UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
INSTITUTO DE MATEMÁTICA
CADERNOS DE MATEMÁTICA E ESTATÍSTICA
SÉRIE B: TRABALHO DE APOIO DIDÁTICO

ÁRVORES DE DECISÃO

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DECISION MODELS FOR BUSINESS

PART III- DECISION TREES

1. Introduction to Probability

Consider any situation which can have several different possible outcomes where we are uncertain about which outcome will actually occur. For example:

1. How many heads will I get if I toss a coin 10 times?
2. Will IBM stock close tomorrow at \$92.50, \$92.51, … , \$94.49, or \$94.50?
3. Will the total rainfall in Chapel Hill this year be 30, 31, … , 49, or 50 inches?
4. How long will it be before a particular light bulb burns out?

Each of these uncertain situations is called a *random process* (or *random experiment*), and the value of its outcome is called a *random variable*. Before the random process runs its course, we cannot know exactly what its outcome will be (what value the random variable will take). However, we often have some idea of how likely each particular outcome is:

A *probability* is a number that measures our belief of how likely a particular outcome is. It is standard for us to use 0 for the probability of an event that cannot occur and 1 for the probability of an event that we believe is certain to occur. The higher the probability, the more certain we are that the event will occur.

Notation: X – a random variable
x_1, x_2, … – the different possible values X may take on
P(X = x) [or simply P(x)] –probability that the random variable X takes the value x

There are several different ways in which we can get values for probabilities:

1. Objective ("theoretical") probability – P(heads on a coin toss) = ½ (or 0.5 or 50%)
2. Relative frequency ("estimated") probability – used for example to set insurance rates. Why did your parents' automobile insurance premium increase the day you got your driver's license?
3. Subjective – to estimate the probability of something somewhat unlike anything you've previously seen. What was the probability (at the beginning of October) that the Red Sox would win the World Series this year? Note that different people had different subjective assessments of this!

The interpretation and appropriateness of each of these has led to much philosophical discussion and argument, but fortunately this is not a philosophy class. We will employ whichever view is useful for the problems we'll discuss.

Regardless of where our probabilities come from, they obey certain laws. Let's see what these rules are. Recall we are performing a *random experiment*. The outcome of the experiment is called an *event*, and the set of all possible events is called the *sample space*.

EXAMPLE: Toss a single die and count the number of spots that show. The sample space is {1, 2, 3, 4, 5, 6}. The events we've listed are called "simple" (or "elementary" or "atomic"). They are indivisible into smaller events. In contrast, events like *odd* (= {1, 2, or 3}) or *at least 3* (= {3, 4, 5, 6}) are "composite."

Once we specify the probabilities of simple events, the probabilities of composite events will be derived via the various probability laws we shall develop.

It will often be convenient to schematically represent problems and probability laws with pictures known as *Venn diagrams* (after the English mathematician, John Venn (1834-1923)). We represent the entire sample space (or "universe of discourse") by a box, and events are represented by subsets of the box. Since all probabilities are between 0 and 1, if we let the area of the box be 1, we can think of the probabilities of various events as the areas of the regions representing them. Indeed, as we shall see, probabilities behave very much like areas, and mathematically speaking, *probability theory* is a special case of *measure theory*, which comprises the mathematics of such things as areas and volumes.

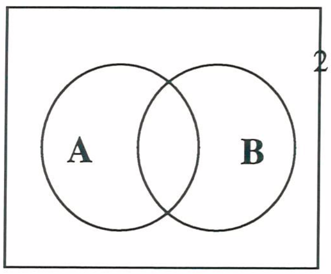
EXAMPLES: 1. tossing a coin – divide the box into two halves, labeled H and T
2. tossing a single die – divide the box into sixths, labeled 1, 2, 3, 4, 5, and 6

PROBABILITY LAWS

1. Law of Complements – associated with any event B is its "complementary event – not B" denoted B′. In tossing coins, the complement of H is T. In rolling a die, the complement of {2 or 3} is {1, 4, 5, or 6}. Since either B occurs or it doesn't, we can see from the diagram that P(B′) = 1 – P(B).

2. Addition Law (law of "unions") – Suppose we know P(A) and P(B) and we want to find P(A or B). From the picture, we have P(A or B) = P(A) + P(B) – P(A and B).

   Why do we subtract P(A and B)?



A special case: Two events, A and B, are *(mutually) exclusive* if at most one of them can occur, in which case P(A and B) = 0, and the addition law simplifies to

$$P(A \text{ or } B) = P(A) + P(B).$$ DRAW A PICTURE!

Likewise, multiple events are mutually exclusive if at most one of them can occur, and in this case the addition law generalizes to

$$P(A_1 \text{ or } A_2 \dots \text{ or } A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

Several events, $A_1, A_2, \dots, A_n$, are *(collectively) exhaustive* if at least one of them must occur, i.e., if $P(A_1 \text{ or } A_2 \dots \text{ or } A_n) = 1$. Exclusive and exhaustive are concepts that have no relationship whatsoever to each other, as shown by the following dice examples:

PRIME = {1, 2, 3, 5} FIVE = {5} EVEN = {2, 4, 6} ODD = {1, 3, 5}

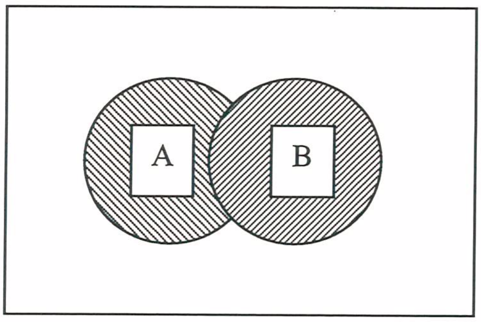
- EVEN and ODD are exclusive and exhaustive
- EVEN and PRIME are exhaustive, but not exclusive (2 is a member of both)
- EVEN and FIVE are exclusive, but not exhaustive (1 and 3 are missing)
- ODD and PRIME are neither exclusive (1, 3, and 5 belong to both) nor exhaustive (4 and 6 are missing)

3. Conditional Probabilities
   - Suppose I roll one die and I ask, "What is P(FIVE)?" Answer: 1/6
   - Now suppose I tell you that an even number of spots are showing and I ask, "What is the probability of *FIVE given ODD*, denoted P(FIVE | ODD)?" Answer: 1/3
   - What is P(FIVE | PRIME)? Answer: ¼
   - What is P(FIVE | EVEN)? Answer: 0

Notice how the answer changes when we are given the additional information! A probability of the form P(A | B) is called a *conditional probability* and B is called the *conditioning event*. The rule for computing such conditional probabilities is

$$P(A | B) = P(A \text{ and } B)/P(B) \quad (\text{provided } P(B) \neq 0)$$

This follows directly from the Venn diagram below. Given that B has occurred, the universe of discourse is no longer the big rectangle – it's just the circle for B. Now relative to that, what's the probability that A has also occurred? Answer: the crossed hatched area, P(A and B), as a fraction of the area of the B circle, P(B).



Obviously the formula doesn't work if $P(B) = 0$, but that's OK because conditioning relative to an impossible event isn't meaningful. Suppose we toss a black die and a red die and let B, R, and S represent the number of spots showing on the black die, the red die, and both dice together. Questions like, "Find $P(B = 5 \mid S = 13)$" are just silly!

Two events are said to be *(statistically) independent* if $P(A \mid B) = P(A)$. Loosely speaking, events are independent if knowledge about one of them tells you absolutely nothing about the other. Continuing our dice example:

$P(R = 6) = 1/6$
$P(R = 6 \mid S = 4) = 0$, so $R = 6$ and $S = 4$ are *dependent* events
$P(R = 6 \mid S = 11) = ½$, so $R = 6$ and $S = 11$ are *dependent* events
$P(R = 6 \mid S = 7) = 1/6$, so $R = 6$ and $S = 7$ are *independent* events

Neophytes often confuse the concepts of *independent* events and *exclusive* events. However, please note that exclusive events are highly dependent – if A and B are mutually exclusive, then $P(A \mid B) = 0$, which is usually not $P(B)$.

4. Multiplication Law:

$$P(A \text{ and } B) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

The probability of two events both occurring is the probability that one of them occurs time the probability that the second occurs, given that the first has occurred.

If you look closely at the definition of conditional probability and the multiplication law, you'll se that they really are the same equation, so it looks like we're running around in circles: To get the conditional probability, $P(A \mid B)$, you need the "joint" probability, $P(A \text{ and } B)$, but to get the joint probability you need the conditional probability, but .... Fortunately, in various problems you'll have one of them, so you can get the other.

Example: In drawing two cards from a standard deck, *without* replacement, let A_i be the event that the *i*th card is an ace. Find:

- $P(A_1 \text{ and } A_2)$ [frequently denoted $P(A_1A_2)$]
  Answer: $P(A_1)P(A_2 \mid A_1) = (4/52)(3/51) = 1/221$
- $P(A_2)$ Note that this is easy *with* replacement – $P(A_2) = 4/52 = 1/13$
  Answer: $A_2 = A_1A_2$ or A'_1A_2, so
  $P(A_2) = P(A_1A_2) + P(A'_1A_2) = 1/221 + (48/52)(4/51) = 1/221 + 16/221 = 1/13$

Example:

KEYSER SöZE'S M&M PROBLEM

Keyser Söze stared at the bowl of M&M's. So many choices, but so little control! The little candies seemed to be mocking him. Sometimes he got a red one, but then sometimes he got a green one. He almost never got one of his favorite blue ones. Keyser was infuriated that the color seemed so unpredictable. He understood that there were probably more critical issues in his life, but the M&M color was still important to him. He needed to take some control over the situation.

Keyser realized he needed to gather some data on the problem. He reached into the bowl of M&M's and randomly selected 100 candies. He carefully sorted them by color, counted up the number of each color, and then ate the lot. The results are given in the following table:

Color:	Blue	Brown	Green	Orange	Red	Yellow
Number of candies:	10	25	15	13	20	17

Suppose Keyser randomly selects *one* M&M out of a bowl containing an *immense* number of M&M's. Keyser has a number of M&M questions that he needs some help answering:

a) What is the probability that the candy is *brown*?

b) What is the probability that the candy is a *warm* color (orange, red, or yellow)?

c) What is the probability that the candy is a *cool* color (blue, brown, or green)?

d) What is the probability that the candy is an *RGB* color (red, green, or blue)?

e) What is the probability that the candy is both warm *and* RGB?

f) What is the probability that the candy is warm *or* RGB?

g) What is the probability that the candy is warm *and* brown?

h) What is the probability that the candy is warm *or* brown?

i) Given that the candy is *RGB*, what is the probability that it is *warm*?

j) Given that the candy is *warm*, what is the probability that it is *RGB*?

k) Are the events given in (i) and (j) *independent* or *dependent*? Why or why not?

KEYSER SöZE'S M&M PROBLEM--SOLUTIONS

Note: This case was written to illustrate the terrifying "probability laws." These laws are nothing more than a set of rules which can be used to calculate the probability of certain events if you are fortunate enough to know the probabilities of certain *other* events. Alternatively, all of the problems in this case can be solved by going back to the original probability distribution (the "sample space") and applying simple logic. If you worked a particular problem in this fashion and got the right answer, then your logic almost surely was correct. On the other hand, if you got the wrong answer, then your logic was definitely incorrect.

a) P(brown) = .25

b) P(warm) = .13 + .20 + .17 = .50

c) P(cool) = 1 - P(warm) = .50

d) P(RGB) = .10 + .15 + .20 = .45

e) P(warm and RGB) = P(red) = .20

f) P(warm or RGB) = P(warm) + P(RGB) - P(warm and RGB) = .50 + .45 - .20 = .75

also = 1 - P(brown)

g) P(warm and brown) = 0
mutually exclusive

h) P(warm or brown) = P(warm) + P(brown) - P(warm and brown) = .50 + .25 – 0 = .75

i) P(warm | RGB) = P(warm and RGB)/P(RGB) = .20/.45 = .44

j) P(RGB | warm) = P(RGB and warm)/P(warm) = .20/.50 = .40

k) For the two events to be independent, we need to show any of the following:

(1) P(warm | RGB) = P(warm) not true since .44 ≠ .50

(2) P(RGB | warm) = P(RGB) not true since .40 ≠ .45

(3) P(warm and RGB) = P(warm) x P(RGB) not true since .20 ≠ .50 x .45

Thus any *one* of these calculations shows that the two events are *dependent*.

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2. Expected Values & Decision Making

Suppose we have a random variable, X, which can take on the possible values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$. The expected value of X is defined as

$$E(X) = p_1x_1 + p_2x_2 + \ldots + p_nx_n = \Sigma p_ix_i$$

E(X) is the *probability weighted* average value of X and is also the long-run average value of X if we did the experiment that produces X lots of times.

Examples

- Roll a die. How many spots do we expect to see? (1/6)(1) + (1/6)(2) + ... + (1/6)(6) = 21/6 = 3.5
  IMPORTANT NOTE: Obviously you can't get 3.5 spots on any given roll! The expected value of a random variable need not be a value that the random variable ever takes on.
- Flip a coin 3 times. Let X = the number of heads observed. What is E(X)?

  (1/8)(0) + (3/8)(1) + (3/8)(2) + (1/8)(3) = 1.5

  Does this make sense? The probability of a head on any flip is ½, so we'd expect to see heads on half of the flips.

For the next several days, we will operate under the following assumption:

> Faced with the choice among several actions, we will choose the one with the highest expected monetary value (EMV).

Example: Find a volunteer and offer to play the following game –

- Version 1: I'm going to flip a coin. If it comes up heads, I'll give you \$3. If it comes up tails, I'll give you nothing. Are you willing to play? Why?

  Of course the student will play – there's something to gain and nothing to lose

- Version 2: Oops, I forgot. There's an entrance fee. You have to pay me \$1 to play. So, if the coin comes up heads, you have a net gain of \$2 (the \$3 I give you less the \$1 entry fee). If it comes up tails, you lose your entry fee. Are you willing to play? Why?

  Again, the typical student will play – with a 50% chance of winning \$2 and a 50% chance of losing \$1, on average s/he will win 50¢ per play of the game. (The expected value of playing – 50¢ – is more than the expected value of not playing - 0¢.)

- Version 3: Let's ratchet up the ante a bit. The entry fee is now \$10. If a head is tossed you get \$30 (netting \$20), but if a tail comes up, you lose your \$10. Will you play? Why?

The student will probably still play (the EMV is \$5), but s/he may think about it for a while. If so, ask why the pause.

- Version 4: Now the entry fee is \$100; a head gets a return of \$300 (netting \$200); a tail means that the \$100 entry fee is lost. Same questions.

  Eventually, the stakes will be so high that the student won't play. When you point out that the EMV is still positive ($\$0.5x10^{v-2}$, where v is the version number), the student will probably say something like, "But I can't afford to lose $\$10^{v-2}$. So our assumption doesn't always hold.

Although it's easy to think of situations where this does not hold, the assumption will simplify some of our analysis. There are ways to relax our assumption – using the economist's concepts of utility functions and risk preferences – but we won't discuss them.

MAKING DECISIONS WITH EXPECTED VALUES

1. Doyle's of Dublin, an Irish syndicate of insurance underwriters, has been asked to insure a Norwegian oil-drilling platform against storm damage for the coming year. Doyle's meteorologists and economists have estimated the following distribution for the potential claims (in millions of dollars) against the policy:

Claim	0	20	50	100
Probability	0.990	0.007	0.002	0.001

   i. What level of claims against the policy can Doyle expect?
   ii. Lars Hansen, the Controller of OljeNord - the platform's owner, offered to pay Doyle's a $325,000 premium for issuing the insurance policy. Should Doyle's be interested in this offer? Explain.

2. At a distance of 100 yards, Mr. Winchester hits the target an average of 60 percent of the time. In a local rifle contest, each contestant will attempt 3 shots and receive the following reward depending upon the number of targets that he hits: $50 for all 3 targets; $25 for 2 targets; and $10 for 1 target. If a contestant misses all 3 targets, he must pay $100. Should Mr. Winchester enter the contest?

3. A population of candidates for accident insurance contains 80 percent type X candidates and 20 percent type Y candidates. Type X individuals have probability .01 of having one accident in a year; for type Y this probability is .05. Neither type will ever have more than one accident in a year. The policy in question costs $20/year and pays $1000 if an accident occurs.

   i. An individual is chosen at random from the population. What is the insurance company's expected profit on a one-year policy?

   ii. What is the company's expected profit on a one-year renewal policy for an individual who had an accident during the first year? (You may assume that individual types and accident rates are constant over time.)

   iii. What is the company's expected profit on a one-year renewal policy for an individual who didn't have an accident during the first year? (You may assume that individual types and accident rates are constant over time.)

MAKING DECISIONS WITH EXPECTED VALUES

SOLUTIONS

1. i. The expected claim is

   $0(0.990) + 20(0.007) + 50(0.002) + 100\,(0.001) = 0.00 + 0.14 + 0.10 + 0.10 = 0.34$, i.e., \$340,000

   ii. A premium of \$325,000 doesn't even cover the expected loss, let alone cover Doyle's expenses or give them some margin for profit. Doyle's should decline Hansen's offer, but they might consider a counter-offer of something in excess of \$350,000.

2. Let H indicate a hit and M indicate a miss. Then the probability that Mr. Winchester

   misses all three targets $= P(MMM) = (0.4)^3 = 0.064$
   hits exactly one target $= P(MMH \text{ or } MHM \text{ or } HMM) = 3(0.4)^2(0.6) = 0.288$
   hits exactly two targets $= P(MHH \text{ or } HMH \text{ or } HHM) = 3(0.4)(0.6)^2 = 0.432$
   hits all three targets $= P(HHH) = (0.6)^3 = 0.216$

   Thus the expected payoff for entering the contest is

   $-100(0.064) + 10(0.288) + 25(0.432) + 50(0.216) = -6.40 + 2.88 + 10.80 + 10.80 = \$18.08 > 0$

   Since the expected payoff is positive, he should enter the contest.

3. i. $P(\text{accident}) = P(X)P(\text{accident} \mid X) + P(Y)P(\text{accident} \mid Y) = 0.8(0.01) + 0.2(0.05) = 0.018$

   Hence the company's expected profit is $20 - 1000(0.018) = \$2.00$

   ii. $P(X \mid \text{accident in year 1}) = P(X \text{ and accident in year 1})/P(\text{accident in year 1})$
   $= P(X)P(\text{accident in year 1} \mid X)/P(\text{accident in year 1})$
   $= 0.8(0.01)/0.018 = 0.4444$
   $P(Y \mid \text{accident in year 1}) = 1 - P(X \mid \text{accident in year 1})$
   $= 1 - 0.4444 = 0.5556$
   Hence, $P(\text{accident in 2nd year} \mid \text{accident in } 1^{st} \text{ year})$
   $= P(X \mid \text{accident in year 1})\, P(\text{accident} \mid X) + P(Y \mid \text{accident in year 1})\, P(\text{accident} \mid Y)$
   $= 0.4444(0.01) + 0\ 5556(0.05) = 0.0322,$

   (The following table might be useful if you don't like the formulas I used above:)

	Accident	No accident	P(Type)
X	0.8(0.01) = 0.008	0.8(0.99) = 0.792	0.8
Y	0.2(0.05) = 0.010	0.2(0.95) = 0.190	0.2
P(Result)	0.018	0.982	

   and the company's expected profit on the renewal is $20 - 1000(0.0322) = -\$12.20$.

And that's why they raise your premiums if you have an accident! Of course, if you don't have an accident in the first year, then their expected profit in the second year is \$2.26. So one might suppose that they should reward you for your good year by reducing the renewal premium.

iii. I leave it to you to verify the computation of the \$2.26 figure.

3. Introduction to Decision Trees

Example: **AN OIL DRILLING PROBLEM**

An oil wildcatter must decide whether or not to drill at a given site before his option expires. H uncertain about the extent of the oil or gas at the site. He has available the objective records of similar not-quite-so-similar drillings in this same basin, and he has discussed the peculiar features of this particu deal with his geologist, his geophysicist, and his land agent. He can gain further relevant information (still not perfect information) about the underlying geophysical structure at this site by conducting seism soundings. This information, however, is quite costly, and his problem is to decide whether or not to coll this information before he makes his final decision: to drill or not to drill.

The wildcatter is uncertain whether the hole is dry, wet, or soaking. The payoffs, as well as unconditional probabilities of the three states, are given in Table 1. We assume here that the cost of drill is $70,000. The net return associated with the (Wet, Drill) pair is $50,000, which is interpreted as a return $120,000 less the $70,000 cost of drilling. Similarly, the $200,000 amount is a net figure; it represent return of $270,000 less the cost of drilling.

Table 1 Monetary Payoffs and State Probabilities

		Drill	Don't Drill	Probability of the state
	Dry	- $70,000	$0	0.45
State	Wet	$50,000	$0	0.30
	Soaking	$200,000	$0	0.25

1. Assuming the wildcatter decides not to conduct seismic soundings, what strategy should he follow?

For a fee, our wildcatter could take seismic soundings which will help determine the underlying geological structure at the site. The soundings will disclose whether the terrain below has (a) no structu – that's bad, or (b) open structure – that's so-so, or (c) closed structure – that's really hopeful. The experts have kindly provided us with Table 2, which shows the state probabilities conditional on the seismic outcomes, and which shows the probabilities of the three seismic outcomes. From this table w can see, for example, that the probability of a dry hole given a seismic outcome of no structure is 0.75, while the probability of no structure is 0.40.

Table 2 Conditional State Probabilities and Seismic Outcome Probabilities				
		Seismic outcome		
		No Structure	Open Structure	Closed Structure
	Dry	0.75	0.40	0.10
State	Wet	0.15	0.40	0.40
	Soaking	0.10	0.20	0.50
	Probability of the seismic outcome	0.40	0.30	0.30

2. Assuming the wildcatter decides to take seismic soundings, what strategy should he follow?

3. What is the maximum amount he should be willing to pay for seismic soundings?

4. How much should he be willing to pay for absolutely reliable information indicating which state v occur?

a. Before taking the possibility of the seismic soundings into account, we can represent the wildcatter's decision problem by the following **decision tree**:

```
  Don't drill
 +---------------------------------------------------------  0
 |                 0.45  Dry
 |                  +----------------------------- -70
 |                  |
 | Drill            | 0.30  Wet
[ ]-----------------------( )--------------------------- 50
                    |
                    | 0.25  Soaking
                    +----------------------------- 200
```

- A **[ ]** represents a **decision** (or **action**) **node**. Here you, the decision maker are in control, and you must decide among several possible **actions** (or **decisions**). There is a **branch** for each possible action at the node.
- A **( )** represents a **chance** (or **event**) **node**. Here you are not in control ("nature rolls the dice"), and there several possible **events** (or **outcomes**, or **states of nature**) that can occur. There is a **branch** for each possible outcome at the node.
- The time sequence of the tree runs from **left to right**, and **it is exceedingly important to ge that sequence correct!** (Suppose you had your choice between the scenario above and one which you were told beforehand whether the land was dry, wet, or soaking, and then you ha to decide whether to drill or not– you'll use it later when calculating EVPI.) Which would y prefer? Why?)
- In general, these trees will structure our problems for us and indicate what computations mu be done to decide what actions to take.

- We put the return (or the cost) from each possible sequence of actions and events at the end (the **top**) of the tree. We then analyze the tree as follows:
  - **Chance nodes:** Calculate the expected value of the return. Recall that the expected va[l] the probability-weighted average of the returns for each of the outcom[e] This is called "**averaging out**."
  - **Decision nodes:** Choose that action which has the highest expected payoff (**maximum** expected **return**, **minimum** expected **cost**). "**Prune**" the other branch the node. This is called "**folding back**".

When you are done, the tree tells you what decision should be made and what is your expected return if you follow this **optimal strategy** (or **policy**).

In the case of the oil drilling problem, the EMV of drilling is

$$0.45(-70) + 0.30(50) + 0.25(200) = 33.5 \text{ (i.e., \$33,500)}$$

Since this is greater than 0 (the EMV of not drilling), the wildcatter should drill the well.

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b. Now let's make the problem more complex. Prior to drilling, the wildcatter can take seismi soundings to get more information about his prospects. (A large dynamite blast is set off, a a seismograph is used to record the shock waves that the blast produces in the ground. Site: with different geologic structures are more or less likely to have underlying oil deposits. Tl shock waves are affected by the geologic structure of the land, so the seismograph reading gives information about how likely it is that the site has underlying oil.)

Given that he takes the soundings, the tree looks like:

```
                 Don't drill
               +-------------------------------------------------------  0
               |                    0.75  Dry
               |                   +----------------------------- -70
               |                   |
  0.40    No structure  |  Drill            | 0.15  Wet
+------------------------[ ]-----------------------( )--------------------------  50
|                                  |
|                                  | 0.10  Soaking
|                                  +----------------------------  200
|
|               Don't drill
|              +-------------------------------------------------------  0
|              |                    0.40  Dry
|              |                   +----------------------------- -70
|              |                   |
| 0.30   Open structure  |  Drill          | 0.40  Wet
( )----------------------[ ]-----------------------( )--------------------------  50
|                                  |
|                                  | 0.20  Soaking
|                                  +----------------------------  200
|
|               Don't drill
|              +-------------------------------------------------------  0
|              |                    0.10  Dry
|              |                   +----------------------------- -70
|              |                   |
| 0.30 Closed structure  |  Drill          | 0.40  Wet
+------------------------[ ]-----------------------( )--------------------------  50
                                   |
                                   | 0.50  Soaking
                                   +----------------------------  200
```

We can now compute the EMV's of the drill options, which are:

- With no structure: 0.75(–70) + 0.15(50) + 0.10(200) = –25 → don't drill
- With open structure: 0.40(–70) + 0.40(50) + 0.20(200) = 32 → drill
- With closed structure: 0.10(–70) + 0.40(50) + 0.50(200) = 113 → drill

Now we average out the chance node to find that the EMV if we take the soundings is:

0.40(0) + 0.30(32) + 0.30(113) = 43.5 (i.e., $43,500)

c. Notice that the EMV(with soundings) > the EMV(without soundings). Why **must** this be so?

The difference between EMV(with soundings) and EMV(without soundings) is called the **expected value of sample information** (**EVSI**), and it tells us the maximum amount the wildcatter would be willing to pay for the soundings. Why is this so? Let's put together the tre for the whole problem:

```
    No Soundings
  +------------------ 33.5
  |
 [ ]
  | Soundings
  +------------------ 43.5
```

He's $10K better off if he takes the soundings, so he'd be willing to pay up to $10K to get the information the soundings provide. So long as the cost of the soundings is less than EVSI, he's better off if he takes the soundings.

(NOTE: At a cost of $10K, he's indifferent to taking the soundings or not, so to avoid sillines like, "He'd be willing to pay up to $9999.99," we take EVSI as the maximum amour to pay for the information.)

d. The computation of EVSI is often quite messy, but it turns out that we can sometimes avoid tha work. As the quality of the sample information gets better and better (i.e., more accurate or reliable), it stands to reason that its value goes up. So suppose we could somehow get absolute reliable (or **perfect**) information about the state of nature. The value of that perfect informatio (EVPI) **would be an upper bound on EVSI for any information whatsoever**. Since we wou get that information before we had to decide whether or not to drill, the appropriate tree is

```
                 Don't drill
                +------------ 0
   0.45 Dry     |
\  +-----------------[ ]
   |            | Drill
   |            +------------ -70
   |
   |             Don't drill
   |            +------------ 0
   | 0.30 Wet   |
  ( )----------------[ ]
   |            | Drill
   |            +------------ 50
   |
   |             Don't drill
   |            +------------ 0
   | 0.25 Soaking |
   +-----------------[ ]
                | Drill
                +------------ 200
```

Folding back and averaging out, we see that the EMV(with perfect information) is

$$0.45(0) + 0.30(50) + 0.25(200) = 65 \text{ (i.e., \$65,000)}$$

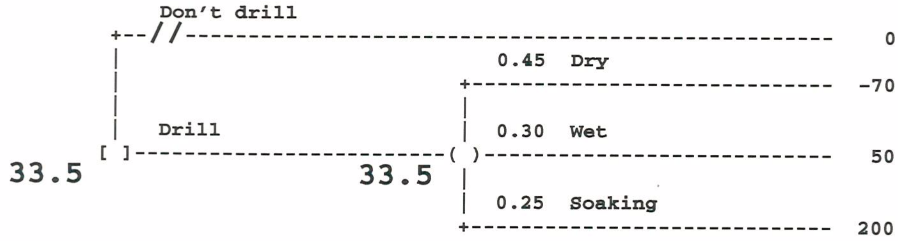
so the EVPI is 65 – 33.5 = 31.5 (i.e., $31,500)

e. This question addresses the issue of **sensitivity analysis** – how do the optimal policy and optimal objective value vary when the costs, returns, and probabilities vary. This is important because of uncertainty about data, changing circumstances, etc. It gives us the ability to do "What if?" analyses. We shall return to this topic shortly.

See the complete decision trees on the next page.

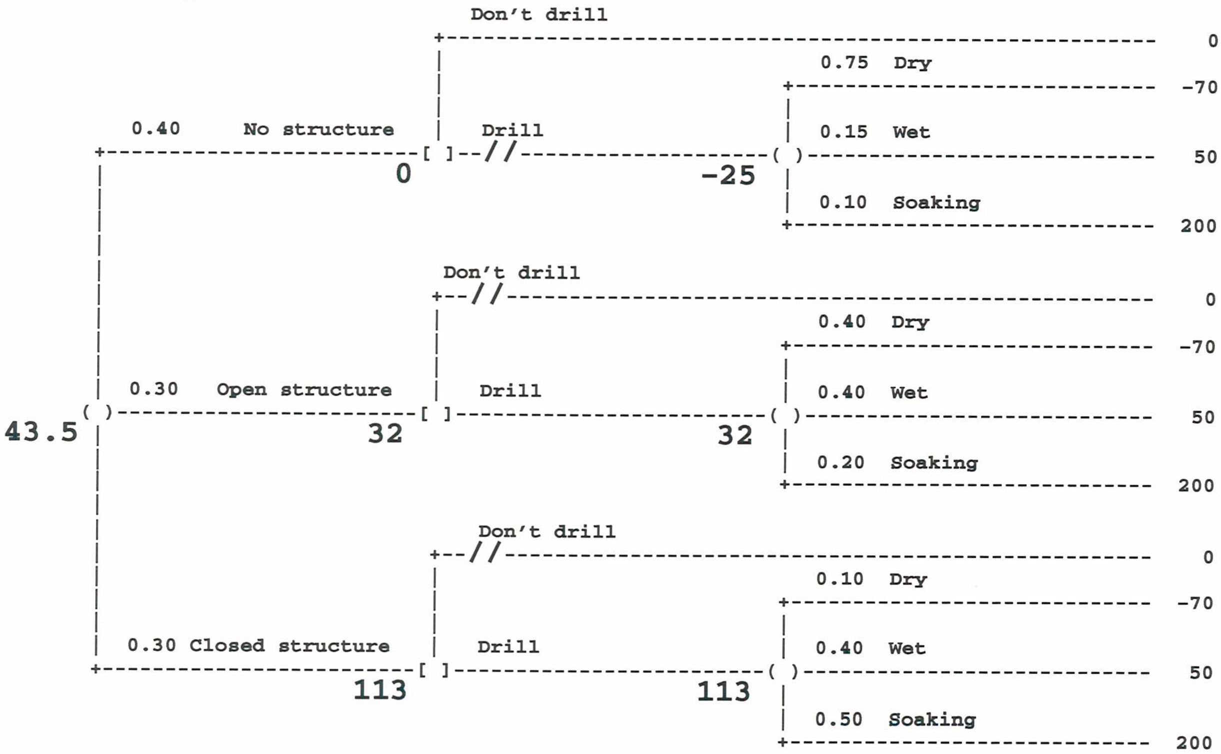
AN OIL DRILLING PROBLEM SOLUTIONS

1) Without soundings



He should drill, with EMV = $33,500

2) With soundings



With no structure, he should let the option expire; with open or closed structure, he should drill The resulting EMV is $43,500.

3) EVSI – will take soundings so long as cost does not exceed 10 (43.5 – 33.5), i.e., $10,000

```
           No Soundings
       +-------------------    33.5
       |
      [ ]
       |   Soundings
       +-------------------    43.5
```

4) EVPI – willing to pay up to $31,500 (31.5 = 65 – 33.5) for perfect information

```
                                  Don't drill
                              +-------------     0
           0.45  Dry          |
    +----------------[ ]
    |                    0 |  Drill
    |                      +--//--------  -70
    |
    |                          Don't drill
    |                      +--//--------     0
    |  0.30  Wet           |
   ( )----------------[ ]
65  |                   50 |  Drill
    |                      +-------------    50
    |
    |                          Don't drill
    |                      +--//--------     0
    |  0.25  Soaking       |
    +----------------[ ]
                   200 |  Drill
                       +-------------   200
```

4. Bayes' Law

When we did the Oil Drilling Problem, the probabilities of the states of nature (Dry, Wet, or Soaking) changed on the basis of the information obtained from the seismic soundings. In that problem, and in the Snow Fun Ski Resort problem in the Decision Trees note, we were given the "revised" probabilities, but sometimes we need to compute them. Let's see how that is done.

Consider the following scenarios:

1. I have 2 urns, A and B. One contains 100 black beads; the other contains 100 white beads.

   A. Q: With no further information, what is P(Urn A contains 100 black beads)?
   A: ½

   B. Q: I pick a bead at random from urn A. It's black.
   What is P(Urn A contains 100 black beads | a black bead was drawn)?
   A: 1

   We took a **sample** and it gave us **information** which enabled us to **revise** our probabilities. In fact, in this example, the sample (or experimental) information removed all uncertainty from the situation.

2. Now suppose I have 2 urns, A and B. One contains 90 black and 10 white beads; the other contains 30 black and 70 white beads.

   A. Q: With no further information, what is P(Urn A contains 90 black and 10 white beads)?
   A: ½

   B. Q: I pick a bead at random from urn A. It's black.
   What is P(Urn A contains 90 black and 10 white beads | a black bead was drawn)?
   A: I don't know exactly, but intuition says it should be more than ½.

   Again, our **sample** gives **information** which should enable us to **revise** our probabilities. Let's how we compute such **revised** (or **posterior**) probabilities and where they might be useful.

- We are interested in situations where there are several possible **states of nature** ($S_1, S_2, \ldots, S_n$
  1. (Urn example): A has 90B, 10W **or** A has 30B, 70W
  2. (Oil drilling): The parcel of land is dry, wet, **or** soaking
  3. (Snow Fun): Snowfall will be > 40", between 20" and 40", **or** < 20"

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- We don't know which is the **true** state of nature (which one actually occurs), but we have a probability assessment (**prior probabilities**, $P(S_j)$).
- We can **perform an experiment** (**take a sample**) to get more information about the true state of nature:
  1. (Urn example): Draw a bead from Urn A.
  2. (Oil drilling): Take seismic soundings.
  3. (Snow Fun): Buy a weather forecast.

- The experiment can have different **results** (**experimental outcomes**) ($E_1, E_2, \ldots, E_m$):
  1. (Urn Example): The chosen bead is black **or** white.
  2. (Oil drilling): The soundings indicate no structure, open structure, **or** closed structure.
  3. (Snow Fun): The weather forecast will be for above normal **or** below normal snowfall.

In examples 2 and 3, we were given the **revised (** or **posterior)** probabilities:

$P(S_j \mid E_i)$ = probability of the state of nature (S_j), conditional on the experimental outcome (E_i)

But example 1 is different. What's going on there?

- We don't know the revised probabilities ($P(S_j \mid E_i)$), but we do know the **likelihoods**:

$P(E_i \mid S_j)$ = probability of the experimental outcome (E_i), conditional on the states of nature (S_j

In our example, P(drawing a **black** bead | urn A contains 90 black and 100 white beads) = 0.9, More generally, these likelihoods will be known either from past experience or, as in our exam from first principles.

- Given the results of the experiment, we want to compute the revised probabilities. The rule for computing these revised probabilities is called **Bayes's Law** and is derived in the diagram on tl next page. (See page 13 of the Course Pack and sheet 2 of Bayes.xls in the Notes directory.)

All quantities in the final equation are known to us, because they're all either **prior** probabilities ($P(S_j)$) or likelihoods ($P(E_i \mid S_j)$). The formula is somewhat foreboding, but it's easy to apply (by hand or in Excel) if we tabulate it as follows:

STATE	PRIOR	LIKELIHOOD	JOINT [$P(S_j \& E_i)$]	REVISED [$P(S_j \mid E_i)$]
S_1	$P(S_1)$	$P(E_i \mid S_1)$	$P(S_1)P(E_i \mid S_1)$	$P(S_1 \& E_i)/P(E_i)$
S_2	$P(S_2)$	$P(E_i \mid S_2)$	$P(S_2)P(E_i \mid S_2)$	$P(S_2 \& E_i)/P(E_i)$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
S_n	$P(S_n)$	$P(E_i \mid S_n)$	$P(S_n)P(E_i \mid S_n)$	$P(S_n \& E_i)/P(E_i)$
			$P(E_i) = \Sigma_j P(S_j)P(E_i \mid S_j)$	
Known			Computed	

Let's apply this to our urn example:

- S_1: urn A contains 90 black and 10 white beads
- S_2: urn A contains 30 black and 70 white beads
- E: a black bead was drawn

State	Prior	Likelihood	Joint	Revised
90B, 10W	1/2	9/10	9/20	9/12 = 3/4
30B, 70W	1/2	3/10	3/20	3/12 = 1/4
			12/20 = P(B)	

Suppose we replaced the black bead, shook the urn and drew another bead, which also turned out t be black. How does this further revise our probabilities?

First of all, note that the revised probabilities from the first sample are now the prior probabilities going into the second sample!

State	Prior	Likelihood	Joint	Revised
90B, 10W	3/4	9/10	27/40	27/30 = 9/10
30B, 70W	1/4	3/10	3/40	3/30 = 1/10
			$30/40 = P(B_2 \mid B_1)$	

Now let's apply this to the Medical Diagnosis Problem (see the example on page 25)

States of nature	Prior probabilities
S_1: Person is HIV positive	P(HIV +) = 0.01
S_2: Person is not HIV positive	P(HIV –) = 0.99

Experimental outcomes	Likelihoods	
	$P(E_i \vert S_1)$	$P(E_i \vert S_2)$
E_1 = + : Test gives + indication	0.99	0.05
E_2 = – : Test gives – indication	0.01	0.95

a) Revised probabilities given a + indication on the test:

State	Prior	Likelihood	Joint	Revised
HIV +	0.01	0.99	0.0099	99/594 = 0.1667
HIV –	0.99	0.05	0.0495	495/594 = 0.8333
			0.0594 = P(+)	

So there's only about 1 chance in 6 that he actually is HIV positive.

b) Revised probabilities given a – indication on the test:

State	Prior	Likelihood	Joint	Revised
HIV +	0.01	0.01	0.0001	1/9406 = 0.0001
HIV –	0.99	0.95	0.9405	9405/9406 = 0.9999
			0.9406 = P(–)	

Let's look at the quality control example (see the example on page 25):

States of nature	Prior probabilities
S_1: Good operating condition	P(Good) = 0.75
S_2: OK operating condition	P(OK) = 0.24
S_3: Bad operating condition	P(Bad) = 0.01

	Likelihoods		
Experimental outcomes	$P(E_i \mid S_1)$	$P(E_i \mid S_2)$	$P(E_i \mid S_3)$
E_1: Selected diskette satisfactory	0.999	0.998	0.995
E_2: Selected diskette defective	0.001	0.002	0.005

a) Revised probabilities given a defective diskette is selected:

State	Prior	Likelihood	Joint	Revised
Good	0.75	0.001	0.00075	75/128 = 0.5859
OK	0.24	0.002	0.00048	48/128 = 0.3750
Bad	0.01	0.005	0.00005	5/128 = 0.0391
			0.00128 = P(defective)	

So P(Good or OK | defective) = 0.9609 > 0.90 → he shouldn't recalibrate.

b) Revised probabilities given two defective diskettes are selected:

State	Prior	Likelihood	Joint	Revised
Good	0.75	0.000001	0.00000075	75/196 = 0.38625
OK	0.24	0.000004	0.00000096	96/196 = 0.48980
Bad	0.01	0.000025	0.00000025	25/196 = 0.12755
			0.00000196 = P(2 of 2 defective)	

So P(Good or OK | 2 of 2 are defective) = 0.87245 < 0.90 → he should recalibrate.

BAYES' LAW EXAMPLES

1. MEDICAL DIAGNOSIS: Assume that one percent of the population of Orange County are HIV positive. The Public Health Department has an inexpensive diagnostic test which it administers to folk who ask to be tested for HIV. The test either results in a positive indication or a negative indication. T diagnostic test has the following characteristics:

   a) The test gives a (true) positive indication 99 percent of the time to people who are indeed HIV positive. Unfortunately, the test gives a (false) negative indication to 1 percent of the people who a HIV positive.

   b) The test gives a (true) negative indication 95 percent of the time to people are HIV negative (and th gives a (false) positive indication 5 percent of the time to these people).

   Suppose a randomly chosen resident of Orange County has just taken the diagnostic test. To his shock, the test has resulted in a positive indication. What do you think is the probability that he really is HIV positive. Suppose the test had resulted in a negative indication. Should he be confident that he really isn't HIV positive?

   How do your answers change if the test becomes 10 times more reliable, that is, if the error rates for fal negatives and false positives, respectively, are reduced to 0.1% and 0.5%?

2. QUALITY CONTROL: The DataSafe Corporation produces 3.5" data diskettes. Their manufacturing process is targeted to produce defects at a rate of only 1/10 of 1%. Sometimes, when the process is started up at the beginning of the day, it slips slightly and produces defects at a rate of 2/10 of 1%. Sin stopping the machinery to recalibrate the process is quite expensive, DataSafe is willing to operate at tl slightly higher rate of defects. On occasion, however, the process drifts so badly that it produces defec at the rate of 5/10 of 1%, which is unacceptable to management. On the basis of past experience, John Parkhill, DataSafe's QC engineer, has noted that on any given day, the probabilities for the true defect rate are:

Operating condition	Good	OK	Bad	
Defect rate		.001	.002	.005
Probability		.75	.24	.01

At the beginning of each day, he samples some of the initial output to decide if the machine needs to be recalibrated. His operating rule is to recalibrate whenever he cannot be at least 90% certain that the defect rate is below 5/10 of 1%.

Suppose that a randomly chosen diskette is defective. Should Parkhill stop the line for recalibration? Wha if both of two randomly chosen diskettes are defective?

SOLUTION TO HIV DIAGNOSIS PROBLEM

With original reliability:

Given a positive indication:

State	Prior	Likelihood	Joint	Revised
HIV +	0.01	0.99	0.01(0.99) = 0.0099	99/594 = 0.1667
HIV -	0.99	0.05	0.99(0.05) = 0.0495	495/594 = 0.8333
			P(+) = 0.0594	

Given a negative indication:

State	Prior	Likelihood	Joint	Revised
HIV +	0.01	0.01	0.01(0.01) = 0.0001	1/9406 = 0.0001
HIV -	0.99	0.95	0.99(0.95) = 0.9405	9405/9406 = 0.9999
			P(+) = 0.9406	

If the test is 10 times more reliable:

Given a positive indication:

State	Prior	Likelihood	Joint	Revised
HIV +	0.01	0.999	0.01(0.999) = 0.00999	999/1494 = 0.6687
HIV -	0.99	0.005	0.99(0.005) = 0.00495	495/1494 = 0.3313
			P(+) = 0.01494	

Given a negative indication:

State	Prior	Likelihood	Joint	Revised
HIV +	0.01	0.001	0.01(0.001) = 0.00001	1/98506 = 0.00001
HIV -	0.99	0.995	0.99(0.995) = 0.98505	98505/98506 = 0.99999
			P(+) = 0.98506	

SOLUTION TO QUALITY CONTROL PROBLEM

States of Nature	Prior Probabilities
S_1: Good operating condition	P(Good) = 0.75
S_2: OK operating condition	P(OK) = 0.24
S_3: Bad operating condition	P(Bad) = 0.01

	Likelihoods		
Experimental outcomes	$P(E_j \mid S_1)$	$P(E_j \mid S_2)$	$P(E_j \mid S_3)$
E_1: Selected diskette satisfactory	0.999	0.998	0.995
E_2: Selected diskette unsatisfactory	0.001	0.002	0.005

a) Revised probabilities given that a defective diskette is selected:

State	Prior	Likelihood	Joint	Revised
Good	0.75	0.001	0.00075	75/128 = 0.5859
OK	0.24	0.002	0.00048	48/128 = 0.3750

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Bad	0.01	0.005	0.00005	5/128 = 0.0391
			0.00128 = P(defective)	

So P(Good or OK | defective) = 0.9609 > 0.90 → he shouldn't recalibrate.

Note the implications of this. If the chosen disk is defective he won't recalibrate, so surely if it isn' defective, he won't recalibrate. Hence sampling only a single disk is useless.

b) Revised probabilities given that two defective diskettes are selected:

State	Prior	Likelihood	Joint	Revised
Good	0.75	0.000001	0.00000075	75/196 = 0.38625
OK	0.24	0.000004	0.00000096	96/196 = 0.48980
Bad	0.01	0.000025	0.00000025	25/196 = 0.12755
			0.00000196 = P(2 of 2 defective)	

So P(Good or OK | 2 of 2 defective) = 0.87245 < 0.90 → he should recalibrate.

I leave it to you to verify (either by running the numbers or by inference from part (a)) that he will n recalibrate if neither sampled diskette is defective or if only one of the two is defective.

4. Decision Trees using Bayes' Law

Example : GARCIA VINEYARDS

Diego Garcia, owner of Garcia Vineyards, located in the San Joaquin Valley near the small towr Hidalgo, California, pondered a difficult decision late in August, 2000. Diego raises Thompson Seedl grapes, and his problem concerned what to do with his "left-over" grapes.

The Thompson Seedless grape is a multiple-use variety that can be utilized for canning, fresh ta consumption, wine production, or for sun-drying into raisins. The acreage of canning and table grapes invariably contracted for at the beginning of the season, and then the remainder of the crop may be shifted l in the season to either wine grapes or to raisins. This is known as "going wet" or "going dry", respectively.

This decision is usually made in August, near the end of the season, and once it has been made, i irrevocable. The weather conditions after the decision is made are critical, and they are difficult to prec Raisins in Hidalgo are sun-dried completely in the open, and rain during this time can inflict heavy losses o farmer who is going dry. If the farmer is going wet, the grapes remain on the vines for several weeks lon and rain does not do as much damage.

Diego Garcia had 100 acres of "uncommitted" grapes and wished to consider two alternatives: allocate all of his acreage to raisins and 2) allocate all of the acreage to wine grapes. As for the weather, felt that he could simplify his problem by assuming that the rainfall situation would be none at all, light, heavy. Diego had available weather records for the past 20 years, and listed the rainfall information in following table:

Year	Rainfall	Year	Rainfall
1979	None	1989	Heavy
1980	Light	1990	None
1981	Heavy	1991	Heavy
1982	Light	1992	None
1983	None	1993	Light
1984	Light	1994	Light
1985	Heavy	1995	None
1986	Heavy	1996	None
1987	None	1997	Heavy
1988	None	1998	Light

He constructed the table below to show his expected dollar profit per acre under the various acre alternatives and weather conditions:

Acreage allocation

		Raisins	Wine
Weather conditions	No rain	$ 60	$ 40
	Light rain	50	30
	Heavy rain	-20	20

As Diego was pondering his decision, a Mr. Pedro T. Barnum arrived at his door, introduced himself as an Elements Prognosticator, and made the following claim:

"Mr. Garcia, I know that the possibility of rain is a crucial aspect of your decision, and I think I can h you. I have devised an elaborate and marvelous weather test, and although it's not perfect, it gives a pr good indication of what type of weather you're going to have. Of course I can't reveal the secret techni details, but the test can indicate whether the end of your growing season will be "arid" or "humid". Sounds good to be true, doesn't it? I'm sure you'll want to learn more about it.

"Now, Diego, I'm not going to try to sell you a bill of goods. I can see you're too smart for that. I'l admit that my test is not perfect. It only indicates "arid" or "humid". In those years with no rain at the end the season, it would have indicated "arid" 90 percent of the time, and "humid" 10 percent of the time. In those years with light rain at the end of the season, it would have indicated "arid" 50 percent of the time, ar "humid" 50 percent of the time. In those years with heavy rain at the end of the season, it would have indicated "arid" 30 percent of the time, and "humid" 70 percent of the time. That's pretty reliable, isn't it?

"I can see from your expression that you're still a bit skeptical. I've saved the best part for last. Sir the test isn't perfect, it's not excessively expensive. In fact, I've got a special introductory deal I'm offering j this week for fine people like you. But let's not talk price right away. I'll leave you this color brochure t summarizes what I've told you, and I'll come back tomorrow and we can talk price."

Diego Garcia raised his eyes searchingly for a moment, and then swiftly strode to the kitchen. grabbed a box of raisins in one hand, and a bottle of wine in the other, and moving quickly to his de resumed analyzing his problem.

1. Assuming Diego decides not to purchase the weather test, what strategy should he follow? Supp your answer with a decision tree.

2. Assuming Diego decides to purchase the weather test, what strategy should he follow? What is maximum amount he should be willing to pay for the test? Support your answer with a decision tree

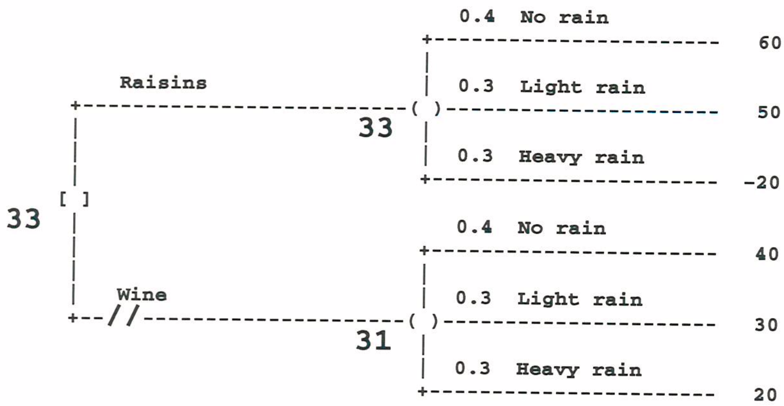
3. Suppose Diego turns Barnum down, but is approached by another Elements Prognosticator nam Claire Voyant. Claire claims to have a guaranteed test which will predict whether the end of season will see no rain, light rain, or heavy rain with perfect reliability. What is the maximum amo Diego should be willing to pay for Claire's test? Support your answer.

1. Garcia Vineyards Review the scenario

- Question 1: No Forecast – evaluate the tree
- Question 2: Forecast:
  - do Bayes's revisions (bottom of slide 1)
  - evaluate tree (bottom of slides 1 and 2)
  - compute EVSI = 100(37.6 – 33) = $460
  - Barnum comes back and hands Garcia some junk about the test ordinarily costing $1000, but since he and Garcia have become "friends," he's willing to sell it at half price. Is Garcia interested?" Of course not! He sends Barnum packing.
- Question 3: Claire Voyante and EVPI:
  - She's selling her absolutely reliable, full money-back guarantee, forecast for $1000.
  - She's an attractive young woman and Garcia accuses her of working for Barnum. ("He thinks I'm going to be a sucker for a good-looking young woman. Well those old sexist stereotypes don't apply to me! Why don't you just follow him out of here?")
  - She assures him, "I have nothing to do with that charlatan Barnum who gives us honest Elements Prognosticators a bad name. My test costs $1000, with no discount for 'friends,' and it has a money-back guarantee. You can't lose."
    - evaluate tree
    - compute EVPI = 100(45-33) = $1200, so Claire's price of $1000 is attractive.

GARCIA VINEYARDS SOLUTIONS

1) Without Barnum's weather test



He should "go dry," i.e., allocate the grapes to raisins, with EMV = $33/acre or $3300 in total.

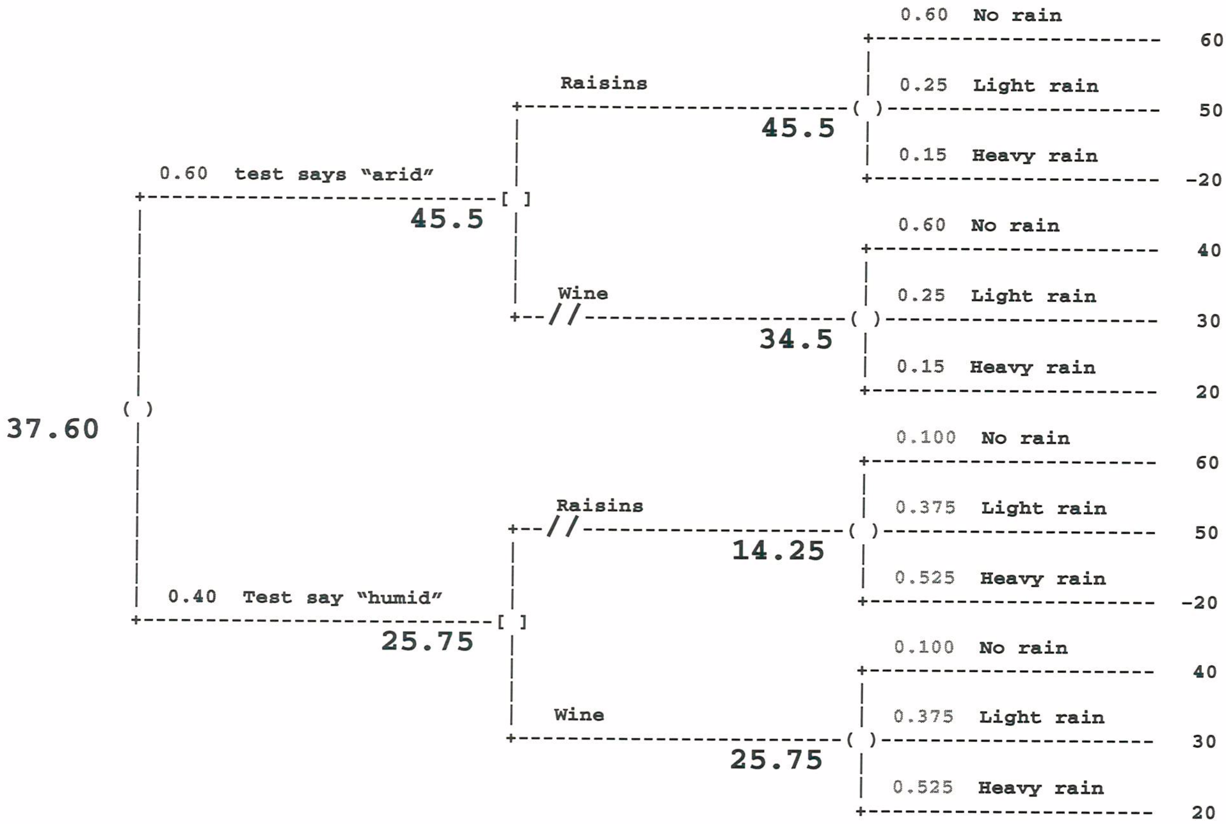
2) With Barnum's weather test, we first need to use Bayes's Law to get the revised probabilities:

If Barnum's test says "arid"

State	Prior	Likelihood	Joint	Revised
No	0.4	0.9	0.36	0.60
Light	0.3	0.5	0.15	0.25
Heavy	0.3	0.3	0.09	0.15
			0.60 = P("arid")	

If Barnum's test says "humid"

State	Prior	Likelihood	Joint	Revised
No	0.4	0.1	0.04	0.100
Light	0.3	0.5	0.15	0.375
Heavy	0.3	0.7	0.21	0.525
			0.40 = P("humid")	



If the test says "arid," he should "go dry." If the test says "humid," he should "go wet."
The resulting EMV is $37.60/acre, or $3760 in total.
He'll pay up to $460 (= $3760 – $3300) for Barnum's test.

3) EVPI – He'd be willing to pay up to $1200 (= $4500 – $3300) for Claire Voyante's test:

```
                                       Raisins
                                     +-------------   60
             0.4  No rain            |
        +--------------------[ ]
        |                     60 |  Wine
        |                        +--//--------   40
        |
        |                              Raisins
        |                        +-------------   50
        |   0.3  Light rain      |
       ( )--------------------[ ]
   45   |                     50 |  Wine
        |                        +--//--------   30
        |
        |                              Raisins
        |                        +--//--------  -20
        |   0.3  Heavy rain      |
        +---------    --------[ ]
                              20 |  Wine
                                 +-------------   20
```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5. Decision Trees: Sensitivity Analysis

1. Sensitivity Analysis in Decision Trees

Frequently we are uncertain about some of the numbers that go into a decision tree, either the payoffs or the probabilities. Such uncertainty can arise because market conditions not under our control lead to changes or because we just "guesstimated" the figures to begin with. In such case it is useful to see how the optimal policy and payoff change as those uncertain figures vary. This called **sensitivity analysis** (or, somewhat more colorfully, **what-if analysis**).

Consider Christie Stem's decision about how to operate Snow Fun Ski Resort below:

Christie Stem, owner and operator of the Snow Fun Ski Resort is trying to decide how the resort should be run in the coming season. Her profits for the skiing season will depend on how much snow falls during th winter months. On the basis of previous experience, she believes the distribution of snowfall and the resulting profit is:

Amount of Snow	Profit	Probability
> 40 inches ("High")	$120,000	0.4
20 - 40 inches ("Medium")	$40,000	0.2
< 20 inches ("Low")	-$40,000	0.4

Christie is contemplating an offer she has just received from a major hotel chain. They will operate the res for the following winter, guaranteeing Christie a $45,000 profit. She has also been considering leasing snc making equipment for the season. If she does so, the resort will be able to operate full-time, regardless of t amount of natural snowfall. If she decides to use snow makers, her profit for the season will be $120,000 minus the cost of leasing and operating the snow makers. The leasing cost will be about $12,000 for the season. The operating cost will be $10,000 ($50,000, $90,000) if the natural snowfall is High (Medium, Low).

A. In the base case:

- the snowmaker lease costs $12K
- the snowmaker operating costs are $10K, $50K, or $90K, depending on the snowfall
- these are all included in the first tree on page 21 of the Course Pack
- the $98K payoff for using the snowmaker when snowfall is high is computed as

120 – 12 – 10 = profit with high snowfall – snowmaker lease cost – snowmaker operating cost

- the $58K & $18K payoffs are similarly calculated (with operating costs of $50K & $90K)

- the optimal decision is for Christie to operate the resort by herself, using the snowmaker.

B. Now suppose Christie is uncertain about the cost of the snowmaker lease, but she's sure it will something between \$0 and \$30K. How does this affect her optimal decision and payoff?

- First of all, notice that this uncertainty about the snowmaker lease cost affects only the branch where she uses the snowmaker! Looking at the other two branches, we see that the \$45K return from letting the hotel chain operate Snow Fun is more than her expected value of \$40K if she operates the resort by herself, without the snowmaking equipment. So the question to be resolved is when to let the hotel chain operate the resort and when to do it herself, using the snowmaker.

- We let the leasing cost be \$1000X, and we now the payoff for using the snowmaker when snowfall is high is computed as

$$120 - X - 10 = 110 - X$$

Similarly, the 58 and 18 payoffs in the medium and low snowfall states become $70 - X$ and $30 - X$. These are shown in blue on the 2nd tree in the SnowFun.doc file.

- Now we just average out and fold back using algebra instead of arithmetic. For the "operate by self, with snow maker" branch, the EMV is

$$0.4(110 - X) + 0.2(70 - X) + 0.4(30 - X) = (44 + 14 + 12) - (0.4 + 0.2 + 0.4)X = 70 - X$$

- The decision depends on the value of X and whether 45 or $70 - X$ is larger:

$0 \leq X \leq 25$ (i.e., lease cost between \$0 and \$25K) → operate by self with snowmaker, EMV = $(70 - X)$ thousand dollars

$25 \leq X \leq 30$ (i.e., lease cost between \$25K and \$30K) → let the hotel chain operate Snow Fun, EMV = \$45K

(Note that at a lease cost of exactly \$25K, she is indifferent between the two policies.)

- In this case, her initial \$12K "guesstimate" is probably good enough for her to be confident that her decision in (A) was correct.

C. Now suppose that the snowmaker operating costs change (increase or decrease) across-the-board by a fixed percent. How does this affect her optimal decision and payoff?

- Once again the uncertainty about the snowmaker operating cost affects only the branch where she uses the snowmaker, so the question to be resolved is when to let the hotel chain operate the resort and when to do it herself, using the snowmaker.

- Now we let the costs be 10X, 50X, and 90X, where, for example, $X = 1$ corresponds to no change, $X = 1.1$ corresponds to a 10% increase in operating costs, and $X = 0.95$ corresponds to a 5% decrease in operating costs.

- The payoff for using the snowmaker when snowfall is high is computed as

$$120 - 12 - 10X = 108 - 10X$$

Similarly, the 58 and 18 payoffs in the medium and low snowfall states become 108 – 50X and 108 – 90X. These are shown in blue on the 3rd tree in the SnowFun.doc file.

- Again, we average out and fold back using algebra instead of arithmetic. For the "operate by self, with snow maker" branch, the EMV is

$$0.4(108 - 10X) + 0.2(108 - 50X) + 0.4(108 - 90X) = 108 - 4X - 10X - 36X = 108 - 50X$$

- The decision again depends on the value of X and whether 45 or $108 - 50X$ is larger:

$0 \le X \le 1.26$ (operating cost increases by at most 26%) → operate by self with snowmaker, EMV = (70 – X) thousand dollars

$X \ge 1.26$ (operating cost increases by at least 26%)) → let the hotel chain operate Snow Fun, EMV = $45K

(Note that at an increase of exactly 26%, she is indifferent between the two policies.)

D. Now suppose that the probability of medium snowfall is fixed at 0.2, but the probability of low snowfall varies between 0 and 0.8 (while that of high snowfall varies, *mutatis mutandis*, between 0.8 and 0). How does this affect her optimal decision and payoff?

- Now the uncertainty about the snowfall probabilities affects both of the branches where she operates the resort by herself, so we cannot, *a priori*, eliminate one of the decisions.

- Let the probabilities of low, medium, and high snowfalls be p, 0.2, and $0.8 - p$. These are shown in blue on the 4th tree in the SnowFun.doc file.

- Again, we average out and fold back using algebra instead of arithmetic. For the "operate by self, without snow maker" branch, the EMV is

$$(0.8 - p)(120) + 0.2(40) + p(-40) = 96 - 120p + 8 - 40p = 104 - 160p$$

and for the "operate by self, with snow maker" branch, the EMV is

$$(0.8 - p)(98) + 0.2(58) + p(18) = 78.4 - 98p + 11.6 + 18p = 90 - 80p$$

- The best way to see what's going on here, when we have to consider all three possible actions is to do a graph of their EMV's vs. p, the probability of low snowfall.

A) Base case

```
PAYOFF         OPERATING DECISION   PAYOFF          SNOWFALL   PROB   PAYOFF
======         ==================   ======          ========   ====   ======

          +--- LET HOTEL OPERATE ------------------------------         45
          |
          |                                    +---   >40"      .4     120
          |    OPERATE BY SELF,                |
  58 --[ ]--    W/O SNOW-MAKER         40 --( )--   20-40"     .2      40
          |                                    |
          |                                    +---   <20"      .4     (40)
          |
          |                                    +---   >40"      .4      98
          |    OPERATE BY SELF,                |
          +---  WITH SNOW-MAKER        58 --( )--   20-40"     .2      58
                                               |
                                               +---   <20"      .4      18
```

B) Cost of snowmaker lease is $1000x

```
PAYOFF         OPERATING DECISION   PAYOFF          SNOWFALL   PROB   PAYOFF
======         ==================   ======          ========   ====   ======

          +--- LET HOTEL OPERATE ------------------------------         45
          |
          |                                    +---   >40"      .4     120
          |    OPERATE BY SELF,                |
  58 --[ ]--    W/O SNOW-MAKER         40 --( )--   20-40"     .2      40
          |                                    |
          |                                    +---   <20"      .4     (40)
          |
          |                                    +---   >40"      .4     ~~98~~ 120 - X - 10
          |    OPERATE BY SELF,     70 - X     |
          +---  WITH SNOW-MAKER     ~~58~~ --( )--   20-40"     .2     ~~58~~ 120 - X - 50
                                               |
                                               +---   <20"      .4     ~~18~~ 120 - X - 90
```

C) Snowmaker operating costs change by a uniform factor

```
PAYOFF          OPERATING DECISION  PAYOFF          SNOWFALL  PROB  PAYOFF
======          ==================  ======          ========  ====  ======

        +--- LET HOTEL OPERATE ------------------------------        45
        |
        |                                       +---   >40"      .4    120
        |    OPERATE BY SELF,                   |
  58 --[ ]-- W/O SNOW-MAKER         40 --( )--  20-40"    .2     40
        |                                       |
        |                                       +---   <20"      .4    (40)
        |
        |                                       +---   >40"      .4    98 120 - 12 - 10X
        |    OPERATE BY SELF,    108 - 50X      |
        +--- WITH SNOW-MAKER            58 --( )--  20-40"    .2     58 120 - 12 - 50X
                                                |
                                                +---   <20"      .4    18 120 - 12 - 90X
```

D) Probability of moderate snowfall (20-40") fixed at .2; probability of low (high) snowfall varies from 0 .8 (.8 to 0).

```
PAYOFF          OPERATING DECISION  PAYOFF          SNOWFALL  PROB  PAYOFF
======          ==================  ======          ========  ====  ======

        +--- LET HOTEL OPERATE ------------------------------        45
        |
        |                                                  .8 - p
        |                                       +---   >40"      .4    120
        |    OPERATE BY SELF, 104 - 160p        |
  58 --[ ]-- W/O SNOW-MAKER         40 --( )--  20-40"    .2     40
        |                                       |                p
        |                                       +---   <20"      .4    (40)
        |
        |                                                  .8 - p
        |                                       +---   >40"      .4    98
        |    OPERATE BY SELF,    90 - 80p       |
        +--- WITH SNOW-MAKER            58 --( )--  20-40"    .2     58
                                                |                p
                                                +---   <20"      .4    18
```


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Decision Trees: Sensitivity Analysis

More sensitivity examples.

DECISION TREE SENSITIVITY PROBLEMS

1. In late August of 1992, Noelle Gripes faced a dilemma. As coordinator for BA 24 during the semester, she needed to hire a grader. There were two potential candidates: Sally Adams graded for Noelle the previous year, and she knew Sally was well qualified for the job. Howe because of her prior experience, Noelle would have to pay Sally $500. Her colleague, D Ribbon, had recommended Mary White as a potential grader. If Mary did qualify for the j Noelle would have to pay her only $400. To qualify as a BA 24 grader, Mary would have to p two tests: a verbal test (V) and a quantitative test (Q). It would cost $35 to administer the ver test and $40 to administer the quantitative test. Noelle estimated that Mary had a 75% chance passing the verbal test and an 80% chance of passing the quantitative test. If she was interestec hiring Mary, Noelle would administer the tests sequentially. If Mary failed the first test, No would hire Sally. If Mary passed the first test, Noelle would give her the second test. Then Mary passed the second test, the grading job would be hers; but if Mary failed the second test, tl Noelle would hire Sally. Because the School's budget was tight, Noelle wanted to <u>minimize</u> cost of getting a grader for BA 24. She constructed the following decision tree to help her dec whether to just hire Sally (S) or to see if Mary (M) was qualified for the job.

```
   Hire S
 +----------------------------------------------------------------------------- 500
 |
 |                          .25  M fails, hire S
 |                         +----------------------------------------------- 535
 |                         |
[ ]            V 1st       |                         .20  M fails, hire S
 |            +---------( )                         +------------------------ 575
 |            |            | .75  M passes, give Q   |
 |            |            +------------------------( )
 |            |                                     | .80  M passes, hire M
 |  Test M    |                                     +------------------------ 475
 +------------ [ ]
              |             .20  M fails, hire S
              |            +----------------------------------------------- 540
              |            |
              | Q 1st      |                         .25  M fails, hire S
              +---------( )                         +------------------------ 575
                           | .80  M passes, give V   |
                           +------------------------( )
                                                    | .75  M passes, hire M
                                                    +------------------------ 475
```

A. What are Noelle's optimal strategy and expected cost?

Answer the following questions independently of each other.

B. The University is considering increasing salaries for experienced graders. They are consider raises of up to 10%. How will Noelle's strategy and expected cost change as Sally's pay vari from \$500 to \$550? (Let her new salary be 500 + x.)

C. Noelle is somewhat uncertain about how likely it is that Mary will pass the quantitative test. How will Noelle's strategy and expected cost change as that probability varies from .50 to 1.0 (Let the new probability be p.)

```
      Hire S
  +--------------------------------------------------------------------- 500
  |
  |                          .25  M fails, hire S
  |                        +-------------------------------------------- 535
  |                        |
 [ ]             V 1st     |                            M fails, hire S
  |            +--------( )                          +------------------ 575
  |            |           | .75  M passes, give Q    |
  |            |           +------------------------( )
  |            |                                       |   M passes, hire M
  |   Test M   |                                       +------------------ 475
  +----------- [ ]
               |                M fails, hire S
               |           +-------------------------------------------- 540
               |           |
               | Q 1st     |                         .25  M fails, hire S
               +--------( )                          +------------------ 575
                           |    M passes, give V     |
                           +------------------------( )
                                                     | .75  M passes, hire M
                                                     +------------------ 475
```

2. You and a friend attend a Carolina basketball game. As one of the players approaches the free throw line for two shots, your friend suggests a friendly wager on the outcome. For every basket the player makes, he will pay you $5; for every basket the player misses, you will pay him $10. Luckily, the opposing team calls a time-out, so you have time to ponder your decision. You kno that this particular player makes both free throws 35% of the time, makes one and misses one 55 of the time, and misses both 10% of the time. On the back of your program, you develop the following decision tree:

```
                                      .35     Player makes both free throws
                                    +----------------------------------------    10
                                    |
           Accept wager             |  .55     Player makes only 1 free throw
 +-------------------------------( )----------------------------------------    -5
 |                                  |
 |                                  |  .10     Player misses both free throws
[ ]                                 +----------------------------------------   -20
 |
 |         Decline wager
 +---------------------------------------------------------------------------    0
```

A. You base your decisions in situations like this on expected monetary value (EMV). Should yo accept or decline the wager? Explain.

B. Suppose your friend is rethinking the bet. He will pay you some amount, X, between $5 and $10 for each shot the player makes, but you still pay him $10 for each shot the player misses. For what values of X would you accept the wager? Complete the tree below and use it to support your answer.

```
                                      .35     Player makes both free throws
                                    +----------------------------------------
                                    |
           Accept wager             |  .55     Player makes only 1 free throw
 +-------------------------------( )----------------------------------------
 |                                  |
 |                                  |  .10     Player misses both free throws
[ ]                                 +----------------------------------------   -20
 |
 |         Decline wager
 +---------------------------------------------------------------------------    0
```

C. Now suppose that you are not entirely sure about the shooting percentages for the player at th foul line. However, you believe that he will make exactly one of the shots with probability .5 Let the probability that he makes both shots be denoted by p. Use the tree below to determin for which values of p (between 0 and .45) you will accept your friend's original wager (that is you receive $5 for each shot the player makes, but lose $10 for each shot he misses).

```
                                    p     Player makes both free throws
                                  +---------------------------------------  10
                                  |
          Accept wager            |  .55  Player makes only 1 free throw
  +-------------------------( )---------------------------------------  -5
  |                               |
  |                               |       Player misses both free throws
 [ ]                              +---------------------------------------  -20
  |
  |       Decline wager
  +-------------------------------------------------------------------  0
```

Another friend is also attending the game. She knows a lot about the game and makes fairly decen predictions about how many shots the player will make. She gives you some historical informatio about the accuracy of her predictions. Although the back of your program is getting pretty cluttere you run her information through Bayes' law, and come up with the following tree:

```
                                                      .700  Player makes both free throws
                                                     +----------------------------------   10
                                                     |
                                  Accept wager       | .275  Player makes only 1 free throw
                                +-------------------( )---------------------------------   -5
             She predicts       |                    |
        .40  2 will be made     |                    | .025  Player misses both free throws
      +-------------------------[ ]                  +----------------------------------  -20
      |                         |
      |                         |   Decline wager
      |                         +-------------------------------------------------------    0
      |
      |                                               .078  Player makes both free throws
      |                                              +----------------------------------   10
      |                                              |
      |                           Accept wager       | .856  Player makes only 1 free throw
      |                         +-------------------( )---------------------------------   -5
      |      She predicts       |                    |
      | .45  1 will be made     |                    | .066  Player misses both free throws
     ( )------------------------[ ]                  +----------------------------------  -20
      |                         |
      |                         |   Decline wager
      |                         +-------------------------------------------------------    0
      |
      |                                               .233  Player makes both free throws
      |                                              +----------------------------------   10
      |                                              |
      |                           Accept wager       | .367  Player makes only 1 free throw
      |                         +-------------------( )---------------------------------   -5
      |      She predicts       |                    |
      | .15  0 will be made     |                    | .400  Player misses both free throws
      +-------------------------[ ]                  +----------------------------------  -20
                                |
                                |   Decline wager
                                +-------------------------------------------------------    0
```

Your friend will predict the results of the two foul shots if you agree to give her half of your expected gains from using the prediction to decide whether or not to accept your first friend's wager. Analyze the tree above to answer the following questions:

D. If you use the prediction, what will your optimal strategy be?

E. How much will you pay your friend for her prediction?

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SOLUTIONS TO DECISION TREE SENSITIVITY PROBLEMS

1. In late August of 1992, Noelle Gripes faced a dilemma. As coordinator for BA 24 during the fall semester, she needed to hire a grader. There were two potential candidates: Sally Adams had graded for Noelle the previous year, and she knew Sally was well qualified for the job. However because of her prior experience, Noelle would have to pay Sally $500. Her colleague, Dave Ribbon, had recommended Mary White as a potential grader. If Mary did qualify for the job, Noelle would have to pay her only $400. To qualify as a BA 24 grader, Mary would have to pass two tests: a verbal test (V) and a quantitative test (Q). It would cost $35 to administer the verbal test and $40 to administer the quantitative test. Noelle estimated that Mary had a 75% chance of passing the verbal test and an 80% chance of passing the quantitative test. If she was interested in hiring Mary, Noelle would administer the tests sequentially. If Mary failed the first test, Noelle would hire Sally. If Mary passed the first test, Noelle would give her the second test. Then, if Mary passed the second test, the grading job would be hers; but if Mary failed the second test, then Noelle would hire Sally. Because the School's budget was tight, Noelle wanted to <u>minimize</u> the cost of getting a grader for BA 24. She constructed the following decision tree to help her decide whether to just hire Sally (S) or to see if Mary (M) was qualified for the job.

```
          Hire S
        +------------------------------------------------------------------ 500
        |
        |                              .25  M fails, hire S
        |                            +--------------------------------------- 535
        |                            |
  500 [ ]                  V 1st     |                        .20  M fails, hire S
        |                +---505-( )                        +-------------------- 575
        |                |           | .75  M passes, give Q  |
        |                |           +----------------495-( )
        |                |                                  | .80  M passes, hire M
        |   Test M       |                                  +-------------------- 475
        +-------505- [ ]
                         |           .20  M fails, hire S
                         |           +--------------------------------------- 540
                         |           |
                         | Q 1st     |                        .25  M fails, hire S
                         +---508-( )                        +-------------------- 575
                                     | .80  M passes, give V  |
                                     +----------------500-( )
                                                            | .75  M passes, hire M
                                                            +-------------------- 475
```

A. What are Noelle's optimal strategy and expected cost?

Therefore Noelle should hire Sally. Her EMV is $500

Answer the following questions independently of each other.

B. The University is considering increasing salaries for experienced graders. They are considering raises of up to 10%. How will Noelle's strategy and expected cost change as Sally's pay varies from \$500 to \$550? (Let her n salary be 500 + x.)

```
     Hire S
 +---------------------------------------------------------------------------------- 500+x
 |
 |                                         .25  M fails, hire S
 |                                        +---------------------------------------- 535+x
 |                                        |
[ ]                        V 1st          |                  .20  M fails, hire S
 |                   +----505+0.4x-( )                      +----------------------- 575+x
 |                   |                    | .75  M passes, give Q |
 |                   |                    +-----------495+0.2x-( )
 |                   |                                      | .80  M passes, hire M
 |  Test M           |                                      +----------------------- 475
 +-505+0.4x-[ ]
             |                             .20  M fails, hire S
             |                            +----------------------------------------- 540+x
             |                            |
             |             Q 1st          |                  .25  M fails, hire S
             +----508+0.4x-( )                              +----------------------- 575+x
                                          | .80  M passes, give V |
                                          +----------500+0.25x-( )
                                                            | .75  M passes, hire M
                                                            +----------------------- 475
```

Hire-Test indifference point: $500 + x = 505 + 0.4x \Rightarrow x = 8.333$
Hence, if $500.000 \le pay \le 508.333$, hire Sally, with expected cost = \$(500 + x)
if $508.333 \le pay \le 550.000$, test Mary, verbal first, with expected cost = \$(505 + 0.4x)

C. Noelle is somewhat uncertain about how likely it is that Mary will pass the quantitative test. How will Noelle's strategy and expected cost change as that probability varies from .50 to 1.00? (Let the new probability be p.)

```
     Hire S
 +----------------------------------------------------------------------------- 500
 |
 |                                    .25  M fails, hire S
 |                                   +------------------------------------------ 535
 |                                   |
[ ]                    V 1st         |              1-p   M fails, hire S
 |               +-565-75p-( )                     +----------------------- 575
 |               |                   | .75  M passes, give Q |
 |               |                   +-----------575-100p-( )
 |               |                                 | p    M passes, hire M
 |  Test M       |                                 +----------------------- 475
 +--------  [ ]
             |                       1-p   M fails, hire S
             |                       +------------------------------------------ 540
             |                       |
             |         Q 1st         |              .25  M fails, hire S
             +-540-40p-( )                         +----------------------- 575
                                     | p    M passes, give V |
                                     +-----------------500-( )
                                                   | .75  M passes, hire M
```

```
+------------------------ 475
```

Note: $540 - 40p \geq 500$ for all $p \leq 1$, so we can ignore the Test M, Q 1st branch.
Then, looking at the other two branches, $565 - 75p = 500 \Rightarrow p = 0.867$
Hence, if $0.500 \leq p \leq 0.867$, hire Sally, expected cost = \$500
if $0.867 \leq p \leq 1.000$, test Mary, verbal 1st, with expected cost = \$(565 – 75p)

2. You and a friend attend a Carolina basketball game. As one of the players approaches the free throw line for two shots, your friend suggests a friendly wager on the outcome. For every basket the player makes, he will pay you $5; for every basket the player misses, you will pay him $10. Luckily, the opposing team calls a time-out, so you have time to ponder your decision. You know that this particular player makes both free throws 35% of the time, makes one and misses one 55% of the time, and misses both 10% of the time. On the back of your program, you develop the following decision tree:

```
                                        .35   Player makes both free throws
                                       +---------------------------------------   10
                                       |
                  Accept wager         | .55   Player makes only 1 free throw
     +-----------------**-1.25**-( )---------------------------------------   -5
     |                                 |
     |                                 | .10   Player misses both free throws
 0   [ ]                               +---------------------------------------  -20
     |
     |      Decline wager
     +--------------------------------------------------------------------------   0
```

A. You base your decisions in situations like this on expected monetary value (EMV). Should you accept or decline the wager? Explain.

Decline – you expect to lose $1.25 if you accept.

B. Suppose your friend is rethinking the bet. He will pay you some amount, X, between $5 and $10 for each shot the player makes, but you still pay him $10 for each shot the player misses. For what values of X would you accept the wager? Complete the tree below and use it to support your answer.

```
                                        .35   Player makes both free throws
                                       +---------------------------------------   2x
                                       |
                  Accept wager         | .55   Player makes only 1 free throw
     +-------------1.25x-7.5-( )---------------------------------------------   x-10
     |                                 |
     |                                 | .10   Player misses both free throws
    [ ]                                +---------------------------------------  -20
     |
     |      Decline wager
     +--------------------------------------------------------------------------   0
```

You are indifferent when 1.25x – 7.5 = 0, i.e., when x = 6
Hence you'd accept the wager for $6 \leq x \leq 10$.

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C. Now suppose that you are not entirely sure about the shooting percentages for the player at the fo line. However, you believe that he will make exactly one of the shots with probability .55. L the probability that he makes both shots be denoted by p. Use the tree below to determine for which values of p (between 0 and 0.45) you will accept your friend's original wager (that is, y receive $5 for each shot the player makes, but lose $10 for each shot he misses).

```
                                          p        Player makes both free throws
                                       +----------------------------------------   10
                                       |
         Accept wager                  |  .55      Player makes only 1 free throw
+-------------30p-11.75-( )----------------------------------------   -5
|                                      |
|                                      | 0.45-p  Player misses both free throws
[ ]                                    +----------------------------------------  -20
|
|        Decline wager
+---------------------------------------------------------------------------   0
```

You are indifferent when $30p - 11.75 = 0$, i.e., when $p = 0.3917$
Hence you'd accept the wager for $0.3917 \leq p \leq 0.45$

Another friend is also attending the game. She knows a lot about the game and makes fairly decen predictions about how many shots the player will make. She gives you some historical informatior about the accuracy of her predictions. Although the back of your program is getting pretty cluttere you run her information through Bayes' law, and come up with the following tree:

```
                                                    .700  Player makes both free throws
                                                  +------------------------------------   10
                                                  |
                                Accept wager      | .275  Player makes only 1 free throw
                             +-------------5.125-( )------------------------------------   -5
                 She predicts |                   |
          .40  2 will be made |                   | .025  Player misses both free throws
          +-----------5.125-[ ]                   +------------------------------------   -20
          |                   |
          |                   |  Decline wager
          |                   +----------------------------------------------------------   0
          |
          |                                         .078  Player makes both free throws
          |                                       +------------------------------------   10
          |                                       |
          |                     Accept wager      | .856  Player makes only 1 free throw
          |                  +------------4.820-( )------------------------------------   -5
          |      She predicts |                   |
          | .45  1 will be made |                 | .066  Player misses both free throws
2.05( )------------------0-[ ]                    +------------------------------------   -20
          |                   |
          |                   |  Decline wager
          |                   +----------------------------------------------------------   0
          |
          |                                         .233  Player makes both free throws
          |                                       +------------------------------------   10
          |                                       |
          |                     Accept wager      | .367  Player makes only 1 free throw
          |                  +-----------7.505-( )-------------------------------------   -5
          |      She predicts |                   |
          | .15  0 will be made |                 | .400  Player misses both free throws
          +------------------0-[ ]                +------------------------------------   -20
                              |
                              |  Decline wager
                              +----------------------------------------------------------   0
```

Your friend will predict the results of the two foul shots if you agree to give her half of your expected gains from using the prediction to decide whether or not to accept your first friend's wager. Analyze the tree above to answer the following questions:

D. If you use the prediction, what will your optimal strategy be?

Accept the wager if she predicts both shots will be made; otherwise decline.

E. How much will you pay your friend for her prediction?

½(2.05) = $1.025

3. Jodan, Maria, and Pete were talking about operating some food concessions at the upcoming Apple Thrill street fair. Jodan had lots of experience running an Italian restaurant and he was sure that pizza and calzones would be a great hit with the fairgoers. Maria was sure that they could make a killing by selling Greek favorites like dolmades and baklava. Jodan and Maria would take care of the food details, while Pe would use his administrative skills to negotiate permits and other arrangements.

Checking with the Chapelboro Department of Recreation, Pete learned that a permit to sell Greek food the fair would cost $1000, while a permit to sell Italian food would cost $1500. However, if he gets bo permits, he will get a 20% reduction and pay only $2000. The amount of food they'll sell will depend the weather the day of the fair. Based on top secret forecasts obtained from his buddies at Rand Corporation, Pete feels that the weather will be rainy, cloudy, or sunny, with respective probabilities 0. 0.3, and 0.4. Jodan has constructed the following decision tree with payoffs given as expected food profits (as estimated by Maria) less the cost of the permits (in hundreds of dollars). The TA's optimal decision is to go with just the Greek food, and their optimal payoff is $750.

```
                 0.3  Rainy
                +---------------  10 - 10 =  0
                |
      Greek      | 0.3  Cloudy
  +-------------------( )-------------  15 - 10 =  5
  |          7.5 |
  |             | 0.4  Sunny
  |             +---------------  25 - 10 = 15
  |
  |              0.3  Rainy
  |             +---------------  10 - 15 = -5
  |             |
  |   Italian    | 0.3  Cloudy
7.5 [ ]-//----------------( )-------------  20 - 15 =  5
  |          6.8 |
  |             | 0.4  Sunny
  |             +---------------  32 - 15 = 17
  |
  |              0.3  Rainy
  |             +---------------  15 - 20 = -5
  |             |
  |   Both       | 0.3  Cloudy
  +--//----------------( )-------------  25 - 20 =  5
             7.4 |
                | 0.4  Sunny
                +--------------- 38.5 - 20 = 18.5
```

Answer the following questions independently of each other.

A. How do their optimal decision and payoff change as the reduction for getting both permits ranges from 10% to 25%? Let 25(1 – X) denote the cost of getting both permits, so that 100X is the percent reduction.

```
                  0.3  Rainy
                 +--------------- 10 - 10 =  0
                 |
      Greek          | 0.3  Cloudy
+--------------------( )------------- 15 - 10 =  5
|             7.5 |
|                 | 0.4  Sunny
|                 +--------------- 25 - 10 = 15
|
|                 0.3  Rainy
|                +--------------- 10 - 15 = -5
|                |
|     Italian        | 0.3  Cloudy
[ ]------------------( )------------- 20 - 15 =  5
|             6.8 |
|                 | 0.4  Sunny
|                 +--------------- 32 - 15 = 17
|
|                 0.3  Rainy
|                +--------------- 15 – 25(1 – X) = -10 + 25X
|                |
|     Both           | 0.3  Cloudy
+--------------------( )------------- 25 - 25(1 – X) =  25X
         2.4 + 25X |
                 | 0.4  Sunny
                 +--------------- 38.5 - 25(1 – X) = 13.5 + 25X
```

They'll switch from Greek to Both when 7.5 = 2.4 + 25X, or when X = 5.1/25 = 0.204.

Range for X	Size of discount	Decision	EMV
$0.100 \leq X \leq 0.204$	10% to 20.4%	Greek	$750
$0.204 \leq X \leq 0.250$	20.4% to 25%	Both	$(240 + 2500X)

B. Now suppose that the probability that the forecast will be for sunny weather is known to be exactly 0.4, but Pete is not sure about the probabilities of rainy and cloudy forecasts. Let *p* be the probability that t forecast will be for rainy weather. Show how the TA's optimal decision and payoff change when *p* var between 0 and 0.6.

```
                      p    Rainy
                    +------------------  10 - 10 =  0
                    |
   Greek            | 0.6 – p  Cloudy
+-------------------( )----------------  15 - 10 =  5
|          9 – 5p   |
|                   | 0.4    Sunny
|                   +------------------  25 - 10 = 15
|
|                     p    Rainy
|                   +------------------  10 - 15 = -5
|                   |
|  Italian          | 0.6 – p  Cloudy
[]------------------( )----------------  20 - 15 =  5
|        9.8 – 10p  |
|                   | 0.4    Sunny
|                   +------------------  32 - 15 = 17
|
|                     p    Rainy
|                   +------------------  15 - 20 = -5
|                   |
|  Both             | 0.6 – p  Cloudy
+-------------------( )----------------  25 - 20 =  5
                    |
       10.4 – 10p   | 0.4      Sunny
                    +------------------ 38.5 - 20 = 18.5
```

Both dominates Italian. They'll switch from Both to Greek when:
$10.4 – 10p = 9 – 5p$, or when $p = 1.4/5 = 0.28$

Range for p	Decision	EMV
$0.00 \le p \le 0.28$	**Both**	**$(1040 – 1000p)**
$0.28 \le p \le 0.60$	**Greek**	**$(900 – 500p)**

C. How do the TA's optimal decision and payoff change when the permit for Greek food varies between $1000 and $1100 (while that for Italian food remains fixed at $1500)? *Remember that Pete gets a 20% reduction on the total permit fees when they sell both Greek and Italian food.* Let 100X denote the cost of the permit for Greek food.

```
                   0.3  Rainy
                 +---------------  10 - X
                 |
   Greek         | 0.3  Cloudy
+----------------( )-------------  15 - X
|    17.5 - X    |
|                | 0.4  Sunny
|                +---------------  25 - X
|
|                  0.3  Rainy
|                +---------------  10 - 15 = -5
|                |
|  Italian       | 0.3  Cloudy
[]---------------( )-------------  20 - 15 = 5
|        6.8     |
|                | 0.4  Sunny
|                +---------------  32 - 15 = 17
|
|                  0.3  Rainy
|                +---------------  15 – 12 – 0.8X =  3 – 0.8X
|                |
|  Both          | 0.3  Cloudy
+----------------( )-------------  25 - 12 – 0.8X =  13 – 0.8X
   15.4 – 0.8X   |
                 | 0.4  Sunny
                 +--------------- 38.5 - 12 – 0.8X = 26.5 – 0.8X
```

From a graph, they'll switch from Greek to Both when

$17.5 - X = 15.4 - 0.8X$, or when $X = 2.1/0.2 = 10.5$

and they'll switch from Both to Italian when

$15.4 - 0.8X = 6.8$, or when $X = 8.6/0.8 = 10.75$

Range for X	Cost of Greek Permit	Decision	EMV
$10 \leq X \leq 10.5$	$1000 to $1050	Greek	$(1750 – 100X)
$10.5 \leq X \leq 10.75$	$1050 to $1075	Both	$(1540 – 80X)
$10.75 \leq X \leq 11$	$1075 to $1100	Italian	$680