2.72E-08**
Student	0.04	0.02	**0.04**	3.15E-08	0.45	0.16	0.00	6.80E-07	0.03	0.02	**0.14**	1.70E-08	0.03	0.02	**0.05**	2.22E-08	0.03	0.02	**0.22**	3.52E-08
Skew-Student	0.04	0.02	**0.06**	3.58E-08	0.44	0.16	0.00	3.67E-07	0.03	0.02	**0.81**	3.34E-08	0.03	0.02	**0.19**	3.03E-08	0.03	0.02	**0.11**	3.22E-08
GED	0.04	0.02	0.00	2.44E-08	0.42	0.16	0.00	4.59E-07	0.03	0.02	0.02	2.47E-08	0.03	0.02	0.01	2.08E-08	0.03	0.02	0.02	3.21E-08
Skew-GED	0.04	0.02	0.00	2.61E-08	0.42	0.17	0.00	3.44E-07	0.03	0.02	**0.53**	2.49E-08	0.03	0.02	0.03	2.14E-08	0.03	0.02	0.03	2.97E-08
Empirical	0.04	0.02	0.00	2.97E-08	0.45	0.18	**0.01**	5.94E-07	0.03	0.02	0.00	3.69E-08	0.03	0.02	0.00	3.23E-08	0.03	0.02	0.00	3.28E-08
GEV	0.04	0.02	0.00	**1.57E-08**	0.40	0.15	0.00	**1.87E-07**	0.03	0.02	0.00	**1.35E-08**	0.03	0.02	0.00	**1.89E-08**	0.03	0.02	0.00	**2.30E-08**
Logistic	0.04	0.02	0.00	2.44E-08	0.39	0.14	0.00	3.31E-07	0.03	0.02	0.07	1.93E-08	0.03	0.02	**0.33**	2.53E-08	0.04	0.02	**0.56**	4.63E-08
Estimation window: 500																				
HS	0.05	0.02	**0.15**	4.09E-08	0.56	0.05	**0.60**	1.11E-06	0.03	0.02	0.02	4.98E-08	0.04	0.02	0.02	3.38E-08	0.04	0.02	0.02	4.25E-08
Normal	0.03	0.02	0.00	**1.71E-08**	0.37	0.13	0.00	**3.02E-07**	0.02	0.01	0.00	**1.60E-08**	0.03	0.02	0.01	**1.77E-08**	0.03	0.02	0.00	**2.67E-08**
Skew-Normal	0.04	0.02	0.00	**1.76E-08**	0.38	0.14	0.00	**2.61E-07**	0.03	0.02	0.00	**1.77E-08**	0.03	0.02	0.11	**2.06E-08**	0.03	0.02	0.02	**2.52E-08**
Student	0.04	0.02	**0.39**	2.75E-08	0.44	0.14	0.00	8.55E-07	0.03	0.02	**0.96**	2.61E-08	0.03	0.02	0.83	2.32E-08	0.03	0.02	**0.41**	3.50E-08
Skew-Student	0.04	0.02	**0.55**	2.79E-08	0.43	0.14	0.00	7.21E-07	0.03	0.02	**0.97**	2.67E-08	0.03	0.02	**0.98**	3.38E-08	0.04	0.02	**0.56**	3.25E-08
GED	0.04	0.02	0.02	2.37E-08	0.42	0.14	0.00	5.51E-07	0.03	0.02	0.43	2.28E-08	0.03	0.02	0.66	2.16E-08	0.03	0.02	0.07	3.10E-08
Skew-GED	0.04	0.02	0.07	2.39E-08	0.42	0.15	0.00	4.94E-07	0.03	0.02	**0.66**	2.29E-08	0.03	0.02	**0.87**	2.29E-08	0.04	0.02	0.23	2.91E-08
Empirical	0.04	0.02	0.09	3.05E-08	0.47	0.17	**0.09**	4.83E-07	0.03	0.02	0.06	2.78E-08	0.03	0.02	0.12	2.91E-08	0.03	0.02	0.24	3.56E-08
GEV	0.03	0.02	0.00	**1.49E-08**	0.39	0.15	0.00	**2.50E-07**	0.02	0.02	0.00	**1.55E-08**	0.03	0.02	0.00	**1.47E-08**	0.03	0.02	0.00	**2.09E-08**
Logistic	0.04	0.02	0.05	2.52E-08	0.39	0.13	0.00	4.02E-07	0.03	0.02	0.29	2.33E-08	0.03	0.02	**0.91**	2.67E-08	0.04	0.02	**0.76**	4.03E-08
Estimation window: 1000																				
HS	0.05	0.02	**0.45**	4.40E-08	0.57	0.02	**0.85**	1.25E-06	0.04	0.01	0.02	4.90E-08	0.04	0.02	0.04	4.04E-08	0.04	0.02	0.08	5.53E-08
Normal	0.03	0.02	0.00	**1.99E-08**	0.37	0.12	0.00	**3.83E-07**	0.02	0.01	0.00	**1.74E-08**	0.03	0.02	0.00	**2.17E-08**	0.03	0.02	0.00	**3.04E-08**
Skew-Normal	0.03	0.02	0.00	**2.16E-08**	0.39	0.12	0.00	**3.32E-07**	0.02	0.02	0.00	**2.03E-08**	0.03	0.02	0.04	**2.50E-08**	0.03	0.02	0.02	**3.22E-08**
Student	0.04	0.02	0.12	2.95E-08	0.44	0.12	0.00	1.02E-06	0.03	0.02	**0.60**	2.92E-08	0.03	0.02	**0.60**	2.99E-08	0.04	0.02	**0.60**	4.11E-08
Skew-Student	0.04	0.02	**0.23**	3.18E-08	0.42	0.12	0.00	8.52E-07	0.03	0.02	**0.90**	3.06E-08	0.03	0.02	**0.82**	4.04E-08	0.04	0.02	**0.79**	4.36E-08
GED	0.04	0.02	0.02	2.65E-08	0.42	0.12	0.00	6.59E-07	0.03	0.02	0.16	2.41E-08	0.03	0.02	0.39	2.73E-08	0.03	0.02	0.26	3.59E-08
Skew-GED	0.04	0.02	0.02	2.81E-08	0.42	0.13	0.00	5.68E-07	0.03	0.02	**0.61**	2.55E-08	0.03	0.02	0.43	3.04E-08	0.04	0.02	0.45	3.86E-08
Empirical	0.04	0.02	**0.27**	3.22E-08	0.48	0.15	**0.32**	6.92E-07	0.03	0.02	0.05	2.73E-08	0.03	0.02	0.12	2.81E-08	0.04	0.02	0.37	5.18E-08
GEV	0.03	0.02	0.01	**1.78E-08**	0.44	0.14	0.00	**2.73E-07**	0.02	0.01	0.00	**1.77E-08**	0.03	0.02	0.00	**2.29E-08**	0.03	0.02	0.00	**2.55E-08**
Logistic	0.04	0.02	0.01	2.95E-08	0.39	0.12	0.00	5.23E-07	0.03	0.02	0.23	2.54E-08	0.03	0.02	**0.87**	3.24E-08	0.04	0.02	**0.71**	4.56E-08

The values in bold correspond to the best results for each criteria.

Table 4.7 – Results for ES forecasting with $\alpha = 1\%$

	$\Lambda_S(X)$				$\Lambda_W(X)$				$S\&P500$				$1/N$				FF			
	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}	Mean	SD	p-value	Z_{ES_α}
Estimation window: 250																				
HS	0.05	0.03	**0.02**	1.48E-08	0.73	0.14	**0.08**	1.77E-07	0.03	0.02	0.03	1.11E-08	0.04	0.02	0.07	7.21E-09	0.04	0.02	**0.11**	**8.05E-09**
Normal	0.04	0.02	0.00	**5.77E-09**	0.43	0.15	0.00	**9.39E-08**	0.03	0.02	0.00	**4.73E-09**	0.03	0.02	0.00	**6.01E-09**	0.03	0.02	0.00	1.05E-08
Skew-Normal	0.04	0.02	0.00	**6.02E-09**	0.44	0.17	0.00	**7.46E-08**	0.03	0.02	0.00	**5.58E-09**	0.03	0.02	0.00	**7.18E-09**	0.03	0.02	0.00	9.75E-09
Student	0.05	0.03	**0.01**	1.48E-08	0.59	0.21	0.00	3.49E-07	0.03	0.02	0.08	7.28E-09	0.03	0.02	**0.21**	8.88E-09	0.03	0.02	**0.18**	1.35E-08
Skew-Student	0.05	0.03	0.00	1.70E-08	0.58	0.22	0.00	1.81E-07	0.03	0.02	**0.77**	1.61E-08	0.04	0.02	**0.12**	7.21E-09	0.03	0.02	0.07	1.23E-08
GED	0.05	0.03	0.00	9.49E-09	0.51	0.19	0.00	1.86E-07	0.03	0.02	**0.16**	9.75E-09	0.03	0.02	0.03	7.74E-09	0.03	0.02	0.02	1.17E-08
Skew-GED	0.05	0.03	0.00	1.01E-08	0.51	0.21	0.00	1.37E-07	0.03	0.02	0.06	9.84E-09	0.04	0.02	0.03	7.83E-09	0.03	0.02	0.01	1.08E-08
Empirical	0.04	0.02	0.00	1.14E-08	0.61	0.25	0.00	1.28E-07	0.03	0.02	0.01	1.31E-08	0.03	0.02	0.00	1.35E-08	0.03	0.02	0.00	**7.19E-09**
GEV	0.04	0.02	**0.04**	**5.47E-09**	0.46	0.19	0.00	**6.38E-08**	0.03	0.02	0.00	**4.73E-09**	0.03	0.02	0.00	**6.73E-09**	0.03	0.02	0.00	**7.99E-09**
Logistic	0.05	0.03	0.00	9.67E-09	0.46	0.16	0.00	1.31E-07	0.03	0.02	**0.14**	7.65E-09	0.04	0.02	**0.18**	1.01E-08	0.04	0.03	**0.40**	1.84E-08
Estimation window: 500																				
HS	0.06	0.03	**0.79**	1.61E-08	0.79	0.11	**0.75**	2.60E-07	0.04	0.02	0.13	2.05E-08	0.04	0.02	0.08	1.18E-08	0.04	0.02	0.09	9.86E-09
Normal	0.04	0.02	0.00	**6.15E-09**	0.43	0.14	0.00	**1.08E-07**	0.03	0.02	0.00	**5.73E-09**	0.03	0.02	0.01	**6.35E-09**	0.03	0.02	0.00	**9.58E-09**
Skew-Normal	0.04	0.02	0.00	**6.34E-09**	0.44	0.15	0.00	**9.29E-08**	0.03	0.02	0.00	**6.37E-09**	0.03	0.02	0.06	**7.41E-09**	0.03	0.02	0.01	**9.03E-09**
Student	0.05	0.03	**0.32**	1.19E-08	0.58	0.17	0.00	4.42E-07	0.03	0.02	**0.95**	1.15E-08	0.04	0.02	**0.66**	9.19E-09	0.03	0.03	**0.18**	1.38E-08
Skew-Student	0.05	0.03	**0.33**	1.20E-08	0.56	0.17	0.00	3.68E-07	0.03	0.02	**0.99**	1.15E-08	0.04	0.02	**0.71**	1.18E-08	0.03	0.03	**0.29**	1.28E-08
GED	0.05	0.03	0.05	8.99E-09	0.51	0.16	0.00	2.21E-07	0.03	0.02	0.32	8.77E-09	0.03	0.02	0.17	7.98E-09	0.03	0.03	0.04	1.14E-08
Skew-GED	0.05	0.03	0.05	9.09E-09	0.51	0.18	0.00	1.98E-07	0.03	0.02	**0.60**	8.72E-09	0.04	0.02	0.43	8.42E-09	0.03	0.03	0.11	1.07E-08
Empirical	0.05	0.02	0.27	1.29E-08	0.67	0.24	**0.26**	3.26E-07	0.03	0.02	0.04	1.25E-08	0.03	0.02	0.02	1.10E-08	0.03	0.02	0.02	1.87E-08
GEV	0.04	0.02	0.02	**5.16E-09**	0.44	0.19	0.00	**8.52E-08**	0.03	0.02	0.00	**5.91E-09**	0.03	0.02	0.00	**5.24E-09**	0.03	0.02	0.01	**7.19E-09**
Logistic	0.05	0.03	0.06	1.00E-08	0.46	0.15	0.00	1.60E-07	0.03	0.02	0.20	9.27E-09	0.04	0.02	**0.52**	1.06E-08	0.04	0.03	**0.50**	1.60E-08
Estimation window: 1000																				
HS	0.07	0.02	**0.38**	1.41E-08	0.79	0.06	**0.79**	3.20E-07	0.05	0.02	0.02	1.74E-08	0.05	0.02	0.02	1.62E-08	0.04	0.02	0.06	1.22E-08
Normal	0.04	0.02	0.00	**7.14E-09**	0.42	0.13	0.00	**1.37E-07**	0.03	0.02	0.00	**6.24E-09**	0.03	0.02	0.00	**7.79E-09**	0.03	0.02	0.01	**1.09E-08**
Skew-Normal	0.04	0.02	0.00	**7.79E-09**	0.44	0.13	0.00	**1.18E-07**	0.03	0.02	0.05	**7.34E-09**	0.03	0.02	0.10	**9.00E-09**	0.03	0.02	0.08	**1.16E-08**
Student	0.05	0.03	0.14	1.23E-08	0.57	0.15	0.00	5.22E-07	0.03	0.02	**0.87**	1.28E-08	0.04	0.02	**0.91**	1.21E-08	0.03	0.03	**0.56**	1.62E-08
Skew-Student	0.05	0.03	**0.20**	1.33E-08	0.55	0.15	0.00	4.31E-07	0.03	0.02	**0.86**	1.29E-08	0.04	0.02	**0.63**	1.62E-08	0.04	0.03	**0.48**	1.73E-08
GED	0.04	0.03	0.01	9.96E-09	0.51	0.14	0.00	2.62E-07	0.03	0.02	0.30	9.14E-09	0.04	0.02	0.40	1.02E-08	0.03	0.03	0.15	1.32E-08
Skew-GED	0.04	0.03	0.03	1.06E-08	0.51	0.15	0.00	2.25E-07	0.03	0.02	**0.40**	9.59E-09	0.04	0.02	0.38	1.13E-08	0.04	0.03	0.27	1.42E-08
Empirical	0.05	0.03	**0.40**	1.34E-08	0.69	0.23	**0.60**	2.62E-07	0.03	0.02	0.07	1.32E-08	0.04	0.02	0.06	1.11E-08	0.04	0.02	0.29	1.76E-08
GEV	0.04	0.02	0.00	**6.21E-09**	0.53	0.21	0.00	**9.29E-08**	0.03	0.01	0.00	**6.25E-09**	0.03	0.02	0.00	**8.07E-09**	0.03	0.02	0.00	**8.80E-09**
Logistic	0.04	0.02	0.03	1.17E-08	0.46	0.14	0.00	2.08E-07	0.03	0.02	0.33	1.01E-08	0.04	0.02	**0.87**	1.29E-08	0.04	0.02	**0.64**	1.81E-08

The values in bold correspond to the best results for each criteria.

The HS is again the worst option among almost all, as occurred with VaR and also with FF as the only exception. The sum portfolio executes its forecast more appropriately with the GEV distribution, only presenting a narrow variation conferred in $\alpha = 5\%$ with the normal distributions. $S\&P500$ performs with higher quality than the normal and GEV distribution, what it is observed either for $1/N$ and FF. Finally, the weighted portfolio is superior in general to the semi-parametric method when the empirical and GEV distributions are applied.

Tables 4.8 to 4.10 expose the results for EVaR forecasting. As this measure is defined by a gain-loss ration, its forecast is only qualified by a loss function. The behavior of the forecasts is similar to the one verified in VaR and in ES, where Λ_S shows results slightly greater than $S\&P500$, yet smaller than the others. As already verified, $1/N$ and FF performs very similar to one another and yet worst than Λ_S and $S\&P500$. In EVaR analysis the results turn more clear the in VaR and ES, where $S\&P500$ and Λ_S present very close losses and far better performs than the others.

Overall, there is a consistent pattern among portfolios and significance levels as seen before. The portfolios tend to produce smaller losses when the significance levels turn greater and the window sizes simultaneously turn smaller. Regarding the distributions overview of EVaR; HS demonstrates once again to be the inferior method for forecasting measures. Λ_S, in general, has minors losses with the GEV and normal, but was very suitable following a student distribution, while $S\&P500$ exposes it via normal and student and Λ_W via GEV and skew-normal. The $1/N$ and the FF portfolios do not show a dominant distribution, but both perform far better with the GEV, student, normal and skew-normal distributions.

After the separated analysis of measures, an overview of the main results aiming to establish the most suitable models and major procedures for each situation is conducted. As there is not an extreme prevailing method through the study, and many results are very similar, some constraints were applied to restrict the analysis, aiding the decision making.

Resembling results are reported for estimates generated by normal and skew-normal distribution in general. To determine a dominance between then, it is admitted that stocks data possess heavy-tailed behavior, usually accompanied with negative skewness and excess kurtosis. As mentioned by Mandelbrot (1963), the normal distribution is insufficient for modeling financial data, and in addition Miller & Liu (2006) argued that normal distribution might lead to a potential model risk. It was therefore decided to work with its skewed variant instead. In contrast, despite allied results, the student distribution displays a mild surpassing performance when compared to the skew student distribution overall.

Table 4.8 – Results for EVaR forecasting with $\alpha = 5\%$

	$\Lambda_S(X)$			$\Lambda_W(X)$			$S\&P500$			$1/N$			FF		
	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}
Estimation window: 250															
HS	0.02	0.01	2.30E-10	0.21	0.02	2.38E-07	0.01	0.01	3.40E-10	0.01	0.01	1.89E-09	0.02	0.01	8.81E-10
Normal	0.01	0.01	**2.06E-11**	0.15	0.09	**9.65E-10**	0.01	0.01	**4.25E-12**	0.01	0.01	**6.10E-10**	0.01	0.01	4.50E-10
Skew-Normal	0.02	0.01	**2.58E-11**	0.16	0.10	2.81E-08	0.01	0.01	1.76E-11	0.01	0.01	8.43E-10	0.01	0.01	**3.87E-10**
Student	0.02	0.01	**1.93E-11**	0.17	0.09	5.76E-09	0.01	0.01	**7.80E-12**	0.01	0.01	**6.06E-10**	0.01	0.01	5.05E-10
Skew-Student	0.02	0.01	4.14E-11	0.16	0.09	3.44E-08	0.01	0.01	2.78E-11	0.01	0.01	1.89E-09	0.01	0.01	**4.15E-10**
GED	0.02	0.01	2.66E-11	0.16	0.10	1.08E-08	0.01	0.01	2.89E-11	0.01	0.01	**6.45E-10**	0.01	0.01	4.90E-10
Skew-GED	0.02	0.01	4.18E-11	0.17	0.11	**8.47E-12**	0.01	0.01	3.18E-11	0.01	0.01	8.76E-10	0.01	0.01	**4.09E-10**
Empirical	0.02	0.01	6.26E-11	0.16	0.10	2.23E-09	0.01	0.01	3.64E-11	0.01	0.01	6.90E-10	0.01	0.01	4.35E-10
GEV	0.02	0.01	4.12E-11	0.17	0.10	5.45E-09	0.01	0.01	1.76E-11	0.01	0.01	6.92E-10	0.01	0.01	4.76E-10
Logistic	0.02	0.01	4.17E-11	0.15	0.09	**1.17E-10**	0.01	0.01	**7.92E-12**	0.01	0.01	6.48E-10	0.01	0.01	7.66E-10
Estimation window: 500															
HS	0.02	0.01	3.48E-10	0.21	0.02	3.34E-07	0.01	0.01	5.09E-10	0.02	0.01	1.92E-09	0.02	0.01	1.10E-09
Normal	0.01	0.01	**2.69E-11**	0.15	0.09	3.17E-09	0.01	0.01	**6.92E-12**	0.01	0.01	7.85E-10	0.01	0.01	4.05E-10
Skew-Normal	0.01	0.01	**3.00E-11**	0.16	0.09	**2.80E-11**	0.01	0.01	**8.13E-12**	0.01	0.01	7.22E-10	0.01	0.01	3.62E-10
Student	0.02	0.01	3.97E-11	0.17	0.09	1.35E-08	0.01	0.01	**5.21E-12**	0.01	0.01	**6.43E-10**	0.01	0.01	4.77E-10
Skew-Student	0.02	0.01	4.22E-11	0.16	0.09	2.07E-09	0.01	0.01	8.81E-12	0.01	0.01	1.92E-09	0.01	0.01	**3.57E-10**
GED	0.02	0.01	4.89E-11	0.17	0.09	1.10E-08	0.01	0.01	2.00E-11	0.01	0.01	**6.69E-10**	0.01	0.01	5.29E-10
Skew-GED	0.02	0.01	5.23E-11	0.17	0.09	**1.37E-09**	0.01	0.01	1.70E-11	0.01	0.01	7.46E-10	0.01	0.01	3.59E-10
Empirical	0.02	0.01	5.32E-11	0.17	0.09	1.48E-09	0.01	0.01	4.79E-11	0.01	0.01	1.02E-09	0.01	0.01	**3.58E-10**
GEV	0.01	0.01	**2.73E-11**	0.16	0.09	**5.17E-10**	0.01	0.01	2.02E-11	0.01	0.01	**5.20E-10**	0.01	0.01	**3.33E-10**
Logistic	0.02	0.01	4.63E-11	0.16	0.09	4.51E-09	0.01	0.01	1.80E-11	0.01	0.01	8.95E-10	0.01	0.01	5.37E-10
Estimation window: 1000															
HS	0.02	0.01	3.45E-10	0.21	0.01	9.43E-07	0.01	0.00	3.93E-10	0.02	0.01	2.03E-09	0.02	0.01	8.92E-10
Normal	0.01	0.01	**2.62E-11**	0.15	0.08	**2.86E-08**	0.01	0.01	**8.25E-12**	0.01	0.01	**7.65E-10**	0.01	0.01	**4.82E-10**
Skew-Normal	0.01	0.01	**2.55E-11**	0.16	0.08	**1.38E-08**	0.01	0.01	2.61E-11	0.01	0.01	8.78E-10	0.01	0.01	**5.50E-10**
Student	0.01	0.01	3.86E-11	0.17	0.08	2.65E-07	0.01	0.01	**9.87E-12**	0.01	0.01	**7.69E-10**	0.01	0.01	5.97E-10
Skew-Student	0.01	0.01	5.54E-11	0.16	0.08	1.54E-07	0.01	0.01	3.35E-11	0.01	0.01	2.03E-09	0.01	0.01	6.90E-10
GED	0.01	0.01	4.38E-11	0.17	0.08	1.73E-07	0.01	0.01	**1.26E-11**	0.01	0.01	**7.78E-10**	0.01	0.01	5.50E-10
Skew-GED	0.01	0.01	6.04E-11	0.17	0.08	1.09E-07	0.01	0.01	3.56E-11	0.01	0.01	1.02E-09	0.01	0.01	6.56E-10
Empirical	0.01	0.01	6.80E-11	0.17	0.09	4.70E-08	0.01	0.01	7.15E-11	0.01	0.01	1.04E-09	0.01	0.01	5.55E-10
GEV	0.01	0.01	**3.15E-11**	0.17	0.08	**1.23E-08**	0.01	0.00	3.01E-11	0.01	0.01	1.00E-09	0.01	0.01	**5.32E-10**
Logistic	0.01	0.01	4.48E-11	0.15	0.08	4.66E-08	0.01	0.01	1.97E-11	0.01	0.01	8.47E-10	0.01	0.01	6.17E-10

The values in bold correspond to the best results for each criteria.

Table 4.9 – Results for EVaR forecasting with $\alpha = 2.5\%$

	$\Lambda_S(X)$			$\Lambda_W(X)$			$S\&P500$			$1/N$			FF		
	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}
Estimation window: 250															
HS	0.03	0.01	3.20E-10	0.28	0.03	2.17E-07	0.02	0.01	4.43E-10	0.02	0.01	1.59E-09	0.02	0.01	8.79E-10
Normal	0.02	0.01	**3.56E-11**	0.18	0.10	4.49E-09	0.01	0.01	**1.39E-11**	0.01	0.01	**4.08E-10**	0.01	0.01	4.02E-10
Skew-Normal	0.02	0.01	**4.18E-11**	0.19	0.10	2.43E-08	0.01	0.01	3.16E-11	0.01	0.01	5.70E-10	0.02	0.01	**3.47E-10**
Student	0.02	0.01	5.84E-11	0.22	0.10	2.15E-08	0.01	0.01	**4.58E-12**	0.01	0.01	**4.38E-10**	0.02	0.01	4.69E-10
Skew-Student	0.02	0.01	9.31E-11	0.21	0.10	3.37E-08	0.01	0.01	7.12E-11	0.01	0.01	1.59E-09	0.02	0.01	3.89E-10
GED	0.02	0.01	5.62E-11	0.20	0.11	2.05E-08	0.01	0.01	5.81E-11	0.01	0.01	4.58E-10	0.02	0.01	4.44E-10
Skew-GED	0.02	0.01	7.52E-11	0.20	0.11	**3.12E-09**	0.01	0.01	6.15E-11	0.01	0.01	6.00E-10	0.02	0.01	**3.74E-10**
Empirical	0.02	0.01	1.18E-10	0.22	0.11	4.70E-09	0.01	0.01	8.58E-11	0.01	0.01	**2.84E-11**	0.02	0.01	4.58E-10
GEV	0.02	0.01	**5.37E-11**	0.19	0.10	**5.08E-10**	0.01	0.01	2.87E-11	0.01	0.01	4.78E-10	0.01	0.01	**3.81E-10**
Logistic	0.02	0.01	7.06E-11	0.19	0.10	**3.58E-09**	0.01	0.01	**2.53E-11**	0.01	0.01	4.84E-10	0.02	0.01	7.22E-10
Estimation window: 500															
HS	0.03	0.01	4.42E-10	0.29	0.02	3.06E-07	0.02	0.01	5.84E-10	0.02	0.01	1.55E-09	0.02	0.01	1.04E-09
Normal	0.02	0.01	**4.25E-11**	0.18	0.09	**8.12E-09**	0.01	0.01	**2.04E-11**	0.01	0.01	5.18E-10	0.01	0.01	3.49E-10
Skew-Normal	0.02	0.01	**4.63E-11**	0.19	0.10	**2.31E-09**	0.01	0.01	**2.42E-11**	0.01	0.01	5.00E-10	0.02	0.01	**3.13E-10**
Student	0.02	0.01	7.21E-11	0.22	0.09	3.79E-08	0.01	0.01	**2.91E-11**	0.01	0.01	**4.64E-10**	0.02	0.01	4.37E-10
Skew-Student	0.02	0.01	7.54E-11	0.21	0.09	1.65E-08	0.01	0.01	3.60E-11	0.01	0.01	1.55E-09	0.02	0.01	3.41E-10
GED	0.02	0.01	7.46E-11	0.21	0.10	2.43E-08	0.01	0.01	4.48E-11	0.01	0.01	**4.73E-10**	0.02	0.01	4.53E-10
Skew-GED	0.02	0.01	7.82E-11	0.20	0.10	9.99E-09	0.01	0.01	4.18E-11	0.01	0.01	5.29E-10	0.02	0.01	**3.28E-10**
Empirical	0.02	0.01	9.65E-11	0.23	0.10	9.20E-09	0.01	0.01	8.82E-11	0.01	0.01	7.15E-10	0.02	0.01	3.46E-10
GEV	0.02	0.01	**3.93E-11**	0.19	0.10	**3.71E-09**	0.01	0.01	3.38E-11	0.01	0.01	**3.45E-10**	0.01	0.01	**2.75E-10**
Logistic	0.02	0.01	7.49E-11	0.19	0.09	1.18E-08	0.01	0.01	4.30E-11	0.01	0.01	6.36E-10	0.02	0.01	5.08E-10
Estimation window: 1000															
HS	0.03	0.01	4.29E-10	0.29	0.01	4.71E-07	0.02	0.01	4.85E-10	0.02	0.01	1.61E-09	0.02	0.01	9.30E-10
Normal	0.02	0.01	**4.50E-11**	0.18	0.08	**1.43E-08**	0.01	0.01	**2.23E-11**	0.01	0.01	**5.26E-10**	0.02	0.01	**4.10E-10**
Skew-Normal	0.02	0.01	**4.75E-11**	0.19	0.09	**6.93E-09**	0.01	0.01	4.50E-11	0.01	0.01	6.13E-10	0.02	0.01	**4.65E-10**
Student	0.02	0.01	7.27E-11	0.21	0.08	1.33E-07	0.01	0.01	**3.83E-11**	0.01	0.01	**5.70E-10**	0.02	0.01	5.40E-10
Skew-Student	0.02	0.01	9.41E-11	0.20	0.08	7.71E-08	0.01	0.01	6.89E-11	0.02	0.01	1.61E-09	0.02	0.01	6.19E-10
GED	0.02	0.01	7.33E-11	0.21	0.08	8.67E-08	0.01	0.01	**3.61E-11**	0.01	0.01	**5.66E-10**	0.02	0.01	4.84E-10
Skew-GED	0.02	0.01	9.23E-11	0.20	0.09	5.47E-08	0.01	0.01	6.32E-11	0.02	0.01	7.31E-10	0.02	0.01	5.71E-10
Empirical	0.02	0.01	1.05E-10	0.23	0.10	2.35E-08	0.01	0.01	9.92E-11	0.02	0.01	7.26E-10	0.02	0.01	5.45E-10
GEV	0.02	0.01	**4.67E-11**	0.21	0.09	**6.14E-09**	0.01	0.01	4.43E-11	0.01	0.01	6.75E-10	0.02	0.01	**4.19E-10**
Logistic	0.02	0.01	7.97E-11	0.19	0.09	2.33E-08	0.01	0.01	4.57E-11	0.01	0.01	6.35E-10	0.02	0.01	5.78E-10

The values in bold correspond to the best results for each criteria.

Table 4.10 – Results for EVaR forecasting with $\alpha = 1\%$

	$\Lambda_S(X)$			$\Lambda_W(X)$			$S\&P500$			$1/N$			FF		
	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}	Mean	SD	Z_{EVaR_α}
Estimation window: 250															
HS	0.03	0.02	3.47E-10	0.41	0.07	1.66E-07	0.02	0.01	4.26E-10	0.02	0.01	1.07E-09	0.03	0.01	7.41E-10
Normal	0.02	0.01	**3.27E-11**	0.21	0.10	**5.48E-09**	0.01	0.01	**1.56E-11**	0.01	0.01	**2.19E-10**	0.02	0.01	2.63E-10
Skew-Normal	0.02	0.01	**3.74E-11**	0.22	0.11	1.55E-08	0.01	0.01	2.95E-11	0.02	0.01	3.06E-10	0.02	0.01	**2.27E-10**
Student	0.02	0.02	8.07E-11	0.29	0.12	3.43E-08	0.02	0.01	**1.24E-11**	0.02	0.01	**2.56E-10**	0.02	0.01	3.26E-10
Skew-Student	0.02	0.01	1.17E-10	0.28	0.13	2.74E-08	0.02	0.01	9.54E-11	0.02	0.01	1.07E-09	0.02	0.01	2.71E-10
GED	0.02	0.01	5.97E-11	0.25	0.12	2.17E-08	0.02	0.01	6.15E-11	0.02	0.01	**2.58E-10**	0.02	0.01	2.96E-10
Skew-GED	0.02	0.01	7.52E-11	0.25	0.13	6.07E-09	0.02	0.01	6.42E-11	0.02	0.01	3.27E-10	0.02	0.01	**2.51E-10**
Empirical	0.02	0.01	1.00E-10	0.30	0.14	1.09E-08	0.01	0.01	9.53E-11	0.02	0.01	3.23E-10	0.02	0.01	2.92E-10
GEV	0.02	0.01	**4.27E-11**	0.23	0.12	**1.37E-09**	0.01	0.01	**2.52E-11**	0.01	0.01	2.59E-10	0.02	0.01	**2.29E-10**
Logistic	0.02	0.01	6.94E-11	0.23	0.11	**6.21E-09**	0.02	0.01	3.10E-11	0.02	0.01	2.90E-10	0.02	0.01	5.17E-10
Estimation window: 500															
HS	0.03	0.02	4.14E-10	0.41	0.04	2.52E-07	0.02	0.01	5.33E-10	0.03	0.01	1.04E-09	0.03	0.01	8.31E-10
Normal	0.02	0.01	**3.76E-11**	0.21	0.10	**8.55E-09**	0.01	0.01	**2.24E-11**	0.02	0.01	2.71E-10	0.02	0.01	2.23E-10
Skew-Normal	0.02	0.01	**4.06E-11**	0.22	0.10	**3.75E-09**	0.01	0.01	**2.68E-11**	0.02	0.01	2.74E-10	0.02	0.01	**2.00E-10**
Student	0.02	0.01	7.69E-11	0.29	0.10	5.58E-08	0.02	0.01	**4.43E-11**	0.02	0.01	**2.70E-10**	0.02	0.01	3.04E-10
Skew-Student	0.02	0.01	7.98E-11	0.27	0.11	3.11E-08	0.02	0.01	5.03E-11	0.02	0.01	1.04E-09	0.02	0.01	2.43E-10
GED	0.02	0.01	6.81E-11	0.25	0.10	2.70E-08	0.02	0.01	4.80E-11	0.02	0.01	**2.66E-10**	0.02	0.01	2.92E-10
Skew-GED	0.02	0.01	7.07E-11	0.25	0.11	1.49E-08	0.02	0.01	4.60E-11	0.02	0.01	2.97E-10	0.02	0.01	**2.22E-10**
Empirical	0.02	0.01	8.71E-11	0.33	0.13	1.81E-08	0.02	0.01	9.93E-11	0.02	0.01	3.93E-10	0.02	0.01	2.66E-10
GEV	0.02	0.01	**3.26E-11**	0.22	0.11	**4.62E-09**	0.01	0.01	4.86E-11	0.02	0.01	**1.82E-10**	0.02	0.01	**1.68E-10**
Logistic	0.02	0.01	7.21E-11	0.23	0.10	1.37E-08	0.02	0.01	4.82E-11	0.02	0.01	3.67E-10	0.02	0.01	3.64E-10
Estimation window: 1000															
HS	0.04	0.01	4.00E-10	0.41	0.02	3.84E-07	0.03	0.01	4.37E-10	0.03	0.01	1.07E-09	0.03	0.01	7.50E-10
Normal	0.02	0.01	**4.12E-11**	0.21	0.09	**1.38E-08**	0.01	0.01	**2.39E-11**	0.02	0.01	**2.85E-10**	0.02	0.01	**2.60E-10**
Skew-Normal	0.02	0.01	**4.51E-11**	0.22	0.09	**7.94E-09**	0.01	0.01	4.16E-11	0.02	0.01	**3.38E-10**	0.02	0.01	**2.94E-10**
Student	0.02	0.01	7.66E-11	0.28	0.09	1.26E-07	0.02	0.01	5.31E-11	0.02	0.01	3.41E-10	0.02	0.01	3.72E-10
Skew-Student	0.02	0.01	9.50E-11	0.27	0.09	7.84E-08	0.02	0.01	7.62E-11	0.02	0.01	1.07E-09	0.02	0.01	4.25E-10
GED	0.02	0.01	6.91E-11	0.25	0.09	6.72E-08	0.01	0.01	**4.15E-11**	0.02	0.01	**3.25E-10**	0.02	0.01	3.18E-10
Skew-GED	0.02	0.01	8.38E-11	0.25	0.09	4.45E-08	0.02	0.01	6.11E-11	0.02	0.01	4.17E-10	0.02	0.01	3.74E-10
Empirical	0.02	0.01	1.04E-10	0.34	0.13	3.20E-08	0.02	0.01	8.96E-11	0.02	0.01	4.10E-10	0.02	0.01	3.93E-10
GEV	0.02	0.01	**3.93E-11**	0.26	0.12	**6.32E-09**	0.01	0.01	**3.76E-11**	0.02	0.01	3.58E-10	0.02	0.01	**2.47E-10**
Logistic	0.02	0.01	7.99E-11	0.23	0.09	2.40E-08	0.01	0.01	5.06E-11	0.02	0.01	3.82E-10	0.02	0.01	4.12E-10

The values in bold correspond to the best results for each criteria.

It is observed that the HS performs worse than every other distribution in the loss function analysis, even though it presents notable results in the backtests for Λ_W and for FF. Based on the assumption that a loss function is a superior form to qualify a forecast, it is evidenced that there is an inferiority of the HS approach when dealing with systemic risk measurement and forecasting. The lower adequacy of HS is corroborated by Pritsker (2006), Christoffersen & Gonçalves (2005), and even by Müller & Righi (2017) which compared VaR, ES, and EVaR based on HS, DCC-GARCH, and copula methods. This indicates a requirement to add heteroscedastic conditional models to deal with volatility clusters existent in financial data, and this was verified with the semiparametric techniques. FHS and GEV perform significantly well in most cases and produce a fine improvement of HS, especially for ES and EVaR loss functions of the weighted portfolio. GEV performs slightly better with $\alpha = 1\%$, in the other cases, a preference was not perceived.

With the intention to better support a final conclusion, Table 4.11 assembles the best overall models for each measure and portfolio. The models selected are the ones that show the prevailing performance for the measure and portfolio in question under significance levels and window sizes. In other words, every model displayed is the one that generates the best forecast for that specific measure and portfolio.

Summarizing what has been discussed, Λ_S and $S\&P500$ exposes resemblant behavior in the aggregate, - as well $1/N$ and FF - meanwhile Λ_W expresses this as less suitable. In relation to measures loss functions, it is viewed that $S\&P500$ outperforms Λ_S. Even though Λ_S yields more losses than $S\&P500$, the difference between them is not disparate enough to truly define superiority. Simultaneously, the VR of the index is even worse than the sum portfolio, enabling inferences about extreme events that are not correctly addressed. Additionally, Λ_S shows dominance over $1/N$ and FF. Regarding the distributions, we can easily see notable patterns among forecasts. While $S\&P500$ demonstrates far better fits with the student and normal distribution, Λ_S, $1/N$ and FF show more frequently a preference for GEV and normal fits, yet also demonstrates suitable results under the student distribution with close outcomes for both. For the Λ_W evaluation, the semi-parametric models (empirical and GEV distributions) prevail for all risk measures, closely followed by the skew-normal outcomes.

Table 4.11 – Best forecasted model for every measure and portfolio under significance levels and window sizes.

	VaR					ES					EVaR				
	Λ_S	Λ_W	$S\&P500$	$1/N$	FF	Λ_S	Λ_W	$S\&P500$	$1/N$	FF	Λ_S	Λ_W	$S\&P500$	$1/N$	FF
Estimation window: 250															
$\alpha = 1\%$	Normal	GEV	Student	Normal	GEV	GEV	GEV	GEV	Normal	Empirical	Normal	GEV	Student	Normal	S-Normal
$\alpha = 2.5\%$	Student	Empirical	Student	Student	GEV	GEV	GEV	Normal	Normal	GEV	Normal	GEV	Student	Empirical	S-Normal
$\alpha = 5\%$	Student	Empirical	Student	Student	Empirical	Normal	GEV	Normal	Normal	GEV	Student	S-GED	Normal	Student	S-Normal
Estimation window: 500															
$\alpha = 1\%$	GEV	Empirical	Normal	GEV	Empirical	GEV	GEV	Normal	Normal	GEV	GEV	S-Normal	Normal	GEV	GEV
$\alpha = 2.5\%$	Student	Empirical	Student	GEV	GEV	GEV	GEV	GEV	GEV	GEV	GEV	S-Normal	Normal	GEV	GEV
$\alpha = 5\%$	Empirical	S-Student	Student	GEV	Empirical	GEV	GEV	GEV	GEV	GEV	Normal	S-Normal	Student	GEV	GEV
Estimation window: 1000															
$\alpha = 1\%$	Normal	GEV	Normal	Normal	Normal	GEV	GEV	Normal	Normal	GEV	GEV	GEV	Normal	Normal	GEV
$\alpha = 2.5\%$	Normal	Empirical	Student	Student	Empirical	GEV	GEV	Normal	Normal	GEV	Normal	GEV	Normal	Normal	Normal
$\alpha = 5\%$	Student	Empirical	Student	Student	Empirical	GEV	GEV	Normal	Normal	GEV	S-Normal	GEV	Normal	Normal	Normal

The notation S is for Skew.

Concerning the comparison of measures, it is hard to determine dominance. VaR disadvantages are indeed dangerous to a systemic risk management since they violate the principle of diversification and do not bother with the potential of losses beyond the quantile point of interest. Although Righi & Ceretta (2015) points out that has not been enough research done to establish the superiority of ES relative to others (due to ES being not often used by the industry and its backtest being more complex than that of the VaR) ES estimation errors can be minimized by larger samples, as applied in this study, and additionally Yamai & Yoshiba (2005) demonstrated how the tail risk of VaR could generate serious problems in real-world cases and where expected shortfall could be more appropriately used instead. Bellini & Bernardino (2017) compared the theoretical and numerical result of VaR, ES, and EVaR, and subsequently argued in favor of expectiles. On the other hand, Emmer et al. (2015) opposed the properties of the same measures and concluded that ES looked to be the best option despite some caveats regarding estimation and backtesting. Meanwhile, any sufficient evidence to replace ES by expectiles was not found by the authors. Our results show that EVaR is more conservative compared to VaR and ES. This is one of the main goals when managing systemic risk, although as EVaR does not possess an easy and clear financial intuition, the use of supporting measures is thus recommended.

Considering this overview, the conclusions are simple. Concerning risk measures, EVaR can be seen as more appropriate and suitable to deal with systemic risk. The Λ_W presents the worst approach to deal with risk, while Λ_S and $S\&P500$ generate close and better results. Finally, corroborating with the literature, the student distributions are demonstrated to be the best distributions to model both Λ_S and $S\&P500$ for VaR, and are a good alternative for ES and EVaR.

5 CONCLUSION

In this study, we deepen the research of systemic risk measurement via aggregation functions. The empirical evaluation consists of modeling and forecasting risk for five distinguished portfolios as a proxy for an economic system, of which two are generated by aggregation functions and the others are a well-known approaches. The measures used are VaR, ES, and EVaR, these being the most disseminated to manage risk in a single-firm view. To assess the performance and evaluate the quality of forecasts of the competing models, we use loss functions and violation backtests as a supplementary tool. To guarantee the robustness of this work, for each risk measure we considered models from distinct distributions, estimation windows, and significance levels.

Regarding the empirical results, we find some patterns associated with the data. The historic simulation demonstrates to be the inferior method by which to approach risk management. This was verified for all scenarios and models, highly corroborating with the literature. The student distribution prevails in general as the foremost distribution, especially for VaR forecasting for Λ_S and $S\&P500$, although some other ones perform similarly such as the skew-normal and the general extreme value. It can nonetheless be affirmed as the more suited distribution for the purpose of this work. The definition of the student as the most suitable is based on the results of it, these results were consistent and very close to the best outcome for every scenario of significance level and window sizes. The $1/N$ and FF portfolios present very similar performances throughout the analysis and show worts results comparing to Λ_S and $S\&P500$ in general, so it can be affirmed that both $1/N$ and FF are inferior to Λ_S and $S\&P500$ regarding this approach to manage systemic risk. Overall, Λ_W presents the worst forecast results, what can be explained by two main reasons. First, as shown in Figure 4.7, the series might not have been correctly modeled by the distributions assumed in this work, since its behavior diverges from what is commonly assumed for financial data. Second, considering Λ_W as the weighted version of Λ_S, the assumptions used for the weighting may not be suitable for this case. The volume of transactions appears to confront information that already has been intrinsically contained in prices, inducing an undefined behavior. The Λ_S portfolio performs quite similarly to the $S\&P500$ index, especially considering that the Standard $\&$ Poor's 500 Index is the most used market index consisted by the larger companies of the U.S., and it can be assumed that the capacity of Λ_S to consider the major characteristics of the system in question. In addition, as the sum portfolio is built by all assets of the American market, it is expected that there is more volatility in general and further difficulty in forecasting what is observed in its descriptive statistics and its measure forecasting.

There is no agreement around the definition of risk, and there is no universal way of choosing the best measure. Measure choice should rely on the purpose of it, and the best manner to evaluate any procedure is by finding out how well they perform for an intended task. Considering EVaR as the most conservative measure among the ones applied in this study, it is rather reasonable to deal with the systemic risk than the others, since the final purpose to manage the systemic risk is to provide and ensure the minimum loss over the aggregate system, especially over crisis periods. The aggregated portfolio of Λ_S demonstrates an aptitude to process the risk of the system selected, as well as superior adequacy in the context that has been applied since this portfolio - considered as a proxy of the U.S. economy - is composed of all the stock of the economy and performs similarly to the main index price of the country itself. We can, therefore, conclude that the aggregation function that generates Λ_S is an adequate method to determine a proxy of an economy and to process the systemic risk inherent in it. In addition, we can consider Λ_W as a tool to study better weighings to forecasts systemic risk, since this measure is created based on Λ_S which show great forecasting performances through this study.

The results from this work can be applied in any firm belonging to the U.S. as a form of managing systemic risk. Since we considered all stocks of the selected economy, the Λ_S possesses a huge advantage as opposed to methods that consider only indexes or small segment stocks, especially to the firms that are not included in those samples. This is seen with both $1/N$ and FF, which are very well-known approaches, where Λ_S present far better results in the loss function analysis. As already said, Λ_S demonstrate a close behavior to $S\&P500$, even though the sum portfolio considerer much more stocks in its compositing, what can highly increase its difficult to model the series and guarantee acceptable forecasts.

As for suggestions for future studies; we emphasize the necessity to expand to alternative aggregation functions to other economies. The Λ_W might be improved by reformulating the weighting or by changing the modeling technique. Moreover, Λ_S

could be applied to stressed scenarios in order to analyze the behavior in distinctive turbulent scenes, and either in a composition of forecasting of distinct models and other aggregation functions. This methodology can also be applied to a specific set of assets and stock, and although it would escape from the systemic context, it may be useful for a sectoral risk management. Furthermore, other single firm risk measures can be applied, especially the ones that consider deviations such as the SDR proposed by Righi & Ceretta (2016). The deviation of a loss measure is a concept that can be linked to the risk of the model, due to the fact that any risk measurement method that presents low deviation is expected to present a better specification of its own model. A well-specified model posses superior adjustment of distribution of the data which could results in lower uncertainty levels, and this is extremely relevant to systemic risk management.

BIBLIOGRAPHY

ACEMOGLU, D.; OZDAGLAR, A.; TAHBAZ-SALEHI, A. Systemic risk and stability in financial networks. *The american economic review*, American Economic Association, v. 105, n. 2, p. 564–608, 2015. 2

ACERBI, C. Spectral measures of risk: a coherent representation of subjective risk aversion. *Journal of Banking & Finance*, Elsevier, v. 26, n. 7, p. 1505–1518, 2002. 2

ACERBI, C. Coherent representations of subjective risk aversion. *Risk Measures for the 21st Century*, p. 147, 2003. 2

ACERBI, C.; SZEKELY, B. General properties of backtestable statistics. *Working paper*, 2017. 18

ACERBI, C.; TASCHE, D. On the coherence of expected shortfall. *Journal of Banking & Finance*, Elsevier, v. 26, n. 7, p. 1487–1503, 2002. 12

ACHARYA, V.; ENGLE, R.; RICHARDSON, M. Capital shortfall: A new approach to ranking and regulating systemic risks. *The American Economic Review*, American Economic Association, v. 102, n. 3, p. 59–64, 2012. 2

ACHARYA, V. V.; PEDERSEN, L. H.; PHILIPPON, T.; RICHARDSON, M. Measuring systemic risk. *The Review of Financial Studies*, Oxford University Press, v. 30, n. 1, p. 2–47, 2017. 2

ADRIAN, T.; BRUNNERMEIER, M. K. Covar. *The American Economic Review*, American Economic Association, v. 106, n. 7, p. 1705–1741, 2016. 2

AIGNER, D. J.; AMEMIYA, T.; POIRIER, D. J. On the estimation of production frontiers: maximum likelihood estimation of the parameters of a discontinuous density function. *International Economic Review*, JSTOR, n. 17, p. 377–396, 1976. 13

ANGELIDIS, T.; BENOS, A.; DEGIANNAKIS, S. The use of garch models in var estimation. *Statistical methodology*, Elsevier, v. 1, n. 1, p. 105–128, 2004. 4

ARTZNER, P.; DELBAEN, F.; EBER, J.-M.; HEATH, D. Coherent measures of risk. *Mathematical finance*, Wiley Online Library, v. 9, n. 3, p. 203–228, 1999. 2, 7, 8, 29

BANULESCU, G.-D.; DUMITRESCU, E.-I. Which are the sifis? *Working paper*, 2012. 3

BELLINI, F.; BERNARDINO, E. D. Risk management with expectiles. *The European Journal of Finance*, Taylor & Francis, v. 23, n. 6, p. 487–506, 2017. 13, 18, 41

BELLINI, F.; KLAR, B.; MÜLLER, A.; GIANIN, E. R. Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, Elsevier, v. 54, p. 41–48, 2014. 13, 18

BENOIT, S. Where is the system? *International Economics*, Elsevier, v. 138, p. 1–27, 2014. 4

BENOIT, S.; COLLETAZ, G.; HURLIN, C.; PÉRIGNON, C. A theoretical and empirical comparison of systemic risk measures. *Working paper*, 2013. 3

BENOIT, S.; COLLIARD, J.-E.; HURLIN, C.; PÉRIGNON, C. Where the risks lie: A survey on systemic risk. *Review of Finance*, Oxford University Press, v. 21, n. 1, p. 109–152, 2017. 2

BERNAL, O.; GNABO, J.-Y.; GUILMIN, G. Assessing the contribution of banks, insurance and other financial services to systemic risk. *Journal of Banking & Finance*, Elsevier, v. 47, p. 270–287, 2014. 4

BERNARDI, M.; MARUOTTI, A.; PETRELLA, L. Multiple risk measures for multivariate dynamic heavy–tailed models. *Journal of Empirical Finance*, Elsevier, v. 43, p. 1–32, 2017. 2

BILLIO, M.; CASARIN, R.; COSTOLA, M.; PASQUALINI, A. An entropy-based early warning indicator for systemic risk. *Journal of International Financial Markets, Institutions and Money*, Elsevier, v. 45, p. 42–59, 2016. 2

BILLIO, M.; GETMANSKY, M.; LO, A. W.; PELIZZON, L. Econometric measures of connectedness and systemic risk in the finance and insurance sectors. *Journal of Financial Economics*, Elsevier, v. 104, n. 3, p. 535–559, 2012. 2

BOLLERSLEV, T. A conditionally heteroskedastic time series model for speculative prices and rates of return. *The review of economics and statistics*, JSTOR, p. 542–547, 1987. 14

BOLLERSLEV, T.; WOOLDRIDGE, J. M. Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances. *Econometric reviews*, Taylor & Francis, v. 11, n. 2, p. 143–172, 1992. 14

BOUCHER, C. M.; DANÍELSSON, J.; KOUONTCHOU, P. S.; MAILLET, B. B. Risk models-at-risk. *Journal of Banking & Finance*, Elsevier, v. 44, p. 72–92, 2014. 4

BROWNLEES, C. T.; CHABOT, B. R.; GHYSELS, E.; KURZ, C. J. Back to the future: Backtesting systemic risk measures during historical bank runs and the great depression. *Working paper*, 2017. 3

BROWNLEES, C. T.; ENGLE, R. F. Volatility, correlation and tails for systemic risk measurement. *Available at SSRN 1611229*, 2012. 2, 3

BRUNNERMEIER, M.; GORTON, G.; KRISHNAMURTHY, A. Liquidity mismatch measurement. *Working paper*, University of Chicago Press, p. 99–112, 2013. 2

CASTRO, C.; FERRARI, S. Measuring and testing for the systemically important financial institutions. *Journal of Empirical Finance*, Elsevier, v. 25, p. 1–14, 2014. 3

CHEN, C.; IYENGAR, G.; MOALLEMI, C. C. An axiomatic approach to systemic risk. *Management Science*, INFORMS, v. 59, n. 6, p. 1373–1388, 2013. 2, 5, 8, 9, 10

CHEN, Q.; GOLDSTEIN, I.; JIANG, W. Payoff complementarities and financial fragility: Evidence from mutual fund outflows. *Journal of Financial Economics*, Elsevier, v. 97, n. 2, p. 239–262, 2010. 2

CHEN, Y. Banking panics: The role of the first-come, first-served rule and information externalities. *Journal of Political Economy*, The University of Chicago Press, v. 107, n. 5, p. 946–968, 1999. 2

CHERUBINI, U.; MULINACCI, S. Contagion-based distortion risk measures. *Applied Mathematics Letters*, Elsevier, v. 27, p. 85–89, 2014. 2

CHRISTOFFERSEN, P.; GONÇALVES, S. Estimation risk in financial risk management. *The Journal of Risk*, Incisive Media Limited, v. 7, n. 3, p. 1, 2005. 16, 39

DANÍELSSON, J. *Financial risk forecasting: the theory and practice of forecasting market risk with implementation in R and Matlab*. [S.l.]: John Wiley & Sons, 2011. v. 588. 16

DANIELSSON, J.; JAMES, K. R.; VALENZUELA, M.; ZER, I. Dealing with systemic risk when we measure it badly. *European Center for Advanced Research in Economics and Statistics Working Paper*, 2012. 2

DANÍELSSON, J.; JORGENSEN, B. N.; SAMORODNITSKY, G.; SARMA, M.; VRIES, C. G. de. Fat tails, var and subadditivity. *Journal of econometrics*, Elsevier, v. 172, n. 2, p. 283–291, 2013. 12, 29

DEGIANNAKIS, S.; FLOROS, C.; DENT, P. Forecasting value-at-risk and expected shortfall using fractionally integrated models of conditional volatility: International evidence. *International Review of Financial Analysis*, Elsevier, v. 27, p. 21–33, 2013. 29

DEGIANNAKIS, S.; POTAMIA, A. Multiple-days-ahead value-at-risk and expected shortfall forecasting for stock indices, commodities and exchange rates: Inter-day versus intra-day data. *International Review of Financial Analysis*, Elsevier, v. 49, p. 176–190, 2017. 5

DIAMANDIS, P. F.; DRAKOS, A. A.; KOURETAS, G. P.; ZARANGAS, L. Value-at-risk for long and short trading positions: Evidence from developed and emerging equity markets. *International Review of Financial Analysis*, Elsevier, v. 20, n. 3, p. 165–176, 2011. 5

EMMER, S.; KRATZ, M.; TASCHE, D. What is the best risk measure in practice? a comparison of standard measures. *Working paper*, 2015. 41

FAMA, E. F.; FRENCH, K. R. A five-factor asset pricing model. *Journal of financial Economics*, Elsevier, v. 116, p. 1–22, 2015. 5, 12

FERNÁNDEZ, C.; STEEL, M. F. On bayesian modeling of fat tails and skewness. *Journal of the American Statistical Association*, Taylor & Francis Group, v. 93, n. 441, p. 359–371, 1998. 15

FÖLLMER, H.; SCHIED, A. Convex measures of risk and trading constraints. *Finance and stochastics*, Springer, v. 6, n. 4, p. 429–447, 2002. 2

GERLACH, R.; WALPOLE, D.; WANG, C. Semi-parametric bayesian tail risk forecasting incorporating realized measures of volatility. *Quantitative Finance*, Taylor & Francis, v. 17, n. 2, p. 199–215, 2017. 18

GIGLIO, S.; KELLY, B.; PRUITT, S. Systemic risk and the macroeconomy: An empirical evaluation. *Journal of Financial Economics*, Elsevier, v. 119, n. 3, p. 457–471, 2016. 2

HUANG, X.; ZHOU, H.; ZHU, H. A framework for assessing the systemic risk of major financial institutions. *Journal of Banking & Finance*, Elsevier, v. 33, n. 11, p. 2036–2049, 2009. 2

KREIS, Y.; LEISEN, D. P. Systemic risk in a structural model of bank default linkages. *Journal of Financial Stability*, Elsevier, 2017. 4

KROMER, E.; OVERBECK, L.; ZILCH, K. Systemic risk measures on general measurable spaces. *Mathematical Methods of Operations Research*, Springer, v. 84, n. 2, p. 323–357, 2016. 3, 10

KUESTER, K.; MITTNIK, S.; PAOLELLA, M. Value-at-risk prediction: A comparison of alternative strategies. *Journal of Financial Econometrics*, 2006. 5, 25

KUSUOKA, S. On law invariant coherent risk measures. *Working paper*, Springer, p. 83–95, 2001. 2

MANDELBROT, B. The variation of certain speculative prices. *The journal of business*, JSTOR, v. 36, n. 4, p. 394–419, 1963. 35

MARINELLI, C.; D'ADDONA, S.; RACHEV, S. T. A comparison of some univariate models for value-at-risk and expected shortfall. *International Journal of Theoretical and Applied Finance*, World Scientific, v. 10, n. 06, p. 1043–1075, 2007. 5

MARKOWITZ, H. Portfolio selection. *The journal of finance*, Wiley Online Library, v. 7, n. 1, p. 77–91, 1952. 1

MARTIN, A.; SKEIE, D.; THADDEN, E.-L. v. Repo runs. *The Review of Financial Studies*, Oxford University Press, v. 27, n. 4, p. 957–989, 2014. 2

MCNEIL, A. J.; FREY, R. Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of empirical finance*, Elsevier, v. 7, n. 3, p. 271–300, 2000. 17, 29

MENSI, W.; HAMMOUDEH, S.; SHAHZAD, S. J. H.; SHAHBAZ, M. Modeling systemic risk and dependence structure between oil and stock markets using a variational mode decomposition-based copula method. *Journal of Banking & Finance*, Elsevier, v. 75, p. 258–279, 2017. 4

MILLER, D. J.; LIU, W.-H. Improved estimation of portfolio value-at-risk under copula models with mixed marginals. *Journal of Futures Markets*, Wiley Online Library, v. 26, n. 10, p. 997–1018, 2006. 35

MÜLLER, F. M.; RIGHI, M. B. Numerical comparison of multivariate models to forecasting risk measures. *Working paper*, 2017. 39

PRITSKER, M. The hidden dangers of historical simulation. *Journal of Banking & Finance*, Elsevier, v. 30, n. 2, p. 561–582, 2006. 29, 39

RIGHI, M. B.; BORENSTEIN, D. A simulation comparison of risk measures for portfolio optimization. *Finance Research Letters*, Elsevier, 2017. 3

RIGHI, M. B.; CERETTA, P. S. A comparison of expected shortfall estimation models. *Journal of Economics and Business*, Elsevier, v. 78, p. 14–47, 2015. 41

RIGHI, M. B.; CERETTA, P. S. Shortfall deviation risk: an alternative for risk measurement. *Journal of Risk*, v. 19, n. 2, p. 81–116, 2016. 44

ROCKAFELLAR, R. T.; URYASEV, S.; ZABARANKIN, M. Generalized deviations in risk analysis. *Finance and Stochastics*, Springer, v. 10, n. 1, p. 51–74, 2006. 2

SINGH, A. K.; ALLEN, D. E.; POWELL, R. J. Value at risk estimation using extreme value theory. *Working paper*, Modelling and Simulation Society of Australia and New Zealand, 2011. 16

SUPERVISION, B. C. on B. *Overview of the amendment to the capital accord to incorporate market risks*. [S.l.]: Bank for International Settlements, 1996. 1

SUPERVISION, B. C. on B. *Fundamental review of the trading book: A revised market risk framework*. [S.l.]: Bank for International Settlements, 2013. 1

SUPERVISION, B. C. on B. *Minimum capital requirements for market risk.* [S.l.]: Bank for International Settlements, 2016. 1

TAYLOR, J. W. Estimating value at risk and expected shortfall using expectiles. *Journal of Financial Econometrics*, Oxford University Press, v. 6, n. 2, p. 231–252, 2008. 16

TELMOUDI, F.; GHOURABI, M. E.; LIMAM, M. On conditional risk estimation considering model risk. *Journal of Applied Statistics*, Taylor & Francis, v. 43, n. 8, p. 1386–1399, 2016. 17

TOLIKAS, K. Unexpected tails in risk measurement: Some international evidence. *Journal of Banking & Finance*, Elsevier, v. 40, p. 476–493, 2014. 5

YAMAI, Y.; YOSHIBA, T. Value-at-risk versus expected shortfall: A practical perspective. *Journal of Banking & Finance*, Elsevier, v. 29, n. 4, p. 997–1015, 2005. 41

YUN, J.; MOON, H. Measuring systemic risk in the korean banking sector via dynamic conditional correlation models. *Pacific-Basin Finance Journal*, Elsevier, v. 27, p. 94–114, 2014. 3

ZHOU, J. Extreme risk measures for reits: A comparison among alternative methods. *Applied Financial Economics*, Taylor & Francis, v. 22, n. 2, p. 113–126, 2012. 4

ZIEGEL, J. F. Coherence and elicitability. *Mathematical Finance*, Wiley Online Library, v. 26, n. 4, p. 901–918, 2016. 1