

SELECTION OF CALIBRATED SUBACTION WHEN TEMPERATURE GOES TO ZERO IN THE DISCOUNTED PROBLEM

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ABSTRACT. Consider $T(x) = dx \pmod{1}$ acting on S^1 , a Lipschitz potential $A : S^1 \rightarrow \mathbb{R}$, $0 < \lambda < 1$ and the unique function $b_\lambda : S^1 \rightarrow \mathbb{R}$ satisfying $b_\lambda(x) = \max_{T(y)=x} \{\lambda b_\lambda(y) + A(y)\}$.

We will show that, when $\lambda \rightarrow 1$, the function $b_\lambda - \frac{m(A)}{1-\lambda}$ converges uniformly to the calibrated subaction $V(x) = \max_{\mu \in \mathcal{M}} \int S(y, x) d\mu(y)$, where S is the Mañé potential, \mathcal{M} is the set of invariant probabilities with support on the Aubry set and $m(A) = \sup_{\mu \in \mathcal{M}} \int A d\mu$.

For $\beta > 0$ and $\lambda \in (0, 1)$, there exists a unique fixed point $u_{\lambda, \beta} : S^1 \rightarrow \mathbb{R}$ for the equation $e^{u_{\lambda, \beta}(x)} = \sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda, \beta}(y)}$. It is known that as $\lambda \rightarrow 1$ the family $e^{[u_{\lambda, \beta} - \sup u_{\lambda, \beta}]}$ converges uniformly to the main eigenfunction ϕ_β for the Ruelle operator associated to βA . We consider $\lambda = \lambda(\beta)$, $\beta(1 - \lambda(\beta)) \rightarrow +\infty$ and $\lambda(\beta) \rightarrow 1$, as $\beta \rightarrow \infty$. Under these hypotheses we will show that $\frac{1}{\beta}(u_{\lambda, \beta} - \frac{P(\beta A)}{1-\lambda})$ converges uniformly to the above V , as $\beta \rightarrow \infty$. The parameter β represents the inverse of temperature in Statistical Mechanics and $\beta \rightarrow \infty$ means that we are considering that the temperature goes to zero. Under these conditions we get selection of subaction when $\beta \rightarrow \infty$.

1. Introduction. Consider $T(x) = dx \pmod{1}$ acting on S^1 and a Lipschitz potential $A : S^1 \rightarrow \mathbb{R}$. We denote by

$$m(A) = \sup \left\{ \int A d\mu, \text{ where } \mu \text{ is invariant for } T \right\}. \quad (1)$$

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Any invariant μ which attains this supremum is called an **A-maximizing probability**.

A **subaction** for A is a continuous function $D : S^1 \rightarrow \mathbb{R}$ such that for all $x \in S^1$,

$$D(T(x)) \geq A(x) + D(x) - m(A).$$

It is called a **calibrated subaction** if for all $y \in S^1$,

$$D(y) = \max_{T(x)=y} \{ A(x) + D(x) - m(A) \}.$$

We refer the reader to [2, 13, 6, 19, 15, 21] for general results on Ergodic Optimization and Thermodynamic Formalism.

Maximizing probabilities and calibrated subactions are dual objects in Ergodic Optimization. On the one hand $m(A)$ satisfies (1), but on the other hand

$$\begin{aligned} m(A) &= \inf_{H \text{ continuous}} \left(\sup_{x \in S^1} (A(x) + H(x) - H(T(x))) \right) \\ &= \sup_{x \in S^1} (A(x) + D(x) - D(T(x))) \end{aligned}$$

for any calibrated subaction D . Furthermore, it is known that a calibrated subaction can help to identify the support of the maximizing probabilities for A (see [6] or [2]).

A natural problem is: how to find subactions? Note that we need to have at hand the exact value $m(A)$ in order to verify if a specific candidate D is indeed a calibrated subaction. The discounted method, which is described below, can be quite useful in order to get a good approximation (via iteration of a contraction) of a calibrated subaction without the knowledge of the value $m(A)$.

For each fixed $\lambda \in (0, 1)$, consider the function $b = b_\lambda : S^1 \rightarrow \mathbb{R}$ satisfying the equation

$$b(x) = \max_{T(y)=x} \{ \lambda b(y) + A(y) \}. \tag{2}$$

This function is unique and we call b_λ the **λ -calibrated subaction** for A (see for instance Theorem 1 in [4], [3] or [17]).

The solution b_λ can be obtained in the following way: consider $\tau_j, j = 1, \dots, d$ the inverse branches of T . For $\lambda < 1$, consider

$$S_{\lambda,A}(x, a) = \sum_{k=0}^{\infty} \lambda^k A((\tau_{a_k} \circ \tau_{a_{k-1}} \circ \dots \circ \tau_{a_0})(x)),$$

where $a \in \{1, 2, \dots, d\}^{\mathbb{N}}$ (see expression (6) in [17] or [22]). Then, the solution of (2) is given by $b_\lambda(x) = \sup_{a \in \{1, 2, \dots, d\}^{\mathbb{N}}} S_\lambda(x, a)$ (see section 3 in [17]). For fixed x , as the function $S_{\lambda,A}(x, \cdot)$ is continuous in the second coordinate, there exists some **a realizing the supremum**, which is called a $b_\lambda(x)$ **realizer**. Denoting $a = a_0 a_1 \dots$ we have that for any k :

$$b_\lambda(\tau_{a_k} \circ \dots \circ \tau_{a_0} x) = \lambda b_\lambda(\tau_{a_{k+1}} \circ \dots \circ \tau_{a_0} x) + A(\tau_{a_{k+1}} \circ \dots \circ \tau_{a_0} x) \tag{3}$$

and

$$b_\lambda(x) = \lambda^{k+1} b_\lambda(\tau_{a_k} \circ \dots \circ \tau_{a_0} x) + A(\tau_{a_0} x) + \dots + \lambda^k A(\tau_{a_k} \circ \dots \circ \tau_{a_0} x),$$

while for a general $a \in \{1, 2, \dots, d\}^{\mathbb{N}}$ we have that

$$b_\lambda(x) \geq \lambda^{k+1} b_\lambda(\tau_{a_k} \circ \dots \circ \tau_{a_0} x) + A(\tau_{a_0} x) + \dots + \lambda^k A(\tau_{a_k} \circ \dots \circ \tau_{a_0} x).$$

The above family $b_\lambda, 0 < \lambda < 1$, is equicontinuous. It is known that any convergent subsequence $b_{\lambda_n} - \sup b_{\lambda_n}, \lambda_n \rightarrow 1$, determines in the limit a calibrated

subaction for A (see Theorem 1 in [4], [1] or [22]). This procedure, known as the discounted method, is borrowed from Control Theory. The function b_λ is obtained as a fixed point of a contraction (see [17]), which is, of course, computationally convenient (note that $m(A)$ does not appear on expression (2)).

In this work the constant $\sup b_\lambda$ will be replaced by $\frac{m(A)}{1-\lambda}$. Our first aim is to describe a certain calibrated subaction as the limit of $b_\lambda - \frac{m(A)}{1-\lambda}$, as $\lambda \rightarrow 1$. In this way the limit will not depend of the convergent subsequence. Later we will consider eigenfunctions of the Ruelle operator and selection of subaction via the limit when temperature goes to zero (see [6] and [2]).

A point x in S^1 is called **A -non-wandering**, if for any $\epsilon > 0$, there exists $z \in S^1$ and $n \geq 1$, such that, $d(z, x) < \epsilon$, $T^n(z) = x$, and $|\sum_{j=0}^{n-1} (A - m(A))(T^j(z))| < \epsilon$. The **Aubry set** for A is the set of A -non-wandering points in S^1 and is denoted by $\Omega(A)$. Any invariant probability with support inside the Aubry set is A -maximizing (see [6], [7] or section 3 in [8]). We denote by \mathcal{M} the set of A -maximizing probabilities.

Consider the function S given by Definition 22 in [6] (see also [12] and [13]):

$$S(y, x) = \limsup_{\epsilon \rightarrow 0} \{S_n(A - m(A))(y') \mid n \geq 1, d(y', y) \leq \epsilon, T^n(y') = x\},$$

where

$$S_n(A - m(A))(y') = \sum_{j=0}^{n-1} (A - m(A))(T^j(y')). \quad (4)$$

This function is called the **Mañé potential**.

Remark 1. For $y \in \Omega(A)$ fixed, the function $S(y, \cdot)$ is a calibrated subaction (see [6], Proposition 5.2 in [13] or [12]) with the same Lipschitz constant as A . As $S(y, y) = 0$ for $y \in \Omega(A)$, we get that for fixed $y \in \Omega(A)$, the function $x \rightarrow S(y, x)$ is bounded.

The function $S(x, y)$ is not continuous as a function of (x, y) (see Proposition 3.5 in [7])

We define, for each $x \in S^1$, the subaction

$$V(x) = \max_{\mu \in \mathcal{M}} \int S(y, x) d\mu(y).$$

Our first goal is to prove the following theorem.

Theorem 1.1. *If A is Lipschitz, when $\lambda \rightarrow 1$, the function $U_\lambda := \left(b_\lambda - \frac{m(A)}{1-\lambda}\right)$ converges uniformly to V .*

The above theorem assures that the subaction V is very special among the possible ones. We will prove that V is calibrated (see corollary 1).

This first part of the paper follows the ideas in [10] and [11] and obtains analogous results in the framework of ergodic optimization.

From now on we describe the second part of our paper which considers a limit when temperature goes to zero - this result has a quite different nature when compared with [10] and [11].

It is known (see for instance section 7 in [4], [19] or [18]) that, for fixed $\beta > 0$ and $\lambda \in (0, 1)$, there exists a unique fixed point $u_{\lambda, \beta} : S^1 \rightarrow \mathbb{R}$ for the equation

$$e^{u_{\lambda, \beta}(x)} = \sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda, \beta}(y)}.$$

For fixed β , the family $u_{\lambda,\beta}$ is equicontinuous in $\lambda \in (0, 1)$. Moreover, $e^{u_{\lambda,\beta} - \sup u_{\lambda,\beta}}$ converges as $\lambda \rightarrow 1$. The limit function ϕ_β is an eigenfunction associated to the main eigenvalue (spectral radius) of the Ruelle operator $L_{\beta A}$ associated to the potential βA (see Lemma 3.2).

A general reference for the Ruelle operator, eigenfunctions and equilibrium states is [21].

The parameter β represents the inverse of temperature in Statistical Mechanics

From [6] we get that $\frac{1}{\beta} \log \phi_\beta$ is equicontinuous and any limit of a convergent subsequence of $\frac{1}{\beta_n} \log \phi_{\beta_n}$, $\beta_n \rightarrow \infty$, is a calibrated subaction. The limit when $\beta \rightarrow \infty$ is known as the limit when temperature goes to zero (see [2]).

In the case the limit $\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \phi_\beta$ exists (not via subsequence) we say that there exists selection of subaction when temperature goes to zero.

For the case of locally constant potentials a precise description is presented in [5] or [16]. The most comprehensive result of this kind of convergence for Lipschitz potentials is Theorem 16 in [14].

In the **standard** terminology we say that there exists selection of subaction at zero temperature if the limit of the function $\frac{1}{\beta} \log \phi_\beta$ exists, when $\beta \rightarrow \infty$ (see [2] for general results).

For a fixed value β , the function $u_{\lambda,\beta}$ is obtained as a fixed point of a contraction. In this way, one can get a computable method (via iteration of the contraction) for getting good approximations of the main eigenfunction ϕ_β (taking λ close to 1).

We denote by $\alpha(\beta)$ the main eigenvalue which is associated to ϕ_β . The pressure $P(\beta A) = \log(\alpha(\beta))$ is equal to

$$\sup_{\mu \text{ invariant for } T} \left\{ \int h(\mu) + \beta \int A d\mu \right\},$$

where $h(\mu)$ is the Kolmogorov entropy of μ (see [21]).

Our second goal is to show:

Theorem 1.2. *Suppose A is Lipschitz and there exists a relation between β and λ so that $\beta(1 - \lambda(\beta)) \rightarrow +\infty$ and $\lambda(\beta) \rightarrow 1$, as $\beta \rightarrow \infty$. Then, when $\beta \rightarrow \infty$, $\frac{1}{\beta}(u_{\lambda,\beta} - \frac{P(\beta A)}{1 - \lambda})$ converges uniformly to V .*

The above result requires a control of the velocity such that $(1 - \lambda(\beta))$ goes to zero, vis-à-vis, the growth of β to infinity. We will show on the end of the paper a counterexample proving that $\frac{1}{\beta} \log \phi_\beta$ does not always converge to V . This shows that is really necessary some kind of joint control of the parameters β and λ (as considered in the above theorem).

The last theorem shows the existence of selection of subaction when temperature goes to zero (but in a sense which is not the standard form).

In the next section we will prove Theorem 1.1 and in the last section we will prove Theorem 1.2.

The results that we get here are also true if one considers $T(x) = \sigma(x)$ acting on Bernoulli space $\{1, \dots, d\}^{\mathbb{N}}$.

2. The limit of the λ -calibrated subaction, when $\lambda \rightarrow 1$. We start with the following result:

Lemma 2.1. *V is a subaction for A and also satisfies:*

1. $\int V d\mu \geq 0$, for any $\mu \in \mathcal{M}$.

2. If w is a calibrated subaction, such that, $\int w d\mu \geq 0$, for any $\mu \in \mathcal{M}$, then $w \geq V$.

Proof. From Remark 1 we get that, for each $y \in \Omega(A)$, the function $S(y, \cdot)$ is Lipschitz continuous and the Lipschitz constant does not depend of y . Therefore V is Lipschitz. For any $x, y \in S^1$ we have $S(y, T(x)) \geq A(x) - m(A) + S(y, x)$. Then, for any $\mu \in \mathcal{M}$ we get $\int S(y, T(x)) d\mu(y) \geq A(x) - m(A) + \int S(y, x) d\mu(y)$ which proves that $V(T(x)) \geq A(x) + V(x) - m(A)$.

Proof of 1. It is known (Prop. 23 page 1395 [6]) that $S(y, x) + S(x, y) \leq 0$. We say that x and y in $\Omega(A)$ are in the same static class if $S(y, x) + S(x, y) = 0$.

For an ergodic maximizing measure μ we claim that for μ -almost every pair x and y these two points are in the same static class.

First note that as A is Lipschitz the function S can be also written as

$$S(y, x) = \limsup_{\epsilon \rightarrow 0} \{S_n(A - m(A))(y') \mid n \geq 1, d(y', y) \leq \epsilon, d(T^n(y'), x) \leq \epsilon\}.$$

Now we will show the claim: given an ergodic probability μ , an integrable function F and a Borel set B , such that $\mu(B) > 0$, denote by \tilde{B} the set of points p in B , such that, for all ϵ , there exists an integer $N > 0$, with the properties:

$$T^N(p) \in B \quad \text{and} \quad \left| \sum_{j=0}^{N-1} F(T^j(p)) - N \int F d\mu \right| < \epsilon.$$

It is known that $\mu(B) = \mu(\tilde{B})$ (see [20]). This results is known as Atkinson's Theorem.

Let $x, y \in \text{supp}(\mu)$ where x is a Birkhoff point for A and fix $\rho > 0$. As $\mu(B(y, \rho)) > 0$, for some n we have $T^n(x) \in B(y, \rho)$. Let $\rho' < \rho$ be such that for any $x' \in B(x, \rho')$ we have $d(T^j(x), T^j(x')) < \rho$, for all $j \in \{0, \dots, n\}$. Consider the set $B = B(x, \rho')$. There exists $p \in B(x, \rho')$, such that, for all ϵ , there exists an integer $N > 0$, satisfying $T^N(p) \in B(x, \rho')$ and $|\sum_{j=0}^{N-1} A(T^j(p)) - N \int A d\mu| < \epsilon$. As this is true for a. e. $p \in B(x, \rho')$ we can suppose that $N > n$, replacing p by $T^N(p)$ in the case it is necessary. It follows that $S(x, y) + S(y, x) = 0$ which proves the claim.

As $S(x, y) = -S(y, x)$ for any $x, y \in \text{supp}(\mu)$, it follows from item 1 of Proposition 3.1 in [7] that there exists $Q > 0$, such that, $-Q < S(x, y) < Q$, for any $x, y \in \text{supp}(\mu)$.

We have $\int \int S(x, y) + S(y, x) d\mu(x) d\mu(y) = 0$, for any $\mu \in \mathcal{M}$ which is ergodic, then

$$\int \int S(x, y) d\mu(x) d\mu(y) = 0,$$

using Fubini's Theorem ($S(x, y)$ is integrable by last paragraph). It follows that

$$\int V(x) d\mu(x) \geq \int \left[\int S(y, x) d\mu(y) \right] d\mu(x) = 0,$$

for any ergodic probability $\mu \in \mathcal{M}$. The same inequality for a general $\mu \in \mathcal{M}$ follows from the ergodic decomposition theorem.

Proof of 2. We know (see Theorem 10 in [12]) that, for any calibrated subaction w and any x ,

$$w(x) = \sup_{y \in \Omega(A)} \{w(y) + S(y, x)\}.$$

Therefore, for $\mu \in \mathcal{M}$, such that, $\int w(y) d\mu(y) \geq 0$, we get

$$w(x) \geq w(x) - \int w(y) d\mu(y) \geq \int S(y, x) d\mu(y).$$

If $\int w(y) d\mu(y) \geq 0$ for any $\mu \in \mathcal{M}$ we obtain $w \geq V$. □

Given λ, y and a $b_\lambda(y)$ realizer $a = a_0 a_1 \dots$, consider the probability

$$\mu_\lambda^y = (1 - \lambda) \sum_{n=0}^\infty \lambda^n \delta_{(\tau_{a_n} \circ \dots \circ \tau_{a_0})(y)}. \tag{5}$$

From (6) in [17] we get that $b_\lambda(y) = S_{\lambda,A}(y, a(y))$, where $a(y)$ is a realizer of y , then for any y we have that

$$b_\lambda(y) = \frac{1}{(1 - \lambda)} \int A d\mu_\lambda^y.$$

We will show that any limit probability of μ_λ^y , as $\lambda \rightarrow 1$, belongs to \mathcal{M} .

Lemma 2.2. *For any continuous function $w : S^1 \rightarrow \mathbb{R}$, and probability μ_λ^y as above, we get*

$$\int w \circ T d\mu_\lambda^y - \int w d\mu_\lambda^y(x) = (1 - \lambda)(w(y) - \int w d\mu_\lambda^y). \tag{6}$$

Proof. Indeed,

$$\begin{aligned} & \int (w \circ T - w) d\mu_\lambda^y \\ &= (1 - \lambda) \left[\sum_{n=0}^\infty \lambda^n w(T((\tau_{a_n} \circ \dots \circ \tau_{a_0})(y))) - \sum_{n=0}^\infty \lambda^n w((\tau_{a_n} \circ \dots \circ \tau_{a_0})(y)) \right] \\ &= (1 - \lambda) \left[w(y) + \sum_{n=1}^\infty \lambda^n w((\tau_{a_{n-1}} \circ \dots \circ \tau_{a_0})(y)) - \sum_{n=0}^\infty \lambda^n w((\tau_{a_n} \circ \dots \circ \tau_{a_0})(y)) \right] \\ &= (1 - \lambda) \left[w(y) + \sum_{n=0}^\infty \lambda^{n+1} w((\tau_{a_n} \circ \dots \circ \tau_{a_0})(y)) - \sum_{n=0}^\infty \lambda^n w((\tau_{a_n} \circ \dots \circ \tau_{a_0})(y)) \right] \\ &= (1 - \lambda) \left[w(y) + (\lambda - 1) \sum_{n=0}^\infty \lambda^n w((\tau_{a_n} \circ \dots \circ \tau_{a_0})(y)) \right] \\ &= (1 - \lambda) \left[w(y) - \int w d\mu_\lambda^y \right]. \end{aligned}$$

□

Lemma 2.3. *Given $y \in S^1$, any accumulation probability μ_∞ , in the weak* topology, of a convergent subsequence $\mu_{\lambda_i}^y, \lambda_i \rightarrow 1$, belongs to \mathcal{M} .*

Proof. It follows from above lemma that μ_∞ is invariant. Moreover, by (3) and definition of μ_λ^y we have that

$$\int b_{\lambda_i}(T(x)) - \lambda_i b_{\lambda_i}(x) - A(x) d\mu_{\lambda_i}^y(x) = 0.$$

Then,

$$\int b_{\lambda_i}(T(x)) - b_{\lambda_i}(x) d\mu_{\lambda_i}^y(x) + \int (1 - \lambda_i) b_{\lambda_i}(x) - A(x) d\mu_{\lambda_i}^y(x) = 0.$$

When, $i \rightarrow \infty$ the left integral converges to zero. Therefore,

$$\lim_{i \rightarrow \infty} \int (1 - \lambda_i) b_{\lambda_i}(x) - A(x) d\mu_{\lambda_i}^y(x) = 0.$$

It is known (see for instance end of Theorem 11 in [3] or [19]) that $(1 - \lambda) \inf b_\lambda \rightarrow m(A)$, uniformly with $\lambda \rightarrow 1$. Therefore,

$$\begin{aligned} \int (m(A) - A) d\mu_\infty &= \lim_{i \rightarrow \infty} \int [(1 - \lambda_i) \inf b_{\lambda_i} - A(x)] d\mu_{\lambda_i}^y(x) \leq \\ \lim_{i \rightarrow \infty} \int [(1 - \lambda_i) b_{\lambda_i}(x) - A(x)] d\mu_{\lambda_i}^y(x) &= 0, \end{aligned}$$

proving the claim. □

Lemma 2.4. *The family of functions*

$$U_\lambda := b_\lambda - \frac{m(A)}{1 - \lambda}$$

is equicontinuous and uniformly bounded. Furthermore, for any maximizing probability $\mu \in \mathcal{M}$ we have

$$\int U_\lambda d\mu \geq 0, \quad \forall \lambda \in (0, 1),$$

and for any subaction w we have

$$U_\lambda(y) \leq w(y) - \int w d\mu_\lambda^y, \quad \forall \lambda \in (0, 1), \forall y \in S^1.$$

Proof. As $b_\lambda(T(z)) - \lambda b_\lambda(z) - A(z) \geq 0$, for any maximizing probability $\mu \in \mathcal{M}$ we have that

$$\int U_\lambda d\mu = \int b_\lambda - \frac{m(A)}{1 - \lambda} d\mu = \frac{1}{1 - \lambda} \int (1 - \lambda) b_\lambda - m(A) d\mu \geq 0. \tag{7}$$

In particular this proves that there exists $x_\lambda \in S^1$, such that, $U_\lambda(x_\lambda) \geq 0$.

On the other hand, if w is a subaction we have that

$$A - m(A) \leq w \circ T - w,$$

therefore, using (6), for any λ and y we have

$$\begin{aligned} U_\lambda(y) &= \frac{1}{1 - \lambda} \int A d\mu_\lambda^y - \frac{m(A)}{1 - \lambda} \leq \frac{1}{1 - \lambda} \left[\int w \circ T d\mu_\lambda^y - \int w d\mu_\lambda^y \right] \\ &= w(y) - \int w d\mu_\lambda^y. \end{aligned}$$

Therefore, the functions $U_{\lambda_n}(x)$ are uniformly bounded above.

As the functions b_λ are equicontinuous in $\lambda < 1$ (see [17]), the family of functions U_λ is equicontinuous. As U_λ are uniformly bounded above and $U_\lambda(x_\lambda) \geq 0$ we conclude that this family is also uniformly bounded. □

Lemma 2.5. *Any limit of $U_\lambda := b_\lambda - \frac{m(A)}{1 - \lambda}$, as $\lambda \rightarrow 1$, is a calibrated subaction.*

Proof. Let U be the limit of the subsequence $U_{\lambda_n} = b_{\lambda_n} - \frac{m(A)}{1 - \lambda_n}$, when $n \rightarrow \infty$. From (2) we get

$$b_\lambda(x) - \frac{m(A)}{1 - \lambda} = \sup_{T(y)=x} \lambda \left[b_\lambda(y) - \frac{m(A)}{1 - \lambda} \right] + A(y) - m(A),$$

that is,

$$U_\lambda(x) = \sup_{T(y)=x} \lambda U_\lambda(y) + A(y) - m(A).$$

Then, as $\lambda_n \rightarrow 1$ we conclude that U is a subaction. Furthermore, for any point $x \in S^1$, there is some point $y_0 \in T^{-1}(x)$ attaining the supremum of $\sup_{T(y)=x} \lambda_n U_{\lambda_n}(y) + A(y) - m(A)$, for infinitely many values of n . In this way we get

$$U(x) = A(y_0) + U(y_0) - m(A).$$

This proves that U is calibrated. □

Proof of Theorem 1.1. We denote U any limit of $U_{\lambda_n} := b_{\lambda_n} - \frac{m(A)}{1-\lambda_n}$, when $n \rightarrow \infty$. We know that U is a calibrated subaction and we want to show that $U = V$.

From lemma 2.4, for any maximizing probability $\mu \in \mathcal{M}$, we have that $\int U d\mu \geq 0$. It follows from lemma 2.1 that $U \geq V$.

Now we will show that $U \leq V$. From lemma 2.1 the subaction V satisfies $\int V d\mu \geq 0$, for any $\mu \in \mathcal{M}$, and from lemma 2.4 we get, for any y and λ , the inequality

$$U_\lambda(y) \leq V(y) - \int V d\mu_\lambda^y.$$

If λ_{n_i} is a subsequence of λ_n , such that, $\mu_{\lambda_{n_i}}^y \rightarrow \mu_\infty$, then, from lemma 2.3 we have that $\mu_\infty \in \mathcal{M}$. Therefore, we finally get that

$$U(y) = \lim_{\lambda_{n_i} \rightarrow 1} U_{\lambda_{n_i}}(y) \leq V(y) - \int V d\mu_\infty \leq V(y).$$

□

Corollary 1. *V is a calibrated subaction.*

Proof. It is a consequence of lemma 2.5 and Theorem 1.1. □

3. Selection for the zero temperature case. Now we will prove Theorem 1.2.

We consider for each $\beta > 0$ (the inverse of the temperature) and for $\lambda < 1$ the operator

$$\mathfrak{S}_{\lambda,\beta}(u)(x) = \log \left(\sum_{T(y)=x} e^{\beta A(y) + \lambda u(y)} \right).$$

It is known that $\mathfrak{S}_{\lambda,\beta}$ is a contraction map (see for instance sections 6 and 7 in [4], [18] or [19]) with a unique fixed point $u_{\lambda,\beta}$ satisfying

$$e^{u_{\lambda,\beta}(x)} = \sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda,\beta}(y)}.$$

For each fixed β , the family $u_{\lambda,\beta}$ is equicontinuous on $0 < \lambda < 1$ with uniform constant given by $\beta Lip(A)$. Therefore, for each β fixed the function $u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda}$ is Lipschitz continuous with Lipschitz constant $\beta Lip(A)$.

Lemma 3.1.

$$\inf u_{\lambda,\beta} \leq \frac{P(\beta A)}{1-\lambda} \leq \sup u_{\lambda,\beta}.$$

Proof. By definition

$$e^{u_{\lambda,\beta}(x)} = \sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda,\beta}(y)}.$$

Then, it follows that

$$\sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda,\beta}(y) - u_{\lambda,\beta}(x)} = 1.$$

Therefore,

$$P(\beta A + \lambda u_{\lambda,\beta} - u_{\lambda,\beta} \circ T) = 0.$$

Let μ_0 be the equilibrium probability for $\beta A + \lambda u_{\lambda,\beta} - u_{\lambda,\beta} \circ T$.

Then,

$$\begin{aligned} (1 - \lambda) \int u_{\lambda,\beta} d\mu_0 &= (1 - \lambda) \int u_{\lambda,\beta} d\mu_0 + \int (\beta A + \lambda u_{\lambda,\beta} - u_{\lambda,\beta} \circ T) d\mu_0 + h(\mu_0) \\ &= \int \beta A d\mu_0 + h(\mu_0) \leq P(\beta A). \end{aligned}$$

It follows that

$$\inf u_{\lambda,\beta} \leq \frac{P(\beta A)}{1 - \lambda}.$$

On the other hand, if μ_1 is the equilibrium probability of βA , then

$$\begin{aligned} (1 - \lambda) \int u_{\lambda,\beta} d\mu_1 &= (1 - \lambda) \int u_{\lambda,\beta} d\mu_1 + P(\beta A + \lambda u_{\lambda,\beta} - u_{\lambda,\beta} \circ T) \\ &\geq (1 - \lambda) \int u_{\lambda,\beta} d\mu_1 + \int (\beta A + \lambda u_{\lambda,\beta} - u_{\lambda,\beta} \circ T) d\mu_1 + h(\mu_1) \\ &= \int \beta A d\mu_1 + h(\mu_1) = P(\beta A). \end{aligned}$$

Therefore,

$$\sup u_{\lambda,\beta} \geq \frac{P(\beta A)}{1 - \lambda}.$$

□

Lemma 3.2. *For each fixed β , the functions $u_{\lambda,\beta}^* := u_{\lambda,\beta} - \frac{P(\beta A)}{1 - \lambda}$ are Lipschitz functions, with the same Lipschitz constant $H = \beta \text{Lip}(A)$, and, moreover, uniformly bounded by $-H$ and H . They also satisfy*

$$e^{u_{\lambda,\beta}^*(x)} = e^{-P(\beta A)} \sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda,\beta}^*(y)}.$$

When $\lambda \rightarrow 1$, any accumulation function of $e^{u_{\lambda,\beta}^*}$ will be an eigenfunction of the Ruelle Operator $L_{\beta A}$ associated to the maximal eigenvalue $e^{P(\beta A)}$.

Proof. As $u_{\lambda,\beta}$ is equicontinuous the same is true for $u_{\lambda,\beta}^*$. Using the equicontinuity (with constant H) we have that for any x :

$$-H \leq u_{\lambda,\beta}(x) - \sup u_{\lambda,\beta} \leq u_{\lambda,\beta}(x) - \frac{P(\beta A)}{1 - \lambda} \leq u_{\lambda,\beta}(x) - \inf u_{\lambda,\beta} \leq H.$$

Furthermore,

$$\begin{aligned} e^{u_{\lambda,\beta}^*(x)} &= e^{u_{\lambda,\beta}(x) - \frac{P(\beta A)}{1 - \lambda}} \\ &= e^{u_{\lambda,\beta}(x) - \lambda \frac{P(\beta A)}{1 - \lambda} - P(\beta A)} \\ &= e^{-P(\beta A)} \sum_{T(y)=x} e^{\beta A(y) + \lambda (u_{\lambda,\beta}(y) - \frac{P(\beta A)}{1 - \lambda})} \end{aligned}$$

$$= e^{-P(\beta A)} \sum_{T(y)=x} e^{\beta A(y)+\lambda(u_{\lambda,\beta}^*(y))}.$$

If u_β is an accumulation function of the family $u_{\lambda,\beta}^*$ (when, $\lambda \rightarrow 1$), then, we have:

$$e^{u_\beta(x)} = e^{-P(\beta A)} \sum_{T(y)=x} e^{\beta A(y)+u_\beta(y)}.$$

□

Remark 2. It is known (Proposition 29 in [6]) that

$$\lim_{\beta \rightarrow \infty} (P(\beta A) - \beta m(A)) = \max_{\mu \in \mathcal{M}} h(\mu).$$

Therefore,

$$\lim_{\beta \rightarrow \infty} \frac{P(\beta A)}{\beta} = m(A),$$

and moreover

$$\frac{P(\beta A)}{\beta(1-\lambda)} - \frac{m(A)}{1-\lambda} \rightarrow 0,$$

when $\lambda \rightarrow 1$, $\beta \rightarrow \infty$ and $\beta(1-\lambda) \rightarrow \infty$.

This Remark will be used on the proof of Lemmas 11 and 12.

Lemma 3.3. Consider a fixed λ . Then, when $\beta \rightarrow \infty$, the unique possible accumulation point of the family $\frac{1}{\beta}u_{\lambda,\beta}$ is the function b_λ defined in (2). Moreover, we get that the unique accumulation point of $\frac{1}{\beta}u_{\lambda,\beta}^*$ is the function $b_\lambda - \frac{m(A)}{1-\lambda}$.

Proof. As $u_{\lambda,\beta}^*$ has Lipschitz constant $\beta Lip(A)$ and is bounded by $-\beta Lip(A)$ and $\beta Lip(A)$, the family $\frac{1}{\beta}u_{\lambda,\beta}^*$ is equicontinuous and uniformly bounded by $Lip(A)$. From the limit $\frac{P(\beta A)}{\beta(1-\lambda)} \rightarrow \frac{m(A)}{1-\lambda}$ (as $\beta \rightarrow \infty$) we conclude that (for fixed λ) the family $\frac{1}{\beta}u_{\lambda,\beta}$ is equicontinuous (with a constant $Lip(A)$) and uniformly bounded. As

$$\frac{1}{\beta}u_{\lambda,\beta}(x) = \frac{1}{\beta} \log \left(e^{u_{\lambda,\beta}(x)} \right) = \frac{1}{\beta} \log \left(\sum_{T(y)=x} e^{\beta A(y)+\lambda u_{\lambda,\beta}(y)} \right),$$

there is a unique accumulation point b of $\frac{1}{\beta}u_{\lambda,\beta}$ which satisfies

$$b(x) = \sup_{T(y)=x} [A(y) + \lambda b(y)],$$

that is $b = b_\lambda$. □

In the previous section we study the limit of $b_\lambda - \frac{m(A)}{1-\lambda}$. Now, we are interested in the limit of $\frac{1}{\beta}(u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda})$, when $\beta \rightarrow \infty$ and $\lambda \rightarrow 1$.

Lemma 3.4. When $\beta \rightarrow \infty$ and $\lambda \rightarrow 1$, $\frac{(1-\lambda)u_{\lambda,\beta}}{\beta}$ converges uniformly to $m(A)$. If U is a limit of some subsequence of the family $\frac{1}{\beta}(u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda})$, as $\beta \rightarrow \infty$ and $\lambda \rightarrow 1$, then U is a calibrated subaction.

Proof. As

$$\left| \frac{1}{\beta}(u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda}) \right| \leq Lip(A)$$

we get

$$\left| \frac{(1-\lambda)u_{\lambda,\beta}}{\beta} - \frac{P(\beta A)}{\beta} \right| \leq (1-\lambda)Lip(A).$$

As $\beta \rightarrow \infty$ and $\lambda \rightarrow 1$, we obtain

$$\frac{(1-\lambda)u_{\lambda,\beta}}{\beta} \rightarrow m(A)$$

uniformly, proving the first claim.

In order to prove the second claim we fix a point $x \in S^1$. Let

$$U = \lim_{\lambda_n \rightarrow 1, \beta_n \rightarrow \infty} \frac{1}{\beta_n} \left(u_{\lambda_n, \beta_n} - \frac{P(\beta_n A)}{1-\lambda_n} \right).$$

As

$$\sum_{b_0} e^{\beta A(\tau_{b_0} x) + \lambda u_{\lambda, \beta}(\tau_{b_0} x) - u_{\lambda, \beta}(x)} = 1,$$

we get, for any $b_0 \in \{1, \dots, d\}$,

$$\begin{aligned} 0 &\leq u_{\lambda_n, \beta_n}(x) - \lambda_n u_{\lambda_n, \beta_n}(\tau_{b_0} x) - \beta_n A(\tau_{b_0} x) \\ &= \left(u_{\lambda_n, \beta_n} - \frac{P(\beta_n A)}{1-\lambda_n} \right)(x) - \left(u_{\lambda_n, \beta_n} - \frac{P(\beta_n A)}{1-\lambda_n} \right)(\tau_{b_0} x) + \\ &\quad (1-\lambda_n)u_{\lambda_n, \beta_n}(\tau_{b_0} x) - \beta_n A(\tau_{b_0} x). \end{aligned}$$

Dividing the right side by β_n , taking $\beta_n \rightarrow \infty$ and $\lambda_n \rightarrow 1$, we get

$$0 \leq U(x) - U(\tau_{b_0} x) + m(A) - A(\tau_{b_0} x).$$

This shows that U is a subaction.

In order to show that U is calibrated, we fix for each λ and β a point $a = a_{\lambda, \beta}$ maximizing $\beta A(\tau_a x) + \lambda u_{\lambda, \beta}(\tau_a x) - u_{\lambda, \beta}(x)$. As

$$\sum_{b_0} e^{\beta A(\tau_{b_0} x) + \lambda u_{\lambda, \beta}(\tau_{b_0} x) - u_{\lambda, \beta}(x)} = 1,$$

we conclude that

$$0 \leq u_{\lambda_n, \beta_n}(x) - \lambda_n u_{\lambda_n, \beta_n}(\tau_a x) - \beta_n A(\tau_a x) \leq \log(d).$$

When $\beta_n \rightarrow \infty$ and $\lambda_n \rightarrow 1$, some $a = a_{\lambda_n, \beta_n}$ will be chosen infinitely many times.

When $\beta_n \rightarrow \infty$ and $\lambda_n \rightarrow 1$, this a will satisfy

$$U(x) - U(\tau_a x) + m(A) - A(\tau_a x) = 0.$$

□

In the last section we proved that the function

$$V(x) = \max_{\mu \in \mathcal{M}} \int S(y, x) d\mu(y)$$

is the unique limit of the family $b_\lambda - \frac{m(A)}{1-\lambda}$. In the present setting, in order to get a similar result, we will assume a certain condition: $\beta \rightarrow \infty$ faster than $\lambda \rightarrow 1$, in the sense that $\beta(1-\lambda) \rightarrow \infty$.

First we need a Lemma.

Lemma 3.5.

$$b_\lambda(x) \leq \frac{1}{\beta} u_{\lambda, \beta} \leq b_\lambda(x) + \frac{\log(d)}{\beta(1-\lambda)}.$$

Proof. From Lemma 3.2 the function $u_{\lambda,\beta}^* = u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda}$ satisfies

$$e^{u_{\lambda,\beta}^*(x)} = e^{-P(\beta A)} \sum_{T(y)=x} e^{\beta A(y) + \lambda u_{\lambda,\beta}^*(y)}.$$

Then, for any $a = (a_0 a_1 a_2 \dots) \in \{1, \dots, d\}^{\mathbb{N}}$,

$$\begin{aligned} \frac{1}{\beta} u_{\lambda,\beta}^*(x) &\geq A(\tau_{a_0} x) - \frac{1}{\beta} P(\beta A) + \lambda \frac{1}{\beta} u_{\lambda,\beta}^*(\tau_{a_0} x) \\ &\geq A(\tau_{a_0} x) + \lambda A(\tau_{a_1} \tau_{a_0}(x)) - \frac{1+\lambda}{\beta} P(\beta A) + \lambda^2 \frac{1}{\beta} u_{\lambda,\beta}^*(\tau_{a_1} \tau_{a_0} x). \end{aligned}$$

By induction, as $u_{\lambda,\beta}^*$ is uniformly bounded and $\lambda < 1$, we obtain

$$\frac{1}{\beta} u_{\lambda,\beta}^*(x) \geq S_{\lambda,A}(x, a) - \frac{P(\beta A)}{(1-\lambda)\beta}.$$

Taking the supremum in a we get

$$\frac{1}{\beta} u_{\lambda,\beta}^*(x) \geq b_{\lambda}(x) - \frac{P(\beta A)}{(1-\lambda)\beta},$$

that is, $\frac{1}{\beta} u_{\lambda,\beta} \geq b_{\lambda}(x)$.

On the other hand

$$\begin{aligned} \frac{1}{\beta} u_{\lambda,\beta}^*(x) &\leq \frac{1}{\beta} \log(d) + \sup_{a_0} [A(\tau_{a_0} x) - \frac{1}{\beta} P(\beta A) + \lambda \frac{1}{\beta} u_{\lambda,\beta}^*(\tau_{a_0} x)] \\ &\leq \frac{1+\lambda}{\beta} \log(d) + \sup_{a_0, a_1} [A(\tau_{a_0} x) + \lambda A(\tau_{a_1} \tau_{a_0}(x)) \\ &\quad - \frac{1+\lambda}{\beta} P(\beta A) + \lambda^2 \frac{1}{\beta} u_{\lambda,\beta}^*(\tau_{a_1} \tau_{a_0} x)]. \end{aligned}$$

Now, we get

$$\begin{aligned} \frac{1}{\beta} u_{\lambda,\beta}^*(x) &\leq \frac{\log(d)}{(1-\lambda)\beta} + \sup_a S_{\lambda,A}(x, a) - \frac{P(\beta A)}{(1-\lambda)\beta} \\ &= \frac{\log(d)}{(1-\lambda)\beta} + b_{\lambda}(x) - \frac{P(\beta A)}{(1-\lambda)\beta}, \end{aligned}$$

that is, $\frac{1}{\beta} u_{\lambda,\beta} \leq b_{\lambda}(x) + \frac{\log(d)}{\beta(1-\lambda)}$. □

Proof of Theorem 1.2. It follows from the above lemma that

$$\begin{aligned} b_{\lambda}(x) - \frac{m(A)}{1-\lambda} &\leq \frac{1}{\beta} (u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda}) + (\frac{P(\beta A)}{\beta(1-\lambda)} - \frac{m(A)}{1-\lambda}) \\ &\leq b_{\lambda}(x) - \frac{m(A)}{1-\lambda} + \frac{\log(d)}{\beta(1-\lambda)}. \end{aligned}$$

Assuming that $\lambda \rightarrow 1$, $\beta \rightarrow \infty$, $\beta(1-\lambda) \rightarrow \infty$, and applying Theorem 1.1 we obtain that $\frac{1}{\beta} (u_{\lambda,\beta} - \frac{P(\beta A)}{1-\lambda}) + (\frac{P(\beta A)}{\beta(1-\lambda)} - \frac{m(A)}{1-\lambda})$ converges uniformly to V . As $P(\beta A) = \beta m(A) + \epsilon_{\beta}$, where $\epsilon_{\beta} \geq 0$ decreases (see [9]), we get that $\frac{P(\beta A)}{\beta(1-\lambda)} - \frac{m(A)}{1-\lambda} = \frac{\epsilon_{\beta}}{\beta(1-\lambda)}$ converges to zero. This concludes the proof. □

We finish this section introducing an example (on the symbolic space) where it is studied the limit of $\frac{1}{\beta} \log(\phi_\beta)$ in a particular case. This limit is not V and this shows that some joint control of β and λ is really necessary.

Example. We consider $X = \{0, 1\}^{\mathbb{N}}$ with the shift map and a potential A depending on two coordinates. More precisely we suppose $A(1, 1) = A(2, 2) = 0$, $A(1, 2) = -5$ and $A(2, 1) = -3$.

Consider the matrix

$$L_\beta = \begin{pmatrix} e^{\beta A(1,1)} & e^{\beta A(1,2)} \\ e^{\beta A(2,1)} & e^{\beta A(2,2)} \end{pmatrix} = \begin{pmatrix} 1 & e^{-5\beta} \\ e^{-3\beta} & 1 \end{pmatrix}$$

that defines the Ruelle Operator associated to βA . We note that the main eigenvalue is given by

$$\alpha_\beta = e^{P(\beta A)} = 1 + e^{-4\beta}.$$

Furthermore, the eigenfunction ϕ_β associated to the Ruelle operator of βA depends on the first coordinate and satisfies

$$\phi_\beta(1) = 1 + e^\beta \text{ and } \phi_\beta(2) = 1 + e^{-\beta}$$

(it can be directly checked that $\phi_\beta L_\beta = \alpha_\beta \phi_\beta$). Any multiple of ϕ_β is also an eigenfunction. When $\beta \rightarrow +\infty$, we get $U = \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log(\phi_\beta)$ which satisfies

$$U(1) = 1, \quad U(2) = 0.$$

Now we will prove that $V \neq U$. Indeed any maximizing measure for A is of the form $\mu_r := r\delta_{1^\infty} + (1-r)\delta_{2^\infty}$. Consider the functions

$$S(y, x) = \limsup_{\varepsilon \rightarrow 0} \sup_n \{S_n(A)(y') : d(y', y) < \varepsilon, T^n(y') = x\}$$

and

$$V(x) = \sup_{\mu_r} \int S(y, x) d\mu_r(y).$$

As A depends only on two coordinates we conclude that $S(y, x)$ depends only on the first coordinate of x . From the analysis of $S(y, x)$ we get

$$S(2^\infty, 1) = -3, \quad S(1^\infty, 1) = 0, \quad S(1^\infty, 2) = -5, \quad S(2^\infty, 2) = 0$$

(for instance, when considered $S(2^\infty, 1)$ the point y' in (4) will contain the word 21, and each word 21 will decrease the value of $S_n(A)(y')$).

Then,

$$\sup_{\mu_r} \int S(y, 1) d\mu_r(y) = \sup_r [r \cdot 0 + (1-r)(-3)] = 0$$

and

$$\sup_{\mu_r} \int S(y, 2) d\mu_r(y) = \sup_r [r(-5) + (1-r)(0)] = 0.$$

This shows that $V \neq U$.

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