Universidade Federal do Rio Grande do Sul

Instituto de Matemática e Estatística

Programa de Pós-Graduação em Matemática

Sistemas de Haar, probabilidades quase invariantes e estados KMS sobre algebras de von Neumann e sobre C^*-algebras em grupoides dinamicamente definidos

Tese de Doutorado

Gabriel Elias Mantovani

Porto Alegre, 21 de março de 2019.

Tese submetida por Gabriel Elias Mantovani [1] como requisito parcial para a obtenção do grau de Doutor em Matemática pelo Programa de Pós-Graduação em Matemática do Instituto de Matemática e Estatística da Universidade Federal do Rio Grande do Sul.

Professor Orientador:
Artur Oscar Lopes (PPGMat-UFRGS)

Banca Examinadora:
Gilles Gonçalves de Castro (D. Mat-UFSC)
Jairo Kras Mengue (PPGMat-UFRGS)
Carlos Felipe Lardizabal (PPGMat-UFRGS)
Leonardo Fernandes Guidi (IME-UFRGS)

Data da Apresentação: 21 de março 2019.

[1]Bolsista do Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq

"If you believe in the "atomic hypothesis", the molecules composing a liter of cold water can be in all kinds of different configurations. In fact, the molecules dance around and the configuration changes all the time. In quantum language, we have a system of many particles, which can be in a very large number of different states. But while these states would look different if you could see microscopic details, they all look the same to the naked eye; in fact, they all look like a liter of cold water.

So, when we refer to a liter of cold water, we refer in fact to something quite ambiguous. Boltzmann's discovery is that the entropy is a measure of this ambiguity. Technically, the right definition is that the entropy of a liter of cold water is the number of digits in the number of "microscopic"states corresponding to this liter of cold water. The definition extends of course to hot water, and to many other systems. "

- David Ruelle, Chance and Chaos.

Resumo

Neste trabalho são analisados sistemas de Haar associados a grupoides obtidos por diversas relações de equivalencia especialmente de carácter dinâmico. A dinâmica básica considerada é a do shift e portanto foca-se em relações sobre conjuntos como $\{1, 2, ..., d\}^{\mathbb{Z}}$, $\{1, 2, ..., d\}^{\mathbb{N}}$ ou $(S^1)^{\mathbb{N}}$. Também são descritas propriedades de funções transversas, probabilidades quase invariantes e estados KMS sobre algebras de von Neumann (e também sobre algebras C^*) associada a estes grupoides. Iremos mostrar que alguns destes estados KMS estão relacionados a estados de Gibbs do formalismo termodinâmico (via medidas quase invariantes) sobre o espaço simbólico $\{1, 2, ..., d\}^{\mathbb{N}}$.

Também é explorado aqui um resultado de Ruelle e Haydn onde é obtida uma relação de equivalencia entre estados KMS de certas algebras C^* e probabilidades de equilibrio do formalismo termodinâmico. Este resultado é obtido aqui no contexto da relação de equivalência homoclínica em $\{1, 2, ..., d\}^{\mathbb{Z}}$ (um setting mais simples do que o considerado por Ruelle e Haydn que considera diffeomorfismos do tipo Axioma A). A vantagem do ponto de vista de considerar o shift agindo no espaço simbólico (seguido no presente trabalho) é que num setting mais simples as principais idéias por trás das diversas demonstrações se tornam mais claras. Elas podem ser entendidas sem a necessidade de ter que enfrentar certas tecnicalidades, por exemplo, associadas a desintegração nas variedades instáveis.

Os estados KMS desempenham na Mecânica Estatística Quântica o papel das medidas de Gibbs na Mecânica Estatística Clássica.

A seção 3.5 descreve as propriedades básicas da integração não comutativa, mais precisamente, a relação entre medidas transversas, funções transversas, cociclos e probabilidades quase invariantes (seguindo a apresentação de [18]). Ressaltamos aqui o fato que conseguimos apresentar uma pequena parte do trabalho "Sur la Theorie commutative de l'integration" by of A. Connes (see [18]) numa linguagem que pode ser mais facilmente entendida pela comunidade de Teoria Ergódica.

Todos os resultados no presente trabalho são expressos na linguagem de Teoria Ergódica.

Abstract

We analyse Haar systems associated to groupoids obtained by certain equivalence relations of dynamical nature on sets like $\{1, 2, ..., d\}^{\mathbb{Z}}$, $\{1, 2, ..., d\}^{\mathbb{N}}$, $S^1 \times S^1$, or $(S^1)^{\mathbb{N}}$, where S^1 is the unitary circle. We also describe properties of transverse functions, quasi-invariant probabilities and KMS states for some examples of von Neumann algebras (and also C^*-Algebras) associated to these groupoids. We relate some of these KMS states with Gibbs states of thermodynamic formalism via quasi-invariant probabilities.

We also explore a result by Ruelle and Haydn where it is shown an equivalence of KMS states of C^*-algebras with equilibrium probabilities of Thermodynamic Formalism. Not surprisingly such result is also obtained in the context of equivalence relations. Such result is obtained here in a simpler setting, with the advantage that in this setting the main ideas of the proofs can be clearly written in the context of measure and ergodic theory.

The KMS states play the role in Quantum Statistical Mechanics of the Gibbs probabilities in Classical Statistical Mechanics.

Section 3.5 describes the basic properties of non commutative integration, more precisely, the relation of transverse measures, transverse functions, cocycles and quasi-invariant probabilities (according to [18]). We point out that we were able to present a small part of the work “Sur la Theorie commutative de l'integration” by of A. Connes (see [18]) in a language more easily understandable for the ergodic theory community.

Agradecimentos

Gostaria de agradecer ao meu orientador Artur Lopes e aos professores Ruy Exel e Gilles Castro pelos comentários que enriqueceram este trabalho e também ao meu orientador de mestrado Ali Tahzibi por comentários relevantes.

Agradeço ao CNPq e a CAPES por me proverem uma bolsa de doutorado o que foi essencial para que este trabalho fosse realizado.

Eu gostaria de expressar meu agradecimento aos professores da UNB Leandro Cioletti e Claus A. H. Matsushigue pela excelente recepção que me foi dada em Brasília. Especialmente ao Leandro pela orientação, que espero algum dia dar frutos.

Finalmente devo agradecer aos meus pais Elenice Fritzsons e Luiz Eduardo Mantovani por todo apoio dado durante o doutorado, o que inclui apoio financeiro. Sem a ajuda deles provavelmente o doutorado não teria ocorrido tão tranquilamente. E por último, e não menos importante agradeço a minha namorada Grasieli Beloni pelo apoio durante o doutorado.

Acknowledgements

I would like to thanks my advisor Artur Oscar Lopes and also express my appreciation of all the work and clarification that professor Gilles Gonçalves de Castro has provided us specially in the field of C^*-algebra and von Neumann algebras. I would also like to thank professor Ruy Exel for many helpful discussions and useful comments during the procedure of writing this thesis. We also thanks Ali Tahzibi for some fruitful remarks.

I also acknowledge both the CNPq and CAPES for providing me the scholarship that was absolute necessary for the completion of this work.

I feel indebt with professors Leandro Cioletti and Claus A. H. Matsushigue from the mathematics departament of the Brasilia University (UNB) for their hearth warming welcome given for me in the time I spent in Brasilia. I hope I will be able to produce in the future a meaningfully mathematical work in the direction Leandro envisioned.

Finally I would like to appreciate all the support of varied nature, including financial support, given to me by my parents, Elenice Fritzsons and Luiz Eduardo Mantovani, without which this project would not reach such smooth conclusion. I also appreciate the support provided to me by my girlfriend Grasieli Beloni de Melo.

Sumário

Capítulo 1

Introduction

The study of dynamical systems can be enriched by the correct choice of an appropriate measure. In this sense several different measures have been defined in the field. Historically the first of these measures where invariant measures defined in the context of Hamiltonian systems [**?**]. Then ergodic measures, a subgroup of the invariant measures, were defined. The utility of ergodic measures became obvious after the proof of the Ergodic Theorem by Birkhoff in 1931 (see [8]).

It is no secret that behind some of these first definitions there was some underling physical intuition coming from the the branch of physics known as Statistical Mechanics [61]. Statistical Mechanics tries to understand the macro behavior of physical systems by looking at how the micro parts of this system behaves. In a general sense this field of physics tries to describe the general behavior of a system by understanding the deterministic behavior of its parts; much in the same way that ergodic theory tries to understand the general behavior of dynamical systems, whose particles follow clear deterministic laws. Relating the two fields of study has not just provided mathematicians with new measures, but also with important concepts such as entropy.

In the sense of exploring further the possible contributions of statistical mechanics for dynamical systems the field of thermodynamic formalism was created, founded mainly by Sinai, Ruelle and Bowen (see [13]). New measures were defined for possible uses in dynamical systems, specially DLR measures, Gibbs measures and equilibrium measures, among others. Gibbs measure, for instance, refer to the Gibbs distribution of Statistical Mechanics, i.e., the state that a system in contact with an infinite heat bath at constant temperature tends to achieve at equilibrium.

In order to apply some of the concepts of Thermodynamic Formalism to Dynamical Systems the concept of disintegration along stable and unstable foliation plays a main role. Despite certain powerful results like Rokhlin's Disintegration Theorem, disintegration in the general sense isn't simple and in some cases is an open problem. This work in some sense follows the approach of endowing the unstable foliations with an algebraic structure, like a C^*-algebra or a von Neumann Algebra, and then defining things that look like measures in this algebraic structure.

The set of DLR probabilities coincides with the set of eigenprobabilities of the dual of the Ruelle operator for the potentials we consider here (see [17]).

We are interested in relating eigenprobabilities of the dual of the Ruelle operator (also called transfer operator) in chapter 3 with KMS states. The role of quasi-invariant probabilities is a key ingredient on this relation. We point out that some probabilities which are absolutely continuous with respect to the eigenprobabilities of the dual of the Ruelle operator can also be quasi-invariant for the groupoid we consider.

Moreover, we will relate equilibrium probabilities in chapter 2 with KMS states (for a certain groupoid).

More precisely in this work we analyze properties of Haar systems, quasi-invariant probabilities, transverse measures, C^*-algebras and KMS states related to Thermodynamic Formalism and Gibbs states. The basis of our algebraic structure will be grupoids obtained by equivalence relations.

One of the most important equivalence relations in the context of dynamical systems is clearly given by $x \sim y$ iff x and y belong to the same unstable foliation. This dynamical relation will not be the only one studied but a special attention will be given to it. We will also consider the *homoclinic* equivalence relation.

In chapter 2 we shall follow D. Ruelle's and Haydn's (see [59] and [23]) footsteps and show an equivalence between KMS states of C^*-algebras and Hölder equilibrium probabilities of Thermodynamic Formalism. We shall present this result in a particularized setting (symbolic space). In this way our results aren't new, but the advantage here is that they are simple, i.e., our examples and demonstrations can be understand with just basic notions of measure theory. Some of the proofs presented here are different from the ones in [23].

Chapter 3 analyze properties of Haar systems, quasi-invariant probabilities, transverse measures, C^*-algebras and KMS states which are related to Thermodynamic Formalism and Gibbs states. We will consider a specific particular setting where the groupoid will be defined by some natural equivalence relations on a set X, where X is $\{1, 2, ..., d\}^{\mathbb{N}}$ or $\{1, 2, ..., d\}^{\mathbb{Z}}$, $S^1 \times S^1$, or $(S^1)^{\mathbb{N}}$. Most of the equivalence relations considered in this text will be of dynamic origin.

Chapters 2 and 3 are completely independent. In this thesis we use a notation and an approach closer to the one commonly used on Ergodic Theory and Thermodynamic Formalism.

For some reason equilibrium probabilities appear in a natural way for the action of the shift on $\{1, 2., , , d\}^{\mathbb{Z}}$ and eigenprobabilities for the dual of the Ruelle operator appear in a natural way for the action of the shift on $\{1, 2., , , d\}^{\mathbb{N}}$.

Section 3.5 describes the basic properties of non comutative integration and its relation with quasi-invariant probabilities. We point out that we were able to present a small part of the work "Sur la Theorie commutative de l'integration" of A. Connes (see [18]) in a language which is more easily understandable for the ergodic theory community.

Classical references on measured groupoids and von Neumann algebras are [33]; [31] and the book [32]. KMS states and C^*-algebras are described on [46]. Results on C^*-algebras and KMS states from the point of view of Thermodynamic Formalism are presented in the papers [37]; [57]; [51]; [65]; [1]; [29] and the books [66]; [30]. The paper [**?**] considers equivalence relations and DLR probabilities for certain interactions on the symbolic space $\{1, 2, ...d\}^{\mathbb{Z}}$ (not in $\{1, 2, ...d\}^{\mathbb{N}}$). Theorem 6.2.18 in Vol II of [12] and [4] describe the relation between KMS states and Gibbs probabilities for interactions on certain spin lattices (on the one-dimensional case corresponds to the space $\{1, 2, .., d\}^{\mathbb{Z}}$). We point out that Lecture 9 in [21] presents a brief introduction to C^*-Algebras and non-commutative integration.

The present work in some sense can be said to be aimed to obtain a better and deeper understanding from the point of view of ergodic theory of the work described in the master dissertation [43] (under the guidance of Prof. Ali Tahzibi in USP São Carlos). This was our inspiration.

Our thesis originated two papers that were submitted for publication:

1) "The KMS Condition for the homoclinic equivalence relation and Gibbs probabilities", A. O. Lopes and G. Mantovani (to appear in São Paulo Journal

of Mathematical Science)

2) "Haar systems, KMS states on von Neumann algebras and C^*-algebras on dynamically defined groupoids and Noncommutative Integration", G. Castro, A. O. Lopes and G. Mantovani

The main contributions of the thesis are:

a) a different proof (for the symbolic space) on chapter 2 of the results presented in [23] concerning KMS states (on the dynamically defined groupoid given by the homoclinic equivalence relation) and equilibrium probabilities on the lattice $\{1, 2, .., d\}^{\mathbb{Z}}$.

b) the relation of DLR probabilities $\{1, 2, .., d\}^{\mathbb{N}}$ with quasi-invariant probabilities for a certain Haar system associated to a groupoid dynamically defined (the bigger than two relation) described in Theorem 3.4.24.

c) questions regarding the relation of the quasi invariant probability and the SBR probability for a non linear of Baker map described in Example 3.3.6.

d) a simplified description of the relation (described on [18]) of non commutative integration with quasi-invariant probabilities which is here presented on section 3.5.

Capítulo 2

The KMS Condition for the homoclinic equivalence relation and Gibbs probabilities

2.1 Introduction

D. Ruelle in [59] considered a general setting (which includes hyperbolic diffeomorphisms on manifolds) where he is able to describe a formulation of the concept of Gibbs state based on **conjugating homeomorphism** in the so called Smale spaces. On this setting he shows a relation of KMS states of C^*-algebras with Hölder equilibrium probabilities of Thermodynamic Formalism. Part of the formulation of this relation requires the use of a non trivial result by N. Haydn (see [23]). Later, the paper [24] by N. Haydn and D. Ruelle presents a shorter proof of the equivalence.

Here we consider similar problems but now on the symbolic space and the dynamics will be given by the shift. We will present a simplified proof of the equivalence mentioned above. The main result of this chapter is Theorem 2.5.4 on section 2.5. One can get a characterization of the equilibrium probability for a potential defined on the lattice $\{1, 2, ..., d\}^{\mathbb{Z}-\{0\}}$ without using the Ruelle operator (which acts on the lattice $\{1, 2, ..., d\}^{\mathbb{N}}$). The probability we get is invariant for the action of the shift τ acting on $\{1, 2, ..., d\}^{\mathbb{Z}-\{0\}}$.

The proof of this result will take several subsequent sections.

In section 3.6 we show the relation of these probabilities with the KMS dynamical C^*-state on the C^*-Algebra associated to the groupoid defined by the homoclinic equivalence relation. On the initial sections we introduce several results which are necessary for the simplification of the final argument on section 3.6.

We present several examples helping the reader on the understanding of the main concepts.

On [60] and also on the beginning of the book [7] it is explained the relation of equilibrium states of Thermodynamic Formalism with the corresponding concept in Statistical Physics. The role of KMS C^*-dynamical states on Quantum Statistical Physics is described on [12]. KMS C^*-dynamical states correspond to the DLR probabilities (see [14] for definition) in Statistical Mechanics.

In section 3.6 we present definitions and properties regarding the C^*-algebra we will consider here.

Working on the symbolic space helps to avoid several technicalities which are required in the case of the study of hyperbolic diffeomorphisms on manifolds (where one have to use stable foliation, the local product structure, etc...).

Our proof consider mainly potentials $A : \{1, 2, ..., d\}^{\mathbb{Z}-\{0\}} \to \mathbb{R}$ which depend on a finite number of coordinates. The case of a general Hölder potential (more technical) can be obtained by adapting our reasoning but we will not address this question here.

On the papers [15] and [42] the authors consider among other things a relation of KMS probabilities with eigenprobabilities for the dual of the Ruelle operator (which are not necessarily invariant for the shift). This problem is analyzed on the lattice $\{1, 2, .., d\}^{\mathbb{N}}$ which is a different setting that the one we consider here. The equivalence relations are also not related. Despite some similarities that can be perceived in the statements of the main results obtained in the two settings we point out that the reasoning on the respective proofs are quite different.

Lecture 9 in [21] presents a brief introduction to C^*-Algebras and the KMS condition.

In [29] and [30] a relation of KMS states in a certain C^*-Algebra and eigenprobabilities of the dual of the Ruelle operator is considered.

In a different setting the paper [9] also considers the homoclinic equivalence relation.

2.2 Conjugating homeomorphisms

In this section $\Omega = \{1, 2, ..., d\}^{\mathbb{Z}-\{0\}}$ and a general point x on Ω is denoted as

$$x = (..., x_{-n}, ..., x_{-2}, x_{-1} \,|\, x_1, x_2, ..., x_n, ...),$$

$x_j \in \{1, 2, .., d\}$, $j \in \mathbb{Z}$.

We consider the dynamics of the shift $\tau : \Omega \to \Omega$, that is,

$$\tau(..., x_{-n}, ..., x_{-2}, x_{-1} \,|\, x_1, x_2, ..., x_n, ...) = (..., x_{-n}, ..., x_{-2}, x_{-1}, x_1 \,|\, x_2, ..., x_n, ...).$$

We also consider the usual metric d on Ω which is defined in such way that for $x, y \in \Omega$ we set

$$d(x, y) = 2^{-N},$$

$N \geq 0$, where for

$$x = (..., x_{-n}, ..., x_{-1} \,|\, x_1, .., x_n, ..)\ \ , y = (..., y_{-n}, ..., y_{-1} \,|\, y_1, .., y_n, ..),$$

we have $x_j = y_j$, for all j, such that, $-N \leq j \leq N$ and, moreover $x_{N+1} \neq y_{N+1}$, or $x_{-N-1} \neq y_{-N-1}$. Given x, y as above we denote $\vartheta(x, y) = N$, therefore $\vartheta(x, y) = -\log_2(d(x, y))$.

Given $x, y \in \Omega$, we say that $x \sim y$ if

$$\lim_{k \to +\infty} d(\tau^k x, \tau^k y) = 0$$

and

$$\lim_{k \to -\infty} d(\tau^k x, \tau^k y) = 0. \tag{2.1}$$

This means there exists an $N \geq 0$ such that $x_j = y_j$ for $j > N$ and $j < -N$ (note that given $\epsilon > 0$, there exists n such that $2^{-n} < \epsilon \leq 2^{-n+1}$, and if $d(x, y) < \epsilon$, then x and y should coincide for coordinates smaller than n). In other words, there are only a finite number of i's such that $x_i \neq y_i$. In this case we say that x and y are homoclinic.

$\sim$ is an equivalence relation and defines the groupoid $G \subset \Omega \times \Omega$ of pairs (x, y) of elements which are related (see for instance [54], [57], [15] or [42]).

Let $\kappa(x, y)$ be the minimum M as above. Therefore $x_{\kappa(x,y)} \neq y_{\kappa(x,y)}$ or $x_{-\kappa(x,y)} \neq y_{-\kappa(x,y)}$. Note that $\vartheta(x, y) \leq \kappa(x, y)$ and could be strictly less. Note that $\kappa(x, y)$ is defined just when $x \sim y$.

Example 2.2.1. *For example in* $\Omega = \{1,2\}^{\mathbb{Z}-\{0\}}$ *take*

$$x = (..., x_{-n}, ..., x_{-7}, 1, 2, 2, 1, 2, 2 \,|\, 1, 2, 1, 2, 1, 1, x_7, ...x_n, ..)$$

and

$$y = (..., y_{-n}, ..., y_{-7}, 1, 2, 2, 1, 2, 2 \,|\, 1, 2, 1, 1, 1, 2, y_7, ...y_n, ..)$$

where $x_j = y_j$ *for* $|j| > 6 = \kappa(x,y)$. *In this case* $d(x,y) = 2^{-3}$ *and* $N = \vartheta(x,y) = 3$.

Given a Hölder function $U : \Omega \to \mathbb{R}$ it is easy to see that if x and y are homoclinic, then the following function is well defined

$$V(x,y) = \sum_{n=-\infty}^{\infty} (U(\tau^n(x)) - U(\tau^n(y))). \tag{2.2}$$

Indded, note that if $x \sim y$, they coincide for large n, then, there exists a constant c, such that, $d(\tau^n(x), \tau^n(y)) \leq c\, 2^{-n}$. If U has Holder exponent α, then, the sum converges absolutely because $\sum_n (2^\alpha)^{-n} < \infty$.

This function satisfies the property

$$V(x,y) + V(y,z) = V(x,z)$$

when $x \sim y \sim z$.

A function V with this property will play an important role in some parts of our reasoning. We will not assume on the first part of this work that V was obtained from a U as above.

Now we will describe a certain class of **conjugating homeomorphism** for the relation $\sim$ (see (2.1)) described above.

Given two fixed points x and y (y in the class of x) we define the open set $\mathcal{O}_{(x,y)} = B_{\frac{1}{2^{\kappa(x,y)}}}(x) = \{z \in \Omega : d(x,z) < 2^{-\kappa(x,y)+1}\}$.

We will define for each such pair (x,y) a conjugating homeomorphisms $\varphi_{(x,y)}$ which has domain on $\mathcal{O}_{(x,y)}$.

We denote for $m, n \in \mathbb{N}$

$$\overline{x_{-m}x_{-m+1}...x_{-1} \,|\, x_1...x_{n-1}x_n} =$$

$$\{z \in \Omega \,|\, z_j = x_j,\ j = -m, -m+1, ..., -1, 1, 2, ..., n-1, n\},$$

and call it the cylinder determined by the finite string

$$x_{-m}x_{-m+1}...x_{-1}\,|\,x_1...x_{n-1}x_n.$$

We will say that a cylinder, or a string, is **symmetric** if $n = m$.

Note that given $x \sim y$

$$\mathcal{O}_{(x,y)} = \overline{x_{-\kappa(x,y)}\; x_{-\kappa(x,y)+1} \ldots x_{-1}\,|\,x_1 \ldots x_{\kappa(x,y)-1}\; x_{\kappa(x,y)}},$$

and $\mathcal{O}_{(x,y)}$ is a symmetric cylinder.

Now we shall define the main kind of **conjugating homeomorphisms** that we will be using. Given $(x, y) \in G$, let $n = \kappa(x, y)$, we define a conjugating $\varphi = \varphi_{(x,y)}$ with domain

$$\mathcal{O}_{(x,y)} = B_{\frac{1}{2^n}}(x) = \{z \in \Omega : d(x, z) < 2^{-n+1}\} = \overline{x_{-n}x_{-n+1}...x_{-1}\,|\,x_1...x_{n-1}x_n},$$

where $\varphi_{(x,y)} : \mathcal{O}_{(x,y)} \to B_{\frac{1}{2^n}}(y)$ is defined by the expression: z of the form

$$z = (...z_{-n-2}z_{-n-1}\; \mathbf{x_{-n}x_{-n+1}...x_{-1}}|\mathbf{x_1...x_n}\; z_{n+1}z_{n+2}...)$$

goes to

$$\varphi_{(x,y)}(z) = ...z_{-n-2}z_{-n-1}\, \mathbf{y_{-n}y_{-n+1}...y_{-1}}|\mathbf{y_1...y_n}\, z_{n+1}z_{n+2}... \tag{2.3}$$

We shall call these transformations the family of **symmetric conjugating homeomorphisms**. We shall denote by S the set of symmetric conjugating homeomorphisms obtained by considering all pairs of related points x and y.

Note that the homeomorphism $\varphi_{(x,y)}$ transforms the cylinder $O_{(x,y)} = \overline{x_{-n}x_{-n+1}...x_{-1}\,|\,x_1...x_{n-1}x_n}$ in the cylinder $\overline{y_{-n}y_{-n+1}...y_{-1}\,|\,y_1...y_{n-1}y_n}$.

The graph of $\varphi_{(x,y)}$ is on G.

A more explicit formulation of the concept of symmetric conjugating homeomorphism will be presented on next section via expressions (2.6) and (2.7).

Example 2.2.2. *Consider*

$$x = (...1\,1\,2\,1\,1\,2\,2\,2\,2\,1\;\; 1\,1|2\,1\;\; 2\,1\,2\,2\,1\,2\,2\,2\,1\,1...)$$

and

$$y = (...1\,1\,2\,1\,1\,2\,2\,2\,2\,1\;\; 1\,2|1\,2\;\; 2\,1\,2\,2\,1\,2\,2\,2\,1\,1...)$$

in this case $\kappa(x, y) = 2$, *and for* z *of the form*

$$z = (...z_{-4}\, z_{-3}\ \ 1\,1|2\,1\ \ z_3\, z_4\, z_5...)$$

we get

$$\varphi_{(x,y)}(z) = (...z_{-4}\, z_{-3}\ \ 1\,2|1\,2\ \ z_3\, z_4\, z_5...).$$

It is easy to see that the family of symmetric conjugating homeomorphisms we define above has the following properties: given $x \sim y$

a) $\varphi_{(x,y)} : \mathcal{O}_{(x,y)} \subset \Omega \to \Omega$ is an homeomorphism over its image

b) $\varphi_{(x,y)}(x) = y$, and

c) $\lim_{k\to\infty} d(\tau^k(z), \tau^k(\varphi_{(x,y)}(z)) = 0$ and $\lim_{k\to-\infty} d(\tau^k(z), \tau^k(\varphi_{(x,y)}(z)) = 0$.

Item c) implies that z and $\varphi_{(x,y)}(z)$ are on the same homoclinic class.

2.3 C^*-Gibbs states and Radon-Nikodym derivative

We consider the groupoid $G \subset \Omega \times \Omega$ of all pair of points which are related by the homoclinic equivalence relation.

We consider on G the topology generated by sets of the form

$$\{\ (z, \varphi_{(x,y)}(z))|\ \text{where } z \in \mathcal{O}_{(x,y)}\ \ \text{with } x \sim y\}.$$

This topology is Hausdorff (see [59]).

Now consider a continuous function $V : G \to \mathbb{R}$ such that

$$V(x, y) + V(y, z) = V(x, z), \tag{2.4}$$

for all related x, y, z. Note that this implies that $V(x, x) = 0$ and $V(x, y) = -V(y, x)$.

Here we call V a modular function.

Under some other notation the function $\delta(x, y) = e^{V(x,y)}$ is called a modular function (or, a cocycle).

Definition 2.3.1. *Given a function $V : G \to \mathbb{R}$ as above we say that a probability measure α on Ω is a C^*-**Gibbs probability** with respect to the parameter $\beta \in \mathbb{R}$ and V, if for any $x \sim y$*

$$\int_{O_{(x,y)}} \exp(-\beta V(z, \varphi_{(x,y)}(z))) f(\varphi_{(x,y)}(z)) d\alpha(z) = \int_{\varphi_{(x,y)}(O_{(x,y)})} f(z) d\alpha(z), \tag{2.5}$$

for every continuous function $f : \Omega \to \mathbb{C}$ (and conjugated homeomorphism $(O_{(x,y)}, \varphi_{(x,y)})$).

We will show on section 3.6 a natural relation of this probability α with the C^*-dynamical state on a certain C^*-algebra. This is the reason for such terminology.

The above definition was taken from [59]. This is a version of the Renault-Radon-Nikodym condition (Def. 1.3.15 in [54]).

It is easy to see that the above definition is equivalent to say that: given any pair of finite strings

$$x_{-n}x_{-n+1}...x_{-1}, x_1... x_{n-1}x_n \quad \text{and} \quad y_{-n}y_{-n+1}...y_{-1}\, y_1...y_{n-1}y_n,$$

$n \in \mathbb{N}$, the transformation

$$\varphi : \overline{x_{-n}x_{-n+1}...x_{-1} \,|\, x_1...x_{n-1}x_n} \to \overline{y_{-n}y_{-n+1}...y_{-1} \,|\, y_1...y_{n-1}y_n} \tag{2.6}$$

defined by the expression:

$$\varphi(z) = (...z_{-n-2}z_{-n-1}\; y_{-n}y_{-n+1}...y_{-1} \,|\, y_1...y_n\; z_{n+1}z_{n+2}...), \tag{2.7}$$

where

$$z = (...z_{-n-2}z_{-n-1}\; z_{-n}z_{-n+1}...z_{-1} \,|\, z_1...z_n\; z_{n+1}z_{n+2}...),$$

is such that for any continuous function $f : \overline{y_{-n}y_{-n+1}...y_{n-1}y_n} \to \mathbb{R}$

$$\int_{\overline{x_{-n}x_{-n+1}...\,|\,...x_{n-1}x_n}} e^{-\beta V(z,\varphi(z))} f(\varphi(z)) d\alpha(z) = \int_{\overline{y_{-n}y_{-n+1}...\,|\,...y_{n-1}y_n}} f(z) d\alpha(z). \tag{2.8}$$

Note in particulary that by taking $f = 1$ we get

$$\int_{\overline{x_{-n}x_{-n+1}...\,|\,...x_{n-1}x_n}} e^{-\beta V(z,\varphi(z))} d\alpha(z) = \int_{\overline{y_{-n}y_{-n+1}...\,|\,...y_{n-1}y_n}} d\alpha(z). \tag{2.9}$$

In the moment we only consider symmetric conjugating homeomorphisms of the form (2.7).

We will show on section 2.5 a relation of the C^***-Gibbs probabilities** α with the **Gibbs (equilibrium) probabilities** of Thermodynamic Formalism.

In a more explicit formulation α is such that given any conjugating homeomorphism $(O_{(x,y)}, \varphi_{(x,y)})$ of the form (2.6), and continuous function $f : \Omega \to \mathbb{C}$

$$\int_{O_{(x,y)}} e^{-\beta V(z,\varphi_{(x,y)}(z))} f(\varphi_{(x,y)}(z)) d\alpha(z) =$$

$$\int_{O_{(x,y)}} e^{-\beta V((\ldots z_{-n},\ldots,z_{-1}|z_1,\ldots,z_n,\ldots),(\ldots z_{-n-1}y_{-n},\ldots,y_{-1}|y_1,\ldots,y_n,z_{n+1},\ldots))} f(\varphi_{(x,y)}(z)) d\alpha(z) =$$

$$\int_{\varphi_{(x,y)}(O_{(x,y)})} f(z) d\alpha(z). \tag{2.10}$$

In this case, clearly the Radon-Nikodym derivative of the change of coordinates φ is

$$e^{-\beta V((\ldots z_{-n},\ldots,z_{-1}|z_1,\ldots,z_n,\ldots),(\ldots z_{-n-1}y_{-n},\ldots,y_{-1}|y_1,\ldots,y_n,z_{n+1},\ldots))}.$$

In order to simplify the notation sometimes on the text we will consider the value $\beta = 1$.

We will consider a larger class of conjugating homeomorphisms.

Definition 2.3.2. *Given n and m and pair of finite strings*

$$x_{-n}x_{-n+1}\ldots x_{-1}, x_1\ldots x_{m-1}x_m \quad \textit{and } y_{-n}y_{-n+1}\ldots y_{-1}\, y_1\ldots y_{n-1}y_m, \tag{2.11}$$

$n, m \in \mathbb{N}$, the transformation

$$\varphi : \overline{x_{-n}x_{-n+1}\ldots x_{m-1}x_m} \to \overline{y_{-n}y_{-n+1}\ldots y_{m-1}y_m} \tag{2.12}$$

defined by the expression:

$$\varphi(z) = (\ldots z_{-n-2}z_{-n-1}\, \mathbf{y_{-n}y_{-n+1}\ldots y_{-1}}\,|\,\mathbf{y_1\ldots y_m}\, z_{m+1}z_{m+2}\ldots), \tag{2.13}$$

where

$$z = (\ldots z_{-n-2}z_{-n-1}\, \mathbf{x_{-n}x_{-n+1}\ldots x_{-1}}\,|\,\mathbf{x_1\ldots x_m}\, z_{m+1}z_{m+2}\ldots),$$

is called a **non-symmetric conjugating homeomorphism** *associated to the pair (2.11).*

Proposition 2.3.4 claims that if α is a C^*-Gibbs probability, then the relation (2.10) is satisfied for a bigger class of φ transformations, i.e. not necessarily symmetric. Before that we shall provide the reader with an example of idea of the proof.

Example 2.3.3. *Consider the non-symmetric conjugating homeomorphism* $\varphi : \overline{0|11} \to \overline{1|10}$ *given by*

$$\varphi(...z_{-3}z_{-2}0|11z_3...) = ...z_{-3}z_{-2}1|10z_3...$$

we shall prove that if α is a C^*-Gibbs measure then relation (2.5) *is valid for φ. This is actually straightforward, first divide the domain and image of the function into symmetric cylinders, and in these cylinders apply relation* (2.10)*. So in this case consider* $\varphi_0 : \overline{00|11} \to \overline{01|10}$*, and* $\varphi_1 : \overline{10|11} \to \overline{11|10}$ *such that*

$$\varphi_a(...z_{-3}a0|11z_3...) = (...z_{-3}a1|10z_3...)$$

for $a = 0$ or $a = 1$. Now notice that

$$\int_{\overline{0|11}} e^{-\beta V(x,\varphi(x))} f(\varphi(x)) d\alpha(x) =$$

$$\int_{\overline{00|11}} e^{-\beta V(x,\varphi(x))} f(\varphi(x)) d\alpha(x) + \int_{\overline{10|11}} e^{-\beta V(x,\varphi(x))} f(\varphi(x)) d\alpha(x) =$$

$$\int_{\overline{00|11}} e^{-\beta V(x,\varphi_0(x))} f(\varphi(x)) d\alpha(x) + \int_{\overline{10|11}} e^{-\beta V(x,\varphi_1(x))} f(\varphi(x)) d\alpha(x) \overset{(2.10)}{=}$$

$$\int_{\overline{01|10}} f(x) d\alpha(x) + \int_{\overline{11|10}} f(x) d\alpha(x) = \int_{\overline{1|10}} f(x) d\alpha(x).$$

This claim proves that relation (2.10) *is valid for this conjugating.*

Proposition 2.3.4. *Assume α is C^*-Gibbs for V as in (2.10), then for any non-simmetric homeomorphism $(\varphi, \mathcal{O})$, as defined on (2.13), we have that for $n, m \in \mathbb{N}$, the transformation*

$$\int_{\overline{x_{-n}x_{-n+1}...x_{-1}\,|\,x_1\,...x_{m-1}x_m}} e^{-\beta V(z,\varphi(z))} f(\varphi(z)) d\alpha(z) =$$

$$\int_{\mathcal{O}} e^{-\beta V((...z_{-n-1}z_{-n},...,z_{-1}|z_1,...,z_m,z_{m+1}...),(...z_{-n-1}y_{-n},...,y_{-1}|y_1,...,y_m,z_{m+1},...))} f(\varphi(z)) d\alpha(z) =$$

$$\int_{\overline{y_{-n}y_{-n+1}\cdots y_{-1}\,|\,y_1\cdots y_{m-1}y_m}} f(z)d\alpha(z). \tag{2.14}$$

We leave the proof (which is similar to the reasoning of example 2.3.3) for the reader.

As a particular case we get

$$\int_{\overline{|\,x_1\ldots x_m}} e^{-\beta V(z,\varphi(z))} f(\varphi(z))d\alpha(z) = \int_{\overline{|\,y_1\ldots y_m}} f(z)d\alpha(z). \tag{2.15}$$

for given $\overline{|\,x_1\ldots x_m}$, $\overline{|\,y_1\ldots y_m}$ and the corresponding conjugating homeomorphism φ.

It is possible to consider more general forms of conjugating homeomorphisms as described on the next example.

Example 2.3.5. *Consider the homeomorphism* $\varphi : \overline{112|2} \to \overline{1|122}$ *given by*

$$\varphi(\ldots z_{-4}112|2z_2z_3z_4\ldots) = (\ldots z_{-4}z_2z_31|122z_4\ldots).$$

Note that $\overline{112|2}$ *is translation by* τ^{-2} *of the set* $\overline{1|122}$.

As in the previous example we will prove that if α *is a* C^**-Gibbs probability then relation* (2.10) *is also valid for such* φ *and* $\mathcal{O} = \overline{112|2}$. *First consider the conjugating homeomorphisms,* φ_1, φ_2, φ_3 *and* φ_4, *given by*

$$\varphi_1(\ldots z_{-4}112|2\,\mathbf{11}z_4\ldots) = (\ldots z_{-4}\,\mathbf{11}1|122z_4\ldots),$$

$$\varphi_2(\ldots z_{-4}112|2\,\mathbf{12}\,z_4\ldots) = (\ldots z_{-4}\,\mathbf{12}\,1|122z_4\ldots),$$

$$\varphi_3(\ldots z_{-4}112|2\,\mathbf{21}\,z_4\ldots) = (\ldots z_{-4}\,\mathbf{21}\,1|122z_4\ldots),$$

$$\varphi_4(\ldots z_{-4}112|2\,\mathbf{22}z_4\ldots) = (\ldots z_{-4}\,\mathbf{22}\,1|122z_4\ldots).$$

Therefore we have that

$$\int_{\overline{112|2}} e^{V(x,\varphi(x))} f(\varphi(x))d\alpha(x) =$$

$$\int_{\overline{112|211}} e^{V(x,\varphi(x))} f(\varphi(x)) + \int_{\overline{112|212}} e^{V(x,\varphi(x))} f(\varphi(x)) +$$

$$\int_{\overline{112|221}} e^{V(x,\varphi(x))} f(\varphi(x)) + \int_{\overline{112|222}} e^{V(x,\varphi(x))} f(\varphi(x)) =$$

$$\int_{\overline{112|211}} e^{V(x,\varphi_1(x))} f(\varphi_1(x)) + \int_{\overline{112|212}} e^{V(x,\varphi_2(x))} f(\varphi_2(x))+$$

$$\int_{\overline{112|221}} e^{V(x,\varphi_3(x))} f(\varphi_3(x)) + \int_{\overline{112|222}} e^{V(x,\varphi_4(x))} f(\varphi_4(x)) =$$

$$\int_{\overline{111|122}} f(x) + \int_{\overline{121|122}} f(x) + \int_{\overline{211|122}} f(x) + \int_{\overline{221|122}} f(x) =$$

$$\int_{\overline{1|122}} f d\alpha(x)$$

where some of the $d\alpha$ where omitted. Since we proved that

$$\int_{\overline{112|2}} e^{V(x,\varphi(x))} f(\varphi(x)) d\alpha(x) = \int_{\overline{1|122}} f d\alpha(x)$$

for any continuous function f then we have that relation (2.10) is satisfied.

In analogous way as in last example one can define a conjugating φ such that
$\varphi : \overline{x_{-n} ... \mathbf{x_{-r}} \,...\, \mathbf{x_{-1}} \,|\, x_1 \,... x_m} \to \overline{x_{-n} x_{-n+1} ... x_{-r-1} \,|\, \mathbf{x_{-r}} \,... \mathbf{x_{-1}} \, x_1 \,... x_m}$.
We will consider such transformation φ in the next result.

Proposition 2.3.6. *Assume α is C^*-Gibbs for V as in (2.10), then for $n, m \in \mathbb{N}$, and $0 < r$, such that, $r \leq n$, we get*

$$\int_{\overline{x_{-n} x_{-n+1} ... x_{-r-1} \; \mathbf{x_{-r}} \, \mathbf{x_{-r+1}} ... \mathbf{x_{-1}} \,|\, x_1 \,... x_{m-1} x_m}} e^{-\beta V(z,\varphi(z))} f(\varphi(z)) d\alpha(z) =$$

$$\int_{\overline{x_{-n} x_{-n+1} ... x_{-r-1} \,|\, \mathbf{x_{-r}} \, \mathbf{x_{-r+1}} ... \mathbf{x_{-1}} \, \mathbf{x_1} \; ... x_{m-1} x_m}} f(z) d\alpha(z), \qquad (2.16)$$

where φ is of the form (2.13).

Proof: The proof is similar to the reasoning of example 2.3.5. One just has to consider the homeomorphisms

$$\varphi(... z_{-n-r-1} \, z_{-n-r} ... z_{-n-1} x_{-n} x_{-n+1} ... x_{-1} \,|\, x_1 \,... x_{m-1} x_m \mathbf{z_{m+1}} ... \mathbf{z_{m+r}} \, z_{m+r+1} ...) =$$

$$(... z_{-n-r} \, \mathbf{z_{m+1}} ... \mathbf{z_{n+r}} \, x_{-n} x_{-n+1} ... x_{-r-1} \,|\, x_{-r} \, x_{-r+1} ...... x_{-1} x_1 ... x_{m-1} x_m \, z_{n+r+1} ...).$$

Note that

$$\tau^{-r}\,(\overline{x_{-n}x_{-n+1}...x_{-1}\,|\,x_1\,...x_{m-1}x_m}\,) =$$
$$\overline{x_{-n}x_{-n+1}...x_{-r-1}\,|\,x_{-r}\,x_{-r+1}...x_{-1}\,x_1\,...x_{m-1}x_m}.$$

□

We want to show that α is C^*-Gibbs for V, then, the pullback $\rho = \tau^*(\alpha)$ is also C^*-Gibbs for V.

The next example will help to understand the main reasoning for the proof of the above claim.

Example 2.3.7. *Suppose $V(x, y)$ is defined when $x \sim y$. Assume that for all x, y on the groupoid we have that $V(x, y) = V(\tau(x), \tau(y))$.*

Given α consider the pull back $\rho = \tau^*(\alpha)$.

Consider

$$\varphi : \overline{11|21} \to \overline{21|12},$$

where

$$\varphi(...x_{-4}x_{-3}\,11|21\,x_3\,x_4...\,) = (...x_{-4}x_{-3}\,21|12\,x_3\,x_4...\,),$$

and

$$\varphi_1 : \overline{112|1} \to \overline{211|2},$$

where

$$\varphi_1(...x_{-5}x_{-4}\,112|1\,x_2\,x_3...\,) = (...x_{-5}x_{-4}\,211|2\,x_2\,x_3...\,).$$

If for any continuous function g we have that

$$\int_{\overline{11|21}} e^{V(x,\varphi(x))} g(\varphi(x))d\alpha(x) = \int_{\overline{21|12}} g(x)d\alpha(x),$$

then, for any continuous function f we have that

$$\int_{\overline{112|1}} e^{V(x,\varphi_1(x))} f(\varphi_1(x))d\rho(x) = \int_{\overline{211|2}} f(x)d\rho(x).$$

In fact both properties are equivalent.

Note first that $\varphi_1 \circ \tau = \tau \circ \varphi$.

Moreover, $V(\tau(x), \varphi_1(\tau(x)) = V(\tau(x), \tau(\varphi_1(x)) = V(x, \varphi_1(x))$ by hypothesis.

Therefore,

$$\int_{\overline{112|1}} e^{V(x,\varphi_1(x))} f(\varphi_1(x))d\rho(x) =$$

$$\int I_{\overline{112|1}}(x)\, e^{V(x,\varphi_1(x))} f(\varphi_1(x)) d\rho(x) =$$

$$\int I_{\overline{112|1}}(\tau(x)) e^{V(\tau(x),\varphi_1(\tau(x)))} f(\varphi_1(\tau(x))) d\alpha(x) =$$

$$\int I_{\overline{112|1}}(\tau(x)) e^{V(x,\varphi_1(x))} f(\varphi_1(\tau(x))) d\alpha(x) =$$

$$\int I_{\overline{112|1}}(\tau(x))\, e^{V(x,\varphi_1(x))} f(\tau(\varphi(x)))) d\alpha(x) =$$

$$\int I_{\overline{11|21}}(x)\, e^{V(x,\varphi_1(x))} f(\tau(\varphi(x)))) d\alpha(x) =$$

$$\int_{\overline{11|21}} e^{V(x,\varphi_1(x))} f(\tau(\varphi(x)))) d\alpha(x) =$$

$$\int_{\overline{21|12}} f(\tau(x)) d\alpha(x) =$$

$$\int I_{\overline{21|12}}\,(x) f(\tau(x)) d\alpha(x) =$$

$$\int I_{\overline{21|12}}\,(\tau^{-1} \circ \tau)(x) f(\tau(x)) d\alpha(x) =$$

$$\int I_{\overline{21|12}}\,(\tau^{-1}(x)) f(x) d\rho(x) =$$

$$\int_{\overline{211|2}} f(x) d\rho(x).$$

Above we took $g = f \circ \tau$.

From the above reasoning we get that both properties are equivalent.

Proposition 2.3.8. *If α is C^*-Gibbs for V, and $V(x,y) = V(\tau(x),\tau(y))$, for all $x, y \in G$, then, the pull back $\rho = \tau^*(\alpha)$ is also C^*-Gibbs for V.*

Proof: Suppose α is C^*-Gibbs for V.

The reasoning of the proof is just a generalization of the argument used on last example.

Consider for $r, s > 0$

$$\varphi : \overline{a_{-r}...a_{-1}|a_1 a_2...a_s} \to \overline{b_{-r}...b_{-1}|b_1 b_2...b_s},$$

where

$$\varphi(\ldots x_{-r+2}x_{-r+1}\, a_{-r}\ldots a_{-1}|a_1a_2\ldots a_s\, x_{s+1}\, x_{s+2}\ldots\,) =$$
$$(\ldots x_{-r+2}x_{-r+1}\, b_{-r}\ldots b_{-1}|b_1b_2\ldots b_s\, x_{s+1}\, x_{s+2}\ldots\,),$$

and

$$\varphi_1 : \overline{a_{-r}\ldots a_{-1}a_1|a_2\ldots a_s} \to \overline{b_{-r}\ldots b_{-1}b_1|b_2\ldots b_s},$$

where

$$\varphi(\ldots x_{-r+2}x_{-r+1}\, a_{-r}\ldots a_{-1}a_1|a_2\ldots a_s\, x_{s+1}\, x_{s+2}\ldots\,) =$$
$$(\ldots x_{-r+2}x_{-r+1}\, b_{-r}\ldots b_{-1}b_1|b_2\ldots b_s\, x_{s+1}\, x_{s+2}\ldots\,),$$

Adapting the argument of last example one can easily show that if for any continuous function g we have that

$$\int_{\overline{a_{-r}\ldots a_{-1}|a_1a_2\ldots a_s}} e^{V(x,\varphi(x))} g(\varphi(x))d\alpha(x) = \int_{\overline{b_{-r}\ldots b_{-1}|b_1b_2\ldots b_s}} g(x)d\alpha(x), \quad (2.17)$$

then, for any continuous function f we have that

$$\int_{\overline{a_{-r}\ldots a_{-1}a_1|a_2\ldots a_s}} e^{V(x,\varphi_1(x))} f(\varphi_1(x))d\rho(x) = \int_{\overline{b_{-r}\ldots b_{-1}b_1|b_2\ldots b_s}} f(x)d\rho(x). \quad (2.18)$$

As α is C^*-Gibbs for V, then (2.18) is true for any f. From (2.18) it follows that ρ is C^*-Gibbs for V.

We point out that it is equivalent to ask the C^*-Gibbs property for V taking symmetric cylinders or taking not symmetric cylinders (this is implicit on the proof of Proposition 2.3.6).

□

2.4 Modular functions and potentials

As we mentioned before given a Hölder function $U : \Omega \to \mathbb{R}$ there is a natural way (described by (2.2)) to get a continuous function V satisfying the property (2.4).

We suppose now that V is such that $V(x,y) = \sum_{k=-\infty}^{\infty}[U(\tau^k(x) - U(\tau^k(y)]$, when $x \sim y$, where $U : \Omega \to \mathbb{R}$ is Hölder (see (2.2)). The function U will sometimes be called a **potential**. We shall also suppose that U is a finite range potential, or equivalently that it depends on a finite number of positive coordinates, that is, there is $k \in \mathbb{N}$ and a function $f : \{1, ..., d\}^k \to \mathbb{R}$, such that, for all $x \in \Omega$ we get

$$U(x) = U\,(...x_{-n}x_{-n+1}...x_{-2}x_{-1} \,|\, x_1\, x_2...x_{m-1}x_m...) = f(x_1, x_2, ..., x_k), \tag{2.19}$$

for this fixed f and $k > 0$, where $U : \Omega \to \mathbb{R}$. In this case we say that U depends on k coordinates.

Note that such V satisfies $V(x, y) = V(\tau(x), \tau(y))$ and then Proposition 2.3.8 can be applied.

Remark 1: By abuse of language we can write $U : \{1, 2, .., d\}^{\mathbb{N}} \to \mathbb{R}$.

If $x \sim y$ it isn't hard to see that there is a finite $M > 0$, such that,

$$V(x, y) = \sum_{k=-\infty}^{\infty} [U(\tau^k(x)) - U(\tau^k(y))] = \sum_{k=-M}^{M} [U(\tau^k(x)) - U(\tau^k(y))].$$

In this way, if $z \sim \varphi(z)$, then,

$$V(z, \varphi(z)) = \sum_{k=-M}^{M} U(\tau^k(z)) - \sum_{k=-M}^{M} U(\tau^k(\varphi(z)) = \sum_{k=-M}^{M} [U(\tau^k(z) - U(\tau^k(\varphi(z))].$$

Therefore, in this case, equation (2.10) means

$$\int_{\overline{x_{-n},...,x_{-1}\,|\,x_1,x_2,...,y_n}} e^{\sum_{k=-M}^{M} U(\tau^k(\varphi(z))] - \sum_{k=-M}^{M} U(\tau^k(z))} f(\varphi(z)) d\alpha(z) =$$

$$\int_{\overline{y_{-n},...,y_{-1}\,|\,y_1,y_2,...,y_n}} f(z) d\alpha(z) \tag{2.20}$$

If α is C^*-Gibbs for V, and $V(z, \varphi(z)) = \sum_{k=-\infty}^{+\infty} U(z) - U(\varphi(z))$ we also say **by abuse of language that α is C^*-Gibbs for $U : \Omega \to \mathbb{R}$.**

Definition 2.4.1. *Given a function $V : G \to \mathbb{R}$, $V(x, y) = \sum_{k=-\infty}^{\infty} [U(\tau^k(x)) - U(\tau^k(y))]$, with U of Hölder class, we say that a probability measure α on Ω is the* ***quasi C^*-Gibbs probability*** *with respect to the parameter $\beta \in \mathbb{R}$*

and U, if there exists constants $d_1 > 0$ and $d_2 > 0$, such that, for any $x \sim y$ and any $O_{(x,y)}$,

$$d_1 \int_{O_{(x,y)}} \exp(-\beta V(z, \varphi_{(x,y)}(z))) g(\varphi_{(x,y)}(z)) d\alpha(z) \leq$$

$$\int_{\varphi_{(x,y)}(O_{(x,y)})} g(z) d\alpha(z) \leq d_2 \int_{O_{(x,y)}} \exp(-\beta V(z, \varphi_{(x,y)}(z))) g(\varphi_{(x,y)}(z)) d\alpha(z) \tag{2.21}$$

for every every continuous function $g : \Omega \to \mathbb{C}$ (and symmetric conjugated homeomorphism $(O_{(x,y)}, \varphi_{(x,y)})$.

In the same way as before one can extend the above property for symmetric conjugated homeomorphisms to non symmetric conjugated homeomorphisms.

A C^*-Gibbs probability is a quasi C^*-Gibbs probability.

We say that a potential $\tilde{U} : \{1, 2.., d\}^{\mathbb{N}} \to \mathbb{R}$ - which depends on a finite number of coordinates - is **normalized**, if for k large enough and for any $(x_1, x_2, .., x_k)$ we get $\sum_{j=1}^{d} e^{\tilde{U}(j, x_1 ..., x_{k-1})} = 1$ - in particularly, we get $e^{\tilde{U}(x)} = e^{\tilde{U}(x_1 ..., x_k)} < 1$ for all $x = (x_1, x_2, ...) \in \{1, 2, ..., d\}^{\mathbb{N}}$.

From this follows that for any $w = (w_1, w_2, ..., w_m, ...) \in \{1, 2, ..., d\}^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$$\sum_{z_1, z_2, .., z_n = 1}^{d} e^{\sum_{j=0}^{n-1} \tilde{U}(\sigma^j(z_1, z_2, .., z_n, w_1, w_2, w_3, ..., w_m, ...)} = 1$$

where σ is the shift acting on $\{1, 2, .., d\}^{\mathbb{N}}$.

Suppose for such U that α is quasi C^*-Gibbs for U (satisfies the double inequality (2.21) for any continuous g). This implies in particular that there exist $d_1, d_2 > 0$, such that, for any cylinders of the form $\overline{|\, x_1^0, x_2^0, ... x_s^0}$ and $\overline{|\, y_1^0, y_2^0, ... y_s^0}$, and a function φ, such that,

$$d_1 \int_{\overline{|\, x_1^0, x_2^0, ... x_s^0}} e^{-\beta V(z, \varphi(z))} g(\varphi_{(x,y)}(z)) d\alpha(z) \leq$$

$$\int_{\overline{|\, y_1^0, y_2^0, ... y_s^0}} g(z) d\alpha(z) \leq d_2 \int_{\overline{|\, x_1^0, x_2^0, ... x_s^0}} e^{-\beta V(z, \varphi(z))} g(\varphi_{(x,y)}(z)) d\alpha(z), \tag{2.22}$$

where $\varphi_{(x,y)}$ is the associated conjugating homeomorphism, such that,

$$\varphi_{(x,y)} : (\overline{| \, x_1^0, x_2^0, ...x_s^0}) \to \overline{| \, y_1^0, y_2^0, ...y_s^0}$$

Example 2.4.2. *Consider the homeomorphism* $\varphi : \overline{112|2} \to \overline{|1122}$ *given by*

$$\varphi(...z_{-4}112|2z_2z_3z_4...) = (...z_{-4}z_2z_3z_4|1122z_5...).$$

Note that $\overline{112|2}$ *is translation by* τ^{-3} *of the set* $\overline{|1122}$.
Consider the conjugating homeomorphisms, φ_1, φ_2, φ_3 *and* φ_4, *given by*

$$\varphi_1(...z_{-4}112|2\,\mathbf{11}z_4...) = (...z_{-4}\,\mathbf{11}|1122z_5...),$$

$$\varphi_2(...z_{-4}112|2\,\mathbf{12}\,z_4...) = (...z_{-4}\,\mathbf{12}\,|1122z_5...),$$
$$\varphi_3(...z_{-4}112|2\,\mathbf{21}\,z_4...) = (...z_{-4}\,\mathbf{21}\,|1122z_5...),$$
$$\varphi_4(...z_{-4}112|2\,\mathbf{22}z_4...) = (...z_{-4}\,\mathbf{22}\,|1122z_5...).$$

Suppose α *is quasi-*C^* *Gibbs and satisfies (2.21).*
Therefore,

$$\int_{\overline{112|2}} e^{V(x,\varphi(x))} f(\varphi(x)) d\alpha(x) =$$

$$\int_{\overline{112|211}} e^{V(x,\varphi(x))} f(\varphi(x)) + \int_{\overline{112|212}} e^{V(x,\varphi(x))} f(\varphi(x)) +$$

$$\int_{\overline{112|221}} e^{V(x,\varphi(x))} f(\varphi(x)) + \int_{\overline{112|222}} e^{V(x,\varphi(x))} f(\varphi(x)) =$$

$$\int_{\overline{112|211}} e^{V(x,\varphi_1(x))} f(\varphi_1(x)) + \int_{\overline{112|212}} e^{V(x,\varphi_2(x))} f(\varphi_2(x)) +$$

$$\int_{\overline{112|221}} e^{V(x,\varphi_3(x))} f(\varphi_3(x)) + \int_{\overline{112|222}} e^{V(x,\varphi_4(x))} f(\varphi_4(x)) \leq$$

$$\frac{1}{d_1} \, [\int_{\overline{11|1122}} f(x) + \int_{\overline{12|1122}} f(x) + \int_{\overline{21|1122}} f(x) + \int_{\overline{22|1122}} f(x) \,] =$$

$$\frac{1}{d_1} \int_{\overline{|1122}} f d\alpha(x),$$

where some of the $d\alpha$ *where omitted. We proved that*

$$\int_{\overline{112|2}} e^{V(x,\varphi(x))} f(\varphi(x)) d\alpha(x) \leq \frac{1}{d_1} \int_{\overline{|1122}} f d\alpha(x),$$

for any measurable function f.

Taking $f = 1$, *we get that*

$$\int_{\overline{112|2}} e^{V(x,\varphi(x))} d\alpha(x) \leq \frac{1}{d_1} \int_{\overline{|1122}} d\alpha(x).$$

As $e^{V(x,\varphi(x))}$ *is strictly positive we get that if* $\alpha(\overline{|1122}) = 0$, *then,* $\alpha(\overline{112|2}) = 0$.

Using the inequality for d_2 *in (2.21) we get in a similar way that if* $\alpha(\overline{112|2}) = 0$, *then,* $\alpha(\overline{|1122}) = 0$.

One can also show that

$$\int_{\overline{|1122}} d\alpha(x) \leq d_2 \int_{\overline{112|2}} e^{V(x,\varphi(x))} d\alpha(x).$$

Proposition 2.4.3. *Suppose* α *is quasi-*C^**-Gibbs for a* ***potential*** U ***that depends on finite coordinates****, then*

$$\alpha(\overline{a_{-r}...a_{-1}|a_1 a_2...a_s}) > 0,$$

if and only if,

$$\alpha(\overline{|a_{-r}...a_{-1}a_1 a_2...a_s}) > 0.$$

Moreover, there exist $b_1 > 0, b_2 > 0$, *such that, for any cylinder set of the form* $\overline{a_{-r}...a_{-1}|a_1 a_2...a_s}$ *we get*

$$b_1 \;\; \alpha(\, \overline{a_{-r}...a_{-1}|a_1 a_2...a_s}\,) \leq \alpha(\, \overline{|\, a_{-r}...a_{-1}a_1 a_2...a_s}\,) \leq$$
$$b_2 \;\alpha(\, \overline{a_{-r}...a_{-1}|a_1 a_2...a_s}\,). \tag{2.23}$$

Proof: We left the proof for the reader which is an adaptation of the reasoning of Example 2.4.2.

□

The next result shows that we can always consider normalized potentials (see Theorem 2.2 in [47] for general results) on the definition of quasi C^*-Gibbs probability.

Theorem 2.4.4. *Suppose the probability α on Ω is C^*-Gibbs for Hölder potential U. Assume, $X : \Omega \to \mathbb{R}$ is such that $X = U + g - g \circ \tau + \lambda$, where $g : \Omega \to \mathbb{R}$ is a Hölder continuous function and λ a constant, then α is quasi C^*-Gibbs for X.*

Proof: Suppose that for any continuous f we have

$$\int_{O_{x,y}} e^{\beta \sum_{k=-\infty}^{\infty} U(\tau^k(\varphi(z))) - U(\tau^k(z))} f(\varphi(z)) d\alpha(z) =$$

$$\int_{\varphi(O_{(x,y)})} f(z) d\alpha(z) \tag{2.24}$$

Note that

$$\sum_{k=-\infty}^{\infty} [\, g(\tau^k(z)) - g(\tau^k(\varphi(z)))]$$

is limited since g is Hölder, actually the summation is absolutely convergent by the same reason. The same can be said of

$$\sum_{k=-\infty}^{\infty} [\, g(\tau^{k+1}(z)) - g(\tau^{k+1}(\varphi(z)))]$$

and of

$$\sum_{k=-\infty}^{\infty} [\, U(\tau^k(z)) - U(\tau^k \varphi(z))\,]$$

.

The absolute convergence allow us to sum the quantities above in any order, the resulting sum is limited since each of the above quantities are.

Therefore,

$$\sum_{k=-\infty}^{\infty} [X(\tau^k(\varphi(z))) - X(\tau^k(z))] =$$

$$[\, \sum_{k=-\infty}^{\infty} U(\tau^k(\varphi(z))] - U(\tau^k(z))\,]+$$

$$[\, \sum_{k=-\infty}^{\infty} g(\tau^k(\varphi(z)) - g(\tau^k(z))\,]-$$

$$[\sum_{k=-\infty}^{\infty} g(\tau^{k+1}(\varphi(z))) - g(\tau^{k+1}(z))\,]$$

is bounded above and below by constants which do not depend on $x \sim y$, $O_{x,y}$ and corresponding $\varphi_{x,y}$.

Then, α is quasi C^*-Gibbs for X.

$\square$

By Proposition 1.2 in [47] given a Hölder potential $U : \Omega \to \mathbb{R}$, one can find W depending on positive coordinates $(1, 2, 3, .., n, ...) \in \{1, 2, ..., d\}^{\mathbb{N}}$ and a continuous function $v : \Omega \to \mathbb{R}$ (which depends on finite coordinates), such that, $W = U + v - v \circ \tau$.

The function V is Hölder and then last theorem can be applied.

More precisely, there exist $\tilde{W} : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ an r, such that,

$$W(...x_{-n-1}\; x_{-n}x_{-n+1}...x_{-1}\,|\,x_1...x_m\, x_{m+1}...) =$$

$$\tilde{W}(x_1...x_m\, x_{m+1}...) = K(x_1...x_r\,),$$

for a certain function $K : \{1, 2, ..., d\}^r \to \mathbb{R}$.

The bottom line is: from Theorem 2.2 in [47], given such $\tilde{W}$ one can find, u and positive constant λ, such that, $\tilde{W} = \tilde{U} + u - u \circ \tau + \lambda$. Moreover, $\tilde{U} : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ and $u : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ both depend on a finite number of coordinates.

Remark 2: Therefore, from Theorem 2.4.4 if α is C^*-Gibbs for a Hölder potential $U : \Omega \to \mathbb{R}$, which depends on a finite number of coordinates, we can assume that α is quasi-C^*-Gibbs for **another** potential, denoted $\tilde{U}$, which is normalized and depending on a finite number of coordinates.

By abuse of language one can write $\tilde{U} : \{1, 2, ..., d\}^{\mathbb{Z}} \to \mathbb{R}$.

2.5 Equivalence between equilibrium measures and C^*-Gibbs measures

First we present two important and well known theorems (see theorems 1.2 and 1.22 in [11] and also [60]).

We will consider without loss of generality that $\beta = 1$.

$\mathcal{M}_\tau(\Omega)$ denotes the set on invariant probabilities for τ acting on Ω.

Theorem 2.5.1. *(see Theorem 1.2 in [11]) Suppose* $U : \Omega \to \mathbb{R}$ *is of Hölder class. Then, there is a unique* $\rho \in \mathcal{M}_\tau(\Omega)$*, for which one can find constants* $C_1 > 0$*,* $C_2 > 0$*, and* P *such that,* ***for all*** $s \geq 0$*, for all cylinder* $\overline{|\, y_1^0, y_2^0, ...y_s^0}$ *we have*

$$C_1 \leq \frac{\rho(\overline{|\, y_1^0, y_2^0, ...y_s^0})}{\exp\left(-P\, s + \sum_{k=0}^{s-1} U(\tau^k x)\right)} \leq C_2, \tag{2.25}$$

where

$$x = (...x_{-k}, x_{-k+1}, ..., x_{-1} \,|\, x_1, ..., x_m, x_{m+1}, ...) \in \overline{|\, y_1^0, y_2^0, ...y_s^0} \subset \Omega,$$

We call (2.25) the Bowen's inequalities.

Definition 2.5.2. *The probability* $\rho = \rho_U$ *of Theorem 2.5.1 is called* ***equilibrium probability*** *for the potential* U*.*

Theorem 2.5.3. *Given* U *as above and* ρ_U *the equilibrium measure for* U*, then* ρ_U *is the unique probability on* $\mathcal{M}_\tau(\Omega)$*, for which*

$$h(\rho_U) + \int U d\rho_U = P(U) := \sup_{\nu \in \mathcal{M}_\tau} \{h(\nu) + \int U d\nu\},$$

where $h(\nu)$ *is the entropy of* ν*.*

For a proof see [47] or [11].

$P(U)$ is called the pressure of U. One can show that the P of (2.25) is equal to such $P(U)$.

Remember that if α is C^*-Gibbs for V, and $V(z, \varphi(z)) = \sum_{k=-\infty}^{+\infty} U(z) - U(\varphi(z))$ we also say by abuse of language that α is C^*-Gibbs for $U : \Omega \to \mathbb{R}$.

Note that if ρ is an equilibrium probability for a Hölder potential U, then, it is also an equilibrium probability for $U + (g \circ \tau) - g + c$, where c is constant and $g : \Omega \to \mathbb{R}$ is Hölder continuous (see [47]). In this way we can assume without lost of generality that ρ_U is an equilibrium probability for a normalized potential U. If U is normalized then $P(U) = 0$.

If α on Ω is C^*-Gibbs for U, then, from Remark 2 we have that α is quasi-C^*-Gibbs for **another** potential U which is normalized.

Note that given U we are dealing with two definitions: C^*-Gibbs and Equilibrium. From the above comments we can assume in either case that U is normalized.

The bottom line is: we can assume (see [47]) that the Hölder potential $\tilde{U} = U + (g \circ \tau) - g + c$ is normalized, depends just on future coordinates $\tilde{U} : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ and has pressure zero.

We will work here (due to Theorem 2.4.4 and the above comments) with the case where the probability α - which is C^*-Gibbs for the potential U - is also a quasi-C^*-Gibbs probability for the potential $\tilde{U}$ satisfying Pressure $P(\tilde{U}) = 0$. In this case, if we want to prove expression (2.25) for such probability α over Ω, this can be simplified just showing that there exist $c_1, c_2 > 0$, such that,

$$c_1 \leq \frac{\alpha(\overline{|\, y_1^0, y_2^0, ...y_s^0})}{\exp\left(\sum_{k=0}^{s-1} \tilde{U}(\sigma^k x)\right)} \leq c_2, \tag{2.26}$$

where σ is the shift acting on $\{1, 2, .., d\}^{\mathbb{N}}$ and where x is of the form

$$x = (y_1^0, y_2^0, ...y_s^0, x_{s+1}, ..., x_m, x_{m+1}, ...) \in \{1, 2, .., d\}^{\mathbb{N}}.$$

Remark 3: Indeed, due to Remark 2 we get that $\tilde{U} = U + (g \circ \tau) - g + c$, where g depends on finite coordinates. Therefore, to show (2.26) - for α **which is C^*-Gibbs for $U : \Omega \to \mathbb{R}$** - is equivalent to prove (see details on the proof of Theorem 2.4.4) that there exists $C_1, C_2 > 0$, such that,

$$C_1 \leq \frac{\alpha(\overline{|\, y_1^0, y_2^0, ...y_s^0})}{\exp\left(\sum_{k=0}^{s-1} U(\tau^k x)\right)} \leq C_2, \tag{2.27}$$

where τ is the shift acting on $\{1, 2, .., d\}^{\mathbb{Z}}$ and where

$$x = (...x_{-2}, x_{-1} \,|\, y_1^0, y_2^0, ...y_s^0, x_{s+1}, ..., x_m, x_{m+1}, ...) \in \{1, 2, .., d\}^{\mathbb{Z}}.$$

It's important to note that the main equivalence (equilibrium and C^*-Gibbs) is still valid in a more general setting of a Hölder potential in a general Smale Space. D. Ruelle proved on the setting of hyperbolic diffeomorphisms that Equilibrium implies C^*-Gibbs in his book [60], see theorems 7.17(b), 7.13(b) and section 7.18). On the other hand Haydn proved in the paper [23] that C^*-Gibbs implies Equilibrium. Later, the paper [24] presents a shorter proof of the equivalence.

On the two next sections we will present the proof of the following theorem.

Theorem 2.5.4. *Given a potential U depending on a finite number of coordinates, then, α is the equilibrium measure for U, if and only if, α is C^*-Gibbs for U. As the equilibrium probability is unique we get that the C^*-Gibbs probability for U is unique.*

2.6 Equilibrium implies C^*-Gibbs

The fact that Equilibrium state implies C^*-Gibbs was proved by Ruelle in a general setting. The proof is in the book [60] (see theorems 7.17(b), 7.13(b) and section 7.18).

For completeness we will explain the proof on our setting.

We drop the (x, y) on $\varphi_{(x,y)}$ and $\mathcal{O}_{(x,y)}$.

Lemma 2.6.1. *Let (Ω, τ) be the shift on the Bernoulli space $\Omega = \{1, 2, ..., d\}^{\mathbb{Z}-\{0\}}$ and ρ_0 be the τ-invariant probability measure which realizes the maximum of the entropy, or, simply the equilibrium state for $U = 0$. If $(\mathcal{O}, \varphi)$ is a conjugating homeomorphism, then for any continuous function f*

$$\int_{\mathcal{O}} f(\varphi(x)) d\rho_0(x) = \int_{\varphi(\mathcal{O})} f(x) d\rho_0(x) \tag{2.28}$$

Proof: Given

$$\mathcal{O} = \overline{x_{-n} x_{-n+1} ... x_{-1} \,|\, x_1 ... x_{m-1} x_m}$$

and

$$\varphi(\mathcal{O}) = \overline{y_{-n} y_{-n+1} ... y_{-1} \,|\, y_1 ... y_{m-1} y_m},$$

we have that for any $r > m$ and $k > n$

$$\rho_0(\overline{x_{-k} x_{-k+1} ... x_{-1} \,|\, x_1 ... x_{r-1} x_r}) = d^{-(r+k)} =$$

$$\rho_0(\overline{y_{-k} y_{-k+1} ... y_{-1} \,|\, y_1 ... y_{r-1} y_r}).$$

We shall prove that equation (2.28) is valid when f is equal to an characteristic function of an arbitrary cylinder. Note that for this purpose is enough to consider f as the characteristic function of cylinders of the form $\overline{y_{-k} y_{-k+1} ... y_{-1} \,|\, y_1 ... y_{r-1} y_r}$. Therefore,

$$\int_{\mathcal{O}} I_{\overline{y_{-k}y_{-k+1}\cdots y_{-1}\,|\,y_1\cdots y_{r-1}y_r}}(\varphi(x))d\rho_0(x) =$$

$$\int_{\varphi(\mathcal{O})} I_{\overline{y_{-k}y_{-k+1}\cdots y_{-1}\,|\,y_1\cdots y_{r-1}y_r}}(y)d\rho_0(y).$$

From this follows the claim.

□

We denote by $C^\alpha(\Omega)$ the set of α Hölder functions on Ω.

Lemma 2.6.2. *(see corollary 7.13 in [60]) Consider the shift space (Ω, τ) and $A, B \in C^\alpha(\Omega)$. Write for integers $a < 0$ and $b > 0$*

$$Z_{[a,b]} = \int e^{\sum_{k=a}^{b-1} B\circ\tau^k} \; d\; \rho_A$$

Then, $Z_{[a,b]}^{-1}$ $(\exp \sum_{k=a}^{b-1} B \circ \tau^k)$ ρ_A tends to ρ_{A+B} in the weak star topology, when $a \to -\infty$ and $b \to +\infty$.

In particular, taking $A = 0$, when $a \to -\infty$ and $b \to +\infty$, we get that

$$Z_{[a,b]}^{-1} \; e^{\sum_{k=a}^{b-1} B\circ\tau^k} \; \rho_0 \;\to\; \rho_B,$$

where

$$Z_{[a,b]} = \int e^{\sum_{k=a}^{b-1} B\circ\tau^k} \; d\; \rho_0$$

Theorem 2.6.3. *If ρ_B is an equilibrium state for a potential B that depends on a finite number of coordinates then it is a C^*-Gibbs state for B.*

Proof: The statement holds for $B = 0$ by Lemma 2.6.1. Moreover, Lemma 2.6.2 allow us to extend this result for all $B \in C^\alpha(\Sigma_N)$ in the following manner: given $\mathcal{O}$ and the associated φ

$$\int_{\varphi(\mathcal{O})} g(x)d\rho_B(x) = \lim_{\substack{a\to-\infty \\ b\to\infty}} Z_{[a,b]}^{-1} \int_{\varphi(\mathcal{O})} \exp\left(\sum_{k=a}^{b-1} B\circ\tau^k(x)\right) g(x)d\rho_0(x) \stackrel{2.6.1}{=}$$

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} Z_{[a,b]}^{-1} \int_{\mathcal{O}} \exp\left(\sum_{k=a}^{b-1} B\circ\tau^k\circ\varphi(x)\right) g\circ\varphi(x)d\rho_0(x) =$$

$$\lim_{\substack{a\to-\infty\\ b\to\infty}} Z^{-1}_{[a,b]}\int_{\mathcal{O}} \exp\left(\sum_{k=a}^{b-1} B\circ\tau^k\circ\varphi(x) - \sum_{k=0}^{b-1} B\circ\tau^k(x)\right)$$

$$\exp\left(\sum_{k=a}^{b-1} B\circ\tau^k(x)\right) g\circ\varphi(x)d\rho_0(x) =$$

$$\lim_{\substack{a\to-\infty\\ b\to\infty}} Z^{-1}_{[a,b]}\int_{\mathcal{O}} e^{(-V(x,\varphi(x)))} g\circ\varphi(x)\exp\left(\sum_{k=a}^{b-1} B\circ\tau^k(x)\right) d\rho_0(x) =$$

$$\int_{\mathcal{O}} e^{(-V(x,\varphi(x)))} g\circ\varphi(x)d\rho_B(x).$$

Since the equality

$$\int_{\varphi(\mathcal{O})} g(x)d\rho_B(x) = \int_{\mathcal{O}} e^{(-V(x,\varphi(x)))} g\circ\varphi(x)d\rho_B(x)$$

was verified for any conjugating homeomorphism φ and any g, then it follows that ρ_B is an C^*-Gibbs state for B.

□

2.7 C^*-Gibbs implies Equilibrium

Given a C^*-Gibbs probability α for a potential U that depends on a finite number of coordinates we will show in this section that α is the equilibrium probability for U. We shall further assume that the potential U depend only on positive coordinates and is normalized according to the Ruelle operator, i.e.

$$\sum_{z_1,z_2,..,z_n=1}^{d} e^{\sum_{j=0}^{n-1}\tilde{U}(\sigma^j(z_1,z_2,..,z_n,w_1,w_2,w_3,...,w_m,...)} = 1, \tag{2.29}$$

for any $w = (w_1, w_2, ..., w_m, ...) \in \{1, 2, ..., d\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Such assuptions aren't restrictive, since given any potential W that depends on a finite number of coordinates, it's possible to find a function g depending on finite coordinates, and a normalized potential $\tilde{W}$ that depends of future coordinates, such that [47]

$$W = \tilde{W} + g - g\circ\tau - \lambda$$

If we show that α is τ-invariant and also satisfies the Bowen's inequalities for U, then, it will follow that α is the equilibrium probability for U by Theorem 2.5.1.

We will show first that a quasi C^*-Gibbs probability α for U satisfies the Bowen's inequalities (2.27) for U.

Later we will show that a C^*-Gibbs probability α is invariant for τ (see Proposition 2.7.5). This will finally show (see Theorem 2.7.6) that "C^*-Gibbs implies Equilibrium".

Note that we want to show (2.27) but due to Remark 3 we just have to show (2.26).

We assume α is such that (2.22) is true, that is, there exists $d_1, d_2 > 0$, such that, for any continuous function g

$$d_1 \int_{\overline{|\, x_1^0, x_2^0, \ldots x_s^0}} e^{-V(z,\varphi_{(x,y)}(z))} g(\varphi_{(x,y)}(z)) d\alpha(z) \leq$$

$$\int_{\overline{|\, y_1^0, y_2^0, \ldots y_s^0}} g(z) d\alpha(z) \leq d_2 \int_{\overline{|\, x_1^0, x_2^0, \ldots x_s^0}} e^{-V(z,\varphi_{(x,y)}(z))} g(\varphi_{(x,y)}(z)) d\alpha(z). \quad (2.30)$$

We denote $\mathcal{U} = \sup_{x\in\Omega} U(x) - \inf_{x\in\Omega} U(x)$.

Lemma 2.7.1. *Given a normalized Hölder potential $U(x) = f(x_1, x_2, .., x_r)$, consider $x_1^0 \ldots x_s^0$ and $y_1^0 \ldots y_s^0$ fixed, and also $a, b \in \{1, 2 \ldots, d\}$ fixed. Let*

$$x = (\ldots x_{-m} x_{-m+1} \ldots x_{-1} \,|\, x_1^0 \ldots x_s^0 \, x_{s+1}, x_{s+2}, \ldots x_{m-1} x_m \ldots) \in \overline{|\, x_1^0, x_2^0, \ldots x_s^0}$$

$$y = (\ldots x_{-m} x_{-m+1} \ldots x_{-1} \,|\, y_1^0 \ldots y_s^0 \, x_{s+1}, x_{s+2}, \ldots x_{m-1} x_m \ldots) \in \overline{|\, y_1^0, y_2^0, \ldots y_s^0},$$

and also

$$x_a = (\ldots x_{-m} x_{-m+1} \ldots x_{-1} \,|\, x_1^0 \ldots x_s^0 \, \mathbf{a} \, x_{s+2} \ldots x_{m-1} x_m \ldots) \in \overline{|\, x_1^0, x_2^0, \ldots x_s^0}$$

$$y_b = (\ldots x_{-m} x_{-m+1} \ldots x_{-1} \,|\, y_1^0 \ldots y_s^0 \, \mathbf{b} \, x_{s+2} \ldots x_{m-1} x_m \ldots) \in \overline{|\, y_1^0, y_2^0, \ldots y_s^0}.$$

Assume that $x \sim y$.

Then,

$$
\begin{gathered}
|\sum_{k=-\infty}^{\infty} U(\tau^k(x_a)) - U(\tau^k(y_b))| \leq \\
2\, r\, \mathcal{U} \; + \; |\sum_{k=-\infty}^{\infty} U(\tau^k(x)) - U(\tau^k(y))|\,.
\end{gathered}
$$

Proof: Let $\mathbb{I}$ the set of indicies for k such that $U(\tau^k(x_a))$ (or, $U(\tau^k(y_b))$) differs from $U(\tau^k(x))$ (or, $U(\tau^k(y))$). It's easy to see that the cardinality of $\mathbb{I}$ is r. Therefore

$$
\begin{gathered}
|\sum_{k\in\mathbb{Z}} U(\tau^k(x_a)) - U(\tau^k(y_b))| \leq \\
|\sum_{k\in\mathbb{Z}\setminus\mathbb{I}} U(\tau^k(x_a)) - U(\tau^k(y_b))| \; + |\sum_{k\in\mathbb{I}} U(\tau^k(x_a)) - U(\tau^k(y_b))| \; = \\
|\sum_{k\in\mathbb{Z}\setminus\mathbb{I}} U(\tau^k(x)) - U(\tau^k(y))| \; + |\sum_{k\in\mathbb{I}} U(\tau^k(x_a)) - U(\tau^k(y_b))| \; \leq \\
|\sum_{k\in\mathbb{Z}\setminus\mathbb{I}} U(\tau^k(x)) - U(\tau^k(y))| \; + r\mathcal{U} \leq \\
|\sum_{k\in\mathbb{Z}} U(\tau^k(x)) - U(\tau^k(y))| \; + 2r\mathcal{U}
\end{gathered}
$$

□

We will adapt the formulation of Proposition 2.1 in [23] to the present situation.

For fixed $\overline{|\, x_1^0, x_2^0, ...x_s^0}$ denote

$$U_a = \overline{|\, x_1^0, x_2^0, ..., x_s^0, a}$$

$a = 1, 2.., d.$

Note that $\sum_a \alpha(U_a) = \;\; \alpha(\overline{|\, x_1^0, x_2^0, ...x_s^0})$, in particular

$$\sum_a \alpha(U_a) < d \;\; \alpha(\overline{|\, x_1^0, x_2^0, ...x_s^0}). \tag{2.31}$$

Consider now a fixed $\overline{|\, y_1^0, y_2^0, ...y_s^0}$ and $\varphi_{a,b}$, $a = 1, 2, ..., d$, $b = 1, 2, ..., d$, denotes the conjugating homeomorphism from U_a to $\overline{|\, y_1^0, y_2^0, ...y_s^0\, b} = \varphi_{a,b}(U_a)$.

Note also that for each a

$$\alpha(\overline{\,|\,y_1^0, y_2^0, ...y_s^0}\,) = \sum_{b=1}^{d} \alpha(\,\varphi_{a,b}\,(U_a)\,). \tag{2.32}$$

Denote

$$K = \sup_{m\in\mathbb{N}}\{\sum_{k=0}^{m-1}[\tilde{U}\tau^k(u) - \tilde{U}\tau^k(v)]\ ,\ \text{where } u, v \in \overline{\,|\,a_1, a_2, ..., a_m}\ ,$$

$$\text{and } (a_1, a_2, ..., a_m) \in \{1, 2, .., d\}^m\,\}.$$

On the above expression we ask that $u \sim v$.
Note that if α is C^*-Gibbs and satisfies (2.30) we get in particular that

$$d_1 \int_{\overline{|\,x_1^0, x_2^0, ...x_s^0}} e^{-V(z,\varphi_{(x,y)}(z))} d\alpha(z) \leq$$

$$\int_{\overline{|\,y_1^0, y_2^0, ...y_s^0}} d\alpha(z) \leq d_2 \int_{\overline{|\,x_1^0, x_2^0, ...x_s^0}} e^{-V(z,\varphi_{(x,y)}(z))} d\alpha(z). \tag{2.33}$$

Proposition 2.7.2. *Suppose α is quasi-C^*-Gibbs for U as above. Then, there exists a constat $c_1 > 0$, such that,*

$$c_1 \leq e^{-\sum_{k=0}^{s-1} U\tau^k(x)}\ \alpha(\overline{\,|\,x_1^0, x_2^0, ...x_s^0}\,)$$

for any cylinder $\overline{|\,x_1^0, x_2^0, ...x_s^0}$ and any x on the cylinder.

The α-probability of any cylinder is positive.

Proof: We assume that (2.33) is true.

Fix a certain cylinder $\overline{|\,x_1^0, x_2^0, ...x_s^0}$ and fix a point $x \in \overline{|\,x_1^0, x_2^0, ...x_s^0}$ then choose another cylinder $\overline{|\,y_1^0, y_2^0, ...y_s^0}$ with non null probability and a point $y \in \overline{|\,y_1^0, y_2^0, ...y_s^0}$. Fix $x \in \overline{|\,x_1^0, x_2^0, ...x_s^0}$ and $y \in \overline{|\,y_1^0, y_2^0, ...y_s^0}$. Choose $a, b \in \{1, 2, ..., d\}$ and define x_a and y_b as

$$x_a = (...x_{-m}x_{-m+1}...x_{-1}\,|\,x_1^0\,...x_s^0\,\mathbf{a}, x_{s+2}, ...x_{m-1}x_m...) \in \overline{|\,x_1^0, x_2^0, ...x_s^0}$$

$$y_b = (...x_{-m}x_{-m+1}...x_{-1}\,|\,y_1^0\,...y_s^0\,\mathbf{b}, x_{s+2}, ...x_{m-1}x_m...) \in \overline{|\,y_1^0, y_2^0, ...y_s^0}.$$

we get from Lemma 2.7.1 that

$$\alpha(\,\varphi_{a,b}\,(U_a)\,) \leq d_2 \int_{U_a} e^{\sum_{k=-\infty}^{\infty} U(\tau^k \varphi(z)) - U(\tau^k(z))} d\alpha(z) \leq$$

$$d_2 \int e^{\sum_{k=0}^{s} U(\tau^k\varphi(z)) - U(\tau^k y_b) + U(\tau^k \varphi(z)) - U(\tau^k x_a)} e^{\sum_{k=0}^{s} U(\tau^k y_b) - U(\tau^k x_a)}$$

$$e^{\sum_{k=s}^{\infty} U(\tau^k\varphi(z)) - U(\tau^k z)} e^{\sum_{k=0}^{\infty} U(\tau^{-k}\varphi(z)) - U(\tau^{-k} z)} d\alpha(z) \leq$$

$$d_2\, e^{2\,K + r\,\mathcal{U}} e^{\sum_{k=0}^{s-1} [\tilde{U}\tau^k(y_b) - \tilde{U}\tau^k(x_a)]}\ \alpha(U_a)\ \leq$$

$$d_2\, e^{2\,K + 3\,r\,\mathcal{U}} e^{\sum_{k=0}^{s-1} [\tilde{U}\tau^k(y) - \tilde{U}\tau^k(x)]}\ \alpha(U_a).$$

Then, from (2.32)

$$\alpha(\overline{|\, y_1^0, y_2^0, ... y_s^0}) = \sum_{b=1}^{d} \alpha(\,\varphi_{a,b}\,(U_a)\,) \leq d_2\, d\, e^{2\,K + 3r\mathcal{U}} e^{\sum_{k=0}^{s-1} [U\tau^k(y) - U\tau^k(x)]}\ \alpha(U_a).$$

From this and from (2.29) we get

$$1 = \sum_{y_1^0, y_2^0, ... y_s^0 = 1}^{d} \alpha(\overline{|\, y_1^0, y_2^0, ... y_s^0}) \leq d_2\, d\, e^{2\,K + 3\,r\mathcal{U}} e^{-\sum_{k=0}^{s-1} U\tau^k(x)}\ \alpha(U_a),$$

and, finally, for $x = (..., x_{-t}, ..., x_{-2}, x_{-1} \,|\, x_1, x_2, .., x_t, ...) \in \overline{|\, x_1^0, x_2^0, ... x_s^0}$

$$d = \sum_{a=1}^{d} \sum_{y_1^0, y_2^0, ... y_s^0 = 1}^{d} \alpha(\overline{|\, y_1^0, y_2^0, ... y_s^0}) \leq \sum_{a=1}^{d} d_2\, d\, e^{2\,K + 3r\mathcal{U}} e^{-\sum_{k=0}^{s-1} U\tau^k(x)}\ \alpha(U_a) =$$

$$d_2\, d\, e^{2\,K + 3\,r\mathcal{U}} e^{-\sum_{k=0}^{s-1} U\tau^k(x)}\ \alpha(\overline{|\, x_1^0, x_2^0, ... x_s^0}).$$

This also shows that the α-probability of any cylinder $\overline{|\, x_1^0, x_2^0, ... x_s^0}$ is positive when α is quasi-C^*-Gibbs.

By Proposition 2.4.3 we get that any cylinder of the form $\overline{x_{-m} ... x_{-1} | x_1 x_2 ... x_s}$ has positive α-probability.

$\square$

Proposition 2.7.3. *There exists a constant $c_2 > 0$, such that,*

$$e^{-\sum_{k=0}^{s-1} U\tau^k(x)}\ \alpha(\overline{|\, x_1^0, x_2^0, ...x_s^0}\,) \leq c_2,$$

for any cylinder $\overline{|\, x_1^0, x_2^0, ...x_s^0}$ and any x on the cylinder.

The α-probability of any cylinder is positive.

Proof: We assume that (2.33) is true.

Again consider fixed $x \in \overline{|\, x_1^0, x_2^0, ...x_s^0}$ and $y \in \overline{|\, y_1^0, y_2^0, ...y_s^0}$. Choose $a, b \in \{1, 2, ..., d\}$ and define x_a and y_b as

$$x_a = (...x_{-m}x_{-m+1}...x_{-1} \,|\, x_1^0 \,...x_s^0\, \mathbf{a}, x_{s+2}, ...x_{m-1}x_m...) \in \overline{|\, x_1^0, x_2^0, ...x_s^0}$$

$$y_b = (...x_{-m}x_{-m+1}...x_{-1} \,|\, y_1^0 \,...y_s^0\, \mathbf{b}, x_{s+2}, ...x_{m-1}x_m...) \in \overline{|\, y_1^0, y_2^0, ...y_s^0}.$$

Using an analogous reasoning as in proposition 2.7.3. But now we use the function $g(z) = e^{V(z,\varphi(z))}$ in the first inequality of (2.30). After some algebraic work similar to the former demonstration we reach

$$\alpha(U_a) \leq \frac{1}{d_1}\, e^{2\,K + r\mathcal{U}} e^{\sum_{k=0}^{s-1}[U\tau^k(x_a) - U\tau^k(y_b)]}\ \alpha(\varphi_{a,b}(U_a)\,) \leq$$

$$\frac{1}{d_1}\, e^{2\,K+3\,r\,\mathcal{U}} e^{\sum_{k=0}^{s-1}[U\tau^k(x) - U\tau^k(y)]}\ \alpha(\, \varphi_{a,b}\,(U_a)\,).$$

Therefore,

$$e^{\sum_{k=0}^{s-1} U\tau^k(y)} \alpha(\overline{|\, x_1^0, x_2^0, ...x_s^0}) = e^{\sum_{k=0}^{s-1} U\tau^k(y)]} \sum_{a=1}^{d} \alpha(U_a) \leq$$

$$\frac{1}{d_1}\, e^{2\,K+3\,r\,\mathcal{U}} e^{\sum_{k=0}^{s-1} U\tau^k(x)}\ \sum_{a=1}^{d} \alpha(\, \varphi_{a,b}\,(U_a)\,). \tag{2.34}$$

Finally, as $\overline{|\, y_1^0, y_2^0, ...y_s^0\, b} = \varphi_{a,b}(U_a)$ we get from (2.29) and (2.34)

$$d\,\alpha(\overline{|\, x_1^0, x_2^0, ...x_s^0}) = \sum_{b=1}^{d}\ \sum_{y_1^0, y_2^0, ...y_s^0 = 1}^{d}\ e^{\sum_{k=0}^{s-1} U\tau^k(y)} \alpha(\overline{|\, x_1^0, x_2^0, ...x_s^0}) \leq$$

$$\frac{e^{2K+3\,r\mathcal{U}}}{d_1} e^{\sum_{k=0}^{s-1} U\tau^k(x)} \sum_{a=1}^{d} \sum_{b=1}^{d} \sum_{y_1^0,y_2^0,\ldots y_s^0=1}^{d} \alpha(\varphi_{a,b}\,(U_a)) = \frac{d\, e^{2K+3\,r\mathcal{U}}}{d_1}\; e^{\sum_{k=0}^{s-1} U\tau^k(x)}.$$

This shows the claim of the proposition.

□

Now we have to show that α is invariant by τ.

Corolary 2.7.4. *If α_1 and α_2 are quasi C^*-Gibbs for U, where*

$$U(...,x_{-n},...,x_{-2},x_{-1}\,|\,x_1,x_2,...,x_r,x_{r+1},...x_m...) = f(x_1,x_2,...,x_r)$$

for some fixed r and function $f : \{1,2,...,d\}^r \to \mathbb{R}$, then α_1 is absolutely continuous with respect to α_2.

Proof: We assume that U is normalized. Suppose α_1 and α_2 are quasi C^*-Gibbs for U.

Expression (2.26) for α_1 and α_2 will determine, respectively, constants d_1^1, d_2^1 and d_1^2, d_2^2.

From last Propositions there exist constants $Y_1 > 0$ and $Y_2 > 0$, such that, for any cylinder $\overline{|\,x_1,x_2,..,x_n}$ and for any point x in this cylinder we get

$$\frac{\alpha_1(\overline{|\,x_1,x_2,..,x_n})}{e^{\sum_{k=0}^{n-1} U(\tau^k(x))}} \le Y_1,$$

and

$$Y_2 \le \frac{\alpha_2(\overline{|\,x_1,x_2,..,x_n})}{e^{\sum_{k=0}^{n-1} U(\tau^k(x))}}.$$

Therefore,

$$\frac{Y_2}{Y_1}\alpha_1(\overline{|\,x_1,x_2,..,x_n}) \le \alpha_2(\overline{|\,x_1,x_2,..,x_n}).$$

Now consider a cylinder set of the form

$$(\overline{x_{-m},...x_{-1}|\,x_1,x_2,..,x_n}).$$

Expression (2.23) for α_1 and α_2 will determine, respectively, constants b_1^1, b_2^1 and b_1^2, b_2^2.

Then, by Proposition 2.4.3 we get that

$$b_1^1\,\alpha_1(\overline{x_{-m},...x_{-1}|\,x_1,x_2,..,x_n}) \le \alpha_1(\overline{|\,x_{-m},...x_{-1}x_1,x_2,..,x_n}) \le$$

$$\frac{Y_1}{Y_2}\alpha_2(\overline{|\, x_{-m},...x_{-1}x_1,x_2,..,x_n}) \leq \frac{Y_1}{Y_2}b_2^2\alpha_2(\overline{x_{-m},...x_{-1}|\, x_1,x_2,..,x_n}). \quad (2.35)$$

The Borel sigma-algebra over Ω is generated by the set of cylinders of the form $\overline{x_{-m},...x_{-1}|\, x_1,x_2,..,x_n}$.

As the probability $\alpha_j(B)$, $j=1,2$, of a Borel set B is obtained, respectively, as an exterior probability using probabilities of the generators we finally get that the analogous inequalities as in (2.35) are true with the same same constants, that is,

$$b_1^1\alpha_1(B) \;\; \leq \;\; \frac{Y_1}{Y_2}b_2^2\alpha_2(B). \quad (2.36)$$

Therefore, α_1 is absolutely continuous with respect to α_2.

□

Proposition 2.7.5. *Assume α is C^*-Gibbs for U, then, α is invariant for τ.*

Proof: From Corollary 2.7.4 we get that any two C^*-Gibbs probabilities for U are absolutely continuous with respect to each other.

Suppose α is C^*-Gibbs, then, $\alpha_1 = \tau^*(\alpha)$ is also C^*-Gibbs by Proposition 2.3.8. If $\alpha \neq \tau^*(\alpha)$ then, following Theorem 2.5 in [24] we get that $\rho_1 = |\alpha_1-\alpha|+\alpha_1-\alpha$ and $\rho_2 = |\alpha_1-\alpha|-\alpha_1+\alpha$ are also C^*-Gibbs. But ρ_1 and ρ_2 are singular with respect to each other and this is a contradiction.

Therefore, $\alpha = \tau^*(\alpha)$.

□

Theorem 2.7.6. *Suppose $U:\Omega\to\mathbb{R}$ is of the form*

$$U(...,x_{-n},...,x_{-2},x_{-1}\,|\,x_1,x_2,...,x_r,x_{r+1},...x_m...) = f(x_1,x_2,...,x_r),$$

for some fixed r and fixed function $f:\{1,2,...,d\}^r\to\mathbb{R}$.

If α is C^*-Gibbs for the potential U then α is the equilibrium state for U.

Proof: As we know by Proposition 2.7.5 that α is τ invariant and, moreover, we also know that α is quasi-C^* invariant for another normalized potential, it follows from Proposition 2.7.2, Proposition 2.7.3 and Theorem 2.5.1 that α is the equilibrium probability for U

□

Another conclusion one can get from the above reasoning is that for potentials that depends on finite coordinates the concepts of quasi C^*-Gibbs and C^*-Gibbs are equivalent on the lattice $\mathbb{Z}$.

2.8 Construction of the C^*-Algebra

Remember that we consider the groupoid $G \subset \Omega \times \Omega$ of all pair of points which are related by the homoclinic equivalence relation.

Remember also that we consider on G the topology generated by sets of the form

$$\{ \ (z, \varphi_{(x,y)}(z)) \,|\, \text{where } z \in \mathcal{O}_{(x,y)} \ \text{ and } x, y \in \Omega \text{ such that } x \sim y\}.$$

This topology is Hausdorff [59].

We denote by $[x]$ the class of $x \in \Omega$. For each x the set of elements on the class $[x]$ is countable.

We now come to the construction of the noncommutative algebra. Let $\mathcal{C}_c(G)$ be the linear space of complex continuous functions with compact support on G. If $A, B \in \mathcal{C}_c(G)$ we define the product $A * B$ by

$$(A * B)(x, y) = \sum_{z \in [x]} A(x, z) B(z, y).$$

Note that if $(x, y) \in G$ then they are conjugated and so the sum is over all z that are conjugated to x and y.

Note that there are only finitely many nonzero terms in the above sum because the functions A, B have compact support [59].

Considering the above, $A * B \in \mathcal{C}_c(G)$ as one checks readily, so that $\mathcal{C}_c(G)$ becomes an associative complex algebra. An **involution** $A \to A^*$ is defined by

$$A^*(x, y) = \overline{A(y, x)}$$

where the bar denotes complex conjugation.

For each equivalence class $[x]$ of conjugated points of Ω there is a representation $\pi_{[x]} \to \mathbb{C}$ in the Hilbert space $l^2([x])$ of square summable functions $[x] \to \mathbb{C}$, such that

$$((\pi_{[x]} A)\xi(y) = \sum_{z \in [x]} A(y, z)\xi(z)$$

for $\xi \in l^2([x])$. Denoting by $\|\pi_{[x]} A\|$ the operator norm, we write

$$\|A\| = \sup_{[x]} \|\pi_{[x]} A\|. \tag{2.37}$$

I_D (the indicator function of the diagonal D) is such that for any $A \in \mathcal{C}_c(G)$ we get $I_D * A = A * I_D = A$.

The completion of $\mathcal{C}_c(G)$ with respect to this norm is separable. It is called the reduced C^*-algebra which is denoted by $C_r^*(G)$. The unity element I_D is contained in this C^* algebra.

Remark 2.8.1. *If $A \in \mathcal{C}_c(G)$ and $t \in \mathbb{R}$, we write*

$$(\sigma^t A)(x,y) = e^{iV(x,y)t} A(x,y) \tag{2.38}$$

defining a one-parameter group (σ^t) of $*$-automorphisms of $\mathcal{C}_c(G)$ and a unique extension to a one parameter group of $*$-automorphisms of $C_r^*(G)$.

We say that $A \in \mathcal{C}_c(G)$ is analytic (a classical terminology on C^*-algebras) if the real variable t on the function $t \to \sigma^t A$ can be extended to the complex variable $z \in \mathbb{C}$. Under our assumptions this will be always the case. Therefore, $\sigma^{-\beta i} A$ is well defined.

Definition 2.8.2. *A state ω on $C_r^*(G)$ is a linear functional $\omega : C_r^*(G) \to \mathbb{C}$, such that, $\omega(A * A^*) \geq 0$, and $\omega(I_D) = 1$ (see [12]).*

Such state ω is sometimes called a dynamical C^*-state.

Definition 2.8.3. *A state ω is invariant if $\omega \circ \sigma^t = \omega$, for all $t \in \mathbb{R}$.*

It is of paramount importance to be able to substitute the above real value t by the complex number $\beta\, i$ (where β is real). We refer the reader to Propositions 5.3.6 e 5.3.7 in [12] for the technical details of this claim.

Definition 2.8.4. *Given a modular function $V : G \to \mathbb{R}$ and the associated σ_t, $t \in \mathbb{R}$, we say that an invariant state $\omega : C_r^*(G) \to \mathbb{C}$ satisfies the **KMS boundary condition** for V and $\beta \in \mathbb{R}$, if for all $A, B \in C_r^*(G)$, there is a continuous function F on $\{z \in \mathbb{C} : 0 \leq \ Im(z) \leq \beta\}$, holomorphic in $\{z \in \mathbb{C} : 0 < \ Im(z) < \beta\}$, and such that for any real t*

$$\omega(\sigma^t A * B) = F(t), \quad \omega(B * \sigma^t A) = F(t + i\beta) \tag{2.39}$$

□

Note that using (2.39) we have that $F(0) = \omega(A \cdot B)$ and

$$F(0) = F(-\beta i + \beta i) = \omega(B * \sigma^{-\beta i} A).$$

Therefore, for any A, B we get

$$\omega(A \, * \, B) = \omega(B * e^{-i\beta} A)$$

which is the classical KMS condition for ω according to [12] (see Propositions 5.3.6 e 5.3.7 there). This condition is equivalent to **KMS boundary condition**.

Theorem 2.8.5. *If μ is a probability measure on Ω then a state $w = \hat{\mu}$ on $C_r^*(G)$ can be defined for any $A \in \mathcal{C}_c(G)$ by*

$$\hat{\mu}(A) = \int A(x,x) d\mu(x) \tag{2.40}$$

Proof:

$\hat{\mu}$ is bounded with respect to the above defined norm.

First note that it's easy to verify that $\hat{\mu}$ is linear, and for any A we have $\hat{\mu}(A * A^*) \geq 0$ and moreover $\hat{\mu}(I_D) = 1$. Now, note that since the diagonal D is a compact set, then any continuous function $A : G \to \mathbb{C}$ has a maximum at D, therefore (2.40) is well defined for continuous function. $\hat{\mu}$ is also well defined on the C^*-algebra.

□

Definition 2.8.6. *A probability ν on Ω is called a* **KMS probability** *for the modular function V if the state $\hat{\nu}$ on $C_r^*(G)$ defined by*

$$\hat{\nu}(A) = \int A(x,x) \nu(dx) \tag{2.41}$$

satisfies the KMS condition for V. Here G is the groupoid given by the homoclinic equivalence relation.

This probability is sometimes called quasi-stationary (see [15]).

The next claim was proved on [59]. For completeness we will present a proof of this claim with full details.

Theorem 2.8.7. *If the probability α on Ω is a C^*-Gibbs probability with respect to V and β, then, $\hat{\alpha}$ is a KMS probability for the modular function βV. The associated $\hat{\alpha}$ is a C^* dynamical state for the $C_r^*(G)$ algebra given by the groupoid obtained by the homoclinic equivalence relation and satisfies the KMS boundary condition.*

Proof: Suppose α is a C^*-Gibbs state with respect to βV. We assume $\beta = 1$.

$\hat{\alpha}$ is σ^t invariant if for all $t \in \mathbb{C}$ it's true that

$$\int \sigma^t A(x,x)\alpha(dx) = \int A(x,x)\alpha(dx)$$

which by definition (2.38) it's equivalent to

$$\int e^{iV(x,x)t} A(x,x)\alpha(dx) = \int A(x,x)\alpha(dx)$$

but since $V(x,x) = 0$ then the state have to be σ^t invariant.

Now we will show that if $A, B \in \mathcal{C}_c(G)$, then

$$\hat{\alpha}(\sigma^t A * B) = \int \alpha(dx) \sum_{y\in[x]} e^{iV(x,y)t} A(x,y) \cdot B(y,x)$$

extends to an entire function (just change t to $z \in \mathbb{C}$). For this purpose we will pick $t_0 \in \mathbb{C}$ and show that

$$\lim_{t\to t_0} \frac{\hat{\alpha}(\sigma^t A * B) - \hat{\alpha}(\sigma^{t_o} A * B)}{t - t_0} \tag{2.42}$$

exist. Indeed, the limit (2.42) is equivalent to

$$\lim_{t\to t_0} \frac{1}{t-t_0} \left(\int \alpha(dx) \sum_{y\in[x]} e^{iV(x,y)t} A(x,y) \cdot B(y,x) - \right.$$

$$\left. \int \alpha(ds) \sum_{y\in[s]} e^{iV(s,y)t_0} A(s,y) \cdot B(y,s) \right) =$$

$$\lim_{t\to t_0} \left(\int \alpha(dx) \sum_{y\in[x]} \frac{(e^{iV(x,y)t} - e^{iV(x,y)t_0})}{t-t_0} A(x,y) \cdot B(y,x) \right). \tag{2.43}$$

Always have in mind that for each x the summation is over finite terms.

Let R be a closed ball of radius 1 centered in t_0. So we can consider the continuous function $f_{t_0} : R\backslash\{t_0\} \times \text{supp}(A) \to \mathbb{C}$

$$f_{t_0}(t,x) = \sum_{y\in[x]} \frac{(e^{iV(x,y)t} - e^{iV(x,y)t_0})}{t-t_0} A(x,y) \cdot B(y,x)$$

To extend f_{t_0} for the case $t = t_0$ we need to solve the limit

$$L_{t_0}(x) = \lim_{t \to t_0} \sum_{y \in [x]} \frac{(e^{iV(x,y)t} - e^{iV(x,y)t_0})}{t - t_0} A(x,y) \cdot B(y,x) = \tag{2.44}$$

$$\sum_{y \in [x]} \lim_{t \to t_0} \frac{(e^{iV(x,y)t} - e^{iV(x,y)t_0})}{t - t_0} A(x,y) \cdot B(y,x) =$$

$$\sum_{y \in [x]} iV(x,y) e^{iV(x,y)t_0} A(x,y) \cdot B(y,x).$$

So define $f_{t_0}(t_0, x) = L_{t_0}(x)$.

In this way f_{t_0} is a continuous function defined on a compact domain. Therefore we may assume that both it's real and imaginary parts are limited by a value M in the domain. Consider a sequence of functions indexed by the t variable, $\{f_{t_0}(t_n, x)\}_{n \in \mathbb{N}^*}$ that converge to $L_{t_0}(x)$ when $n \to \infty$, e.g. $f_{t_0}(t_0 + (1+i)/n, x)$. In this way the dominated convergence theorem assures that the limit (2.42) is equal to the integral:

$$\int \alpha(dx) L_{t_0}(x).$$

Indeed formally what we have is,

$$\int \alpha(dx) L_{t_0}(x) = \int \alpha(dx) \sum_{y \in [x]} iV(x,y) e^{iV(x,y)t_0} A(x,y) \cdot B(y,x) =$$

$$\int \alpha(dx) \lim_{n \to \infty} \sum_{y \in [x]} \frac{(e^{iV(x,y)t_n} - e^{iV(x,y)t_0})}{t_n - t_0} A(x,y) \cdot B(y,x) =$$

$$\lim_{n \to \infty} \int \alpha(dx) \sum_{y \in [x]} \frac{(e^{iV(x,y)t_n} - e^{iV(x,y)t_0})}{t_n - t_0} A(x,y) \cdot B(y,x) =$$

$$\lim_{n \to \infty} \frac{\hat{\alpha}(\sigma^{t_n} A * B) - \hat{\alpha}(\sigma^{t_o} A * B)}{t_n - t_0} \tag{2.45}$$

Now since the sequence was arbitrary we could remake these calculations to any desired convergent sequence with the same result, therefore (2.45) is equal to

$$\lim_{t \to t_0} \frac{\hat{\alpha}(\sigma^{t} A * B) - \hat{\alpha}(\sigma^{t_o} A * B)}{t - t_0},$$

what proves existence of the limit in equation (2.42). This allow us to conclude that $\hat{\alpha}(\sigma^t A * B)$ is an holomorphic function everywhere.

Let $F(t) = \hat{\alpha}(\sigma^t A * B)$. Using a partition of unity on supp A we may write $A = \sum A_j$, where supp $A_j \subset W_j = \{(z, \varphi_j(z)) : z \in \mathcal{O}_j\}$, and $(\mathcal{O}_j, \varphi_j)$ is a conjugating homeomorphism. Since supp A is a compact set then we may assume the summation to occur over a finite amount of elements. Thus

$$F(t) = \int_\Omega \sum_j \alpha(dx) A_j(x, \varphi_j x) B(\varphi_j x, x) \exp(iV(x, \varphi_j x)t) =$$

$$\sum_j \int_{\mathcal{O}_j} \alpha(dx) A_j(x, \varphi_j x) B(\varphi_j x, x) \exp(iV(x, \varphi_j x)t)$$

and therefore

$$F(t+i) = \sum_j \int_{\mathcal{O}_j} [e^{-V(x,\varphi_j x)} \alpha(dx)] A_j(x, \varphi_j x) B(\varphi_j x, x) \exp(iV(x, \varphi_j x)t)$$

If α is an C^*-Gibbs state by (2.5) we have that

$$F(t+\beta i) = \sum_j \int_{\varphi_j(\mathcal{O}_j)} \alpha(dy) B(y, \varphi_j^{-1} y) A_j(\varphi_j^{-1} y, y) \exp(iV(\varphi_j^{-1} y, y)t) =$$

$$\sum_j \int_{\varphi_j(\mathcal{O}_j)} \alpha(dy) B(y, \varphi_j^{-1} y) \sigma^t A_j(\varphi_j^{-1} y, y) =$$

$$\int_\Omega \sum_j \alpha(dy) B(y, \varphi_j^{-1} y) \sigma^t A_j(\varphi_j^{-1} y, y) =$$

$$\int_\Omega \alpha(dy)(B * \sigma^t A)(y, y) = \hat{\alpha}(B * \sigma_t A)$$

so that $\hat{\alpha}$ satisfies the KMS condition.

□

Capítulo 3

Haar systems, KMS states on von Neumann algebras, C^*-algebras on dynamically defined groupoids and Noncommutative Integration

3.1 The groupoid associated to a partition

We will analyze properties of Haar systems, quasi-invariant probabilities, transverse measures, C^*-algebras and KMS states related to Thermodynamic Formalism and Gibbs states. We will consider a specific particular setting where the groupoid will be defined by some natural equivalence relations on the sets of the form $\{1, 2, ..., d\}^{\mathbb{N}}$ or $\{1, 2, ..., d\}^{\mathbb{Z}}$, $S^1 \times S^1$, or $(S^1)^{\mathbb{N}}$. These equivalence relations will be of dynamic origin.

We will denote by X any one of the above sets.

The main point here is that we will use a notation which is more close to the one used on Ergodic Theory and Thermodynamic Formalism.

On section 3.2 we introduce the concept of transverse functions associated to groupoids and Haar systems.

On section 3.3 we consider modular functions and quasi-invariant probabilities on groupoids. In the end of this section we present a new result concerning a (non-)relation of the quasi-invariant probability with the SBR probability of the generalized Baker map.

On section 3.4 we consider a certain von Neumann algebra and the associated KMS states. On proposition 3.4.24 we present a new result concerning the relation between probabilities satisfying the KMS property (quasi-invariant) and Gibbs (DLR) probabilities of Thermodynamic Formalism on the symbolic space $\{1, 2, ..., d\}^{\mathbb{N}}$ for a certain groupoid. Proposition 3.4.25 shows that the KMS probability is not unique on this case.

[33], [31] and [32] are the classical references on measured groupoids and von Neumann algebras. KMS states and C^*-algebras are described on [46].

On section 3.5 we present a natural expression - based on quasi-invariant probabilities - for the integration of a transverse function by a transverse measure. Some basic results on non-commutative integration (see [18] for a detailed description of the topic) are briefly described.

On section 3.6 we present briefly the setting of C^*-algebras associated to groupoids on symbolic spaces. We present the well known and important concept of approximately proper equivalence relation and its relation with the direct inductive limit topology (see [25], [26], [27] and [57]).

On section 3.7 we present several examples of quasi-invariant probabilities for different kinds of groupoids and Haar systems.

Results on C^*-algebras and KMS states from the point of view of Thermodynamic Formalism are presented in [37], [57], [51], [65], [66], [1], [29] and [30].

The paper [9] considers equivalence relations and DLR probabilities for certain interactions on the symbolic space $\{1, 2, ...d\}^{\mathbb{Z}}$ (not in $\{1, 2, ...d\}^{\mathbb{N}}$ like here).

Theorem 6.2.18 in Vol II of [12] and [4] describe the relation between KMS states and Gibbs probabilities for interactions on certain spin lattices (on the one-dimensional case corresponds to the space $\{1, 2, .., d\}^{\mathbb{Z}}$).

We point out that Lecture 9 in [21] presents a brief introduction to C^*-Algebras and non-commutative integration.

We denote $\{1, 2, ..., d\}^{\mathbb{N}} = \Omega$ and consider the compact metric space with metric d where for $x = (x_0, x_1, x_2, ..) \in \Omega$ and $y = (y_0, y_1, y_2, ..) \in \Omega$

$$d(x, y) = 2^{-N},$$

where N is the smallest natural number $j \geq 0$, such that, $x_j \neq y_j$.

We also consider $\{1,2,...,d\}^{\mathbb{Z}} = \hat{\Omega}$ and elements in $\hat{\Omega}$ are denoted by $x = (..., x_{-n}, ..., x_{-1} \,|\, x_0, x_1, .., x_n, ..)$.

We will use the notation $\overleftarrow{\Omega} = \{1,2,...,d\}^{\mathbb{N}}$ and $\overrightarrow{\Omega} = \{1,2,...,d\}^{\mathbb{N}}$.

Given $x = (..., x_{-n}, ..., x_{-1} \,|\, x_0, x_1, .., x_n, ..) \in \hat{\Omega}$, we call $(..., x_{-n}, ..., x_{-1}) \in \overleftarrow{\Omega}$ the past of x and $(x_0, x_1, .., x_n, ..) \in \overrightarrow{\Omega}$ the future of x.

In this way we express $\hat{\Omega} = \overleftarrow{\Omega} \times | \overrightarrow{\Omega}$.

Sometimes we denote

$$(..., a_{-n}, ..., a_{-1} \,|\, b_0, b_1, .., b_n, ..) =< a \,|\, b >,$$

where $a = (..., a_{-n}, ..., a_{-1}) \in \overleftarrow{\Omega}$ and $b = (b_0, b_1, .., b_n, ..) \in \overrightarrow{\Omega}$.

On $\hat{\Omega}$ we consider the usual metric d, in such way that for $x, y \in \hat{\Omega}$ we set

$$d(x,y) = 2^{-N},$$

$N \geq 0$, where for

$$x = (..., x_{-n}, ..., x_{-1} \,|\, x_0, x_1, .., x_n, ..) \;\; , y = (..., y_{-n}, ..., y_{-1} \,|\, y_0, y_1, .., y_n, ..),$$

we have $x_j = y_j$, for all j, such that, $-N+1 \leq j \leq N-1$ and, moreover $x_N \neq y_N$, or $x_{-N} \neq y_{-N}$.

The shift $\hat{\sigma}$ on $\hat{\Omega} = \{1,2,...,d\}^{\mathbb{Z}}$ is such that

$$\hat{\sigma}(..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...) = (..., y_{-n}, ..., y_{-2}, y_{-1}, y_0 \,|\, y_1, ..., y_n, ...).$$

On the other hand the shift σ on $\Omega = \{1,2,...,d\}^{\mathbb{N}}$ is such that

$$\sigma(y_0, y_1, ..., y_n, ...) = (y_1, ..., y_n, ...).$$

A general equivalence relation R on a space X define classes and we will denote by $x \sim y$ when two elements x and y are on the same class. We denote by $[y]$ the class of $y \in X$.

Definition 3.1.1. *Given an equivalence relation $\sim$ on X, where X is any of the sets Ω, $\hat{\Omega}$, $(S^1)^{\mathbb{N}}$, or $S^1 \times S^1$, we denote by G the subset of $X \times X$, containing all pairs (x,y), where $x \sim y$. We call G the groupoid associated to the equivalence relation $\sim$.*

We also denote by G^0 the set $\{(x,x) \,|\, x \in X\} \sim X$, where X denote any of the sets Ω, $\hat{\Omega}$, $(S^1)^{\mathbb{N}}$, or $S^1 \times S^1$.

Remark: There is a general definition of groupoid (see [18]) which assumes more structure but we will not need this here. For all results we will consider there is no need for an additional algebraic structure (on the class of each point). In this way we can consider a simplified definition of groupoid as it is above. Our intention is to study C^*-algebras and Haar systems as a topic on measure theory (intersected with ergodic theory) avoiding questions of algebraic nature.

There is a future issue about the topology we will consider induced on G. One possibility is the product topology, which we call the standard structure, or, a more complex one which will be defined later on section 3.6 (specially appropriate for some C^*-algebras).

We will present several examples of dynamically defined groupoids. The equivalence relation of most of our examples is proper (see definition 3.6.5).

Example 3.1.2. *For example consider on $\{1, 2, ..., d\}^{\mathbb{N}}$ the equivalence relation R such that $x \sim y$, if $x_j = y_j$, for all $j \geq 2$, when $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$. This defines a groupoid G. In this case $G^0 = \Omega = \{1, 2, ..., d\}^{\mathbb{N}}$.*

For a fixed $x = (x_1, x_2, x_3, ...)$ the equivalence class associated to x is the set $\{\, (j, x_2, x_3, ...),\ j = 1, 2.., d\,\}$. We call this relation the **bigger than two relation**.

Example 3.1.3. *Consider an equivalence relation R which defines a partition η_0 of $\{1, 2, ..., d\}^{\mathbb{Z}-\{0\}} = \hat{\Omega}$ such its elements are of the form*

$$a \times \mid \overrightarrow{\Omega} = a \times \{1, 2, ..., d\}^{\mathbb{N}} = (..., a_{-n}, ..., a_{-2}, a_{-1}) \times \mid \{1, 2, ..., d\}^{\mathbb{N}},$$

where $a \in \{1, 2, ..., d\}^{\mathbb{N}} = \overleftarrow{\Omega}$. This defines an equivalence relation $\sim$.

In this way two elements x and y are related if they have the same past.

There exists a bijection of classes of η_0 and points in $\overleftarrow{\Omega}$.

Denote $\pi = \pi_2 : \hat{\Omega} \to \overleftarrow{\Omega}$ the transformation such that takes a point and gives as the result its class.

In this sense

$$\pi^{-1}(x) = \pi^{-1}((..., x_{-n}, ..., x_{-1} \,|\, x_1, .., x_n, ..))) =$$

$$(..., x_{-n}, ..., x_{-2}, x_{-1}) \times \mid \overrightarrow{\Omega} \cong \Omega.$$

The groupoid obtained by this equivalence relation can be expressed as $G = \{(x, y), \pi(x) = \pi(y)\}$. *In this way* $x \sim y$ *if they have the same past.*

In this case the number of elements in each fiber is not finite.

Using the notation of page 46 of [18] we have $Y \subset \hat{\Omega} \times \hat{\Omega}$ *and* $X = \overleftarrow{\Omega}$.

In this case each class is associated to certain $a = (a_{-1}, a_{-2}, ..) \in \overleftarrow{\Omega} = \{1, 2, ..., d\}^{\mathbb{N}}$..

We use the notation $(a \,|\, x)$ *for points on a class of the form*

$$(a \,|\, x) = (..., a_{-n}, ..., a_{-2}, a_{-1} \,|\, x_1, ..., x_n, ...) \ .$$

Example 3.1.4. *A particulary important equivalence relation* R *on* $\hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}}$ *is the following: we say* $x \sim y$ *if*

$$x = (..., x_{-n}, ..., x_{-2}, x_{-1} \,|\, x_0, x_1, ..., x_n, ...),$$

and,

$$y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...)$$

are such that there exists $k \in \mathbb{Z}$, *such that,* $x_j = y_j$, *for all* $j \leq k$.

The groupoid G_u *is defined by this relation* $x \sim y$.

By definition the unstable set of the point $x \in \hat{\Omega}$ *is the set*

$$W^u(x) = \{y \in \hat{\Omega}\,,\ \textit{such that} \ \lim_{n \to \infty} d(\,\hat{\sigma}^{-n}(x), \hat{\sigma}^{-n}(y)\,) \ = 0\,\}.$$

One can show that the unstable manifold of $x \in \hat{\Omega}$ *is the set*

$$W^u(x) = \{y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...) \,|\, \textit{there exists}$$
$$k \in \mathbb{Z}, \ \textit{such that } x_j = y_j, \textit{for all } j \leq k\}.$$

If we denote by G_u *the groupoid defined by the above relation, then,* $x \sim y$, *if and only if,* $y \in W^u(x)$.

An equivalence relation of this sort - for hyperbolic diffeomorphism - was considered on [62] and [43].

Example 3.1.5. *An equivalence relation on* $\overrightarrow{\Omega} = \{1, 2, ..., d\}^{\mathbb{N}}$ *similar to the previous one is the following: we say* $x \sim y$ *if*

$$x = (\,x_0, x_1, ..., x_n, ...),$$

and,

$$y = (y_0, y_1, ..., y_n, ...)$$

are such that there exists $k \in \mathbb{N}$, *such that,* $x_j = y_j$, *for all* $j \geq k$.

Example 3.1.6. *Another equivalence relation on* $\overrightarrow{\Omega}$ *is the following: fix* $k \in \mathbb{N}$, *and we say* $x \sim_k y$, *if when*

$$x = (\, x_0, x_1, ..., x_n, ...),$$

and,

$$y = (y_0, y_1, ..., y_n, ...)$$

we have $x_j = y_j$, *for all* $j \geq k$.

In this case each class has d^k *elements.*

Example 3.1.7. *Given* $x, y \in \hat{\Omega} = \{1, 2, ..., d\}^{\mathbb{Z}}$, *we say that* $x \sim y$ *if*

$$\lim_{k \to +\infty} d(\hat{\sigma}^k x, \hat{\sigma}^k y) = 0$$

and

$$\lim_{k \to -\infty} d(\hat{\sigma}^k x, \hat{\sigma}^k y) = 0. \tag{3.1}$$

This means there exists an $M \geq 0$, *such that,* $x_j = y_j$ *for* $j > M$, *and,* $j < -M$. *In other words, there are only a finite number of* i*'s such that* $x_i \neq y_i$. *This is the same to say that* x *and* y *are homoclinic.*

For example in $\hat{\Omega} = \{1, 2\}^{\mathbb{Z}}$ *take*

$$x = (..., x_{-n}, ..., x_{-7}, 1, 2, 2, 1, 2, 2 \,|\, 1, 2, 1, 2, 1, 1, x_7, ...x_n, ..)$$

and

$$y = (..., y_{-n}, ..., y_{-7}, 1, 2, 2, 1, 2, 2 \,|\, 1, 2, 1, 1, 1, 2, y_7, ...y_n, ..)$$

where $x_j = y_j$ *for* $|j| \geq 7$.

In this case $x \sim y$.

This relation is called the **homoclinic relation** *on* $\hat{\Omega}$ *(see section 2) It was considered for instance by D. Ruelle and N. Haydn in [58] and [24] for hyperbolic diffeomorphisms and also on more general contexts (see also [40], [44] and [9] for the symbolic case).*

Example 3.1.8. *Consider an expanding transformation* $T : S^1 \to S^1$, *of degree two, such that* $\log T'$ *is Holder and* $\log T'(a) > \log \lambda > 0$, $a \in S^1$, *for some* $\lambda > 1$.

Suppose $T(x_0) = 1$, *where* $0 < x_0 < 1$. *We say that* $(0, x_0)$ *and* $(x_0, 1)$ *are the domains of injectivity of* T.

Denote $\psi_1 : [0,1) \to [0,x_0)$ the first inverse branch of T and $\psi_2 : [0,1) \to [x_0,1]$ the second inverse branch of T.

In this case for all y we have $T \circ \psi_1(y) = y$ and $T \circ \psi_2(y) = y$.

The **associated T-Baker map** *is the transformation $F : S^1 \times S^1$ such that satisfies for all a, b the following rule:*

1) if $0 \le b < x_0$

$$F(a,b) = (\psi_1(a), T(b)),$$

and

2) if $x_0 \le b < 1$

$$F(a,b) = (\psi_2(a)\,, T(b)).$$

In this case we take as partition the one associated to (local) unstable manifolds for F, that is, sets of the form $W_a = \{\,(a,b)\,|\,b \in S^1\}$, where $a \in S^1$.

Given two points $z_1, z_2 \in S^1 \times S^1$ we say that they are related if the first coordinate of each is equal.

On $S^1 \times S^1$ we use the distance d which is the product of the usual arc length distance on S^1.

The bijection F expands vertical lines and contract horizontal lines.

As an example one can take $T(a) = 2a \pmod 1$ and we get (the inverse of) the classical Baker map (see [63]).

One can say that the dynamics of such F in some sense looks like the one of an Anosov diffeomorphism.

Example 3.1.9. *The so called generalized XY model consider the space $(S^1)^{\mathbb{N}}$, where S^1 is the unitary circle and the shift acting on it.*

We can consider the equivalence relation R such that $x \sim y$, if $x_j = y_j$, for all $j \ge 2$, when $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$. This defines a groupoid G. In this case $G^0 = (S^1)^{\mathbb{N}}$.

For a fixed $x = (x_1, x_2, x_3, ...)$ the equivalence class associated to x is the set $\{\,(a, x_2, x_3, ...),\, a \in S^1\,\}$. We call this relation the bigger than two relation for the XY model and G the standard XY groupoid over $(S^1)^{\mathbb{N}}$.

3.2 Kernels and transverse functions

A general reference for the material of this section is [18] (see also [34]).

We consider over $G \subset X \times X$ the Borel sigma-algebra $\mathcal{B}$ (on G) induced by the natural product topology on $X \times X$ and the metric d on X ([31] and [32] also consider this sigma algebra). This will be fine for the setting of von Neumann algebras. Later, another sigma-algebra will be considered for the setting C^*-algebras.

We point out that the only sets X which we are interested are of the form $\hat{\Omega}$, Ω, $(S^1)^{\mathbb{N}}$, or $S^1 \times S^1$.

We denote $\mathcal{F}^+(G)$ the space of Borel measurable functions $f : G \to [0, \infty)$ (a function of two variables (a, b)).

$\mathcal{F}(G)$ is the space of Borel measurable functions $f : G \to \mathbb{R}$. Note that $f(x, y)$ just make sense if $x \sim y$.

There is a natural involution on $\mathcal{F}(G)$ which is $f \to \tilde{f}$, where $\tilde{f}(x, y) = f(y, x)$.

We also denote $\mathcal{F}^+(G^0)$ the space of Borel measurable functions $f : G^0 \to [0, \infty)$ (a function of one variable a).

There is a natural identification of functions $f : G^0 \to \mathbb{R}$, of the form $f(x)$, with functions $g : G \to \mathbb{R}$ which depend only on the first coordinate, that is $g(x, y) = f(x)$. This will be used without mention, but if necessary we write $(f \circ P_1)(x, y) = f(x)$ and $(f \circ P_2)(x, y) = f(y)$.

Definition 3.2.1. *A* **measurable groupoid** *G is a groupoid with the topology induced by the product topology over $X \times X$, such that, the following functions are measurable for the Borel sigma-algebra:*

$P_1(x, y) = x$, $P_2(x, y) = y$, $h(x, y) = (y, x)$ and $Z(\,(x, s), (s, y)\,) = (x, y)$, where $Z : \{\,(\,(x, s), (r, y)\,) \mid r = s\,\} \subset G \times G \to G$.

Now, we will present the definition of kernel (see beginning of section 2 in [18]).

Definition 3.2.2. *A* **G-kernel** *ν on the measurable groupoid G is an application of G^0 in the space of measures over the sigma-algebra $\mathcal{B}$, such that,*

1) for any $y \in G^0$, we have that ν^y has support on $[y]$,

and

2) for any $A \in \mathcal{B}$, the function $y \to \nu^y(A)$ is measurable.

The set of all G- kernels is denoted by $\mathcal{K}^+$.

Example 3.2.3. *As an example consider for the case of the groupoid G associated to the bigger than two relation, the measure ν^y, for each $y = (y_1, y_2, y_3, ...)$, such that $\nu(j, y_2, y_3, ...) = 1$, $j = 1, 2, ..., d$. In other words* *we are using the counting measure on each class. We call this the standard G-kernel for the the bigger than two relation.*

More precisely, the **counting measure** *is such that $\nu^y(A) = \#(A \cap [y])$, for any $A \in \mathcal{B}$.*

Example 3.2.4. *Another possibility is to consider the G-kernel such that ν^y, for each $y = (y_1, y_2, y_3, ...)$, is such that $\nu(j, y_2, y_3, ...) = \frac{1}{d}$, We call this the normalized standard G-kernel for the bigger than two relation.*

Example 3.2.5. *Given any groupoid G another example of kernel is the* **delta kernel** *ν which is the one such that for any $y \in G^0$ we have that $\nu^y(dx) = \delta_y(d\,x)$, where δ_y is the delta Dirac on y. We denote by $\mathfrak{d}$ such kernel.*

We denote by $\mathcal{F}_\nu(G)$ the set of ν-integrable functions.

Definition 3.2.6. *Given a G-kernel ν and an integrable function $f \in \mathcal{F}_\nu(G)$ we denote by $\nu(f)$ the function in $\mathcal{F}(G^0)$ defined by*

$$\nu(f)\ (y) = \int f\,(s, y)\,\nu^y(ds), \ \ y \in G^0.$$

A kernel ν is characterized by the law

$$f \in \mathcal{F}_\nu(G) \ \ \to \ \ \nu(f) \in \mathcal{F}(G^0).$$

In other words, for a kernel ν we get

$$\nu : \ \mathcal{F}_\nu(G) \to \mathcal{F}(G^0).$$

By notation given a kernel ν and a positive $f \in \mathcal{F}_\nu(G)$ then the kernel $f\,\nu$ is the one defined by $f(x, y)\nu^y(dx)$. In other words the action of the kernel $f\,\nu$ get rid of the first coordinate:

$$h(x, y) \ \to \ \int h(s, y) f\,(s, y)\,\nu^y(ds).$$

In this way if $f \in \mathcal{F}_\nu(G^0)$ we get $f(x)\nu^y(dx)$.

Note that $\nu(f)$ is a function and $f\,\nu$ is a kernel.

Definition 3.2.7. *A* **transverse function** *is a G-kernel ν, such that, if $x \sim y$, then, the finite measures ν^y and ν^x are the same. The set of transverse functions for G is denoted by $\mathcal{E}^+$. We call probabilistic transverse function any one such that for each $y \in G^0$ we get that ν^y is a probability on the class of y.*

The above means that

$$\int f(a)\nu^x(d\,a) = \int f(a)\nu^y(d\,a),$$

if x and y are related. In the above we have $x \sim y \sim a$.

Remark 3.2.8. *The above equality implies that a transverse function is left (and right) invariant. Together with the conditions defining a G-kernel, we have that ν is* **a Haar system** *(see [54]) in a* **measurable sense**. *In what follows we use* **transverse function and Haar system as synonyms**.

The standard G-kernel for the bigger than two relation (see example 3.2.3) is a transverse function.

The normalized standard G-kernel for the the bigger than two relation (see example 3.2.4) is a probabilistic transverse function.

If we consider the equivalence relation such that each point is related just to itself, then the transverse functions can be identified with the positive functions defined on X.

The difference between a function and a transverse function is that the former takes values on the set of real numbers and the latter on the set of measures.

If ν is transverse, then $\nu^x = \nu^y$ when $x \sim y$, and we have from definition 3.2.6

$$(\nu * f)(x,y) = \int f\,(x,s)\,\nu^x(ds) = \nu(\tilde{f})(x), \;\; \forall y \sim x \tag{3.2}$$

and,

$$(f * \nu)(x,y) = \int f\,(s,y)\,\nu^y(ds) = \nu(f)(y), \;\; \forall x \sim y. \tag{3.3}$$

Definition 3.2.9. *The pair (G, ν), where $\nu \in \mathcal{E}^+$, is called the* **measured groupoid for the transverse function** ν*. We assume any ν we consider is such that ν^y is not the zero measure for any y.*

In the case ν is such that, $\int \nu^y(ds) = 1$, for any $y \in G^0$, the Haar system will be called a probabilistic Haar system.

Note that the delta kernel $\mathfrak{d}$ is not a transverse function.

Given a measured groupoid (G, ν) and two measurable functions $f, g \in \mathcal{F}_\nu(G)$, we define $(f \underset{\nu}{*} g) = h$ in such way that for any $(x, y) \in G$

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y)\ \nu^y(ds) = h(x, y).$$

$(f \underset{\nu}{*} g)$ is called the **convolution** of the functions f, g for the measured groupoid (G, ν).

Example 3.2.10. *Consider the groupoid G of Example 3.1.2 and the family ν^y, $y \in \{1, 2, ..., d\}^{\mathbb{N}}$, of measures (where each measure ν^y has support on the equivalence class of y), such that, ν^y is the counting measure. This defines a transverse function (Haar system) called the* **standard Haar system**.

Example 3.2.11. *Consider the groupoid G of Example 3.1.2 and the normalized standard family ν^y, $y \in \{1, 2, ..., d\}^{\mathbb{N}}$. This defines a transverse function called the* **normalized standard Haar system**.

More precisely the family ν^y, $y \in \{1, 2, ..., d\}^{\mathbb{N}}$, $y = (y_1, y_2, y_3, ...)$, of probabilities on the set

$$\{\ (a, y_2, y_3, ...), a \in \{1, 2, .., d\}\ \},$$

is such that, $\nu^y(\ \{(a, y_2, y_3, ...)\ \}) = \frac{1}{d}$, $a \in \{1, 2, .., d\}$

Example 3.2.12. *In example 3.1.6 in which k is fixed consider the transverse function ν such that for each $y \in G^0$, we get that ν^y is the counting measure on the set of points $x \sim_k y$.*

Example 3.2.13. *Suppose $J : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ is continuous positive function such that for any $x \in \Omega$ we have that $\sum_{a=1}^d J(ax) = 1$. For the groupoid G of Example 3.1.2, the family ν^y, $y \in \{1, 2, ..., d\}^{\mathbb{N}}$, of probabilities on*

$\{\,(a, y_2, y_3, ...), a \in \{1, 2, .., d\}\,\}$, *such that,* $\nu^y(a, y_2, y_3, ...) = J(a, y_2, y_3, ...)$, $a \in \{1, 2, .., d\}$ *defines a Haar system. We call it the* **probability Haar system associated to** J.

Example 3.2.11 is a particular case of the present example.

Example 3.2.14. *On the groupoid over* $\{1, 2, ..., d\}^{\mathbb{Z}}$ *described on example 3.1.3, where we consider the notation: for each class specified by* $a \in \overleftarrow{\Omega}$ *the general element in the class is given by*

$$(a \,|\, x) = (..., a_{-n}, ..., a_{-2}, a_{-1} \,|\, x_1, ..., x_n, ...),$$

where $x \in \overrightarrow{\Omega}$.

Consider a fixed probability μ *on* $\overrightarrow{\Omega}$. *We define the transverse function* $\nu^a(dx) = \mu(dx)$ *independent of* a.

Example 3.2.15. *In the example 3.1.8 we consider the partition of* $S^1 \times S^1$ *given by the sets* $W_a = \{\,(a, b) \,|\, b \in S^1\}$, *where* $a \in S^1$. *For each* $a \in S^1$, *consider a probability* $\nu^a(d\,b)$ *over* $\{\,(a, b) \,|\, b \in S^1\}$ *such that for any Borel set* $K \subset S^1 \times S^1$ *we have that* $a \to \nu^a(K)$ *is measurable. This defines a probabilistic transverse function and a Haar system.*

Consider a continuous function $A : S^1 \times S^1 \to \mathbb{R}$. *For each* $a \in S^1$ *consider the kernel* ν^a *such that* $\int f(b)\nu^a(db) = \int f(b)\, e^{A(a,b)}db$, *where* db *is the Lebesgue measure. This defines a transverse function.*

We call the **standard Haar system on** $\mathbf{S^1} \times \mathbf{S^1}$ *the case where for each* a *we consider as the probability* $\nu^a(d\,b)$ *over* $\{\,(a, b) \,|\, b \in S^1\}$ *the Lebesgue probability on* S^1.

We will present several properties of kernels and transverse functions on Section 3.5.

A question of notation: for a fixed groupoid G we will describe now for the reader the common terminology on the field (see [18], [34], [54] and [57]). It is usual to denote a general pair $(x, y) \in G$ by γ (of related elements x, y). The γ is called the directed arrow from x to y. In this case we call $s(\gamma) = x$ and $r(\gamma) = y$ (see [45] for a more detailed description of the arrow's setting).

Here, for each pair of related elements (x, y) there exist an unique directed arrow γ satisfying $s(\gamma) = x$ and $r(\gamma) = y$. Note that, since we are dealing with equivalence relations, (y, x) denotes another arrow. In category language: there is a unique morphism γ that takes $\{x\}$ to $\{y\}$, whenever x

and y are related, and this morphism is associated in a unique way to the pair (x, y).

In this notation $r^{-1}(y)$ is the set of all arrows that end in y. This is in a bijection with all elements on the same class of equivalence of y. We call $r^{-1}(y)$ the fiber over y. If $x \sim y$, then $r^{-1}(y) = r^{-1}(x)$.

We adapt the notation in [18] and [34] to our notation. We use here the expression (s, y) instead of $\gamma\,\gamma'$. This makes sense considering that $\gamma = (x, y)$ and $\gamma' = (s, x)$. We use the expression (y, s) for $(\gamma')^{-1}\,\gamma$, where in this case, $\gamma = (x, y)$ and $\gamma' = (s, y)$, and, finally, $\nu^y(\gamma')$ means $\nu^y(ds)$ for $\gamma' = (s, y)$.

In the case of the groupoid G associated to the bigger than two relation we have for each $x = (x_1, x_2, x_3, ...)$ the property $r^{-1}(x) = \{\,(j, x_2, x_3, ...),\ j = 1, 2.., d\,\}$.

The terminology of arrows will not be essentially used here. It was introduced just for the reader to make a parallel (a dictionary) with the one commonly used on papers on the topic.

Using the terminology of arrows Definition 3.2.7 is equivalent to say that: if, $\gamma = (x, y) = (s(\gamma), r(\gamma))$, then,

$$\nu^y = \gamma\,\nu^x.$$

3.3 Quasi-invariant probabilities

Definition 3.3.1. *A function $\delta : G \to \mathbb{R} \setminus \{0\}$ such that*

$$\delta(x, z) = \delta(x, y)\,\delta(y, z),$$

for any (x, y), $(y, z) \in G$ is called a **modular function** *(also called a multiplicative* **cocycle***).*

In the arrow notation this is equivalent to say that

$$\delta(\gamma_1\gamma_2) = \delta(\gamma_1)\delta(\gamma_2).$$

Note that $\delta(x, y)\,\delta(y, y) = \delta(x, y)$ and it follows that for any y we have $\delta(y, y) = 1$. Moreover, $\delta(x, y)\,\delta(y, x) = \delta(x, x) = 1$ is true. Therefore, we get $\tilde{\delta} = \delta^{-1}$.

Example 3.3.2. *Given $W : G^0 \to \mathbb{R}$, $W(x) > 0, \forall x$, a natural way to get a modular function is to consider $\delta(x,y) = \frac{W(x)}{W(y)}$. In this case we say that the modular function is derived from W.*

Example 3.3.3. *In the case of example 3.1.8 the equivalence relation is: given two points $z_1, z_2 \in S^1 \times S^1$ they are related if the first coordinate is equal.*

Consider a expanding transformation T and the associated Baker map F. Note que $F^n(a,b) = (*, T^n(b))$ for some point $*$.

Given two points $z_1 \sim z_2$, for each n there exist z_1^n and z_2^n, such that, respectively, $F^n(z_1^n) = z_1$ and $F^n(z_2^n) = z_2$, and $z_1^n \sim z_2^n$.

For each pair $z_1 = (a, b_1)$ and $z_2 = (a, b_2)$, and $n \geq 0$, the elements z_1^n, z_2^n are of the form $z_1^n = (a^n, b_1^n)$, $z_2^n = (a^n, b_2^n)$.

In this case $T^n(b_1^n) = b_1$ and $T^n(b_2^n) = b_2$.

Note also that $T^n(a) = a^n$.

The distances between b_1^n and b_2^n are exponentially decreasing with n.

We denote

$$\delta(z_1, z_2) = \Pi_{j=1}^{\infty} \, \frac{T'(b_1^n)}{T'(b_2^n)} < \infty.$$

This product is well defined because

$$\sum_n \log \frac{T'(b_1^n)}{T'(b_2^n)} = \sum_n [\log T'(b_1^n) \, - \, \log T'(b_2^n)\,]$$

converges. This is so because $\log T'$ is Hölder and for all n we have $|b_1^n - b_2^n| < \lambda^{-n}$, where $T'(x) > \lambda > 1$ for all x.

This δ is a cocycle.

In the case of example 3.2.15 considered a Holder continuous function $A(a,b)$, where $A : S^1 \times S^1 \to \mathbb{R}$.

Define for $z_1 = (a, b_1)$ and $z_2 = (a, b_2)$

$$\delta(z_1, z_2) = \Pi_{j=1}^{\infty} \, \frac{e^{A(z_1^n)}}{e^{A(z_2^n)}}.$$

The modular function $\delta(z_1, z_2)$ is well defined because A is Hölder.

We will show that δ can be expressed in the form of example 3.3.2. Indeed, fix a certain $b_0 \in S^1$, then, taking $z_1 = (a, b_1)$ consider $z_0 = (a, b_0)$. We denote in an analogous way z_1^n and z_0^n the ones such that $F^n(z_1^n) = z_1$ and $F^n(z_0^n) = z_0$.

Define $V : G^0 \to \mathbb{R}$ by

$$V(z_1) = \Pi_{j=1}^{\infty} \; \frac{e^{A(z_1^n)}}{e^{A(z_0^n)}}. \tag{3.4}$$

V is well defined and if $z_1 \sim z_2$ we get that

$$\delta(z_1, z_2) = \frac{V(z_1)}{V(z_2)}.$$

We will show later (see Proposition 3.7.4) that $V(a, b)$ does not depend on a, and then we can write $V(b)$, and finally

$$\delta(z_1, z_2) = \frac{V(b_1)}{V(b_2)}.$$

Example 3.3.4. *Consider a fixed Holder function $\hat{A} : \{1, 2, ..., d\}^{\mathbb{Z}} \to \mathbb{R}$ and the groupoid given by the equivalence relation of example 3.1.4. Denote for any (x, y)*

$$\delta(x, y) = \Pi_{j=1}^{\infty} \; \frac{\hat{A}(\hat{\sigma}^{-j}(s(\gamma)))}{\hat{A}(\hat{\sigma}^{-j}(r(\gamma)))} = \Pi_{j=1}^{\infty} \; \frac{\hat{A}(\hat{\sigma}^{-j}(x))}{\hat{A}(\hat{\sigma}^{-j}(y))}.$$

The modular function δ is well defined because $\hat{A}$ is Holder. Indeed, this follows from the bounded distortion property.

In a similar way as in the last example one can show that such δ can be expressed on the form of example 3.3.2.

Definition 3.3.5. *Given a measured groupoid G for the transverse function ν we say that a probability M on G^0 is* **quasi-invariant** *for ν if there exist a modular function $\delta : G \to \mathbb{R}$, such that, for any integrable function $f : G \to \mathbb{R}$ we have*

$$\int \int f(s, x) \nu^x(ds) dM(x) = \int \int f(x, s) \, \delta^{-1}(x, s) \, \nu^x(ds) dM(x). \tag{3.5}$$

In a more accurate way we say that M is quasi-invariant for the transverse function ν and the modular function δ.

Note that if $\delta(x,s) = \frac{B(x)}{B(s)}$ we get that the above condition (3.5) can be written as

$$\int \int f(s,x) B(s) \nu^x(ds) dM(x) = \int \int f(x,s)\, B(s)\, \nu^x(ds) dM(x). \quad (3.6)$$

Indeed, in (3.5) replace $f(s,x)$ by $B(s)f(s,x)$.

Quasi-invariant probabilities will be also described as the ones which satisfies the so called KMS condition (on the setting of von Neumann algebras, or C^*-algebras) as we will see later on section 3.4.

As an extreme example consider the equivalence relation such that each point is related to just itself. In this case a modular function δ takes only the value 1. Given any transverse function ν the condition

$$\int \int f(s,x) \nu^x(ds) dM(x) = \int \int f(x,s)\, \delta^{-1}(x,s)\, \nu^x(ds) dM(x) \quad (3.7)$$

is satisfied by any probability M on X. In this case the set of probabilities is the set of quasi-invariant probabilities.

Example 3.3.6. Quasi invariant probability and the SBR probability for the Baker map

We will present a particular example where we will compare the probability M satisfying the quasi invariant condition with the so called SBR probability. We will consider a different setting of the case described on [62] (considering Anosov systems) which, as far as we know, was never published.

We will show that the **quasi invariant probability is not the SBR probability**.

We will address later on the end of this example the kind of questions discussed on [62] and [43].

We will consider the groupoid of example 3.1.8, that is, we consider the equivalence relation: given two points $z_1, z_2 \in S^1 \times S^1$ they are related if the first coordinate is equal.

In example 3.1.8 we consider an expanding transformation $T : S^1 \to S^1$ and F denotes the associated T-Baker map. The associated SBR probability is the only absolutely continuous F-invariant probability over $S^1 \times S^1$.

The dynamical action of F in some sense looks like the one of an Anosov diffeomorphism.

Consider the measured groupoid $(G\,,\nu)$ where in each vertical fiber over the point a we set ν^a as the Lebesgue probability db over the class (a,b), $0 \leq b \leq 1$.

This groupoid corresponds to the local unstable foliation for the transformation F.

We fix a certain point $b_0 \in (0,1)$. For each pair $x=(a,b)$ and $y=(a,\,b_0)$, where $a,b \in S^1$, and $n \geq 0$, the elements z_1^n, z_2^n, $n \in \mathbb{N}$, are such that $F^n(z_1^n) = x = (a,b)$ and $F^n(z_2^n) = y = (a,b_0)$. Note that they are of the form $z_1^n = (a^n, b^n)$, $z_2^n = (a^n, s^n)$. We use the notation $z_1^n(x)$, $b^n(x)$, $n \in \mathbb{N}$, to express the dependence on x.

We denote for $x \in S^1 \times S^1$

$$V(x) = V(a,b) = \Pi_{n=1}^{\infty}\ \frac{T'(b^n(x))}{T'(s^n)} = \Pi_{n=1}^{\infty}\ \frac{T'(b^n(a,b))}{T'(s^n)} < \infty.$$

This is finite because s^n and $b^n(x)$ are on the same domain of injectivity of T for all n and T' is of Holder class.

In a similar fashion as in [62] we define δ by the expression

$$\delta(\,(a,y_1)\,,\,(a,y_2)\,) = \frac{V(a,y_1)}{V(a,y_2)} = \frac{V(y_1)}{V(y_2)} = \Pi_{n=1}^{\infty}\ \frac{T'(b^n(a,y_1))}{T'(b^n(a,y_2))},$$

where $(a,y_1) \sim (a,y_2)$.

Consider the probability M on $S^1 \times S^1$ given by

$$dM(a,b) = \frac{V(a,b)}{\int V(a,c)dc}\, db\, da.$$

The density $\psi(a,b) = \frac{V(a,b)}{\int V(a,c)dc}$ satisfies the equation

$$\psi(a,b)\frac{1}{T'(b)} = \psi(F(a,b)). \tag{3.8}$$

Denote $F(a,b) = (\tilde{a},\tilde{b})$, then, it is known that the density $\varphi(a,b)$ of the SBR probability for F satisfies

$$\varphi(a,b)\frac{T'(\tilde{a})}{T'(b)} = \varphi(F(a,b)). \tag{3.9}$$

This follows from the F-invariance of the SBR

Therefore, M is not the SBR probability - by uniqueness of the SBR.

We will show that M satisfies the quasi invariant condition.
Note that

$$\int \int f(\,(a,b),(a,s)\,)\ \nu^a(d\,s)\ dM(a,b) =$$

$$\int \int \int f(\,(a,b),(a,s)\,)\ \ \frac{V(a,b)}{\int V(a,c)dc}\, ds\, db\, da.$$

On the other hand

$$\int \int f(\,(a,s),(a,b)\,)\,\frac{V(a,s)}{V(a,b)}\ \nu^a(d\,s)\ dM(a,b) =$$

$$\int \int \int f(\,(a,s),(a,b)\,)\,\frac{V(a,s)}{V(a,b)}\ \ \frac{V(a,b)}{\int V(a,c)dc}\, ds\, db\, da =$$

$$\int \int \int f(\,(a,s),(a,b)\,)\ \ \frac{V(a,s)}{\int V(a,c)dc}\, ds\, db\, da.$$

If we exchange the variables b and s, and using Fubini's theorem, we get that M satisfies the quasi invariant condition.

The relation of quasi-invariant probabilities and transverse measures is described on section 3.5.

The result considered on Theorem 6.18 in [62] for an Anosov diffeomorphism concerns transverse measures and cocycles. [62] did not mention quasi-invariant probabilities.

Note that from equations (3.8) and (3.9) one can get that the conditional disintegration along unstable leaves of both the SRB and the quasi-invariant probability M are equal (see page 533 in [39]).

Using the relation of quasi-invariant probabilities, cocycles and transverse measures one can say that one of the main claims in [62] (see Theorem 6.18) and [43] (both considering the case of Anosov Systems) can be expressed in some sense via the above mentioned property about conditional disintegration along unstable leaves (using the analogy with the case of the above Baker map F).

In section 3.7 we will present more examples of quasi-stationary probabilities.

3.4 von Neumann Algebras derived from measured groupoid

We refer the reader to [2], [34] and [18] as general references for von Neumann algebras related to groupoids.

Here $X \sim G^0$ will be either $\hat{\Omega}$, Ω or $S^1 \times S^1$. We will denote by G a general groupoid obtained by an equivalence relation R.

Definition 3.4.1. *Given a measured groupoid G for the transverse function ν and two measurable functions $f, g \in \mathcal{F}_\nu(G)$, we define the convolution $(f \underset{\nu}{*} g) = h$, in such way that, for any $(x, y) \in G$*

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y)\ \nu^y(ds) = h(x, y).$$

In the case there exists a multiplicative neutral element for the operation $*$ we denote it by $\mathbb{1}$.

The above expression in some sense resembles the way we get a matrix as the product of two matrices.

For a fixed Haar system ν the product $\underset{\nu}{*}$ defines an algebra on the vector space of ν-integrable functions $\mathcal{F}_\nu(G)$.

As usual function of the form $f(x, x)$ are identified with functions $f : G^0 \to \mathbb{R}$ of the from $f(x)$.

Example 3.4.2. *In the particular case where ν^y is the counting measure on the fiber over y then*

$$(f \underset{\nu}{*} g)(x, y) = \sum_s g(x, s)\, f(s, y).$$

Denote by I_Δ the indicator function of the diagonal on $G^0 \times G^0$. In this case, I_Δ is the neutral element for the product $\underset{\nu}{*}$ operation.

In this case $\mathbb{1} = I_\Delta$.

Note that I_Δ is measurable but generally not continuous. This is fine for the von Neumann algebra setting. However, we will need a different topology (and σ-algebra) on $G^0 \times G^0$ - other than the product topology - when considering the unit $\mathbb{1} = I_\Delta$ for the C^*-algebra setting (see [20], [56], [57]). This will be more carefully explained on section 3.6.

Remark: The indicator function of the diagonal on $G^0 \times G^0$ is not always the multiplicative neutral element on the von Neumann algebra obtained from a general Haar system (G, ν).

Example 3.4.3. *Another example: consider the standard Haar system of example 3.2.10.*

In this case

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\, f(s,y) \nu^y(ds) =$$

$$\frac{1}{d} \sum_{a=1}^{d} g(x, \,(\, a, x_2, x_3, ...)\,)\; f(\,(a, x_2, x_3, ...)\,, y\,) \; = h(x,y).$$

The neutral element is $d\, I_\Delta = \mathbf{1}$.

Example 3.4.4. *Suppose* $J : \{1, 2, ..., d\}^{\mathbb{N}} \to \mathbb{R}$ *is a continuous positive function such that for any* $x \in \Omega$ *we have that* $\sum_{a=1}^{d} J(ax) = 1$*. The measured groupoid* (G, ν) *of Example 3.2.13, where* ν^y, $y \in \{1, 2, ..., d\}^{\mathbb{N}}$*, is such that given* $f, g : G \to \mathbb{R}$*, we have for any* $(x, y) \in G$, $x = (x_1, x_2, x_3, ...)$, $y = (y_1, x_2, x_3, ..)$ *that*

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\, f(s,y) \nu^y(ds) =$$

$$\sum_{a=1}^{d} g(x, \,(\, a, x_2, x_3, ...)\,)\; f(\,(a, x_2, x_3, ...)\,, y\,) \; J(a, x_2, x_3, ...) = h(x,y).$$

Note that $x_j = y_j$ *for* $j \geq 2$.

Suppose that f *is such that for any string* $(x_2, x_3, ...)$ *and* $a \in \{1, 2, ., d\}$ *we get*

$$f(\,(a, x_2, x_3, ...)\,, (a, x_2, x_3, ...)\,) \; = \; \frac{1}{J(a, x_2, x_3, ...)},$$

and, $a, b \in \{1, 2, ., d\}$, $a \neq b$

$$f(\,(a, x_2, x_3, ...)\,, (b, x_2, x_3, ...)\,) \; = \; 0.$$

In this case the neutral multiplicative element is $\mathbf{1}(x,y) = \frac{1}{J(x)} I_\Delta(x,y)$.

Consider a measured groupoid (G, ν), $\nu \in \mathcal{E}$, then, given two functions ν-integrable $f, g : G \to \mathbb{R}$, we had defined before an algebra structure on $\mathcal{F}_\nu(G)$ in such way that $(f \underset{\nu}{*} g) = h$, if

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y) \nu^y(ds) = h(x, y),$$

where $(x, y) \in G$ and $(s, y) \in G$.

To define the von Neumann algebra associated to (G, ν), we work with complex valued functions $f : G \to \mathbb{C}$. The product is again given by the formula

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y) \nu^y(ds).$$

The involution operation $*$ is the rule $f \to \tilde{f} = f^*$, where $\tilde{f}(x, y) = \overline{f(y, x)}$. The functions $f \in \mathcal{F}(G^0)$ are of the form $f(x) = f(x, x)$ are such that $\tilde{f} = f$.

Following Hahn [33], we define the I-norm

$$\|f\|_I = \max \left\{ \left\| y \mapsto \int |f(x, y)|\, \nu^y(dx) \right\|_\infty, \left\| y \mapsto \int |f(y, x)|\, \nu^y(dx) \right\|_\infty \right\},$$

and the algebra $I(G, \nu) = \{f \in L^1(G, \nu) : \|f\|_I < \infty\}$ with the product and involution as above. An element $f \in I(G, \nu)$ defines a bounded operator L_f of left convolution multiplication by a fixed f on $L^2(G, \nu)$. This gives the left regular representation of $I(G, \nu)$.

Definition 3.4.5. *Given a measured groupoid (G, ν), we define the* **von Neumann Algebra associated to** (G, ν)*, denoted by $W^*(G, \nu)$, as the von Neumann generated by the left regular representation of $I(G, \nu)$, that is, $W^*(G, \nu)$ is the closure of $\{L_f : f \in I(G, \nu)\}$ in the weak operator topology.*

The multiplicative unity is denoted by $\imath$.

In the case ν is such that, $\int \nu^y(ds) = 1$, for any $y \in G^0$, we say that the von Neumann algebra is normalized.

In the setting of von Neumann Algebras we do not require that $\imath$ is continuous.

Definition 3.4.6. *We say an element* $h \in W^*(G\,,\nu)$ *is positive if there exists a* g *such that* $h = g * \tilde{g}$.

This means

$$h(x,y) = (g \underset{\nu}{*} \tilde{g})(x,y) = \int g(x,s)\,\overline{g(y,s)}\nu^y(ds) = h(x,y).$$

Note que $h(x,x) = (g \underset{\nu}{*} \tilde{g})(x,x) \geq 0$.

Example 3.4.7. *Consider over the set* $G^0 = \{1,2..,d\}$ *the equivalence relation where all points are related. In this case* $G = \{1,2..,d\} \times \{1,2..,d\}$. *Take* ν *as the counting measure. A function* $f : G \to \mathbb{R}$ *is denoted by* $f(i,j)$, *where* $i \in \{1,2..,d\}, j \in \{1,2..,d\}$.

The convolution product is

$$(f \underset{\nu}{*} g)(i,j) = \sum_k g(i,k)\, f(k,j).$$

In this case the associated von Neumann algebra (the set of functions $f : G \to \mathbb{C}$*) is identified with the set of matrices, the convolution is the product of matrices and the identity matrix is the unit* $\mathbb{1}$. *The involution operation is to take the hermitian of a matrix.*

Example 3.4.8. *For the groupoid* G *of Example 3.1.2 and the counting measure, given* $f, g : G \to \mathbb{C}$, *we have that*

$$(f \underset{\nu}{*} g)(x,y) =$$

$$\sum_{a \in \{1,2,..,d\}} g(\,(\,x_1, x_2, ...\,)\,,\,(\,a, x_2, x_3, ..)\,)\; f(\,(\,a, x_2, x_3, ..)\,,\,(\,y_1, x_2, ...\,)\,).$$

We call standard von Neumann algebra on the groupoid G *(of Example 3.1.2) the associated von Neumann algebra. For this* $W^*(G\,,\nu)$ *the neutral element* $\mathbb{1}$ *(or, more formally* $L_{\mathbb{1}}$*) is the indicator function of the diagonal (a subset of* G*). In this case* $\mathbb{1}$ *is measurable but not continuous.*

Example 3.4.9. *For the probabilistic Haar system* (G, ν) *of Example 3.2.13, given* $f, g : G \to \mathbb{C}$, *we get*

$$(f \underset{\nu}{*} g)(x,y) =$$

$$\sum_{a\in\{1,2,...,d\}} \varphi(a,x_2,x_3,...)\, g((\,x_1,x_2,...\,),(\,a,x_2,x_3..))\; f((\,a,x_2,x_3,...),(\,y_1,x_2,...\,)),$$

where φ is Hölder and such that $\sum_{a\in\{1,2,...,d\}} \varphi(a,x_1,x_2,...) = 1$, for all $x = (x_1,x_2,...)$.

This φ is a Jacobian.

The neutral element is described in example 3.4.4.

Example 3.4.10. *In the case $\nu^y = \delta_{x_0}$ for a fixed x_0 independent of y, then*

$$(f \underset{\nu}{*} g)(x,y) = g(x,x_0)\, f(x_0,y).$$

Proposition 3.4.11. *If (G,ν) is a measured groupoid, then for $f,g \in I(G,\lambda)$.*

$$(f \underset{\nu}{*} g)^{\sim} = \tilde{g} \underset{\nu}{*} \tilde{f}.$$

Proof: Remember that for (x,y) in G

$$(f \underset{\nu}{*} g)(x,y) = \int g(x,s)\, f(s,y)\nu^y(ds) = h(x,y).$$

Then,

$$(f \underset{\nu}{*} g)^{\sim}(x,y) = \int \overline{g(y,s)\, f(s,x)}\nu^x(ds).$$

On the other hand

$$(\tilde{g} \underset{\nu}{*} \tilde{f})(y,x) = \int \overline{f(s,x)}\,\overline{g(y,s)}\nu^y(ds).$$

As $\nu^y = \nu^x$ we get that the two expressions are equal.

□

Then by proposition 3.4.11 we have for the involution $*$ it is valid the property

$$(f \underset{\lambda}{*} g)^* = g^* \underset{\lambda}{*} f^*.$$

For more details about properties related to this definition we refer the reader to chapter II in [54] and section 5 in [34].

We say that $c : G \to \mathbb{R}$ is a linear cocycle function if $c(x,y) + c(y,z) = c(x,z)$, for all x,y,z which are related. If c is a linear cocycle then e^{δ} is a modular function (or, a multiplicative cocycle).

Definition 3.4.12. *Consider the von Neumann algebra* $W^*(G,\nu)$ *associated to* (G,ν)*.*

Given a continuous cocycle function $c: G \to \mathbb{R}$ *we define the* **group homomorphism** $\alpha: \mathbb{R} \to Aut(W^*(G,\nu))$*, where for each* $t \in \mathbb{R}$ *we have that* $\alpha_t \in Aut(W^*(G,\nu))$ *is defined by: for each fixed* $t \in \mathbb{R}$ *and* $f: G \to \mathbb{R}$ *we set* $\alpha_t(f) = e^{t\,i\,c} f$*.*

Remark: Observe that in the above definition that for each fixed $t \in \mathbb{R}$ and any $f: G^0 \to \mathbb{R}$, we have $\alpha_t(f) = f$, since $c(x,x) = 0$ for all $x \in G^0$.

We are particularly interested here in the case where $G^0 = \Omega$ or $G^0 = \hat{\Omega}$.

The value t above is related to temperature and not time. We are later going to consider complex numbers z in place of t. Of particular interest is $z = \beta i$ where β is related to the inverse of temperature in Thermodynamic Formalism (or, Statistical Mechanics).

Definition 3.4.13. *Consider the von Neumann Algebra* $W^*(G,\nu)$ *with unity* $\mathbf{1}$ *associated to* (G,ν)*. A* **von Neumann dynamical state** *is a linear functional* w *(acting on the linear space* $W^*(G,\nu)$*) of the form* $w: W^*(G,\nu) \to \mathbb{C}$*, such that,* $w(a) \geq 0$*, if* a *is a positive element of* $W^*(G,\nu)$*, and* $w(\mathbf{1}) = 1$*.*

Example 3.4.14. *Consider over* $\Omega = \{1,2,..,d\}^{\mathbb{N}}$ *the equivalence relation* R *of Example 3.1.2 and the Haar system* (G,ν) *associated to the counting measure in each fiber* $r^{-1}(x) = \{(a,x_2,x_3,...)\,|a \in \{1,2,...,d\}\}$*, where* $x = (x_1,x_2,...)$*.*

Given a probability μ *over* Ω *we can define a von Neumann dynamical state* φ_μ *in the following way: for* $f: G \to \mathbb{C}$ *define*

$$\varphi_\mu(f) \;=\; \int f(x,x)\,d\mu(x) = \int\; f((\,x_1,x_2,x_3,\dots),(\,x_1,x_2,x_3,...))\;d\mu(x). \tag{3.10}$$

If h *is positive, that is, of the form* $h(x,y) = \int\, g(x,s)\,\overline{g(y,s)}\nu^y(ds)$*, then*

$$\varphi_\mu(h) \;=\; \int (\;\int\; \|g(x,s)\|^2\nu^x(ds)\;)\;d\mu(x) \geq 0.$$

Note that $\varphi_\mu \mathbf{1} = 1$*.*

Then, φ_μ *is indeed a von Neumann dynamical state.*

In this case given $f,g: G \to \mathbb{C}$

$$\varphi_\mu(f \underset{\nu}{*} g) \;=$$

$$\int \sum_{a\in\{1,2,..,d\}} f((\,x_1,x_2,...\,),(\,a,x_2,x_3..)) \; g((\,a,x_2,x_3..),(\,x_1,x_2,...\,))\, d\mu(x).$$

It seems natural to try to obtain dynamical states from probabilities M on G^0 (adapting the reasoning of the above example). Then, given a cocycle c it is also natural to ask: what we should assume on M in order to get a KMS state for c?

Example 3.4.15. *For the von Neumann algebra of complex matrices of example 3.4.7 taking $p_1, p_2, .., p_d \geq 0$, such that $p_1 + p_2 + .. + p_d = 1$, and $\mu = \sum_{j=1}^d \delta_j$, we consider φ_μ such that*

$$\varphi_\mu(A) = A_{11}p_1 + A_{22}p_2 + ... + A_{dd}p_d,$$

where A_{ij} are the entries of A.

Note first that $\varphi_\mu(I) = 1$.

If $B = A\,A^*$, then the entries $B_{jj} \geq 0$, for $j = 1, 2, ..., d$.

Therefore, φ_μ is a dynamical state on this von Neumann algebra.

Example 3.4.16. *Consider over $\Omega = \{1, 2, .., d\}^{\mathbb{N}}$ the equivalence relation R of Example 3.2.13 and the associated probability Haar system ν.*

Given a probability μ over Ω we can define a von Neumann dynamical state φ_μ in the following way: given $f : G \to \mathbb{C}$ we get $\varphi_\mu(f) = \int f(x,x)\, J(x)\, d\mu(x)$. In this way given f, g we have

$$\varphi_\mu(f \underset{\nu}{*} g) =$$

$$\int \sum_{a\in\{1,2,..,d\}} J(a,x_2,..)\, g((\,x_1,x_2,...\,),(\,a,x_2,..)) \; f((\,a,x_2,..),(\,x_1,x_2...\,))\, J(x) d\mu(x).$$

For the neutral multiplicative element $\mathbb{1}(x,y) = \frac{1}{J(x)} I_\Delta(x,y)$ we get

$$\varphi_\mu(\mathbb{1}) = \int \frac{1}{J(x)} I_\Delta(x,x)\, J(x) d\mu(x) = 1.$$

Consider G a groupoid and a von Neumann Algebra $W^*(G,\nu)$, where ν is a transverse function, with the algebra product $f \underset{\nu}{*} g$ and involution $f \to \tilde{f}$.

Given a continuous cocycle $c : G \to \mathbb{R}$ we consider $\alpha : \mathbb{R} \to \mathrm{Aut}(W^*(G,\nu))$, $t \mapsto \alpha_t$, the associated homomorphism according to definition 3.4.12: for each fixed $t \in \mathbf{R}$ and $f : G \to \mathbb{R}$ we set $\alpha_t(f) = e^{t\,i\,c}\, f$.

Definition 3.4.17. *An element $a \in W^*(G,\nu)$ is said to be* **analytical** *with respect to α if the map $t \in \mathbb{R} \mapsto \alpha_t(a) \in W^*(G,\nu)$ has an analytic continuation to the complex numbers.*

More precisely, there is a map $\varphi : \mathbb{C} \to W^*(G,\nu)$, such that, $\varphi(t) = \alpha_t(a)$, for all $t \in \mathbb{R}$, and moreover, for every $z_0 \in \mathbb{C}$, there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $W^*(G,\nu)$, such that, $\varphi(z) = \sum_{n=0}^{\infty}(z-z_0)^n a_n$ in a neighborhood of z_0.

The analytical elements are dense on the von Neumann algebra (see [48]).

Definition 3.4.18. *We say that a von Neumann dynamical state w is a* **KMS state for** β **and** c *if*

$$w\,(\,b \underset{\nu}{*} (\alpha_{i\,\beta}(a)\,)\,) = w\,(a \underset{\nu}{*} b\,),$$

for any b and any analytical element a.

It follows from general results (see [48]) that it is enough to verify: for any $f, g \in I(G,\nu)$ and $\beta \in \mathbb{R}$ we get

$$w\,(\,g \underset{\nu}{*} \alpha_{\beta\,i}(\,f\,)\,) = w\,(\,g \underset{\nu}{*} (\,e^{-\beta\,c}\,f\,)\,) = w\,(f \underset{\nu}{*} g\,). \tag{3.11}$$

Consider the functions

$$u(x,y) = (f * g)(x,y) = \int g(x,s)\, f(s,y)\, \nu^y(ds),$$

and

$$v(x,y) = (\,g * (e^{-\beta\,c}\,f)\,)\,(x,y) = \int e^{-\beta c(x,s)} f(x,s)\, g(s,y)\, \nu^y(ds).$$

Equation (3.11) means

$$w(\,u(x,y)\,) = w(\,v(x,y)\,). \tag{3.12}$$

Note that equation (3.11) implies that a KMS von Neumann (or, C^*)-dynamical state w satisfies:

a) for any $f : G^0 \to \mathbb{C}$ and $g : G \to \mathbb{C}$:

$$w\,(\,g \underset{\nu}{*} f\,) = w\,(f \underset{\nu}{*} g\,). \tag{3.13}$$

This follows from the fact that for any $t \in \mathbb{R}$ and any $f : G^0 \to \mathbb{R}$, we have that $\alpha_t(f) = f$.

b) if the function $\mathbf{1}$ depends just on $x \in G^0$, then, for any β

$$\alpha_{i\,\beta}(\mathbf{1}) = \mathbf{1}.$$

c) w is invariant for the group α_t, $t \in \mathbb{R}$. Indeed,

$$w(\,\alpha_t(f)) = w(\mathbf{1} \underset{\nu}{*} \alpha^t(f)) = w(f \underset{\nu}{*} \mathbf{1}) \;=\; w(f).$$

Example 3.4.19. *For the von Neumann algebra (C^*-algebra) of complex matrices of examples 3.4.7 and 3.4.15 consider the dynamical evolution $\sigma_t = e^{i\,t\,H}$, $t \in \mathbb{R}$, where H is a diagonal matrix with entries the real numbers $H_{11} = U_1, H_{22} = U_2, ..., H_{dd} = U_d$. The KMS state ρ for β is*

$$\rho(A) = A_{1\,1}\rho_1 + A_{2\,2}\rho_2 + ... + A_{d\,d}\rho_d,$$

where $\rho_i = \frac{e^{-\beta U_i}}{\sum_{j=1}^{d} e^{-\beta U_j}}$, $i = 1, 2, ..., d$, and $A_{i,j}$, $i, j = 1, 2, ..., d$, are the entries of the matrix A (see [57]).

The probability μ of example 3.4.14 corresponds in some sense to the probability $\mu = (\rho_1, \rho_2, .., \rho_d)$ on $\{1, 2, ..., d\}$. That is, $\rho = \varphi_\mu$.

This is a clear indication that the μ associated to the KMS state has in some sense a relation with Gibbs probabilities. This property will appear more explicitly on Theorem 3.4.24 for the case of the bigger than two equivalence relation.

Remember that if c is a cocycle, then $c(x,z) = c(x,y) + c(y,z)$, $\forall x \sim y \sim z$, and, therefore,

$$\delta(x,y) = e^{\,\beta c(x,y)} = e^{\,-\beta c(y,x)}$$

is a modular function.

Definition 3.4.20. *Given a cocycle $c : G \to \mathbb{R}$ we say that a probability M over G^0 satisfies the (c, β)-KMS condition for the groupoid (G, ν), if for any $h \in I(G, \nu)$, we have*

$$\int \int h(s,x)\nu^x(ds)dM(x) = \int \int h(x,s)e^{-\beta\, c(x,s)}\, \nu^x(ds)dM(x), \qquad (3.14)$$

where $\beta \in \mathbb{R}$.

In this case we will say that M is a **KMS probability**.

The above means that M **is quasi-invariant for** ν **and** $\delta(x,s) = e^{-\beta c(s,x)}$.

When $\beta = 1$ and c is of the form $c(s,x) = V(x) - V(s)$ the above condition means

$$\int \int h(s,x)\nu^x(ds)e^{V(x)}dM(x) = \int \int h(x,s)e^{V(x)}\, \nu^x(ds)dM(x). \qquad (3.15)$$

Proposition 3.4.21. *(J. Renault - Proposition II.5.4 in [54]) Suppose that the state w is such that for a certain probability μ on G^0 we have that for any $h \in I(G, \nu)$ we get $w(h) = \int h(x,x)d\mu(x)$. Then, to say that μ satisfies the (c, β)-KMS condition for (G, ν) according to Definition 3.4.20 is equivalent to say that w is KMS for (G, ν), c and β, according to equation* (3.11).

Proof: Note that for any f, g

$$(\, f \underset{\nu}{*}\, g\,)(x,y) = \int g(x,s)\, f(s,y)d\nu^x(s)$$

and

$$(\, g \underset{\nu}{*}\, (\, e^{-\beta\, c}\, f\,)\,)(x,y) = \int f(x,s)\, g(s,y)\, e^{-\beta\, c(x,s)}\, d\nu^x(s).$$

We have to show that $\int u(x,x)d\mu(x) = \int v(x,x)d\mu(x)$ (see equation (3.12)).

Then, if the (c, β)-KMS condition for M is true, we take $h(s,x) = g(x,s)\, f(s,x)$ and we got equation (3.12) for such w.

By the other hand if (3.12) is true for such w and any f, g, then take $f(s,x) = h(s,x)$ and $g(s,x) = 1$.

□

Example 3.4.22. *In the case for each y we have that ν^y is the counting measure we get that to say that a probability M over $\hat{\Omega}$ satisfies the (c,β)-KMS condition means: for any $h : G \to \mathbb{C}$*

$$\sum_{y\sim x}\int h(x,y)e^{-\beta\, c(x,y)}dM(x) = \sum_{x\sim y}\int h(x,y)\, dM(y). \tag{3.16}$$

In the notation of [55] we can write the above in an equivalent way as

$$\int h\, e^{-\beta\, c} d(s^*(M)) = \int h\, d(r^*(M)).$$

Note that in [55] it is considered $r(x,y) = x$ and $s(x,y) = y$.

Suppose $c(x,y) = \varphi(x) - \varphi(y)$. Then, taking $h(x,y) = k(x,y)\, e^{\beta\,\varphi(x)}$ we get an equivalent expression for (3.16): for any $k(x,y)$

$$\sum_{y\sim x}\int k(x,y)e^{\beta\,\varphi(y)}dM(x) = \sum_{x\sim y}\int k(x,y)\, e^{\beta\,\varphi(x)}dM(y). \tag{3.17}$$

For a Hölder continuous potential $A : \{1,2..,d\}^{\mathbb{N}} \to \mathbb{R}$ the Ruelle operator $\mathcal{L}_A$ acts on continuous functions $v : \{1,2..,d\}^{\mathbb{N}} \to \mathbb{R}$ by means of $\mathcal{L}_A(v) = w$, if

$$\mathcal{L}_A(v)(x_1,x_2,x_3,...) = \sum_{a=1}^{d} e^{A(a,x_1,x_2,x_3,...)}v(a,x_1,x_2,x_3,...) = w(x).$$

For a Hölder continuous potential $A : \{1,2..,d\}^{\mathbb{N}} \to \mathbb{R}$ there exist a continuous positive eigenfunction f, such that, $\mathcal{L}_A(f) = \lambda\, f$, where λ is positive and also the spectral radius of $\mathcal{L}_A$ (see [47]).

The dual $\mathcal{L}_A^*$ of $\mathcal{L}_A$ acts on probabilities by Riesz Theorem (see [47]). We say that the probability m on $\{1,2..,d\}^{\mathbb{N}}$ is **Gibbs for the potential** A, if $\mathcal{L}_A^*(m) = \lambda\, m$ (same λ as above). In this case we say that m is an eigenprobability for A.

Gibbs probabilities for Hölder potentials A are also **DLR probabilities** on $\{1,2,...d\}^{\mathbb{N}}$ (see [17]).

Gibbs probabilities for Hölder potentials A can be also obtained via Thermodynamic Limit from boundary conditions (see [17]).

We say that the **potential A is normalized** if $\mathcal{L}_A(1) = 1$. In this case a probability μ is Gibbs (equilibrium) for the normalized potential A if it is a fixed point for the dual of the Ruelle operator, that is, $\mathcal{L}_A^*(\mu) = \mu$.

Suppose $\Omega = \{-1, 1\}^{\mathbb{N}}$ and $A : \Omega \to \mathbb{R}$ is of the form

$$A(x_0, x_1, x_2, ...) = x_0 a_0 + x_1 a_1 + x_2 a_2 + x_3 a_3 + ... + x_n a_n + ...$$

where $\sum a_n$ is absolutely convergent.

In [16] the explicit expression of the eigenfunction for $\mathcal{L}_A$ and the eigenprobability for the dual $\mathcal{L}_A^*$ of the Ruelle operator $\mathcal{L}_A$ is presented. The eigenprobability is not invariant for the shift.

Example 3.4.23. *For the Haar system of examples 3.1.6 and 3.2.12 where k is fixed consider a normalized potential Hölder $A : \{1, 2.., d\}^{\mathbb{N}} \to \mathbb{R}$. Denote by μ the equilibrium probability associated to such A.*

Consider

$$\delta(x, y) = \frac{e^{A(y)+A(\sigma(y))+...+A(\sigma^{k-1}(y))}}{e^{A(x)+A(\sigma(x)+...+A(\sigma^{k-1}(x))}}.$$

We claim that μ satisfies the (c, β)-KMS condition (3.15) for such δ when $\beta = 1$.

For each cylinder set $\overline{a_1, a_2, .., a_k}$ the transformation $\sigma^k : \overline{a_1, a_2, .., a_k} \to \{1, 2.., d\}^{\mathbb{N}}$ is a bijection. The pull back by σ^k of the probability μ with respect to μ has Radon-Nykodin derivative

$$\phi_{\overline{a_1,a_2,..,a_k}}(x) = e^{A(x)+A(\sigma(x)+...+A(\sigma^{k-1}(x))}.$$

Denote $\varphi_{\overline{a_1,a_2,..,a_k}} : \{1, 2.., d\}^{\mathbb{N}} \to \overline{a_1, a_2, .., a_k}$ the inverse of σ^k (restricted to $\{1, 2.., d\}^{\mathbb{N}}$).

Consider the cylinders $K = \overline{a_1, a_2, .., a_k}$, $L = \overline{b_1, b_2, .., b_k}$.

Note that it follows from the use of the change of coordinates $y \to x = \varphi_{\overline{b_1,b_2,..,b_k}} \circ (\varphi_{\overline{a_1,a_2,..,a_k}})^{-1}(y)$ that

$$\int_K e^{A(y)+A(\sigma(y))+...+A(\sigma^{k-1}(y))} d\mu(y) = \int_L e^{A(x)+A(\sigma(x)+...+A(\sigma^{k-1}(x))} d\mu(x)$$

For each class the number of elements s on K or L is the same.

This means that

$$\sum_{s\in L} \int_K e^{A(y)+A(\sigma(y))+...+A(\sigma^{k-1}(y))} dM(y) =$$

$$\sum_{s\in K} \int_L e^{A(x)+A(\sigma(x))+...+A(\sigma^{k-1}(x))} dM(x). \tag{3.18}$$

Remark: *Given $y \in X$, consider the function $f(x,s)$, where $f : [y] \times [y] \to \mathbb{C}$.*

For each pair $(i,j) \in [y] \times [y]$, denote $z_{i,j} = f(i,j)$.

Then, $f : [y] \times [y] \to \mathbb{C}$ can be written as

$$\sum_{i,j \in [y]} z_{i,j} I_i \, I_j.$$

Then, any function $f(x,s)$, $f : [y] \times [y] \to \mathbb{C}$, is a linear combination of functions which are the product of two functions: one depending just on x and the other just on s.

Then, expression (3.18) means that for such f we have

$$\sum_s \int f(s,y) e^{A(y)+A(\sigma(y))+\ldots+A(\sigma^{k-1}(y))} dM(y) =$$

$$\sum_s \int f(x,s) e^{A(x)+A(\sigma(x))+\ldots+A(\sigma^{k-1}(x))} \, dM(x), \tag{3.19}$$

The (c,β)-KMS condition (3.15) for the probability M and for any continuous function f means

$$\sum_s \int f(s,y) e^{\beta(A(y)+A(\sigma(y))+\ldots+A(\sigma^{k-1}(y))} dM(y) =$$

$$\sum_s \int f(x,s) e^{\beta(A(x)+A(\sigma(x))+\ldots+A(\sigma^{k-1}(x)))} \, dM(x), \tag{3.20}$$

Expression (3.20) follows from (3.19) and the above remark. Therefore, such M satisfies the KMS condition for such δ.

In example 3.4.8 consider $\Omega = \{1,2\}^{\mathbb{N}}$ and take ν^y the counting measure on the class of y. Consider the von Neumann algebra associated to this measured groupoid (G, ν) where G is given by the bigger than two relation.

In this case $\imath(x,y) = I_\Delta(x,y)$.

Consider $c(x,y) = \varphi(x) - \varphi(y)$, where φ is Hölder. We do not assume that φ is normalized.

A natural question is: the eigenprobability μ for such potential φ is such that $f \to \varphi_\mu(f) = \int f(x,x) \, d\mu(x)$ defines the associated KMS state? For each modular function c?

The purpose of the next results is to analyze this question when $c(x,y) = \varphi(x) - \varphi(y)$.

Consider the equivalence relation on $\Omega = \{1, 2..., d\}^{\mathbb{N}}$ which is

$$x = (x_1, x_2, x_3, ..) \sim y = (y_1, y_2, y_3, ...) \text{ , if an only if } , x_j = y_j \text{ for all } j \geq 2.$$

In this case the class $[x]$ of $x = (x_1, x_2, x_3, ..)$ is

$$[x] = \{ (1, x_2, x_3, ..), (2, x_2, x_3, ..), ..., (d, x_2, x_3, ..) \}.$$

The associated groupoid by $G \subset \Omega \times \Omega$, is

$$G = \{(x,y) \,|\, x \sim y\}.$$

G is a closed set on the compact set $\Omega \times \Omega$. We fix the measured groupoid (G, ν) where ν^x is the counting measure. The results we will get are the same if we take the Haar system as the one where each point y on the class of x has mass $1/d$.

In this case equation (3.14) means

$$\sum_j \int f(\,(j, x_2, x_3, .., x_n, ..)\,,\,(x_1, x_2, x_3, .., x_n, ..))dM(x) =$$

$$\sum_j \int f(\,(x_1, x_2, x_3, ..)\,,\,(j, x_2, x_3, ..))e^{-c(j,x_2,x_3,..)\,,\,(x_1,x_2,x_3,..))}dM(x). \quad (3.21)$$

The first question: given a cocycle c does there exist M as above?

Suppose $c(x,y) = \varphi(y) - \varphi(x)$.

In this case equation (3.21) means

$$\sum_j \int f(\,(j, x_2, x_3, .., x_n, ..)\,,\,(x_1, x_2, x_3, .., x_n, ..))dM(x) =$$

$$\sum_j \int f(\,(x_1, x_2, x_3, ..)\,,\,(j, x_2, x_3, ..))e^{-\varphi(j,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}dM(x). \quad (3.22)$$

Among other things we will show later that if we assume that φ depends just on the first coordinate then we can take M as the independent probability (that is, such independent M satisfies the KMS condition (3.22)).

In section 3.4 in [57] and in [37] the authors present a result concerning quasi-invariant probabilities and Gibbs probabilities on $\{1,2,..d\}^{\mathbb{N}}$ which has a different nature when compared to the next one. The groupoid is different from the one we will consider (there elements are of the form (x,n,y), $n \in \mathbb{Z}$). In [57] and [37] for just one value of β you get the existence of the quasi invariant probability. Moreover, the KMS state is unique (here this will not happen as we will show on Theorem 3.4.25)

In [58], [24], [44] and [40] the authors present results which have some similarities with the next theorem. They consider Gibbs (quasi-invariant) probabilities in the case of the symbolic space $\{1,2,..d\}^{\mathbb{Z}}$ and not $\{1,2,..d\}^{\mathbb{N}}$ like here. In all these papers the quasi-invariant probability is unique and invariant for the shift. In [9] the authors consider DLR probabilities for interactions in $\{1,2,..d\}^{\mathbb{Z}}$. The equivalence relation (the homoclinic relation of Example 3.1.7) in all these cases is quite different from the one we will consider.

Theorem 3.4.24. *Consider the Haar system with the counting measure ν for the bigger than two relation on $\{1,2,..d\}^{\mathbb{N}}$. Suppose that φ depends just on the first k coordinates,*

$$\varphi(x_1,x_2,...,x_k,x_{k+1},x_{k+2},..) = \varphi(x_1,x_2,...,x_k).$$

Then, the eigenprobability μ (a DLR probability) for the potential $-\varphi$ (that is, $\mathcal{L}^*_{-\varphi}(\mu) = \lambda\mu$, for some positive λ) satisfies the KMS condition (is quasi-invariant) for the associated modular function $c(x,y) = \varphi(y) - \varphi(x)$.

The same result is true, of course, for βc, where $\beta > 0$.

Proof: We are going to show that the Gibbs probability μ for the potential $-\varphi$ satisfies the KMS condition.

We have to show that (3.22) is true when $M = \mu$. That is, μ is a KMS probability for the Haar system and the modular function.

Denote for any finite string $a_1, a_2, ...a_n$ and any n

$$p_{a_1,a_2,...a_n} = \frac{e^{-[\varphi(a_1,a_2,...a_n,1^\infty)+\varphi(a_2,...a_n,1^\infty)+...+\varphi(a_n,1^\infty)]}}{\sum_{b_1,b_2,...b_n} e^{-[\varphi(b_1,b_2,...b_n,1^\infty)+\varphi(b_2,...b_n,1^\infty)+...+\varphi(b_n,1^\infty)]}}.$$

Note that for $n > k$ we have that

$$e^{-[\varphi(a_1,a_2,...a_n,1^\infty)+\varphi(a_2,...a_n,1^\infty)+...+\varphi(a_n,1^\infty)]} =$$

$$e^{-[\varphi(a_1,a_2,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,1,..,1}_{k-1})+(n-k)\varphi(\underbrace{1,1,..,1}_{k-1})]}.$$

Therefore,

$$\sum_{b_1,b_2,...b_n} e^{-[\varphi(b_1,b_2,...b_n,1^\infty)+\varphi(b_2,...b_n,1^\infty)+...+\varphi(b_n,1^\infty)]} =$$

$$e^{-(n-k)\varphi(\underbrace{1,1,..,1}_{k-1})} \sum_{b_1,b_2,...b_n} e^{-[\varphi(b_1,b_2,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,1,..,1}_{k-1})]}.$$

Consider the probability μ_n, such that,

$$\mu_n = \sum_{a_1,a_2,...a_n} \delta_{(a_1,a_2,...a_n,1^\infty)}\, p_{a_1,a_2,...a_n} =$$

$$\sum_{a_1,...a_n} \delta_{(a_1,...a_n,1^\infty)} \frac{e^{-[\varphi(a_1,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n} e^{-[\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}}.$$

and μ such that $\mu = \lim_{n\to\infty} \mu_n$.
Note that

$$p_{a_1,a_2,..,a_n} = \frac{e^{-[\varphi(a_1,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n} e^{-[\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}}.$$

If φ is Hölder it is known that the above probability μ is the eigenprobability for the dual of the Ruelle operator $\mathcal{L}_{-\varphi}$ (a DLR probability). That is, there exists $\lambda > 0$ such that $\mathcal{L}^*_{-\varphi}(\mu) = \lambda\mu$. This follows from the Thermodynamic Limit with boundary condition property as presented in [17].

We claim that the above probability μ satisfies the KMS condition.

Indeed, note that

$$\sum_j \int f(\,(j,x_2,x_3,..,x_n,..)\,,\,(x_1,x_2,x_3,..,x_n,..))d\mu(x) =$$

$$\lim_{n\to\infty} \sum_j \sum_{a_1,a_2,...a_n} f(\,(j,a_2,a_3,..,a_n,\,1^\infty)\,,\,(a_1,a_2,a_3,..,a_n,\,1^\infty)\,)p_{a_1,a_2,...,a_n} =$$

$$\lim_{n\to\infty}\sum_j\sum_{a_1}\sum_{a_2,...a_n} f((j,a_2,a_3,..,a_n,\,1^\infty)(a_1,a_2,a_3,..,a_n,1^\infty)p_{a_1,a_2,...,a_n}. \tag{3.23}$$

On the other hand

$$\sum_j\int f(\,(x_1,x_2,x_3,..)\,,\,(j,x_2,x_3,..))e^{-\varphi(j,x_2,x_3,...)\,+\,\varphi(x_1,x_2,x_3,..)}d\mu(x) =$$

$$\lim_{n\to\infty}\sum_j\sum_{a_1,a_2,...,a_n} f(\,(a_1,..,a_n,\,1^\infty)\,,\,(j,a_2,..,a_n,\,1^\infty)\,)e^{-\varphi(j,a_2,...,a_k)\,+\,\varphi(a_1,a_2,...a_k)}p_{a_1,a_2,...,a_n} =$$

$$\lim_{n\to\infty}\sum_j\sum_{a_1}\sum_{a_2,...a_n} f(\,(a_1,..,a_n,\,1^\infty)\,,\,(j,a_2,..,a_n,\,1^\infty)\,)e^{-\varphi(j,a_2,...,a_k)\,+\,\varphi(a_1,a_2,...a_k)}$$

$$\frac{e^{-\,[\,\varphi(a_1,a_2,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n} e^{-[\,\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}} =$$

$$\lim_{n\to\infty}\sum_j\sum_{a_1}\sum_{a_2,...a_n} f(\,(a_1,..,a_n,\,1^\infty)\,,\,(j,a_2,..,a_n,\,1^\infty)\,)$$

$$\frac{e^{-\,[\,\varphi(j,a_2,,...a_k)+\varphi(a_2,...a_{k+1})+...+\varphi(a_n,\underbrace{1,..,1}_{k-1})]}}{\sum_{b_1,...b_n} e^{-[\,\varphi(b_1,...b_k)+\varphi(b_2,...b_{k+1})+...+\varphi(b_n,\underbrace{1,..,1}_{k-1})]}}. =$$

$$\lim_{n\to\infty}\sum_j\sum_{a_1}\sum_{a_2,...a_n} f(\,(a_1,..,a_n,\,1^\infty)\,,\,(j,a_2,..,a_n,\,1^\infty)\,)p_{j,a_2,...,a_n}.$$

On this last equation if we exchange coordinates j and a_1 we get expression (3.23).

Then, such μ satisfies the KMS condition.

□

The above theorem can be extended to the case the potential φ is Hölder. We refer the reader to [42] for more general results.

We will show now that under the above setting the KMS probability is not unique.

Proposition 3.4.25. *Suppose μ satisfies the KMS condition for the measured groupoid (G,ν) where $c(x,y)=\varphi(y)-\varphi(x)$. Suppose φ is normalized for the Ruelle operator, where $\varphi: G^0=\Omega\to\mathbb{R}$. Consider $v(x_1,x_2,x_3,..)$ which does not depend of the first coordinate. Then, $v(x)d\mu(x)$ also satisfies the KMS condition for the measured groupoid (G,ν).*

Proof: Suppose μ satisfies the (c,β)-KMS condition for the measured groupoid (G,ν). This means: for any $g\in I(G,\nu)$

$$\int \sum_{a\in\{1,2,..,d\}} g((\,a,y_2,y_3..)\,,\,(\,y_1,y_2,\dots\,))\,e^{\beta\varphi(\,a,y_2,y_3..)}\,d\mu(y)=$$

$$\int \sum_{a\in\{1,2,..,d\}} g((\,x_1,x_2,\dots\,),(\,a,x_2,x_3..))\,e^{\beta\varphi(\,a,x_2,x_3..)}\,d\mu(x). \tag{3.24}$$

Take

$$h(x_1,x_2,x_3,..),(y_1,y_2,y_3,..))=$$
$$k(\,(x_1,x_2,x_3,..),(y_1,y_2,y_3,..))\,v(x_1,x_2,x_3,..)\,)=$$
$$k(\,(x_1,x_2,x_3,..),(y_1,y_2,y_3,..))\,v(x_2,x_3,..)\,).$$

From the hypothesis about μ we get that

$$\int \sum_{a\in\{1,2,..,d\}} h((\,a,y_2,y_3..)\,,\,(\,y_1,y_2,\dots\,))\,e^{\beta\varphi(\,a,y_2,y_3..)}\,d\mu(y)=$$

$$\int \sum_{a\in\{1,2,..,d\}} h((\,x_1,x_2,x_3,\dots\,),(\,a,x_2,x_3..))\,e^{\beta\varphi(\,a,x_2,x_3..)}\,d\mu(x).$$

This means, for any continuous k the equality

$$\int \sum_{a\in\{1,2,..,d\}} k((\,a,y_2,y_3..)\,,\,(\,y_1,y_2,\dots\,))\,e^{\beta\varphi(\,a,y_2,y_3..)}\,v(y_2,y_3,\dots)d\mu(y)=$$

$$\int \sum_{a\in\{1,2,..,d\}} k((\,x_1,x_2,x_3,\dots\,),(\,a,x_2,x_3..))\,e^{\beta\varphi(\,a,x_2,x_3..)}\,v(x_2,x_3,..)d\mu(x).$$

Therefore, $v(x)d\mu(x)$ also satisfies the (c,β)-KMS condition for the measured groupoid (G,ν).

□

It follows from the above result that the probability that satisfies the KMS condition for c and the measured groupoid (G, ν) is not always unique.

A probability ρ satisfies the Bowen condition for the potential $-\varphi$ if there exists constants $c_1, c_2 > 0$, and P, such that, for every

$$x = (x_1, ..., x_m, ...) \in \Omega = \{1, 2, ...d\}^{\mathbb{N}},$$

and all $m \geq 0$,

$$c_1 \leq \frac{\rho\{y : y_i = x_i, \quad \forall i = 1, ..., m\}}{\exp\left(-Pm - \sum_{k=1}^{m} \varphi(\sigma^k(x)\right)} \leq c_2. \tag{3.25}$$

Suppose φ is Hölder, then, if ρ is the equilibrium probability (or, if ρ is the eigenprobability for the dual of Ruelle operator $\mathcal{L}_{-\varphi}$) one can show that it satisfies the Bowen condition for $-\varphi$.

In the case v is continuous and does not depend on the first coordinate then $v(x)d\mu(x)$ also satisfies the Bowen condition for φ. The same is true for the probability $\hat{\rho}$ of example 3.4.26 on the case $-\varphi = \log J$.

There is an analogous definition of the Bowen condition on the space $\{1, 2, ...d\}^{\mathbb{Z}}$ but it is a much more strong hypothesis on this case (see section 5 in [40]).

Example 3.4.26. *We will show an example where the probability μ of theorem 3.4.24 (the eigenprobability for the potential $-\varphi$) is such that if f is a function that depends just on the first coordinate, then, $f\,\mu$ does not necessarily satisfies the KMS condition.*

Suppose $\varphi = -\log J$, where $J(x_1, x_2, x_3, ..) = J(x_1, x_2) > 0$, and $\sum_i P_{i,j} = 1$, for all i. In other words the matrix P, with entries $P_{i,j}$, $i, j \in \{1, 2.., d\}$, is a column stochastic matrix. The Ruelle operator for $-\varphi$ is the Ruelle operator for $\log J$. *The potential* $\log J$ *is normalized for the Ruelle operator.*

We point out that in Stochastic Processes it is usual to consider line stochastic matrices which is different from our setting.

There exists a unique right eigenvalue probability vector π for P (acting on vectors on the right). The Markov chain determined by the matrix P and the initial vector of probability $\pi = (\pi_1, \pi_2, .., \pi_d)$ determines an stationary process, that is, a probability ρ on the Bernoulli space $\{1, 2, ..., d\}^{\mathbb{N}}$, which is invariant for the shift acting on $\{1, 2, ..., d\}^{\mathbb{N}}$.

For example, we have that $\rho(\overline{21}) = P_{21}\pi_1$.

We point out that such ρ is the eigenprobability for the $\mathcal{L}^*_{\log J}$ (associated to the eigenvalue 1). Therefore, ρ satisfies the KMS condition from the above results.

The Markov Process determined by the matrix P and the initial vector of probability $\pi = (1/d, 1/d, ..., 1/d)$ defines a probability $\hat{\rho}$ on the Bernoulli space $\{1, 2, ..., d\}^{\mathbb{N}}$, which is not invariant for the shift acting on $\{1, 2, ..., d\}^{\mathbb{N}}$.

In this case, for example, $\hat{\rho}(\overline{21}) = P_{21}\, 1/d$.

Note that the probability ρ satisfies $\rho = u\, \hat{\rho}$ where u depends just on the first coordinate.

Note that unless P is double stochastic is not true that for any j_0 we have that $\sum_k P_{j_0,k} = 1$.

Assume that there exists j_0 such that $\sum_k P_{j_0,k} \neq 1$.

We will check that, in this case $\hat{\rho}$ does not satisfies the KMS condition for the function $f(x, y) = I_{X_1=i_0}(x)\, I_{X_1=j_0}(y)$.

Indeed, equation (3.22) means

$$\sum_j \int f(\,(j, x_2, x_3, .., x_n, ..)\,,\,(x_1, x_2, x_3, .., x_n, ..))d\hat{\rho}(x) =$$

$$\sum_j \int I_{X_1=i_0}(j, x_2, x_3, ..)\, I_{X_1=j_0}(x_1, x_2, x_3...)d\hat{\rho}(x) =$$

$$\int I_{X_1=i_0}(i_0, x_2, x_3, ..)\, I_{X_1=j_0}(x_1, x_2, x_3...)d\hat{\rho}(x) =$$

$$\int I_{X_1=j_0}(x_1, x_2, x_3...)d\hat{\rho}(x) = \hat{\rho}(\overline{j_0}) = 1/d =$$

$$\sum_j \int f(\,(x_1, x_2, x_3, ..)\,,\,(j, x_2, x_3, ..))e^{-\varphi(j,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\sum_j \int I_{X_1=i_0}(x_1, x_2, x_3, ..)\, I_{X_1=j_0}(j, x_2, x_3...)e^{-\varphi(j,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\int I_{X_1=i_0}(x_1, x_2, x_3, ..)\, I_{X_1=j_0}(j_0, x_2, x_3...)e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\int I_{X_1=i_0}(x_1, x_2, x_3, ..)\, e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(x_1,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\int_{X_1=i_0} e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(i_0,x_2,x_3,..)}d\hat{\rho}(x) =$$

$$\sum_k \int_{X_1=i_0\,,\,X_2=k} e^{-\varphi(j_0,x_2,x_3,..)\,+\,\varphi(i_0,x_2,x_3,..)} d\hat{\rho}(x) =$$

$$\sum_k \int_{X_1=i_0\,,\,X_2=k} P_{j_0,k}\; P_{i_0,k}^{-1} d\hat{\rho}(x) =$$

$$\sum_k P_{j_0,k}\; P_{i_0,k}^{-1}\, P_{i_0,k} 1/d =$$

$$\sum_k P_{j_0,k}\; 1/d\; \neq 1/d = \hat{\rho}(\overline{j_0}).$$

Therefore, $\hat{\rho}$ does not satisfies the KMS condition.

Example 3.4.27. *Consider $\Omega = \{1,2\}^{\mathbb{N}}$, a Jacobian J and take ν^y the probability on each class y given by $\sum_a J(a, y_2, y_3, ..)\delta_{(a,y_2,y_3,..)}$.*

Note first that $\varphi = \log J$ is a normalized potential. Does the equilibrium probability for $\log J$ satisfies the KMS condition? We will show that this in not always true.

The question means: is it true that for any function k is valid

$$\int \sum_{a\in\{1,2\}} k(\,(a, y_2, ..),\,(y_1, y_2, ..)\,)\, e^{\varphi(a,y_2,...)}\, d\mu(y) =$$

$$\int \sum_{a\in\{1,2\}} k(\,(a, x_2, ..),\,(x_1, x_2, ..)\,)\, e^{\varphi(a,x_2,...)}\, d\mu(x) =$$

$$\int \sum_{a\in\{1,2\}} k(\,(x_1, x_2, ..),\,(a, x_2, ..)\,)\, e^{\varphi(a,x_2,...)}\, d\mu(x)? \tag{3.26}$$

Consider the example: take $c(x, y) = \varphi(x) - \varphi(y)$, for $\varphi : \{1,2\}^{\mathbb{N}} \to \mathbb{R}$, such that,

$$\varphi(a,\ .\,,\ .\,, ...) = \log p$$

where $p = p_a$, for $a \in \{1,2\}$, and $p_1 + p_2 = 1$, $p_1, p_2 > 0$.

The Gibbs probability μ for such φ is the independent probability associated to p_1, p_2.

Given such probability μ over Ω we can define a dynamical state φ_μ in the following way: given $f : G \to \mathbb{R}$ we get $\varphi_\mu(f) = \int f(x, x)\; d\mu(x)$.

Take $\beta = 1$. We will show that φ_μ is not KMS for c.

The equation (3.26) for such μ means for any $k(x,y)$

$$\int \sum_{a\in\{1,2\}} p_a \, k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\;d\mu(y) =$$

$$\int \sum_{a\in\{1,2\}} k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\,p(a,y_2,...)\,d\mu(y) =$$

$$\int \sum_{b\in\{1,2\}} k(\,(x_1,x_2,..),\,(b,x_2,..)\,)\,p(b,x_2,...)\,d\mu(x) =$$

$$\int \sum_{b\in\{1,2\}} p_b \, k(\,(x_1,x_2,..),\,(b,x_2,..)\,)\;d\mu(x).$$

It is not true that μ is Gibbs for the potential $\log p$.
Indeed, given k consider the function

$$g(y_1,y_2,y_3,y_4,...) = k(\,(y_1,y_3,...)\,,\,(y_2,y_3,...)\,).$$

Note that

$$\mathcal{L}_{\log p}(g)(y_1,y_2,y_3,..) = \sum_{a\in\{1,2\}} p(a,y_1,y_2,y_3,...)\,g(a,y_1,y_2,...)$$

$$\sum_{a\in\{1,2\}} p_a \, k(\,(a,y_2,y_3,..),\,(y_1,y_2,..)\,).$$

Then,

$$\int \sum_{a\in\{1,2\}} p_a \, k(\,(a,y_2,..),\,(y_1,y_2,..)\,)\;d\mu(y) =$$

$$\int \mathcal{L}_{\log p}(g)(y_1,y_2,y_3,..)\,d\mu(y) = \int k(\,(y_1,y_3,...)\,,\,(y_2,y_3,...)\,)d\mu(y).$$

Now, given k consider the function

$$h(x_1,x_2,x_3,x_4,...) = k(\,(x_2,x_3,...)\,,\,(x_1,x_3,x_4,...)\,).$$

Then,

$$\int \mathcal{L}_{\log p}(h)(x_1,x_2,x_3,..)\,d\mu(y) = \int k(\,(x_2,x_3,...)\,,\,(x_1,x_3,...)\,)\,d\mu(x).$$

For the Gibbs probability μ for $\log p$ is not true that for all k

$$\int k(\,(x_2,x_3,...)\,,\,(x_1,x_3,...)\,)\,d\mu(x) = \int k(\,(x_1,x_3,...)\,,\,(x_2,x_3,...)\,)d\mu(x).$$

3.5 Noncommutative integration and quasi-invariant probabilities

In non-commutative integration the transverse measures are designed to integrate transverse functions (see [18] or [34]).

In the same way we can say that a function can be integrated by a measure resulting in a real number we can say that the role of a transverse measure is to integrate transverse functions (producing a real number).

The main result here is Theorem 3.5.8 which describes a natural way to define a transverse measure from a modular function δ and a Haar system $(G\,,\hat{\nu})$.

As a motivation for the topic of this section consider a foliation of the two dimensional torus where we denote each leaf by l. This partition defines a groupoid with a quite complex structure. Each leave is a class on the associated equivalence relation. This motivation is explained with much more details in [19]

We consider in each leave l the intrinsic Lebesgue measure on the leave which will be denoted by ρ_l.

A random operator q is the association of a bounded operator $q(l)$ on $\mathcal{L}^2(\rho_l)$ for each leave l. We will avoid to describe several technical assumptions which are necessary on the theory (see page 51 in [19]).

The set of all random operators defines a von Neumann algebra under some natural definitions of the product, etc... (see Proposition 2 in page 52 in [19]) (*).

This setting is the formalism which is natural on **noncommutative geometry** (see [19]).

Important results on the topic are for instance the characterization of when such von Neumann algebra is of type I, etc... (see page 53 in [19]). There is a natural trace defined on this von Neumann algebra.

A more abstract formalism is the following: consider a fixed groupoid G. Given a transverse function λ one can consider a natural operator $L_\lambda : \mathcal{F}^+(G) \to \mathcal{F}^+(G)$, which satisfies

$$f \to \lambda * f = L_\lambda(f)\,.$$

L_λ acts on $\mathcal{F}^+(G)$ and can be extended to a linear action on the von Neumann algebra $\mathcal{F}(G)$. This defines a Hilbert module structure (see section 3.2 in [38] or [34]).

Given λ we can also define the operator $R_\lambda : \mathcal{F}^+(G) \to \mathcal{F}^+(G)$ by

$$h(x,y) = R_\lambda(f)(\gamma) \; = \; \int f(s,y) d\lambda^x(d\,s),$$

for any (x,y).

Definition 3.5.1. *Given two G-kernels λ_1 and λ_2 we get a new G-kernel $\lambda_1 * \lambda_2$, called the convolution of λ_1 and λ_2, where given the function $f(x,y)$, we get the rule*

$$(\lambda_1 * \lambda_2)\,(f) \; = g \in \mathcal{F}(G^0),$$

given by

$$g(y) = \int (\; \int f(s,y) \; \lambda_2^x(d\,s)\,) \quad \lambda_1^y(dx).$$

In the above $x \sim y \sim s$.

In other words $(\lambda_1 * \lambda_2)$ is such that for any y we have

$$(\lambda_1 * \lambda_2)^y(dx) = \int \lambda_2^x(d\,s\,) \;\; \lambda_1^y(dx). \tag{3.27}$$

Note that

$$R_{\lambda_1 * \lambda_2} = R_{\lambda_1} \circ R_{\lambda_2}.$$

For a given fixed transverse function λ, for each class $[y]$ on the groupoid G, we get that R_λ defines an operator acting on functions $f(r,s)$, where $f : [y] \times [y] \to \mathbb{C}$, and where $R_\lambda(f) = h$.

In this way, each transverse function λ defines a random operator q, where $q([y])$ acts on $\mathcal{L}^2(\lambda^y)$ via R_λ.

A transverse measure can be seen as an integrator of transverse functions or as an integrator of random operators (which are elements on the von Neumann algebra (*) we mention before).

First we will present the basic definitions and results that we will need later on this section.

Remember that $\mathcal{E}^+$ is the set of transverse functions for the groupoid $G \subset X \times X$ associated to a certain equivalence relation $\sim$.

$\mathcal{F}^+(G)$ denotes the space of Borel measurable functions $f : G \to [0,\infty)$ (a real function of two variables (a,b)).

Remember that given a G kernel ν and an integrable function $f \in \mathcal{F}_\nu(G)$ we can define two functions on G:

$$(x,y) \to (\nu * f)(x,y) = \int f\,(x,s)\,\nu^y(ds)$$

and

$$(x,y) \to (f * \nu)(x,y) = \int f\,(s,y)\,\nu^x(ds).$$

Note that $\nu * 1 = 1$ if ν^y is a probability for all y. Also note that $(f * \nu)(y,y) = \nu(f)\ (y)$ (see definition 3.2.6).

About (3.27) we observe that

$$(\lambda_1 * \lambda_2)\,(f) = \lambda_1(f * \lambda_2).$$

A kind of analogy of the above concept of convolution (of kernels) with integral kernels is the following: given the kernels $K_1(s,x)$ and $K_2(x,y)$ we define the kernel

$$\hat{K}(s,y) = \int K_2(s,x)\,K_1(x,y)\,dx.$$

This is a kind of convolution of integral kernels.

This defines the operator

$$f(x,y) \to g(y) = \int f(s,y)\hat{K}(s,y)ds = \int (\int f(s,y)\,K_2(s,x)\,d\,s\,)\ K_1(x,y)\,dx.$$

Example 3.5.2. *Given any kernel ν we have that $\mathfrak{d} * \nu = \nu$, where $\mathfrak{d}$ is the delta kernel of Example 3.2.5.*

Indeed, for any $f \in \mathcal{F}(G)$

$$\int f\,(\,\mathfrak{d} * \nu\,)^y = \int \int f(s,y)\,\nu^x(d\,s)\,\mathfrak{d}^{\,y}(d\,x) = \int f(s,y)\,\nu^y(d\,s)\ = \int f\,\nu^y$$

In the same way for any ν we have that $\nu * \mathfrak{d} = \nu$.

Example 3.5.3. *Given a fixed positive function $h(x,y)$ and a fixed kernel ν, we get that the kernel $\nu * (\,h\,\mathfrak{d}\,)$, where $\mathfrak{d}$ is the Dirac kernel, is such that given any $f(x,y)$,*

$$(\nu\,*(\,h\,\mathfrak{d}\,))(f)(y) = \int (\ \int f(s,y)\,h(s,x)\,\mathfrak{d}^x(d\,s)\,)\quad \nu^y(dx) =$$

$$\int f(x,y)\, h(x,x) \quad \nu^y(dx).$$

Particularly, taking $h=1$, we get $\nu * \mathfrak{d} = \nu$.

Example 3.5.4. *For the bigger than two equivalence relation of example 3.1.9 on $(S^1)^{\mathbb{N}}$, where S^1 is the unitary circle, the equivalence classes are of the form $\{\, (a, x_2, x_3, ...),\, a \in S^1 \,\}$, where $x_j \in S^1$, $j \geq 2$, is fixed.*

Given $x = (x_1, x_2, x_3, ...)$ we define $\nu^x(da)$ the Lebesgue probability on S^1, which can be identified with $S^1 \times (x_2, x_3, .., x_n, ...)$. This defines a transverse function where $G^0 = (S^1)^{\mathbb{N}}$. We call it the **standard XY Haar system**.

In this case given a function $f(x,y) = f(\,(x_1, x_2, x_3, ..), (y_1, y_2, y_3, ..)\,)$

$$(\nu * f)(x,y) = \int f(x,s)\, \nu^x(d\,s) = \int f\,(\,(x_1, x_2, x_3, ..), (s, x_2, x_3, ..)\,)\, ds,$$

where $s \in S^1$. Note that in the present example the information on y was lost after convolution.

Such ν is called in [6] the a priori probability for the Ruelle operator. Results about Ruelle operators and Gibbs probabilities for such kind of XY models appear in [6] and [41].

After Proposition 3.5.14 we will present several properties of convolution of transverse function (we will need soon some of them).

Note that if ν is transverse and λ is a kernel, then $\nu * \lambda$ is transverse.

Remember that given a kernel λ and a fixed y the property $\lambda^y(1) = 1$ means $\int \lambda^y(dx) = 1$.

Definition 3.5.5. *A transverse measure Λ over the modular function $\delta(x,y)$, $\delta : G \to \mathbb{R}$, is a linear function $\Lambda : \mathcal{E}^+ \to \mathbb{R}^+$, such that, for each kernel λ which satisfies the property $\lambda^y(1) = 1$, for any y, if ν_1 and ν_2 are transverse functions such that $\nu_1 * (\delta\lambda) = \nu_2$, then,*

$$\Lambda(\nu_1) = \Lambda(\nu_2). \tag{3.28}$$

A measure produces a real number from the integration of a classical function (which takes values on the real numbers), and, on the other hand, the transverse measure produces a real number from a transverse function ν (which takes values on measures).

The assumptions on the above definition are necessary (for technical reasons) when considering the abstract concept of integral of a transverse function by Λ (as is developed in [18]). We will show later that there is a more simple expression providing the real values of such process of integration by Λ which is related to quasi-invariant probabilities.

If one consider the equivalence relation such that each point is related just to itself, any cocycle is constant equal 1 and the only kernel satisfying $\lambda^y(1) = 1$, for any y, is the delta Dirac kernel $\mathfrak{d}$. In this case if ν_1 and ν_2 are such that $\nu_1 * (\delta\lambda) = \nu_2$, then, $\nu_1 = \nu_2$ (see Example 3.5.2). Moreover, $\mathcal{E}^+$ is just the set of positive functions on X. Finally, we get that the associated transverse measure Λ is just a linear function $\Lambda : \mathcal{E}^+ \to \mathbb{R}^+$

Example 3.5.6. *Given a probability μ over G^0 we can define*

$$\Lambda(\nu) = \int\int \nu^y(dz)d\mu(y).$$

Suppose that λ satisfies $\lambda^x(1) = 1$, for any x, and

$$\nu_1 * \lambda = \nu_2.$$

Then, $\Lambda(\nu_1) = \Lambda(\nu_2)$. This means that Λ is invariant by translation on the right side.

Indeed, note that,

$$\Lambda(\nu_1) = \int\int \nu_1^y(dz)d\mu(y),$$

and, moreover

$$\Lambda(\nu_2) = \int\int \nu_2^y(dz)d\mu(y) =$$

$$\int [\int (\int \lambda^x(d\,s)\,)\,)\ \nu_1^y(dx))\,]\,d\mu(y) =$$

$$\int (\int \nu_1^y(dx))\ d\mu(y).$$

Therefore, Λ is a transverse measure of modulus $\delta = 1$.

In this way for each measure μ on G^0 we can associate a transverse measure of modulus 1 by the rule $\nu \to \Lambda(\nu) = \int\int \nu^y(dz)d\mu(y) \in \mathbb{R}$.

The condition

$$\nu_1 * (\delta\lambda) = \nu_2$$

means for any f we get

$$\int f(x,y)\,(\,\nu_1 * (\delta\lambda\,))^y\,(d\,x) \;=\int (\;\int f(s,y)\,[\,\delta(s,x)\lambda^x(ds)\,]\,)\quad \nu_1^y(dx) =$$

$$\int f(x,y)\;\nu_2^y(dx). \tag{3.29}$$

We define before (see (3.5)) the concept of quasi-invariant probability for a given modular function δ, a groupoid G and a fixed transverse function ν.

For reasons of notation we use a slight variation of that definition. **In this section we say that M is a quasi invariant probability for δ and** ν if for any $f(x,y)$

$$\int\int f(y,x)\,\nu^y(x)dM(y) = \int\int f(x,y)\delta(x,y)^{-1}\nu^y(x)dM(y). \tag{3.30}$$

Proposition 3.5.7. *Given a modular function δ, a groupoid G and a fixed transverse function $\hat{\nu}$ denote by M the quasi invariant probability for δ.*

Assume that $\int \hat{\nu}^y(dr) \neq 0$ for all y.

If $\hat{\nu} * \lambda_1 = \hat{\nu} * \lambda_2$, where λ_1, λ_2 are kernels, then,

$$\int \delta^{-1}\lambda_1(1)\;d\,M = \int \delta^{-1}\lambda_2(1)\;d\,M.$$

This is equivalent to say that

$$\int\int \delta^{-1}(s,y)\lambda_1^y(ds)dM(y) = \int\int \delta^{-1}(s,y)\lambda_2^y(ds)dM(y).$$

Proof: By hypothesis $g(y) = (\hat{\nu} * \lambda_1)(\delta^{-1})\,(y) = (\hat{\nu} * \lambda_2)(\delta^{-1})(y)$.

Then, we assume that (see (3.27))

$$\int g(y)\frac{1}{\int \hat{\nu}^y(dr)}dM(y) = \int\int\int \frac{1}{\int \hat{\nu}^y(dr)}\delta^{-1}(s,y)\lambda_1^x(ds)\hat{\nu}^y(dx)\,dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}\delta^{-1}(s,y)\lambda_2^x(ds)\hat{\nu}^y(dx)\,dM(y). \tag{3.31}$$

Therefore,

$$\int\int \delta^{-1}(s,y)\lambda_1^y(ds)dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^x(dr)}\delta(y,s)\lambda_1^y(ds)\hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}[\,\delta(y,x)\,\delta(x,s)\,\lambda_1^y(ds)\,]\hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}[\,\delta(x,y)^{-1}\,\delta(x,s)\,\lambda_1^y(ds)\,]\hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^x(dr)}[\,\delta(y,x)^{-1}\,\delta(y,s)\,\lambda_1^x(ds)\,]\,\delta^{-1}(x,y)\hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^x(dr)}\delta(y,s)\lambda_1^x(ds)\hat{\nu}^y(dx)\, dM(y) \tag{3.32}$$

On the above from the fourth to the fifth line we use the quasi-invariant expression (3.30) for M taking

$$f(y,x) = \int \frac{1}{\int \hat{\nu}^y(dr)}\,\delta(x,y)^{-1}\,\delta(x,s)\,\lambda_1^y(ds).$$

Note that if $\hat{\nu}$ is transverse $\int \hat{\nu}^x(dr)$ does not depend on x on the class $[y]$.

Finally, from the above equality (3.32) (and replacing λ_1^x by λ_2^x) it follows that

$$\int\int \delta^{-1}(s,y)\lambda_1^y(ds)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}\delta(y,s)\lambda_1^y(ds)\hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int\int \frac{1}{\int \hat{\nu}^y(dr)}\delta(y,s)\lambda_2^y(ds)\hat{\nu}^y(dx)\, dM(y) =$$

$$\int\int \delta^{-1}(s,y)\lambda_2^y(ds)\, dM(y).$$

□

From now on we assume that $\int \hat{\nu}^y(dr) \neq 0$ for all y.

Theorem 3.5.8. *Given a modular function δ and a Haar system $(G\,,\hat{\nu})$, suppose M is quasi invariant for δ.*

We shall see in Proposition 3.5.16 that given a transverse function $\hat{\nu}$ there exists a kernel ρ such that $\nu = \hat{\nu} * \rho$. With this result in mind we define Λ on the following way: Set

$$\Lambda(\nu) = \int\int \delta(x,y)^{-1}\rho^y(dx)dM(y). \tag{3.33}$$

Then, Λ is well defined and it is a transverse measure.

Proof: Λ is well defined by proposition 3.5.7.

We have to show that if $\lambda^x(1) = 1$, for any x, and ν_1 and ν_2 are such that $\nu_1 * (\delta\lambda) = \nu_2$, then, $\Lambda(\nu_1) = \Lambda(\nu_2)$.

Suppose $\nu_1 = \hat{\nu} * \lambda_1$, then, $\nu_2 = \hat{\nu} * (\,\lambda_1 * (\delta\,\lambda)\,)$.

Note that

$$\Lambda(\nu_1) = \int\int \delta(x,y)^{-1}\,\lambda_1^y(dx)dM(y).$$

On the other hand from (3.27)

$$\Lambda(\nu_2) = \int\int \delta(s,y)^{-1}\,(\,\lambda_1 * (\delta\,\lambda)\,)^y(ds)dM(y) =$$

$$\int\int\int \delta(s,y)^{-1}\,\delta(s,x)\,\lambda^x(ds)\,\lambda_1^y(dx)\,dM(y) =$$

$$\int\int\int \delta(x,y)^{-1}\,\lambda^x(ds)\,\lambda_1^y(dx)\,dM(y) =$$

$$\int\int \delta(x,y)^{-1}\;(\int \lambda^x(ds)\,)\;\lambda_1^y(dx)\,dM(y) =$$

$$\int\int \delta(x,y)^{-1}\;\lambda_1^y(dx)\,dM(y) = \Lambda(\nu_1).$$

$\square$

Remark: The last proposition shows that given a quasi invariant probability M - for a transverse function $\hat{\nu}$ and a cocycle δ - there is a natural way to define a transverse measure Λ (associated to a groupoid G and a modular function δ).

One can ask the question: given transverse measure Λ (associated to a groupoid G and a modular function δ) is it possible to associate a probability on G_0? In the affirmative case, is this probability quasi invariant? We will elaborate on that.

Definition 3.5.9. *Given a transverse measure Λ for δ we can associate by Riesz Theorem to a transverse function $\hat{\nu}$ a measure M on G^0 by the rule: given a non-negative continuous function $h : G^0 \to \mathbb{R}$ we will consider the transverse function $h(x)\,\hat{\nu}^y(dx)$ and set*

$$h \to \Lambda(h\,\hat{\nu}) = \int h(x)dM(x).$$

Such M is a well defined measure (a bounded linear functional acting on continuous functions) and we denote such M by $\Lambda_{\hat{\nu}}$.

$\Lambda_{\hat{\nu}}$ means the rule $h \to \Lambda(h\,\hat{\nu}) = \Lambda_{\hat{\nu}}(h)$.

Proposition 3.5.10. *Given any transverse measure Λ associated to the modular function δ and any transverse functions ν and ν' we have for any continuous f that*

$$\Lambda_{\nu'}(\nu(\tilde{\delta f})) = \Lambda(\nu(\tilde{\delta f})\nu') = \Lambda(\nu'(\tilde{f})\nu) = \Lambda_{\nu}(\nu'(\tilde{f})).$$

Proof: If $\lambda^y(1) = 1\ \forall y$, that is, $\int 1\lambda^y(ds) = 1\ \ \forall y$, then $\Lambda(\nu * \delta\lambda) = \Lambda(\nu)$. If $g(x) = \lambda^x(1) = \int 1\lambda^x(ds) \neq 1$, then we can write $\lambda'^x(ds) = \frac{1}{g(x)}\lambda^x(ds)$, where λ and λ' are just kernels. In this way $(\nu * \delta\lambda) = (g\nu) * \delta\lambda'$. Indeed, for $h(x,y)$,

$$\int h(x,y)\,(\nu * \delta\lambda)^y(dx) = \int h(s,y)\delta(s,x)\lambda^x(ds)\nu^y(dx)$$

$$= \int h(s,y)\delta(s,x)\lambda'^x(ds)g(x)\nu^y(dx) = \int h(x,y)((g\nu) * \delta\lambda')^y(dx).$$

Denoting $\lambda(1)(x) = g(x) = \lambda^x(1) = \int 1\lambda^x(ds)$, it follows that

$$\Lambda(\nu * \delta\lambda) = \Lambda(g\nu * \delta\lambda') = \Lambda(g\nu) = \Lambda_\nu(g) = \Lambda_\nu(\lambda(1)) = \Lambda_\nu(\int 1\lambda^x(ds)). \tag{3.34}$$

From, (3.2) if ν is a kernel and $f = f(x,y)$

$$(\nu * f)(x,y) = \nu(\tilde{f})(x),$$

and, from the future Lemma 3.5.17, if λ is a kernel and ν is a transverse function, then, for any $f = f(x,y)$,

$$\lambda * (f\nu) = (\lambda * f)\nu.$$

It follows that, for transverse functions ν and ν', we get

$$\nu * [(\delta \tilde{f})\nu'] = [\nu * (\delta \tilde{f})]\nu' = [\nu(\tilde{\delta} f)]\nu'.$$

As a consequence

$$\Lambda_{\nu'}(\nu(\tilde{\delta} f)) = \Lambda([\nu(\tilde{\delta} f)]\nu') = \Lambda(\nu * [(\delta \tilde{f})\nu']) = \Lambda(\nu * \delta(\tilde{f}\nu')) =$$

$$\Lambda_{\nu}((\tilde{f}\nu')(1)) = \Lambda_{\nu}(\int 1 \cdot \tilde{f}(s,y)\nu'^{y}(ds)) = \Lambda_{\nu}(\nu'(\tilde{f})).$$

Above we use equation (3.34) with $\lambda = \tilde{f}\nu'$. □

Corolary 3.5.11. *If $\nu \in \mathcal{E}^{+}$, then for any f*

$$\Lambda(\nu(\tilde{f})\,\nu\,) \;=\; \Lambda(\nu(\tilde{\delta}\, f)\;\nu) \tag{3.35}$$

Proof: Just take $\nu = \nu'$ on last Proposition.

□

Among other things we are interested on a modular function δ, a transverse function $\hat{\nu}$ and a transverse measure Λ (of modulo δ) such that $M_{\Lambda,\hat{\nu}} = M$ is Gibbs for a Jacobian J. What conditions are required from M?

The main condition of the next theorem is related to the KMS condition of definition 3.4.20

Proposition 3.5.12. *Given a transverse measure Λ associated to the modular function δ, and a* **transverse function** *$\hat{\nu}$, consider the associated $M = \Lambda_{\hat{\nu}}$. Then, M is quasi invariant for δ. That is, M satisfies for all g*

$$\int\int g(s,x)\hat{\nu}^{x}(ds)dM(x) = \int\int g(x,s)\delta(x,s)\,\hat{\nu}^{x}(ds)dM(x). \tag{3.36}$$

Proof: First we point out that (3.36) is consistent with (3.30) (we are just using different variables).

A transverse function $\hat{\nu}$ defines a function of $f \in \mathcal{F}(G) \to \mathcal{F}(G^0)$.

The probability M associated to $\hat{\nu}$ satisfies for any continuous function $h(x)$, where $h : G^0 \to \mathbb{R}$ the rule

$$h \to \Lambda(h\,\hat{\nu}) = \int h(x)dM(x),$$

where $h(x)\,\hat{\nu}^y(dx) \in \mathcal{E}^+$.

From proposition 3.5.11 we have that for the continuous function $f(s,x) = \tilde{g}(s,x)$, where $f : G \to \mathbb{R}$, the expression

$$\Lambda(\hat{\nu}\ (g)\,\hat{\nu}\,) \ = \Lambda(\hat{\nu}\ (\tilde{f})\,\hat{\nu}\,) \ = \ \Lambda(\hat{\nu}(\delta^{-1}\,f)\ \hat{\nu}\ ) = \Lambda(\hat{\nu}(\delta^{-1}\,\tilde{g})\ \hat{\nu}\ )$$

For a given function $g(s,x)$ it follows from the above that

$$\Lambda(\hat{\nu}\ (g)\,\hat{\nu}\,) \ = \int \hat{\nu}\ (g)(x)\,dM(x) = \int\int g(s,x)\hat{\nu}^x(ds)\,dM(x).$$

On the other hand

$$\Lambda(\hat{\nu}(\delta^{-1}\,\tilde{g})\ \hat{\nu}\ ) = \int \hat{\nu}(\delta^{-1}\,\tilde{g})(x)d\,M(x) = \int g(x,s)\delta^{-1}(s,x)\,\hat{\nu}^x(ds)\,dM(x).$$

□

Proposition 3.5.13. *Given a modular function δ, a grupoid G, a transverse measure Λ and a transverse function $\hat{\nu}$, suppose for any ν, such that $\nu = \hat{\nu} * \rho$, we have that*

$$\Lambda(\nu) = \Lambda(\hat{\nu} * \rho) \ = \int\int \delta(s,x)^{-1}\rho^x(ds)d\mu(x) = \int \delta^{-1}\rho(1)\,d\,\mu.$$

Then, $\mu = \Lambda_{\hat{\nu}}$.

Proof: Given $f \in \mathcal{F}(G_0)$ consider λ the kernel such that $\lambda^x(ds) = f(x)\delta_x(ds)$, where δ_x is the Delta Dirac on x.

Then, using the fact that $\delta(x,x) = 0$ we get that the kernel $f(x)\hat{\nu}^y(dx)$ is equal to $\hat{\nu} * \delta\,\lambda$.

Then, taking $\rho = \delta\,\lambda$ on the above expression we get

$$\Lambda(\,f\hat{\nu}) = \Lambda(\hat{\nu} *\ (\delta\,\lambda)\,) =$$

$$\int \delta^{-1}\rho(1)d\mu = \int \delta^{-1}\,(\delta\lambda)(1)d\mu = \int \lambda(1)d\mu = \int f(x)d\mu(x).$$

Therefore, $\Lambda_{\hat{\nu}} = \mu$.

□

Now we present a general procedure to get transverse measures.

Proposition 3.5.14. *For a fixed modular function δ we can associate to any given probability μ over G^0 a transverse measure Λ by the rule*

$$\nu \to \Lambda(\nu) = \int\int \delta(s,x)^{-1}\, \nu^x(ds)\ \ d\mu(x). \tag{3.37}$$

Proof:

Consider $\nu' \in \mathcal{E}^+$ and λ, such that, $\int \lambda^r(ds) = 1$, for all r, and moreover that $\nu' = \nu * (\delta\lambda)$.

We will write

$$(\nu *\, \delta\lambda)\,(\delta^{-1}) = \int\int \delta^{-1}(s,x)\,\delta(s,r)\,\lambda^r(ds)\nu^x(dr)$$

which is a function of x

Then

$$\Lambda(\nu') = \int\int \delta(s,x)^{-1}\, \nu'^x(ds)\ \ d\mu(x) = \int \nu'(\delta^{-1})(x)d\mu(x) =$$

$$= \int \Big(\nu * (\delta\lambda)\Big)(\delta^{-1})(x)d\mu(x) = \int\int\int \delta(s,x)^{-1}\delta(s,r)\lambda^r(ds)\nu^x(dr)d\mu(x) =$$

$$= \int\int\int \delta(r,x)^{-1}\lambda^r(ds)\nu^x(dr)d\mu(x) = \int\int \delta(r,x)^{-1}\nu^x(dr)d\mu(x) = \Lambda(\nu).$$

□

This last transverse measure is defined in a quite different way that the one described on Theorem 3.5.8.

Now we will present some general properties of convolution of transverse functions.

Lemma 3.5.15. *Suppose $\nu \in \mathcal{E}^+$ is a transverse function, ν_0 a kernel, and $g \in \mathcal{F}^+(G)$ is such that $\int g(s,x)\nu_0^y(dx) = 1$, for all s, y. Then, $\nu_0 * (g\,\nu)\ = \nu$, where $g\,\nu$ is a kernel.*

Remark The condition $\int g(s,x)\nu_0^y(dx) = 1$, for all s, y means $(\nu_0 * g)(s,y) = 1$ for all s, y, that is $\nu_0 * g \equiv 1$ (See lemma 3 below).

Proof:

$$z(y) = \int\ f(s,y)\ \nu^y(ds) =$$

$$\int f(s,y)\,[\int g(s,x)\nu_0^y(dx)]\,\nu^x(ds) \quad =$$

$$\int \int f(s,y)\,[g(s,x)\nu^x(ds)] \quad \nu_0^y(dx) =$$

$$\int f(s,y)\,(g\,\nu)^x(ds)\nu_0^y(dx) = \int f(s,y)\,(\,\nu_0 * (g\,\nu))^y(ds).$$

□

We say that the kernel ν is fidel if $\int \nu_0^y(ds) \neq 0$ for all y.

Proposition 3.5.16. *For a fixed transverse function ν_0 we have that for each given transverse function ν there exists a kernel λ, such that, $\nu_0 * \lambda = \nu$.*

Proof: Given the kernel ν_0 take $g_0(s) = \frac{1}{\int 1 \nu_0^s(dr)} \geq 0$. Note that $g_0(v)$ is constant for $v \in [s]$. Then $\nu_0(g) = 1$, that is, for each s we get that $\int g_0(s)\nu_0^s(dx) = 1$.

We can take $\lambda = g_0\,\nu$ as a solution. Indeed, in a similar way as last lemma we get

$$z(y) = \int f(s,y)\ \nu^y(ds) =$$

$$\int f(s,y)\,[\int g_0(s)\nu_0^s(dx)]\,\nu^x(ds) \quad =$$

$$\int \int f(s,y)\,[g_0(s)\nu^x(ds)] \quad \nu_0^y(dx) =$$

$$\int f(s,y)\,(g_0\,\nu)^x(ds)\nu_0^y(dx) = \int f(s,y)\,(\,\nu_0 * (g_0\,\nu))^y(ds) =$$

$$\int f(s,y)\,(\,\nu_0 * \lambda)^y(ds).$$

□

The next Lemma is just a more general form of Lemma 3.5.15.

Lemma 3.5.17. *Suppose $\nu \in \mathcal{E}^+$, $g \in \mathcal{F}^+(G)$ and λ a kernel, then $\lambda * (g\,\nu) = (\lambda * g)\ \nu$, where $g\,\nu$ is a kernel and $\lambda * g$ is a function.*

Proof:
Given $f \in \mathcal{F}(G)$ we get

$$(\lambda * (g\,\nu))(f)(y) = \int f(x,y)\,(\lambda * (g\,\nu))^y(dx)$$

$$= \int\int f(s,y)[(g\nu)^x(ds)]\,\lambda^y(dx) = \int\int f(s,y)[g(s,x)\nu^x(ds)]\,\lambda^y(dx).$$

On the other hand

$$[(\lambda * g)\ \nu](f)(y) = \int f(s,y)[(\lambda * g)\,\nu]^y(ds) = \int f(s,y)[(\lambda * g)(s,y)]\nu^y(ds)$$

$$= \int f(s,y)[\int g(s,x)\lambda^y(dx)]\nu^y(ds) = \int\int f(s,y)g(s,x)\nu^x(ds)\lambda^y(dx).$$

□

Proposition 3.5.18. *Suppose ν and λ are transverse. Given $f \in \mathcal{F}^+(G)$, we have that*

$$\lambda\,(\nu * f) \,=\, \nu\,(\lambda * \tilde{f}).$$

Proof:
Indeed, by definition $(\nu * f)(x,y) = g(x,y) = \int f(x,s)\nu^y(ds)$, and by definition 3.2.6

$$\lambda\,(\nu * f)(y) = \lambda(g)(y) = \int g(x,y)\lambda^y(dx) = \int\int f(x,s)\nu^y(ds)\lambda^y(dx).$$

By the same arguments $(\lambda * \tilde{f})(x,y) = h(x,y) = \int \tilde{f}(x,s)\lambda^y(ds)$, and

$$\nu\,(\lambda * \tilde{f})(y) = \nu(h)(y) = \int h(x,y)\nu^y(dx) = \int\int \tilde{f}(x,s)\lambda^y(ds)\nu^y(dx) =$$

$$= \int\int f(s,x)\lambda^y(ds)\nu^y(dx) = \lambda\,(\nu * f)(y),$$

if we exchange the coordinates x and s.

Note that in the case $f \in \mathcal{F}(G^0)$ we denote $f(x,s) = f(x)$. In the same way

$$\lambda\,(\nu * f) \,=\, \nu\,(\lambda * \tilde{f})$$

in the following sense:

$$\int\int f(x)\nu^y(ds)\lambda^y(dx) = \int\int f(s)\lambda^y(ds)\nu^y(dx).$$

$\square$

3.6 C^*-Algebras derived from Haar Systems

In this section the functions $f : G \to \mathbb{R}$ will be required to be continuous (not just measurable).

An important issue here is to have suitable hypotheses in such way that the indicator of the diagonal $\imath$ belongs to the underlying space we consider. On von Neumann algebras the unit is just measurable and not continuous (this is good enough). We want to consider another setting (certain C^*-algebras associated to Haar Systems) where the unit will be required to be a continuous function. In general terms, given a groupoid $G \subset \Omega \times \Omega$, as we will see, we will need another topology on the set G for the C^*-Algebra formalism and for defining KMS states.

We will begin with some more examples. The issue here is to set a certain appropriate topology.

Example 3.6.1. *For $n \in \mathbb{N}$ we define the partition η_n over $\overrightarrow{\Omega} = \{1, 2, ..., d\}^{\mathbb{N}}$, $d \geq 2$, such that two elements $x \in \overrightarrow{\Omega}$ and $y \in \overrightarrow{\Omega}$ are on the same element of the partition, if and only if, $x_j = y_j$, for all $j > n$. This defines an equivalence relation denoted by R_n.*

Example 3.6.2. *We define a partition η over $\overrightarrow{\Omega}$, such that two elements $x \in \overrightarrow{\Omega}$ and $y \in \overrightarrow{\Omega}$ are on the same element of the partition, if and only if, there exists an n such that $x_j = y_j$, for all $j > n$. This defines an equivalence relation denoted by R_∞.*

Example 3.6.3. *For each fixed $n \in \mathbb{Z}$ consider the equivalence relation on $\hat{\Omega}$: $x \sim y$ if*

$$y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...)$$

is such that $x_j = y_j$ for all $j \leq n$, where $\hat{\Omega} = \overleftarrow{\Omega} \times \overrightarrow{\Omega}$.

This defines a groupoid.

Example 3.6.4. *Recall that by definition the unstable set of the point $x \in \hat{\Omega}$ is the set*

$$W^u(x) = \{y \in \hat{\Omega}\,,\ \textit{such that}\ \lim_{n\to\infty} d(\hat{\sigma}^{-n}(x), d(\hat{\sigma}^{-n}(y)\,)\ = 0\,\}$$

One can show that the unstable manifold of $x \in \hat{\Omega}$ is the set

$$W^u(x) = \{y = (..., y_{-n}, ..., y_{-2}, y_{-1} \,|\, y_0, y_1, ..., y_n, ...) \,|\, \textit{there exists}$$

$$k \in \mathbb{Z},\ \textit{such that}\ x_j = y_j, \textit{for all}\ j \geq k\}.$$

If we denote by G_u the groupoid defined by the above relation, then, $x \sim y$, if and only if $y \in W^u(x)$.

Definition 3.6.5. *Given the equivalence relation R, when the quotient $\hat{\Omega}/R$ (or, $\overrightarrow{\Omega}/R$) is Hausdorff and locally compact we say that R is a proper equivalence.*

For more details about proper equivalence see section 2.6 in [57].

On the set $X = \overrightarrow{\Omega}$, if we denote $x = (x_1, x_2, ..x_n, ..)$, the family $U_x(m) = \{y \in \overrightarrow{\Omega}$, such that, $y_1 = x_1, y_2 = x_2, ..., y_m = x_m\}$, $m = 1, 2, ...$, is a fundamental set of open neighbourhoods on Ω.

Considering the relations R_m and R_∞ we get the corresponding groupoids

$$G_1 \subset G_2 \subset ... \subset G_m \subset ... \subset G_\infty \subset \overrightarrow{\Omega} \times \overrightarrow{\Omega} = X \times X.$$

The equivalence relation described in example 3.6.2 (and also 3.1.4) is not proper if we consider the product topology on $\overrightarrow{\Omega}$ (respectively on $\hat{\Omega}$). The equivalence relation described in example 3.6.1 (and also 3.6.3) is proper if we consider the product topology on $\overrightarrow{\Omega}$ (respectively on $\hat{\Omega}$) (see [29]).

We consider over G_n the quotient topology.

Lemma 3.6.6. *Given $X = \overrightarrow{\Omega}$, for each n the map defined by the canonical projection $X \to G_n$ is open.*

Proof: Given an open set $U \subset X$ take $V = \{y \in X \,|$ there exists $x \in X$, satisfying $y \sim x$ for the relation $R_n\}$. We will show that V is open.

Consider $y \in V$, $y \in U$, such that, $y \sim x$ for the relation R_n. There exists $m > n$, such that, $U_x(m) \subset U$. Then, $U_y(m) \subset V$. Indeed, if $z \in U_y(m)$, take $z' \in X$, such that $z'_j = x_j$, when $1 \leq j \leq m$, and $z'_j = z_j$, when $j > m$.

Then, $z' \sim z$ for the relation R_n. But, as $z \in U_y(m)$, this implies that $z_j = y_j$, when $1 \leq j \leq m$, and $y \sim x$, for R_n, implies that $y_j = x_j$, when $j > n$. Then, $z'_j = x_j$, if $1 \leq j \leq m$. Therefore, $z' \in U_x(m) \subset U$.

$\square$

Lemma 3.6.7. *Given $X = \overrightarrow{\Omega}$, for each n the map defined by the canonical projection $X \to G_\infty$ is open.*

Proof: Given an open set $U \subset X$ take $V = \{y \in X \mid$ there exists $x \in X$, satisfying $y \sim x$ for the relation $R_n\}$ and $V_\infty = \{y \in X \mid$ there exists $x \in X$, satisfying $y \sim x$ for the relation $R_\infty\}$. Then, $V_\infty = \cup_{n=1}^\infty V_n$ is open.

$\square$

Lemma 3.6.8. *Given $X = \overrightarrow{\Omega}$, for each $n = 1, 2..., n, ...$, the set G_n is Hausdorff.*

Proof: Given a fixed n, and $x, y \in X$, such that x and y are not related by R_n, then, there exists $m > n$ such that $x_m \neq y_m$. From this follows that no element of $U_x(m)$ is equivalent by R_n to an element of $U_y(m)$. By lemma 3.6.6 it follows that G_n is Hausdorff.

$\square$

Lemma 3.6.9. *Given $X = \overrightarrow{\Omega}$ the set G_∞ is not Hausdorff.*

Proof: If $x_m = (\underbrace{1, 1, ..., 1}_{m}, d, d, d...)$, then $\lim_{n\to\infty} x_n = (1, 1, 1..., 1, ...)$ and $(\underbrace{1, 1, ..., 1}_{m}, d, d, d...) \sim (d, d, d, ..., d, ...)$, for the relation R_∞. Note, however, that $(1, 1, 1..., 1, ...)$ is not in the class $(d, d, d, ..., d, ...)$ for the relation R_∞.

$\square$

Lemma 3.6.10. *Given $X = \overrightarrow{\Omega}$ denote by D the diagonal set on $X \times X$. Then, D is open on G_n for any n, where we consider on D the topology induced by $X \times X$.*

Proof: Given $x \in X$, we have that $U_x(n) \times U_x(n)$ is an open set of $X \times X$ which contains (x, x).

Consider $y, z \in U_x(n)$ such that y and z are related by R_n. Then, $y_j = x_j = z_j$, when $1 \leq j \leq n$, and $y_j = z_j$, when $j > n$. Therefore, $y = z$.

From this we get that

$$U_x(n) \times U_x(n) \cap G_n \subset D$$

$\square$

Definition 3.6.11. *An equivalence relation R on a compact Hausdorff space X is said to be approximately proper if there exists an increasing sequence of proper equivalence relations R_n, $n \in \mathbb{R}$, such that $R = \cup_n R_n$, $n \in \mathbb{N}$. This in the sense that if $x \sim_R y$, then there exists an n such that $x \sim_{R_n} y$.*

Example 3.6.12. *Consider the equivalence relation R_∞ of example 3.6.2 and R_n the one of example 3.6.1. For each n the equivalence relation R^n is proper.*

Then, $R_\infty = \cup_n R_n$, $n \in \mathbb{N}$ is approximately proper (see [29]).

Definition 3.6.13. *Consider a fixed set K, a sequence of subsets $W_0 \subset W_1 \subset W_2 \subset ... \subset W_n \subset ... \subset K$ and a topology $\mathcal{W}_n$ for each set $W_n \subset K$.*

By the direct inductive limit

$$t - \lim_{n \to \infty} \mathcal{W}_n = \mathcal{K}$$

we understand the set K endowed with the largest topology $\mathcal{K}$ turning the identity inclusions $W_n \to K$ into continuous maps.

The topology of $t - \lim_{n \to \infty} \mathcal{W}_n = \mathcal{K}$ can be easily described: it consists of all subsets $U \subset K$ whose intersection $U \cap W_n$ is in $\mathcal{W}_n$ for all n.

We call $\mathcal{K}$ the direct limit topology over K.

For more details about the inductive limit (see section 2.6 in [57]).

In the case $W_n = G_n$ we consider as $\mathcal{W}_n$ the product topology.

Lemma 3.6.14. *Given $X = \overrightarrow{\Omega}$ if we consider over $K = G_\infty$ the inductive limit topology defined by the sequence of the $G_n \subset X \times X$, then, the indicator function $\mathbb{1}$ on the diagonal is continuous.*

Proof: By lemma 3.6.10 the diagonal D is an open set.

Moreover, $(G_\infty - D) \cap G_n = (\,(X \times X) - D) \cap G_\infty) \cap G_n = (\,(X \times X\,) - D) \cap G_n$ is open on G_n for all n. Then, $(G_\infty - D)$ is open on G_∞.

□

Remark: Note that on G_∞ we have that D is not open on the induced topology by $X \times X$. Indeed, consider $a = (1, 1, 1, .., 1, ..)$ and $b_m = (\underbrace{1, 1, ..., 1}_{m-1}, d, 1, 1, 1, ..1, ...)$. Then, $\lim_{m\to\infty}(a, b_m) = (a, a) \in D$, and $(a, b_m) \in G_\infty$ but (a, b_m) is not on D, for all m.

Example 3.6.15. *In the above definition 3.6.13 consider $W_n = G_n \subset \overleftarrow{\Omega} \times \overrightarrow{\Omega}$, $n \in \mathbb{N}$, which is the groupoid associated to the equivalence relation R_n (see example 3.6.1). Then, $\cup_n W_n = K = G \subset \overleftarrow{\Omega} \times \overrightarrow{\Omega}$, where G is the groupoid associated to the equivalence relation R^∞. Consider on W_n the topology $\mathcal{W}_n$ induced by the product topology on $\overrightarrow{\Omega} \times \overrightarrow{\Omega}$.*

For a fixed x the set $U = \{y \,|\, x_j = y_j \text{ for all } j \leq n\} \cap G_n$ is open on G_n, that is, an element on $\mathcal{W}_n$.

Note that $G_n \cap (U \times U)$ is a subset of the diagonal.

Points of the form

$$(\,(x_1, x_2, ..., x_n, z_{n+1}, z_{n+2}, ...), (x_1, x_2, ..., x_n, z_{n+1}, z_{n+2}, ...)\,)$$

are on this intersection.

Then, the diagonal $\{(y, y)\,,\, y \in \overrightarrow{\Omega}\}$ is an open set in the inductive limit topology $\mathcal{K}$ over G

From this follows that the indicator function of the diagonal, that is, I_Δ, where $\Delta = \{(x, x) |\, x \in \overrightarrow{\Omega}\}$, is a continuous function.

Example 3.6.16. *Consider the partition η_n, $n \in \mathbb{Z}$, over $\hat{\Omega}$ of Example 3.6.3, $W_n = G_n$, for all n, and $K = G_u$.*

We consider the topology $\mathcal{W}_n$ over G_n induced by the product topology. In this way $A \in \mathcal{W}_n$ if

$$A = B \cap G_n,$$

where B is an open set on the product topology for $\hat{\Omega} \times \hat{\Omega}$.

In this way A is open on $t - \lim_{n\to-\infty} \mathcal{W}_n = \mathcal{K}$ if for all n we have that

$$A \cap X_n \in \mathcal{X}_n.$$

Denote by D the diagonal on $\hat{\Omega} \times \hat{\Omega}$ and consider the indicator function $I_D : \hat{\Omega} \times \hat{\Omega} \to \mathbb{R}$.

The function I_D is continuous over the inductive limit topology $\mathcal{K}$ over $K = G_u$.

Here G^0 will be the set $\hat{\Omega} = \overleftarrow{\Omega} \times \overrightarrow{\Omega}$. We will denote by G a general groupoid obtained by an equivalence relation R.

The measures we consider on this section are defined over the sigma-algebra generated by the inductive limit topology.

Definition 3.6.17. *Given a Haar system (G, ν), where $G^0 = \hat{\Omega} = \overleftarrow{\Omega} \times \overrightarrow{\Omega}$ is equipped with the inductive limit topology, considering two continuous functions with compact support $f, g \in C_C(G)$, we define $(f \underset{\nu}{*} g) = h$ in such way that for any $(x, y) \in G$*

$$(f \underset{\nu}{*} g)(x, y) = \int g(x, s)\, f(s, y)\ \nu^y(ds) = h(x, y).$$

The closure of the operators of left multiplication by elements of $C_C(G)$, $\{L_f : f \in C_C(G)\} \subseteq B(L^2(G, \nu))$, with respect to the norm topology is called the reduced C^*-algebra associated to (G, ν) and denoted by $C_r^*(G, \nu)$.

Remark: There is another definition of a C*-algebra associated to (G, ν) called the full C*-algebra. For a certain class of groupoid, namely the amenable groupoids, the full and reduced C*-algebras coincide. See [3] for more details.

As usual function of the form $f(x, x)$ are identified with functions $f : G^0 \to \mathbb{C}$ of the form $f(x)$.

The collection of these functions is commutative sub-algebra of the C^*-algebra $C_r^*(G, \nu)$.

We denote by $\imath$ the indicator function of the diagonal on $G^0 \times G^0$. Then, $\imath$ is the neutral element for the product $\underset{\nu}{*}$ operation. Note that $\imath$ is continuous according to example 3.6.16.

In the case there exist a neutral multiplicative element we say the C^*-Algebra is unital.

Similar properties to the von Neumann setting can also be obtained.

We can define in analogous way to definition 3.4.13 the concept of C^*-dynamical state (which requires an unit $\imath$) and the concept of KMS state for a continuous modular function δ.

General references on the C^*-algebra setting are [54], [57], [25], [26], [27], [29], [30], [35], [52], [37] and [1] .

3.7 Examples of quasi-stationary probabilities

On this section we will present several examples of measured groupoids, modular functions and the associated quasi-stationary probability (KMS probability).

Example 3.7.1. *Considering the example 3.1.3 we get that each*

$$a \in \{1, 2, ..., d\}^{\mathbb{N}} = \overleftarrow{\Omega}$$

defines a class of equivalence

$$a \times | \overrightarrow{\Omega} = a \times \{1, 2, ..., d\}^{\mathbb{N}} = (..., a_{-n}, ..., a_{-2}, a_{-1}) \times | \{1, 2, ..., d\}^{\mathbb{N}}.$$

On next theorem we will denote by G such groupoid.

Given a Haar system ν over such $G \subset \hat{\Omega} \times \hat{\Omega}$, note that if $z_1 =< a|b_1 >$ and $z_2 =< a|b_2 >$, then $\nu^{z_1} = \nu^{z_2}$. In this way it is natural to index the Haar system by ν^a, where $a \in \overleftarrow{\Omega}$. In other words, we have

$$\nu^{<a|b>}(d < a|\tilde{b} >) = \nu^a(d\,\tilde{b}). \tag{3.38}$$

Consider $V : G \to \mathbb{R}$, m a probability over $\overleftarrow{\Omega}$ and the modular function $\delta(x, y) = \frac{e^{V(x)}}{e^{V(y)}}$, where $(x, y) \in G$.

Finally we denote by $\mu_{m,\nu,V}$ the probability on $G^0 = \hat{\Omega}$, such that, for any function $g : \hat{\Omega} \to \mathbb{R}$ and $y =< a\,|\,b >$

$$\int g(y) d\mu_{m,\nu,V}(y) = \int_{\overleftarrow{\Omega}} (\int_{\overrightarrow{\Omega}} g(< a|b >)\, e^{V(<a|b>)}\, d\nu^a(d\,b)\,)\; d\,m(da).$$

Note that $\hat{\nu} = e^V \nu$ is a G-kernel but maybe not transverse. The next theorem will provide a large class of examples of quasi-invariant probabilities for such groupoid G.

Theorem 3.7.2. *Consider a Haar System (G, ν) for the groupoid of example 3.7.1. Then, given m, V, using the notation above we get that $M = \mu_{m,\nu,V}$ is quasi-invariant for the modular function $\delta(x, y) = \frac{e^{V(x)}}{e^{V(y)}}$.*

Proof:
From (3.6) we just have to prove that for any $f : G \to \mathbb{R}$

$$\int\int f(x,y)e^{V(x)}\nu^y(dx)d\mu_{m,\nu,V}(dy) = \int\int f(y,x)e^{V(x)}\nu^y(dx)d\mu_{m,\nu,V}(dy). \tag{3.39}$$

We denote $y =< a|b >$ and $x =< \tilde{a}|\tilde{b} >$. Note that if $y \sim x$, then $a = \tilde{a}$. Note that, from (3.38)

$$\int (\int f\,(x, y)\, e^{V(x)}\nu^y(dx)\ )\ d\mu_{m,\nu,V}(dy) =$$

$$\int (\int f\,(< \tilde{a}|\tilde{b} >, < a|b >)\, e^{V(< \tilde{a}|\tilde{b} >)}\nu^{< a|b >}(d < \tilde{a}|\tilde{b} >)\ d\mu_{m,\nu,V}(d\ < a|b >) =$$

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int\int f\,(< \tilde{a}|\tilde{b} >, < a|b >)\ e^{V(< \tilde{a}|\tilde{b} >)}\nu^{< a|b >}(d < \tilde{a}|\tilde{b} >)\ e^{V(< a|b >)}\, d\nu^a(d\,b)\,)\,]\,d\,m(da) =$$

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int\int f\,(< a|\tilde{b} >, < a|b >)\ e^{V(< a|\tilde{b} >)}\ e^{V(< a|b >)}\,\nu^{a}(d\,\tilde{b})\,d\nu^a(d\,b)\,)\,]\,d\,m(da).$$

In the above expression we can exchange the variables b and $\tilde{b}$, and, finally, as $a = \tilde{a}$, we get

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int\int f\,(< a|b >, < a|\tilde{b} >)\ e^{V(< a|b >)}\ e^{V(< a|\tilde{b} >)}\,\nu^{a}(d\,b)\,d\nu^a(d\,\tilde{b})\,)\,]\,d\,m(da) =$$

$$\int_{\overleftarrow{\Omega}} [\int_{\overrightarrow{\Omega}} (\int\int f\,(< a|b >, < a|\tilde{b} >)\, e^{V(< a|\tilde{b} >)}\ d\nu^a(d\,\tilde{b})\ \ e^{V(< a|b >)}\,\nu^{a}(d\,b)\ )\,]\,d\,m(da) =$$

$$(\int\int f\,(y,\, x)\, e^{V(x)}\ d\nu^y(dx)\ d\mu_{m,\nu,V}(dy).$$

This shows the claim.

□

Example 3.7.3. *Consider G associated to the equivalence relation given by the unstable manifolds for $\hat{\sigma}$ acting on $\hat{\Omega}$ (see example 3.1.4). Let's fix a certain $x_0 \in \overrightarrow{\Omega}$. Note that in the case $x =< a^1|b^1 >$ and $y =< a^2|b^2 >$ are on the same unstable manifold, then there exists an $N > 0$ such that $a^1_j = a^2_j$,* *for any $j < -N$. Therefore, when $\hat{A} : \hat{\Omega} \to \mathbb{R}$ is Hölder and $(x, y) \in G$ then it is well defined*

$$\delta(x,y) = \Pi_{i=1}^{\infty} \frac{e^{\hat{A}(\hat{\sigma}^{-i}(x))}}{e^{\hat{A}(\hat{\sigma}^{-i}(y))}} = \Pi_{i=1}^{\infty} \frac{e^{\hat{A}(\hat{\sigma}^{-i}(<a^1|b^1>))}}{e^{\hat{A}(\hat{\sigma}^{-i}(<a^2|b^2>))}}.$$

Fix a certain $x_0 =< a^0, b^0 >$, then the above can also be written as

$$\delta(x,y) = \frac{e^{V(x)}}{e^{V(y)}} = \frac{e^{V(<a^1|b^1>)}}{e^{V(<a^2|b^2>)}},$$

where

$$e^{V(<a|b>)} = \Pi_{i=1}^{\infty} \frac{e^{\hat{A}(\hat{\sigma}^{-i}(<a|b>))}}{e^{\hat{A}(\hat{\sigma}^{-i}(<a^0|b^0>))}}.$$

Then, in this case δ is also of the form of example 3.3.2.

In this case, given any Haar system ν and any probability m, Theorem 3.7.2 can be applied and we get examples of quasi-invariant probabilities.

The next result has a strong similarity with the reasoning of [43] and [62].

Proposition 3.7.4. *Given the modular function δ of example 3.3.3 consider the probability $M(d\,a, d\,b) = W(b)\,d\,b\,d\,a$ on $S^1 \times S^1$. Assume ν^y, $y = (a_0, b_0)$, is the Lebesgue probability db on the fiber $(a_0, b), 0 \le b < 1$, then, M satisfies for all f*

$$\int \int f(s,y) \nu^y(ds) dM(y) = \int \int f(y,s) \delta^{-1}(y,s)\, \nu^y(ds) dM(y). \qquad (3.40)$$

Proof:

We consider the equivalence relation: given two points $z_1, z_2 \in S^1 \times S^1$ they are related if the first coordinate is equal.

In the case of example 3.3.3 we take the a priori transverse function $\nu^{z_1}(d\,b) = \nu^a(d\,b)$, $z_1 = (a, \tilde{b})$, constant equal to $d\,b$ in each fiber. This corresponds to the Lebesgue probability on the fiber.

For each pair $z_1 = (a, b)$ and $z_2 = (a,\, s)$, and $n \ge 0$, the elements z_1^n, z_2^n, $n \in \mathbb{N}$, such that $F^n(z_1^n) = z_1 = (a, b)$ and $F^n(z_2^n) = z_2 = (a, s)$, are of the form $z_1^n = (a^n, b^n)$, $z_2^n = (a^n, s^n)$.

We define the cocycle

$$\delta(z_1, z_2) = \Pi_{j=1}^{\infty} \; \frac{A(z_1^n)}{A(z_2^n)}.$$

Fix a certain point $z_0 = (a, c)$ and define V by

$$V(z_1) = \Pi_{j=1}^{\infty} \; \frac{A(z_1^n)}{A(z_0^n)}.$$

Note that we can write

$$\delta(z_1, z_2) = \frac{V(z_1)}{V(z_2)},$$

for such function V.

Remember that by notation x_0 is a point where $(0, x_0)$ and $(x_0, 1)$ are intervals which are domains of injectivity of T.

Remark: Note the important point that if $x = (a, b)$ and $x' = (a', b)$, with $x_0 \leq a \leq a'$, we get that $b_n(x) = b_n(x')$. In the same way if $0 \leq a \leq x_0$ we get that $b_n(x) = b_n$. In this way the b_n does not depend on a.

This means, there exists W such that we can write

$$\delta(z_1, z_2) = \delta^{-1}(\,(a, b), (a, s)\,) = Q(s, b) = \frac{W(s)}{W(b)},$$

where $b, s \in S^1$.

Condition (3.40) for y of the form $y = (a, b)$ means for any f:

$$\int \int f(\,(a, b), (a, s)\,) \; \nu^a(d\,s) \; dM(a, b) =$$

$$\int \int f(\,(a, s), (a, b)\,) \; \delta^{-1}(\,(a, b), (a, s)\,) \; \nu^a(d\,s) \; dM(a, b) =$$

$$\int \int f(\,(a, s), (a, b)\,) \; Q(s, b) \; \nu^a(d\,s) \; dM(a, b).$$

Now, considering above $f(\,(a, b), (a, s)\,)V(s)$ instead of $f(\,(a, b), (a, s)\,)$, we get the equivalent condition: for any f:

$$\int \int f(\,(a, b), (a, s)\,) \, W(s) \, d\,s \; dM(a, b) =$$

$$\int\int f(\,(a,s),(a,b)\,)\;W(s)\;d\,s\;dM(a,b).$$

As $dM = W(b)d\,b\,da$ we get the alternative condition

$$\int\int f(\,(a,b),(a,s)\,)\,W(s)\,d\,s\;W(b)d\,b\,d\,a\;=$$

$$\int\int f(\,(a,s),(a,b)\,)\;W(s)\;d\,s\;W(b)d\,b\,d\,a\,, \tag{3.41}$$

which is true because we can exchange the variables b and s on the first term above.

□

Example 3.7.5. *Consider the groupoid G associated to the equivalence relation of example 3.1.5. In this case x and y are on the same class when there exists an $N>0$ such that $x_j = y_j$, for any $j \geq N$. Each class has a countable number of elements.*

Consider a Hölder potential $A: \overrightarrow{\Omega} \to \mathbb{R}$.

For $(x,y) \in G$ it is well defined

$$\delta(x,y) = \Pi_{i=0}^{\infty} \frac{e^{A(\sigma^i(x))}}{e^{A(\sigma^i(y))}}.$$

Consider the counting Haar system ν on each class.

We say $f: G \to \mathbb{R}$ is admissible if for each class there exist a finite number of non zero elements.

The quasi-invariant condition (3.5) for the probability M on $\overrightarrow{\Omega}$ means: for any admissible integrable function $f: G \to \mathbb{R}$ we have

$$\sum_s \int f(s,x)dM(x) = \sum_s \int f(x,s)\,\Pi_{i=0}^{\infty} \frac{e^{A(\sigma^i(s))}}{e^{A(\sigma^i(x))}}\,dM(x). \tag{3.42}$$

Suppose B is such that $B = A + \log h - \log(g \circ \sigma) - c$. This expression is called a coboundary equation for A and B. Under this assumption, as $x \sim s$, we get

$$\sum_s \int f(x,s)\,\Pi_{i=0}^{\infty} \frac{e^{B(\sigma^i(x))}}{e^{B(\sigma^i(s))}}\,dM(x) =$$

$$\sum_s \int f(x,s)\,\Pi_{i=0}^{\infty} \frac{e^{A(\sigma^i(x))}}{e^{A(\sigma^i(s))}} \frac{h(x)}{h(s)}\,dM(x).$$

Take $f(s,x) = g(s,x)\,h(x)$, then, as M is quasi-invariant for A, we get that

$$\sum_s \int g(x,s)\, \Pi_{i=0}^{\infty} \frac{e^{B(\sigma^i(x))}}{e^{B(\sigma^i(s))}}\, h(x)\, dM(x) = \sum_s \int g(s,x)\, h(x)\, dM(x). \quad (3.43)$$

As $g(x,s)$ is a general function we get that $h(x)\, dM(x)$ is quasi-invariant for B.

Any Hölder function A is coboundary to a normalized Hölder potential. In this way, if we characterize the quasi-invariant probability M for any given normalized potential A, then, we will be able to determine, via the corresponding coboundary equation, the quasi-invariant probability for any Hölder potential.

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