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A supnorm estimate for one-dimensional porous medium equations with advection

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Abstract. We give a short derivation of supnorm estimates for solutions of one-dimensional porous medium equations of the form

$$u_t + (f(t,u))_x = (|u|^{\alpha} u_x)_x,$$

assuming initial data $u(\cdot,0) \in L^{p_0}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ for some $1 \leq p_0 < \infty$.

Key-words. Porous Medium Equation, Supnorm Estimate, Comparison Theorem

1 Introduction

There are a number of physical applications where the porous medium equation describes processes involving fluid flow, heat transfer or diffusion [5]. The porous medium equation without advection is given by

$$\frac{\partial u}{\partial t} = (|u|^{m-1}u)_{xx} + f, \;\; m > 1, \tag{1}$$

where $f = f(x,t)$ is a source term.

Here we consider the following initial-value problem

$$\begin{cases} u_t + (f(t,u))_x = (|u|^{\alpha} u_x)_x, \;\; x \in \mathbb{R}, \, t > 0, \\ \\ u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}), \;\; 1 \leq p_0 < \infty, \end{cases} \tag{2}$$

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where $\alpha \geq 0$ and $f \in C^1([0,\infty)\times\mathbb{R})$ are given. The solutions of (2) are known to be defined for all $t > 0$ and decay as $t \to \infty$ in several spaces. In this work, we derive a supnorm estimate for the solutions of (2) when considering $u(\cdot, 0)$ in $L^p(\mathbb{R})$, $p = p_0+\alpha/2$. By solution in some interval $[0, T^*)$, $0 < T^* \leq \infty$, we mean a measurable function $u : \mathbb{R} \times [0, T^*) \to \mathbb{R}$ which is bounded in each strip $\mathbb{R} \times [0, T]$, $0 < T < T^*$, and which solves the equation (2) in distributional sense.

2 Preliminary

An important result to obtain a supnorm estimate also for negative solutions is the following Theorem.

Theorem 2.1. ***(Theorem of comparison:)***
Let $u(\cdot, t)$, $v(\cdot, t)$ solutions of the equation (1), with initial value u_0, $v_0 \in L^\infty(\mathbb{R})$, respectively, both defined for $0 < t < T$ and limited in the strip $\mathbb{R} \times [0, T]$. Also, if

$$|f(x, t, u) - f(x, t, v)| \leq K_f(M, T)|u - v|, \ \forall\, x \in \mathbb{R}, \forall\, t,\ 0 \leq t \leq T,$$

then

$$u_0(x) \leq v_0(x)\ a.e.\, x \in \mathbb{R} \Rightarrow u(x, t) \leq v(x, t)\, \forall x \in \mathbb{R}, \tag{3}$$

for all t, $0 < t \leq T$.

The proof of this Theorem is in [2].

2.1 Some importants inequalities

The following inequalities will be important througout this work.

- For any p, q and r such that $0 < p \leq r \leq \infty$, $1 \leq q \leq \infty$:

$$\|w\|_{L^r(\mathbb{R})} \leq \tilde{K}(r, q, p)\|w\|^{1-\tilde{\theta}}_{L^p(\mathbb{R})}\|w_x\|^{\tilde{\theta}}_{L^q(\mathbb{R})}\ \ \forall\, w \in C^1_0(\mathbb{R}), \tag{4}$$

  where $\tilde{\theta} = \frac{1-p/r}{1+p\cdot(1-1/q)}$, $\tilde{K}(r, q, p) = (2\theta)^{-\tilde{\theta}}$ and $\theta = \frac{1}{1+p\cdot(1-1/q)}$.

- $\forall\ \beta_0 > 0$:

$$\|u\|_{L^\infty(\mathbb{R})} \leq \left(\frac{2 + \beta_0}{4}\right)\|u\|^{1-\theta}_{L^{\beta_0}(\mathbb{R})}\|u_x\|^{\theta}_{L^2(\mathbb{R})}, \tag{5}$$

  where $\theta = \frac{1}{1+\frac{\beta_0}{2}}$.

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2.2 Basic Result

Theorem 2.2. *If $u(\cdot,t) \in L^{\infty}_{loc}([0,T^*), L^{\infty}(\mathbb{R}))$ solves problem (2) then*

1) $u(\cdot,t) \in L^{p_0}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) \;\; \forall\, t, \;\; 0 < t < T^*$

2) $\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})} \;\; \forall\, t, \;\; 0 < t < T^* \;\; (\forall\, q, \;\; p_0 \leq q \leq \infty)$

3) $\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u(\cdot,t_0)\|_{L^q(\mathbb{R})} \;\; \forall\, t, \;\; 0 \leq t_0 < t < T^*, \;\; \forall\, q, \;\; p_0 \leq q \leq \infty.$

Proof of (1). For simplicity, we will consider the case of positive solutions, which are known to be smooth. Let $\zeta \in C^2(\mathbb{R})$ be such that $\zeta(x) = 1 \;\forall\,|x| \leq 1$, $\zeta(x) = 0 \;\forall\,|x| \geq 2$, $0 \leq \zeta(x) \leq 1 \;\forall\, x \in \mathbb{R}$. Given $R > 0$, let ζ_R be the cut-off function given by $\zeta_R(x) = \zeta\left(\dfrac{x}{R}\right)$.

Let $p_0 \leq q < \infty$. Multipliyng the PDE at the initial value problem (2) by $qu^{q-1}\zeta_R(x)$ we have

$$\frac{\partial}{\partial t} u^q \zeta_R(x) + f(t,u)_x q u^{q-1} \zeta_R(x) = q u^{q-1} (u^{\alpha} u_x)_x \zeta_R(x).$$

Integrating on $\mathbb{R} \times [0,t]$,

$$\int_{|x|<2R} u(x,t)\zeta_R(x)dx + q(q-1)\int_0^t \int_{|x|<2R} u^{q+\alpha-2} u_x^2 \zeta_R(x) dx d\tau = \int_{|x|<2R} u_0(x)^q \zeta_R(x) dx$$
$$+ \;\; \frac{q}{q+\alpha}\int_0^t \int_{|x|<2R} u^{q+\alpha}\zeta_R''(x) dx d\tau - \int_0^t \int_{|x|<2R} f(t,u)_x q u^{q-1} \zeta_R(x) dx d\tau.$$

Next, integrating by parts and then, letting $R \to \infty$, we get the result.

Proof of (2) and (3). Again, we consider the simpler case of positive solutions. Defining $F(t,U) = \displaystyle\int_0^U f'(t,v) v^{q-1} dv$, then equation (4) can be written as

$$\int_{|x|<2R} u(x,t)\zeta_R(x)dx + q(q-1)\int_0^t \int_{|x|<2R} u^{q+\alpha-2} u_x^2 \zeta_R(x) dx d\tau = \int_{|x|<2R} u_0(x)^q \zeta_R(x) dx$$
$$+ \;\; \frac{q}{q+\alpha}\int_0^t \int_{|x|<2R} u^{q+\alpha}\zeta_R''(x) dx d\tau + q\int_0^t \int_{|x|<2R} F(t,u)\, \zeta_R'(x) dx d\tau.$$

Observe that

$$\begin{aligned} \int_0^t \int_{R<|x|<2R} F(t,u)\zeta_R'(x) dx d\tau &\leq \int_0^t \int_{R<|x|<2R} |F(t,u)||\zeta_R'(x)| dx d\tau \\ &\leq \frac{M}{R}\int_0^t \int_{R<|x|<2R} |u(x,\tau)|^q dx d\tau \to 0, \end{aligned}$$

when $R \to \infty$, where M is a constant. Then

$$\begin{aligned} \int_{\mathbb{R}} u(x,t) dx &\leq \int_{\mathbb{R}} u(x,t)^q dx + q(q-1)\int_0^t \int_{\mathbb{R}} u(x,\tau)^{q+\alpha-2} u_x^2 dx d\tau \\ &\leq \int_{\mathbb{R}} u_0(x)^q dx. \end{aligned}$$

Therefore, we get

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})} \ \forall\, q,\ p_0 \leq q < \infty,\ \forall\, t,\ 0 < t < T^*,$$

and

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \ \forall\, t,\ 0 < t < T^*.$$

as claimed. In particular, solutions of the initial-value problem (2) are globally defined (i.e., $T^* = \infty$).

3 Main Theorems

Theorem 3.1. *If $u(\cdot,t) \in L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R}))$ solves problem (2) with $u_0 > 0$, then*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\|u(\cdot,t_0)\|^{\frac{2p_0}{\alpha+2p_0}}_{L^{p_0}(\mathbb{R})}(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \qquad \forall\, t,\ 0 \leq t_0 < t, \tag{6}$$

where $K(\alpha,p_0)$ is a constant that only depends on α and p_0.

Proof. Let $\psi \in C^1(\mathbb{R})$ be monotonically increasing such that $\psi(u) = 1\ \forall\ u \geq 1$, $\psi(0) = 0$ and $\psi(u) = -1,\ \forall\ u \leq -1$. Taking $\delta > 0$, let us define $\psi_\delta(u) = \psi(\frac{u}{\delta})$ and $\phi_\delta(u) = L_\delta(u)^q$, $q \geq 2$, where $L_\delta(u) = \int_0^u \psi_\delta(v)dv,\ L_\delta \in C^2(\mathbb{R})$. Let $\gamma > 0$. Multiplying the equation in (2) above by $(t-t_0)^\gamma \phi'_\delta(u)$ and integrating in $\mathbb{R}\times[t_0,t]$, we get

$$\begin{aligned} &\int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^\gamma\phi'_\delta(u(x,\tau))u(x,\tau)_\tau dxd\tau + \int_{t_0}^t\int_{\mathbb{R}}(t-t_0)^\gamma\phi'_\delta(u(x,\tau))(f(\tau,u))_x dxd\tau \\ = &\int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^\gamma\phi'_\delta(u(x,\tau))(|u|^\alpha u_x)_x dxd\tau \end{aligned}$$

By Fubini's theorem, integrating by parts, using an appropriate cut-off function and taking $\delta \to 0$, this gives

$$\begin{aligned} &(t-t_0)^\gamma\|u(x,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\int_{t_0}^t(\tau-t_0)^\gamma\int_{\mathbb{R}}|u(x,\tau)|^{\alpha+q-2}(u_x)^2dxd\tau \\ \leq\ &\gamma\int_{t_0}^t(\tau-t_0)^{\gamma-1}\|u(x,\tau)\|^q_{L^q(\mathbb{R})}d\tau \end{aligned}$$

Introducing

$$v^{[q]}(x,t) := \begin{cases} u(x,t) \text{ se } \sigma = \alpha+q = 2, \\ \\ |u(x,t)|^{\sigma/2},\ \sigma = \alpha+q > 2, \end{cases}$$

we then have

$$\begin{aligned} &(t-t_0)^\gamma\|v^{[q]}(\cdot,t)\|^{2q/\sigma}_{L^{2q/\sigma}(\mathbb{R})} + \frac{4q(q-1)}{(\alpha+q)^2}\int_{t_0}^t(\tau-t_0)^\gamma\|v^{[q]}_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}d\tau \\ \leq\ &\gamma\int_{t_0}^t(\tau-t_0)^{\gamma-1}\|v^{[q]}(\cdot,\tau)\|^{2q/\sigma}_{L^{2q/\sigma}(\mathbb{R})}d\tau \end{aligned}$$

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Using Hölder, Nirenberg-Sobolev-Gagliardo II (5), with $\beta_0 = 2q/\sigma$ and $q = 2p_0$, and Nirenberg-Sobolev-Gagliardo I (4) inequalities, we obtain the supnorm estimate (6). □

Let $w(\cdot,t)$ be the solution of (2) with initial condition $w_0 = u_0^+ + \epsilon\zeta$ for some $\epsilon > 0$, where u_0^+ denotes the positive part of u_0 and $\zeta \in C^0(\mathbb{R}) \cap L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. That is, $w_0 \geq u_0$. Then, by the Theorem of Comparison (2.1), $u(\cdot,t) \leq w(\cdot,t)$, for all $0 \leq t < T$ and

$$\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\|w(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})}^{\frac{2p_0}{\alpha+2p_0}}(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \quad \forall\, t,\ 0 \leq t_0 < t, \tag{7}$$

Now let $z(\cdot,t)$ be the solution of (2) with initial condition $z_0 = -u_0^- - \epsilon\zeta$ for some $\epsilon > 0$, where u_0^- denotes the negative part of u_0. That is, $z_0 \leq u_0$. Then, by the Theorem of Comparison (2.1), $u(\cdot,t) \geq w(\cdot,t)$, for all t, $0 \leq t < T$ and

$$\|z(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\|z(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})}^{\frac{2p_0}{\alpha+2p_0}}(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \quad \forall\, t,\ 0 \leq t_0 < t, \tag{8}$$

By (7) and (8), we have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\max\{\|u_0^+\|,\|u_0^-\|\}_{L^{p_0}(\mathbb{R})}^{\frac{2p_0}{\alpha+2p_0}}(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \quad \forall\, t,\ 0 \leq t_0 < t.$$

This proves the following theorem:

Theorem 3.2. *If $u(\cdot,t) \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}))$ solves problem (2), then*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\max\{\|u_0^+\|,\|u_0^-\|\}_{L^{p_0}(\mathbb{R})}^{\frac{2p_0}{\alpha+2p_0}}(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \quad \forall\, t,\ 0 \leq t_0 < t,$$

where $K(\alpha,p_0)$ is a constant that only depends on α and p_0 and u_0^+ and u_0^- denote the positive and negative part of u_0, respectively.

4 Conclusions

We derived a supnorm estimate for the solution of the porous medium equation (2) with no restriction on the sign of u_0.

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