Felipe Lopes Castro

On Partial (Co)Actions on Coalgebras: Globalizations and Some Galois Theory

Porto Alegre
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PhD thesis presented by Felipe Lopes Castro[1] in partial fulfillment of the requirements for the degree of Doctor in Mathematics at Universidade Federal do Rio Grande do Sul.

Universidade Federal do Rio Grande de Sul - UFRGS
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To my son Henri and my wife Erica.

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"Pure mathematics is, in its
way, the poetry of logical ideas."

Albert Einstein

Resumo

Módulo coálgebra parcial e comódulo coálgebra parcial são noções duais de módulo álgebra parcial, e estas estruturas são bem relacionadas. Módulo álgebra parcial foi definido por Caenepeel e Janssen em [11] e desenvolvido numa certa direção por Alves e Batista em [1–3]. Estamos interessados em algumas construções de Alves e Batista, especificamente: globalização para módulos álgebra parciais, um contexto de Morita relacionando a subálgebra dos invariantes e o produto smash parcial, e Teoria de Galois.

Neste trabalho, introduzimos a noção de globalização para módulo coálgebra parcial e para comódulo coálgebra parcial. Mostramos que todo módulo coálgebra é globalizável construindo uma globalização, chamada *standard*. No caso de comódulo coálgebra parcial, precisamos supor um tipo de racionalidade para obter uma globalização correspondente.

Mais ainda, para um comódulo coálgebra parcial, construímos um contexto de Morita-Takeuchi relacionando a coálgebra dos coinvariantes e o coproduto smash parcial, definimos uma coextensão de Galois e obtemos algumas propriedades, relacionando coextensões de Galois para comódulo coálgebras parciais com extensões de Galois para coações parciais em álgebras, estendendo resultados de Dăscălescu, Raianu e Zhang obtidos em [20].

Palavas-Chave: Álgebras de Hopf, Ação Parcial, Coação Parcial, Globalização, Teoria de Galois.

Abstract

Partial module coalgebra and partial comodule coalgebra are the dual notions of partial module algebra, and all these structures are close related. Partial module algebra was defined by Caenepeel and Janssen in [11] and developed in a certain direction by Alves and Batista in [1–3]. We are interested in some constructions made by Alves and Batista, namely: globalization for partial module algebras, a Morita context relating the invariant subalgebra and the partial smash product, and Galois theory.

In this work, we introduce the notion of globalization for partial module coalgebra and for partial comodule coalgebra. We show that every partial module coalgebra is globalizable constructing a globalization, named *standard*. For the case of partial comodule coalgebra we need assume some kind of rationality condition to obtain a correspondent globalization.

Moreover, for a partial comodule coalgebra, we construct a Morita-Takeuchi context relating the coinvariant coalgebra and the partial smash coproduct, and we define a Galois coextension and show some properties, relating the Galois coextension for partial comodule coalgebra with the Galois extension for partial coaction on algebras, extending results of Dăscălescu, Raianu, and Zhang obtained in [20].

Key-Words: Hopf Algebras, Partial Action, Partial Coaction, Globalization, Galois Theory.

Contents

Introduction

Partial action of groups was defined for the first time by Exel [23] in the context of C^*-algebras. After that, Dockuchaev and Exel [21] developed such a theory from a purely algebraic point of view, generalizing some classical results of group actions to the setting of partial actions of groups. A great development of this theory has been done since then. As an example, we can refer the development of a Galois theory for partial actions of groups.

Actions of groups on algebras was extended to actions of Hopf algebras on algebras (cf. [27, 30]), where a great theory was developed, including Galois theory and some categorical relations. Since the theory of group actions was successfully extended in two different directions, a natural question arose: *Is it possible to extend such a notion to the setting of partial action of Hopf algebras and recover the classical results in this new context?*

In order to answer this question Caenepeel and Janssen [11] introduced the notion of partial action of Hopf algebras and, after that, Alves and Batista have explored this new structure (see [1–3]) having constructed a beautiful related theory in which we would like to highlight some special aspects for our work, namely: 1 - the existence of a globalization for a given partial module algebra, extending the already done for partial action of groups (see [21]); 2 - the construction of a Morita context connecting the invariant subalgebra and the partial smash product, extending the classical Morita context (cf. [27, Theorem 4.5.3]); and 3 - a Galois theory for partial action of Hopf algebras and its relation with the Morita context, extending the classical case (see [27, Theorem 8.3.3]).

Partial module coalgebra and partial comodule coalgebra are the dual notions of partial module algebra and partial comodule algebra respectively. All these four partial structures are close related. Batista and Vercruysse studied these dual structures, showing some interesting properties between these new objects. Our aim in this work is to study the above mentioned aspects in the setting of partial module coalgebra and partial comodule coalgebra, relating our results with the corresponding results obtained by Alves and Batista.

This work is organized as follows. In the first chapter we recall some preliminary results which are necessary for a good understanding of this work, that is, some basic facts on linear algebra, as well as basic Hopf algebra theory. Finally, we recall some important aspects from the theory of partial module algebra developed by Alves and Batista.

The second chapter is devoted to the study of partial action on coalgebras. We will see the correspondence between this new structure and the partial module algebra, some examples and useful properties. Our aim in this chapter is to discuss the existence of a globalization for partial action on coalgebras, and to do this we introduce the notion of induced partial action on coalgebras. For this purpose we need to construct a comultiplicative projection satisfying an special condition (see Proposition 2.1.15). After that, we define a globalization for partial module coalgebras (see Definition 2.2.1) and also we discuss about the existence of some relations between this our globalization with the well known notion of globalization for partial module algebra, so obtaining a direct relation between such globalizations. Finally, we show that every partial module coalgebra has a globalization by building such a globalization named *standard* (see Theorem 2.2.5).

In the third chapter, we study partial coaction on coalgebras. We present some examples and important properties related to this new partial structure. We show a correspondence among all of these four partial objects studied in this work, sometimes asking for special conditions like density of the finite dual of a Hopf algebra on its dual; or finite dimensionality of the involved Hopf algebra. Similar to the Chapter 2, we define induced partial coaction under the existence of a comultiplicative projection satisfying an special condition (see Proposition 3.2.1). After that, we define globalization for partial comodule coalgebra (see Definition 3.2.2) and we relate it with the structures already defined in Chapter 2 (see Theorem 3.2.5). We show that, under a kind of rationality hypothesis, every partial comodule coalgebra is globalizable, constructing the *standard* globalization for it (see Theorem 3.2.6).

Given a comodule coalgebra C, we can associate two new coalgebras, namely, the smash coproduct and the coinvariant coalgebra that is a quotient of C by a coideal. Dăscălescu, Raianu, and Zhang in [20] constructed a Morita-Takeuchi context related with these two coalgebras and, after that, they developed the theory of Galois coextensions, obtaining some interesting properties.

In the last chapter, we extend the above mentioned results to the setting of partial comodule coalgebras. In the first section, we show that given a partial comodule coalgebra C we can consider two coalgebras associated with C, namely, the partial smash coproduct and the coinvariant coalgebra. In this case, the coinvariant coalgebra is a quotient coalgebra of C by a suitable coideal. With these two coalgebras we construct a Morita-Takeuchi context (see Theorem 4.1.14). In the second section, we define the notion of Galois coextension for partial (co)action on coalgebras (see Definition 4.2.3) and show that Galois extension implies Galois coextension (see Theorem 4.2.8). In the third section, we show that dualizing the Morita-Takeuchi context obtained in the first section, we get the classical Morita context for partial module algebras (see Theorem 4.3.3).

Chapter 1

Preliminaries

With the purpose to make this text as self-contained as possible, we include this first chapter where we will recall some basic facts from linear algebra and from Hopf algebras, including its actions on algebras, and also from Morita theory. Some references will be given in an opportune moment. The reader who has enough familiarity with these subjects can go directly to the next chapter.

1.1 Basic Linear Algebra

In this section we will remember some definitions and properties from linear algebra that will be useful in this entire work. We refer [24–26] and [28] for more details, if necessary.

During all this work $\Bbbk$ is a field, all objects are $\Bbbk$-vector spaces (i.e. algebra, coalgebra, Hopf algebra, etc mean $\Bbbk$-algebra, $\Bbbk$-coalgebra, $\Bbbk$-Hopf algebra, etc, respectively), linear map means $\Bbbk$-linear map and unadorned tensor product means $\otimes_{\Bbbk}$.

Given V, W two vector spaces and a linear map $\tau\colon V \to W$, we can consider the dual (or transpose) map which is defined by

$$\begin{aligned} \tau^*\colon W^* &\longrightarrow V^* \\ f &\longmapsto \tau^*(f) = f \circ \tau, \end{aligned}$$

and with this notation we have the following properties:

Proposition 1.1.1. *Given V, W two vector spaces, $\tau\colon V \to W$ a linear map and $\tau^*\colon W^* \to V^*$ its dual map, then we have that the following properties hold:*

(1) τ is injective if and only if τ^* is surjective;

(2) τ is surjective if and only if τ^* is injective;

(3) τ is bijective if and only if τ^* is bijective;

If V and W are coalgebras, then we have that the following additional property holds (see Definitions 1.2.4, 1.2.7 and 1.2.9):

(4) τ is a coalgebra morphism if and only if τ^* is an algebra morphism.

Given two vector spaces V and W we have the natural immersion which is defined by

$$\begin{aligned} \imath_{V,W} : V^* \otimes W^* &\longrightarrow (V \otimes W)^* \\ f \otimes g &\longmapsto \imath(f \otimes g)\left(\sum_{i=1}^{n} v_i \otimes w_i\right) = \sum_{i=1}^{n} f(v_i)\, g(w_i), \end{aligned}$$

for all $f \in V^*$, $g \in W^*$ and any $\sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W$.

Proposition 1.1.2. *Let V be a vector space, $W \subseteq V$ a subspace and $\pi_W : V \to {}^{V}\!/_{W}$ the canonical projection. Then $W^{\perp} := \{f \in V^* \mid f(W) = 0\} \simeq ({}^{V}\!/_{W})^*$ via the dual map π_W^*, i. e.,*

$$W^{\perp} = \pi_W^*(({}^{V}\!/_{W})^*).$$

1.2 Hopf Algebras

In this section we will remember some classical properties of Hopf algebras that will be useful in next chapters. As good references for Hopf algebra theory we refer to [19, 29] and [30].

Definition 1.2.1 (Algebra)**.** An *algebra* is a triple (A, m, u), where A is a vector space, $m\colon A \otimes A \longrightarrow A$ and $u\colon \Bbbk \longrightarrow A$ are a linear maps, so called respectively *multiplication* and *unity*, such that the following diagrams are commutative:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "I\otimes m"] \arrow[d, "m\otimes I"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \otimes A \arrow[d, "m"] & \\ \Bbbk \otimes A \arrow[ur, "u\otimes I"] \arrow[dr, "\simeq"'] \arrow[r, phantom, "\circlearrowleft"] & A & A \otimes \Bbbk \arrow[ul, "I\otimes u"'] \arrow[dl, "\simeq"] \arrow[l, phantom, "\circlearrowleft"] \end{tikzcd}$$

For simplicity, we will say that A is an algebra.

Remark 1.2.2. Denoting by $m(a \otimes b) = a\,b$ and by $u(1_{\Bbbk}) = 1_A$, we can translate the diagrams in Definition 1.2.1 in the following conditions, for all a, $b \in A$:

Associativity: $(a\,b)c = a(b\,c)$;

Unity: $1_A\, a = a = a\, 1_A$.

Definition 1.2.3 (Subalgebra)**.** A subspace A' of an algebra A is called a *subalgebra* of A if the image of u is contained in A' and the image of the restriction of m to $A' \otimes A'$ is contained in A', i.e.,

$$u(\Bbbk) \subseteq A'$$
$$m(A' \otimes A') \subseteq A'.$$

Definition 1.2.4. Let A, B be two algebras. Then a map $f\colon A \longrightarrow B$ is an *algebra morphism* if it is a linear map that respects the multiplication and the unity maps, that is, if the following diagrams are commutative:

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "f"] & \circlearrowleft & B \\ & \Bbbk \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

Remark 1.2.5. Using the notation for multiplication of elements of an algebra by concatenation, then the diagrams in definition of algebra morphism can be translated as

$$f(ab) = f(a)f(b)$$

and

$$f(1_A) = 1_B,$$

for all $a, b \in A$. If the map satisfy just the first equality, then it is called a *multiplicative map*.

Definition 1.2.6 (Module)**.** Given an algebra A, a vector space M and a linear map $\triangleright\colon A \otimes M \to M$, we say that $(M, \triangleright)$ is a *left A-module* if the following diagrams are commutative:

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "I \otimes \triangleright"] \arrow[d, "m_A \otimes I"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes M \arrow[d, "\triangleright"] \\ A \otimes M \arrow[r, "\triangleright"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \otimes M \arrow[dd, "\triangleright"] \\ \Bbbk \otimes M \arrow[ur, "u \otimes I"] \arrow[dr, "\simeq"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & M \end{tikzcd}$$

Equivalently, one can see the above diagrams in terms of elements, that is, denoting $\triangleright(a \otimes m) = a \triangleright m$, the above diagrams can be translated as

$$a \triangleright (b \triangleright m) = (ab) \triangleright m$$
$$1_A \triangleright m = m.$$

The notion of coalgebra is the dual notion of algebra and, for a better comprehension about such a dualization, the definition of algebra by diagrams as given before make it more clear, as follows.

Definition 1.2.7. A *coalgebra* is a triple (C, Δ, ε), where C is a vector space, $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to \Bbbk$ are linear maps, called *comultiplication* and *counity*, respectively, such that the following diagrams are commutative:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \otimes C \arrow[d, "I\otimes\Delta"] \\
C \otimes C \arrow[r, "\Delta\otimes I"] & C \otimes C \otimes C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & & C \otimes C \arrow[dll, "\varepsilon\otimes I"'] \arrow[drr, "I\otimes\varepsilon"] & & \\
\Bbbk \otimes C \arrow[r, phantom, "\circlearrowleft"] & & & & C \otimes \Bbbk \arrow[l, phantom, "\circlearrowleft"'] \\
 & & C \arrow[uu, "\Delta"] \arrow[ull, "\simeq"] \arrow[urr, "\simeq"'] & &
\end{tikzcd}$$

For simplicity, we will say that C is a coalgebra.

Remark 1.2.8. The first diagram above means that the coalgebra C is *coassociative* and the second one tell us that the coalgebra C is *counital*. Thus, if we do not assume the property derived from this last diagram we obtain a *non counital coalgebra*, and this notion correspond the dual notion of non unital algebras.

Since the tensor product $C \otimes C$ is a quotient of the vector space generated by the set $C \times C$ (cf. [26]), it follows that a typical element of $C \otimes C$ is a equivalence class which is presented as a finite sum of simple tensor of elements in C, i.e., if $x \in C \otimes C$ then there exist finite elements in C, denoted by x_i and x_i', $i = 1, \dots n$, such that

$$x = \sum_{i=1}^{n} x_i \otimes x_i'.$$

Therefore, given an element c in a coalgebra C, we have that the comultiplication of c is an element in $C \otimes C$ that can be write as

$$\Delta(c) = \sum_{i=1}^{n} c_i \otimes c_i'$$

and so the diagrams in Definition 1.2.7 can be translated in the following equalities, for all $c \in C$,

$$\textstyle\sum_{i,j} c_i \otimes (c_i')_j \otimes (c_i')_j' = \sum_{i,l} (c_i)_l \otimes (c_i)_l' \otimes c_i'$$

and

$$\begin{aligned} c &= \textstyle\sum_i c_i \varepsilon(c_i') \\ &= \textstyle\sum_i \varepsilon(c_i) c_i'. \end{aligned}$$

As it is clear from above computations, this notation is not so good to be used when we are working with Hopf algebras. Thus, since all coalgebras are coassociative, we will use the *Sigma notation*, also called *Heyneman-Sweedler notation* (cf. [30]), where the comultiplication of an element will be denoted by

$$\Delta(c) = c_1 \otimes c_2,$$

where the summation is understood.

Therefore, under this new notation, the diagrams in Definition 1.2.7 can be translated as

$$c_1 \otimes c_2 \otimes c_3 = c_1 \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes c_2 \tag{1.1}$$

$$c = \varepsilon(c_1)c_2 = c_1\varepsilon(c_2). \tag{1.2}$$

Definition 1.2.9 (Coalgebra map)**.** Let C, D be two coalgebras and $f\colon C \to D$ a linear map. We say that f is a *coalgebra morphism* if the following diagrams are commutative:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & \circlearrowright & D \arrow[dl, "\varepsilon_D"] \\ & \Bbbk & \end{tikzcd}$$

One can see that, the above diagrams can be translated, as follow, for any $c \in C$:

$$f(c)_1 \otimes f(c)_2 = f(c_1) \otimes f(c_2)$$
$$\varepsilon(f(c)) = \varepsilon(c)$$

If the map f satisfy just the first condition, then it is called a *comultiplicative map.*

Definition 1.2.10 (Coideals)**.** Let J be a subspace of a coalgebra C. Then we say that:

(1) J is a subcoalgebra of C if $\Delta(J) \subseteq J \otimes J$;

(2) J is a left (resp. right) coideal of C if $\Delta(J) \subseteq C \otimes J$ (resp. $\Delta(J) \subseteq J \otimes C$);

(3) J is a coideal of C if $\Delta(J) \subseteq J \otimes C + C \otimes J$ and $\varepsilon(J) = 0$.

Remark 1.2.11. Given a coalgebra C, its dual $C^* = \mathrm{Hom}(H, \Bbbk)$ is an algebra with product and unity given by, for any $\alpha, \beta \in C^*$, $c \in C$,

$$(f * g)(c) := f(c_1)g(c_2)$$
$$1_{C^*} := \varepsilon_C.$$

The next result is a classical one from coalgebra theory, and a proof for this can be found in [30, Propositions 1.4.3, 1.4.5 and 1.4.6].

Proposition 1.2.12. *Let C be a coalgebra, J a subspace of C and consider $J^\perp = \{f \in C^* \mid f(J) = 0\}$ that is a subspace of C^*. Then the following statements hold:*

(1) J is a subcoalgebra of C if and only if $J^\perp$ is an ideal of C^*;

(2) J is a left (resp. right) coideal of C if and only if $J^\perp$ is a left (resp. right) ideal of C^*;

(3) J is a coideal of C if and only if $J^\perp$ is a subalgebra of C^*.

Proposition 1.2.13 (Quotient Coalgebra)**.** *Let C be a coalgebra, $J \subseteq C$ a coideal and $\pi_J\colon C \to {}^C/_J$ the canonical linear projection. Then there exists a unique coalgebra structure on ${}^C/_J$ such that π_J is a coalgebra map.*

In this case, denoting by $\pi_J(c) = \overline{c}$, we have that

$$(\overline{c})_1 \otimes (\overline{c})_2 = \overline{(c_1)} \otimes \overline{(c_2)}$$
$$\varepsilon(\overline{c}) = \varepsilon_C(c)$$

This coalgebra is called the quotient coalgebra *of C by J.*

Definition 1.2.14 (Comodule)**.** Let C be a coalgebra, M a vector space and $\rho\colon M \to M \otimes C$ a linear map. We say that M is a *right comodule* if the following diagrams are commutative:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & M \otimes C \arrow[d, "I\otimes\Delta"] \\ M \otimes C \arrow[r, "\rho\otimes I"'] & M \otimes C \otimes C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M \arrow[rr, "\rho"] \arrow[dr, "\simeq"'] & \arrow[d, phantom, "\circlearrowright"] & M \otimes C \arrow[dl, "I\otimes\varepsilon"] \\ & M \otimes \Bbbk & \end{tikzcd}$$

The next result is important to the definition of bialgebra.

Proposition 1.2.15. *Let B be a vector space that have, at same time, a structure of algebra (B, m, u) and a structure of coalgebra (B, Δ, ε). Then the following conditions are equivalent:*

(1) Δ and ε are algebra morphisms;

(2) m and u are coalgebra morphism.

In this situation we say that the algebra and the coalgebra structure of B are compatible.

Definition 1.2.16 (Bialgebra)**.** A *bialgebra* is a quintuple $(B, m, u, \Delta, \varepsilon)$, where (B, m, u) is an algebra, (B, Δ, ε) is a coalgebra, such that the algebra and the coalgebra structures are compatible in the sense of Proposition 1.2.15.

Definition 1.2.17. Let B, B' be two bialgebras and $f\colon B \to B'$ a linear map. We say that f is a *bialgebra morphism* if it is an algebra and a coalgebra morphism.

Definition 1.2.18 (Module algebra)**.** Let B be a bialgebra, A an algebra and $\triangleright\colon B \otimes A \to A$ a linear map. We say that $(A, \triangleright)$ is a *left B-module algebra* if the following conditions hold:

(MA1) $1_B \triangleright a = a$;

(MA2) $b \triangleright (a'a) = (b_1 \triangleright a')(b_2 \triangleright a)$;

(MA3) $b \triangleright (c \triangleright a) = (bc) \triangleright a$;

(MA4) $b \triangleright 1_A = \varepsilon(b)1_A$.

Given a coalgebra $C = (C, \Delta, \varepsilon)$ and an algebra $A = (A, m, u)$, one can consider a new algebra, so called *the convolution algebra* of C and A, defined as $(Hom(C, A), *, u \circ \varepsilon)$, where $*$ is called *the convolution product* and it is given by $f * g = m \circ (f \otimes g) \circ \Delta$, for every $f, g \in Hom(C, A)$. Under our notations, if $c \in C$ and $f, g \in Hom(C, A)$ then we will write $(f * g)(c) = f(c_1)g(c_2)$ to denote the image of the element c in A, under the map $f * g$. We observe that when $A = \Bbbk$ then the convolution algebra of C and $\Bbbk$ is exactly the dual algebra of C (see Remark 1.2.11).

Now we are in position to present the notion of Hopf algebra.

Definition 1.2.19 (Hopf algebra)**.** A *Hopf algebra* is a sextuple $(H, m, u, \Delta, \varepsilon, S)$, where $(H, m, u, \Delta, \varepsilon)$ is a bialgebra and the linear map $S\colon H \to H$ is a convolutive inverse to $I\colon H \to H$ in the convolution algebra $Hom(H, H)$, i.e., if S satisfy the following equality:

$$\varepsilon(h)1_H = S(h_1)h_2 = h_1S(h_2).$$

The map S in the above definition is called *antipode* of H. Thus, a Hopf algebra is a bialgebra that has an antipode. In the next result we summarize some useful properties of the antipode in a Hopf algebra.

Proposition 1.2.20. *Let H be a Hopf algebra with antipode S. Then the following properties hold:*

(1) $S(hg) = S(g)S(h)$*;*

(2) $S(1_H) = 1_H$*;*

(3) $\Delta(S(h)) = S(h_2) \otimes S(h_1)$*;*

(4) $\varepsilon(S(h)) = \varepsilon(h)$*;*

Remark 1.2.21 ([19, Proposition 6.1.4])**.** When H is a Hopf algebra instead of a bialgebra, it follows that the last item in Definition 1.2.18 is a consequence from the others axioms, as it is easy to see. In fact, let $\in H$, so

$$
\begin{aligned}
\varepsilon(h)1_A &\overset{(MA1)}{=} \varepsilon(h)1_H \triangleright 1_A \\
&= h_1 S(h_2) \triangleright 1_A \\
&\overset{(MA3)}{=} h_1 \triangleright (S(h_2) \triangleright 1_A) \\
&= h_1 \triangleright [1_A(S(h_2) \triangleright 1_A)] \\
&\overset{(MA2)}{=} (h_1 \triangleright 1_A)[h_2 \triangleright (S(h_3) \triangleright 1_A)] \\
&\overset{(MA3)}{=} (h_1 \triangleright 1_A)(h_2 S(h_3) \triangleright 1_A) \\
&= (h_1 \triangleright 1_A)(\varepsilon(h_2)1_H \triangleright 1_A) \\
&\overset{(MA1)}{=} (h_1 \varepsilon(h_2) \triangleright 1_A)1_A \\
&= h \triangleright 1_A
\end{aligned}
$$

Definition 1.2.22. Let H be a Hopf algebra and $t \neq 0$ an element in H. Then t is said to be a *left integral in* H if, for all $h \in H$, we have that

$$h\,t = \varepsilon(h)\,t.$$

It is possible to show that if exists a left integral in H, then H is finite dimensional and, moreover, the set of all left integrals in H is an one dimensional subspace of H which is a left ideal of H. Denoting this space of left integrals in H by $\int_l^H$, we have the following classical property:

Proposition 1.2.23. *Let H be a finite dimensional Hopf algebra, then $H \simeq \int_l^H \otimes H^*$ via*

$$
\begin{aligned}
\gamma\colon \int_l^H \otimes H^* &\longrightarrow H \\
t \otimes f &\longmapsto t \rightharpoonup f = f_1 f_2(t).
\end{aligned}
$$

Remark 1.2.24. Given t a left integral in H, since $\dim \int_l^H = 1$ and $t\,h$ lies in $\int_l^H$, for any $h \in H$, hence there exists $\alpha \in H^*$ such that

$$t\,h = t\alpha(h).$$

This element is an algebra morphism in H^*, called the *distinguished element* associated to t.

Corollary 1.2.25. *Fixing an element $0 \neq T \in \int_l^{H^*}$, we have the following isomorphism between H and H^**

$$
\begin{aligned}
\gamma\colon H &\longrightarrow H^* \\
h &\longmapsto h \rightharpoonup T = T_1 T_2(h).
\end{aligned}
$$

1.3 Partial Module Algebra

Partial module algebras arose from a study of Caenepeel and Janssen [11] as a generalization of partial action of groups, and it was developed in a sequence of papers authored by Alves and Batista (see [1–3]), among others. In this section we will recall some definitions and properties that will be useful to the development of the next chapters. The proofs for the results presented here can be found in the above mentioned references.

Definition 1.3.1 ([11, Proposition 4.5])**.** Given a Hopf algebra H, an algebra A and a linear map $\rightarrow\colon H \otimes A \to A$. We say that $(A, \rightarrow)$ is a *left partial H-module algebra*, if the following conditions are satisfied:

(PMA1) $1_H \rightarrow a = a$;

(PMA2) $h \rightarrow (ab) = (h_1 \rightarrow a)(h_2 \rightarrow b)$; and

(PMA3) $h \rightarrow (k \rightarrow a) = (h_1 \rightarrow 1_A)(h_2 k \rightarrow a)$.

We say that the partial action $\rightarrow$ is *symmetric* if, in addition, we have the following condition:

(PMA4) $h \rightarrow (k \rightarrow a) = (h_1 k \rightarrow a)(h_2 \rightarrow 1_A)$.

We observe that in the above definition we are assuming that A is a unitary algebra, but it is not necessary to be assumed, because we can replace the conditions $PMA2$ and $PMA3$ by the following:

$$h \rightarrow (a(g \rightarrow b)) = (h_1 \rightarrow a)(h_2 g \rightarrow b), \forall h, g \in H; a, b \in A.$$

Proposition 1.3.2 ([2, Proposition 1])**.** *Let H be a Hopf algebra and B a (non necessarily unital) H-module algebra via $\triangleright\colon H \otimes B \to B$. Let A be a right ideal of B such that A is also a unital algebra. Then, the linear map $\rightarrow\colon H \otimes A \to A$ given by*

$$h \rightarrow a = 1_A(h \triangleright a)$$

is a partial action of H on A, called induced partial action.

Definition 1.3.3. Let H be a Hopf algebra and A a partial H-module algebra. We say that a pair (B, θ) is a *globalization of A*, where B is a (non necessarily unital) H-module algebra via $\triangleright\colon H \otimes B \to B$, $\theta\colon A \to B$ is an algebra morphism, and the following conditions hold:

(GMA1) $\theta(A)$ is a right ideal of B;

(GMA2) the partial action on A is equivalent to the partial action induced by $\triangleright$ on $\theta(A)$, that is, $\theta(h \rightarrow a) = h \rightarrow \theta(a) = \theta(1_A)(h \triangleright \theta(a))$; and

(GMA3) B is the H-module algebra generated by $\theta(A)$, that is, $B = H \triangleright \theta(A)$.

Theorem 1.3.4 ([2, Theorem 1])**.** *Any partial H-module algebra has a globalization*

Alves and Batista shown the above theorem constructing the standard globalization as a subalgebra of $Hom(H, A)$.

Now we recall the notion of Morita context between two algebras A and B. This is a useful tool from ring theory which allows us to decide when the category of modules over A and over B are equivalent or, at least, to find a connection between them. In this work we will be interested in an special Morita context.

Definition 1.3.5 (Morita Context)**.** A *Morita context* is a sextuple (A, B, M, N, τ, μ), where A and B are algebras, M is an (A, B)-bimodule, N is an (B, A)-bimodule, $\tau\colon M \otimes_B N \to A$ is an A-bimodule map and $\mu\colon N \otimes_A M \to B$ is an B-bimodule map, such that the following diagrams are commutative:

$$\begin{tikzcd} M \otimes_B N \otimes_A M \arrow[r, "I\otimes\mu"] \arrow[d, "\tau\otimes I"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes_B B \arrow[d, "\triangleleft"] \\ A \otimes_A M \arrow[r, "\triangleright"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} N \otimes_A M \otimes_B N \arrow[r, "I\otimes\tau"] \arrow[d, "\mu\otimes I"'] \arrow[dr, phantom, "\circlearrowleft"] & N \otimes A \arrow[d, "\triangleleft"] \\ B \otimes_B N \arrow[r, "\triangleright"] & N \end{tikzcd}$$

We say that this context is *strict (or A is Morita equivalent to B)* if τ and μ are bijective. In this case we denote by $A \simeq_{\mathcal{M}} B$.

Proposition 1.3.6. *Let (A, B, M, N, τ, μ) be a Morita context between A and B (both unitary algebras), and suppose that τ and μ are surjective. Then we have that τ and μ are bijective and, therefore, $A \simeq_{\mathcal{M}} B$.*

Moreover, in the above case, we have that the category of right A-modules is equivalent to the category of right B-modules via

$$F\colon \mathfrak{M}_A \to \mathfrak{M}_B \qquad \textit{given by} \qquad F = \_\_ \otimes_A M$$
$$G\colon \mathfrak{M}_B \to \mathfrak{M}_A \qquad \textit{given by} \qquad G = \_\_ \otimes_B N$$

Given a partial H-module algebra A, let $A^{\underline{H}} = \{a \in A \mid h \to a = a(h \to 1_A) = (h \to 1_A)a,\ \forall h \in H\}$ be the *invariant subalgebra* of A. Also, we can consider the *smash product* $A\#H$ that is the vector space $A \otimes H$ with structure of associative algebra given by

$$(a \otimes h)(b \otimes k) = a(h_1 \to b) \otimes h_2 k$$

and left unity $1_A \otimes 1_H$. Since $A\#H$ is an associative algebra with left unity $1_A\#1_H$, it follows that we can consider the *partial smash product*, $A\underline{\#}H$, as the following unital subalgebra

$$A\underline{\#}H = (A\#H)(1_A\#1_H).$$

Therefore, given a partial H-module algebra we have two associated algebras, namely, the invariant subalgebra $A^{\underline{H}}$ and the partial smash product $A\underline{\#}H$. In [1], Alves and Batista constructed a Morita context between $A^{\underline{H}}$ and $A\underline{\#}H$, whenever H is a finite dimensional Hopf algebra, and now we will presented the Morita context developed by those authors.

Lemma 1.3.7. *Let H be a finite dimensional Hopf algebra and A a symmetric partial H-module algebra. Then we have the following:*

1. *A is an $(A\underline{\#}H, A^{\underline{H}})$-bimodule via*

$$(a\underline{\#}h) \triangleright b = a(h \rightarrow b)$$
$$a \triangleleft c = ac.$$

2. *A is an $(A^{\underline{H}}, A\underline{\#}H)$-bimodule via*

$$a \triangleleft (b\underline{\#}h) = \alpha(h_2)S^{-1}(h_1) \rightarrow ab$$
$$c \triangleright a = ca,$$

where α is the distinguished element associated to a fixed left integral $t \in H$ (see Remark 1.2.24).

Lemma 1.3.8. *With the above notation, we have*

$$\begin{aligned} [\ \ ,\ \ ] \colon A \otimes_{A^{\underline{H}}} A &\longrightarrow A\underline{\#}H \\ a \otimes b &\longmapsto [a,b] = (a\underline{\#}1_H)(1\underline{\#}t)(b\underline{\#}1_H) \end{aligned}$$

that is an $A\underline{\#}H$-bilinear map, and

$$\begin{aligned} (\ \ ,\ \ ) \colon A \otimes_{A\underline{\#}H} A &\longrightarrow A^{\underline{H}} \\ a \otimes b &\longmapsto (a,b) = t \rightarrow (ab) \end{aligned}$$

that is an $A^{\underline{H}}$-bilinear map.

With the above notations we have the desired Morita context as follows.

Theorem 1.3.9 ([1, Theorem 1])**.** *Let H be a finite dimensional Hopf algebra and A a symmetric partial H-module algebra. Then, with the above definitions, we have that*

$$(A^{\underline{H}}, A\underline{\#}H, A, A, [,], (,))$$

is a Morita context.

Chapter 2

Partial Actions on Coalgebras

2.1 Partial Modules Coalgebras

In the setting of partial action of group on algebras, Dokuchaev and Exel [21] have considered the problem of existence of the globalization of partial actions of groups on algebras, having found necessary and sufficient conditions for such a partial action to have a globalization (see [21, Theorem 4.5]). Inspired in this work, Alves and Batista [2] studied this subject of globalization for partial actions of Hopf algebras. It was proved in [2] that every partial H-module algebra has a globalization, where H is a Hopf algebra (see [2, Theorem 1]).

In this section we discuss about the existence of a globalization for partial module coalgebras. In particular, we prove that it always exists and we also construct the standard one. Moreover, we will present a close relationship between the globalizations for partial H-module algebras and for partial H-module coalgebras.

We will start presenting the concept of (partial) H-module coalgebras. Also, we will give some examples of this structures and discuss about the relations between the correspondent global and partial structures.

2.1.1 Definitions and Correspondences

Our aims in this subsection is to define the (partial) module coalgebra, to see some interesting examples and properties, and to relate the partial module coalgebra with the global one.

Definition 2.1.1 ((Global) Module Coalgebra)**.** Let H be a Hopf algebra. A coalgebra D is a *right H-module coalgebra* via $\blacktriangleleft\colon D \otimes H \to D$ if the following properties hold:

(MC1) $d \blacktriangleleft 1_H = d$;

(MC2) $\Delta(d \blacktriangleleft h) = d_1 \blacktriangleleft h_1 \otimes d_2 \blacktriangleleft h_2$;

(MC3) $(d \blacktriangleleft h) \blacktriangleleft g = d \blacktriangleleft hg$.

In this case we say that the Hopf algebra H act on D via $\blacktriangleleft$, or that $\blacktriangleleft$ is a global action of H on D.

Since H is an Hopf algebra, it follows that the category of right H-modules $(\mathfrak{M}_H, \otimes_{\Bbbk}, \Bbbk)$ is a monoidal category. Thus, the above definition can be seen in a categorical sense, as follows(cf. [29, Definition 11.2.8]): *a $\Bbbk$-vector space D is said a right H-module coalgebra if it is a coalgebra object in the category of right H-modules.*

Note that from this categorical approach, we need to require a fourth property on D for it becomes an H-module coalgebra. More precisely, we need to require that the following equality be true:

$$\varepsilon_C(c \blacktriangleleft h) = \varepsilon_C(c)\varepsilon_H(h), \forall h \in H, d \in D.$$

In the case when H is simply a bialgebra, in fact this fourth condition needs to be required. But under a more strong hypothesis that H is a Hopf algebra (and so it has an antipode), this additional condition is a consequence from the others axioms of Definition 2.1.1, as the next result shows.

Proposition 2.1.2. *Let H be a Hopf algebra and D a right H-module coalgebra. Then $\varepsilon_D(d \blacktriangleleft h) = \varepsilon_D(d)\varepsilon_H(h)$, for every $d \in D, h \in H$.*

Proof. Let $d \in D$ and $h \in H$, so

$$\begin{aligned}
\varepsilon_D(d)\varepsilon_H(h) &\overset{(MC1)}{=} \varepsilon_D(d \blacktriangleleft 1_H)\varepsilon_H(h) \\
&= \varepsilon_D(d \blacktriangleleft 1_H\varepsilon_H(h)) \\
&= \varepsilon_D(d \blacktriangleleft h_1 S(h_2)) \\
&\overset{(MC3)}{=} \varepsilon_D((d \blacktriangleleft h_1) \blacktriangleleft S(h_2)) \\
&= \varepsilon_D(\varepsilon_D(d \blacktriangleleft h_1)_1(d \blacktriangleleft h_1)_2 \blacktriangleleft S(h_2)) \\
&\overset{(MC2)}{=} \varepsilon_D(\varepsilon_D(d_1 \blacktriangleleft h_1)(d_2 \blacktriangleleft h_2) \blacktriangleleft S(h_3)) \\
&= \varepsilon_D(d_1 \blacktriangleleft h_1)\varepsilon_D((d_2 \blacktriangleleft h_2) \blacktriangleleft S(h_3)) \\
&= \varepsilon_D(d_1 \blacktriangleleft h_1)\varepsilon_D(d_2 \blacktriangleleft h_2 S(h_3)) \\
&= \varepsilon_D(d_1 \blacktriangleleft h_1)\varepsilon_D(d_2 \blacktriangleleft 1_H\varepsilon_H(h_2)) \\
&= \varepsilon_D(d_1 \blacktriangleleft h_1)\varepsilon_D(d_2 \blacktriangleleft 1_H)\varepsilon_H(h_2) \\
&\overset{(MC1)}{=} \varepsilon_D(d_1 \blacktriangleleft h_1)\varepsilon_D(d_2)\varepsilon_H(h_2) \\
&= \varepsilon_D(d \blacktriangleleft h).
\end{aligned}$$

□

Remark 2.1.3. From Definition 2.1.1 and Proposition 2.1.2 it follows that we can define, without loss of generality, module coalgebra for non-counital coalgebras, and both definitions coincide when the coalgebra is counital.

Now we will present some examples of module coalgebras. The first one is the canonical example that any Hopf algebra is a module coalgebra over itself. Also, we present a standard method to construct new module coalgebras from any given module coalgebra.

Example 2.1.4. Any Hopf algebra H is a right H-module coalgebra via right multiplication.

Example 2.1.5. If C is a coalgebra and D is a right H-module coalgebra, then $C \otimes D$ is a right H-module coalgebra with action given by $(c \otimes d) \blacktriangleleft h = c \otimes (d \blacktriangleleft h)$.

From the definition of partial actions of group, Caenepeel and Janssen developed the notion of partial module algebras generalizing the global ones. Since a module coalgebra can be seen as a dualization of a module algebra, then we can ask if there exists some partial structure that is the dual of a partial module algebra extending the concept of module coalgebra. In fact, it already there exists and it was introduced in the literature by Batista and Vercruysse in [4]. In this same paper, beyond the authors define partial module coalgebras, they also exhibit some examples and present certain relations of these new structures with the others partial structures.

Definition 2.1.6 (Partial Module Coalgebra[4, Definition 5.1])**.** A coalgebra C is a *right partial H-module coalgebra*, via the linear map $\leftharpoondown: C \otimes H \to C$, if the following properties hold:

(PMC1) $c \leftharpoondown 1_H = c$;

(PMC2) $\Delta(c \leftharpoondown h) = c_1 \leftharpoondown h_1 \otimes c_2 \leftharpoondown h_2$;

(PMC3) $(c \leftharpoondown h) \leftharpoondown g = \varepsilon(c_1 \leftharpoondown h_1)(c_2 \leftharpoondown h_2 g)$.

A partial module coalgebra is said to be *symmetric* if the following additional condition holds:

(PMC4) $(c \leftharpoondown h) \leftharpoondown g = (c_1 \leftharpoondown h_1 g)\varepsilon(c_2 \leftharpoondown h_2)$.

One can define left partial module coalgebra in analogous way.

Proposition 2.1.7. *Let C be a right partial H-module coalgebra. Then the action of H on C is global if and only if the following equality holds:*

$$\varepsilon_C(c \leftharpoondown h) = \varepsilon_C(c)\varepsilon_H(h).$$

Proof. Supposing that $\varepsilon_C(c \leftharpoondown h) = \varepsilon_C(c)\varepsilon_H(h)$, since C is a right partial H-module coalgebra, we just need to show the property MC3.

$$\begin{aligned} (c \leftharpoondown h) \leftharpoondown k &\overset{(PMC3)}{=} \varepsilon(c_1 \leftharpoondown h_1)(c_2 \leftharpoondown h_2 k) \\ &= \varepsilon(c_1)\varepsilon(h_1)(c_2 \leftharpoondown h_2 k) \\ &= c \leftharpoondown hk. \end{aligned}$$

The converse follows from Proposition 2.1.2. □

The definition of right partial H-module coalgebra can be extended for non-counital coalgebras as follows:

Definition 2.1.8. Let C be a (non necessarily counital) coalgebra and H a Hopf algebra. We say that $(C, \leftharpoonup)$ is a *right partial H-module coalgebra* if:

(NCPMC1) $c \leftharpoonup 1_H = c$;

(NCPMC2) $(c \leftharpoonup h)_1 \otimes ((c \leftharpoonup h)_2 \leftharpoonup k) = (c_1 \leftharpoonup h_1) \otimes (c_2 \leftharpoonup h_2 k)$.

Moreover, it is called *symmetric* if:

(NCPMC3) $((c \leftharpoonup h)_1 \leftharpoonup k) \otimes (c \leftharpoonup h)_2 = (c_1 \leftharpoonup h_1 k) \otimes (c_2 \leftharpoonup h_2)$.

The next result shows that these last two definitions coincide when the considered coalgebra is counital.

Proposition 2.1.9. *If C is a counital coalgebra, then Definition 2.1.8 is equivalent to Definition 2.1.6.*

Proof. First note that conditions PMC1 and NCPMC1 are the same in both definitions.

Now suppose that C is a right partial H-module coalgebra in the sense of Definition 2.1.8. Then:

(PMC2): Let $c \in C$ and $h \in H$, so

$$\begin{aligned}
\Delta(c \leftharpoonup h) &= (c \leftharpoonup h)_1 \otimes (c \leftharpoonup h)_2 \\
&\overset{(NCPMC1)}{=} (c \leftharpoonup h)_1 \otimes ((c \leftharpoonup h)_2 \leftharpoonup 1_H) \\
&\overset{(NCPMC2)}{=} c_1 \leftharpoonup h_1 \otimes c_2 \leftharpoonup h_2
\end{aligned}$$

(PMC3): Let $c \in C$ and $h, k \in H$. Applying $(\varepsilon \otimes I)$ in both sides of (NCPMC2), we have the desired result.

Analogously we can obtain (PMC4) from (NCPMC3).

Conversely, suppose that C is a right partial H-module coalgebra in the sense of Definition 2.1.6. Then:

(NCPMC2): Let $c \in C$ and $h, k \in H$, so

$$\begin{aligned}
(c \leftharpoonup h)_1 \otimes ((c \leftharpoonup h)_2 \leftharpoonup k) &\overset{(PMC2)}{=} (c_1 \leftharpoonup h_1) \otimes ((c_2 \leftharpoonup h_2) \leftharpoonup k) \\
&\overset{(PMC3)}{=} (c_1 \leftharpoonup h_1) \otimes (\varepsilon(c_2 \leftharpoonup h_2) c_3 \leftharpoonup h_3 k) \\
&= (c_1 \leftharpoonup h_1 \varepsilon(c_2 \leftharpoonup h_2)) \otimes (c_3 \leftharpoonup h_3 k) \\
&\overset{(PMC2)}{=} ((c_1 \leftharpoonup h_1)_1 \varepsilon((c_1 \leftharpoonup h_1)_2)) \otimes (c_2 \leftharpoonup h_2 k) \\
&= (c_1 \leftharpoonup h_1) \otimes (c_2 \leftharpoonup h_2 k).
\end{aligned}$$

Analogously as made before, we can obtain NCPMC3 from PMC4. $\square$

Now we present some examples of partial module coalgebras.

Example 2.1.10. Every global right H-module coalgebra is a partial one.

Example 2.1.11. Let H be a Hopf algebra and $\alpha \in H^*$. Then $\Bbbk$ is a right partial H-module coalgebra via α if and only if the following conditions hold:

(i) $\alpha(1_H) = 1_\Bbbk$;

(ii) $\alpha(h)\alpha(k) = \alpha(h_1)\alpha(h_2 k)$.

Note that in this case we have $1_\Bbbk \leftharpoonup h = \alpha(h)$.

This example give us a way to produce structures of partial module coalgebra on the groundfield $\Bbbk$. Now we will consider a especial case of this last example.

Let G be a group and $H = \Bbbk G$. Consider $\alpha \in \Bbbk G^*$ and let $N = \{g \in G | \ \alpha(g) \neq 0\}$. Then $\Bbbk$ is a partial $\Bbbk G$-module coalgebra if and only if N is a subgroup of G. In this case, we have that

$$\alpha(g) = \begin{cases} 1, & \text{if } g \text{ lies in } N \\ 0, & \text{otherwise.} \end{cases}$$

In [4], Batista and Vercruysse introduced the notion of partial action of groups on coalgebras, in the following way.

Definition 2.1.12. A partial action of a group G on a coalgebra C is a family $\{C_g, \theta_g\}_{g\in G}$, where C_g are subcoalgebras of C and $\theta_g \colon C_{g^{-1}} \to C_g$ are coalgebra isomorphisms such that

(1) For each $g \in G$, there exist a comultiplicative map $P_g \colon C \twoheadrightarrow C_g$ such that

$$P_g(c) = c_1 \varepsilon(P_g(c_2)) = \varepsilon(P_g(c_1))c_2;$$

(2) $C_e = C$ and $\theta_e = P_e = I$;

(3) For all $g, h \in G$, we have that

$$P_g \circ P_h = P_h \circ P_g;$$
$$\theta_{h^{-1}} \circ P_h \circ P_{g^{-1}} = P_{(gh)^{-1}} \circ \theta_{h^{-1}} \circ P_{g^{-1}};$$
$$\theta_g \circ \theta_h \circ P_{(gh)^{-1}} \circ P_{h^{-1}} = \theta_{gh} \circ P_{(gh)^{-1}} \circ P_{h^{-1}}.$$

With the above definition, Batista and Vercruysse shown that a group G acts partially on a coalgebra C if and only if C is a symmetric partial left $\Bbbk G$-module coalgebra [4, Theorem 5.7]. In this case, $g \cdot c = \theta_g(P_{g^{-1}}(c))$ and $P_g(c) = \varepsilon(g^{-1} \cdot c_1)c_2 = c_1 \varepsilon(g^{-1} \cdot c_2)$.

Now we would like to discuss about the existence of certain relation between right partial module coalgebras and left partial module algebras which will be useful in the construction of a globalization of partial module coalgebras in the next section. Batista and Vercruysse considered this same relation between these partial actions in [4], using non-degenerated dual pairing between an algebra A and a coalgebra C. Here we will consider an special case, when the considered algebra is the dual of a given coalgebra.

Proposition 2.1.13 ([4, Theorem 5.14])**.** *Let H be a Hopf algebra. Then every right partial H-module coalgebra induces a left partial H-module algebra.*

Proof. Let C be a right partial H-module coalgebra and consider the following linear map

$$\begin{aligned} \rightarrow : H \otimes C^* &\longrightarrow C^* \\ h \otimes \varphi &\longmapsto (h \rightarrow \varphi)(c) = \varphi(c \leftharpoonup h) \end{aligned}$$

It is clear that $\rightarrow$ is well-defined and, moreover, we have:
(PMA1): Let $\varphi \in C^*$. Then $(1_H \rightarrow \varphi)(c) = \varphi(c \leftharpoonup 1_H) = \varphi(c),\ \forall\ c \in C$.
(PMA2): Let $h \in H$ and $\varphi, \psi \in C^*$. Then

$$\begin{aligned} [h \rightarrow (\varphi * \psi)](c) &= (\varphi * \psi)(c \leftharpoonup h) \\ &= \varphi((c \leftharpoonup h)_1)\psi((c \leftharpoonup h)_2) \\ &\overset{(PMC2)}{=} \varphi(c_1 \leftharpoonup h_1)\psi(c_2 \leftharpoonup h_2) \\ &= [(h_1 \rightarrow \varphi)(c_1)][(h_2 \rightarrow \psi)(c_2)] \\ &= [(h_1 \rightarrow \varphi) * (h_2 \rightarrow \psi)](c). \end{aligned}$$

(PMA3): Let $h, k \in H$ and $\varphi \in C^*$. Then

$$\begin{aligned} [h \rightarrow (k \rightarrow \varphi)](c) &= (k \rightarrow \varphi)(c \leftharpoonup h) \\ &= \varphi((c \leftharpoonup h) \leftharpoonup k) \\ &\overset{(PMC3)}{=} \varepsilon(c_1 \leftharpoonup h_1)\varphi(c_2 \leftharpoonup h_2k) \\ &= [(h_1 \rightarrow \varepsilon)(c_1)][(h_2k \rightarrow \varphi)(c_2)] \\ &= [(h_1 \rightarrow \varepsilon) * (h_2k \rightarrow \varphi)](c). \end{aligned}$$

Therefore C^* is a left partial H-module algebra. □

Proposition 2.1.14. *Assume that H is a finite dimensional Hopf algebra. Then the converse of Proposition 2.1.13 is true.*

Proof. It is straightforward. □

Now, our main interest is to build a globalization for partial action on coalgebras. For this, first we need to be able to induce a partial action from a global one. Thus, we consider the following situation.

Let D be a right H-module coalgebra and $C \subseteq D$ a subcoalgebra. Since D is a right H-module, we can try to induce the desired partial action simply by restriction of the action of D to C. But we observe that the image of this restriction does not need to be contained in C and, therefore, we need to project it on C. By this way, let $\pi\colon D \twoheadrightarrow C$ be a projection of D over C, as a vector space, and then consider the following map

$$\,_{\imath}\!\!\leftharpoondown\ : C \otimes H \xrightarrow{\blacktriangleleft} D \xrightarrow{\pi} C,$$

where $\blacktriangleleft$ denote the global action of H on D on the right. To work with this composite map, we need to find necessary conditions for this map be a partial action of H on C. This is exactly what we will do in the sequel.

(PMC1): Let $c \in C$, so $c \,_{\imath}\!\!\leftharpoondown 1_H = \pi(c \blacktriangleleft 1_H) = \pi(c) = c$, where the last equality holds because π is a projection.

Supposing that π is a comultiplicative map (i.e., $\Delta \circ \pi = (\pi \otimes \pi) \circ \Delta$), then we have that

(PMC2): Let $c \in C$ and $h \in H$, so

$$\begin{array}{rcl} \Delta(c \,_{\imath}\!\!\leftharpoondown h) & = & \Delta(\pi(c \blacktriangleleft h)) \\ & = & (\pi \otimes \pi)(\Delta(c \blacktriangleleft h)) \\ & \overset{(MC2)}{=} & (\pi \otimes \pi)(c_1 \blacktriangleleft h_1 \otimes c_2 \blacktriangleleft h_2) \\ & = & \pi(c_1 \blacktriangleleft h_1) \otimes \pi(c_2 \blacktriangleleft h_2) \\ & = & c_1 \,_{\imath}\!\!\leftharpoondown h_1 \otimes c_2 \,_{\imath}\!\!\leftharpoondown h_2. \end{array}$$

Then, just remains to show that PMC3 also holds, but for this we will need to assume an special technical condition on π. Suppose that π satisfy the following condition

$$\pi[\pi(d) \blacktriangleleft h] = \pi[\varepsilon(\pi(d_1))d_2 \blacktriangleleft h],$$

for all $d \in D$. If it is the case, then we can deduce what we need, as it is showed below.

(PMC3): Let $h, k \in H$ and $c \in C$, so

$$\begin{array}{rcl} \varepsilon(c_1 \,_{\imath}\!\!\leftharpoondown h_1)(c_2 \,_{\imath}\!\!\leftharpoondown h_2 k) & = & \varepsilon[\pi(c_1 \blacktriangleleft h_1)]\pi(c_2 \blacktriangleleft h_2 k) \\ & \overset{(MC3)}{=} & \varepsilon[\pi(c_1 \blacktriangleleft h_1)]\pi[(c_2 \blacktriangleleft h_2) \blacktriangleleft k] \\ & \overset{(MC2)}{=} & \varepsilon[\pi((c \blacktriangleleft h)_1)]\pi[((c \blacktriangleleft h)_2) \blacktriangleleft k] \\ & = & \pi[\varepsilon[\pi((c \blacktriangleleft h)_1)](c \blacktriangleleft h)_2 \blacktriangleleft k] \\ & = & \pi[\pi(c \blacktriangleleft h) \blacktriangleleft k] \\ & = & (c \,_{\imath}\!\!\leftharpoondown h) \,_{\imath}\!\!\leftharpoondown k. \end{array}$$

Therefore, under the hypotheses assumed above, we have that C is a partial module coalgebra, which will be called the *induced right partial H-module coalgebra*. Thus, our above argumentation shows the following result.

Proposition 2.1.15 (Induced Partial Module Coalgebra)**.** *Let H be a Hopf algebra and D a right H-module coalgebra. Suppose that $C \subseteq D$ is a subcoalgebra and let $\pi\colon D \to C$ be a comultiplicative projection satisfying*

$$\pi[\pi(d) \blacktriangleleft h] = \pi[\varepsilon(\pi(d_1))d_2 \blacktriangleleft h], \forall d \in D. \tag{2.1}$$

If ${}_{\imath}\!\leftharpoonup : C \otimes H \to C$ is the linear map given by

$$c \,{}_{\imath}\!\leftharpoonup h = \pi(c \blacktriangleleft h), \tag{2.2}$$

then C becomes a right partial H-module coalgebra via ${}_{\imath}\!\leftharpoonup$. □

With the above construction we have the necessary tools to define a globalization for a partial module coalgebra. It will be the subject of the next section.

2.2 Globalization for Partial Modules Coalgebras

Inspired in Alves and Batista [2], who first defined what would be a globalization for partial H-module algebras, we present now the concept of globalization for partial H-module coalgebras as follows.

Definition 2.2.1. Let H be a Hopf algebra. Given a right partial H-module coalgebra $(C, \leftharpoonup)$, a *globalization* of C is a triple (D, θ, π), where D is a right H-module coalgebra via $\blacktriangleleft$, $\theta\colon C \to D$ is a coalgebra monomorphism and π is a comultiplicative projection from D onto $\theta(C)$, such that:

(GMC1) $\pi[\pi(d) \blacktriangleleft h] = \pi[\varepsilon(\pi(d_1))d_2 \blacktriangleleft h]$, $\forall d \in D, h \in H$;

(GMC2) $\theta(c \leftharpoonup h) = \theta(c) \,{}_{\imath}\!\leftharpoonup h$, $\forall c \in C, h \in H$;

(GMC3) D is the H-module generated by $\theta(C)$, that is, $D = \theta(C) \blacktriangleleft H$.

where ${}_{\imath}\!\leftharpoonup$ is defined as in Proposition 2.1.15.

Before to continue, we would like to observe some facts that can be derived from these conditions given in the above definition.

Remark 2.2.2. The first condition of Definition 2.2.1 says that we can induce an structure of partial module coalgebra on $\theta(C)$. The second one says that this induced partial action on $\theta(C)$ is equivalent to the partial action on C. The last one says that do not exists any submodule coalgebra of D containing $\theta(C)$.

The next remark show us the motivation to the projection π be a comultiplicative maps instead of a coalgebra map.

Remark 2.2.3. If the projection π from the Definition 1.3.3 is a coalgebra map, then the partial action on C is global. In fact, applying $I \otimes \varepsilon$ in the equality GMA2, we have

$$\begin{aligned}
\varepsilon_C(c \leftharpoonup h) &= \varepsilon_D(\theta(c \leftharpoonup h)) \\
&\overset{(GMA2)}{=} \varepsilon_D(\theta(c) \leftharpoonup h) \\
&= \varepsilon_D(\pi(\theta(c) \blacktriangleleft h)) \\
&= \varepsilon_D(\theta(c) \blacktriangleleft h) \\
&\overset{(2.1.2)}{=} \varepsilon_D(\theta(c))\varepsilon(h) \\
&= \varepsilon_C(c)\varepsilon(h),
\end{aligned}$$

therefore follow from Proposition 2.1.7 that the partial action on C is global.

The concept of globalization above defined was thought as a dual notion of the globalization defined by Alves and Batista in [2]. Thus, it would be interesting to explore a little more their relationships, if any. This is the subject of the next subsection.

2.2.1 Correspondence Between Globalizations

Our aim in this subsection is to establish relations between the globalization for partial module coalgebras, as defined in Definition 2.2.1, and for partial module algebras, as defined in [2] (see Definition 1.3.3). For this we will use the fact that a partial module coalgebra C naturally induces an structure of partial module algebra on C^*.

First of all, we recall from basic linear algebra that given a linear map $T\colon V \to W$ we have the dual linear map $T^*\colon W^* \to V^*$, given by $T^*(f) = f \circ T$. Moreover, remember that T is injective (resp. surjective) if and only if T^* is surjective (resp. injective). Also, if V and W are coalgebras, then T is a coalgebra map if and only if T^* is an algebra map (see Proposition 1.1.1).

Given a right partial H-module coalgebra C, it follows from Proposition 2.1.13 that the dual C^* is a left partial H-module algebra in such a way that the involved partial actions respect the following rule

$$\alpha(c \leftharpoonup h) = (h \rightarrow \alpha)(c), \forall \alpha \in C^*, c \in C, h \in H,$$

where $\leftharpoonup$ denotes the partial action of H on C and $\rightarrow$ denotes the correspondent partial action of H on C^*. The same is true for (global) module coalgebras (cf. [29, Theorem 11.2.10(b)]).

With the above considerations, take C a right partial H-module coalgebra, D a right H-module coalgebra, $\theta\colon C \to D$ a coalgebra monomorphism and $\pi\colon D \to \theta(C)$ a comultiplicative projection. Consider the linear map $\varphi\colon C^* \to D^*$, given by the dual map of $\theta^{-1} \circ \pi$, i.e.,

$$\begin{aligned}
\varphi\colon C^* &\longrightarrow D^* \\
\alpha &\longmapsto \varphi(\alpha) = (\theta^{-1} \circ \pi)^*(\alpha) = \alpha \circ \theta^{-1} \circ \pi,
\end{aligned}$$

that is a injective map since $\theta^{-1} \circ \pi$ is a surjective map ($c = (\theta^{-1} \circ \pi)(\theta(c))$, for any $c \in C$).

Then, by Proposition 2.1.13 and Proposition 1.1.1, we have the following properties:

(1) C^* is a left partial H-module algebra;

(2) D^* is a left H-module algebra;

(3) φ is a multiplicative monomorphism.

As a consequence, we obtain the next result.

Theorem 2.2.4. *With the above notations, we have that $(\theta(C) \blacktriangleleft H, \theta, \pi)$ is a globalization for C if and only if $(H \triangleright \varphi(C^*), \varphi)$ is a globalization for C^*.*

Proof. Suppose that $(\theta(C) \blacktriangleleft H, \theta, \pi)$ is a globalization for C. Since φ is a multiplicative monomorphism, it follows that

$$\begin{aligned}
[\varphi(\varepsilon_C) * (h \triangleright \varphi(\alpha))](d) &= (\varphi(\varepsilon_C)(d_1))((h \triangleright \varphi(\alpha))(d_2)) \\
&= \varepsilon_C(\theta^{-1}\pi(d_1))\varphi(\alpha)(d_2 \blacktriangleleft h) \\
&= \varepsilon_{\theta(C)}(\pi(d_1))\alpha(\theta^{-1}\pi(d_2 \blacktriangleleft h)) \\
&= \alpha(\theta^{-1}\pi(\varepsilon_{\theta(C)}(\pi(d_1))d_2 \blacktriangleleft h)) \\
&\overset{(GMC1)}{=} \alpha(\theta^{-1}\pi(\pi(d) \blacktriangleleft h)) \\
&\overset{(2.2)}{=} \alpha(\theta^{-1}(\pi(d) \leftharpoonup h)) \\
&\overset{(GMC2)}{=} \alpha[(\theta^{-1}\pi(d)) \leftharpoonup h] \\
&\overset{(2.1.13)}{=} (h \rightarrow \alpha)(\theta^{-1}\pi(d)) \\
&= \varphi(h \rightarrow \alpha)(d),
\end{aligned}$$

for every $h \in H, \alpha \in C^*$ and $d \in D$. Thus, it means that $\varphi(C^*)$ is a right ideal of $H \triangleright \varphi(C^*)$ and, moreover, $h \rightarrow \varphi(\alpha) = \varphi(h \rightarrow \alpha)$. Therefore $(H \triangleright \varphi(C^*), \varphi)$ is a globalization for C^*, as desired.

Conversely, if $(H \triangleright \varphi(C^*), \varphi)$ is a globalization for C^*, then for any $\alpha \in C^*, h \in H$ and $d \in D$, we have

(GMC1):

$$\begin{aligned}
\alpha(\theta^{-1}\pi(\pi(d) \blacktriangleleft h)) &= \alpha(\theta^{-1}\pi(\pi(d) \leftharpoonup h)) \\
&= \varphi(\alpha)[\pi(d) \leftharpoonup h] \\
&\overset{(2.1.13)}{=} [h \rightarrow \varphi(\alpha)]\pi(d) \\
&\overset{(GMA2)}{=} \varphi(h \rightarrow \alpha)\pi(d) \\
&= (h \rightarrow \alpha)[\theta^{-1}\pi(\pi(d))] \\
&= (h \rightarrow \alpha)[\theta^{-1}(\pi(d))] \\
&= \varphi(h \rightarrow \alpha)(d) \\
&= [\varphi(\varepsilon_C) * (h \triangleright \varphi(\alpha))](d) \\
&= \varphi(\varepsilon_C)(d_1)(h \triangleright \varphi(\alpha))(d_2) \\
&= \varphi(\varepsilon_C)(d_1)\varphi(\alpha)(d_2 \blacktriangleleft h) \\
&= \varphi(\varepsilon_C)(d_1)\alpha(\theta^{-1}\pi(d_2 \blacktriangleleft h)) \\
&= \varepsilon_C(\theta^{-1}\pi(d_1))\alpha(\theta^{-1}\pi(d_2 \blacktriangleleft h)) \\
&= \varepsilon_{\theta(C)}(\pi(d_1))\alpha(\theta^{-1}\pi(d_2 \blacktriangleleft h)) \\
&= \alpha(\theta^{-1}\pi[\varepsilon(\pi(d_1))d_2 \blacktriangleleft h])
\end{aligned}$$

Since the above equality holds for any $\alpha \in C^*$, we conclude that

$$\theta^{-1}\pi(\pi(d) \blacktriangleleft h) = \theta^{-1}\pi[\varepsilon(\pi(d_1))d_2 \blacktriangleleft h].$$

Moreover, since $Im(\pi) = Im(\theta) = dom(\theta^{-1})$, where $dom(\theta^{-1})$ means the domain of θ^{-1}, it follows that

$$\theta \circ \theta^{-1} = I|_{dom(\theta^{-1})} = I|_{Im(\pi)}$$

and, therefore, we have that

$$\pi(\pi(d) \blacktriangleleft h) = \pi[\varepsilon(\pi(d_1))d_2 \blacktriangleleft h].$$

(GMC2): Let $\alpha \in C^*$, $h \in H$ and $c \in C$. Then

$$\begin{aligned}
(\alpha \circ \theta^{-1})[\theta(c) \leftharpoondown h] &= \alpha\ \theta^{-1}\pi[\theta(c) \blacktriangleleft h] \\
&= \varphi(\alpha)[\theta(c) \blacktriangleleft h] \\
&= [h \triangleright \varphi(\alpha)]\theta(c) \\
&= [h \triangleright \varphi(\alpha)]\theta(\varepsilon_C(c_1)\ c_2) \\
&= \varepsilon_C(c_1)\ [h \triangleright \varphi(\alpha)]\theta(c_2) \\
&= \varepsilon_C\theta^{-1}\theta(c_1)\ [h \triangleright \varphi(\alpha)]\theta(c_2) \\
&= \varepsilon_C\theta^{-1}\pi\theta(c_1)\ [h \triangleright \varphi(\alpha)]\theta(c_2) \\
&= \varphi(\varepsilon_C)\theta(c_1)\ [h \triangleright \varphi(\alpha)]\theta(c_2) \\
&= \varphi(\varepsilon_C)\theta(c)_1\ [h \triangleright \varphi(\alpha)]\theta(c)_2 \\
&= \{\varphi(\varepsilon_C) * [h \triangleright \varphi(\alpha)]\}\ \theta(c) \\
&= [h \rightarrow \varphi(\alpha)]\ \theta(c) \\
&= \varphi(h \rightarrow \alpha)\ \theta(c) \\
&= (h \rightarrow \alpha)\ \theta^{-1}\pi\theta(c) \\
&= (h \rightarrow \alpha)\ \theta^{-1}\theta(c) \\
&= (h \rightarrow \alpha)(c) \\
&= \alpha\ (c \leftharpoondown h) \\
&= \alpha\ \theta^{-1}\theta(c \leftharpoondown h) \\
&= (\alpha \circ \theta^{-1})\theta(c \leftharpoondown h).
\end{aligned}$$

Since $\alpha \in C^*$ is arbitrary, we obtain that $\theta(c) \leftharpoondown h = \theta(c \leftharpoondown h)$. Therefore, $(\theta(C) \blacktriangleleft H, \theta, \pi)$ is a globalization to C, and the proof is complete. □

2.2.2 The Standard Globalization

Now we will show that every partial module coalgebra has a globalization and we will also construct the standard globalization for a given partial module coalgebra.

Let C be a right partial H-module coalgebra and consider the coalgebra $C \otimes H$ with the natural structure of a tensor coalgebra. Consider the coalgebra monomorphism from C into $C \otimes H$ as the natural embedding

$$\begin{aligned}
\varphi\colon C &\longrightarrow C \otimes H \\
c &\longmapsto c \otimes 1_H.
\end{aligned}$$

Clearly, φ is a well-defined linear map and, moreover, it is injective. Then consider the comultiplicative projection from $C \otimes H$ onto $\varphi(C)$ given as follows

$$\begin{aligned} \pi\colon C \otimes H &\longrightarrow \varphi(C) \\ c \otimes h &\longmapsto (c \leftharpoonup h) \otimes 1_H. \end{aligned}$$

Clearly π is a well-defined linear map. From the property PMC1 and since $\pi(c \otimes 1_H) = \varphi(c)$, for all $c \in C$, one conclude that π is in fact a projection.

We claim that π is comultiplicative. In fact, let $c \in C$ and $h \in H$. Then we have

$$\begin{aligned} \Delta(\pi(c \otimes h)) &= \Delta((c \leftharpoonup h) \otimes 1_H) \\ &= (c \leftharpoonup h)_1 \otimes 1_H \otimes (c \leftharpoonup h)_2 \otimes 1_H \\ &\overset{(PMC2)}{=} c_1 \leftharpoonup h_1 \otimes 1_H \otimes c_2 \leftharpoonup h_2 \otimes 1_H \\ &= \pi(c_1 \otimes h_1) \otimes \pi(c_2 \otimes h_2) \\ &= (\pi \otimes \pi)\Delta(c \otimes h), \end{aligned}$$

therefore π is comultiplicative, as claimed.

With the above noticed we are able to construct a globalization for every partial H-module coalgebra.

Theorem 2.2.5. *Let H be a Hopf algebra. Every right partial H-module coalgebra has a globalization.*

Proof. From Examples 2.1.4 and 2.1.5, we know that $C \otimes H$ is an H-module coalgebra, with action given by right multiplication in H. By the above noticed, we already have the applications $\varphi\colon C \to C \otimes H$ and $\pi\colon C \otimes H \to \varphi(C)$, as required in Definition 2.2.1. Then we only need to show that the conditions GMC1 and GMC2 hold.

(GMC1): For every $h, k \in H$ and $c \in C$, we have

$$\begin{aligned} \pi[\varepsilon(\pi((c \otimes h)_1))(c \otimes h)_2 \blacktriangleleft k] &\overset{(PMC2)}{=} \pi[\varepsilon(\pi(c_1 \otimes h_1))(c_2 \otimes h_2) \blacktriangleleft k] \\ &= \varepsilon(c_1 \leftharpoonup h_1)\pi[(c_2 \otimes h_2) \blacktriangleleft k] \\ &\overset{(MC2)}{=} \varepsilon(c_1 \leftharpoonup h_1)\pi[(c_2 \otimes h_2 k)] \\ &= \varepsilon(c_1 \leftharpoonup h_1)(c_2 \leftharpoonup h_2 k) \\ &\overset{(PMC3)}{=} (c \leftharpoonup h) \leftharpoonup k \otimes 1_H \\ &= \pi[(c \leftharpoonup h) \otimes k] \\ &= \pi[((c \leftharpoonup h) \otimes 1_H) \blacktriangleleft k] \\ &= \pi[\pi(c \otimes h) \blacktriangleleft k] \end{aligned}$$

(GMC2): Let $h \in H$ and $c \in C$, then

$$\begin{aligned}\varphi(c) \leftharpoonup h &= \pi[\varphi(c) \blacktriangleleft h]\\ &= \pi[(c \otimes 1_H) \blacktriangleleft h]\\ &= \pi[c \otimes h]\\ &= c \leftharpoonup h \otimes 1_H\\ &= \varphi(c \leftharpoonup h).\end{aligned}$$

Moreover, by the definition of π, φ and $\blacktriangleleft$ it follows that $\varphi(C) \blacktriangleleft H = C \otimes H$. Therefore $C \otimes H$ is a globalization for C. $\square$

The globalization above constructed is called the *standard globalization* and it is close related with the standard globalization for partial module algebras defined by Alves and Batista in [2], as we will see below.

Given a right partial H-module coalgebra C one can construct the standard globalization $(C \otimes H, \varphi, \pi)$, as above. Now consider the multiplicative map

$$\begin{aligned}\phi \colon C^* &\longrightarrow (C \otimes H)^*\\ \alpha &\longmapsto \alpha \circ \varphi^{-1} \circ \pi.\end{aligned}$$

Thus, by the Theorem 2.2.4, we have that $(H \triangleright \phi(C^*), \phi)$ is a globalization for C^*, where the action on $(C \otimes H)^*$ is given by $(h \triangleright \xi)(c \otimes k) = \xi(c \otimes k\, h)$, for every $\xi \in (C \otimes H)^*$. Now, consider the following algebra isomorphism given by the adjoint isomorphism

$$\begin{aligned}\Psi \colon (C \otimes H)^* &\longrightarrow \mathrm{Hom}(H, C^*)\\ \xi &\longmapsto [\Psi(\xi)(h)](c) = \xi(c \otimes h)\end{aligned}$$

which is an H-module morphism. In fact, let $h, k \in H$, $c \in C$ and $\xi \in (C \otimes H)^*$. Then

$$\begin{aligned}\{[\Psi(h \triangleright \xi)](k)\}(c) &= [(h \triangleright \xi)](c \otimes k)\\ &= \xi(c \otimes k\, h)\\ &= \{[\Psi(\xi)](k\, h)\}(c)\\ &= \{[h \triangleright \Psi(\xi)](k)\}(c)\end{aligned}$$

which shows that Ψ is in fact an H-module map. Moreover, composing Ψ with ϕ we obtain

$$\begin{aligned}\{[\Psi\phi(\alpha)](h)\}(c) &= \phi(\alpha)(c \otimes h)\\ &= \alpha(\varphi^{-1}(\pi(c \otimes h)))\\ &= \alpha(\varphi^{-1}(c \leftharpoonup h \otimes 1_H))\\ &= \alpha(c \leftharpoonup h)\\ &= (h \rightarrow \alpha)(c)\\ &= [\Phi(\alpha)(h)](c),\end{aligned}$$

where $\Phi\colon C^* \to \mathrm{Hom}(H, C^*)$ given by $\Phi(\alpha)(h) = h \rightarrow \alpha$, for all $h \in H$ and $\alpha \in C^*$, is the multiplicative map that appear in the construction of the standard globalization of C^* (see [2, Theorem 1]). We finish this subsection by presenting a result that summarizes what was discussed above.

Theorem 2.2.6. *Let H be a Hopf algebra and C a right partial H-module coalgebra. The standard globalization for C is dual to the standard globalization for the left partial H-module algebra C^*.* $\square$

Chapter 3

Partial Coactions on Coalgebras

3.1 Partial Comodules Coalgebras

Comodule coalgebra is a dual notion of module algebra, as well as comodule is a dual notion of module and coalgebra is a dual notion of algebra.

Our aim in this chapter is to develop the theory of partial comodule coalgebras introduced by Batista and Vercruysse in [4]. In particular, we will discuss about possible relations between the partial and the correspondent global structures. Also, some examples will be constructed and, moreover, interesting properties of comodule coalgebras will be pointed out. After that, we will relate the four partial structures: partial module algebras, partial comodule algebras, partial module coalgebras and partial comodule coalgebras.

As the main aim, we will develop the notion of globalization for partial comodule coalgebras, defining the notion of induced partial coaction. We will also show the relations among this new globalization and the others already defined before, as well as we will construct a globalization for partial comodule coalgebras under a condition of rationality.

Given two vector spaces V and W, we will write $\tau_{V,W}$ to denote the standard isomorphism between $V \otimes W$ and $W \otimes V$. We start this section with the following well know concept.

Definition 3.1.1 ((Global) Comodule Coalgebra)**.** Let H be a Hopf algebra, D a coalgebra and $\lambda \colon D \to H \otimes D$ a linear map. We say that D is a *left H-comodule coalgebra* via λ if the following conditions hold:

(CC1) $(\varepsilon_H \otimes I)\lambda(d) = d$;

(CC2) $(I \otimes \Delta_D)\lambda(d) = (m_H \otimes I \otimes I)(I \otimes \tau_{D,H} \otimes I)(\lambda \otimes \lambda)\Delta_D(d)$;

(CC3) $(I \otimes \lambda)\lambda(d) = (\Delta_H \otimes I)\lambda(d)$.

Since H is a $\Bbbk$-algebra, it follows that the category of left H-comodules $({}^H\mathfrak{M}, \otimes_{\Bbbk}, \Bbbk)$ is a monoidal category. Thus, we can see the above definition in a categorical approach, in the following sense(cf. [29, Definition 11.3.7]): *a $\Bbbk$-vector space D is a left H-comodule coalgebra if it is a coalgebra object in the category of left H-comodules.*

By this categorical point of view, we need to require an additional condition in Definition 3.1.1, that is

$$(I \otimes \varepsilon_D)\lambda(d) = \varepsilon_D(d)1_H. \tag{3.1}$$

Also in this context of coactions, exactly as it happens in the case of actions, when H is simply a bialgebra, then the above condition need to be assumed as an axiom in the definition of comodule coalgebra over H. But, in the case when H is a Hopf algebra (and then it has an antipode), this extra condition is, in fact, a consequence from the others axioms of Definition 3.1.1, as it is showed in the next result.

Proposition 3.1.2. *Let D be a left H-comodule coalgebra in the sense of Definition 3.1.1, where H is a Hopf algebra. Then $(I \otimes \varepsilon_D)\lambda(d) = \varepsilon_D(d)1_H$, for any $d \in D$.*

Proof. Let $d \in D$. Thus we have

$$\begin{aligned}
\varepsilon_D(d)1_H &\overset{(CC1)}{=} 1_H\varepsilon_H(d^{-1})\varepsilon_D(d^{-0}) \\
&= d^{-1}{}_1S(d^{-1}{}_2)\varepsilon_D(d^{-0}) \\
&\overset{(CC3)}{=} d^{-1}S(d^{-0-1})\varepsilon_D(d^{-0-0}) \\
&= d^{-1}S([\varepsilon(d^{-0}{}_1)d^{-0}{}_2]^{-1})\varepsilon_D([\varepsilon(d^{-0}{}_1)d^{-0}{}_2]^{-0}) \\
&= d^{-1}\varepsilon_D(d^{-0}{}_1)S(d^{-0}{}_2{}^{-1})\varepsilon_D(d^{-0}{}_2{}^{-0}) \\
&\overset{(CC2)}{=} d_1{}^{-1}d_2{}^{-1}\varepsilon_D(d_1{}^{-0})S(d_2{}^{-0-1})\varepsilon_D(d_2{}^{-0-0}) \\
&= d_1{}^{-1}\varepsilon_D(d_1{}^{-0})d_2{}^{-1}S(d_2{}^{-0-1})\varepsilon_D(d_2{}^{-0-0}) \\
&\overset{(CC3)}{=} d_1{}^{-1}\varepsilon_D(d_1{}^{-0})d_2{}^{-1}{}_1S(d_2{}^{-1}{}_2)\varepsilon_D(d_2{}^{-0}) \\
&= d_1{}^{-1}\varepsilon_D(d_1{}^{-0})\varepsilon_H(d_2{}^{-1})\varepsilon_D(d_2{}^{-0}) \\
&= d_1{}^{-1}\varepsilon_D(d_1{}^{-0})\varepsilon_D(d_2{}^{-0}\varepsilon_H(d_2{}^{-1})) \\
&\overset{(CC1)}{=} d_1{}^{-1}\varepsilon_D(d_1{}^{-0})\varepsilon_D(d_2) \\
&= d^{-1}\varepsilon_D(d^{-0}).
\end{aligned}$$

Therefore, $(I \otimes \varepsilon_D)\lambda(d) = \varepsilon_D(d)1_H$, for any $d \in D$, as desired. □

Remark 3.1.3. From the above proposition it follows that one can define comodule coalgebra for non-counital coalgebras D in such a way that if the considered coalgebra D is counital, then both definitions coincide.

Now we will exhibit some classical examples of comodule coalgebras, for more examples see [29].

Example 3.1.4. Let H be a Hopf algebra. Then H becomes an H-comodule coalgebra with $\lambda\colon H \to H \otimes H$ defined by $\lambda(h) = h_1S(h_3) \otimes h_2$.

Example 3.1.5. Given a coalgebra D and a Hopf algebra H, then it follows that D is an H-comodule coalgebra, via $\lambda\colon D \to H \otimes D$ defined by $\lambda(d) = 1_H \otimes d$, for every $d \in D$.

Example 3.1.6. Let H be a finite dimensional Hopf algebra. Then H^* is an H-comodule coalgebra with structure given by $\lambda\colon H^* \to H \otimes H^*$ defined as $\lambda(f) = \sum\limits_{i=1}^{n} h_i \otimes f * h_i^*$, where $\{h_i\}_{i=1}^n$ and $\{h_i^*\}_{i=1}^n$ are dual basis for H and H^*, respectively.

Example 3.1.7. Let H be a Hopf algebra and consider a left H-comodule coalgebra C with structure given by a linear map λ. Given any coalgebra D, it follows that $C \otimes D$ is a left H-comodule coalgebra via $\lambda \otimes I_D$.

3.1.1 Definitions and Correspondences

In this subsection we will construct some important examples and we will discuss about some properties of partial comodule coalgebras. We will describe some relations among the four partial structures. In particular, these relations discussed here will be useful to construct a globalization of partial comodule coalgebras in the next section. We start this subsection with the following definition.

Definition 3.1.8 (Partial Comodule Coalgebra [4, Definition 6.1])**.** Let H be a Hopf algebra, C a coalgebra and $\lambda'\colon C \to H \otimes C$ a linear map. We say that C is a *left partial H-comodule coalgebra* via λ', if the following conditions hold:

(PCC1) $(\varepsilon_H \otimes I)\lambda'(c) = c$;

(PCC2) $(I \otimes \Delta_C)\lambda'(c) = (m_H \otimes I \otimes I)(I \otimes \tau_{C,H} \otimes I)(\lambda' \otimes \lambda')\Delta_C(c)$;

(PCC3) $(I \otimes \lambda')\lambda'(c) = (m_H \otimes I \otimes I)\{\nabla \otimes [(\Delta_H \otimes I)\lambda']\}\Delta_C(c)$,

where $\nabla\colon C \to H$ is defined by $\nabla(c) = (I \otimes \varepsilon_C)\lambda'(c)$.

We say that C is a *symmetric left partial H-comodule coalgebra* if the following additional condition holds:

(PCC4) $(I \otimes \lambda')\lambda'(c) = (m_H \otimes I \otimes I)(I \otimes \tau_{H\otimes C,H})\{[(\Delta_H \otimes I)\lambda'] \otimes \nabla\}\Delta_C(c)$.

Remark 3.1.9. One can see the above definition in terms of commutative diagrams, as it is showed below.

$$\begin{tikzcd}
C \arrow[rr, "I"] \arrow[dr, "\lambda'"'] & PCC1 & C \\
& H \otimes C \arrow[ur, "\varepsilon\otimes I"'] &
\end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd}
C \arrow[rr, "\Delta"] \arrow[d, "\lambda'"'] & & C \otimes C \arrow[d, "\lambda' \otimes \lambda'"] \\
H \otimes C \arrow[d, "I \otimes \Delta"'] & PCC2 & H \otimes C \otimes H \otimes C \arrow[d, "I \otimes \tau_{C,H} \otimes I"] \\
H \otimes C \otimes C & & H \otimes H \otimes C \otimes C \arrow[ll, "m_H \otimes I \otimes I"]
\end{tikzcd} \tag{3.3}$$

$$\begin{tikzcd}
C \arrow[rr, "\Delta"] \arrow[d, "\lambda'"'] & & C \otimes C \arrow[d, "\lambda' \otimes \lambda'"] \\
H \otimes C \arrow[d, "I \otimes \lambda'"'] & PCC3 & H \otimes C \otimes H \otimes C \arrow[d, "I \otimes \varepsilon \otimes \Delta \otimes I"] \\
H \otimes H \otimes C & & H \otimes H \otimes H \otimes C \arrow[ll, "m_H \otimes I \otimes I"]
\end{tikzcd} \tag{3.4}$$

Remark 3.1.10. Denoting the partial coaction of an element by $\lambda'(c) = c^{-\overline{1}} \otimes c^{-\overline{0}}$, we can write the axioms of Definition 3.1.8 in terms of its elements, as follow:

(PCC1) $c = \varepsilon(c^{-\overline{1}})c^{-\overline{0}}$;

(PCC2) $c^{-\overline{1}} \otimes {c^{-\overline{0}}}_1 \otimes {c^{-\overline{0}}}_2 = {c_1}^{-\overline{1}}{c_2}^{-\overline{1}} \otimes {c_1}^{-\overline{0}} \otimes {c_2}^{-\overline{0}}$;

(PCC3) $c^{-\overline{1}} \otimes c^{-\overline{0}-\overline{1}} \otimes c^{-\overline{0}-\overline{0}} = {c_1}^{-\overline{1}}\varepsilon({c_1}^{-\overline{0}})\, {{c_2}^{-\overline{1}}}_1 \otimes {{c_2}^{-\overline{1}}}_2 \otimes {c_2}^{-\overline{0}}$;

(PCC4) $c^{-\overline{1}} \otimes c^{-\overline{0}-\overline{1}} \otimes c^{-\overline{0}-\overline{0}} = {{c_1}^{-\overline{1}}}_1\, {c_2}^{-\overline{1}}\varepsilon({c_2}^{-\overline{0}}) \otimes {{c_1}^{-\overline{1}}}_2 \otimes {c_1}^{-\overline{0}}$.

Proposition 3.1.11. *Given a left partial H-comodule coalgebra C via λ', the following equalities hold*

$$c^{-\overline{1}} \otimes c^{-\overline{0}} = \nabla(c_1){c_2}^{-\overline{1}} \otimes {c_2}^{-\overline{0}} = {c_1}^{-\overline{1}}\nabla(c_2) \otimes {c_1}^{-\overline{0}} \tag{3.5}$$

$$\nabla(c_1)\nabla(c_2) = \nabla(c) \tag{3.6}$$

Proof. It follows directly from the definition of ∇ and PCC2. □

From the Definitions 3.1.1 and 3.1.8, it follows that every comodule coalgebra is a partial comodule coalgebra, so that this last concept is a generalization of the first one. We will register this fact as an example.

Example 3.1.12. Let H be a Hopf algebra. Every left H-comodule coalgebra is a left partial H-comodule coalgebra.

The next result give us a simple method to construct new examples of comodule coalgebras. The proof is straightforward and it will be omitted.

Proposition 3.1.13. *Let $\lambda' \colon \Bbbk \to H \otimes \Bbbk$ be a linear map, where $\lambda'(1_{\Bbbk}) = h \otimes 1_{\Bbbk}$, for some $h \in H$. Then $\Bbbk$ is a left partial H-comodule coalgebra if and only if the following conditions hold:*

1. $\varepsilon_H(h) = 1_{\Bbbk}$;
2. $h \otimes h = (h \otimes 1_H)\Delta(h)$.

One can see as an immediate consequence of this definition that $h^2 = h$.

As an application of the above result, we present the next example.

Example 3.1.14. Let G be a group and denote by $x = \sum_{g \in G} \alpha_g g$ an arbitrary element of $\Bbbk G$. With this notation, consider $N = \{g \in G \mid \alpha_g \neq 0\}$. Then $\Bbbk$ is a left partial $\Bbbk G$-comodule coalgebra if and only if N is a finite subgroup of G. In this case we have

$$\alpha_g = \begin{cases} \frac{1}{|N|}, & \text{if } g \in N \\ 0, & \text{otherwise.} \end{cases}$$

The next result show us that the equality (3.1) is a necessary and sufficient condition to a partial comodule coalgebra be global.

Proposition 3.1.15. *Let H be a Hopf algebra and consider a left partial H-comodule coalgebra C via λ'. Then C is a (global) H-comodule coalgebra if and only if $\nabla(c) = \varepsilon_C(c)1_H,\ \forall c \in C$, where ∇ is as defined in Definition 3.1.8.*

Proof. By Proposition 3.1.2, it is clear that if C is an H-comodule coalgebra then $\nabla(c) = \varepsilon_C(c)1_H,\ \forall c \in C$.

Conversely, note that

$$\begin{aligned}
(I \otimes \lambda')\lambda'(c) &= (m_H \otimes I \otimes I)\{\nabla \otimes [(\Delta_H \otimes I)\lambda']\}\Delta_C(c) \\
&= (m_H \otimes I \otimes I)\{\nabla(c_1) \otimes [(\Delta_H \otimes I)\lambda'(c_2)]\} \\
&\overset{(3.1)}{=} (m_H \otimes I \otimes I)\{1_H\varepsilon_C(c_1) \otimes [(\Delta_H \otimes I)\lambda'(c_2)]\} \\
&= (m_H \otimes I \otimes I)\{1_H \otimes [(\Delta_H \otimes I)\lambda'(\varepsilon_C(c_1)c_2)]\} \\
&= (m_H \otimes I \otimes I)\{1_H \otimes [(\Delta_H \otimes I)\lambda'(c)]\} \\
&= (\Delta_H \otimes I)\lambda'(c)
\end{aligned}$$

which turns C into a left H-comodule coalgebra. $\square$

Now we will relate these four partial structures, but first we need to define the finite dual of a Hopf algebra (or dual Hopf algebra). Let A be an algebra, and consider the finite dual of A, defined by

$$\begin{aligned} A^0 &= \{f \in A^* \mid f(I) = 0, \text{ for some } I \text{ ideal of } A \text{ of finite codimension}\} \\ &= \{f \in A^* \mid \exists \textstyle\sum_{i=0}^n f_i \otimes f_i' \in A^* \otimes A^* \text{ with } f(ab) = \sum_i f_i(a) f_i'(b), \forall a, b \in A\}, \end{aligned}$$

that is a coalgebra with structure given by $\Delta\colon f \mapsto f_1 \otimes f_2$ such that $f(ab) = f_1(a)f_2(b)$, for all $a, b \in A$ and $\varepsilon\colon f \mapsto \varepsilon(f) = f(1_A)$ (cf. [29, Section 2.5]). Moreover, when A is finite dimensional, it follows that $A^0 = A^*$. Let H be a Hopf algebra, then its finite dual H^0 is a Hopf algebra too (cf. [29, Section 7.4]). We say that H^0 *separate points* if

$$f(h) = 0, \text{ for every } f \in H^0 \implies h = 0.$$

Initially, we will see that from a given coaction we can consider two induced actions, without any kind of restriction. Under the additional hypothesis that H^0 separate points, we will show that partial comodule coalgebras, partial module coalgebras and partial module algebras are close related. Moreover, when H is finite dimensional, then all these partial structures are equivalent. Batista and Vercruysse have obtained these same relations, but in a different way, using dual pairings between algebras and coalgebras (see [4]). The arguments presented here are more direct.

Given a coalgebra C and any linear map $\lambda'\colon C \to H \otimes C$, denoted by $\lambda'(c) = c^{-\overline{1}} \otimes c^{-\overline{0}}$. We have two induced linear maps ${}_{\lambda'}\!\leftharpoonup\colon C \otimes H^* \to C$ and $\rightarrow_{\lambda'}\colon H^* \otimes C^* \to C^*$, given respectively by

$$c \;{}_{\lambda'}\!\leftharpoonup f = f(c^{-\overline{1}}) c^{-\overline{0}}, \forall c \in C, f \in H^* \tag{3.7}$$

$$(f \rightarrow_{\lambda'} \alpha)(c) = f(c^{-\overline{1}})\alpha(c^{-\overline{0}}), \forall f \in H^*, \alpha \in C^*, c \in C. \tag{3.8}$$

In this context, if C is a left partial H-comodule coalgebra by λ', then we can restrict ${}_{\lambda'}\!\leftharpoonup$ and $\rightarrow_{\lambda'}$ to the subspaces $C \otimes H^0$ and $H^0 \otimes C^*$, respectively, obtaining the following results.

Theorem 3.1.16. *With the above notations, if H is a Hopf algebra and C is a left partial H-comodule coalgebra via λ', then we have that:*

(1) C is a right partial H^0-module coalgebra via ${}_{\lambda'}\!\leftharpoonup$;

(2) C^* is a left partial H^0-module algebra via $\rightarrow_{\lambda'}$.

Proof. To prove (1), we need to verify the conditions given in Definition 2.1.6. Thus,
(PMC1): Let $c \in C$, then

$$\begin{aligned} c \;{}_{\lambda'}\!\leftharpoonup 1_{H^0} &= c \;{}_{\lambda'}\!\leftharpoonup \varepsilon_H \\ &= c^{-\overline{0}} \varepsilon_H(c^{-\overline{1}}) \\ &\overset{(PCC1)}{=} c \end{aligned}$$

(PMC2): Let $c \in C$ and $f \in H^0$, then

$$\begin{array}{rcl} \Delta(c \,{}_{\lambda'}\!\leftharpoonup f) & = & \Delta(f(c^{-\bar{1}})c^{-\bar{0}}) \\ & = & f(c^{-\bar{1}}){c^{-\bar{0}}}_1 \otimes {c^{-\bar{0}}}_2 \\ & \overset{(PCC2)}{=} & f({c_1}^{-\bar{1}}{c_2}^{-\bar{1}}){c_1}^{-\bar{0}} \otimes {c_2}^{-\bar{0}} \\ & = & f_1({c_1}^{-\bar{1}}){c_1}^{-\bar{0}} \otimes f_2({c_2}^{-\bar{1}}){c_2}^{-\bar{0}} \\ & = & c_1 \,{}_{\lambda'}\!\leftharpoonup f_1 \otimes c_2 \,{}_{\lambda'}\!\leftharpoonup f_2 \end{array}$$

(PMC3): Let $c \in C$ and f, $g \in H^0$, then

$$\begin{array}{rcl} (c \,{}_{\lambda'}\!\leftharpoonup f) \,{}_{\lambda'}\!\leftharpoonup g & = & f(c^{-\bar{1}})c^{-\bar{0}} \,{}_{\lambda'}\!\leftharpoonup g \\ & = & f(c^{-\bar{1}})g(c^{-0-1})c^{-0-0} \\ & \overset{(PCC3)}{=} & f({c_1}^{-\bar{1}}\varepsilon_C({c_1}^{-\bar{0}}){{c_2}^{-\bar{1}}}_1)g({{c_2}^{-\bar{1}}}_2){c_2}^{-\bar{0}} \\ & = & f_1({c_1}^{-\bar{1}}\varepsilon_C({c_1}^{-\bar{0}}))f_2 * g({c_2}^{-\bar{1}}){c_2}^{-\bar{0}} \\ & = & \varepsilon_C(f_1({c_1}^{-\bar{1}}){c_1}^{-\bar{0}})f_2 * g({c_2}^{-\bar{1}}){c_2}^{-\bar{0}} \\ & = & \varepsilon_C(c_1 \,{}_{\lambda'}\!\leftharpoonup f_1)[c_2 \,{}_{\lambda'}\!\leftharpoonup f_2 * g] \end{array}$$

Therefore, C is a right partial H^0-module coalgebra with structure given by ${}_{\lambda'}\!\leftharpoonup$.

To prove (2) we need to show that the linear map $\rightarrow_{\lambda'}$ satisfy the conditions given in Definition 1.3.1. Thus,

(PMA1): Let $\alpha \in C^*$, then

$$\begin{array}{rcl} (1_{H^0} \rightarrow_{\lambda'} \alpha)(c) & = & (\varepsilon_H \rightarrow_{\lambda'} \alpha)(c) \\ & = & \varepsilon_H(c^{-\bar{1}})\alpha(c^{-\bar{0}}) \\ & = & \alpha(\varepsilon_H(c^{-\bar{1}})c^{-\bar{0}}) \\ & \overset{(PCC1)}{=} & \alpha(c) \end{array}$$

(PMA2): Let α, $\beta \in C^*$ and $f \in H^0$, then

$$\begin{array}{rcl} (f \rightarrow_{\lambda'} (\alpha * \beta))(c) & = & f(c^{-\bar{1}})(\alpha * \beta)(c^{-\bar{0}}) \\ & = & f(c^{-\bar{1}})\alpha({c^{-\bar{0}}}_1)\beta({c^{-\bar{0}}}_2) \\ & \overset{(PCC2)}{=} & f({c_1}^{-\bar{1}}{c_2}^{-\bar{1}})\alpha({c_1}^{-\bar{0}})\beta({c_2}^{-\bar{0}}) \\ & = & f_1({c_1}^{-\bar{1}})f_2({c_2}^{-\bar{1}})\alpha({c_1}^{-\bar{0}})\beta({c_2}^{-\bar{0}}) \\ & = & f_1({c_1}^{-\bar{1}})\alpha({c_1}^{-\bar{0}})f_2({c_2}^{-\bar{1}})\beta({c_2}^{-\bar{0}}) \\ & = & (f_1 \rightarrow_{\lambda'} \alpha)(c_1)(f_2 \rightarrow_{\lambda'} \beta)(c_2) \\ & = & (f_1 \rightarrow_{\lambda'} \alpha) * (f_2 \rightarrow_{\lambda'} \beta)(c) \end{array}$$

(PMA3): Let $\alpha \in C^*$ and $f, g \in H^0$, then

$$\begin{aligned}
(f \rightharpoonup_{\lambda'} (g \rightharpoonup_{\lambda'} \alpha))(c) &= f(c^{-\bar{1}})(g \rightharpoonup_{\lambda'} \alpha)(c^{-\bar{0}}) \\
&= f(c^{-\bar{1}})g(c^{-\bar{0}-\bar{1}})\alpha(c^{-\bar{0}-\bar{0}}) \\
&\overset{(PCC3)}{=} f({c_1}^{-\bar{1}}\varepsilon_C({c_1}^{-\bar{0}}){c_2}^{-\bar{1}}{}_1)g({c_2}^{-\bar{1}}{}_2)\alpha({c_2}^{-\bar{0}}) \\
&= f_1({c_1}^{-\bar{1}}\varepsilon_C({c_1}^{-\bar{0}}))[f_2 * g]({c_2}^{-\bar{1}})\alpha({c_2}^{-\bar{0}}) \\
&= f_1({c_1}^{-\bar{1}})\varepsilon_C({c_1}^{-\bar{0}})[f_2 * g]({c_2}^{-\bar{1}})\alpha({c_2}^{-\bar{0}}) \\
&= (f_1 \rightharpoonup_{\lambda'} \varepsilon_C)(c_1)(f_2 * g \rightharpoonup_{\lambda'} \alpha)(c_2) \\
&= [(f_1 \rightharpoonup_{\lambda'} \varepsilon_C) * (f_2 * g \rightharpoonup_{\lambda'} \alpha)](c)
\end{aligned}$$

Therefore, C is a left partial H^0-module algebra with structure given by $\rightharpoonup_{\lambda'}$. □

In general, we do not know if the converse of the above theorem is true. However, if H^0 separate points then the converse holds, as the next result shows.

Theorem 3.1.17. *With the above notations, if H^0 separate points, then the following conditions are equivalent:*

(1) C is a right partial H^0-module coalgebra via ${}_{\lambda'}\!\leftharpoonup$;

(2) C is a left partial H^0-module algebra via $\rightharpoonup_{\lambda'}$;

(3) C is a left partial H-comodule coalgebra via λ'.

Proof. Note that for every $f \in H^0$, $c \in C$ and $\alpha \in C^*$, we have

$$(f \rightharpoonup_{\lambda'} \alpha)(c) = f(c^{-\bar{1}})\alpha(c^{-\bar{0}}) = \alpha(f(c^{-\bar{1}})c^{-\bar{0}}) = \alpha(c \;{}_{\lambda'}\!\leftharpoonup f).$$

Then, by Propositions 2.1.13 and 2.1.14, it is clear that the conditions (1) and (2) are equivalent.

Thus, by Theorem 3.1.16 it is enough to show that (1) or (2) implies (3).

In fact, supposing that the condition (1) holds, we obtain

(PCC1): For $c \in C$,

$$\begin{aligned}
(\varepsilon_H \otimes I)\bar{\lambda}(c) &= \varepsilon_H(c^{-\bar{1}})c^{-\bar{0}} \\
&= c \;{}_{\lambda'}\!\leftharpoonup \varepsilon_H \\
&= c \;{}_{\lambda'}\!\leftharpoonup 1_{H^0} \\
&\overset{(PMC1)}{=} c
\end{aligned}$$

(PCC2): For any $f \in H^0$ and any $c \in C$, we have

$$
\begin{aligned}
&(f \otimes I \otimes I)(m_H \otimes I \otimes I)(I \otimes \tau_{C,H} \otimes I)(\lambda' \otimes \lambda')\Delta_C(c) \\
&\quad = f({c_1}^{-\overline{1}}{c_2}^{-\overline{1}}){c_1}^{-\overline{0}} \otimes {c_2}^{-0} \\
&\quad = f_1({c_1}^{-\overline{1}})f_2({c_2}^{-\overline{1}}){c_1}^{-\overline{0}} \otimes {c_2}^{-0} \\
&\quad = f_1({c_1}^{-\overline{1}}){c_1}^{-\overline{0}} \otimes f_2({c_2}^{-\overline{1}}){c_2}^{-0} \\
&\quad = c_1 \;{}_{\lambda'}\!\leftharpoonup f_1 \otimes c_2 \;{}_{\lambda'}\!\leftharpoonup f_2 \\
&\quad \overset{(PMC2)}{=} \Delta(c \;{}_{\lambda'}\!\leftharpoonup f) \\
&\quad = \Delta(f(c^{-\overline{1}})c^{-\overline{0}}) \\
&\quad = f(c^{-\overline{1}})\Delta(c^{-\overline{0}}) \\
&\quad = (f \otimes I \otimes I)(c^{-\overline{1}} \otimes {c^{-\overline{0}}}_1 \otimes {c^{-\overline{0}}}_2) \\
&\quad = (f \otimes I \otimes I)(I \otimes \Delta_C)\lambda'(c)
\end{aligned}
$$

(PCC3): Take $f, g \in H^0$ and $c \in C$, so we have

$$
\begin{aligned}
&(f \otimes g \otimes I)(m_H \otimes I \otimes I)\{\nabla \otimes [(\Delta_H \otimes I)\lambda']\}\Delta_C(c) \\
&\quad = f_1(\nabla(c_1){{c_2}^{-\overline{1}}}_1)g({{c_2}^{-\overline{1}}}_2){c_2}^{-\overline{0}} \\
&\quad = f({c_1}^{-\overline{1}}\varepsilon_c({c_1}^{-\overline{0}}){{c_2}^{-\overline{1}}}_1)g({{c_2}^{-\overline{1}}}_2){c_2}^{-\overline{0}} \\
&\quad = f({c_1}^{-\overline{1}}{{c_2}^{-\overline{1}}}_1)\varepsilon_c({c_1}^{-\overline{0}})g({{c_2}^{-\overline{1}}}_2){c_2}^{-\overline{0}} \\
&\quad = f_1({c_1}^{-\overline{1}})\varepsilon_c({c_1}^{-\overline{0}})f_2({{c_2}^{-\overline{1}}}_1)g({{c_2}^{-\overline{1}}}_2){c_2}^{-\overline{0}} \\
&\quad = f_1({c_1}^{-\overline{1}})\varepsilon_c({c_1}^{-\overline{0}})[f_2 * g]({c_2}^{-\overline{1}}){c_2}^{-\overline{0}} \\
&\quad = \varepsilon_C(c_1 \;{}_{\lambda'}\!\leftharpoonup f_1)(c_2 \;{}_{\lambda'}\!\leftharpoonup f_2 * g) \\
&\quad \overset{(PMC3)}{=} c \;{}_{\lambda'}\!\leftharpoonup f \;{}_{\lambda'}\!\leftharpoonup g
\end{aligned}
$$

and

$$
\begin{aligned}
(f \otimes g \otimes I)(I \otimes \lambda')\lambda'(c) &= (f \otimes g \otimes I)(c^{-\overline{1}} \otimes c^{-\overline{0}-\overline{1}} \otimes c^{-\overline{0}-\overline{0}}) \\
&= f(c^{-\overline{1}})g(c^{-\overline{0}-\overline{1}})c^{-\overline{0}-\overline{0}} \\
&= f(c^{-\overline{1}})(c^{-\overline{0}} \;{}_{\lambda'}\!\leftharpoonup g) \\
&= (f(c^{-\overline{1}})c^{-\overline{0}} \;{}_{\lambda'}\!\leftharpoonup g) \\
&= (c \;{}_{\lambda'}\!\leftharpoonup f) \;{}_{\lambda'}\!\leftharpoonup g
\end{aligned}
$$

Therefore, C is a partial H-comodule coalgebra with structure given by λ', as desired. $\square$

The above theorem can be translated in the following commutative diagram:

$$
\begin{tikzcd}
(C, \lambda', H) \arrow[rr, shift left, leftarrow, dashed, "\mathrm{H}^0\ \text{sep points}"] \arrow[rr, shift right] \arrow[d, shift right, leftarrow, dashed, "\mathrm{H}^0\ \text{sep points}"'] \arrow[d, shift left] & & (C^*, \rightharpoonup_{\lambda'}, H^0) \arrow[dll, <->] \\
(C, {}_{\lambda'}\!\leftharpoonup, H^0) & &
\end{tikzcd}
\tag{3.9}
$$

The Theorems 3.1.16 and 3.1.17 above show relations between partial comodule coalgebra, partial module coalgebra and partial module algebra, whenever we start from a partial coaction λ'. In general, we can not start from an action and induce a coaction. To do this we will require a more strong hypothesis on H. More precisely, we will assume that H is finite dimensional.

Note that, if H is finite dimensional, then $H^0 = H^*$ (and so H^0 separate points). Moreover, given a linear map $\leftharpoonup\colon C \otimes H^* \to C$ and assuming that $\{h_i, h_i^*\}$ is dual basis of H and H^*, then we can induce a linear map $\lambda'_{\leftharpoonup}\colon C \to H \otimes C$, by

$$\lambda'_{\leftharpoonup}(c) = \sum_i h_i \otimes c \leftharpoonup h_i^*$$

and it is clear that

$$f(c^{-\overline{1}})c^{-\overline{0}} = c \leftharpoonup f$$

for all $c \in C$ and $f \in H^*$.

Thus, under the hypothesis that H is finite dimensional, we can induce a coaction of H on a coalgebra C from a given action of H^* on C. Also, as we have seen before, we know how to induce actions from coactions too. Therefore, we can start from a right action $\leftharpoonup$ of H^* on C, and then to induce a left coaction $\lambda'_{\leftharpoonup}$ of H on C and, again, to induce a right action ${}_{(\lambda'_{\leftharpoonup})}\leftharpoonup$ of H^* on C. After this process we will reobtain the original action $\leftharpoonup$, i.e., ${}_{(\lambda'_{\leftharpoonup})}\!\leftharpoonup = \leftharpoonup$. Analogously, we can repeat these constructions but now starting from a coaction λ' and we will have that $\lambda'_{({}_{\lambda'}\leftharpoonup)} = \lambda'$.

In [1], there is a similar construction for (right) partial H-comodule algebras and (left) partial H^*-module algebras. Thus, in the finite dimensional case, we have that these four partial structures are close related, in the following sense:

Theorem 3.1.18. *Let C be a coalgebra and H a finite dimensional Hopf algebra. Then the following statements are equivalent:*

(1) C is a left partial H-comodule coalgebra;

(2) C^* is a right partial H-comodule algebra;

(3) C is a right partial H^*-module coalgebra;

(4) C^* is a left partial H^*-module algebra.

where the relations between the correspondent actions and coactions are given in the following way: for any $c \in C,\ \alpha \in C^*$ and $f \in H^*$, we have

$$\alpha(c^{-\bar{1}})c^{-\bar{0}} = \alpha^{+\bar{0}}(c)\alpha^{+\bar{1}} \tag{3.10}$$

$$(f \rightharpoonup \alpha)(c) = \alpha(c \leftharpoonup f) \tag{3.11}$$

$$c \leftharpoonup f = f(c^{-\bar{1}})c^{-\bar{0}} \tag{3.12}$$

$$f \rightharpoonup \alpha = \alpha^{+\bar{0}} f(\alpha^{+\bar{1}}), \tag{3.13}$$

where $\lambda' \colon c \mapsto c^{-\bar{1}} \otimes c^{-\bar{0}}$ and $\rho' \colon \alpha \mapsto \alpha^{+\bar{0}} \otimes \alpha^{+\bar{1}}$ are the partial coactions on C and C^*, respectively. $\square$

After the Theorem 3.1.18, we can extend the Diagram 3.9 to the following commutative diagram, under the hypothesis that H is a finite dimensional Hopf algebra:

$$\begin{tikzcd}
(C, \lambda', H) \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & (C^*, \rightharpoonup_{\lambda'}, H^0) \arrow[d, leftrightarrow] \\
(C, {}_{\lambda'}\!\leftharpoonup, H^0) \arrow[rr, leftrightarrow] & & (C^*, \rho', H)
\end{tikzcd} \tag{3.14}$$

Now, we would like to discuss about the existence of a globalization for a partial comodule coalgebra like it was done in Chapter 2 for partial module coalgebras. This will be the subject of the next section.

3.2 Globalization for Partial Comodules Coalgebras

Our main goal in this section is to introduce the concept of globalization for partial comodules coalgebras. Following the same steps made in the case of partial module coalgebras in the last chapter, we will need previously introduce the notion of induced partial coaction. Thus, we start this section discussing about conditions for the existence of an induced partial coaction.

Let H be a Hopf algebra and consider a left H-comodule coalgebra D via $\lambda \colon d \mapsto d^{-1} \otimes d^{-0} \in H \otimes D$. Let C be a subcoalgebra of D. In order to induce a coaction of H on C we can restrict the coaction of D to C, but, in general, we do not have that $\lambda(C) \subseteq H \otimes C$. Thus, our strategy consist in take a linear map $\pi \colon D \to C$ and then to consider the following composite map

$$\begin{aligned} \lambda' \colon C &\longrightarrow H \otimes C \\ c &\longmapsto c^{-1} \otimes \pi(c^{-0}) \overset{not}{=} c^{-\bar{1}} \otimes c^{-\bar{0}}. \end{aligned} \tag{3.15}$$

Now we need to find sufficient conditions that we must require on π for that λ' defines a left partial H-comodule coalgebra structure on C. We will do this exactly studying the conditions given in Definition 3.1.8 which λ' must satisfy to provide a left partial H-comodule coalgebra structure on C.

First of all, observe that if π is a linear projection from D onto C, then λ' satisfies the condition $PCC1$ of Definition 3.1.8. In fact, given $c \in C$, we have

$$\begin{aligned}(\varepsilon \otimes I)\lambda'(c) &= \varepsilon(c^{-\overline{1}})\pi(c^{-\overline{0}}) \\ &= \pi(\varepsilon(c^{-\overline{1}})c^{-\overline{0}}) \\ &\overset{(CC1)}{=} \pi(c) \\ &= c,\end{aligned}$$

where the last equality holds since π is a projection (and so $\pi(c) = c$, for all $c \in C$).

Supposing now that π is a comultiplicative map then it follows that λ' satisfies the condition PCC2 of Definition 3.1.8. In fact, let $c \in C$, so

$$\begin{aligned}(I \otimes \Delta_C)\lambda'(c) &= c^{-1} \otimes \Delta(\pi(c^{-0})) \\ &= c^{-1} \otimes (\pi \otimes \pi)(\Delta(c^{-0})) \\ &= c^{-1} \otimes \pi(c^{-0}{}_1) \otimes \pi(c^{-0}{}_2) \\ &\overset{(CC2)}{=} c_1{}^{-1}c_2{}^{-1} \otimes \pi(c_1{}^{-0}) \otimes \pi(c_2{}^{-0}) \\ &= (m_H \otimes I \otimes I)(I \otimes \tau_{C,H} \otimes I)(\lambda' \otimes \lambda')\Delta_C(c).\end{aligned}$$

We recall that to define the induced partial module coalgebra in the last chapter, we needed to suppose that the comultiplicative projection used there should satisfy an special condition, namely, the equation (2.1). The same occurs in the setting of partial comodule coalgebras, as we will see below. In order to ensure that the condition PCC3 is satisfied by λ', we will assume the following condition on π:

$$\pi(d)^{-1} \otimes \pi(\pi(d)^{-0}) = d_2{}^{-1} \otimes \varepsilon(\pi(d_1))\pi(d_2{}^{-0}). \tag{3.16}$$

Thus, we suppose additionally that π satisfies (3.16). Let $c \in C$, so

$$\begin{aligned}(I \otimes \lambda')\lambda'(c) &= c^{-\overline{1}} \otimes c^{-\overline{0}-\overline{1}} \otimes c^{-\overline{0}-\overline{0}} \\ &= c^{-1} \otimes \pi(c^{-0})^{-1} \otimes \pi(\pi(c^{-0})^{-0}) \\ &\overset{(3.16)}{=} c^{-1} \otimes c^{-0}{}_2{}^{-1} \otimes \varepsilon(\pi(c^{-0}{}_1))\pi(c^{-0}{}_2{}^{-0}) \\ &\overset{(CC2)}{=} c_1{}^{-1}c_2{}^{-1} \otimes c_2{}^{-0-1} \otimes \varepsilon(\pi(c_1{}^{-0}))\pi(c_2{}^{-0-0}) \\ &\overset{(CC3)}{=} c_1{}^{-1}\varepsilon(\pi(c_1{}^{-0}))c_2{}^{-1}{}_1 \otimes c_2{}^{-1}{}_2 \otimes \pi(c_2{}^{-0}) \\ &= c_1{}^{-\overline{1}}\varepsilon(c_1{}^{-\overline{0}})c_2{}^{-\overline{1}}{}_1 \otimes c_2{}^{-\overline{1}}{}_2 \otimes c_2{}^{-\overline{0}} \\ &= (m_H \otimes I \otimes I)\{\nabla \otimes [(\Delta_H \otimes I)\lambda']\}\Delta_C(c).\end{aligned}$$

Therefore, given D a left H-comodule coalgebra, C a subcoalgebra of D and $\pi\colon D \to C$ a comultiplicative projection satisfying the equation (3.16), we have that C is a partial H-comodule coalgebra, called the *induced partial comodule coalgebra*, with partial coaction given by the equation (3.15).

We will summarize our above discussion in the next result.

Proposition 3.2.1 (Induced Partial Comodule Coalgebra)**.** *Let H be a Hopf algebra, D a left H-comodule coalgebra via λ and $C \subseteq D$ a subcoalgebra. Then, with the above notations, the linear map $\lambda'\colon C \to H \otimes C$, defined as in* (3.15)*, induces a structure of left partial H-comodule coalgebra on C.* □

After we have introduced the notion of induced partial comodule coalgebra, we are able to propose a definition of what would be a globalization for a partial comodule coalgebra. The next definition contemplates this.

Definition 3.2.2. Let H be a Hopf algebra and C a partial H-comodule coalgebra. Then a triple (D, θ, π) is a globalization for C, where D is an H-comodule coalgebra, θ is a coalgebra monomorphism from C into D and π is a comultiplicative projection from D onto $\theta(C)$, if the following conditions hold:

(GCC1) $\pi(d)^{-1} \otimes \pi(\pi(d)^{-0}) = {d_2}^{-1} \otimes \varepsilon(\pi(d_1))\pi({d_2}^{-0})$;

(GCC2) θ is an equivalence of partial H-comodule coalgebras;

(GCC3) D is the H-comodule coalgebra generated by $\theta(C)$.

We will explain the meaning of these conditions in the above definition.

Remark 3.2.3. The first item in Definition 3.2.2 tell us that it is possible to define the induced partial comodule coalgebra on $\theta(C)$. The second one tell us that this induced partial coaction coincide with the original, and this fact can be translated in the commutative diagram bellow:

$$\begin{tikzcd}
C \arrow[r, "\lambda'"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\circlearrowleft"] & H \otimes C \arrow[d, "I\otimes\theta"] \\
\theta(C) \arrow[r, "\lambda'"] & H \otimes \theta(C),
\end{tikzcd} \tag{3.17}$$

moreover, the second condition could be seen as

$$\theta(c)^{-1} \otimes \pi(\theta(c)^{-0}) = c^{\overline{-1}} \otimes \theta(c^{\overline{0}}). \tag{3.18}$$

Finally, the last condition of Definition 3.2.2 tell us that there is no subcomodules of D containing $\theta(C)$.

Remark 3.2.4. Analogously to the noticed for partial module coalgebras, if the map π in Definition 3.2.2 is a coalgebra map, then the partial coaction on C is global. To show it, one just need to calculate $I \otimes \varepsilon$ in the condition GCC2.

3.2.1 Correspondence Between Globalizations

Let H be a Hopf algebra and suppose that C is a left partial H-comodule coalgebra. From Theorem 3.1.16, item 1, we can induce an structure of right partial H^0-module coalgebra on C. The same is true for a (global) comodule coalgebra, of course inducing a (global) H^0-module coalgebra. Therefore, given a globalization (D, θ, π) for a partial H-comodule coalgebra C, one can ask: *Is there some relation between D and C when viewed as H^0-module coalgebras (global and partial, respectively)?* Now we will study a little bit more these structures and answer the above question. The notations used previously will be kept.

Let H be a Hopf algebra and C a left partial H-comodule coalgebra with structure given by λ'. Suppose that $((D, \lambda), \theta, \pi)$ is a globalization for (C, λ'). From Theorem 3.1.16, we have that C is a right partial H^0-module coalgebra with partial action given by

$$c \leftharpoondown f = f(c^{-\overline{1}})c^{-\overline{0}}$$

and the same is true for D, i.e., we have an structure of H^0-module coalgebra on D given by

$$d \blacktriangleleft f = f(d^{-1})d^{-0}.$$

Since θ is a coalgebra monomorphism from C into D and π is a comultiplicative projection from D onto $\theta(C)$, in order to induce an structure of partial H^0-module coalgebra on $\theta(C)$ we just need to check the equation (2.1). For this, let $d \in D$ and $f \in H^0$, so

$$\begin{aligned}
\pi(\pi(d) \blacktriangleleft f) &= f(\pi(d)^{-1})\pi(\pi(d)^{-0}) \\
&\overset{(3.16)}{=} f(d_2{}^{-1})\pi(\varepsilon(\pi(d_1))d_2{}^{-0}) \\
&= \pi(\varepsilon(\pi(d_1))d_2{}^{-0}f(d_2{}^{-1})) \\
&= \pi(\varepsilon(\pi(d_1))d_2 \blacktriangleleft f)
\end{aligned}$$

therefore, we can induce an structure of left partial H^0-module coalgebra on $\theta(C)$.

Now we will show that θ is in fact a morphism of partial action. For this, let $c \in C$, so

$$\begin{aligned}
\theta(c) \leftharpoondown f &= \pi(\theta(c) \blacktriangleleft f) \\
&= f(\theta(c)^{-1})\pi(\theta(c)^{-0}) \\
&\overset{(3.18)}{=} f(c^{-\overline{1}})\theta(c^{-\overline{0}}) \\
&= \theta(f(c^{-\overline{1}})c^{-\overline{0}}) \\
&= \theta(c \leftharpoondown f).
\end{aligned}$$

The next result gives conditions to the existence of a globalization for C viewed as partial module coalgebra.

Theorem 3.2.5. *Let H be a Hopf algebra, C a left partial H-comodule coalgebra and (D, θ, π) a globalization for C. Supposing that H^0 separate points, then (D, θ, π) is also a globalization for C as right partial H^0-module coalgebra.*

Proof. From our discussion above, we just need to show the condition GMC3 in Definition 2.2.1.

Let M be any H^0-submodule coalgebra of D containing $\theta(C)$. We need to show that $M = D$ and for this, we will show that M is an H-subcomodule coalgebra of D. Thus, since M contain $\theta(C)$ and D is a globalization for C viewed as left partial H-comodule coalgebra, it follows that $M = D$, as desired.

In fact, take $f \in H^0$, $m \in M$, and consider $\{h_k\}$ a basis of H. Let $\{h_k^*\}$ be the set whose elements are the dual applications of the h_k's. Then write $\lambda(m) \in H \otimes D$ is terms of the basis of H, i.e.,

$$\lambda(m) = \sum_{i=0}^{n} h_i \otimes m_i,$$

where the m_i's are nonzero elements, at least, in D.

Since D is an H-comodule, so it is an H^*-module via the same action of H^0. Moreover, the action of H^0 on D is a restriction of the action of H^*. Since H^0 separate points, it follows, by Jacobson density theorem, that give $m \in M$ there exists $\{h^0_{(m)i} \in H^0\}$ such that $m \blacktriangleleft h^0_{(m)i} = m \blacktriangleleft h_i^*$, for each i. Then

$$m \blacktriangleleft h^0_{(m)j} = m \blacktriangleleft h_j^* = \sum_{i=0}^{n} h_j^*(h_i) m_i = m_j$$

and so the m_i's lies in M. Therefore M is an H-subcomodule of D and, consequently, M is an H-subcomodule coalgebra of D containing $\theta(C)$. Since D is a globalization for C as partial comodule coalgebra, it follows that $M = D$ and, therefore, (D, θ, π) is a globalization for C as partial H^0-module coalgebra. The proof is complete now. □

3.2.2 Constructing a Globalization

Now we will construct a globalization for a left partial comodule coalgebra C, in an special situation. First of all, remember that if M is a right H^0-module and H^0 separate points, then we have a linear map $\alpha \colon M \to \mathrm{Hom}(H^0, M)$ given by $\alpha(m)(f) = m \cdot f$ and an injective linear map $\beta \colon H \otimes M \to \mathrm{Hom}(H^0, M)$ given by $\beta(h \otimes m)(f) = f(h)m$.

In the above situation, we say that M is a *rational H^0-module* if $\alpha(M) \subseteq \beta(H \otimes M)$. Notice that, given a rational H^0-module then we have an structure of H-comodule via $\lambda \colon M \to H \otimes M$ satisfying

$$\lambda(m) = \sum h_i \otimes m_i \iff m \cdot f = \sum f(h_i) m_i, \ \forall f \in H^0. \tag{3.19}$$

This definition can be seen pictorially in the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Hom}(H^0, M) & \\ M \arrow[ur, "\alpha"] \arrow[rr, dashed, "\lambda"] & & H \otimes M \arrow[ul, hook, "\beta"'] \end{tikzcd} \tag{3.20}$$

From Theorem 3.2.5, it follows that exists a natural way to try to find a globalization for a partial coaction of H on C, which is to consider the structure in the standard globalization for the related partial action of H^0 on C, under the hypothesis that H^0 separate points. Thus, given a left partial H-comodule coalgebra (C, λ') we have, from Theorem 2.2.5, that $(C \otimes H^0, \theta, \pi)$ is a globalization for the partial action of H^0 on C.

In this point we would like to obtain that $C \otimes H^0$ is an H-comodule coalgebra, but, in general, this can not be true. In order to overcome this problem we will assume one more condition, namely, we will suppose that $C \otimes H^0$ is a rational H^0-module. In this particular case, we have that $C \otimes H^0$ is an H-comodule with coaction satisfying the equation (3.19), i.e., in our situation, we have the following rule, for any $c \otimes f \in C \otimes H^0$ and $g \in H^0$,

$$\lambda(c \otimes f) = \sum h_i \otimes c_i \otimes f_i \iff c \otimes (f * g) = \sum g(h_i) c_i \otimes f_i. \tag{3.21}$$

Therefore, in a similar way to Theorem 3.1.17, we have that $C \otimes H^0$ is an H-comodule coalgebra.

Thus, keeping our notations used before, we already know that $\theta\colon C \to C \otimes H^0$ is a coalgebra map and $\pi\colon C \otimes H^0 \twoheadrightarrow \theta(C)$ is a comultiplicative projection. Therefore, we are in the position to obtain a globalization for a left partial H-comodule coalgebra, as follows.

Theorem 3.2.6. *Let H be a Hopf algebra and C a left partial H-comodule coalgebra. With the above notations, if $C \otimes H^0$ is a rational H^0-module and H^0 separate points, then $(C \otimes H^0, \theta, \pi)$ is a globalization for C.*

Proof. By the above discussion, we can show directly that the conditions GCC1-GCC3 hold. Let $c \otimes f \in C \otimes H^0$, and any $g \in H^0$, so

$$\begin{aligned}
(g \otimes I)[\pi(c \otimes f)^{-1} \otimes \pi(\pi(c \otimes f)^{-0})] &= g(\pi(c \otimes f)^{-1})\pi[\pi(c \otimes f)^{-0}] \\
&= \pi[g(\pi(c \otimes f)^{-1})\pi(c \otimes f)^{-0}] \\
&= \pi[\pi(c \otimes f) \blacktriangleleft g] \\
&= \pi[\varepsilon(\pi((c \otimes f)_1))(c \otimes f)_2 \blacktriangleleft g] \\
&= \varepsilon(\pi(c \otimes f)_1)\pi[g((c \otimes f)_2{}^{-1})(c \otimes f)_2{}^{-0}] \\
&= g[(c \otimes f)_2{}^{-1}\pi(\varepsilon(\pi((c \otimes f)_1))(c \otimes f)_2{}^{-0})] \\
&= (g \otimes I)[(c \otimes f)_2{}^{-1}\pi(\varepsilon(\pi((c \otimes f)_1))(c \otimes f)_2{}^{-0})].
\end{aligned}$$

Since H^0 separate points, therefore the condition GCC1 is satisfied.

To proof the condition GCC2 we take $c \in C$ and observe that

$$\begin{aligned}(g \otimes I)[\theta(c)^{-1} \otimes \pi(\theta(c)^{-0})] &= g(\theta(c)^{-1})\pi(\theta(c)^{-0}) \\ &= \pi(g(\theta(c)^{-1})\theta(c)^{-0}) \\ &= \pi(\theta(c) \blacktriangleleft g) \\ &= \theta(c \leftharpoonup g) \\ &= g(c^{-\overline{1}})\theta(c^{-\overline{0}}) \\ &= (g \otimes I)[c^{-\overline{1}} \otimes \theta(c^{-\overline{0}})]\end{aligned}$$

since H^0 separate points, then θ is an equivalence of partial coactions, i.e., GCC2 holds.

Finally, to show that $C \otimes H^0$ is generated by $\theta(C)$, we consider M a subcomodule coalgebra of $C \otimes H^0$ containing $\theta(C)$. By Theorem 3.1.16, M is an H^0-submodule coalgebra of $C \otimes H^0$ containing $\theta(C)$. Thus, it follows from condition GMC3 that $M = C \otimes H^0$.

Therefore $C \otimes H^0$ is a globalization for C as partial H-comodule coalgebra. $\square$

The globalization above constructed is called the *standard globalization* for a partial comodule coalgebra.

Remark 3.2.7. When the Hopf algebra is finite dimensional, so $H^0 = H^*$ and it separate points and, moreover, $C \otimes H^*$ is an H^*-rational module with coaction given by

$$\lambda \colon c \otimes f \longmapsto \sum_{i=0}^{n} h_i \otimes c \otimes f * h_i^*,$$

where $\{h_i, h_i^*\}$ is a dual basis for H and H^*.

Chapter 4

Galois Theory and Morita-Takeuchi Correspondence

4.1 Morita-Takeuchi Context for Partial Comodules Coalgebras

Takeuchi developed the notion of Morita context for coalgebras, called *Morita-Takeuchi context*. This tool allow us to compare the category of comodules over two given coalgebras. Moreover, as shown by Takeuchi in [31], if we have an strict Morita-Takeuchi context between two coalgebras, then the category of comodules over these coalgebras are equivalent.

Our aim in this section is to construct a Morita-Takeuchi context connecting two coalgebras that are obtained from a given left partial H-comodule coalgebra. First of all we need to remember the definition of Morita-Takeuchi context, or pre-equivalence data as defined by Takeuchi in [31]. For this we start recalling the notion of cotensor product.

Definition 4.1.1. Let X be a coalgebra, M a right X-comodule via ρ_M and N a left X-comodule via λ_N. Then we define the *tensor coproduct* $M \square_X N$ as the subspace of $M \otimes N$ spanned by

$$\left\{ \sum m_i \otimes n_i \mid \sum \rho_M(m_i) \otimes n_i = \sum m_i \otimes \lambda_N(n_i) \right\}.$$

Notice that $M \simeq M \square_X X$ via ρ_M and $\lambda_X = \Delta$, and analogously $N \simeq X \square_X N$ via λ_N and $\rho_X = \Delta$.

Definition 4.1.2. A *Morita-Takeuchi context* between two coalgebras X and Y is a sextuple *(X, Y, M, N, μ, τ)*, where M is a (Y, X)-bicomodule, N is a (X, Y)-bicomodule, $\mu \colon Y \to M \square_X N$ is a map of Y-bicomodules and $\tau \colon X \to N \square_Y M$ is a map of X-bicomodules, such that the following diagrams commute:

$$\begin{tikzcd} M \square_X N \square_Y M & M \square_X X \arrow[l, "I_M \otimes \tau"'] \\ Y \square_Y M \arrow[u, "\mu \otimes I_M"] & M \arrow[l, "\lambda_M"] \arrow[u, "\rho_M"'] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd}
N \,\square_Y M \,\square_X N & & N \,\square_Y Y \arrow[ll, "I_N\otimes\mu"'] \\
X \,\square_X N \arrow[u, "\tau\otimes I_N"] \arrow[rr, phantom, "\circlearrowleft"] & & N \arrow[ll, "\lambda_N"] \arrow[u, "\rho_N"']
\end{tikzcd} \tag{4.2}$$

A Morita-Takeuchi context is said *strict* if μ and τ are bijections.

As noticed by Takeuchi, if the maps μ and τ are injections, then they are bijections (cf. [31, 2.5 Theorem]).

Given an H-comodule coalgebra C, Dăscălescu, Raianu, and Zhang constructed in [20] a Morita-Takeuchi context between the smash coproduct $C \rtimes H$ and the quotient ${}^{C}/_{\mathcal{J}}$, where $\mathcal{J}$ is defined by $\mathcal{J} = C \blacktriangleleft \operatorname{Ker}(\varepsilon_{H^*})$, being $\blacktriangleleft$ the action of H^* on C induced by the structure of comodule coalgebra on C.

If we take C as a left partial H-comodule coalgebra, a natural question arises from the construction given in [20]: *Is it possible to construct a Morita-Takeuchi context related to a partial comodule coalgebra?* Our aim in this section in to answer this question affirmatively, but for this we will need to make some constructions.

For simplicity, since all comodule coalgebra will be partial in this chapter, we will use the global notation. First of all, we need to discuss the existence of a partial smash coproduct and a coideal for then construct the quotient coalgebra.

Batista and Vercruysse constructed in [4] the partial smash coproduct associated to a partial comodule coalgebra in the following way.

Given a left partial H-comodule coalgebra $(C, \bar{\lambda})$ we define the smash coproduct $C \rtimes H$ as the vector space $C \otimes H$ with the following linear maps

$$\begin{aligned} \Delta\colon C \rtimes H &\longrightarrow C \rtimes H \otimes C \rtimes H \\ c \rtimes h &\longmapsto c_1 \rtimes c_2{}^{-1}h_1 \otimes c_2{}^{-0} \rtimes h_2 \end{aligned}$$

and

$$\begin{aligned} \varepsilon\colon C \rtimes H &\longrightarrow \Bbbk \\ c \rtimes h &\longmapsto \varepsilon_C(c)\varepsilon_H(h) \end{aligned}$$

With the above maps, $C \rtimes H$ becomes a coassociative coalgebra with left counity.

In fact, let $c \rtimes h \in C \rtimes H$, so

$$\begin{aligned}
(I \otimes \Delta)\Delta(c \rtimes h) &= c_1 \rtimes c_2{}^{-1}h_1 \otimes \Delta(c_2{}^{-0} \rtimes h_2) \\
&= c_1 \rtimes c_2{}^{-1}h_1 \otimes c_2{}^{-0}{}_1 \rtimes c_2{}^{-0}{}_2{}^{-1}h_2 \otimes c_2{}^{-0}{}_2{}^{-0} \rtimes h_3 \\
&\overset{(PCC2)}{=} c_1 \rtimes c_2{}^{-1}c_3{}^{-1}h_1 \otimes c_2{}^{-0} \rtimes c_3{}^{-0-1}h_2 \otimes c_3{}^{-0-0} \rtimes h_3 \\
&\overset{(PCC3)}{=} c_1 \rtimes c_2{}^{-1}\nabla(c_3)c_4{}^{-1}{}_1h_1 \otimes c_2{}^{-0} \rtimes c_4{}^{-1}{}_2h_2 \otimes c_4{}^{-0} \rtimes h_3 \\
&\overset{(3.5)}{=} c_1 \rtimes c_2{}^{-1}c_3{}^{-1}{}_1h_1 \otimes c_2{}^{-0} \rtimes c_3{}^{-1}{}_2h_2 \otimes c_3{}^{-0} \rtimes h_3 \\
&= \Delta(c_1 \rtimes c_2{}^{-1}h_1) \otimes c_2{}^{-0} \rtimes h_2 \\
&= (\Delta \otimes I)\Delta(c \rtimes h)
\end{aligned}$$

and, moreover,

$$\begin{aligned}(\varepsilon \otimes I)\Delta(c \rtimes h) &= \varepsilon(c_1)\varepsilon({c_2}^{-1}h_1){c_2}^{-0} \rtimes h_2\\ &= \varepsilon(c_1)\varepsilon({c_2}^{-1}){c_2}^{-0} \rtimes \varepsilon(h_1)h_2\\ &= \varepsilon(c_1)c_2 \rtimes h\\ &= c \rtimes h.\end{aligned}$$

Note that in our case we do not have a right counity. Moreover, ε is a right counity for $C \rtimes H$ if and only if the partial comodule coalgebra C is global.

In fact, supposing C a comodule coalgebra it follows clearly that ε is a right counity. Conversely, supposing that ε is a right counity for $C \rtimes H$ so $(I \otimes \varepsilon)\Delta(c \rtimes 1_H) = c \rtimes 1_H$. Then $c \rtimes 1_H = c_1 \rtimes {c_2}^{-1}\varepsilon({c_2}^{-0})$. Therefore, applying $\varepsilon \otimes I$ in the above equality, we obtain that $\varepsilon(c)1_H = c^{-1}\varepsilon(c^{-0})$ and by Proposition 3.1.15 the coaction is global.

In the case of partial module algebra, the smash product is an associative algebra with left unity, so we can consider the subalgebra generate by this left unity. In our case, given a coassociative coalgebra with left counit we can consider the subcoalgebra generated by this left counity, in the following way:

Proposition 4.1.3. *Let X be a vector space with a coassociative comultiplication $\Delta\colon X \to X \otimes X$ and considerate a linear map $\varepsilon\colon X \to \Bbbk$. If ε is a left counity for X, then it is a right counity for the subcoalgebra $\overline{X} = (I \otimes \varepsilon)\Delta(X)$.*

Moreover, denoting $\bar{x} = (I \otimes \varepsilon)\Delta(x)$, we have that $\Delta(\bar{x}) = \overline{x_1} \otimes \overline{x_2}$.

Proof. First of all, we need to check that $\overline{X}$ is a subcoalgebra of X.

Let $\bar{x} = x_1\varepsilon(x_2) \in \overline{X}$, so

$$\begin{aligned}\Delta(\bar{x}) &= \Delta(x_1)\varepsilon(x_2)\\ &= x_1 \otimes x_2\varepsilon(x_3)\\ &= x_1 \otimes \varepsilon(x_2)x_3\varepsilon(x_3) && \text{since } \varepsilon \text{ is a left counity}\\ &= x_1\varepsilon(x_2) \otimes x_3\varepsilon(x_3)\\ &= \overline{x_1} \otimes \overline{x_2}\end{aligned}$$

Then $\overline{X}$ is a subcoalgebra of X. Now just remains to show that ε is a right counity for $\overline{X}$. In fact,

$$\begin{aligned}(I \otimes \varepsilon)\Delta(\bar{x}) &= \overline{x_1}\varepsilon(\overline{x_2})\\ &= x_1\varepsilon(x_2)\varepsilon(x_3\varepsilon(x_4))\\ &= x_1\varepsilon(x_2)\varepsilon(x_3)\varepsilon(x_4)\\ &= x_1\varepsilon(\varepsilon(\varepsilon(x_2)x_3)x_4)\\ &= x_1\varepsilon(x_2)\\ &= \bar{x}\end{aligned}$$

Where in the last equalities we use the fact that ε is a left counity in X. $\square$

Since, for a partial comodule coalgebra, we have a coassociative coproduct and a left counity in $C \rtimes H$, hence we can consider the subcoalgebra generated by the left counity, in the same way as done in Proposition 4.1.3.

Corollary 4.1.4. *Given a partial H-comodule coalgebra C, the partial smash coproduct $C \bar{\rtimes} H = (I \otimes \varepsilon)\Delta(C \rtimes H)$ is a coalgebra, with structure given by*

$$\begin{aligned} \Delta \colon C \bar{\rtimes} H &\longrightarrow C \bar{\rtimes} H \otimes C \bar{\rtimes} H \\ c \bar{\rtimes} h &\longmapsto c_1 \bar{\rtimes} {c_2}^{-1} h_1 \otimes {c_2}^{-0} \bar{\rtimes} h_2 \end{aligned} \tag{4.3}$$

$$\begin{aligned} \varepsilon \colon C \bar{\rtimes} H &\longrightarrow \Bbbk \\ c \bar{\rtimes} h &\longmapsto \varepsilon_C(c)\varepsilon_H(h) \end{aligned} \tag{4.4}$$

Proof. Denote a simple element in $C \bar{\rtimes} H$ by $c \bar{\rtimes} h = (I \otimes \varepsilon)\Delta(c \rtimes h) = c_1 \rtimes \nabla(c_2)h$. Therefore the result follows directly from Proposition 4.1.3. □

In [20] the authors shown that $C \blacktriangleleft \ker(\varepsilon_{H^*})$ is a coideal, and so the correspondent quotient coalgebra was taken to construct the Morita-Takeuchi context. In the case of partial comodule coalgebra, this is not true. Moreover, if $C \leftharpoonup \ker(\varepsilon_{H^*})$ is a coideal of C then C is a (global) module coalgebra.

In fact, since $H^* = \Bbbk\varepsilon_H \oplus \ker \varepsilon_{H^*}$, then any element $f \in H^*$ can be write in the following way $f = f(1_H)\varepsilon_H + f_{\ker}$, where $f_{\ker}$ lies in $\ker(\varepsilon_{H^*})$. Supposing that $C \leftharpoonup \ker(\varepsilon)$ is a coideal of C, so $\varepsilon_C(C \leftharpoonup \ker(\varepsilon_{H^*})) = 0$. Then, for any $f \in H^*$ and $c \in C$ we have that

$$\begin{aligned} \varepsilon_C(c \leftharpoonup f) &= \varepsilon_C(c \leftharpoonup f(1_H)\varepsilon_H) + \varepsilon_C(c \leftharpoonup f_{\ker}) \\ &= \varepsilon_C(c \leftharpoonup f(1_H)\varepsilon_H) \\ &\overset{(PMC1)}{=} \varepsilon_C(c\ \varepsilon_{H^*}(f)) \\ &= \varepsilon_C(c)\varepsilon_{H^*}(f). \end{aligned}$$

Therefore, by Proposition 2.1.7, C is a global H^*-module coalgebra.

Now we will construct a coideal $\mathcal{J}$ of C which will play the role of the coideal considered in [20] in the construction of the Morita-Takeuchi context. From now on, we will suppose that H is finite dimensional. Remember from Theorem 3.1.16 that, if C is a left partial H-comodule coalgebra, then C^* is a left partial H^*-module algebra via

$$(f \rightarrow \alpha)(c) = f(c^{-1})\alpha(c^{-0}),$$

hence we can consider the subalgebra of invariants, i.e.,

$$C^{*\underline{H^*}} = \{\psi \in C^* \mid f \rightarrow \psi = (f \rightarrow \varepsilon_C) * \psi = \psi * (f \rightarrow \varepsilon_C),\ \forall f \in H^*\}$$

Moreover, remember, from classical coalgebra theory (see [30]), the relation between a subspace and its orthogonal.

Proposition 4.1.5. *Let C be a coalgebra and $V \leq C$ a subspace, so V is a coideal of C iff $V^{\perp}$ is a subalgebra of C^*.* $\square$

As a consequence of the above proposition, we have that

$$\mathcal{J} := \left(C^{*\underline{H^*}}\right)^{\perp} \tag{4.5}$$

is a coideal of C. Therefore, we can consider the quotient coalgebra ${}^{C}/_{\mathcal{J}}$ with induced structure (see Proposition 1.2.13).

Remark 4.1.6. In the case when C be a (global) H-comodule coalgebra, one can note that

$$C \blacktriangleleft \ker(\varepsilon_{H^*}) = \{c(c^{-1})c^{-0} - f(1_H)c \mid f \in H^*, c \in C\}.$$

Moreover, it is easy to see that $C \blacktriangleleft \ker(\varepsilon_{H^*}) = (C^{*H^*})^{\perp}$. Then, the coideal defined by Equation 4.5 is exactly the same of $C \blacktriangleleft \ker(\varepsilon_{H^*})$. Therefore the coalgebra quotient above considered is the same of ${}^{C}/_{CH^+}$, whenever the coaction is global.

In the sequel, we have an important property of ${}^{C}/_{\mathcal{J}}$ that will be very useful to construct the Morita-Takeuchi context.

Theorem 4.1.7. *Let C be a left symmetric partial H-comodule coalgebra, then we have for all $c \in C$,*

$$c^{-1} \otimes \overline{c^{-0}} = \nabla(c_2) \otimes \overline{c_1} = \nabla(c_1) \otimes \overline{c_2}, \tag{4.6}$$

where $\bar{c}$ denotes the class of $c \in C$ in ${}^{C}/_{\mathcal{J}}$.

Proof. Let $c \in C$, so for all $\psi \in C^{*\underline{H^*}}$ and $f \in H^*$ we have

$$\begin{aligned}
\psi(f(c^{-1})c^{-0}) &= f(c^{-1})\psi(c^{-0}) \\
&= (f \rightarrow \psi)(c) \\
&= [\psi * (f \rightarrow \varepsilon)](c) \\
&= \psi(c_1)f(c_2{}^{-1})\varepsilon(c_2{}^{-0}) \\
&= \psi(c_1 f(\nabla(c_2))).
\end{aligned}$$

Thus, $\psi[f(c^{-1})c^{-0} - c_1 f(\nabla(c_2))] = 0$, for all $\psi \in C^{*\underline{H^*}}$, and then $f(c^{-1})c^{-0} - c_1 f(\nabla(c_2))$ lies in $\left(C^{*\underline{H^*}}\right)^{\perp} = \mathcal{J}$, for all $f \in H^*$.

By the above observations, $f(c^{-1})\overline{c^{-0}} = f(\nabla(c_2))\overline{c_1}$ in ${}^{C}/_{\mathcal{J}}$, for all $f \in H^*$. Therefore, for all $c \in C$, we have

$$c^{-1} \otimes \overline{c^{-0}} = \nabla(c_2) \otimes \overline{c_1}.$$

The other equality holds from a similar computation. $\square$

To construct the bicomodules we will need to suppose the symmetry, therefore, from now on, C will be a left symmetric partial H-comodule coalgebra.

Now we will construct the bicomodules which will be necessary in our context. Take $M = C$ and consider the following maps

$$\begin{aligned} \lambda_M \colon M &\longrightarrow C \mathbin{\bar{\rtimes}} H \otimes M \\ c &\longmapsto c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1} \otimes {c_2}^{-0} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \rho_M \colon M &\longrightarrow M \otimes {}^{C}/_{\mathcal{J}} \\ c &\longmapsto c_1 \otimes \overline{c_2}. \end{aligned} \tag{4.8}$$

For the next result we write ${}^{C \bar{\rtimes} H}\mathcal{M}^{C/\mathcal{J}}$ to denote the category of all $(C \mathbin{\bar{\rtimes}} H, {}^{C}/_{\mathcal{J}})$-bicomodules.

Proposition 4.1.8. *With the above structure maps, $M \in {}^{C \bar{\rtimes} H}\mathcal{M}^{C/\mathcal{J}}$.*

Proof. First we will show that M is a left $C \mathbin{\bar{\rtimes}} H$-comodule, so let $c \in M$

$$\begin{aligned} (\varepsilon \otimes I)\lambda_M(c) &= \varepsilon(c_1)\varepsilon({c_2}^{-1}){c_2}^{-0} \\ &\overset{(PCC1)}{=} \varepsilon(c_1)c_2 \\ &= c \end{aligned}$$

and

$$\begin{aligned} (I \otimes \lambda_M)\lambda_M(c) &= c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1} \otimes \lambda({c_2}^{-0}) \\ &= c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1} \otimes {{c_2}^{-0}}_1 \mathbin{\bar{\rtimes}} {{{c_2}^{-0}}_2}^{-1} \otimes {{{c_2}^{-0}}_2}^{-0} \\ &= c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1}{c_3}^{-1} \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} {c_3}^{-0-1} \otimes {c_3}^{-0-0} \\ &= c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1}\nabla(c_3){{c_4}^{-1}}_1 \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} {{c_4}^{-1}}_2 \otimes {c_4}^{-0} \\ &\overset{(3.5)}{=} c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1}{{c_3}^{-1}}_1 \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} {{c_3}^{-1}}_2 \otimes {c_3}^{-0} \\ &= \Delta(c_1 \mathbin{\bar{\rtimes}} {c_2}^{-1}) \otimes {c_2}^{-0} \\ &= (\Delta \otimes I)\lambda(c). \end{aligned}$$

Now to check that M is a right ${}^{C}/_{\mathcal{J}}$-comodule, we take $c \in M$ and observe that

$$(I \otimes \bar{\varepsilon})\rho_M(c) = c_1\bar{\varepsilon}(\overline{c_2}) = c_1\varepsilon(c_2) = c$$

and

$$\begin{aligned} (\rho_M \otimes I)\rho_M(c) &= \rho_M(c_1) \otimes \overline{c_2} \\ &= c_1 \otimes \overline{c_2} \otimes \overline{c_3} \\ &= c_1 \otimes \bar{\Delta}(\overline{c_2}) \\ &= (I \otimes \bar{\Delta})\rho_M(c). \end{aligned}$$

Now just remains to show the compatibility relation between λ_M and ρ_M. So let $c \in M$,

$$\begin{aligned}
(\lambda_M \otimes I)\rho_M(c) &= \lambda(c_1) \otimes \overline{c_2} \\
&= c_1 \,\bar{\rtimes}\, {c_2}^{-1} \otimes {c_2}^{-0} \otimes \overline{c_3} \\
&\overset{(3.5)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1} \nabla(c_3) \otimes {c_2}^{-0} \otimes \overline{c_4} \\
&\overset{(4.6)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1}{c_3}^{-1} \otimes {c_2}^{-0} \otimes \overline{{c_3}^{-0}} \\
&\overset{(PCC2)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1} \otimes {{c_2}^{-0}}_1 \otimes \overline{{{c_2}^{-0}}_2} \\
&\overset{(PCC2)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1} \otimes \rho({c_2}^{-0}) \\
&= (I \otimes \rho_M)\lambda_M(c)
\end{aligned}$$

and therefore M is a $(C \,\bar{\rtimes}\, H, {}^{C}/_{\mathcal{J}})$-bicomodule. $\square$

To construct the $({}^{C}/_{\mathcal{J}}, C \,\bar{\rtimes}\, H)$-bicomodule we will recall once more the definitions of integral and distinguished group-like elements (see Section 1.2).

An element $0 \neq T \in H^*$ is said an *left integral* in H^*, if for all $f \in H^*$ hold that $f * T = \varepsilon_{H^*}(f)T$ or equivalently if

$$h_1 T(h_2) = T(h) 1_H, \tag{4.9}$$

for all $h \in H$. In this case, we denote that $T \in \int_l^{H^*}$.

Then, by the classical theory, given $0 \neq T$ a left integral in H^* there exists $\hat{\lambda} \in \mathrm{G}(H^{**})$, called the *distinguished grouplike associated* associated to T, satisfying $T * f = \hat{\lambda}(f)T$, that is

$$T(h_1)h_2 = \lambda T(h) \tag{4.10}$$

associating H with H^{**} by the natural isomorphism.

Another interesting classical property between T and λ is the following

$$T(h) = T(S^{-1}(h)\lambda), \tag{4.11}$$

that will be useful in this section.

Then, taking $N = C$, we define the following maps

$$\begin{aligned}
\lambda_N \colon N &\longrightarrow {}^{C}/_{\mathcal{J}} \otimes N \\
c &\longmapsto \overline{c_1} \otimes c_2
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
\rho_N \colon N &\longrightarrow N \otimes C \,\bar{\rtimes}\, H \\
c &\longmapsto {c_1}^{-0} \otimes {c_2}^{-0} \,\bar{\rtimes}\, S^{-1}({c_1}^{-1}{c_2}^{-1})\lambda.
\end{aligned} \tag{4.13}$$

Analogously as before, in the nest result we will write ${}^{C/\mathcal{J}}\mathcal{M}^{C \,\bar{\rtimes}\, H}$ to denote the category of all $({}^{C}/_{\mathcal{J}},\, C \,\bar{\rtimes}\, H)$-bicomodules.

Proposition 4.1.9. *With the above structure maps, $N \in {}^{C/\mathcal{J}}\mathcal{M}^{C\bar{\rtimes}H}$.*

Proof. Let $c \in C$. Then we have

$$\begin{aligned}
(I\otimes\Delta)\rho_N(c) &= c_1{}^{-0}\otimes\Delta(c_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda)\\
&= c_1{}^{-0}\otimes c_2{}^{-0}{}_1\bar{\rtimes}c_2{}^{-0}{}_2{}^{-1}S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2)\lambda\\
&\qquad\otimes c_2{}^{-0}{}_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1)\lambda\\
&\overset{(PCC2)}{=} c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}c_3{}^{-0-1}S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2c_3{}^{-1}{}_2)\lambda\\
&\qquad\otimes c_3{}^{-0-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1c_3{}^{-1}{}_1)\lambda\\
&= c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}c_3{}^{-0-1}S^{-1}(c_1{}^{-1}{}_2(c_2{}^{-1}c_3{}^{-1})_2)\lambda\\
&\qquad\otimes c_3{}^{-0-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1(c_2{}^{-1}c_3{}^{-1})_1)\lambda\\
&\overset{(PCC3)}{=} c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}c_4{}^{-1}{}_2S^{-1}(c_1{}^{-1}{}_2(c_2{}^{-1}\nabla(c_3)c_4{}^{-1}{}_1)_2)\lambda\\
&\qquad\otimes c_4{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1(c_2{}^{-1}\nabla(c_3)c_4{}^{-1}{}_1)_1)\lambda\\
&\overset{(3.5)}{=} c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}c_3{}^{-1}{}_3S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2c_3{}^{-1}{}_2)\lambda\\
&\qquad\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1c_3{}^{-1}{}_1)\lambda\\
&= c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}c_3{}^{-1}{}_3S^{-1}(c_3{}^{-1}{}_2)S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2)\lambda\\
&\qquad\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1c_3{}^{-1}{}_1)\lambda\\
&= c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2)\lambda\\
&\qquad\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1c_3{}^{-1})\lambda
\end{aligned}$$

and, on the other side,

$$\begin{aligned}
&(\rho_N\otimes I)\rho_N(c) = \rho_N(c_1{}^{-0})\otimes c_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda\\
&= c_1{}^{-0}{}_1{}^{-0}\otimes c_1{}^{-0}{}_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-0}{}_1{}^{-1}c_1{}^{-0}{}_2{}^{-1})\lambda\otimes c_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda\\
&= c_1{}^{-0-0}\otimes c_2{}^{-0-0}\bar{\rtimes}S^{-1}(c_1{}^{-0-1}c_2{}^{-0-1})\lambda\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}c_2{}^{-1}c_3{}^{-1})\lambda\\
&= c_1{}^{-0-0}\otimes c_2{}^{-0-0}\bar{\rtimes}S^{-1}(c_1{}^{-0-1}c_2{}^{-0-1})\lambda\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}c_2{}^{-1}c_3{}^{-1})\lambda\\
&\overset{(PCC4)}{=} c_1{}^{-0}\otimes c_3{}^{-0-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_2c_3{}^{-0-1})\lambda\otimes c_4{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1\nabla(c_2)c_3{}^{-1}c_4{}^{-1})\lambda\\
&\overset{(3.5)}{=} c_1{}^{-0}\otimes c_2{}^{-0-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_2c_2{}^{-0-1})\lambda\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}c_3{}^{-1})\lambda\\
&\overset{(PCC4)}{=} c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2)\lambda\otimes c_4{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1\nabla(c_3)c_4{}^{-1})\lambda\\
&\overset{(3.5)}{=} c_1{}^{-0}\otimes c_2{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_2c_2{}^{-1}{}_2)\lambda\otimes c_3{}^{-0}\bar{\rtimes}S^{-1}(c_1{}^{-1}{}_1c_2{}^{-1}{}_1c_3{}^{-1})\lambda
\end{aligned}$$

Then $\rho^2 = (I\otimes\Delta)\rho$. Moreover, for any $c \in C$, we have

$$\begin{aligned}
(I\otimes\varepsilon)\rho_N(c) &= c_1{}^{-0}\varepsilon(c_2{}^{-0})\varepsilon(S^{-1}(c_1{}^{-1}c_2{}^{-1}))\varepsilon(\lambda)\\
&= c_1{}^{-0}\varepsilon(c_2{}^{-0})\varepsilon(c_1{}^{-1})\varepsilon(c_2{}^{-1})\\
&\overset{(PCC1)}{=} c_1\varepsilon(c_2)\\
&= c
\end{aligned}$$

Therefore, N is a right $C \bar{\rtimes} H$-comodule.

Clearly N is a left ${}^{C}/_{\mathcal{J}}$-comodule, so just remains to show the compatibility relation between λ_N and ρ_N.

$$\begin{aligned}
(\lambda_N \otimes I)\rho_N(c) &= \lambda_N(c_1{}^{-0}) \otimes c_2{}^{-0} \bar{\rtimes} S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda \\
&= \overline{c_1{}^{-0}{}_1} \otimes c_1{}^{-0}{}_2 \otimes c_2{}^{-0} \bar{\rtimes} S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda \\
&\overset{(PCC2)}{=} \overline{c_1{}^{-0}} \otimes c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} S^{-1}(c_1{}^{-1}c_2{}^{-1}c_3{}^{-1})\lambda \\
&\overset{(4.6)}{=} \overline{c_1} \otimes c_3{}^{-0} \otimes c_4{}^{-0} \bar{\rtimes} S^{-1}(\nabla(c_2)c_3{}^{-1}c_4{}^{-1})\lambda \\
&\overset{(3.5)}{=} \overline{c_1} \otimes c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} S^{-1}(c_2{}^{-1}c_3{}^{-1})\lambda \\
&= \overline{c_1} \otimes \rho_N(c_2) \\
&= (I \otimes \rho_N)\lambda_N(c)
\end{aligned}$$

Therefore N is a $({}^{C}/_{\mathcal{J}}, C \bar{\rtimes} H)$-bicomodule. □

Now we are able to define the following linear map

$$\begin{aligned}
\mu\colon C \rtimes H &\longrightarrow M \,\Box_{\,C/\mathcal{J}} N \\
c \rtimes h &\longmapsto c_1 \,\Box\, c_2{}^{-0}T(c_2{}^{-1}h).
\end{aligned} \tag{4.14}$$

Note that $\mu(c \bar{\rtimes} h) = \mu(c \rtimes h)$. In fact,

$$\begin{aligned}
\mu(c \bar{\rtimes} h) &= \mu(c_1 \bar{\rtimes} \nabla(c_2)h) \\
&= c_1 \,\Box\, c_2{}^{-0}T(c_2{}^{-1}\nabla(c_3)h) \\
&\overset{(3.5)}{=} c_1 \,\Box\, c_2{}^{-0}T(c_2{}^{-1}h) \\
&= \mu(c \rtimes h).
\end{aligned}$$

Proposition 4.1.10. *With the above notation, we have $\mu\colon C \rtimes H \longrightarrow M \,\Box_{\,C/\mathcal{J}} N$ is a well-defined map in ${}^{C \bar{\rtimes} H}\mathcal{M}^{C \bar{\rtimes} H}$.*

Proof. First of all, to show that μ is well-defined we need to proof that $Im(\mu) \subseteq M \,\Box_{\,C/\mathcal{J}} N$. In fact,

$$\begin{aligned}
(\rho_M \otimes I)\mu(c \bar{\rtimes} h) &= \rho_M(c_1) \otimes c_2{}^{-0}T(c_2{}^{-1}h) \\
&= c_1 \otimes \overline{c_2} \otimes c_3{}^{-0}T(c_3{}^{-1}h) \\
&\overset{(3.5)}{=} c_1 \otimes \overline{c_2} \otimes c_4{}^{-0}T(\nabla(c_3)c_4{}^{-1}h) \\
&\overset{(4.6)}{=} c_1 \otimes \overline{c_2{}^{-0}} \otimes c_3{}^{-0}T(c_2{}^{-1}c_3{}^{-1}h) \\
&\overset{(PCC2)}{=} c_1 \otimes \overline{c_2{}^{-0}{}_1} \otimes c_2{}^{-0}{}_2T(c_2{}^{-1}h) \\
&= c_1 \otimes \lambda_N(c_2{}^{-0}T(c_2{}^{-1}h)) \\
&= (I \otimes \lambda_N)\mu(c \bar{\rtimes} h).
\end{aligned}$$

Therefore $Im(\mu) \subseteq M \square_{C/\mathcal{J}} N$ and μ is well-defined. Now, we will proof that μ is $C \bar{\rtimes} H$-bicolinear. Let $c \bar{\rtimes} h \in C \bar{\rtimes} H$, so

$$\begin{aligned}
(I \otimes \rho_N)\mu(c \bar{\rtimes} h) &= c_1 \square \rho_N(c_2{}^{-0})T(c_2{}^{-1}h) \\
&= c_1 \square c_2{}^{-0}{}_1{}^{-0} \otimes c_2{}^{-0}{}_2{}^{-0} \bar{\rtimes} S^{-1}(c_2{}^{-0}{}_1{}^{-1}c_2{}^{-0}{}_2{}^{-1})\lambda T(c_2{}^{-1}h) \\
&\overset{(PCC2)}{=} c_1 \square c_2{}^{-0-0} \otimes c_3{}^{-0-0} \bar{\rtimes} S^{-1}(c_2{}^{-0-1}c_3{}^{-0-1})\lambda T(c_2{}^{-1}c_3{}^{-1}h) \\
&\overset{(PCC4)}{=} c_1 \square c_2{}^{-0} \otimes c_4{}^{-0-0} \bar{\rtimes} S^{-1}(c_2{}^{-1}{}_2c_4{}^{-0-1})\lambda T(c_2{}^{-1}{}_1\nabla(c_3)c_4{}^{-1}h) \\
&\overset{(3.5)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0-0} \bar{\rtimes} S^{-1}(c_2{}^{-1}{}_2c_3{}^{-0-1})\lambda T(c_2{}^{-1}{}_1c_3{}^{-1}h) \\
&\overset{(PCC4)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} S^{-1}(c_2{}^{-1}{}_2c_3{}^{-1}{}_2)\lambda T(c_2{}^{-1}{}_1c_3{}^{-1}{}_1\nabla(c_4)h) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} S^{-1}(c_2{}^{-1}{}_2c_3{}^{-1}{}_2)\lambda T(c_2{}^{-1}{}_1c_3{}^{-1}{}_1c_4{}^{-1}h)\varepsilon(c_4{}^{-0}) \\
&\overset{(4.10)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} S^{-1}(c_2{}^{-1}{}_3c_3{}^{-1}{}_3)c_2{}^{-1}{}_2c_3{}^{-1}{}_2c_4{}^{-1}{}_2h_2T(c_2{}^{-1}{}_1c_3{}^{-1}{}_1c_4{}^{-1}{}_1h_1)\varepsilon(c_4{}^{-0}) \\
&= c_1 \square c_2{}^{-0} \otimes \\
&\qquad \otimes c_3{}^{-0} \bar{\rtimes} S^{-1}((c_2{}^{-1}{}_2c_3{}^{-1}{}_2)_2)(c_2{}^{-1}{}_2c_3{}^{-1}{}_2)_1c_4{}^{-1}{}_2h_2T(c_2{}^{-1}{}_1c_3{}^{-1}{}_1c_4{}^{-1}{}_1h_1)\varepsilon(c_4{}^{-0}) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} c_4{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}c_4{}^{-1}{}_1h_1)\varepsilon(c_4{}^{-0}) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0}{}_1 \bar{\rtimes} \nabla(c_3{}^{-0}{}_2)c_4{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}c_4{}^{-1}{}_1h_1)\varepsilon(c_4{}^{-0}) \\
&\overset{(PCC2)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} \nabla(c_4{}^{-0})c_5{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}c_4{}^{-1}c_5{}^{-1}{}_1h_1)\varepsilon(c_5{}^{-0}) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} c_4{}^{-0-1}c_5{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}c_4{}^{-1}c_5{}^{-1}{}_1h_1)\varepsilon(c_5{}^{-0})\varepsilon(c_4{}^{-0-0}) \\
&\overset{(PCC3)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} c_5{}^{-1}{}_2c_6{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}\nabla(c_4)c_5{}^{-1}{}_1c_6{}^{-1}{}_1h_1)\varepsilon(c_6{}^{-0})\varepsilon(c_5{}^{-0}) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} (c_5{}^{-1}c_6{}^{-1})_2h_2T(c_2{}^{-1}c_3{}^{-1}\nabla(c_4)(c_5{}^{-1}c_6{}^{-1})_1h_1)\varepsilon(c_6{}^{-0})\varepsilon(c_5{}^{-0}) \\
&\overset{(PCC2)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} c_5{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}\nabla(c_4)c_5{}^{-1}{}_1h_1)\varepsilon(c_5{}^{-0}{}_2)\varepsilon(c_5{}^{-0}{}_1) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} c_5{}^{-1}{}_2h_2T(c_2{}^{-1}c_3{}^{-1}\nabla(c_4)c_5{}^{-1}{}_1h_1)\varepsilon(c_5{}^{-0}) \\
&\overset{(PCC3)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} c_4{}^{-0-1}h_2T(c_2{}^{-1}c_3{}^{-1}c_4{}^{-1}h_1)\varepsilon(c_4{}^{-0-0}) \\
&= c_1 \square c_2{}^{-0} \otimes c_3{}^{-0} \bar{\rtimes} \nabla(c_4{}^{-0})h_2T(c_2{}^{-1}c_3{}^{-1}c_4{}^{-1}h_1) \\
&\overset{(PCC2)}{=} c_1 \square c_2{}^{-0} \otimes c_3{}^{-0}{}_1 \bar{\rtimes} \nabla(c_3{}^{-0}{}_2)h_2T(c_2{}^{-1}c_3{}^{-1}h_1) \\
&= c_1 \square c_2{}^{-0}T(c_2{}^{-1}c_3{}^{-1}h_1) \otimes c_3{}^{-0} \bar{\rtimes} h_2 \\
&= \mu(c_1 \bar{\rtimes} c_2{}^{-1}h_1) \otimes c_2{}^{-0} \bar{\rtimes} h_2 \\
&= (\mu \otimes I)\Delta(c \bar{\rtimes} h)
\end{aligned}$$

Therefore μ is right $C \bar{\rtimes} H$-colinear. Moreover, to show that μ is left $C \bar{\rtimes} H$-colinear we need to show that $(I \otimes \mu)\Delta(c \bar{\rtimes} h) = (\lambda_M \otimes I)\mu(c \bar{\rtimes} h)$ for all $c \bar{\rtimes} h \in C\ PxpH$.

In fact,

$$\begin{aligned}
(I \otimes \mu)\Delta(c \bar{\rtimes} h) &= c_1 \bar{\rtimes} c_2{}^{-1}h_1 \otimes \mu(c_2{}^{-0} \bar{\rtimes} h_2) \\
&= c_1 \bar{\rtimes} c_2{}^{-1}h_1 \otimes c_2{}^{-0}{}_1 \square c_2{}^{-0}{}_2{}^{-0}T(c_2{}^{-0}{}_2{}^{-1}h_2)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_M \otimes I)\mu(c \,\bar{\rtimes}\, h) &= \lambda_N(c_1) \,\square\, {c_2}^{-0}T({c_2}^{-1}h) \\
&= c_1 \,\bar{\rtimes}\, {c_2}^{-1} \otimes {c_2}^{-0} \,\square\, {c_3}^{-0}T({c_3}^{-1}h) \\
&\overset{(4.9)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1}({c_3}^{-1}h)_1T(({c_3}^{-1}h)_2) \otimes {c_2}^{-0} \,\square\, {c_3}^{-0} \\
&= c_1 \,\bar{\rtimes}\, {c_2}^{-1}{c_3}^{-1}{}_1h_1 \otimes {c_2}^{-0} \,\square\, {c_3}^{-0}T({c_3}^{-1}{}_2h_2) \\
&\overset{(3.5)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1}\nabla(c_3){c_4}^{-1}{}_1h_1 \otimes {c_2}^{-0} \,\square\, {c_4}^{-0}T({c_4}^{-1}{}_2h_2) \\
&\overset{(PCC3)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1}{c_3}^{-1}h_1 \otimes {c_2}^{-0} \,\square\, {c_3}^{-0-0}T({c_3}^{-0-1}h_2) \\
&\overset{(PCC2)}{=} c_1 \,\bar{\rtimes}\, {c_2}^{-1}h_1 \otimes {c_2}^{-0}{}_1 \,\square\, {c_2}^{-0}{}_2{}^{-0}T({c_2}^{-0}{}_2{}^{-1}h_2).
\end{aligned}$$

Then μ is left $C \,\bar{\rtimes}\, H$-colinear and consequently a $C \,\bar{\rtimes}\, H$-bicolinear map. $\square$

In [20], Dăscălescu, Raianu, and Zhang have considered the quotient of a (global) H-comodule coalgebra C by the coideal $C \blacktriangleleft \mathrm{Ker}(\varepsilon_{H^*})$. In the partial case, as already said before, the set $C \leftharpoonup \mathrm{Ker}(\varepsilon_{H^*})$ is not a coideal of C, hence we can not considerate the correspondent quotient coalgebra. Also, to construct a map from that quotient coalgebra, those authors used the fact that the coideal $C \blacktriangleleft \mathrm{Ker}(\varepsilon_{H^*})$ can be seen as the following set

$$\{f(c^{-1})c^{-0} - f(1_H)c \mid f \in H^* \text{ and } c \in C\}.$$

In our case, we take the coideal $\mathcal{J} = \left(C^{*\underline{H^*}}\right)^{\perp}$ and we need to define a map from ${}^{C}/_{\mathcal{J}}$. Inspired in the paper [20] we would like to describe $\mathcal{J}$ by its elements. For do this, we observe that it follows from Theorem 4.1.7 that the vector space

$$X = \{f(c^{-1})c^{-0} - f(\nabla(c_2))c_1 |\ c \in C \textit{ and } f \in H^*\}$$

is contained in $\mathcal{J}$. In fact, also the reverse inclusion holds, as it is shown in the next result.

Proposition 4.1.11. *With the above notations, we have that $X = \mathcal{J}$.*

Proof. From the proof of Theorem 4.1.7 it follows that $X \subseteq \mathcal{J}$, so that we have $\mathcal{J}^{\perp} \subseteq X^{\perp}$.

Now, let $\psi \in X^{\perp}$. Thus, $0 = \psi(f(c^{-1})c^{-0} - f(\nabla(c_2))c_1)$, for all $c \in C$ and $f \in H^*$. Then

$$\begin{aligned}
(f \rightarrow \psi)(c) &= f(c^{-1})\psi(c^{-0}) \\
&= \psi(f(c^{-1})c^{-0}) \\
&= \psi(f(\nabla(c_2))c_1) \\
&= f(\nabla(c_2))\psi(c_1) \\
&= \psi(c_1)f({c_2}^{-1})\varepsilon({c_2}^{-0}) \\
&= [\psi * (f \rightarrow \varepsilon)](c).
\end{aligned}$$

Hence, ψ lies in $C^{*\underline{H^*}}$. Therefore, $\mathcal{J} = \left(C^{*\underline{H^*}}\right)^{\perp} \subseteq X^{\perp\perp} = X$. $\square$

Now, with this representation of the coideal $\mathcal{J}$, we are in position to define a map from the quotient coalgebra ${}^{C}/_{\mathcal{J}}$ to the cotensor product $N \,\square_{\,C \bar{\rtimes} H} M$. We start defining the following linear map

$$\begin{aligned} \tilde{\tau}\colon C &\longrightarrow N \otimes M \\ c &\longmapsto {c_1}^{-0} \otimes {c_2}^{-0} T({c_1}^{-1}{c_2}^{-1}). \end{aligned}$$

Remark 4.1.12. Note that $\mathcal{J} \subseteq \ker \tilde{\tau}$.

In fact, let $c \in C$ and $f \in H^*$, so

$$\begin{aligned} \tilde{\tau}(f(c^{-1})c^{-0}) &= {c^{-0}}_1{}^{-0} \otimes {c^{-0}}_2{}^{-0} T({c^{-0}}_1{}^{-1}{c^{-0}}_2{}^{-1}) f(c^{-1}) \\ &\overset{(PCC2)}{=} {c_1}^{-0-0} \otimes {c_2}^{-0-0} T({c_1}^{-0-1}{c_2}^{-0-1}) f({c_1}^{-1}{c_2}^{-1}) \\ &\overset{(PCC4)}{=} {c_1}^{-0} \otimes {c_3}^{-0-0} T({c_1}^{-1}{}_2{c_3}^{-0-1}) f({c_1}^{-1}{}_1 \nabla(c_2){c_3}^{-1}) \\ &\overset{(3.5)}{=} {c_1}^{-0} \otimes {c_2}^{-0-0} T({c_1}^{-1}{}_2{c_2}^{-0-1}) f({c_1}^{-1}{}_1{c_2}^{-1}) \\ &\overset{(PCC4)}{=} {c_1}^{-0} \otimes {c_2}^{-0} T({c_1}^{-1}{}_2{c_2}^{-1}{}_2) f({c_1}^{-1}{}_1{c_2}^{-1}{}_1 \nabla(c_3)) \\ &= {c_1}^{-0} \otimes {c_2}^{-0} f(({c_1}^{-1}{c_2}^{-1})_1 T(({c_1}^{-1}{c_2}^{-1})_2) \nabla(c_3)) \\ &= {c_1}^{-0} \otimes {c_2}^{-0} f(T({c_1}^{-1}{c_2}^{-1}) \nabla(c_3)) \\ &= \tilde{\tau}[c_1 f(\nabla(c_2))]. \end{aligned}$$

Then $\tilde{\tau}(\mathcal{J}) = 0$.
Therefore, we have the well-defined map from the quotient ${}^{C}/_{\mathcal{J}}$ to $N \otimes M$ as follow

$$\begin{aligned} \tau\colon {}^{C}/_{\mathcal{J}} &\longrightarrow N \,\square_{\,C \bar{\rtimes} H} M \\ \bar{c} &\longmapsto {c_1}^{-0} \,\square\, {c_2}^{-0} T({c_1}^{-1}{c_2}^{-1}) = {c^{-0}}_1 \,\square\, {c^{-0}}_2 T(c^{-1}). \end{aligned} \tag{4.15}$$

Proposition 4.1.13. *With the above notation, τ is well-defined ${}^{C}/_{\mathcal{J}}$-bicolinear map.*

Proof. First of all, we need to see that τ is well-defined. By the already shown we just need to show that $Im(\tau) \subseteq N \,\square_{\,C \bar{\rtimes} H} M$.

$$\begin{aligned} (I \otimes \lambda_M)\tau(\bar{c}) &= {c_1}^{-0} \otimes \lambda_M({c_2}^{-0}) T({c_1}^{-1}{c_2}^{-1}) \\ &= {c_1}^{-0} \otimes {c_2}^{-0}{}_1 \bar{\rtimes} {c_2}^{-0}{}_2{}^{-1} \otimes {c_2}^{-0}{}_2{}^{-0} T({c_1}^{-1}{c_2}^{-1}) \\ &\overset{(PCC2)}{=} {c_1}^{-0} \otimes {c_2}^{-0} \bar{\rtimes} {c_3}^{-0-1} \otimes {c_3}^{-0-0} T({c_1}^{-1}{c_2}^{-1}{c_3}^{-1}) \\ &\overset{(PCC3)}{=} {c_1}^{-0} \otimes {c_2}^{-0} \bar{\rtimes} {c_4}^{-1}{}_2 \otimes {c_4}^{-0} T({c_1}^{-1}{c_2}^{-1} \nabla(c_3){c_4}^{-1}{}_1) \\ &\overset{(3.5)}{=} {c_1}^{-0} \otimes {c_2}^{-0} \bar{\rtimes} {c_3}^{-1}{}_2 \otimes {c_3}^{-0} T({c_1}^{-1}{c_2}^{-1}{c_3}^{-1}{}_1) \end{aligned}$$

and

$$
\begin{aligned}
&(\rho_N \otimes I)\tau(\bar{c}) = \\
&\quad = \quad \rho_N({c_1}^{-0}) \otimes {c_2}^{-0} T({c_1}^{-1}{c_2}^{-1}) \\
&\quad = \quad {{c_1}^{-0}}_1{}^{-0} \otimes {{c_1}^{-0}}_2{}^{-0} \mathbin{\bar{\rtimes}} S^{-1}({{c_1}^{-0}}_1{}^{-1}{{c_1}^{-0}}_2{}^{-1})\lambda \otimes {c_2}^{-0} T({c_1}^{-1}{c_2}^{-1}) \\
&\quad \overset{(PCC2)}{=} {c_1}^{-0-0} \otimes {c_2}^{-0-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-0-1}{c_2}^{-0-1})\lambda \otimes {c_3}^{-0} T({c_1}^{-1}{c_2}^{-1}{c_3}^{-1}) \\
&\quad \overset{(PCC4)}{=} {c_1}^{-0} \otimes {c_3}^{-0-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-1}{}_2{c_3}^{-0-1})\lambda \otimes {c_4}^{-0} T({c_1}^{-1}{}_1\nabla(c_2){c_3}^{-1}{c_4}^{-1}) \\
&\quad \overset{(3.5)}{=} \quad {c_1}^{-0} \otimes {c_2}^{-0-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-1}{}_2{c_2}^{-0-1})\lambda \otimes {c_3}^{-0} T({c_1}^{-1}{}_1{c_2}^{-1}{c_3}^{-1}) \\
&\quad \overset{(PCC4)}{=} {c_1}^{-0} \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-1}{}_2{c_2}^{-1}{}_2)\lambda \otimes {c_4}^{-0} T({c_1}^{-1}{}_1{c_2}^{-1}{}_1\nabla(c_3){c_4}^{-1}) \\
&\quad \overset{(3.5)}{=} \quad {c_1}^{-0} \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-1}{}_2{c_2}^{-1}{}_2)\lambda T({c_1}^{-1}{}_1{c_2}^{-1}{}_1{c_3}^{-1}) \otimes {c_3}^{-0} \\
&\quad \overset{(4.10)}{=} \quad {c_1}^{-0} \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-1}{}_2{c_2}^{-1}{}_2)({c_1}^{-1}{}_1{c_2}^{-1}{}_1{c_3}^{-1})_2 T(({c_1}^{-1}{}_1{c_2}^{-1}{}_1{c_3}^{-1})_1) \otimes {c_3}^{-0} \\
&\quad = \quad {c_1}^{-0} \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} S^{-1}({c_1}^{-1}{}_3{c_2}^{-1}{}_3){c_1}^{-1}{}_2{c_2}^{-1}{}_2{c_3}^{-1}{}_2 T({c_1}^{-1}{}_1{c_2}^{-1}{}_1{c_3}^{-1}{}_1) \otimes {c_3}^{-0} \\
&\quad = \quad {c_1}^{-0} \otimes {c_2}^{-0} \mathbin{\bar{\rtimes}} {c_3}^{-1}{}_2 T({c_1}^{-1}{c_2}^{-1}{c_3}^{-1}{}_1) \otimes {c_3}^{-0}
\end{aligned}
$$

Now we have a well-defined map, then just remains to proof the bicollinearity.

$$
\begin{aligned}
(I \otimes \rho_M)\tau(\bar{c}) &= {c_1}^{-0} \,\Box\, \rho_M({c_2}^{-0}) T({c_1}^{-1}{c_2}^{-1}) \\
&= {c_1}^{-0} \,\Box\, {c_2}^{-0}{}_1 \otimes \overline{{c_2}^{-0}{}_2} T({c_1}^{-1}{c_2}^{-1}) \\
&\overset{(PCC2)}{=} {c_1}^{-0} \,\Box\, {c_2}^{-0} \otimes \overline{{c_3}^{-0}} T({c_1}^{-1}{c_2}^{-1}{c_3}^{-1}) \\
&\overset{(4.6)}{=} {c_1}^{-0} \,\Box\, {c_2}^{-0} \otimes \overline{c_4} T({c_1}^{-1}{c_2}^{-1}\nabla(c_3)) \\
&\overset{(3.5)}{=} {c_1}^{-0} \,\Box\, {c_2}^{-0} \otimes \overline{c_3} T({c_1}^{-1}{c_2}^{-1}) \\
&\overset{(PCC2)}{=} {c_1}^{-0}{}_1 \,\Box\, {c_1}^{-0}{}_2 \otimes \overline{c_2} T({c_1}^{-1}) \\
&= \tau(\overline{c_1}) \otimes \overline{c_2} \\
&= (\tau \otimes I)\Delta(\bar{c})
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_N \otimes I)\tau(\bar{c}) &= \lambda_N({c_1}^{-0}) \,\Box\, {c_2}^{-0} T({c_1}^{-1}{c_2}^{-1}) \\
&= \overline{{c_1}^{-0}{}_1} \otimes {c_1}^{-0}{}_2 \,\Box\, {c_2}^{-0} T({c_1}^{-1}{c_2}^{-1}) \\
&\overset{(PCC2)}{=} \overline{{c_1}^{-0}} \otimes {c_2}^{-0} \,\Box\, {c_3}^{-0} T({c_1}^{-1}{c_2}^{-1}{c_3}^{-1}) \\
&\overset{(4.6)}{=} \overline{c_1} \otimes {c_3}^{-0} \,\Box\, {c_4}^{-0} T(\nabla(c_2){c_3}^{-1}{c_4}^{-1}) \\
&\overset{(3.5)}{=} \overline{c_1} \otimes {c_2}^{-0} \,\Box\, {c_3}^{-0} T({c_2}^{-1}{c_3}^{-1}) \\
&\overset{(PCC2)}{=} \overline{c_1} \otimes {c_2}^{-0}{}_1 \,\Box\, {c_2}^{-0}{}_2 T({c_2}^{-1}) \\
&= \overline{c_1} \otimes \tau(\overline{c_2}) \\
&= (I \otimes \tau)\Delta(\bar{c})
\end{aligned}
$$

Therefore τ is ${}^{C}/_{\mathcal{J}}$-bicolinear. $\square$

Now we have all elements necessary to construct our Morita-Takeuchi context between ${}^{C}/_{\mathcal{J}}$ and $C \,\bar{\rtimes}\, H$. Then consider the following Theorem.

Theorem 4.1.14. *Let H be a finite dimensional Hopf algebra and C a left symmetric partial H-comodule coalgebra. Then, with the above notations, $({}^{C}/_{\mathcal{J}}, C \,\bar{\rtimes}\, H, M, N, \mu, \tau)$ is a Morita-Takeuchi context.*

Proof. By the construction already done, just remains to show that the following diagrams are commutative:

$$\begin{tikzcd}
M \square_{{}^{C}/_{\mathcal{J}}} N \square_{C \bar{\rtimes} H} M & M \square_{{}^{C}/_{\mathcal{J}}} {}^{C}/_{\mathcal{J}} \arrow[l, "I_M \otimes \tau"'] \\
C \,\bar{\rtimes}\, H \square_{C \bar{\rtimes} H} M \arrow[u, "\mu \otimes I_M"] \arrow[ur, phantom, "\circlearrowleft"] & M \arrow[l, "\lambda_M"] \arrow[u, "\rho_M"']
\end{tikzcd}
\qquad
\begin{tikzcd}
N \square_{C \bar{\rtimes} H} M \square_{{}^{C}/_{\mathcal{J}}} N & N \square_{C \bar{\rtimes} H} C \,\bar{\rtimes}\, H \arrow[l, "I_N \otimes \mu"'] \\
{}^{C}/_{\mathcal{J}} \square_{{}^{C}/_{\mathcal{J}}} N \arrow[u, "\tau \otimes I_N"] \arrow[ur, phantom, "\circlearrowleft"] & N \arrow[l, "\lambda_N"] \arrow[u, "\rho_N"']
\end{tikzcd}$$

In fact,

$$\begin{aligned}
(I \otimes \tau)\rho_M(c) &= c_1 \otimes \tau(\overline{c_2}) \\
&= c_1 \otimes c_2{}^{-0} \square c_3{}^{-0} T(c_2{}^{-1} c_3{}^{-1}) \\
&= c_1 \otimes c_2{}^{-0} T(c_2{}^{-1} c_3{}^{-1}) \square c_3{}^{-0} \\
&= \mu(c_1 \,\bar{\rtimes}\, c_2{}^{-1}) \square c_2{}^{-0} \\
&= (\mu \otimes I)\lambda_M(c)
\end{aligned}$$

and

$$\begin{aligned}
(I \otimes \mu)\rho_N(c) &= c_1{}^{-0} \otimes \mu(c_2{}^{-0} \,\bar{\rtimes}\, S^{-1}(c_1{}^{-1} c_2{}^{-1})\lambda) \\
&= c_1{}^{-0} \otimes c_2{}^{-0}{}_1 \square c_2{}^{-0}{}_2{}^{-0} T[c_2{}^{-0}{}_2{}^{-1} S^{-1}(c_1{}^{-1} c_2{}^{-1})\lambda] \\
&\overset{(PCC2)}{=} c_1{}^{-0} \otimes c_2{}^{-0} \square c_3{}^{-0-0} T[c_3{}^{-0-1} S^{-1}(c_1{}^{-1} c_2{}^{-1} c_3{}^{-1})\lambda] \\
&\overset{(PCC3)}{=} c_1{}^{-0} \otimes c_2{}^{-0} \square c_4{}^{-0} T[c_4{}^{-1}{}_2 S^{-1}(c_1{}^{-1} c_2{}^{-1} \nabla(c_3) c_4{}^{-1}{}_1)\lambda] \\
&\overset{(3.5)}{=} c_1{}^{-0} \otimes c_2{}^{-0} \square c_3{}^{-0} T[c_3{}^{-1}{}_2 S^{-1}(c_1{}^{-1} c_2{}^{-1} c_3{}^{-1}{}_1)\lambda] \\
&= c_1{}^{-0} \otimes c_2{}^{-0} \square c_3{}^{-0} T[c_3{}^{-1}{}_2 S^{-1}(c_3{}^{-1}{}_1) S^{-1}(c_1{}^{-1} c_2{}^{-1})\lambda] \\
&\overset{(PCC1)}{=} c_1{}^{-0} \otimes c_2{}^{-0} \square c_3 T[S^{-1}(c_1{}^{-1} c_2{}^{-1})\lambda] \\
&\overset{(4.11)}{=} c_1{}^{-0} \otimes c_2{}^{-0} \square c_3 T(c_1{}^{-1} c_2{}^{-1}) \\
&= \tau(\overline{c_1}) \square c_2 \\
&= (\tau \otimes I)\lambda_M(c).
\end{aligned}$$

Then $({}^{C}/_{\mathcal{J}}, C \,\bar{\rtimes}\, H, M, N, \mu, \tau)$ is a Morita-Takeuchi context between ${}^{C}/_{\mathcal{J}}$ and $C \,\bar{\rtimes}\, H$. $\square$

4.2 CoGalois Coextensions

4.2.1 Galois Coextension for Partial Modules Coalgebras

By Proposition 2.1.13, given a right partial H-module coalgebra C we have that C^* is a left partial H-module algebra, so we can considerate the subalgebra of invariants

$$C^{*\underline{H}} = \{\alpha \in C^* \mid h \rightarrow \alpha = (h \rightarrow \varepsilon_C) * \alpha, \ \forall h \in H\},$$

therefore we have that $\mathcal{J} = \left(C^{*\underline{H}}\right)^{\perp}$ is a coideal of C. Then we define the invariant coalgebra as the quotient of C by this coideal, i.e.,

$$C^{\underline{H}} = {}^{C}/_{\mathcal{J}}.$$

In this case we say that $C/C^{\underline{H}}$ *is an H-coextension.* The following result follows directly from Proposition 1.1.2.

Proposition 4.2.1. *Let C be a partial H-module coalgebra. Then $C/C^{\underline{H}}$ is an H-coextension if and only if $C^{*\underline{H}} \subseteq C^*$ is an H-extension*

We will now construct the definition of Hopf-Galois coextension for partial H-module coalgebras, which will be simply called of coGalois.

Let C be a right partial H-module coalgebra, and define the following linear map

$$\begin{aligned} \tilde{\beta} \colon C \otimes H &\longrightarrow C \otimes C \\ c \otimes h &\longmapsto c_1 \otimes (c_2 \leftharpoonup h). \end{aligned}$$

One can show, in a similar way as it was proved in Propositions 4.1.8 and 4.1.9, that C is a ${}^{C}/_{\mathcal{J}}$-bicomodule via $\lambda(c) = \overline{c_1} \otimes c_2$ and $\rho(c) = c_1 \otimes \overline{c_2}$. Thus, we have the following.

Proposition 4.2.2. *With the above notations, the image of $\tilde{\beta}$ is contained in $C \,\square_{\,C/\mathcal{J}} C$.*

Proof. In fact, note that for all $\psi \in C^{*\underline{H}}$, $\alpha \in C^*$, $c \in C$ and $h \in H$, we have

$$\begin{aligned} \psi(c_1 \leftharpoonup h_1 \alpha(c_2 \leftharpoonup h_2)) &= \psi(c_1 \leftharpoonup h_1)\alpha(c_2 \leftharpoonup h_2) \\ &= [(h_1 \rightarrow \psi) * (h_2 \rightarrow \alpha)](c) \\ &= [\psi * (h_1 \rightarrow \varepsilon_C) * (h_2 \rightarrow \alpha)](c) \qquad \text{since } \psi \in C^{*\underline{H}} \\ &= [\psi * (h \rightarrow \alpha)](c) \\ &= \psi(c_1)\alpha(c_2 \leftharpoonup h) \\ &= \psi(c_1 \alpha(c_2 \leftharpoonup h)). \end{aligned}$$

Since ψ is any element in $C^{*\underline{H}}$, then $c_1 \leftharpoonup h_1 \alpha(c_2 \leftharpoonup h_2) - c_1\alpha(c_2 \leftharpoonup h)$ lies in $\left(C^{*\underline{H}}\right)^{\perp}$.

It means that $\overline{c_1 \leftharpoondown h_1}\ \alpha(c_2 \leftharpoondown h_2) = \overline{c_1}\ \alpha(c_2 \leftharpoondown h)$ in ${}^{C}/_{\mathcal{J}}$. Since it holds for any $\alpha \in c^*$, then

$$\overline{c_1 \leftharpoondown h_1} \otimes (c_2 \leftharpoondown h_2) = \overline{c_1} \otimes (c_2 \leftharpoondown h) \tag{4.16}$$

in ${}^{C}/_{\mathcal{J}} \otimes C$. Then,

$$\begin{aligned} (I \otimes \lambda)\tilde{\beta}(c) &= c_1 \otimes \lambda(c_2 \leftharpoondown h) \\ &= c_1 \otimes \overline{c_2 \leftharpoondown h_1} \otimes (c_3 \leftharpoondown h_2) \\ &\overset{(4.16)}{=} c_1 \otimes \overline{c_2} \otimes (c_3 \leftharpoondown h) \\ &= \rho(c_1) \otimes (c_2 \leftharpoondown h) \\ &= (\rho \otimes I)\tilde{\beta}(c) \end{aligned}$$

and it means that the image of $\tilde{\beta}$ is contained in $C \,\square_{\,C/\mathcal{J}} C$. □

Now consider the subspace $C \bar{\otimes} H$ of $C \otimes H$ spanned by the elements of the form $\{c_1 \otimes h_1 \varepsilon_C(c_2 \leftharpoondown h_2) \mid c \in C \text{ and } h \in H\}$. Denoting an element of $C \bar{\otimes} H$ by $c \bar{\otimes} h = c_1 \otimes h_1 \varepsilon_C(c_2 \leftharpoondown h_2)$, consider β as the restriction of $\tilde{\beta}$ to $C \bar{\otimes} H$. Then

$$\begin{aligned} \beta(c \bar{\otimes} h) &= \tilde{\beta}(c_1 \otimes h_1)\varepsilon_C(c_2 \leftharpoondown h_2) \\ &= c_1 \otimes (c_2 \leftharpoondown h_1)\varepsilon_C(c_3 \leftharpoondown h_2) \\ &\overset{(PMC2)}{=} c_1 \otimes (c_2 \leftharpoondown h), \end{aligned}$$

i.e., we have a well-defined linear map

$$\begin{aligned} \beta\colon C \bar{\otimes} H &\longrightarrow C \,\square_{\,C/\mathcal{J}} C \\ c \bar{\otimes} h &\longmapsto c_1 \,\square\, (c_2 \leftharpoondown h). \end{aligned}$$

Definition 4.2.3. Let C be a right partial H-module coalgebra. Then the coextension $C/C^{\underline{H}}$ is said an H-*Galois coextension (or* H*-coGalois)* if β is bijective.

Remark 4.2.4. If C is a (global) module coalgebra, then $C \bar{\otimes} H = C \otimes H$ and $\mathcal{J} = \left(C^{*\underline{H}}\right)^{\perp} = C \blacktriangleleft \ker \varepsilon$. Hence, the Definition 4.2.3 recover the classical definition of H-coextension (cf. [20, p. 403]).

4.2.2 Galois Coextension for Partial Comodules Coalgebras

Given a left partial H-comodule coalgebra C we have a coideal $\mathcal{J}$ defined as in the equation (4.5), and then we define the coalgebra of coinvariants as the coalgebra quotient, i.e.,

$$C^{\underline{\mathrm{coH}}} = {}^{C}/_{\mathcal{J}}.$$

In the above situation, we have that $C/C^{\underline{\mathrm{coH}}}$ is an H^*-coextension. Since C is a left partial H-comodule coalgebra, hence C is a right partial H^*-module coalgebra, then we have that

$$C^{\underline{\mathrm{coH}}} = C^{\underline{H^*}}.$$

Proposition 4.2.5. *In the above situation, the following are equivalents:*

1. $C/C^{\underline{\mathrm{coH}}}$ *is an* H^**-coextension;*
2. ${C^*}^{\underline{H^*}} \subseteq C^*$ *is an* H*-extension.*

Proof. It follows directly from Proposition 4.2.1. $\square$

Let C be a left partial H-comodule coalgebra, then we recall that $C \,\bar{\otimes}\, H^*$ is the subspace of $C \otimes H^*$ spanned by the set

$$\{c_1 \otimes f_1 \varepsilon_C(c_2 \leftharpoonup f_2) \mid c \in C \text{ and } f \in H^*\}.$$

Notice that, $c_1 \otimes f_1 \varepsilon_C(c_2 \leftharpoonup f_2) = c_1 \otimes f_1 f_2({c_2}^{-1})\varepsilon_C({c_2}^{-0}) = c_1 \otimes f_1 f_2(\nabla(c_2))$. Thus, a typical element of $C \,\bar{\otimes}\, H^*$ can be writen in the following way

$$c \,\bar{\otimes}\, f = c_1 \otimes \nabla(c_2) \rightharpoonup f.$$

Hence, we have the canonical map

$$\begin{aligned} \beta \colon C \,\bar{\otimes}\, H^* &\longrightarrow C \,\square_{C/J}\, C \\ c \,\bar{\otimes}\, f &\longmapsto c_1 \,\square\, c_2 \leftharpoonup (\nabla(c_3) \rightharpoonup f) = c_1 \,\square\, f({c_2}^{-1}){c_2}^{-0}, \end{aligned}$$

where the equality is true because

$$\begin{aligned} c_1 \,\square\, c_2 \leftharpoonup (\nabla(c_3) \rightharpoonup f) &= c_1 \,\square\, c_2 \leftharpoonup (f_1 \; f_2(\nabla(c_3))) \\ &= c_1 \,\square\, f_1({c_2}^{-1}) f_2(\nabla(c_3)){c_2}^{-0} \\ &= c_1 \,\square\, f({c_2}^{-1}\nabla(c_3)){c_2}^{-0} \\ &\overset{(3.5)}{=} c_1 \,\square\, f({c_2}^{-1}){c_2}^{-0}. \end{aligned}$$

Therefore, we have that the H^*-coextension $C/C^{\underline{\mathrm{H}^*}}$ is Galois (or H^*-coGalois) if the above defined map β is bijective.

Now we will relate the Morita-Takeuchi context provide by Theorem 4.1.14 and the H^*-coextension $C/C^{\underline{\mathrm{coH}}}$ be Galois. But, for this, we need to make some remarks.

Remember from [30] that for any finite dimensional Hopf algebra H and for a left integral $T \neq 0$ in H^*, we have the isomorphism

$$\begin{aligned} \gamma \colon H &\longrightarrow H^* \\ h &\longmapsto h \rightharpoonup T = T_1 T_2(h). \end{aligned}$$

Since C is a left partial H-comodule coalgebra then we have the partial smash coproduct $C \bar{\rtimes} H$. Thus, restricting $I \otimes \gamma$ to $C \bar{\rtimes} H$, we have

$$\begin{aligned}
(I \otimes \gamma)(c \bar{\rtimes} h) &= (I \otimes \gamma)(c_1 \otimes \nabla(c_2)h) \\
&= c_1 \otimes \gamma(\nabla(c_2)h) \\
&= c_1 \otimes \nabla(c_2)h \rightharpoonup T \\
&= c_1 \otimes \nabla(c_2) \rightharpoonup (h \rightharpoonup T) \\
&= c_1 \otimes \nabla(c_2) \rightharpoonup \gamma(h) \\
&= c \bar{\otimes} \gamma(h).
\end{aligned}$$

Proposition 4.2.6. *With the above notations, we have that $C \bar{\rtimes} H \simeq C \bar{\otimes} H^*$ as vector spaces.*

Proof. Define Γ as the restriction of $I \otimes \gamma$ to $C \bar{\rtimes} H$, i.e.

$$\begin{aligned}
\Gamma\colon C \bar{\rtimes} H &\longrightarrow C \bar{\otimes} H^* \\
c \bar{\rtimes} h &\longmapsto c \bar{\otimes} \gamma(h).
\end{aligned}$$

Since Γ is the restriction of $I \otimes \gamma$ and γ is injective, hence Γ is injective too. Moreover, by the definition of Γ and by the surjectivity of γ, it follows that Γ is surjective. Therefore Γ is bijective, as desired. $\square$

Theorem 4.2.7. *Let C be a left symmetric partial H-comodule coalgebra. The following statements are equivalent:*

(1) C is H^*-coGalois;

(2) β is injective;

(3) μ is injective;

(4) μ is bijective.

Proof. The implication (1) $\Longrightarrow$ (2) is a tautology.

In [31], Takeuchi has shown that μ injective implies μ bijective. Then it is clear that (3) $\Longleftrightarrow$ (4).

Now consider the following commutative diagram

$$\begin{tikzcd}
& C \,\square_{C/\mathcal{J}}\, C \arrow[d, phantom, "\circlearrowleft"] & \\
C \bar{\rtimes} H \arrow[ur, "\mu"] \arrow[rr, "\Gamma"] & & C \bar{\otimes} H^* \arrow[ul, "\beta"']
\end{tikzcd}$$

In fact, let $c \,\bar{\rtimes}\, h \in C \,\bar{\rtimes}\, H$. Then we have

$$\begin{aligned}\beta[\Gamma(c \,\bar{\rtimes}\, h)] &= \beta[c \,\bar{\otimes}\, \gamma(h)]\\ &= \beta[c \,\bar{\otimes}\, (h \rightharpoonup T)]\\ &= c_1 \,\Box\, (h \rightharpoonup T)({c_2}^{-1}){c_2}^{-0}\\ &= c_1 \,\Box\, T_1({c_2}^{-1})T_2(h){c_2}^{-0}\\ &= c_1 \,\Box\, T({c_2}^{-1}h){c_2}^{-0}\\ &= \mu(c \,\bar{\rtimes}\, h).\end{aligned}$$

Since Γ is an isomorphism, it follows that (2) $\iff$ (3) and (4) $\implies$ (1). $\square$

Now we will present some relations between this notion of Galois coextension, for partial coaction on coalgebras, and Galois extension, for partial coaction on algebras.

First, we need to remember the definition of Galois extension for partial coaction on algebras (cf. [1, 11]). Given K an Hopf algebra and A a right partial K-comodule algebra via $\rho\colon a \mapsto a^{+0} \otimes a^{+1}$, consider the subspace of $A \otimes K$ given by the right multiplication for $\rho(1_A)$, i.e.

$$A \,\underline{\otimes}\, K = (A \otimes K)\rho(1_A),$$

where a typical element is of form $a \,\underline{\otimes}\, k = a1^{+0} \otimes k1^{+1}$. Thus, we can consider the canonical map

$$\begin{aligned}\mathrm{can}\colon A \otimes_{A^{\underline{\mathrm{coK}}}} A &\longrightarrow A \,\underline{\otimes}\, K\\ a \otimes b &\longmapsto a\, b^{+0} \,\underline{\otimes}\, b^{+1}\end{aligned}$$

and $A^{\underline{\mathrm{coK}}} \subseteq A$ is said a K-Galois extension when the canonical maps is a bijection.

Since H is a finite dimensional Hopf algebra and C is a left partial H-comodule coalgebra, then, by Theorem 3.1.18, C^* is a right partial H-comodule algebra and the coactions satisfy the equation (3.10), that is,

$$\alpha^{+0}(c)\alpha^{+1} = \alpha(c^{-0})c^{-1}$$

for $c \in C$ and $\alpha \in C^*$. In this context, we have the subalgebra of coinvariant, that is,

$$C^{*\underline{\mathrm{coH}}} = \{\psi \in C^* \mid \rho(\psi) = (\psi \otimes 1_H)\rho(\varepsilon_C) = \psi * \varepsilon^{+0} \otimes \varepsilon^{+1}\}.$$

We will show that H^*-Galois implies H^*-coGalois, but first we will need to build some morphisms connecting $C^* \otimes_{C^{*\underline{H}^*}} C^*$ and $(C \,\Box\,_{C/\mathcal{J}} C)^*$ and connecting $C^* \,\underline{\otimes}\, H$ and $(C \,\bar{\otimes}\, H^*)^*$.

First we define the map

$$\begin{aligned}\tilde{\Phi}\colon C^* \times C^* &\longrightarrow (C \,\Box\,_{C/\mathcal{J}} C)^*\\ (\varphi, \alpha) &\longmapsto \tilde{\Phi}(\varphi, \alpha)(c \,\Box\, d) = \varphi(c)\alpha(d).\end{aligned}$$

and notice that we have, by Theorem 1.1.2, that given $\psi \in C^{*\underline{H^*}}$ there exists $\bar{\psi} \in ({}^{C}/_{\mathcal{J}})^*$ such that $\psi = \bar{\psi} \circ \pi_{\mathcal{J}}$. Then, by the above noted, we have

$$\begin{aligned}
\tilde{\Phi}(\varphi, \psi * \alpha)(c \,\square\, d) &= \varphi(c)\psi(d_1)\alpha(d_2) \\
&= \varphi(c)\bar{\psi}[\pi_{\mathcal{J}}(d_1)]\alpha(d_2) \\
&= \varphi(c)\bar{\psi}(\overline{d_1})\alpha(d_2) \\
&= \varphi(c_1)\bar{\psi}(\overline{c_2})\alpha(d) \qquad\qquad \text{since } \square_{\,C/\mathcal{J}} \\
&= \varphi(c_1)\psi(c_2)\alpha(d) \\
&= \tilde{\Phi}(\varphi * \psi, \alpha)(c \,\square\, d)
\end{aligned}$$

and, consequently, $\tilde{\Phi}$ is $C^{*\underline{H^*}}$-balanced. Therefore, we have the following well-defined linear map

$$\begin{aligned}
\Phi \colon C^* \otimes_{C^{*\underline{H^*}}} C^* &\longrightarrow (C \,\square_{\,C/\mathcal{J}} C)^* \\
\varphi \otimes \alpha &\longmapsto \Phi(\varphi \otimes \alpha)(c \,\square\, d) = \varphi(c)\alpha(d).
\end{aligned}$$

Moreover, we also define

$$\begin{aligned}
\tilde{\Psi} \colon C^* \otimes H &\longrightarrow (C \,\bar{\otimes}\, H^*)^* \\
\alpha \otimes h &\longmapsto \Psi(\alpha \otimes h)(c \,\bar{\otimes}\, f) = \alpha(c_1) f(h\nabla(c_2))
\end{aligned}$$

which is clearly well-defined.

Now observe that, taking $\alpha \,\underline{\otimes}\, h \in C^* \,\underline{\otimes}\, H$, we have

$$\begin{aligned}
\tilde{\Psi}(\alpha \,\underline{\otimes}\, h)(c \,\bar{\otimes}\, f) &= \tilde{\Psi}(\alpha * \varepsilon_C{}^{+0} \otimes h\, \varepsilon_C{}^{+1})(c \,\bar{\otimes}\, f) \\
&= [\alpha * \varepsilon_C{}^{+0}](c_1) f(h\, \varepsilon_C{}^{+1}\nabla(c_2)) \\
&= \alpha(c_1)\varepsilon_C{}^{+0}(c_2) f(h\, \varepsilon_C{}^{+1}\nabla(c_3)) \\
&= \alpha(c_1) f(h\, \varepsilon_C{}^{+0}(c_2)\varepsilon_C{}^{+1}\nabla(c_3)) \\
&\overset{(3.10)}{=} \alpha(c_1) f(h\, \varepsilon_C(c_2{}^{-0}) c_2{}^{-1}\nabla(c_3)) \\
&= \alpha(c_1) f(h\, \nabla(c_2)\nabla(c_3)) \\
&\overset{(3.6)}{=} \alpha(c_1) f(h\, \nabla(c_2)) \\
&= \tilde{\Psi}(\alpha \otimes h)(c \,\bar{\otimes}\, f).
\end{aligned}$$

Then we can consider the following well-defined linear map

$$\begin{aligned}
\Psi \colon C^* \,\underline{\otimes}\, H &\longrightarrow (C \,\bar{\otimes}\, H^*)^* \\
\alpha \,\underline{\otimes}\, h &\longmapsto \Psi(\alpha \,\underline{\otimes}\, h)(c \,\bar{\otimes}\, f) = \alpha(c_1) f(h\nabla(c_2)).
\end{aligned}$$

To proof Theorem 4.2.8 below, we need to show that Ψ is surjective. Given $\xi \in (C \bar{\otimes} H^*)^*$ and $\{h_i\}_{i=1}^n$ a basis for H, we define for each i

$$\begin{aligned}\xi_i\colon C &\longrightarrow \Bbbk \\ c &\longmapsto \xi(c \bar{\otimes} h_i^*)\end{aligned}$$

and so

$$\begin{aligned}\Psi(\sum_{i=1}^n \xi_i \underline{\otimes} h_i)(c \bar{\otimes} f) &= \sum_{i=1}^n \xi_i(c_1) f(h_i \nabla(c_2)) \\ &= \sum_{i=1}^n \xi(c_1 \bar{\otimes} h_i^*) f_1(h_i) f_2(\nabla(c_2)) \\ &= \xi(c_1 \bar{\otimes} \sum_{i=1}^n h_i^* f_1(h_i)) f_2(\nabla(c_2)) \\ &= \xi(c_1 \bar{\otimes} f_1) f_2(\nabla(c_2)) \\ &= \xi(c_1 \bar{\otimes} f_1 f_2(\nabla(c_2))) \\ &= \xi(c_1 \bar{\otimes} \nabla(c_2) \rightharpoonup f) \\ &= \xi(c \bar{\otimes} f),\end{aligned}$$

and, therefore, Ψ is surjective, as desired. Moreover, one can show that Ψ is injective.

With the above considerations, we have the following theorem.

Theorem 4.2.8. *Let H be a finite dimensional Hopf algebra and C a left partial H-comodule coalgebra. If C^* is H^*-Galois, then C is H^*-coGalois.*

Proof. We have the canonical map

$$\begin{aligned}\mathrm{can}\colon C^* \otimes_{C^{*\underline{H}^*}} C^* &\longrightarrow C^* \underline{\otimes} H \\ \varphi \otimes \alpha &\longmapsto \varphi * \alpha^{+0} \otimes \alpha^{+1}\end{aligned}$$

that is bijective by hypothesis. And, by the already discussed, we have the following diagram

$$\begin{tikzcd} C^* \otimes_{C^{*\underline{H}^*}} C^* \arrow[r, "\mathrm{can}"] \arrow[d, "\Phi"'] & C^* \underline{\otimes} H \arrow[d, "\Psi"] \\ (C \,\square_{\,C/\mathcal{J}} C)^* \arrow[r, "\beta^*"] & (C \bar{\otimes} H^*)^* \end{tikzcd} \tag{4.17}$$

which is commutative. In fact,

$$\begin{aligned}
[\beta^*\Phi(\varphi\otimes\alpha)](c\,\bar{\otimes}\,f) &= \Phi(\varphi\otimes\alpha)(\beta(c\,\bar{\otimes}\,f))\\
&= \Phi(\varphi\otimes\alpha)(c_1\,\square\, f({c_2}^{-1}){c_2}^{-0}))\\
&= \varphi(c_1)\alpha(f({c_2}^{-1}){c_2}^{-0})\\
&= \varphi(c_1)f({c_2}^{-1}\alpha({c_2}^{-0}))\\
&\overset{(3.5)}{=} \varphi(c_1)f({c_2}^{-1}\nabla(c_3)\alpha({c_2}^{-0}))\\
&= \varphi(c_1)f({c_2}^{-1}\alpha({c_2}^{-0})\nabla(c_3))\\
&\overset{(3.10)}{=} \varphi(c_1)f(\alpha^{+1}\alpha^{+0}(c_2)\nabla(c_3))\\
&= \varphi(c_1)\alpha^{+0}(c_2)f(\alpha^{+1}\nabla(c_3))\\
&= [\varphi*\alpha^{+0}](c_1)f(\alpha^{+1}\nabla(c_2))\\
&= \Psi(\varphi*\alpha^{+0}\,\underline{\otimes}\,\alpha^{+1})(c\,\bar{\otimes}\,f)\\
&= [\Psi(\mathrm{can}(\varphi\otimes\alpha))](\mathrm{c}\,\bar{\otimes}\,\mathrm{f}).
\end{aligned}$$

Since can and Ψ are surjective, it follows that β^* is surjective, and so, by Proposition 1.1.1, β is injective. Therefore, from Theorem 4.2.7, C is H^*-coGalois. □

Remark 4.2.9. In general, we do not know if the converse of the Theorem 4.2.8 is true. In the global case, Dăscălescu, Raianu, and Zhang find several sufficient conditions for the converse be true (cf. [20, Proposition 1.5]).

4.3 Revisiting the Classical Morita Context

The aim in this section is to relate the Morita-Takeuchi context presented in Theorem 4.1.14 with the Morita context for partial module algebras presented in [1] by Alves and Batista. To develop this relation we need first to show that the dual of the partial smash coproduct is isomorphic, as algebras, to the partial smash product, i.e.,

$$(C\,\bar{\rtimes}\,H)^*\simeq C^*\underline{\#}H^*,$$

where C is a left partial H-comodule coalgebra.

Proposition 4.3.1. *Let H be a finite dimensional Hopf algebra and C a partial H-comodule coalgebra. Then $C^*\underline{\#}H^*\simeq(C\,\bar{\rtimes}\,H)^*$ as algebras.*

Proof. Consider the linear map

$$\begin{aligned}
\tilde{\theta}\colon C^*\otimes H^* &\longrightarrow (C\,\bar{\rtimes}\,H)^*\\
\varphi\otimes f &\longmapsto \theta(\varphi\otimes f)(c\,\bar{\rtimes}\,h)=\varphi(c_1)f(\nabla(c_2)h),
\end{aligned}$$

and, since

$$\begin{aligned}
\tilde{\theta}(\varphi\underline{\#}f)(c\,\bar{\rtimes}\,h) &= \tilde{\theta}(\varphi * (f_1 \rightharpoonup \varepsilon_C)\otimes f_2)(c\,\bar{\rtimes}\,h)\\
&= [\varphi * (f_1 \rightharpoonup \varepsilon_C)](c_1)f_2(\nabla(c_2)h)\\
&= \varphi(c_1)\varepsilon_C({c_2}^{-0})f_1({c_2}^{-1})f_2(\nabla(c_3)h)\\
&= \varphi(c_1)\varepsilon_C({c_2}^{-0})f({c_2}^{-1}\nabla(c_3)h)\\
&\overset{(3.5)}{=} \varphi(c_1)\varepsilon_C({c_2}^{-0})f({c_2}^{-1}h)\\
&= \varphi(c_1)f({c_2}^{-1}\varepsilon_C({c_2}^{-0})h)\\
&= \varphi(c_1)f(\nabla(c_2)h)\\
&= \tilde{\theta}(\varphi\#f)(c\,\bar{\rtimes}\,h),
\end{aligned}$$

then we can restrict $\tilde{\theta}$ and obtain the well-defined linear map

$$\begin{aligned}
\theta\colon C^*\underline{\#}H^* &\longrightarrow (C\,\bar{\rtimes}\,H)^*\\
\varphi\underline{\#}f &\longmapsto \theta(\varphi\underline{\#}f)(c\,\bar{\rtimes}\,h) = \varphi(c_1)f(\nabla(c_2)h).
\end{aligned}$$

We can also consider

$$\begin{aligned}
\sigma\colon (C\,\bar{\rtimes}\,H)^* &\longrightarrow C^*\underline{\#}H^*\\
\xi &\longmapsto \sum_{i=1}^{n}\xi_i\underline{\#}h_i^*,
\end{aligned}$$

where $\{h_i\}_{i=1}^n$ is a basis for H and, for each i, we have $\xi_i\colon C \to \Bbbk$ defined by $\xi_i(c) = \xi(c\,\bar{\rtimes}\,h_i)$. Now we have two well-defined linear maps that are inverses one each other.

In fact, if $\xi \in (C\,\bar{\rtimes}\,H)^*$, then we have that

$$\begin{aligned}
\theta(\sigma(\xi))(c\,\bar{\rtimes}\,h) &= \theta(\sum_{i=1}^{n}\xi_i\underline{\#}h_i^*)(c\,\bar{\rtimes}\,h)\\
&= \sum_{i=1}^{n}\xi_i(c_1)h_i^*(\nabla(c_2)h)\\
&= \sum_{i=1}^{n}\xi(c_1\,\bar{\rtimes}\,h_i)h_i^*(\nabla(c_2)h)\\
&= \xi(c_1\,\bar{\rtimes}\sum_{i=1}^{n}h_ih_i^*(\nabla(c_2)h))\\
&= \xi(c_1\,\bar{\rtimes}\,\nabla(c_2)h)\\
&= \xi(c\,\bar{\rtimes}\,h)
\end{aligned}$$

and, also

$$\begin{aligned}
\sigma(\theta(\varphi\underline{\#}f)) &= \sum_{i=1}^{n}[\theta(\varphi\underline{\#}f)]_i\underline{\#}h_i^*\\
&= \sum_{i=1}^{n}[\theta(\varphi\underline{\#}f)]_i * ({h_i^*}_1 \rightharpoonup \varepsilon_C)\otimes {h_i^*}_2.
\end{aligned}$$

Considering $\imath\colon C^*\otimes H^* \to (C\otimes H)^*$ the canonical embedding, then we have the following

$$
\begin{aligned}
\iota[\sigma(\theta(\varphi\underline{\#}f))](c\otimes k) &= \iota\left(\sum_{i=1}^{n}[\theta(\varphi\underline{\#}f)]_i * (h^*_{i\,1}\to\varepsilon_C)\otimes h^*_{i\,2}\right)(c\otimes k)\\
&= \sum_{i=1}^{n}[[\theta(\varphi\underline{\#}f)]_i * (h^*_{i\,1}\to\varepsilon_C)](c)h^*_{i\,2}(k)\\
&= \sum_{i=1}^{n}[\theta(\varphi\underline{\#}f)]_i(c_1)(h^*_{i\,1}\to\varepsilon_C)(c_2)h^*_{i\,2}(k)\\
&= \sum_{i=1}^{n}\theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\,h_i)h^*_{i\,1}(c_2{}^{-1})\varepsilon_C(c_2{}^{-0})h^*_{i\,2}(k)\\
&= \sum_{i=1}^{n}\theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\,h_i)\varepsilon_C(c_2{}^{-0})h^*_{i}(c_2{}^{-1}k)\\
&= \theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\sum_{i=1}^{n}h_ih^*_{i}(c_2{}^{-1}k))\varepsilon_C(c_2{}^{-0})\\
&= \theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\,c_2{}^{-1}k)\varepsilon_C(c_2{}^{-0})\\
&= \theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\,\varepsilon_C(c_2{}^{-0})c_2{}^{-1}k)\\
&= \theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\,\nabla(c_2)k)\\
&= \varphi(c_1)f(\nabla(c_2)\nabla(c_3)k)\\
&\overset{(3.6)}{=} \varphi(c_1)f(\nabla(c_2)k)\\
&= \varphi(c_1)f_1(\nabla(c_2))f_2(k)\\
&= \varphi(c_1)\varepsilon_C(c_2{}^{-0})f_1(c_2{}^{-1})f_2(k)\\
&= \varphi(c_1)(f_1\to\varepsilon_C)(c_2)f_2(k)\\
&= [\varphi * (f_1\to\varepsilon_C)](c)f_2(k)\\
&= \iota(\varphi\underline{\#}f)(c\otimes k),
\end{aligned}
$$

and, therefore, $\sigma(\theta(\varphi\underline{\#}f)) = \varphi\underline{\#}f$. So just remains to show that θ is an algebra morphism. Let $\varphi\underline{\#}f$ and $\alpha\underline{\#}g$ be elements in $C^*\underline{\#}H^*$ and $c\,\bar{\rtimes}\,h\in C\,\bar{\rtimes}\,H$, respectively. Thus,

$$
\begin{aligned}
\theta((\varphi\underline{\#}f)(\alpha\underline{\#}g))(c\,\bar{\rtimes}\,h) &= \theta(\varphi * (f_1\to\alpha)\underline{\#}f_2 * g)(c\,\bar{\rtimes}\,h)\\
&= [\varphi * (f_1\to\alpha)](c_1)(f_2 * g)(\nabla(c_2)h)\\
&= \varphi(c_1)(f_1\to\alpha)(c_2)(f_2 * g)(\nabla(c_3)h)\\
&= \varphi(c_1)(f_1\to\alpha)(c_2)(f_2 * g)(c_3{}^{-1}h)\varepsilon_C(c_3{}^{-0})\\
&= \varphi(c_1)\alpha(c_2{}^{-0})f_1(c_2{}^{-1})f_2(c_3{}^{-1}{}_1h_1)g(c_3{}^{-1}{}_2h_2)\varepsilon_C(c_3{}^{-0})\\
&= \varphi(c_1)\alpha(c_2{}^{-0})f(c_2{}^{-1}c_3{}^{-1}{}_1h_1)g(c_3{}^{-1}{}_2h_2)\varepsilon_C(c_3{}^{-0})\\
&\overset{(3.5)}{=} \varphi(c_1)\alpha(c_2{}^{-0})f(c_2{}^{-1}\nabla(c_3)c_4{}^{-1}{}_1h_1)g(c_4{}^{-1}{}_2h_2)\varepsilon_C(c_4{}^{-0})\\
&\overset{(PCC3)}{=} \varphi(c_1)\alpha(c_2{}^{-0})f(c_2{}^{-1}c_3{}^{-1}h_1)g(c_3{}^{-0-1}h_2)\varepsilon_C(c_3{}^{-0-0})\\
&= \varphi(c_1)\alpha(c_2{}^{-0})f(c_2{}^{-1}c_3{}^{-1}h_1)g(\nabla(c_3{}^{-0})h_2)\\
&\overset{(PCC2)}{=} \varphi(c_1)\alpha(c_2{}^{-0}{}_1)f(c_2{}^{-1}h_1)g(\nabla(c_2{}^{-0}{}_2)h_2)\\
&\overset{(3.5)}{=} \varphi(c_1)\alpha(c_3{}^{-0}{}_1)f(\nabla(c_2)c_3{}^{-1}h_1)g(\nabla(c_3{}^{-0}{}_2)h_2)
\end{aligned}
$$

by the other side, we have

$$\begin{aligned}[\theta(\varphi\underline{\#}f) * \theta(\alpha\underline{\#}g)](c \,\bar{\rtimes}\, h) &= \theta(\varphi\underline{\#}f)(c_1 \,\bar{\rtimes}\, {c_2}^{-1}h_1)\theta(\alpha\underline{\#}g)({c_2}^{-0} \,\bar{\rtimes}\, h_2)\\ &= [\varphi(c_1)f(\nabla(c_2){c_3}^{-1}h_1)][\alpha({c_3}^{-0}{}_1)g(\nabla({c_3}^{-0}{}_2)h_2)]\\ &= \varphi(c_1)\alpha({c_3}^{-0}{}_1)f(\nabla(c_2){c_3}^{-1}h_1)g(\nabla({c_3}^{-0}{}_2)h_2)\end{aligned}$$

and

$$\begin{aligned}\theta(\varepsilon_C\underline{\#}\varepsilon_H)(c \,\bar{\rtimes}\, h) &= \varepsilon_C(c_1)\varepsilon_H(\nabla(c_2)h)\\ &= \varepsilon_C(c_1)\varepsilon_H({c_2}^{-1}\varepsilon_C({c_2}^{-0})h)\\ &= \varepsilon_C(c_1)\varepsilon_H({c_2}^{-1})\varepsilon_C({c_2}^{-0})\varepsilon_H(h)\\ &= \varepsilon_C(c)\varepsilon_H(h)\\ &= \varepsilon_{C \,\bar{\rtimes}\, H}(c \,\bar{\rtimes}\, h)\end{aligned}$$

so that θ is an algebra isomorphism, as desired. $\square$

Let C be a left partial H-comodule coalgebra and remember some considerations that we had made before. We have the algebra isomorphism

$$\begin{aligned}\theta\colon C^*\underline{\#}H^* &\longrightarrow (C \,\bar{\rtimes}\, H)^*\\ \varphi\underline{\#}f &\longmapsto \theta(\varphi\underline{\#}f)(c \,\bar{\rtimes}\, h) = \varphi(c_1)f(\nabla(c_2)\, h)\end{aligned}$$

and we have the Morita-Takeuchi context (cf. Theorem 4.1.14)

$$({}^{C}\!/\!_{\mathcal{J}}, C \,\bar{\rtimes}\, H, C, C, \mu, \tau)$$

with structure maps given by

$$\begin{aligned}\rho(c) &= c_1 \otimes \overline{c_2}\\ \lambda(c) &= \overline{c_1} \otimes c_2\\ \rho_{\bar{\rtimes}}(c) &= {c_1}^{-0} \otimes {c_2}^{-0} \,\bar{\rtimes}\, S^{-1}({c_1}^{-1}{c_2}^{-1})\lambda\\ \lambda_{\bar{\rtimes}}(c) &= c_1 \,\bar{\rtimes}\, {c_2}^{-1} \otimes {c_2}^{-0}\\ \mu(c \,\bar{\rtimes}\, h) &= c_1 \,\square\, {c_2}^{-0}T({c_2}^{-1}h)\\ \tau(\bar{c}) &= {c_1}^{-0} \,\square\, {c_2}^{-0}T({c_1}^{-1}{c_2}^{-1}).\end{aligned}$$

From linear algebra theory, given two vector spaces V, W, we have the natural embedding

$$\imath_{V,W}\colon V^* \otimes W^* \longrightarrow (V \otimes W)^*,$$

and, given any subspace $X \leq V$, we have the natural projection

$$\begin{aligned}\pi_X\colon V &\longrightarrow {}^{V}\!/\!_{X}\\ v &\longmapsto \overline{v}\end{aligned}$$

such that $X^{\perp} \simeq ({}^{V}\!/\!{}_{X})^{*}$, via π_X^*.

Now we will construct the Morita context between $C^{*\underline{H^*}}$ and $C^*\underline{\#}H^*$ induced by the Morita-Takeuchi context constructed before.

We know that

$$\mathcal{I} = C^{*\underline{H^*}} = \pi_{\mathcal{J}}^* \left(({}^{C}\!/\!{}_{\mathcal{J}})^*\right)$$

so that we can define the following linear maps $\triangleright_{\mathcal{I}} \colon \mathcal{I} \otimes C^* \longrightarrow C^*$, $\triangleleft_{\mathcal{I}} \colon C^* \otimes \mathcal{I} \longrightarrow C^*$, $\triangleright_{\underline{\#}} \colon C^*\underline{\#}H^* \otimes C^* \longrightarrow C^*$ and $\triangleleft_{\underline{\#}} \colon C^* \otimes C^*\underline{\#}H^* \longrightarrow C^*$, respectively by

$$\begin{aligned}
\triangleright_{\mathcal{I}} &= \lambda^* \circ \imath \circ (\pi_{\mathcal{J}}^{*-1} \otimes I) \\
\triangleleft_{\mathcal{I}} &= \rho^* \circ \imath \circ (I \otimes \pi_{\mathcal{J}}^{*-1}) \\
\triangleright_{\underline{\#}} &= \lambda^*_{\overline{\rtimes}} \circ \imath \circ (\theta \otimes I) \\
\triangleleft_{\underline{\#}} &= \rho^*_{\overline{\rtimes}} \circ \imath \circ (I \otimes \theta).
\end{aligned}$$

Proposition 4.3.2. *With the above notations, the maps above defined are the (left and right) actions of* $C^{*\underline{H^*}}$ *and* $C^*\underline{\#}H^*$ *on* C^**.*

Proof. Let $\psi \in \mathcal{I}$, $\varphi \in C^*$, $f \in H^*$ and $c \in C$. Thus, we have

$$\begin{aligned}
(\varphi \triangleleft_{\mathcal{I}} \psi)(c) &= \lambda^*\{\imath[(\pi_{\mathcal{J}}^{*-1} \otimes I)(\varphi \otimes \psi)]\}(c) \\
&= \{\imath[(\pi_{\mathcal{J}}^{*-1} \otimes I)(\varphi \otimes \psi)]\}\lambda(c) \\
&= \{\imath[(\pi_{\mathcal{J}}^{*-1} \otimes I)(\varphi \otimes \psi)]\}(\overline{c_1} \otimes c_2) \\
&= \pi_{\mathcal{J}}^{*-1}(\varphi)(\overline{c_1})\psi(c_2) \\
&= \pi_{\mathcal{J}}^{*-1}(\varphi)(\pi_{\mathcal{J}}(c_1))\psi(c_2) \\
&= \varphi[\pi_{\mathcal{J}}^{*}\pi_{\mathcal{J}}^{*-1}(c_1)]\psi(c_2) \\
&= \varphi(c_1)\psi(c_2) \\
&= [\varphi * \psi](c)
\end{aligned}$$

and

$$\begin{aligned}
(\psi \triangleright_{\mathcal{I}} \varphi)(c) &= \rho^*\{\imath[(I \otimes \pi_{\mathcal{J}}^{*-1})(\psi \otimes \varphi)]\}(c) \\
&= \{\imath[(I \otimes \pi_{\mathcal{J}}^{*-1})(\psi \otimes \varphi)]\}\rho(c) \\
&= \{\imath[(I \otimes \pi_{\mathcal{J}}^{*-1})(\psi \otimes \varphi)]\}(c_1 \otimes \overline{c_2}) \\
&= \psi(c_1)\pi_{\mathcal{J}}^{*-1}(\varphi)(\pi_{\mathcal{J}}(c_2)) \\
&= \psi(c_1)\varphi[\pi_{\mathcal{J}}^{*}\pi_{\mathcal{J}}^{*-1}(c_2)] \\
&= \psi(c_1)\varphi(c_2) \\
&= (\psi * \varphi)(c)
\end{aligned}$$

and

$$
\begin{aligned}
[(\varphi\underline{\#}f)\triangleright_{\underline{\#}}\alpha](c) &= \lambda^*_{\bar{\rtimes}}\{\imath[(\theta\otimes I)(\varphi\underline{\#}f\otimes\alpha)]\}(c)\\
&= \imath(\theta(\varphi\underline{\#}f)\otimes\alpha)\lambda_{\bar{\rtimes}}(c)\\
&= \imath(\theta(\varphi\underline{\#}f)\otimes\alpha)(c_1\,\bar{\rtimes}\,c_2{}^{-1}\otimes c_2{}^{-0})\\
&= \theta(\varphi\underline{\#}f)(c_1\,\bar{\rtimes}\,c_2{}^{-1})\alpha(c_2{}^{-0})\\
&= \varphi(c_1)f(\nabla(c_2)c_3{}^{-1})\alpha(c_3{}^{-0})\\
&\overset{(3.5)}{=} \varphi(c_1)f(c_2{}^{-1})\alpha(c_2{}^{-0})\\
&\overset{(3.5)}{=} [\varphi*(f\rightarrow\alpha)](c)
\end{aligned}
$$

and

$$
\begin{aligned}
[\alpha\triangleleft_{\underline{\#}}(\varphi\underline{\#}f)](c) &= \rho^*_{\bar{\rtimes}}\{\imath[(I\otimes\theta)(\alpha\otimes\varphi\underline{\#}f)]\}(c)\\
&= \imath[(\alpha\otimes\theta)(\varphi\underline{\#}f)]\rho_{\bar{\rtimes}}(c)\\
&= \imath[\alpha\otimes\theta(\varphi\underline{\#}f)](c_1{}^{-0}\otimes c_2{}^{-0}\,\bar{\rtimes}\,S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda)\\
&= \alpha(c_1{}^{-0})\theta(\varphi\underline{\#}f)(c_2{}^{-0}\,\bar{\rtimes}\,S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda)\\
&= \alpha(c_1{}^{-0})\varphi(c_2{}^{-0}{}_1)f[\nabla(c_2{}^{-0}{}_2)S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda]\\
&\overset{(PCC2)}{=} \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(\nabla(c_3{}^{-0})S^{-1}(c_1{}^{-1}c_2{}^{-1}c_3{}^{-1})\lambda)\\
&= \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(c_3{}^{-0-1}S^{-1}(c_1{}^{-1}c_2{}^{-1}c_3{}^{-1})\lambda)\varepsilon(c_3{}^{-0-0})\\
&\overset{(PCC3)}{=} \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(c_4{}^{-1}{}_2S^{-1}(c_1{}^{-1}c_2{}^{-1}\nabla(c_3)c_4{}^{-1}{}_1)\lambda)\varepsilon(c_4{}^{-0})\\
&\overset{(3.5)}{=} \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(c_3{}^{-1}{}_2S^{-1}(c_1{}^{-1}c_2{}^{-1}c_3{}^{-1}{}_1)\lambda)\varepsilon(c_3{}^{-0})\\
&= \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(c_3{}^{-1}{}_2S^{-1}(c_3{}^{-1}{}_1)S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda)\varepsilon(c_3{}^{-0})\\
&= \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda)\varepsilon(c_3)\\
&= \alpha(c_1{}^{-0})\varphi(c_2{}^{-0})f(S^{-1}(c_1{}^{-1}c_2{}^{-1})\lambda)\\
&\overset{(PCC2)}{=} \alpha(c^{-0}{}_1)\varphi(c^{-0}{}_2)f(S^{-1}(c^{-1})\lambda)\\
&= \alpha(c^{-0}{}_1)\varphi(c^{-0}{}_2)f_1(S^{-1}(c^{-1}))f_2(\lambda)\\
&= \alpha(c^{-0}{}_1)\varphi(c^{-0}{}_2)S^{-1*}(f_1)(c^{-1})f_2(\lambda)\\
&= (\alpha*\varphi)(c^{-0})S^{-1}_{H^*}(f_1)(c^{-1})f_2(\lambda)\\
&= [\hat{\lambda}(f_2)\ S^{-1}_{H^*}(f_1)\rightarrow(\alpha*\varphi)](c).
\end{aligned}
$$

Thus, we recover the standard actions (cf. [1, Lemma 2]). $\square$

To construct the bimodule structures necessaries to construct the desired Morita context, we need first to define some maps, as follow.

We start defining the following map

$$
\begin{aligned}
\tilde{\Phi}_{\bar{\rtimes}} : C^*\times C^* &\longrightarrow (C\,\square_{\,C\,\bar{\rtimes}\,H}C)^*\\
(\varphi,\alpha) &\longmapsto \Phi_{\bar{\rtimes}}(\varphi,\alpha)(c\,\square\, d) = \varphi(c)\alpha(d)
\end{aligned}
$$

that is $C^*\underline{\#}H^*$-balanced.

In fact, let $\varphi, \alpha \in C^*$, $\varphi'\underline{\#}f \in C^*\underline{\#}H^*$ and $c \,\square\, d \in C \,\square\, C$. Thus,

$$\begin{aligned}\tilde{\Phi}_{\bar{\rtimes}}(\varphi, (\varphi'\underline{\#}f)\triangleright_{\underline{\#}}\alpha)(c \,\square\, d) &= \varphi(c)((\varphi'\underline{\#}f)\triangleright_{\underline{\#}}\alpha)(d)\\ &= \varphi(c)([\lambda^*_{\bar{\rtimes}} \circ \imath \circ (\theta \otimes I)]((\varphi'\underline{\#}f) \otimes \alpha))(c)\\ &= \varphi(c)[\imath \circ (\theta \otimes I)]((\varphi'\underline{\#}f) \otimes \alpha)](d_1 \,\bar{\rtimes}\, {d_2}^{-1} \otimes {d_2}^{-0})\\ &= \varphi(c)\imath(\theta(\varphi'\underline{\#}f) \otimes \alpha)(d_1 \,\bar{\rtimes}\, {d_2}^{-1} \otimes {d_2}^{-0})\end{aligned}$$

and

$$\begin{aligned}\tilde{\Phi}_{\bar{\rtimes}}(\varphi\triangleleft_{\underline{\#}}(\varphi'\underline{\#}f), \alpha)(c \,\square\, d) &= (\varphi\triangleleft_{\underline{\#}}(\varphi'\underline{\#}f))(c)\alpha(d)\\ &= [\rho_{\bar{\rtimes}} \circ \imath \circ (I \otimes \theta)(\varphi \otimes \varphi'\underline{\#}f)](c)\alpha(d)\\ &= [\imath(\varphi \otimes \theta(\varphi'\underline{\#}f))](\rho_{\bar{\rtimes}}(c))\alpha(d)\\ &= [\imath(\varphi \otimes \theta(\varphi'\underline{\#}f))]({c_1}^{-0} \otimes {c_2}^{-0} \,\bar{\rtimes}\, S^{-1}({c_1}^{-1}{c_2}^{-1})\lambda)\alpha(d)\\ &= \varphi({c_1}^{-0})\theta(\varphi'\underline{\#}f)({c_2}^{-0} \,\bar{\rtimes}\, S^{-1}({c_1}^{-1}{c_2}^{-1})\lambda)\alpha(d)\\ &= \varphi({c_1}^{-0})\imath(\theta(\varphi'\underline{\#}f) \otimes \alpha)({c_2}^{-0} \,\bar{\rtimes}\, S^{-1}({c_1}^{-1}{c_2}^{-1})\lambda \otimes d)\\ &= \varphi(c)\imath(\theta(\varphi'\underline{\#}f) \otimes \alpha)(d_1 \,\bar{\rtimes}\, {d_2}^{-1} \otimes {d_2}^{-0}),\end{aligned}$$

where the last equality holds since $c \,\square\, d$ lies in $C \,\square_{C \bar{\rtimes} H} C$. Hence,

$$c \otimes (d_1 \,\bar{\rtimes}\, {d_2}^{-1}) \otimes {d_2}^{-0} = {c_1}^{-0} \otimes ({c_2}^{-0} \,\bar{\rtimes}\, S^{-1}({c_1}^{-1}{c_2}^{-1})\lambda) \otimes d.$$

Therefore, we have the following well-defined linear map

$$\begin{aligned}\Phi_{\bar{\rtimes}} : C^* \otimes_{C^*\underline{\#}H^*} C^* &\longrightarrow (C \,\square_{C \bar{\rtimes} H} C)^*\\ \varphi \otimes \alpha &\longmapsto \Phi_{\bar{\rtimes}}(\varphi \otimes \alpha)(c \,\square\, d) = \varphi(c)\alpha(d).\end{aligned}$$

Now we define the following map

$$\begin{aligned}\tilde{\Phi} \colon C^* \times C^* &\longrightarrow (C \,\square_{C/\mathcal{J}} C)^*\\ (\varphi, \alpha) &\longmapsto \tilde{\Phi}(\varphi, \alpha)(c \,\square\, d) = \varphi(c)\alpha(d).\end{aligned}$$

that is $\mathcal{I}$-balanced. In fact, let $\varphi, \alpha \in C^*$, $\psi \in \mathcal{I}$ and $c \,\square\, d$. Thus,

$$\begin{aligned}\tilde{\Phi}(\varphi, \psi \triangleright_{\mathcal{I}} \alpha)(c \,\square\, d) &= \varphi(c)(\psi \triangleright_{\mathcal{I}} \alpha)(d)\\ &= \varphi(c)\ \lambda^*[\imath(\pi^{*-1}_{\mathcal{J}} \otimes I)](\psi \otimes \alpha)(d)\\ &= \varphi(c)\ \lambda^*[\imath(\pi^{*-1}_{\mathcal{J}}(\psi) \otimes \alpha)](d)\\ &= \varphi(c)[\imath(\pi^{*-1}_{\mathcal{J}}(\psi) \otimes \alpha)](\lambda(d))\\ &= \varphi(c)\pi^{*-1}_{\mathcal{J}}(\psi)(\overline{d_1})\alpha(d_2)\end{aligned}$$

and

$$\begin{aligned}\tilde{\Phi}(\varphi \triangleleft_{\mathcal{I}} \psi, \alpha)(c \,\square\, d) &= (\varphi \triangleleft_{\mathcal{I}} \psi)(c)\alpha(d)\\ &= \rho^*[\imath(\varphi \otimes \pi_{\mathcal{J}}^{*-1}(\psi))](c)\alpha(d)\\ &= [\imath(\varphi \otimes \pi_{\mathcal{J}}^{*-1}(\psi))](\rho(c))\alpha(d)\\ &= \varphi(c_1)\pi_{\mathcal{J}}^{*-1}(\psi)(\overline{c_2})\alpha(d)\\ &= \varphi(c)\pi_{\mathcal{J}}^{*-1}(\psi)(\overline{d_1})\alpha(d_2),\end{aligned}$$

where the last equation since the cotensor product is over ${}^{C}/_{\mathcal{J}}$. Therefore, we have the following well-defined linear map

$$\begin{aligned}\Phi\colon C^* \otimes_{\mathcal{I}} C^* &\longrightarrow (C \,\square\, {}_{C/\mathcal{J}} C)^*\\ \varphi \otimes \alpha &\longmapsto \Phi(\varphi \otimes \alpha)(c \,\square\, d) = \varphi(c)\alpha(d).\end{aligned}$$

Finally, we consider the maps $[,]\colon C^* \otimes_{\mathcal{I}} C^* \longrightarrow C^* \underline{\#} H^*$ and $(,)\colon C^* \otimes_{C^* \underline{\#} H^*} C^* \longrightarrow \mathcal{I}$, respectively given by

$$\begin{aligned}[,] &= \theta^{-1} \circ \mu^* \circ \Phi\\ (,) &= \pi_{\mathcal{J}}^* \circ \tau^* \circ \Phi_{\,\bar{\rtimes}}\,.\end{aligned}$$

With this constructions above made, we are in position to present the following interesting result.

Theorem 4.3.3. *Let H be a finite dimensional Hopf algebra and C a left partial H-comodule coalgebra. Then the Morita-Takeuchi context in Theorem 4.1.14 recovers the classical Morita context between $C^{*\underline{H}^*}$ and $C^* \underline{\#} H^*$ (cf.* [1, Theorem 1]*).*

Proof. By our previous discussion, we just need to show that $[,]$ and $(,)$ are the same maps that appear in [1].

In fact, let $\varphi, \alpha \in C^*$ and $c \,\bar{\rtimes}\, k \in C \,\bar{\rtimes}\, H$. Then we have

$$\begin{aligned}\theta([,](\varphi \otimes \alpha))(c \,\bar{\rtimes}\, k) &= \theta(\theta^{-1} \circ \mu^* \circ \Phi(\varphi \otimes \alpha))(c \,\bar{\rtimes}\, k)\\ &= \mu^* \circ \Phi(\varphi \otimes \alpha)(c \,\bar{\rtimes}\, k)\\ &= \Phi(\varphi \otimes \alpha)(\mu(c \,\bar{\rtimes}\, k))\\ &= \Phi(\varphi \otimes \alpha)(c_1 \,\square\, {c_2}^{-0} T({c_2}^{-1} k))\\ &= \varphi(c_1)\alpha({c_2}^{-0})T({c_2}^{-1} k)\\ &\overset{(3.5)}{=} \varphi(c_1)\alpha({c_2}^{-0})T({c_2}^{-1}\nabla(c_3)k)\\ &= \varphi(c_1)\alpha({c_2}^{-0})T_1({c_2}^{-1})T_2(\nabla(c_3)k)\\ &= \varphi(c_1)(T_1 \rightarrow \alpha)(c_2)T_2(\nabla(c_3)k)\\ &= [\varphi * (T_1 \rightarrow \alpha)](c_1)T_2(\nabla(c_2)k)\\ &= \theta(\varphi * (T_1 \rightarrow \alpha)\underline{\#} T_2)(c \,\bar{\rtimes}\, k)\\ &= \theta((\varphi \underline{\#} 1_{H^*})(1_{C^*} \underline{\#} T)(\alpha \underline{\#} 1_{H^*}))(c \,\bar{\rtimes}\, k)\end{aligned}$$

and, since θ is isomorphism, we have that $[,](\varphi \otimes \alpha) = (\varphi\underline{\#}1_{H^*})(1_{C^*}\underline{\#}T)(\alpha\underline{\#}1_{H^*})$, as desired.

Moreover, let $\varphi, \alpha \in C^*$ and $c \in C$. Thus,

$$\begin{aligned}
(,)(\varphi \otimes \alpha)(c) &= \pi_{\mathcal{J}}^*(\tau^*(\Phi_{\bar{\rtimes}}(\varphi \otimes \alpha)))(c) \\
&= (\tau^*(\Phi_{\bar{\rtimes}}(\varphi \otimes \alpha)))(\bar{c}) \\
&= (\Phi_{\bar{\rtimes}}(\varphi \otimes \alpha))\tau(\bar{c}) \\
&= \Phi_{\bar{\rtimes}}(\varphi \otimes \alpha)(c_1{}^{-0} \,\square\, c_2{}^{-0}T(c_1{}^{-1}c_2{}^{-1})) \\
&= \varphi(c_1{}^{-0})\alpha(c_2{}^{-0}T(c_1{}^{-1}c_2{}^{-1})) \\
&\overset{(PCC2)}{=} \varphi(c^{-0}{}_1)\alpha(c^{-0}{}_2)T(c^{-1}) \\
&= (\varphi * \alpha)(c^{-0})T(c^{-1}) \\
&= [T \rightarrow (\varphi * \alpha)](c).
\end{aligned}$$

Therefore, we recover the classical Morita context, as desired. $\square$

References

[1] M. Alves and E. Batista, *Partial Hopf actions, partial invariants and a Morita context*, Algebra Discrete Math. **3** (2009), 1–19.

[2] ______, *Enveloping actions for partial Hopf actions*, Comm. Algebra **38** (2010), no. 8, 2872–2902.

[3] ______, *Globalization theorems for partial Hopf (co)actions, and some of their applications*, Groups, Algebras and Applications, Contemp. Math., vol. 537, Amer. Math. Soc., Providence, RI, 2011, pp. 13–30.

[4] E. Batista and J. Vercruysse, *Dual Constructions for Partial Actions of Hopf Algebras*, Preprint (2014).

[5] T. Brzeziński, *The structure of corings with a grouplike element*, Noncommutative geometry and quantum groups (Warsaw, 2001), Banach Center Publ., vol. 61, Polish Acad. Sci., Warsaw, 2003, pp. 21–35.

[6] T. Brzeziński and P. Hajac, *Coalgebra extensions and algebra coextensions of Galois type*, Comm. Algebra **27** (1999), no. 3, 1347–1367.

[7] T. Brzezinski and R. Wisbauer, *Corings and comodules*, London Mathematical Society Lecture Note Series, vol. 309, Cambridge University Press, Cambridge, 2003.

[8] S. Caenepeel, *Brauer groups, Hopf algebras and Galois theory*, K-Monographs in Mathematics, vol. 4, Kluwer Academic Publishers, Dordrecht, 1998.

[9] ______, *Galois corings from the descent theory point of view*, Galois theory, Hopf algebras, and semiabelian categories, Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 163–186.

[10] S. Caenepeel and E. De Groot, *Galois corings applied to partial Galois theory*, Proceedings of the International Conference on Mathematics and its Applications (ICMA 2004), Kuwait Univ. Dep. Math. Comput. Sci., Kuwait, 2005, pp. 117–134.

[11] S. Caenepeel and K. Janssen, *Partial (co)actions of Hopf algebras and partial Hopf-Galois theory*, Comm. Algebra **36** (2008), no. 8, 2923–2946.

[12] S. Caenepeel, J. Vercruysse, and S. Wang, *Morita theory for corings and cleft entwining structures*, J. Algebra **276** (2004), no. 1, 210–235.

[13] F. Castro, D. Freitas, G. Quadros, and A. Paques, *Galois Correspondences for partial Hopf-Galois Azumaya extensions*, Preprint.

[14] F. Castro and G. Quadros, *Partial (Co)Module Coalgebras: Globalization Theorems*, Preprint.

[15] ______, *Galois Theory for Partial Comodule Coalgebras*, Preprint.

[16] S. Chase, D. Harrison, and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. No. **52** (1965), 15–33.

[17] S. Chase and M. Sweedler, *Hopf algebras and Galois theory*, Lect. Notes in Math. **97** (1969).

[18] M. Cohen, D. Fischman, and S. Montgomery, *Hopf-Galois extensions, smash products, and Morita equivalence*, J. Algebra **133** (1990), no. 2, 351–372.

[19] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu, *Hopf algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 235, Marcel Dekker, Inc., New York, 2001. An introduction. MR1786197 (2001j:16056)

[20] S. Dăscălescu, Ş. Raianu, and Y. Zhang, *Finite Hopf-Galois coextensions, crossed coproducts, and duality*, J. Algebra **178** (1995), no. 2, 400–413.

[21] M. Dokuchaev and R. Exel, *Associativity of crossed products by partial actions, enveloping actions and partial representations*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 1931–1952.

[22] M. Dokuchaev, M. Ferrero, and A. Paques, *Partial actions and Galois theory*, J. Pure Appl. Algebra **208** (2007), no. 1, 77–87.

[23] R. Exel, *Circle actions on C^*-algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence*, J. Funct. Anal. **122** (1994), no. 2, 361–401.

[24] K. Hoffman and R. Kunze, *Linear algebra*, Second edition, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971. MR0276251 (43 #1998)

[25] T. Lam, *A first course in noncommutative rings*, 2nd ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR1838439 (2002c:16001)

[26] ———, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR1653294 (99i:16001)

[27] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[28] J. Rotman, *An introduction to homological algebra*, Pure and Applied Mathematics, vol. 85, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1979. MR538169 (80k:18001)

[29] D. Radford, *Hopf algebras*, Series on Knots and Everything, vol. 49, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. MR2894855

[30] M. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

[31] M. Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 3, 629–644.

[32] F. Van Oystaeyen and Y. Zhang, *Finite-dimensional Hopf algebras coacting on coalgebras*, Bull. Belg. Math. Soc. Simon Stevin **5** (1998), no. 1, 1–14.