

Glauber Rodrigues de Quadros

**Partial (co)actions of weak Hopf algebras:
globalizations, Galois theory and Morita
theory**

Porto Alegre
2015

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PhD thesis presented by Glauber Rodrigues de Quadros¹ in partial fulfillment of the requirements for the degree of Doctor in Mathematics at Universidade Federal do Rio Grande do Sul.

Universidade Federal do Rio Grande de Sul - UFRGS
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To my wife Maiane.

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*“The good that men do is oft
interred with their bones. But the
evil that men do lives on.”*

Iron Maiden
Paraphrasing Shakespeare

Abstract

In this work we introduce the notion of partial actions of weak Hopf algebras on algebras. This new subject arises in order to unify the notions of partial group action [29], partial Hopf action ([2],[3],[18]) and partial groupoid action [7]. We also develop some fundamental tools in order to construct the partial smash product and the globalization of a partial action, as well as, we establish a connection between partial and global smash products via a surjective Morita context. In particular, in the case that the globalization is unital, these smash products are Morita equivalent.

We show that it is possible to connect globalizable partial groupoid actions and symmetric partial groupoid algebra actions, extending similar results for group actions [18].

We also introduce the concept of partial coactions of weak Hopf algebras. In this context, we show that every partial comodule algebra comes from a global one and also that the reduced tensor product is a coring.

Moreover, we give a complete description of all partial (co)actions of a weak Hopf algebra on its ground field, which suggests a method to construct more general examples.

Finally, we explore the Morita theory in two different ways. The first one is made using partial actions. In this approach we show that, under some special conditions on the weak Hopf algebra, the obtained Morita context is a generalization of the one given in [37]. As an application, we develop a Galois theory connecting this Morita context, the canonical map, and the Galois coordinates.

The second one is via partial coactions. We construct the reduced tensor product and show that it is a coring. Motivated by corresponding results in [16], we construct the Morita theory obtaining some new additional Galois equivalences.

Resumo

Neste trabalho introduzimos a noção de ações parciais de álgebras de Hopf fracas em álgebras. Este novo conceito surge com o intuito de unificar as noções de ação parcial de grupos [29], ação parcial de álgebras de Hopf ([2],[3],[18]) e ação parcial de grupóides [7]. Também desenvolvemos ferramentas fundamentais para a construção do produto smash parcial e para a globalização da ação parcial, bem como estabelecemos uma conexão entre o produto smash parcial e global através de um contexto de Morita sobrejetor. Em particular, no caso em que a globalização é unitária, estes produtos smash são Morita equivalentes.

Mostramos que é possível conectar ação parcial de grupóide e ação parcial simétrica da álgebra de grupóide, extendendo resultados similares para ação parcial de grupos [18].

Nós também introduzimos o conceito de coação parcial de álgebras de Hopf fracas. Neste contexto, mostramos que todo comódulo álgebra parcial vem de um global e também que o produto tensorial reduzido é um coanel.

Mais ainda, damos uma descrição completa de todas as (co)ações parciais de uma álgebra de Hopf fraca no seu corpo base, o que sugere um método de construir exemplos mais gerais.

Finalmente, exploramos a teoria de Morita de duas maneiras distintas. A primeira é feita usando-se ações parciais. Nesta abordagem, mostramos que sob certas condições para a álgebra de Hopf fraca, o contexto de Morita obtido é uma generalização daquele dado em [37]. Como uma aplicação, desenvolvemos a teoria de Galois conectando esse contexto de Morita, a aplicação canônica e as coordenadas de Galois.

A segunda abordagem é feita através de coações parciais. Construímos o produto tensorial reduzido e mostramos que ele é um coanel. Motivados por resultados em [16], construímos a teoria de Morita, obtendo algumas novas equivalências de Galois.

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Introduction

Partial (co)actions

Partial action of groups on algebras was introduced in the literature by R. Exel in [31]. His main purpose in that paper was to develop a method that allowed to describe the structure of C^* -algebras under actions of the circle group. The first approach of partial group actions on algebras, in a purely algebraic context, appeared later in a paper by M. Dokuchaev and R. Exel [29].

Partial group actions can be easily obtained by restriction from the global ones, and this fact stimulated the interest in knowing under what conditions (if any) a given partial group action is of this type. In the topological context this question was dealt with by F. Abadie in [1]. The algebraic version of a globalization (or enveloping action) of a partial group action, as well as the study about its existence, was also considered by Dokuchaev and Exel in [29]. A nice approach on the relevance of the relationship between partial and global group actions, in several branches of mathematics, can be seen in [28].

As a natural task, S. Caenepeel and K. Janssen [18] extended the notion of partial group action to the setting of Hopf algebras and developed a theory of partial (co)actions of Hopf algebras, as well as a partial Hopf-Galois theory. Based on the Caenepeel-Janssen's work, E. Batista and M. Alves in [2, 3] showed that every partial action of a Hopf algebra has a globalization and that the corresponding partial and global smash products are related by a surjective Morita context. At almost the same time, D. Bagio and A. Paques developed a theory of partial groupoid actions extending, in particular, results of Exel and Dokuchaev in [29], about partial group actions and their globalizations.

Actions of weak Hopf algebras is the precise context to unify these above mentioned theories, and this is our main purpose.

In this work we deal with actions of weak Hopf algebras and extend to this setting many of the results above mentioned. As it is well known, Hopf algebras and groupoid algebras are perhaps the simplest examples of weak Hopf algebras. The weak Hopf algebra theory has been started at the end of the 90s by G. Böhm, F. Nill and K. Szlachányi [9–11, 39, 43].

In Chapter 2 we present the definition of the partial action of a weak Hopf algebra on an algebra. One of our main goals is to show that the notion of globalization can be extended to partial module algebras over weak Hopf algebras. We succeed to prove that every partial module algebra over weak Hopf algebras has a globalization (also called enveloping action),

extending the corresponding results on partial Hopf actions obtained by E. Batista e M. Alves in [2] and [3]. We also prove the existence of minimal globalizations and that any two of them are isomorphic, as well as that any globalization is a homomorphic preimage of a minimal one (see Theorem 2.4.9).

Another one is to ensure that the partial version of the smash product, as introduced by D. Nikshych in [38], can also be obtained in the partial context (see Section 2.5). The hardest task here is to show that such a partial smash product is well-defined, just because the tensor product, in this case, is not over the ground field. The usual theory does not fit into our context since the definitions for partial structures are a bit different. The existence of partial smash products allows us to construct a surjective Morita context relating them with the corresponding global ones (see Theorem 2.6.6).

As an application, we describe completely all the partial actions of a weak Hopf algebra on its ground field, which also suggests the construction of other examples of these partial actions, different from the canonical ones (see Lemma 2.3.1). We also analyze the relation between partial groupoid actions, as introduced in [7], and partial actions of groupoid algebras, showing how partial group actions, in particular, and partial groupoid actions, in general, fit into this new context (see Theorem 2.2.5).

In Chapter 3, we study the partial coactions of a weak Hopf algebra on an algebra. In the same way as we did in Chapter 2, we describe the partial coactions on the ground field (see Theorem 3.2.1) and, moreover, we show that any partial coaction can be obtained from a global one (see Theorem 3.3.3). As a last task in this chapter, we show that the reduced tensor product $A \otimes H$ is an A -coring (see Proposition 3.4.1). However, our main goal in this chapter is to build a necessary theory that we will need in Chapter 4.

Galois and Morita

Évariste Galois left us a rich and prosperous theory, intending to describe the relations between roots of a polynomial equation. A modern approach of the Galois ideas, developed by E. Artin, R. Dedekind, L. Kronecker and several other mathematicians, deals with field extensions and their automorphism groups. It is well known that there are several equivalent conditions for a field extension to be a Galois extension.

The extension of the field Galois theory to the commutative ring context was started by M. Auslander and O. Goldman in [6] and continued by S. U. Chase, D. K. Harrison and A. Rosenberg in [25]. In this last paper several others equivalent conditions for a ring extension to be Galois were introduced.

Recently, a Galois theory for partial group actions on rings was introduced in the literature by M. Dokuchaev, M. Ferrero and A. Paques generalizing the Galois theory developed by Chase-Harrison-Rosenberg, and stimulating many others contributions to the developing of this new subject.

On the other side, K. Morita has developed his own theory relating two rings by means of two bimodules and two morphisms of bimodules. The Morita context is useful to relate the category of modules over these rings. Indeed, it is well known that if they are unital

rings and the morphisms are surjective then we can show that the correspondent categories of modules are equivalent.

We merge both such Morita and Galois theories in order to obtain a theorem which gives necessary and sufficient conditions for an extension to be Galois, in the context of partial actions of weak Hopf algebras.

In Chapter 4, we construct a Morita context relating the partial smash product and the subalgebra of invariants (see Theorem 4.1.10). Moreover, under some additional hypothesis, we show that the context match with the one given by M. Cohen, D. Fischman and S. Montgomery in [27] (see Subsection 4.1.2). We also develop a Galois theory, relating the canonical map, the Morita context and the Galois coordinates (see Theorem 4.2.3).

We continue the chapter showing that the category of the relative partial Hopf module is equivalent to the category of comodules over the A -coring $A \otimes H$, where A is a partial H -module algebra (see Proposition 4.3.5). Finally, we present a new Morita context for partial H -comodule algebras relating the subalgebra of coinvariants and an specific subalgebra of $\text{Hom}(H, A)$, unifying results obtained by S. Caenepeel, E. De Groot and K. Janssen in [14, 15, 18] (see Theorem 4.3.11).

Conventions

Throughout, H will denote a weak Hopf algebra over a field \mathbb{k} . Every \mathbb{k} -algebra is assumed to be associative and unital, unless otherwise stated. Unadorned \otimes means $\otimes_{\mathbb{k}}$. We will adopt the Sweedler notation for the comultiplication of H , that is, $\Delta(h) = h_1 \otimes h_2$ (summation understood), for any $h \in H$. We will also denote by \cdot any partial action and by \triangleright any global one. In a similar way, for partial coactions we will use the notation $\bar{\rho}$ and for the global one ρ . The Sweedler notation for right partial coaction will be written as $\bar{\rho}(a) = a^{\bar{0}} \otimes a^{\bar{1}}$ and $\rho(a) = a^0 \otimes a^1$ for the global one (summation understood).

Chapter 1

Preliminaries

In this chapter we recall some basic concepts which will be necessary along the whole work. For the proofs of the results in Section 1.2 see [14] and [16].

1.1 Weak Hopf algebras

We start recalling the definition and some properties of a weak Hopf algebra over a field \mathbb{k} . For more about it we refer to [9].

Definition 1.1.1. A sextuple $(H, m, u, \Delta, \varepsilon, S)$ is a weak Hopf algebra, with antipode S , if:

$$(WHA1) \quad (H, m, u) \text{ is a } \mathbb{k}\text{-algebra,}$$

$$(WHA2) \quad (H, \Delta, \varepsilon) \text{ is a } \mathbb{k}\text{-coalgebra,}$$

$$(WHA3) \quad \Delta(kh) = \Delta(k)\Delta(h), \quad \forall h, k \in H,$$

$$(WHA4) \quad \varepsilon(kh_1)\varepsilon(h_2g) = \varepsilon(khg) = \varepsilon(kh_2)\varepsilon(h_1g),$$

$$(WHA5) \quad (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) = \Delta^2(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)),$$

$$(WHA6) \quad h_1 S(h_2) = \varepsilon_L(h),$$

$$(WHA7) \quad S(h_1)h_2 = \varepsilon_R(x),$$

$$(WHA8) \quad S(h) = S(h_1)h_2S(h_3),$$

where $\varepsilon_L : H \rightarrow H$ and $\varepsilon_R : H \rightarrow H$ are defined by $\varepsilon_L(h) = \varepsilon(1_1h)1_2$ and $\varepsilon_R(h) = 1_1\varepsilon(h1_2)$. We will denote $H_L = \varepsilon_L(H)$ and $H_R = \varepsilon_R(H)$. It is clear from these definitions that H_L and H_R are both finite dimensional over \mathbb{k} . Algebras H satisfying only the five first statements enumerated above are simply called weak bialgebras.

Note that item (WHA5) can be written as $1_1 \otimes 1'_1 1_2 \otimes 1'_2 = 1_1 \otimes 1_2 \otimes 1_3 = 1_1 \otimes 1_2 1'_1 \otimes 1'_2$.

Example 1.1.2. Every Hopf algebra is a weak Hopf algebra.

Example 1.1.3. Let G be a groupoid (see Definition 2.2.1) such that the order of G_0 is finite. Then the groupoid algebra $\mathbb{k}G$ with basis $\{\delta_g \mid g \in G\}$ is a weak Hopf algebra with the maps

$$\begin{aligned}
m: \mathbb{k}G \otimes \mathbb{k}G &\rightarrow \mathbb{k}G \\
\delta_g \otimes \delta_h &\mapsto \begin{cases} \delta_{gh} & \text{if } \exists gh \\ 0 & \text{otherwise} \end{cases} \\
u: \mathbb{k} &\rightarrow \mathbb{k}G \\
1_{\mathbb{k}} &\mapsto \sum_{e \in G_0} \delta_e \\
\Delta: \mathbb{k}G &\rightarrow \mathbb{k}G \otimes \mathbb{k}G \\
\delta_g &\mapsto \delta_g \otimes \delta_g \\
\varepsilon: \mathbb{k}G &\rightarrow \mathbb{k} \\
\delta_g &\mapsto 1_{\mathbb{k}} \\
S: \mathbb{k}G &\rightarrow \mathbb{k}G \\
\delta_g &\mapsto \delta_{g^{-1}}.
\end{aligned}$$

In this case the algebras H_L and H_R are the subalgebra of $\mathbb{k}G$ generated by the elements in G_0

Example 1.1.4. If H is a finite dimensional weak Hopf algebra, then H^* is a weak Hopf algebra with the maps

$$\begin{aligned}
m: H^* \otimes H^* &\rightarrow H^* \\
f \otimes g &\mapsto f * g : H \rightarrow \mathbb{k} \\
&[f * g](h) = f(h_1)g(h_2) \\
u: \mathbb{k} &\rightarrow H^* \\
1_{\mathbb{k}} &\mapsto \varepsilon_H \\
\Delta: H^* &\rightarrow H^* \otimes H^* \\
f &\mapsto \sum_{i=1}^n f_i \otimes g_i
\end{aligned}$$

where $f(hk) = \sum_{i=1}^n f_i(h)g_i(k)$,

$$\begin{aligned}
\varepsilon: H^* &\rightarrow \mathbb{k} \\
f &\mapsto f(1_H) \\
S_{H^*}: H &\rightarrow H \\
f &\mapsto f \circ S_H.
\end{aligned}$$

Example 1.1.5. Let H and L be two weak Hopf algebras, then the tensor product $H \otimes L$ is a weak Hopf algebra.

Remark 1.1.6. Note that a Hopf algebra is also a weak Hopf algebra, then for a Hopf algebra H the tensor algebra $H \otimes L$ is a weak Hopf algebra (which is not a Hopf algebra) if and only if L is a weak Hopf algebra (which is not a Hopf algebra).

Based on it, the whole amount of examples of Hopf algebras can generate weak Hopf algebras simply using the tensor product.

It is usual to use the next proposition to determine when a weak Hopf algebra is a Hopf algebra.

Proposition 1.1.7. A weak Hopf algebra is a Hopf algebra if one of the following equivalent conditions hold:

1. $\Delta(1_H) = 1_H \otimes 1_H$;
2. $\varepsilon(hk) = \varepsilon(h)\varepsilon(k) \forall h, k \in H$;
3. $h_1S(h_2) = \varepsilon(h)1_H \forall h \in H$;
4. $S(h_1)h_2 = \varepsilon(h)1_H \forall h \in H$;
5. $H_L = H_R = \mathbb{k}1_H$.

Many of the basic properties of a weak Hopf algebra proved in the finite dimensional case (see [9, 11, 17]) also hold in the general case. We start by enumerating some of these properties which can be verified using arguments similar to those used in the finite dimensional case. These properties will be very useful along this work.

Lemma 1.1.8. Let H be a weak Hopf algebra. Then,

$$\varepsilon_L \circ \varepsilon_L = \varepsilon_L \tag{1.1}$$

$$\varepsilon_R \circ \varepsilon_R = \varepsilon_R \tag{1.2}$$

$$\varepsilon(h\varepsilon_L(k)) = \varepsilon(hk) \tag{1.3}$$

$$\varepsilon(\varepsilon_R(h)k) = \varepsilon(hk) \tag{1.4}$$

$$\varepsilon_L(h\varepsilon_L(k)) = \varepsilon_L(hk) \tag{1.5}$$

$$\varepsilon_R(\varepsilon_R(h)k) = \varepsilon_R(hk) \tag{1.6}$$

for all h, k in H .

Proof. We will show (1.1), (1.3) and (1.5). The remaining ones follow in a similar way.

(1.1) Let $h \in H$

$$\begin{aligned}
\varepsilon_L(\varepsilon_L(h)) &= \varepsilon_L(\varepsilon(1_1 h)1_2) \\
&= \varepsilon(1_1 h)\varepsilon_L(1_2) \\
&= \varepsilon(1_1 h)\varepsilon(1'_1 1_2)1'_2 \\
&= \varepsilon(1_1 h)\varepsilon(1_2)1_3 \\
&= \varepsilon(1_1 h)1_2 \\
&= \varepsilon_L(h).
\end{aligned}$$

Then $\varepsilon_L \circ \varepsilon_L = \varepsilon_L$.

(1.3) Let $h, k \in H$

$$\begin{aligned}
\varepsilon(h\varepsilon_L(k)) &= \varepsilon(h\varepsilon(1_1 k)1_2) \\
&= \varepsilon(h1_2)\varepsilon(1_1 k) \\
&= \varepsilon(h1_H k) \\
&= \varepsilon(hk).
\end{aligned}$$

Then $\varepsilon(h\varepsilon_L(k)) = \varepsilon(hk)$.

(1.5) Let $h, k \in H$

$$\begin{aligned}
\varepsilon_L(h\varepsilon_L(k)) &= \varepsilon(1_1 h\varepsilon_L(k))1_2 \\
&\stackrel{(1.3)}{=} \varepsilon(1_1 hk)1_2 \\
&= \varepsilon_L(hk).
\end{aligned}$$

Then $\varepsilon_L(h\varepsilon_L(k)) = \varepsilon_L(hk)$. □

Lemma 1.1.9. *Let H be a weak Hopf algebra. Then, $\Delta(1_H) \in H_R \otimes H_L$.*

Proof. Note that

$$\begin{aligned}
\Delta(1_H) &= 1_1 \otimes 1_2 \\
&= 1_1 \otimes \varepsilon(1_2)1_3 \\
&= 1_1 \otimes \varepsilon(1'_1 1_2)1'_2 \\
&= 1_1 \otimes \varepsilon_L(1_2)
\end{aligned}$$

so $\Delta(1_H) \in H \otimes H_L$.

On the other hand

$$\begin{aligned}
\Delta(1_H) &= 1_1 \otimes 1_2 \\
&= 1_1 \varepsilon(1_2) \otimes 1_3 \\
&= 1_1 \varepsilon(1'_1 1_2) \otimes 1'_2 \\
&= \varepsilon_R(1_1) \otimes 1_2
\end{aligned}$$

so $\Delta(1_H) \in H_R \otimes H$.

Then $\Delta(1_H) \in (H_R \otimes H) \cap (H \otimes H_L) = H_R \otimes H_L$. □

Lemma 1.1.10. *The following statement holds:*

$$z \in H_L \Leftrightarrow \Delta(z) = 1_1 z \otimes 1_2 \quad (1.7)$$

and in this case $\Delta(z) = z1_1 \otimes 1_2$.

For H_R there is a similar statement:

$$w \in H_R \Leftrightarrow \Delta(w) = 1_1 \otimes w1_2 \quad (1.8)$$

and in this case $\Delta(w) = 1_1 \otimes 1_2 w$.

Moreover $\Delta(H_L) \subset H \otimes H_L$ and $\Delta(H_R) \subset H_R \otimes H$.

Proof. Let h be in H_L . Then

$$\begin{aligned} \Delta(h) &= \Delta(\varepsilon_L(h)) \\ &= \Delta(\varepsilon(1_1 h)1_2) \\ &= \varepsilon(1_1 h)\Delta(1_2) \\ &= \varepsilon(1_1 h)1_2 \otimes 1_3 \\ &= \varepsilon(1'_1 h)1_1 1'_2 \otimes 1_2 \\ &= 1_1 \varepsilon(1'_1 h)1'_2 \otimes 1_2 \\ &= 1_1 \varepsilon_L(h) \otimes 1_2 \\ &= 1_1 h \otimes 1_2. \end{aligned}$$

Conversely, if $\Delta(h) = 1_1 h \otimes 1_2$, thus $h = (\varepsilon \otimes I)\Delta(h) = \varepsilon(1_1 h)1_2 = \varepsilon_L(h) \in H_L$.
Moreover, in the above case,

$$\begin{aligned} \Delta(h) &= \Delta(\varepsilon_L(h)) \\ &= \Delta(\varepsilon(1_1 h)1_2) \\ &= \varepsilon(1_1 h)\Delta(1_2) \\ &= \varepsilon(1_1 h)1_2 \otimes 1_3 \\ &= \varepsilon(1_1 h)1_2 1'_1 \otimes 1'_2 \\ &= \varepsilon_L(h)1_1 \otimes 1_2 \\ &= h1_1 \otimes 1_2. \end{aligned}$$

Also, if $h \in H_L$, then we have

$$\begin{aligned} \Delta(h) &= 1_1 h \otimes 1_2 \\ &= 1_1 h \otimes \varepsilon(1_2)1_3 \\ &= 1_1 h \otimes \varepsilon(1'_1 1_2)1'_2 \\ &= 1_1 h \otimes \varepsilon_L(1_2). \end{aligned}$$

and so $\Delta(H) \subset H \otimes H_L$.

The remaining items are analogous. □

Proposition 1.1.11. *Let h, k in H . The following properties hold:*

$$h_1 \otimes h_2 S(h_3) = 1_1 h \otimes 1_2 \quad (1.9)$$

$$S(h_1) h_2 \otimes h_3 = 1_1 \otimes h 1_2 \quad (1.10)$$

$$h \varepsilon_L(k) = \varepsilon(h_1 k) h_2 \quad (1.11)$$

$$\varepsilon_R(h) k = k_1 \varepsilon(h k_2) \quad (1.12)$$

Proof. We will show (1.9) and (1.11). The items (1.10) and (1.12) work in a similar way.

(1.9) For $h \in H$,

$$\begin{aligned} h_1 \otimes h_2 S(h_3) &= h_1 \otimes \varepsilon_L(h_2) \\ &= h_1 \otimes \varepsilon(1_1 h_2) 1_2 \\ &= h_1 \varepsilon(1_1 h_2) \otimes 1_2 \\ &= (1h)_1 \varepsilon(1_1 (1h)_2) \otimes 1_2 \\ &= 1'_1 h_1 \varepsilon(1_1 1'_2 h_2) \otimes 1_2 \\ &= 1_1 h_1 \varepsilon(1_2 h_2) \otimes 1_3 \\ &= (1_1 h)_1 \varepsilon((1_1 h)_2) \otimes 1_2 \\ &= 1_1 h \otimes 1_2. \end{aligned}$$

(1.11) For $h, k \in H$,

$$\begin{aligned} h \varepsilon_L(k) &= h \varepsilon(1_1 k) 1_2 \\ &= \varepsilon(h_1) \varepsilon(1_1 k) h_2 1_2 \\ &\stackrel{(1.10)}{=} \varepsilon(h_1) \varepsilon(h_2 S(h_3) k) h_4 \\ &= \varepsilon(1h_1) \varepsilon(h_2 S(h_3) k) h_4 \\ &= \varepsilon(1h_1 S(h_2) k) h_3 h_4 \\ &= \varepsilon(\varepsilon_R(h_1) k) h_2 \\ &\stackrel{(1.4)}{=} \varepsilon(h_1 k) h_2. \end{aligned}$$

□

Lemma 1.1.12. *H_L and H_R are subalgebras of H with unit 1_H . Moreover, if $z \in H_L$ and $w \in H_R$, we have that $zw = wz$.*

Proof. Let $h, h' \in H_L$. Then

$$\begin{aligned} hh' &= h \varepsilon_L(h') \\ &\stackrel{(1.11)}{=} \varepsilon(h_1 h') h_2 \\ &\stackrel{(1.7)}{=} \varepsilon(1_1 h h') 1_2 \\ &= \varepsilon_L(h h') \end{aligned}$$

and it means $hh' \in H_L$.

Analogously, if $k, k' \in H_R$, then we have

$$\begin{aligned}
k'k &= \varepsilon_R(k')k \\
&\stackrel{(1.12)}{=} k_1\varepsilon(k'k_2) \\
&\stackrel{(1.8)}{=} 1_1\varepsilon(k'k_2) \\
&= \varepsilon_R(k'k)
\end{aligned}$$

and it means $k'k \in H_R$.

Moreover,

$$\varepsilon_L(1_H) = \varepsilon(1_1)1_2 = 1_H = 1_1\varepsilon(1_2) = \varepsilon_R(1_H)$$

hence 1_H lies in both H_R and H_L .

Now, picking up $h \in H_L$ and $k \in H_R$ we have

$$\begin{aligned}
hk &= \varepsilon_L(h)\varepsilon_R(k) \\
&= \varepsilon(1_1h)1_21'_1\varepsilon(k1'_2) \\
&= \varepsilon(1_1h)1_2\varepsilon(k1_3) \\
&= \varepsilon(1_1h)1'_11_2\varepsilon(k1'_2) \\
&= 1'_1\varepsilon(k1'_2)\varepsilon(1_1h)1_2 \\
&= \varepsilon_R(k)\varepsilon_L(h) \\
&= kh.
\end{aligned}$$

□

Lemma 1.1.13. *Let H be a weak Hopf algebra. Then,*

$$\varepsilon_L(\varepsilon_L(h)k) = \varepsilon_L(h)\varepsilon_L(k) \tag{1.13}$$

$$\varepsilon_R(h\varepsilon_R(k)) = \varepsilon_R(h)\varepsilon_R(k) \tag{1.14}$$

$$\varepsilon_L(h) = \varepsilon(S(h)1_1)1_2 \tag{1.15}$$

$$\varepsilon_R(h) = 1_1\varepsilon(1_2S(h)) \tag{1.16}$$

$$\varepsilon_L(h) = S(1_1)\varepsilon(1_2h) \tag{1.17}$$

$$\varepsilon_R(h) = \varepsilon(1_1h)S(1_2), \tag{1.18}$$

for all $h, k \in H$.

Proof. (1.13)

$$\begin{aligned}
\varepsilon_L(\varepsilon_L(h)k) &\stackrel{(1.5)}{=} \varepsilon_L(\varepsilon_L(h)\varepsilon_L(k)) \\
&\stackrel{(1.1.12)}{=} \varepsilon_L(h)\varepsilon_L(k).
\end{aligned}$$

(1.14)

$$\begin{aligned}
\varepsilon_R(h\varepsilon_R(k)) &\stackrel{(1.6)}{=} \varepsilon_R(\varepsilon_R(h)\varepsilon_R(k)) \\
&\stackrel{(1.1.12)}{=} \varepsilon_R(h)\varepsilon_R(k).
\end{aligned}$$

(1.15)

$$\begin{aligned}
\varepsilon_L(h) &= \varepsilon(1_1 h) 1_2 \\
&\stackrel{(1.3)}{=} \varepsilon(1_1 \varepsilon_L(h)) 1_2 \\
&\stackrel{(1.1.12)}{=} \varepsilon(\varepsilon_L(h) 1_1) 1_2 \\
&= \varepsilon(h_1 S(h_2) 1_1) 1_2 \\
&\stackrel{(1.4)}{=} \varepsilon(\varepsilon_R(h_1) S(h_2) 1_1) 1_2 \\
&= \varepsilon(S(h_1) h_2 S(h_3) 1_1) 1_2 \\
&= \varepsilon(S(h) 1_1) 1_2.
\end{aligned}$$

(1.16)

$$\begin{aligned}
\varepsilon_R(h) &= 1_1 \varepsilon(h 1_2) \\
&\stackrel{(1.4)}{=} 1_1 \varepsilon(\varepsilon_R(h) 1_2) \\
&\stackrel{(1.1.12)}{=} 1_1 \varepsilon(1_2 \varepsilon_R(h)) \\
&= 1_1 \varepsilon(1_2 S(h_1) h_2) \\
&\stackrel{(1.3)}{=} 1_1 \varepsilon(1_2 S(h_1) \varepsilon_L(h_2)) \\
&= 1_1 \varepsilon(1_2 S(h_1) h_2 S(h_3)) \\
&= 1_1 \varepsilon(1_2 S(h)).
\end{aligned}$$

(1.17)

$$\begin{aligned}
S(1_1) \varepsilon(1_2 h) &= S(1_1) 1_2 S(1_3) \varepsilon(1_4 h) \\
&= \varepsilon_R(1_1) S(1_2) \varepsilon(1_3 h) \\
&\stackrel{(1.1.12)}{=} 1_1 S(1_2) \varepsilon(1_3 h) \\
&= \varepsilon_L(1_1) \varepsilon(1_2 h) \\
&= \varepsilon(1'_1 1_1) 1'_2 \varepsilon(1_2 h) \\
&= \varepsilon(1'_1 1_1) \varepsilon(1_2 h) 1'_2 \\
&= \varepsilon(1'_1 h) 1'_2 \\
&= \varepsilon_L(h).
\end{aligned}$$

(1.18)

$$\begin{aligned}
\varepsilon(h 1_1) S(1_2) &= \varepsilon(h 1_1) S(1_2) 1_3 S(1_4) \\
&= \varepsilon(h 1_1) S(1_2) \varepsilon_L(1_3) \\
&\stackrel{(1.1.12)}{=} \varepsilon(h 1_1) S(1_2) 1_3 \\
&= \varepsilon(h 1_1) \varepsilon_R(1_2) \\
&= \varepsilon(h 1_1) 1'_1 \varepsilon(1_2 1'_2) \\
&= 1'_1 \varepsilon(h 1_1) \varepsilon(1_2 1'_2) \\
&= 1_1 \varepsilon(h 1_2) \\
&= \varepsilon_R(h).
\end{aligned}$$

□

Lemma 1.1.14. *Let H be a weak Hopf algebra. Then,*

$$\varepsilon_L \circ S = \varepsilon_L \circ \varepsilon_R = S \circ \varepsilon_R \quad (1.19)$$

$$\varepsilon_R \circ S = \varepsilon_R \circ \varepsilon_L = S \circ \varepsilon_L \quad (1.20)$$

$$S(1_1) \otimes S(1_2) = 1_2 \otimes 1_1 \quad (1.21)$$

$$S(hk) = S(k)S(h) \quad (1.22)$$

$$S(h)_1 \otimes S(h)_2 = S(h_2) \otimes S(h_1) \quad (1.23)$$

$$\varepsilon \circ S = \varepsilon \quad (1.24)$$

$$S(1_H) = 1_H \quad (1.25)$$

$$h_1 \otimes S(h_2) h_3 = h_1 \otimes S(1_2) \quad (1.26)$$

$$h_1 S(h_2) \otimes h_3 = S(1_1) \otimes 1_2 h, \quad (1.27)$$

for all $h, k \in H$.

Proof. (1.19) For the first equality:

$$\begin{aligned} \varepsilon_L(S(h)) &\stackrel{(1.17)}{=} S(1_1)\varepsilon(1_2S(h)) \\ &= S(1_1\varepsilon(1_2S(h))) \\ &\stackrel{(1.16)}{=} S(\varepsilon_R(h)). \end{aligned}$$

For the second:

$$\begin{aligned} \varepsilon_L(\varepsilon_R(h)) &\stackrel{(1.16)}{=} \varepsilon_L(1_1\varepsilon(1_2S(h))) \\ &= \varepsilon_L(1_1)\varepsilon(1_2S(h)) \\ &= \varepsilon(1'_1 1_1)1'_2\varepsilon(1_2S(h)) \\ &= \varepsilon(1'_1 1_1)\varepsilon(1_2S(h))1'_2 \\ &= \varepsilon(1'_1 S(h))1'_2 \\ &= \varepsilon_L(S(h)). \end{aligned}$$

(1.20) The first equality holds because:

$$\begin{aligned} \varepsilon_R(S(h)) &\stackrel{(1.18)}{=} \varepsilon(S(h)1_1)S(1_2) \\ &= S(\varepsilon(S(h)1_1)1_2) \\ &\stackrel{(1.15)}{=} S(\varepsilon_L(h)). \end{aligned}$$

For the second:

$$\begin{aligned} \varepsilon_R(\varepsilon_L(h)) &\stackrel{(1.15)}{=} \varepsilon_R(\varepsilon(S(h)1_1)1_2) \\ &= \varepsilon(S(h)1_1)\varepsilon_R(1_2) \\ &= \varepsilon(S(h)1_1)1'_1\varepsilon(1_21'_2) \\ &= 1'_1\varepsilon(S(h)1_1)\varepsilon(1_21'_2) \\ &= 1'_1\varepsilon(S(h)1'_2) \\ &= \varepsilon_R(S(h)). \end{aligned}$$

(1.21)

$$\begin{aligned}
S(\mathbf{1}_1) \otimes S(\mathbf{1}_2) &= S(\varepsilon_R(\mathbf{1}_1)) \otimes S(\mathbf{1}_2) \\
&\stackrel{(1.20)}{=} \varepsilon_L(\varepsilon_R(\mathbf{1}_1)) \otimes S(\mathbf{1}_2) \\
&= \varepsilon_L(\mathbf{1}_1) \otimes S(\mathbf{1}_2) \\
&= \varepsilon(\mathbf{1}_{1'} \mathbf{1}_1) \mathbf{1}_{2'} \otimes S(\mathbf{1}_2) \\
&= \mathbf{1}_{2'} \otimes \varepsilon(\mathbf{1}_{1'} \mathbf{1}_1) S(\mathbf{1}_2) \\
&\stackrel{(1.18)}{=} \mathbf{1}_{2'} \otimes \varepsilon_R(\mathbf{1}_{1'}) \\
&= \mathbf{1}_{2'} \otimes \mathbf{1}_{1'}.
\end{aligned}$$

(1.22)

$$\begin{aligned}
S(hk) &= S((hk)_1)(hk)_2 S((hk)_3) \\
&= S(h_1 k_1) h_2 k_2 S(h_3 k_3) \\
&= S(h_1 k_1) \varepsilon_L(h_2 k_2) \\
&\stackrel{(1.5)}{=} S(h_1 k_1) \varepsilon_L(h_2 \varepsilon_L(k_2)) \\
&\stackrel{(1.11)}{=} S(h_1 k_1) \varepsilon_L(h_3) \varepsilon(h_2 \varepsilon_L(k_2)) \\
&= S(h_1 k_1) h_3 S(h_4) \varepsilon(h_2 \varepsilon_L(k_2)) \\
&= S(h_1 k_1) \varepsilon(h_2 \varepsilon_L(k_2)) h_3 S(h_4) \\
&\stackrel{(1.11)}{=} S(h_1 k_1) h_2 \varepsilon_L(k_2) S(h_3) \\
&= S(h_1 k_1) h_2 k_2 S(k_3) S(h_3) \\
&= \varepsilon_R(h_1 k_1) S(k_2) S(h_2) \\
&\stackrel{(1.4)}{=} \varepsilon_R(\varepsilon_R(h_1) k_1) S(k_2) S(h_2) \\
&\stackrel{(1.12)}{=} \varepsilon_R(k_1) \varepsilon(\varepsilon_R(h_1) k_2) S(k_3) S(h_2) \\
&= S(k_1) k_2 \varepsilon(\varepsilon_R(h_1) k_3) S(k_4) S(h_2) \\
&\stackrel{(1.12)}{=} S(k_1) \varepsilon_R(h_1) k_2 S(k_3) S(h_2) \\
&\stackrel{(1.12)}{=} S(k_1) \varepsilon_R(h_1) \varepsilon_L(k_2) S(h_2) \\
&\stackrel{(1.1.12)}{=} S(k_1) \varepsilon_L(k_2) \varepsilon_R(h_1) S(h_2) \\
&= S(k_1) k_2 S(k_3) S(h_1) h_2 S(h_3) \\
&= S(k) S(h).
\end{aligned}$$

(1.23)

$$\begin{aligned}
\Delta(S(h)) &= \Delta(S(h_1) h_2 S(h_3)) \\
&= \Delta(S(h_1) \varepsilon_L(h_2)) \\
&\stackrel{(1.22)}{=} \Delta(S(h_1)) \Delta(\varepsilon_L(h_2)) \\
&= \Delta(S(h_1)) [1_1 \varepsilon_L(h_2) \otimes 1_2] \\
&= \Delta(S(h_1)) [1_1 h_2 S(h_3) \otimes 1_2] \\
&\stackrel{(1.9)}{=} \Delta(S(h_1)) [h_2 S(h_3) \otimes h_3 S(h_4)]
\end{aligned}$$

$$\begin{aligned}
&= \Delta(S(h_1))\Delta(h_2)[S(h_4) \otimes S(h_3)] \\
&= \Delta(S(h_1))\Delta(h_2)[S(h_4) \otimes S(h_3)] \\
&\stackrel{(1.22)}{=} \Delta(S(h_1) h_2)[S(h_4) \otimes S(h_3)] \\
&= \Delta(\varepsilon_R(h_1))[S(h_3) \otimes S(h_2)] \\
&= [1_1 \otimes 1_2 \varepsilon_R(h_1)][S(h_3) \otimes S(h_2)] \\
&= 1_1 S(h_3) \otimes 1_2 \varepsilon_R(h_1) S(h_2) \\
&= 1_1 S(h_4) \otimes 1_2 S(h_1)h_2S(h_3) \\
&= 1_1 S(h_2) \otimes 1_2 S(h_1) \\
&\stackrel{(1.21)}{=} S(1_2) S(h_2) \otimes S(1_1) S(h_1) \\
&\stackrel{(1.22)}{=} S(h_2 1_2) \otimes S(h_1 1_1) \\
&= S(h_2) \otimes S(h_1).
\end{aligned}$$

(1.24)

$$\begin{aligned}
\varepsilon(S(h)) &= \varepsilon(S(h_1)h_2S(h_3)) \\
&= \varepsilon(S(h_1)\varepsilon_L(h_2)) \\
&\stackrel{(1.3)}{=} \varepsilon(S(h_1)h_2) \\
&= \varepsilon(\varepsilon_R(h)) \\
&\stackrel{(1.4)}{=} \varepsilon(h).
\end{aligned}$$

Moreover, $S(1_H) = S(\varepsilon_L(1_H)) \stackrel{(1.20)}{=} \varepsilon_R(\varepsilon_L(1_H)) = 1_H$.

(1.26)

$$\begin{aligned}
h_1 \otimes S(h_2) h_3 &= h_1 \otimes \varepsilon_L(h_2) \\
&\stackrel{(1.18)}{=} h_1 \otimes \varepsilon(h_2 1_1) S(1_2) \\
&= h_1 \varepsilon(h_2 1_1) \otimes S(1_2) \\
&= h_1 1_{1'} \varepsilon(h_2 1_{2'} 1_1) \otimes S(1_2) \\
&= h_1 1_1 \varepsilon(h_2 1_2) \otimes S(1_3) \\
&= h 1_1 \otimes S(1_2).
\end{aligned}$$

(1.27)

$$\begin{aligned}
h_1 S(h_2) \otimes h_3 &= \varepsilon_L(h_1) \otimes h_2 \\
&\stackrel{(1.17)}{=} S(1_1) \varepsilon(1_2 h_1) \otimes h_2 \\
&= S(1_1) \otimes \varepsilon(1_2 h_1) h_2 \\
&= S(1_1) \otimes \varepsilon(1_2 1_{1'} h_1) 1_{2'} h_2 \\
&= S(1_1) \otimes \varepsilon(1_2 h_1) 1_3 h_2 \\
&= S(1_1) \otimes 1_2 h.
\end{aligned}$$

□

The finite dimension of H_L and H_R is enough to prove the following lemma, which allows us to obtain results in the sequel without to resort, as it is usual, to the bijectivity of the antipode.

Lemma 1.1.15. *Let H be a weak Hopf algebra. Then $S(H_L) = H_R$ and $S(H_R) = H_L$. Moreover, $S_L = S|_{H_L}$ and $S_R = S|_{H_R}$ are bijections between H_L and H_R .*

Proof. We proceed by showing that the linear maps $S_L = S|_{H_L} : H_L \rightarrow H_R$ and $S_R = S|_{H_R} : H_R \rightarrow H_L$ are surjective and $\dim(H_L) = \dim(H_R)$.

Since $H_L = \varepsilon_L(H)$, it follows by (1.20) that $S(H_L) \subseteq H_R$. Similarly, $S(H_R) \subseteq H_L$. Conversely, if $z \in H_L$, then $z = \varepsilon_L(z) \stackrel{(1.17)}{=} S(1_1)\varepsilon(1_2z) \in S(H_R)$. And, if $w \in H_R$, then $w = \varepsilon_R(w) \stackrel{(1.18)}{=} \varepsilon(w1_1)S(1_2) \in S(H_L)$.

Now, take the linear map $\varphi_L : H_L \rightarrow H_R^*$, given by $\varphi_L(z)(w) = \varepsilon(wz)$, for all $z \in H_L$ and $w \in H_R$.

Notice that if $z \in \text{Ker}(\varphi_L)$, then $z = \varepsilon_L(z) = \varepsilon(1_1z)1_2 = \varphi_L(z)(1_1)1_2 = 0$. Hence, φ_L is injective and $\dim(H_L) \leq \dim(H_R^*) = \dim(H_R)$. Similarly, we also get that $\dim(H_R) \leq \dim(H_L)$. \square

Lemma 1.1.16. *For all $z \in H_L$ and $w \in H_R$, we have:*

$$1_1 S_R^{-1}(z) \otimes 1_2 = 1_1 \otimes 1_2 z \quad (1.28)$$

$$1_1 \otimes S_L^{-1}(w) 1_2 = w 1_1 \otimes 1_2 \quad (1.29)$$

Proof. Indeed,

(1.28):

$$\begin{aligned} 1_1 S_R^{-1}(z) \otimes 1_2 &\stackrel{(1.9)}{=} S_R^{-1}(z)_1 \otimes S_R^{-1}(z)_2 S(S_R^{-1}(z)_3) \\ &= S_R^{-1}(z)_1 \otimes \varepsilon_L(S_R^{-1}(z)_2) \\ &= 1_1 \otimes \varepsilon_L(1_2 S_R^{-1}(z)) && \text{(since } S_R^{-1}(z) \in H_R) \\ &\stackrel{(1.13)}{=} 1_1 \otimes 1_2 \varepsilon_L(S_R^{-1}(z)) \\ &= 1_1 \otimes 1_2 \varepsilon_L(\varepsilon_R(S_R^{-1}(z))) && \text{(since } S_R^{-1}(z) \in H_R) \\ &\stackrel{(1.19)}{=} 1_1 \otimes 1_2 \varepsilon_L(S(S_R^{-1}(z))) \\ &= 1_1 \otimes 1_2 \varepsilon_L(S|_{H_R}(S_R^{-1}(z))) \\ &= 1_1 \otimes 1_2 \varepsilon_L(z) \\ &= 1_1 \otimes 1_2 z \end{aligned}$$

(1.29):

$$\begin{aligned} 1_1 \otimes S_L^{-1}(w) 1_2 &\stackrel{(1.10)}{=} S(S_L^{-1}(w)_1) S_L^{-1}(w)_2 \otimes S_L^{-1}(w)_3 \\ &= \varepsilon_R(S_L^{-1}(w)_1) \otimes S_L^{-1}(w)_2 \\ &= \varepsilon_R(S_L^{-1}(w) 1_1) \otimes 1_2 && \text{(since } S_L^{-1}(w) \in H_L) \\ &\stackrel{(1.14)}{=} \varepsilon_R(S_L^{-1}(w)) 1_1 \otimes 1_2 \\ &= \varepsilon_R(\varepsilon_L(S_L^{-1}(w))) 1_1 \otimes 1_2 && \text{(since } S_L^{-1}(w) \in H_L) \\ &\stackrel{(1.20)}{=} \varepsilon_R(S(S_L^{-1}(w))) 1_1 \otimes 1_2 \\ &= \varepsilon_R(S|_{H_L}(S_L^{-1}(w))) 1_1 \otimes 1_2 \\ &= \varepsilon_R(w) 1_1 \otimes 1_2 \\ &= w 1_1 \otimes 1_2 \end{aligned}$$

\square

Lemma 1.1.17. *For all $z \in H_L$ and $w \in H_R$, we have:*

$$zS(1_1) \otimes 1_2 = S(1_1) \otimes 1_2z \quad (1.30)$$

$$1_1 \otimes S(1_2)w = w1_1 \otimes S(1_2) \quad (1.31)$$

Proof. Indeed,

(1.30):

$$\begin{aligned} S(1_1) \otimes 1_2z &\stackrel{(1.28)}{=} S(1_1S_R^{-1}(z)) \otimes 1_2 \\ &= S(S_R^{-1}(z))S(1_1) \otimes 1_2 \\ &= zS(1_1) \otimes 1_2 \end{aligned}$$

(1.31):

$$\begin{aligned} w1_1 \otimes S(1_2) &\stackrel{(1.29)}{=} w1_1 \otimes S(1_2S_L^{-1}(w)) \\ &= w1_1 \otimes S(S_L^{-1}(w))S(1_2) \\ &= w1_1 \otimes wS(1_2) \end{aligned}$$

□

1.2 Corings

Definition 1.2.1. Let A be a unital ring. A set \mathcal{C} is an A -coring if \mathcal{C} is an A -bimodule and there are morphisms of A -bimodules $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$ such that

$$(CR1) \quad (\Delta_{\mathcal{C}} \otimes_A I)\Delta_{\mathcal{C}} = (I \otimes_A \Delta_{\mathcal{C}})\Delta_{\mathcal{C}};$$

$$(CR2) \quad (I \otimes_A \varepsilon_{\mathcal{C}})\Delta_{\mathcal{C}}(c) = c \otimes_A 1_A \simeq c \quad \forall c \in \mathcal{C};$$

$$(CR3) \quad (\varepsilon_{\mathcal{C}} \otimes_A I)\Delta_{\mathcal{C}}(c) = 1_A \otimes_A c \simeq c \quad \forall c \in \mathcal{C}.$$

Example 1.2.2. *If A is a commutative ring then any A -coalgebra is an A -coring.*

Definition 1.2.3. Let A be a unital ring. An A -bimodule R is said to be an A -ring if there exist morphisms of A -bimodules $\mathfrak{m} : R \otimes_A R \rightarrow A$ and $\iota : A \rightarrow R$ such that for all $a \in A$ and $r \in R$,

$$1. \quad (I \otimes_A \mathfrak{m})\mathfrak{m} = (\mathfrak{m} \otimes_A I)\mathfrak{m};$$

$$2. \quad \mathfrak{m}(\iota \otimes_A I)(a \otimes_A r) = 1_A \otimes_A \iota(a)r;$$

$$3. \quad \mathfrak{m}(I \otimes_A \iota)(r \otimes_A a) = 1_A \otimes_A r\iota(a).$$

Note that if \mathcal{C} is an A -coring then ${}^*\mathcal{C} = {}_A\text{Hom}(\mathcal{C}, A)$ is an A -ring with multiplication $\#$ given by $(f\#g)(c) = g(c_1f(c_2))$ and $\iota : A \rightarrow {}^*\mathcal{C}$ given by $\iota(a)(c) = \varepsilon_{\mathcal{C}}(c)a$.

Definition 1.2.4. An element x in an A -coring \mathcal{C} is a grouplike element if $\Delta_{\mathcal{C}}(x) = x \otimes_A x$ and $\varepsilon_{\mathcal{C}}(x) = 1_A$.

From now on we will consider the A -coring \mathcal{C} with a fixed grouplike x .

Definition 1.2.5. Let M be a vector space over \mathbb{k} and A be a \mathbb{k} -algebra. M is a right comodule over the A -coring \mathcal{C} if M is a right A -module and there exist a map $\rho : M \rightarrow M \otimes_A \mathcal{C}$ such that

- (CM1) ρ is right A -linear;
- (CM2) $(I \otimes_A \varepsilon_{\mathcal{C}})\rho(m) = m \otimes_A 1_A \simeq m$;
- (CM3) $(\rho \otimes_A I)\rho = (I \otimes_A \Delta_{\mathcal{C}})\rho$.

We will denote the category of the right \mathcal{C} -comodules by $\mathcal{M}^{\mathcal{C}}$.

The set of elements of a \mathcal{C} -comodule M invariant by the coaction ρ is defined by

$$M^{co\mathcal{C}} = \{m \in M \mid \rho(m) = m \otimes_A x\}.$$

Note that A is an object in $\mathcal{M}^{\mathcal{C}}$ with the coaction $\rho : A \rightarrow A \otimes_A \mathcal{C} \simeq \mathcal{C}$ defined by $\rho(a) = 1_A \otimes_A xa$. Hence $A^{co\mathcal{C}} = \{a \in A \mid xa = ax\}$ which is clearly a subring of A .

Definition 1.2.6. Let \mathcal{C} be an A -coring and consider the map

$$\begin{aligned} can : A \otimes_{A^{co\mathcal{C}}} A &\rightarrow \mathcal{C} \\ a \otimes b &\mapsto axb. \end{aligned}$$

We say that \mathcal{C} is a Galois coring if can is bijective.

Let now $Q = \{q \in {}^*\mathcal{C} \mid c_1q(c_2) = q(c)x\}$. Q is an $({}^*\mathcal{C}, A^{co\mathcal{C}})$ -bimodule via

$$f \rightarrow q \triangleleft a = f \# q \# \iota(a).$$

Moreover, A is an $(A^{co\mathcal{C}}, {}^*\mathcal{C})$ -bimodule with actions $b \triangleright a = ba$ and $a \triangleleft f = f(xa)$.

The maps

$$\begin{aligned} \tau : A \otimes_{{}^*\mathcal{C}} Q &\rightarrow A^{co\mathcal{C}} \\ a \otimes q &\mapsto q(xa) \end{aligned}$$

and

$$\begin{aligned} \mu : Q \otimes_{A^{co\mathcal{C}}} A &\rightarrow {}^*\mathcal{C} \\ q \otimes a &\mapsto q \# \iota(a) \end{aligned}$$

determine the Morita context $(A^{co\mathcal{C}}, {}^*\mathcal{C}, A, Q, \tau, \mu)$.

Finally, consider the map

$$\begin{aligned} {}^*can : {}^*\mathcal{C} &\rightarrow {}^*(A \otimes_{A^{co\mathcal{C}}} A) \simeq {}_{A^{co\mathcal{C}}}End(A)^{op} \\ f &\mapsto {}^*can(f)(a) = f(xa), \end{aligned}$$

and the functors

$$\begin{aligned} F_{\mathcal{C}} : \mathcal{M}_{A^{co\mathcal{C}}} &\rightarrow \mathcal{M}^{\mathcal{C}} \\ N &\mapsto N \otimes_{A^{co\mathcal{C}}} A \end{aligned}$$

and

$$\begin{aligned} G_{\mathcal{C}} : \mathcal{M}^{\mathcal{C}} &\rightarrow \mathcal{M}_{A^{\text{co}\mathcal{C}}} \\ M &\mapsto M^{\text{co}\mathcal{C}} \end{aligned}$$

where $\mathcal{M}_{A^{\text{co}\mathcal{C}}}$ denote the category of right $A^{\text{co}\mathcal{C}}$ -modules and $\mathcal{M}^{\mathcal{C}}$ the category of right \mathcal{C} -comodules.

Theorem 1.2.7. *Let \mathcal{C} be a coring with fixed grouplike x such that \mathcal{C} is a left A -progenerator. The following statements are equivalent:*

1. *can is bijective and A is faithfully flat as left $A^{\text{co}\mathcal{C}}$ -module;*
2. **can is bijective and A is a left $A^{\text{co}\mathcal{C}}$ -progenerator;*
3. *The Morita context $(A^{\text{co}\mathcal{C}}, {}^*\mathcal{C}, A, Q, \tau, \mu)$ is strict;*
4. *$(F_{\mathcal{C}}, G_{\mathcal{C}})$ is an equivalence of categories.*

Chapter 2

Partial actions of weak Hopf algebras on Algebras

2.1 Partial actions of weak Hopf algebras

Hereafter, all actions of a weak Hopf algebra on any algebra will be considered only on the left side. Actions on the right side can be defined in a similar way, and corresponding results similar to the ones we will deal with along this text can be obtained as well. Recall that in this work all algebras are assumed to be associative and unital, unless otherwise stated. Furthermore, in order to avoid confusion, we will always denote by \cdot any partial action and by \triangleright any global one (see, in particular, Section 2.4). Throughout, H will always denote a weak Hopf algebra, without any more explicit mention, unless otherwise required.

The usual definition for (global) actions of weak Hopf algebras on algebras is the following.

Definition 2.1.1. Let A be an algebra. A (global) *action* of H on A is a \mathbb{k} -linear map $\triangleright: H \otimes A \rightarrow A$ such that the following properties hold for all $a, b \in A$ and $h, k \in H$:

- (i) $1_H \triangleright a = a$,
- (ii) $h \triangleright ab = (h_1 \triangleright a)(h_2 \triangleright b)$,
- (iii) $h \triangleright (k \triangleright a) = hk \triangleright a$.

In this case, A is called an *H -module algebra*.

Note that in this definition we do not need to require A to be unital. However, in the literature we find the definition of an action of a weak bialgebra H on an algebra A with unit 1_A , where the conditions (i)-(iii) have to be satisfied, as well as the fourth condition

$$h \triangleright 1_A = \varepsilon_L(h) \triangleright 1_A.$$

Nevertheless, in the case that H is a weak Hopf algebra, this fourth condition is implied by the three previous ones, as we will see in the following lemma.

Lemma 2.1.2. *Let A be an H -module algebra. Then,*

$$h \triangleright 1_A = \varepsilon_L(h) \triangleright 1_A,$$

for all $h \in H$.

Proof. In fact,

$$\begin{aligned} \varepsilon_L(h) \triangleright 1_A &= h_1 S(h_2) \triangleright 1_A \\ &\stackrel{\text{(iii)}}{=} h_1 \triangleright (1_A (S(h_2) \triangleright 1_A)) \\ &\stackrel{\text{(ii)}}{=} (h_1 \triangleright 1_A) (h_2 S(h_3) \triangleright 1_A) \\ &\stackrel{\text{(1.9)}}{=} (1_1 h \triangleright 1_A) (1_2 \triangleright 1_A) \\ &\stackrel{\text{(iii)}}{=} (1_1 \triangleright (h \triangleright 1_A)) (1_2 \triangleright 1_A) \\ &\stackrel{\text{(ii)}}{=} 1_H \triangleright (h \triangleright 1_A) 1_A \\ &\stackrel{\text{(i)}}{=} h \triangleright 1_A. \end{aligned}$$

□

In the setting of partial actions we have the following.

Definition 2.1.3. Let A be an algebra. A *partial action* of H on A is a \mathbb{k} -linear map $\cdot : H \otimes A \rightarrow A$ such that the following properties hold for all $a, b \in A$ and $h, k \in H$:

$$\text{(PMA1)} \quad 1_H \cdot a = a;$$

$$\text{(PMA2)} \quad h \cdot ab = (h_1 \cdot a)(h_2 \cdot b);$$

$$\text{(PMA3)} \quad h \cdot (k \cdot a) = (h_1 \cdot 1_A)(h_2 k \cdot a).$$

In this case A is called a *partial H -module algebra*.

Moreover, we say that \cdot is *symmetric* (or, A is a *symmetric partial H -module algebra*) if the additional condition also holds:

$$\text{(PMA4)} \quad h \cdot (k \cdot a) = (h_1 k \cdot a)(h_2 \cdot 1_A).$$

Remark 2.1.4. *Observe that, assuming the condition (PMA1), the conditions (PMA2) and (PMA3) in Definition 2.1.3 are equivalent to*

$$h \cdot (a(k \cdot b)) = (h_1 \cdot a)(h_2 k \cdot b).$$

In a similar way, conditions (PMA2) and (PMA4) are equivalent to

$$h \cdot ((k \cdot a)b) = (h_1 k \cdot a)(h_2 \cdot b).$$

Example 2.1.5. Let $H_4 = \mathbb{k} \langle g, x \mid g^2 = 1, x^2 = 0 \text{ and } gx = -xg \rangle$ be the Sweedler's algebra over a field \mathbb{k} with characteristic different of 2. Define $\lambda \in H_4^*$ by

$$\begin{aligned} \lambda: H_4 &\rightarrow \mathbb{k} \\ 1 &\mapsto 1_{\mathbb{k}} \\ g &\mapsto 0 \\ x &\mapsto r \\ xg &\mapsto -r \end{aligned}$$

where $r \in \mathbb{k}$.

For a weak Hopf algebra H , the map

$$\begin{aligned} \cdot: H \otimes H_4 \otimes H &\rightarrow H \\ h \otimes v \otimes h' &\mapsto h_1 h' S(h_2) \lambda(v) \end{aligned}$$

defines in H a structure of partial $H \otimes H_4$ -module algebra.

It is immediate to check that any (global) action is a particular example of a partial one. The following proposition tells us under what condition a partial action is global.

Lemma 2.1.6. Let A be a partial H -module algebra. Then, A is an H -module algebra if and only if $h \cdot 1_A = \varepsilon_L(h) \cdot 1_A$, for all $h \in H$.

Proof. From Lemma 2.1.2, if A is an H -module algebra then $h \cdot 1_A = \varepsilon_L(h) \cdot 1_A$, for all $h \in H$.

Conversely, suppose that $h \cdot 1_A = \varepsilon_L(h) \cdot 1_A$, for all $h \in H$. Then,

$$\begin{aligned} h \cdot (g \cdot a) &= (h_1 \cdot 1_A)(h_2 g \cdot a) \\ &= (\varepsilon_L(h_1) \cdot 1_A)(h_2 g \cdot a) \\ &= (h_1 S(h_2) \cdot 1_A)(h_3 g \cdot a) \\ &\stackrel{(1.27)}{=} (S(1_1) \cdot 1_A)(1_2 h g \cdot a) \\ &= (\varepsilon_L(S(1_1)) \cdot 1_A)(1_2 h g \cdot a) \\ &\stackrel{(1.19)}{=} (\varepsilon_L(\varepsilon_R(1_1)) \cdot 1_A)(1_2 h g \cdot a) \\ &= (\varepsilon_R(1_1) \cdot 1_A)(1_2 h g \cdot a) \\ &\stackrel{(1.1.9)}{=} (1_1 \cdot 1_A)(1_2 h g \cdot a) \\ &= 1_H \cdot (h g \cdot a) \\ &= h g \cdot a, \end{aligned}$$

for all $g, h \in H$. □

In the next lemmas we will see some technical properties of partial actions, which will be very useful tools in the sequel.

Lemma 2.1.7. *Let A be a partial H -module algebra. If $w \in H_R$ (or, $w \in H_L$ and the partial action is symmetric), then*

$$w \cdot (h \cdot a) = wh \cdot a,$$

for every $h \in H$ and $a \in A$.

Proof. Suppose first that $w \in H_R$. Thus,

$$\begin{aligned} w \cdot (h \cdot a) &= (w_1 \cdot 1_A)(w_2 h \cdot a) \\ &\stackrel{(1.8)}{=} (1_1 \cdot 1_A)(1_2 w h \cdot a) \\ &= 1_H \cdot (w h \cdot a) \\ &= w h \cdot a. \end{aligned}$$

Now, assuming that the partial action is symmetric and $w \in H_L$, we have

$$\begin{aligned} w \cdot (h \cdot a) &= (w_1 h \cdot a)(w_2 \cdot 1_A) \\ &\stackrel{(1.7)}{=} (1_1 w h \cdot 1_A)(1_2 \cdot a) \\ &= 1_H \cdot (w h \cdot a) \\ &= w h \cdot a. \end{aligned}$$

□

Lemma 2.1.8. *Let A be a partial H -module algebra, $h, k \in H$ and $a, b \in A$. Then,*

$$(h \cdot a)(k \cdot b) = (1_1 h \cdot a)(1_2 k \cdot b).$$

Proof. In fact,

$$\begin{aligned} (h \cdot a)(k \cdot b) &= 1_H \cdot [(h \cdot a)(k \cdot b)] \\ &= (1_1 \cdot h \cdot a)(1_2 \cdot k \cdot b) \\ &= (1_1 \cdot h \cdot a)(1_2 \cdot 1_A)(1_3 k \cdot b) \\ &= (1_1 \cdot (h \cdot a) 1_A)(1_2 k \cdot b) \\ &= (1_1 \cdot h \cdot a)(1_2 k \cdot b) \\ &= (1_1 h \cdot a)(1_2 k \cdot b) \end{aligned}$$

where the last equality follows by Lemmas 2.1.7 and 1.1.9. □

Lemma 2.1.9. *Let A be a partial H -module algebra, $a, b \in A$ and $z \in H$.*

(i) *If $z \in H_L$, then $(z \cdot a)b = z \cdot ab$.*

(ii) *If $z \in H_R$, then $a(z \cdot b) = z \cdot ab$.*

In particular, $(H_L \cdot A)$ is a right ideal of A and $(H_R \cdot A)$ is a left ideal of A .

Proof. (i) If $z \in H_L$,

$$\begin{aligned}
(z \cdot a)b &= (z \cdot a)(1_H \cdot b) \\
&\stackrel{(2.1.8)}{=} (1_1 z \cdot a)(1_2 \cdot b) \\
&\stackrel{(1.7)}{=} (z_1 \cdot a)(z_2 \cdot b) \\
&= z \cdot ab
\end{aligned}$$

(ii) If $z \in H_R$,

$$\begin{aligned}
a(z \cdot b) &= (1_H \cdot a)(z \cdot b) \\
&\stackrel{(2.1.8)}{=} (1_1 \cdot a)(1_2 z \cdot b) \\
&\stackrel{(1.8)}{=} (z_1 \cdot a)(z_2 \cdot b) \\
&= z \cdot ab
\end{aligned}$$

The last assertion is obvious. □

From the above lemma, we have the following immediate consequences.

Corollary 2.1.10. *Let A be a partial H -module algebra, $h, z \in H$ and $a \in A$.*

(i) *If $z \in H_L$, we have that $(z \cdot 1_A)(h \cdot a) = z \cdot (h \cdot a)$. If, in addition, the action is symmetric, then $z \cdot (h \cdot a) = zh \cdot a$.*

(ii) *If $z \in H_R$, then $(h \cdot a)(z \cdot 1_A) = zh \cdot a$* □

Lemma 2.1.11. *Let A be a partial H -module algebra. The following assertions hold for all $h, k \in H$ and $a, b \in A$:*

(i) $(h \cdot a)(k \cdot b) = h_1 \cdot (a(S(h_2)k \cdot b))$.

(ii) *If the action is symmetric and the antipode S is invertible then,*

$$(h \cdot a)(k \cdot b) = k_2 \cdot ((S^{-1}(k_1)h \cdot a)b).$$

Proof. Let $h, k \in H$ and $a, b \in A$. Then,

(i)

$$\begin{aligned}
h_1 \cdot a(S(h_2)k \cdot b) &= (h_1 \cdot a)(h_2 \cdot (S(h_3)k \cdot b)) \\
&= (h_1 \cdot a)(h_2 \cdot 1_A)(h_3 S(h_4)k \cdot b) \\
&= (h_1 \cdot a)(h_2 S(h_3)k \cdot b) \\
&\stackrel{(1.9)}{=} (1_1 h \cdot a)(1_2 k \cdot b) \\
&\stackrel{(2.1.8)}{=} (h \cdot a)(k \cdot b).
\end{aligned}$$

(ii) Since S is invertible, we obtain from (1.27) that

$$k_2 S^{-1}(k_1) \otimes k_3 = 1_1 \otimes 1_2 k, \tag{2.1}$$

for all $k \in H$.

Thus,

$$\begin{aligned}
k_2 \cdot (S^{-1}(k_1)h \cdot a)b &= (k_2 \cdot (S^{-1}(k_1)h \cdot a))(k_3 \cdot b) \\
&\stackrel{\text{(PMA4)}}{=} (k_2 S^{-1}(k_1)h \cdot a)(k_3 \cdot 1_A)(k_4 \cdot b) \\
&= (k_2 S^{-1}(k_1)h \cdot a)(k_3 \cdot b) \\
&\stackrel{\text{(2.1)}}{=} (1_1 h \cdot a)(1_2 k \cdot b) \\
&\stackrel{\text{(2.1.8)}}{=} (h \cdot a)(k \cdot b)
\end{aligned}$$

□

2.2 Partial groupoid actions

Partial groupoid actions were introduced in the literature by D. Bagio and A. Paques in [7]. Our main purpose in this section is to prove that, given a groupoid G such that the set G_0 of all its identities is finite, there is a one to one correspondence between the symmetric partial actions of the groupoid algebra $\mathbb{k}G$ on a G_0 -graded algebra A and the globalizable partial actions of the groupoid G on A (see Theorem 2.2.5).

Definition 2.2.1. A *groupoid* is a non-empty set G equipped with a partially defined binary operation, for which the usual axioms of a group hold whenever they make sense, that is:

- (i) For every $g, h, l \in G$, $g(hl)$ exists if and only if $(gh)l$ exists and in this case they are equal.
- (ii) For every $g, h, l \in G$, $g(hl)$ exists if and only if gh and hl exist.
- (iii) For each $g \in G$ there exist (unique) elements $d(g), r(g) \in G$ such that $gd(g)$ and $r(g)g$ exist and $gd(g) = g = r(g)g$.
- (iv) For each $g \in G$ there exists an element $g^{-1} \in G$ such that $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$.

The uniqueness of the element g^{-1} is an immediate consequence of the above definition, and $(g^{-1})^{-1} = g$, for all $g \in G$. The element gh exists if and only if $d(g) = r(h)$ if and only if exists $h^{-1}g^{-1}$ and, in this case, $(gh)^{-1} = h^{-1}g^{-1}$, $r(gh) = r(g)$ and $d(gh) = d(h)$.

An element $e \in G$ is said to be an *identity* of G if there exists $g \in G$ such that $e = d(g)$ (and so $e = r(g^{-1})$). Let G_0 denote the set of all identities of G . Note that $e = e^{-1} = d(e) = r(e)$, for all $e \in G_0$. For more about groupoid's properties we refer to [35].

Definition 2.2.2. [7] A *partial action* of a groupoid G on an algebra A is a pair

$$\alpha = (\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$$

where, for each $e \in G_0$ and $g \in G$, D_e is an ideal of A , D_g is an ideal of $D_{r(g)}$, and $\alpha_g: D_{g^{-1}} \rightarrow D_g$ is an algebra isomorphism such that:

- (i) α_e is the identity map I_{D_e} of D_e ,
- (ii) $\alpha_h^{-1}(D_{g^{-1}} \cap D_h) \subseteq D_{(gh)^{-1}}$,
- (iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for all $x \in \alpha_{h^{-1}}(D_{g^{-1}} \cap D_h)$,

for all $e \in G_0$ and $g, h \in G$ such that $d(g) = r(h)$.

For the proof of the following lemma see [7, Lemma 1.1].

Lemma 2.2.3. *Let $\alpha = (\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$ be a partial action of a groupoid G on an algebra A . Then,*

- (i) $\alpha_g^{-1} = \alpha_{g^{-1}}$, for all $g \in G$,
- (ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$, if $d(g) = r(h)$. □

Given a groupoid G , the groupoid algebra $\mathbb{k}G$ is a \mathbb{k} -vector space with basis $\{\delta_g \mid g \in G\}$, and multiplication given by the rule

$$\delta_g \delta_h = \begin{cases} \delta_{gh}, & \text{if } d(g) = r(h) \\ 0, & \text{otherwise} \end{cases}$$

for all $g, h \in G$. It is easy to see that $\mathbb{k}G$ is an algebra, and has an identity element, given by $1_{\mathbb{k}G} = \sum_{e \in G_0} \delta_e$, if and only if G_0 is finite [36]. Moreover, $\mathbb{k}G$ has a coalgebra structure given by

$$\Delta(\delta_g) = \delta_g \otimes \delta_g \quad \text{and} \quad \varepsilon(\delta_g) = 1_{\mathbb{k}},$$

for all $g \in G$. It is well known that $\mathbb{k}G$, with the algebra and coalgebra structures above described, and antipode S given by $S(\delta_g) = \delta_{g^{-1}}$, for all $g \in G$, is a weak Hopf algebra.

From now on we will assume that G_0 is finite and $\alpha = (\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$ is a partial action of G on $A = \bigoplus_{e \in G_0} D_e$. We also assume that each ideal D_g has a unit, denoted by 1_g . Notice that, in this case, each 1_g is a central element of A (in particular, D_g is also an ideal of A), and the conditions (ii) and (iii) of Definition 2.2.2 imply the following:

$$\alpha_g(\alpha_h(x1_{h^{-1}})1_{g^{-1}}) = \alpha_{gh}(x1_{(gh)^{-1}})1_g, \tag{2.2}$$

for all $x \in A$, whenever $d(g) = r(h)$.

Lemma 2.2.4. *With the notations and assumptions given above, the map*

$$\begin{aligned} \cdot : \mathbb{k}G \otimes A &\rightarrow A \\ \delta_g \otimes a &\mapsto \alpha_g(a1_{g^{-1}}) \end{aligned}$$

is a symmetric partial action of $\mathbb{k}G$ on A .

Proof. Indeed, \cdot is a well-defined linear map. Furthermore,

(i) for all $a \in A$,

$$\begin{aligned} 1_{\mathbb{k}G} \cdot a &= \sum_{e \in G_0} \delta_e \cdot a \\ &= \sum_{e \in G_0} \alpha_e(a1_{e^{-1}}) \\ &= \sum_{e \in G_0} a1_e \\ &= a1_A = a. \end{aligned}$$

(ii) for all $g \in G$ and $a, b \in A$,

$$\begin{aligned} \delta_g \cdot ab &= \alpha_g(ab1_{g^{-1}}) \\ &= \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}) \\ &= (\delta_g \cdot a)(\delta_g \cdot b). \end{aligned}$$

(iii) for all $g, h \in G$ and $a \in A$, if $d(g) \neq r(h)$ then $D_{g^{-1}} \cap D_h = 0 = \delta_g \delta_h$, and

$$\delta_g \cdot (\delta_h \cdot a) = \alpha_g(\alpha_h(a1_{h^{-1}})1_{g^{-1}}) = 0 = (\delta_g \cdot 1_A)(\delta_g \delta_h \cdot a).$$

Otherwise, if $d(g) = r(h)$ then

$$\begin{aligned} \delta_g \cdot (\delta_h \cdot a) &= \alpha_g(\alpha_h(a1_{h^{-1}})1_{g^{-1}}) \\ &\stackrel{(2.2)}{=} \alpha_{gh}(a1_{(gh)^{-1}})1_g \\ &= (\delta_g \cdot 1_A)(\delta_{gh} \cdot a) \\ &= (\delta_g \cdot 1_A)(\delta_g \delta_h \cdot a). \end{aligned}$$

The symmetry of \cdot is obvious for, as noticed above, $1_g = \delta_g \cdot 1_A$ is central in A , for all $g \in G$, and the groupoid algebra is cocommutative. \square

The converse of Lemma 2.2.4 is given in the following theorem, which in particular generalizes [32, Proposition 2.2].

Theorem 2.2.5. *Let A be an algebra and G a groupoid such that G_0 is finite. The following statements are equivalent:*

(i) *There exists a partial action $\alpha = (\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$ of G on A such that the ideals D_g are unital and $A = \bigoplus_{e \in G_0} D_e$.*

(ii) *A is a symmetric partial $\mathbb{k}G$ -module algebra.*

Proof. (i) \Rightarrow (ii) It follows from Lemma 2.2.4.

(ii) \Rightarrow (i) Let $D_g = \delta_g \cdot A$, $1_g = \delta_g \cdot 1_A$, and $\alpha_g: D_{g^{-1}} \rightarrow D_g$ given by $\alpha_g(x) = \delta_g \cdot x$, for all $g \in G$ and $x \in D_{g^{-1}}$. We will proceed by steps.

To show that $\alpha = (\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$ is a partial action of G on A we need to check that, for every $g \in G$ and $e \in G_0$, D_g is an ideal of $D_{r(g)}$, D_e is an ideal of A , and α_g is an algebra isomorphism, which will be done in the steps 1, 2, and 3. We also show in the step 1 that the ideals D_g , $g \in G$, are all unital. In the step 4, we show that the conditions (i)-(iii) of Definition 2.2.2 hold. Finally, in the step 5 we show that $A = \bigoplus_{e \in G_0} D_e$.

Step 1: First of all, 1_g is a central idempotent of A and $D_g = 1_g A$, which implies that D_g is a unital ideal of A , for all $g \in G$.

Indeed, $(1_g)^2 = (\delta_g \cdot 1_A)(\delta_g \cdot 1_A) = \delta_g \cdot 1_A = 1_g$, and

$$\begin{aligned}
1_g a &= (\delta_g \cdot 1_A) a \\
&= 1_{\mathbb{k}G} \cdot (\delta_g \cdot 1_A) a \\
&= \sum_{e \in G_0} (\delta_e \cdot \delta_g \cdot 1_A) (\delta_e \cdot a) \\
&= \sum_{e \in G_0} (\delta_e \delta_g \cdot 1_A) (\delta_e \cdot 1_A) (\delta_e \cdot a) \\
&= \sum_{e \in G_0} (\delta_e \delta_g \cdot 1_A) (\delta_e \cdot a) \\
&= (\delta_{r(g)} \delta_g \cdot 1_A) (\delta_{r(g)} \cdot a) \\
&= (\delta_{r(g)g} \cdot 1_A) (\delta_{r(g)} \cdot a) \\
&= (\delta_g \cdot 1_A) (\delta_g \delta_{g^{-1}} \cdot a) \\
&= \delta_g \cdot \delta_{g^{-1}} \cdot a \\
&= (\delta_g \delta_{g^{-1}} \cdot a) (\delta_g \cdot 1_A) \\
&= (\delta_{r(g)} \cdot a) (\delta_{r(g)g} \cdot 1_A) \\
&= (\delta_{r(g)} \cdot a) (\delta_{r(g)} \delta_g \cdot 1_A) \\
&= \sum_{e \in G_0} (\delta_e \cdot a) (\delta_e \delta_g \cdot 1_A) \\
&= \sum_{e \in G_0} \delta_e \cdot (a (\delta_g \cdot 1_A)) \\
&= 1_{\mathbb{k}G} \cdot (a (\delta_g \cdot 1_A)) \\
&= a (\delta_g \cdot 1_A) \\
&= a 1_g.
\end{aligned}$$

Note that the above sequence of equalities gives an important and useful relation for the partial action of G on A , that is,

$$(\delta_g \cdot 1_A) a = \delta_g \cdot \delta_{g^{-1}} \cdot a = a (\delta_g \cdot 1_A), \quad (2.3)$$

for all $g \in G$ and $a \in A$, which implies

$$D_g = \delta_g \cdot A = (\delta_g \cdot 1_A) (\delta_g \cdot A) \subseteq 1_g A = (\delta_g \cdot 1_A) A \stackrel{(2.3)}{=} \delta_g \cdot \delta_{g^{-1}} \cdot A \subseteq \delta_g \cdot A = D_g,$$

hence $D_g = 1_g A$. The last assertion is immediate.

Step 2: $D_g = D_{r(g)} 1_g$, in particular D_g is an ideal of $D_{r(g)}$, for all $g \in G$.

It follows from (2.3) and the symmetry of \cdot that

$$\begin{aligned}
D_g &= (\delta_g \cdot 1_A)A \\
&\stackrel{(2.3)}{=} \delta_g \cdot \delta_{g^{-1}} \cdot A \\
&= (\delta_g \delta_{g^{-1}} \cdot A)(\delta_g \cdot 1_A) \\
&= (\delta_{r(g)} \cdot A)(\delta_g \cdot 1_A) \\
&= D_{r(g)}1_g.
\end{aligned}$$

Step 3: α_g is an isomorphism of algebras, for all $g \in G$.

It is clear from the above that α_g is a well-defined linear map. Thus, it is enough to show that α_g is multiplicative and $\alpha_g^{-1} = \alpha_{g^{-1}}$, for all $g \in G$.

For all $a, b \in A$, we have

$$\begin{aligned}
\alpha_g(ab1_{g^{-1}}) &= \delta_g \cdot ab1_{g^{-1}} \\
&= (\delta_g \cdot a1_{g^{-1}})(\delta_g \cdot b1_{g^{-1}}) \\
&= \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}).
\end{aligned}$$

In order to show that α_g is an isomorphism, we need first to show that α_e is the identity in D_e for all $e \in G_0$.

Notice that for any $h \in G$ and $a, b \in A$,

$$(\delta_h \cdot a)b = (\delta_h \cdot a)(\delta_h \cdot 1_A)b \stackrel{(2.3)}{=} (\delta_h \cdot a)(\delta_h \cdot \delta_{h^{-1}} \cdot b) = \delta_h \cdot (a(\delta_{h^{-1}} \cdot b)). \quad (2.4)$$

Therefore, for $e \in G_0$ and $a \in A$ we have

$$\begin{aligned}
a1_e &= (1_{\mathbb{k}G} \cdot a)1_e \\
&= \sum_{e' \in G_0} (\delta_{e'} \cdot a)(\delta_e \cdot 1_A) \\
&\stackrel{(2.4)}{=} \sum_{e' \in G_0} \delta_{e'} \cdot (a(\delta_{e'} \cdot \delta_e \cdot 1_A)) \quad (\text{taking } b = \delta_e \cdot 1_A) \\
&= \sum_{e' \in G_0} \delta_{e'} \cdot (a(\delta_{e'} \delta_e \cdot 1_A)(\delta_{e'} \cdot 1_A)) \\
&= \delta_e \cdot (a(\delta_e \cdot 1_A)) \\
&= \delta_e \cdot a1_e \\
&= \alpha_e(a1_e).
\end{aligned}$$

Finally, for all $a \in D_g$ we have

$$\begin{aligned}
\alpha_g(\alpha_{g^{-1}}(a)) &= \delta_g \cdot (\delta_{g^{-1}} \cdot a) \\
&= (\delta_g \delta_{g^{-1}} \cdot a)(\delta_g \cdot 1_A) \\
&= (\delta_{r(g)} \cdot a)1_g \\
&= (\delta_{r(g)} \cdot a1_{r(g)})1_g \\
&= a1_{r(g)}1_g \\
&= a.
\end{aligned}$$

For the equality $\alpha_{g^{-1}}\alpha_g(a) = a$, for all $a \in D_{g^{-1}}$, one proceeds in a similar way.

Step 4: $\alpha = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ is a partial groupoid action of G on A .

We only need to check that α satisfies the conditions (i)-(iii) of Definition 2.2.2.

(i) It follows from Step 3.

(ii) For all $h, g \in G$ such that $d(g) = r(h)$ and $a \in A$, we have

$$\begin{aligned} \alpha_{h^{-1}}(a1_{g^{-1}}1_h) &= \alpha_{h^{-1}}(a1_h)\alpha_{h^{-1}}(1_{g^{-1}}1_h) \\ &\stackrel{(2.2)}{=} \alpha_{h^{-1}}(a1_h)1_{h^{-1}}1_{(gh)^{-1}} \in D_{(gh)^{-1}}, \end{aligned}$$

thus $\alpha_{h^{-1}}(D_{g^{-1}} \cap D_h) \subseteq D_{(gh)^{-1}}$.

(iii) It follows from (ii) that if $x \in \alpha_{h^{-1}}(D_{g^{-1}} \cap D_h)$ then $\alpha_h(x) \in D_{g^{-1}}$ and $x = a1_{h^{-1}}1_{(gh)^{-1}}$, for some $a \in A$. Hence, the elements $\alpha_g\alpha_h(x)$ and $\alpha_{gh}(x)$ exist and lie in $D_g \cap D_{gh}$, and

$$\begin{aligned} \alpha_g(\alpha_h(x)) &= \alpha_g(\delta_h \cdot a1_{h^{-1}}1_{(gh)^{-1}}) \\ &= \delta_g \cdot (\delta_h \cdot a1_{h^{-1}}1_{(gh)^{-1}}) \\ &= (\delta_g\delta_h \cdot a1_{h^{-1}}1_{(gh)^{-1}})(\delta_g \cdot 1_A) \\ &= (\delta_{gh} \cdot a1_{h^{-1}}1_{(gh)^{-1}})(\delta_g \cdot 1_A) \\ &= \alpha_{gh}(x)1_g \\ &= \alpha_{gh}(x). \end{aligned}$$

Step 5: $A = \bigoplus_{e \in G_0} D_e$.

Indeed, notice that

$$A = 1_{\mathbb{k}G} \cdot A = \sum_{e \in G_0} \delta_e \cdot A = \sum_{e \in G_0} D_e,$$

and, since

$$1_e 1_f = (\delta_e \cdot 1_A)(\delta_f \cdot 1_A) \stackrel{(2.3)}{=} \delta_e \cdot (\delta_e \cdot \delta_f \cdot 1_A) = \delta_e \cdot ((\delta_e \cdot 1_A)(\delta_e \delta_f \cdot 1_A)) = 0,$$

for all $e \neq f$ in G_0 , it easily follows that $D_e \cap (\sum_{\substack{f \in G_0 \\ f \neq e}} D_f) = 0$. □

2.3 Partial actions on the ground field

Partial actions of a weak Hopf algebra H on the ground field \mathbb{k} provide a large amount of examples of partial actions. In this section we give the necessary and sufficient conditions for an action of H on \mathbb{k} to be partial and, as an application, we describe all the partial actions of a groupoid algebra on \mathbb{k} . The reader is invited to compare the results presented here with the results in [4].

It is clear that any action of H on \mathbb{k} is, in particular, a \mathbb{k} -linear map from H on \mathbb{k} . The question is: under what conditions a \mathbb{k} -linear map from H on \mathbb{k} defines a partial action of H on \mathbb{k} ? The answer is given in the following lemma.

Lemma 2.3.1. *Let $\lambda: H \rightarrow \mathbb{k}$ be a \mathbb{k} -linear map. Then, λ defines a partial action of H on \mathbb{k} , via*

$$h \cdot 1_{\mathbb{k}} = \lambda(h), \quad \text{for all } h \in H,$$

if and only if

$$\lambda(1_H) = 1_{\mathbb{k}} \quad \text{and} \quad \lambda(h)\lambda(g) = \lambda(h_1)\lambda(h_2g), \quad \text{for all } g, h \in H.$$

Proof. Assume that \cdot is a partial action. Then,

$$\lambda(1_H) = 1_H \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}}$$

and

$$\lambda(h)\lambda(g) = h \cdot (g \cdot 1_{\mathbb{k}}) = (h_1 \cdot 1_{\mathbb{k}})(h_2g \cdot 1_{\mathbb{k}}) = \lambda(h_1)\lambda(h_2g).$$

Conversely, note that $1_H \cdot 1_{\mathbb{k}} = \lambda(1_H) = 1_{\mathbb{k}}$ and $h \cdot (g \cdot 1_{\mathbb{k}}) = \lambda(h)\lambda(g) = \lambda(h_1)\lambda(h_2g) = (h_1 \cdot 1_{\mathbb{k}})(h_2g \cdot 1_{\mathbb{k}})$. Taking $g = 1_H$ in this last equality we have the third required condition. \square

It is well known that, in the setting of Hopf algebra actions, the only global action on \mathbb{k} is given by the counit ε . This is not true for actions of weak Hopf algebras. In the following proposition we will give necessary and sufficient conditions to obtain a global action of a weak Hopf algebra H on \mathbb{k} .

Proposition 2.3.2. *Let $\lambda: H \rightarrow \mathbb{k}$ be a \mathbb{k} -linear map and $\triangleright: H \otimes \mathbb{k} \rightarrow \mathbb{k}$ the \mathbb{k} -linear map given by $h \triangleright 1_{\mathbb{k}} = \lambda(h)$. Then, \triangleright is a global action of H on \mathbb{k} if and only if λ is a convolutional idempotent in $\text{Alg}_{\mathbb{k}}(H, \mathbb{k}) = \{f \in \text{Hom}_{\mathbb{k}}(H, \mathbb{k}) \mid f \text{ is multiplicative}\}$.*

Proof. Assume that \triangleright is global. Then, $\lambda(h)\lambda(g) = h \triangleright g \triangleright 1_{\mathbb{k}} = hg \triangleright 1_{\mathbb{k}} = \lambda(hg)$ and $\lambda(1_H) = 1_H \triangleright 1_{\mathbb{k}} = 1_{\mathbb{k}}$, for all $g, h \in H$. Thus, $\lambda \in \text{Alg}_{\mathbb{k}}(H, \mathbb{k})$. Moreover, $\lambda * \lambda(h) = \lambda(h_1)\lambda(h_2) = (h_1 \triangleright 1_{\mathbb{k}})(h_2 \triangleright 1_{\mathbb{k}}) = h \triangleright 1_{\mathbb{k}} = \lambda(h)$, for all $h \in H$, that is, $\lambda * \lambda = \lambda$.

Conversely, since λ is a map of \mathbb{k} -algebras, for all a, b in \mathbb{k} and g, h in H , we have

$$(i) \quad 1_H \triangleright 1_{\mathbb{k}} = \lambda(1_H) = 1_{\mathbb{k}},$$

$$(ii) \quad h \triangleright ab = ab\lambda(h) = ab(\lambda * \lambda)(h) = (h_1 \triangleright a)(h_2 \triangleright b),$$

$$(iii) \quad h \triangleright g \triangleright a = a\lambda(h)\lambda(g) = a\lambda(hg) = hg \triangleright a.$$

\square

The example below illustrates this previous result.

Example 2.3.3. *Let G be a groupoid given by a disjoint union of finite groups G_1, \dots, G_n . Choose G_j one of these subgroups and define $\lambda: \mathbb{k}G \rightarrow \mathbb{k}$ by $\lambda(g) = 1$ if $g \in G_j$ and $\lambda(g) = 0$ otherwise. It is straightforward to check that λ is a convolutional idempotent in $\text{Alg}_{\mathbb{k}}(\mathbb{k}G, \mathbb{k})$. \square*

Note that the counit ε of a weak Hopf algebra is not an algebra homomorphism, so it does not turn \mathbb{k} on an H -module algebra. The next proposition gives a necessary and sufficient condition for ε to define a partial (so, global) action on \mathbb{k} .

Proposition 2.3.4. *A weak Hopf algebra H is a Hopf algebra if and only if ε defines a partial action on \mathbb{k} . In this case, ε is the unique convolutional idempotent in $\text{Alg}_{\mathbb{k}}(H, \mathbb{k})$.*

Proof. If H is a Hopf algebra, then ε is an idempotent element in $\text{Alg}_{\mathbb{k}}(H, \mathbb{k})$ and so it defines an action on \mathbb{k} . Conversely, if ε defines a partial action on \mathbb{k} , then $\varepsilon(1_H) = 1_{\mathbb{k}}$ and $\varepsilon(h)\varepsilon(g) = \varepsilon(h_1)\varepsilon(h_2g) = \varepsilon(\varepsilon(h_1)h_2g) = \varepsilon(hg)$, which implies that H is a Hopf algebra.

For the last assertion, it is enough to see that if H is a Hopf algebra then $\text{Alg}_{\mathbb{k}}(H, \mathbb{k})$ is a group with the convolution product and ε is its unit. \square

We end this section presenting a complete description of all partial actions of a groupoid algebra $\mathbb{k}G$ on \mathbb{k} , as well as, all partial actions of $\mathbb{k}G^*$ on \mathbb{k} when G is a finite groupoid.

Let G be a groupoid. We say that a set $V \subset G$ is a group in G if V is a group with the operation of G .

Proposition 2.3.5. *Let G be a groupoid such that G_0 is finite, and $\lambda: \mathbb{k}G \rightarrow \mathbb{k}$ a \mathbb{k} -linear map. Then, λ defines a partial action of $\mathbb{k}G$ on \mathbb{k} , as characterized in Lemma 2.3.1, if and only if the set*

$$V = \{v \in G \mid \delta_v \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}} = \delta_{r(v)} \cdot 1_{\mathbb{k}}\}$$

is a group in G and $\delta_g \cdot 1_{\mathbb{k}} = 0$, for all $g \in G \setminus V$.

Proof. Assume that λ defines a partial action on \mathbb{k} , given by $\delta_g \cdot 1_{\mathbb{k}} = \lambda(g)$, for all $g \in G$. Then, it follows from the equality

$$(\delta_g \cdot 1_{\mathbb{k}})(\delta_h \cdot 1_{\mathbb{k}}) = (\delta_g \cdot 1_{\mathbb{k}})(\delta_g \delta_h \cdot 1_{\mathbb{k}}),$$

that

$$\begin{aligned} \delta_g \cdot 1_{\mathbb{k}} &= (\delta_g \cdot 1_{\mathbb{k}})(1_{\mathbb{k}G} \cdot 1_{\mathbb{k}}) \\ &= \sum_{e \in G_0} (\delta_g \cdot 1_{\mathbb{k}})(\delta_e \cdot 1_{\mathbb{k}}) \\ &= \sum_{e \in G_0} (\delta_g \cdot 1_{\mathbb{k}})(\delta_g \delta_e \cdot 1_{\mathbb{k}}) \\ &= (\delta_g \cdot 1_{\mathbb{k}})(\delta_g \delta_{d(g)} \cdot 1_{\mathbb{k}}) \\ &= (\delta_g \cdot 1_{\mathbb{k}})^2. \end{aligned}$$

Thus, $\delta_g \cdot 1_{\mathbb{k}}$ is an idempotent in \mathbb{k} , and therefore equal to either $1_{\mathbb{k}}$ or 0 , for all $g \in G$. Furthermore, the equality $\lambda(1_{\mathbb{k}G}) = 1_{\mathbb{k}}$ ensures that $V \neq \emptyset$.

Now, we show that V is a group.

(i) For all $g, h \in V$, the product gh exists and lies in V . This is an immediate consequence of the following expression

$$1_{\mathbb{k}} = \lambda(\delta_g)\lambda(\delta_h) = \lambda(\delta_g)\lambda(\delta_g \delta_h) = \lambda(\delta_g \delta_h).$$

(ii) For all $g \in V$, the element g^{-1} lies in V . Indeed, since $g \in V$ we have

$$\lambda(\delta_{g^{-1}}) = \lambda(\delta_g)\lambda(\delta_{g^{-1}}) = \lambda(\delta_g)\lambda(\delta_{gg^{-1}}) = \lambda(\delta_g)\lambda(\delta_{r(g)}) = 1_{\mathbb{k}}.$$

Conversely, assume that V is group and let e_V denote its identity element. Also, assume that $\delta_g \cdot 1_{\mathbb{k}} = 0$, for all $g \in G \setminus V$. Under these assumptions we have, in particular, that $r(g) = d(g) = e_V$, for all $g \in V$, and $\delta_e \cdot 1_{\mathbb{k}} = 0$, for all $e \in G_0$, $e \neq e_V$. Thus, $\lambda(1_{\mathbb{k}G}) = \sum_{e \in G_0} \lambda(\delta_e) = \lambda(\delta_{e_V}) = 1_{\mathbb{k}}$, and it is straightforward to check that $\lambda(\delta_g)\lambda(\delta_h) = \lambda(\delta_g)\lambda(\delta_g\delta_h)$. \square

In the above proposition, if we assume \mathbb{k} a field of characteristic 0, then $\delta_v \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}}$ implies $\delta_{r(v)} \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}}$. Later in Section 3.2 we will suppose that this characteristic is 0 to simplify the calculations. Similar results can be obtained for any field.

Example 2.3.6. Let $G = G_1 \cup G_2$ be the groupoid given by the disjoint union of two groups G_1 and G_2 . Any subgroup V of G_1 (or G_2) defines a partial action of $\mathbb{k}G$ on \mathbb{k} , given by $\lambda(\delta_g) = \delta_{g,V}$, for all g in G , where $\delta_{g,V} = 1_{\mathbb{k}}$ if $g \in V$ and $\delta_{g,V} = 0$ otherwise.

Proposition 2.3.7. Let G be a finite groupoid, $\mathbb{k}G^*$ the dual Algebra of $\mathbb{k}G$ with basis $\{p_g \mid g \in G\}$ and $\cdot : \mathbb{k}G^* \otimes \mathbb{k} \rightarrow \mathbb{k}$ a \mathbb{k} -linear map. Take the set $V = \{v \in G \mid p_{v^{-1}} \cdot 1_{\mathbb{k}} \neq 0 \text{ and } p_v \cdot 1_{\mathbb{k}} \neq 0\}$ and suppose that the characteristic of the field does not divide the cardinality of V . Then \cdot is a partial action of $\mathbb{k}G^*$ on \mathbb{k} if and only if V is a group in G , and in this case it is defined by

$$p_g \cdot 1_{\mathbb{k}} = \begin{cases} \frac{1}{|V|}, & \text{if } g \in V, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Supposing that λ defines a partial action, then for any $g, h \in V$ we have that

$$\begin{aligned} p_g \cdot (p_h \cdot 1_{\mathbb{k}}) &= \sum_{l \in G} (p_l \cdot 1_{\mathbb{k}})((p_{l^{-1}g} * p_h) \cdot 1_{\mathbb{k}}) \\ &= (p_{gh^{-1}} \cdot 1_{\mathbb{k}})(p_h \cdot 1_{\mathbb{k}}) \end{aligned}$$

and, on the other side, since $p_h \cdot 1_{\mathbb{k}}$ lies in \mathbb{k}

$$p_g \cdot (p_h \cdot 1_{\mathbb{k}}) = (p_g \cdot 1_{\mathbb{k}})(p_h \cdot 1_{\mathbb{k}}).$$

But $h \in V$ means that $p_h \cdot 1_{\mathbb{k}} \neq 0$. Hence $p_g \cdot 1_{\mathbb{k}} = p_{gh^{-1}} \cdot 1_{\mathbb{k}}$ for any $g, h \in V$.

Therefore, it is clear that there exists the product of any two elements in V and it lies in V .

So, the above stated says that if g lies in V (and so g^{-1}) then

$$p_{g^{-1}} \cdot 1_{\mathbb{k}} = p_{g^{-1}g} \cdot 1_{\mathbb{k}}$$

which means that $d(g)$ also lies in V .

Moreover, there is a unique unit in V because for any $g, h \in V$, $d(g) = r(h^{-1}) = d(h)$. Then V is a group in G .

It just remains to show that $p_g \cdot 1_{\mathbb{k}} = \frac{1}{|V|}$. It follows directly from

$$\begin{aligned} 1_{\mathbb{k}G^*} \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}} &\Leftrightarrow \sum_{g \in G} p_g \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}} \\ &\Leftrightarrow \sum_{g \in V} p_g \cdot 1_{\mathbb{k}} = 1_{\mathbb{k}} \end{aligned}$$

and since $p_g \cdot 1_{\mathbb{k}} = p_h \cdot 1_{\mathbb{k}}$ for any $g, h \in V$, we have the desired result.

The converse follows by a simple calculation. \square

2.4 Globalization of partial actions

In this section we show that any partial action of a weak Hopf algebra can be obtained from a global one. Particularly, in this section, the notations \cdot for partial actions and \triangleright for global ones are crucial.

First of all, given a global action we will see how to construct a partial one from it. The method to do this is given in the following lemma.

Lemma 2.4.1. *Let B be an H -module algebra via $\triangleright: H \otimes B \rightarrow B$. Let A be a right ideal of B which is also an algebra with unit 1_A . Then, the \mathbb{k} -linear map $\cdot: H \otimes A \rightarrow A$ given by*

$$h \cdot a = 1_A(h \triangleright a)$$

is a partial action of H on A .

Proof. For every $a, b \in A$ and $g, h \in H$, we have

- (i) $1_H \cdot a = 1_A(1_H \triangleright a) = 1_A a = a$.
- (ii) $h \cdot (ab) = 1_A(h \triangleright ab) = 1_A(h_1 \triangleright a)(h_2 \triangleright b) = 1_A(h_1 \triangleright a)1_A(h_2 \triangleright b) = (h_1 \cdot a)(h_2 \cdot b)$.
- (iii) $h \cdot (g \cdot a) = 1_A(h \triangleright (g \cdot a)) = 1_A(h \triangleright 1_A(g \triangleright a)) = 1_A(h_1 \triangleright 1_A)(h_2 g \triangleright a) = 1_A(h_1 \triangleright 1_A)1_A(h_2 g \triangleright a) = (h_1 \cdot 1_A)(h_2 g \cdot a)$.

\square

The partial action \cdot of H on A , obtained by the method given above, is called *induced* by the action \triangleright .

Definition 2.4.2. Let A be a partial H -module algebra. We say that a pair (B, θ) is a *globalization* of A if B is an H -module algebra via $\triangleright: H \otimes B \rightarrow B$, and

- (i) $\theta: A \rightarrow B$ is a monomorphism of algebras such that $\theta(A)$ is a right ideal of B ,

(ii) the partial action on A is equivalent to the partial action induced by \triangleright on $\theta(A)$, that is, $\theta(h \cdot a) = h \cdot \theta(a) = \theta(1_A)(h \triangleright \theta(a))$,

(iii) B is the H -module algebra generated by $\theta(A)$, that is, $B = H \triangleright \theta(A)$.

Notice that in the above definition as well as in Lemma 2.4.1 we do not need to require B to be unital.

The existence of such a globalization will be ensured by the construction presented in the sequel.

We start by taking the convolution algebra $\mathcal{F} = \text{Hom}(H, A)$, which is an H -module algebra with the action given by $(h \triangleright f)(k) = f(kh)$, for all $f \in \mathcal{F}$ and $h, k \in H$. Let $\varphi: A \rightarrow \mathcal{F}$ be the map given by $\varphi(a): h \mapsto h \cdot a$, for all $a \in A$ and $h \in H$. Put $B = H \triangleright \varphi(A)$.

Proposition 2.4.3. *The pair (B, φ) is a globalization of A .*

Proof.

(i) φ is an algebra monomorphism such that $\varphi(h \cdot a) = \varphi(1_A) * (h \triangleright \varphi(a))$, for all $h \in H$ and $a \in A$. Indeed,

- φ is clearly \mathbb{k} -linear,
- φ is injective because $1_H \cdot a = a$,
- $\varphi(ab)(h) = h \cdot ab = (h_1 \cdot a)(h_2 \cdot b) = \varphi(a)(h_1)\varphi(b)(h_2) = [\varphi(a) * \varphi(b)](h)$, for all $a, b \in A$ and $h \in H$,
- $(\varphi(1_A) * (h \triangleright \varphi(a)))(k) = (k_1 \cdot 1_A)(k_2 h \cdot a) = k \cdot h \cdot a = \varphi(h \cdot a)(k)$, for all $k \in H$.

(ii) $\varphi(A)$ is a right ideal of B . Indeed,

$$\begin{aligned} \varphi(b) * (h \triangleright \varphi(a)) &= \varphi(b) * \varphi(1_A) * (h \triangleright \varphi(a)) \\ &= \varphi(b) * \varphi(h \cdot a) \\ &= \varphi(b(h \cdot a)), \end{aligned}$$

for all $a, b \in A$ and $h \in H$.

(iii) B is an H -module algebra.

In fact, B is clearly a vector subspace of \mathcal{F} which is invariant under the action \triangleright , and

$$\begin{aligned} (h \triangleright \varphi(a)) * (k \triangleright \varphi(b)) &\stackrel{(2.1.11)}{=} h_1 \triangleright (\varphi(a) * (S(h_2)k \triangleright \varphi(b))) \\ &= h_1 \triangleright (\varphi(a) * \varphi(1_A) * (S(h_2)k \triangleright \varphi(b))) \\ &= h_1 \triangleright (\varphi(a) * \varphi(S(h_2)k \cdot b)) \\ &= h_1 \triangleright (\varphi(a(S(h_2)k \cdot b))), \end{aligned}$$

for all $a, b \in A$ and $h, k \in H$. □

The pair (B, φ) , as constructed above, is called the *standard globalization* of A .

Proposition 2.4.4. *With the above notations, a partial action on A is symmetric if and only if $\varphi(A)$ is an ideal of B .*

Proof. Suppose that a partial action of H on A is symmetric. Then,

$$\begin{aligned}
((h \triangleright \varphi(a)) * \varphi(b))(k) &= (h \triangleright \varphi(a))(k_1) \varphi(b)(k_2) \\
&= \varphi(a)(k_1 h) \varphi(b)(k_2) \\
&= (k_1 h \cdot a)(k_2 \cdot b) \\
&\stackrel{(\text{PMA4})}{=} k \cdot ((h \cdot a) b) \\
&= \varphi((h \cdot a) b)(k)
\end{aligned}$$

for all $h, k \in H$ and $a, b \in A$. As $\varphi(A)$ is a right ideal of B , the required follows.

Conversely, since $\varphi(A)$ is an ideal of B we have that $\varphi(1_A)$ is central in B . Then,

$$\begin{aligned}
k \cdot (h \cdot a) &= \varphi(h \cdot a)(k) \\
&= (h \cdot \varphi(a))(k) \\
&= (\varphi(1_A) * (h \triangleright \varphi(a)))(k) \\
&= ((h \triangleright \varphi(a)) * \varphi(1_A))(k) \\
&= (h \triangleright \varphi(a))(k_1) \varphi(1_A)(k_2) \\
&= \varphi(a)(k_1 h) \varphi(1_A)(k_2) \\
&= (k_1 h \cdot a)(k_2 \cdot 1_A)
\end{aligned}$$

for all $h, k \in H$ and $a \in A$. □

By a homomorphism between two globalizations of a same partial H -module algebra we mean a multiplicative linear map that commutes with the respective actions. If such a homomorphism is bijective we say that such globalizations are *isomorphic*.

Proposition 2.4.5. *With the above notations, any globalization of A is a homomorphic preimage of the standard one.*

Proof. Let (B', θ) be a globalization of the partial H -module algebra A and define the following map

$$\begin{aligned}
\Phi: \quad B' &\rightarrow B \\
\sum_{i=0}^n h_i \blacktriangleright \theta(a_i) &\mapsto \sum_{i=0}^n h_i \triangleright \varphi(a_i)
\end{aligned}$$

In order to prove that Φ is well-defined, it is enough to check that if $\sum_{i=0}^n h_i \blacktriangleright \theta(a_i) = 0$

then $\sum_{i=0}^n h_i \triangleright \varphi(a_i) = 0$. Assume that $\sum_{i=0}^n h_i \blacktriangleright \theta(a_i) = 0$. Then, for all $k \in H$ we have

$$\begin{aligned}
0 &= \theta(1_A)(k \blacktriangleright \sum_{i=0}^n h_i \blacktriangleright \theta(a_i)) \\
&= \theta(1_A)(\sum_{i=0}^n kh_i \blacktriangleright \theta(a_i)) \\
&= \sum_{i=0}^n kh_i \cdot \theta(a_i) \\
&= \theta\left(\sum_{i=0}^n kh_i \cdot a_i\right)
\end{aligned}$$

and, as θ is injective, we get $\sum_{i=0}^n kh_i \cdot a_i = 0$.

Hence, for any $k \in H$,

$$\begin{aligned}
\left(\sum_{i=0}^n h_i \triangleright \varphi(a_i)\right)(k) &= \sum_{i=0}^n \varphi(a_i)(kh_i) \\
&= \sum_{i=0}^n kh_i \cdot a_i \\
&= 0.
\end{aligned}$$

Clearly, Φ is surjective and $\Phi(g \blacktriangleright b') = g \triangleright \Phi(b')$, for all $b' \in B'$ and $g \in H$.

Finally, for all $h, k \in H$ and $a, b \in A$,

$$\begin{aligned}
\Phi((h \blacktriangleright \theta(a))(k \blacktriangleright \theta(b))) &= \Phi(1_H \blacktriangleright ((h \blacktriangleright \theta(a))(k \blacktriangleright \theta(b)))) \\
&= \Phi((1_1 \blacktriangleright h \blacktriangleright \theta(a))(1_2 \blacktriangleright k \blacktriangleright \theta(b))) \\
&= \Phi((1_1 h \blacktriangleright \theta(a))(1_2 k \blacktriangleright \theta(b))) \\
&= \Phi((h_1 \blacktriangleright \theta(a))(h_2 S(h_3) k \blacktriangleright \theta(b))) \\
&= \Phi(h_1 \blacktriangleright (\theta(a)(S(h_2) k \blacktriangleright \theta(b)))) \\
&= \Phi(h_1 \blacktriangleright \theta(a(S(h_2) k \cdot b))) \\
&= h_1 \triangleright (\varphi(a(S(h_2) k \cdot b))) \\
&= h_1 \triangleright (\varphi(a) * (S(h_2) k \triangleright \varphi(b))) \\
&= (h_1 \triangleright \varphi(a)) * (h_2 S(h_3) k \triangleright \varphi(b)) \\
&= (1_1 h \triangleright \varphi(a)) * (1_2 k \triangleright \varphi(b)) \\
&= (1_1 \triangleright h \triangleright \varphi(a)) * (1_2 \triangleright k \triangleright \varphi(b)) \\
&= 1_H \triangleright ((h \triangleright \varphi(a)) * (k \triangleright \varphi(b))) \\
&= \Phi(h \blacktriangleright \theta(a)) * \Phi(k \blacktriangleright \theta(b)).
\end{aligned}$$

□

Definition 2.4.6. Let (B, θ) be a globalization of a partial H -module algebra A . We say that B is *minimal* if for every H -submodule M of B such that $\theta(1_A)M = 0$ we have $M = 0$.

The concept of minimal globalization was introduced by M. Alves and E. Batista in [2], where they shown that the globalization for a partial action of the group algebra are ever minimal.

Proposition 2.4.7. *The standard globalization (B, φ) of A is minimal.*

Proof. It is enough to prove that the minimal condition holds for any cyclic submodule of B . Let $m = \sum_{i=0}^n h_i \triangleright \varphi(a_i)$ be an element in B . Suppose that $\varphi(1_A) * \langle m \rangle = 0$, where $\langle m \rangle$ is the H -submodule of B generated by m , that is, $\langle m \rangle = H \triangleright m$.

Then, for all $k \in H$,

$$\begin{aligned} 0 &= \varphi(1_A) * (k \triangleright m) \\ &= \varphi(1_A) * \left(k \triangleright \sum_{i=0}^n h_i \triangleright \varphi(a_i) \right) \\ &= \varphi(1_A) * \left(\sum_i k h_i \triangleright \varphi(a_i) \right) \\ &= \left(\sum_i k h_i \cdot \varphi(a_i) \right) \\ &= \varphi \left(\sum_i k h_i \cdot a_i \right) \end{aligned}$$

which implies $\sum_i k h_i \cdot a_i = 0$, since φ is a monomorphism.

Since $m \in B \subseteq \sum_i Hom(H, A)$ we have

$$\begin{aligned} m(k) &= \left(\sum_{i=0}^n h_i \triangleright \varphi(a_i) \right)(k) \\ &= \left(\sum_i \varphi(a_i) \right)(k h_i) \\ &= \sum_i k h_i \cdot a_i \\ &= 0 \end{aligned}$$

for all $k \in H$. Therefore, $m = 0$. □

Proposition 2.4.8. *Any two minimal globalizations of a partial H -module algebra A are isomorphic.*

Proof. Let (B', θ) be a minimal globalization of A , (B, φ) the standard one, and $\Phi: B' \rightarrow B$ as defined in Proposition 2.4.5. It is enough to prove that Φ is injective.

Suppose $\Phi(\sum_i h_i \triangleright \theta(a_i)) = 0$. Thus, $0 = (\sum_i h_i \triangleright \varphi(a_i))(g) = \sum_i g h_i \cdot a_i$, for all $g \in H$, and so $0 = \theta(\sum_i g h_i \cdot a_i) = \sum_i g h_i \cdot \theta(a_i) = \theta(1_A)(\sum_i g h_i \triangleright \theta(a_i)) = \theta(1_A)(g \triangleright \sum_i h_i \triangleright \theta(a_i))$.

Now, if M denotes the H -submodule of B' generated by $\sum_i h_i \triangleright \theta(a_i)$ we have that $\theta(1_A)M = 0$, hence $M = 0$. Therefore, Φ is injective. □

We end this section summarizing all the above main results in the following theorem.

Theorem 2.4.9. *Let A be a partial H -module algebra.*

- (i) *A has a minimal globalization.*
- (ii) *Any two minimal globalizations of A are isomorphic.*
- (iii) *Any globalization of A is a homomorphic preimage of a minimal one.*

□

2.5 Partial smash product

In this section we construct the smash product for a partial H -module algebra.

The smash product already exists for an H -module algebra and even for a partial H -module algebra when H is a Hopf algebra. So, it is natural to ask if it still works for a partial H -module algebra when H is a weak Hopf algebra. The hard task here is to get the good definition of smash product. In fact, the smash product for Hopf algebra actions (partial or global) is, by construction, a tensor product over the ground field, which makes it easy to show that it is well-defined. This does not occur when dealing with weak Hopf algebra actions because the tensor product, in this case, is not anymore over the ground field but over the algebra H_L .

In order to get our aim we need first a right H_L -module structure for a partial H -module algebra.

Proposition 2.5.1. *Let A be a partial H -module algebra. Then, A is a right H_L -module via $a \triangleleft z = S_R^{-1}(z) \cdot a = a(S_R^{-1}(z) \cdot 1_A)$, for all $a \in A$ and $z \in H_L$.*

Proof. As $1_H \in H_L$ and $S_R^{-1}(1_H) = 1_H$, it follows that $a \triangleleft 1_H = 1_H \cdot a = a$.

Let $a \in A$, $h, g \in H_L$, so

$$\begin{aligned}
 (a \triangleleft h) \triangleleft g &= S_R^{-1}(g) \cdot (S_R^{-1}(h) \cdot a) \\
 &= (S_R^{-1}(g)_1 \cdot 1_A)(S_R^{-1}(g)_2 S_R^{-1}(h) \cdot a) \\
 &\stackrel{(1.8)}{=} (1_1 \cdot 1_A)(1_2 S_R^{-1}(g) S_R^{-1}(h) \cdot a) \\
 &= 1_H \cdot (S_R^{-1}(hg) \cdot a) \\
 &= a \triangleleft hg.
 \end{aligned}$$

The equality $S_R^{-1}(z) \cdot a = a(S_R^{-1}(z) \cdot 1_A)$ holds by 2.1.9(ii). □

Notice that the action, by restriction, of H_R on a partial H -module algebra usually behaves like a global action. The *Lemma 2.1.7* is a good example of it. In fact, a partial H -module algebra does not become an H_R -module algebra simply because H_R is not a coalgebra. However, the partial action in A generates on it a structure of H_R -module. The next lemma shows one more property for the action of H_R on a partial H -module algebra that works like a global action.

Lemma 2.5.2. *Let A be a partial H -module algebra. If h belongs to H_R , then $\varepsilon_L(h) \cdot 1_A = h \cdot 1_A$.*

Proof. Let $h \in H_R$

$$\begin{aligned}
\varepsilon_L(h) \cdot 1_A &= h_1 S(h_2) \cdot 1_A \\
&\stackrel{(1.8)}{=} 1_1 S(1_2 h) \cdot 1_A \\
&\stackrel{(2.1.7)}{=} 1_1 \cdot (S(1_2 h) \cdot 1_A) \\
&\stackrel{(1.8)}{=} h_1 \cdot (S(h_2) \cdot 1_A) \\
&= (h_1 \cdot 1_A)(h_2 S(h_3) \cdot 1_A) \\
&\stackrel{(1.9)}{=} (1_1 h \cdot 1_A)(1_2 \cdot 1_A) \\
&\stackrel{(2.1.7)}{=} (1_1 \cdot (h \cdot 1_A))(1_2 \cdot 1_A) \\
&= 1_H \cdot (h \cdot 1_A) 1_A \\
&= h \cdot 1_A.
\end{aligned}$$

□

There is also another useful characterization for the right action of H_L on A .

Lemma 2.5.3. *If $z \in H_L$, then $a \triangleleft z = a(z \cdot 1_A)$.*

Proof. Let $z \in H_L$.

$$\begin{aligned}
a \triangleleft z &= a(S_R^{-1}(z) \cdot 1_A) \\
&\stackrel{(2.5.2)}{=} a(\varepsilon_L(S_R^{-1}(z)) \cdot 1_A) \\
&= a(\varepsilon_L(\varepsilon_R(S_R^{-1}(z))) \cdot 1_A) \quad \text{since } S_R^{-1}(z) \in H_R \\
&\stackrel{(1.19)}{=} a(\varepsilon_L(S(S_R^{-1}(z))) \cdot 1_A) \\
&= a(\varepsilon_L(z) \cdot 1_A) \\
&= a(z \cdot 1_A) \quad \text{since } z \in H_L.
\end{aligned}$$

□

Now we are able to define the smash product for a partial H -module algebra A .

First, notice that H has a natural structure of a left H_L -module via its multiplication. We start by considering the \mathbb{k} -vector space given by the tensor product $A \otimes_{H_L} H$, and also denoted by $A \# H$, with the multiplication defined by

$$(a \# h)(b \# g) = a(h_1 \cdot b) \# h_2 g.$$

Theorem 2.5.4. *This above multiplication is well-defined, associative, and $1_A \# 1_H$ is a left unit.*

Proof. The well-definition:

It is enough to show that the map $\tilde{\mu}: A \times H \times A \times H \rightarrow A \# H$ given by $\tilde{\mu}(a, h, b, g) = a(h_1 \cdot b) \# h_2 g$ is (H_L, \mathbb{k}, H_L) -balanced. In fact, for all $a, b \in A$, $h, g \in H$, $z \in H_L$ and $r \in \mathbb{k}$ we have:

$$\begin{aligned}
\tilde{\mu}(a, h, b \triangleleft z, g) &= a(h_1 \cdot (b \triangleleft z)) \# h_2 g \\
&= a(h_1 \cdot (S_R^{-1}(z) \cdot b)) \# h_2 g \\
&= a(h_1 \cdot 1_A)(h_2 S_R^{-1}(z) \cdot b) \# h_3 g \\
&= a(h_1 \cdot 1_A)(h_2 1_1 S_R^{-1}(z) \cdot b) \# h_3 1_2 g \\
&\stackrel{(1.28)}{=} a(h_1 \cdot 1_A)(h_2 1_1 \cdot b) \# h_3 1_2 z g \\
&= a(h_1 \cdot 1_A)(h_2 \cdot b) \# h_3 z g \\
&= a(h_1 \cdot 1_A b) \# h_2 z g \\
&= a(h_1 \cdot b) \# h_2 z g \\
&= \tilde{\mu}(a, h, b, z g).
\end{aligned}$$

It is clear that $\tilde{\mu}(a, hr, b, g) = \tilde{\mu}(a, h, rb, g)$, and

$$\begin{aligned}
\tilde{\mu}(a \triangleleft z, h, b, g) &= (a \triangleleft z)(h_1 \cdot b) \# h_2 g \\
&= a(S_R^{-1}(z) \cdot 1_A)(h_1 \cdot b) \# h_2 g \\
&\stackrel{(2.1.8)}{=} a(1_1 S_R^{-1}(z) \cdot 1_A)(1_2 h_1 \cdot b) \# h_2 g \\
&\stackrel{(1.28)}{=} a(1_1 \cdot 1_A)(1_2 z h_1 \cdot b) \# h_2 g \\
&= a(1_1 \cdot 1_A)(1_2 \cdot 1_A)(1_3 z h_1 \cdot b) \# h_2 g \\
&= a(1_1 \cdot 1_A)(1_2 \cdot z h_1 \cdot b) \# h_2 g \\
&= a(1_H \cdot 1_A(z h_1 \cdot b)) \# h_2 g \\
&= a(z 1_1 h_1 \cdot b) \# 1_2 h_2 g \\
&\stackrel{(1.7)}{=} a(z_1 h_1 \cdot b) \# z_2 h_2 g \\
&= \tilde{\mu}(a, zh, b, g).
\end{aligned}$$

The associativity:

$$\begin{aligned}
((a \# h)(b \# g))(c \# k) &= (a(h_1 \cdot b) \# h_2 g)(c \# k) \\
&= a(h_1 \cdot b)(h_2 g_1 \cdot c) \# h_3 g_2 k \\
&= a(h_1 \cdot b 1_A)(h_2 g_1 \cdot c) \# h_3 g_2 k \\
&= a(h_1 \cdot b)(h_2 \cdot 1_A)(h_3 g_1 \cdot c) \# h_4 g_2 k \\
&= a(h_1 \cdot b)(h_2 \cdot (g_1 \cdot c)) \# h_3 g_2 k \\
&= a(h_1 \cdot b(g_1 \cdot c)) \# h_2 g_2 k \\
&= (a \# h)(b(g_1 \cdot c) \# g_2 k) \\
&= (a \# h)((b \# g)(c \# k)).
\end{aligned}$$

The left unit:

$$\begin{aligned}
(1_A \# 1_H)(a \# h) &= 1_A(1_1 \cdot a) \# 1_2 h \\
&= (1_1 \cdot a) \# 1_2 h \\
&\stackrel{(1.1.9)}{=} S_R^{-1}(S(1_1) \cdot a) \# 1_2 h \\
&= a \triangleleft S(1_1) \# 1_2 h \\
&= a \# S(1_1) 1_2 h \\
&= a \# \varepsilon_L(1_H) h \\
&= a \# h.
\end{aligned}$$

□

It follows from the above theorem that

$$A\#H = (A\#H)(1_A\#1_H)$$

is an algebra with $1_A\#1_H$ as its unit. This algebra is called the *partial smash product* of A by H .

The following example illustrates that, in general, $1_A\#1_H$ is not a unit of $A\#H$.

Example 2.5.5. *Let G be a finite groupoid which is not a group and $G_e = \{g \in G \mid d(g) = e = r(g)\}$ the isotropy group associated to e , for some $e \in G_0$. It is easy to see that \mathbb{k} is a partial $\mathbb{k}G$ -module algebra via*

$$\begin{aligned} \cdot : \mathbb{k}G \otimes \mathbb{k} &\rightarrow \mathbb{k} \\ g \otimes 1_{\mathbb{k}} &\mapsto \delta_{g, G_e}, \end{aligned}$$

where $\delta_{g, G_e} = 1_{\mathbb{k}}$ if $g \in G_e$ and 0 otherwise. In this case, $1_{\mathbb{k}}\#1_{\mathbb{k}G}$ is not a right unit for the smash product $\mathbb{k}\#\mathbb{k}G$. Indeed, since G is not a group, there exists an element x in $G \setminus G_e$. Thus, $x \cdot 1_{\mathbb{k}} = 0$ and, consequently, $(1_{\mathbb{k}}\#x)(1_{\mathbb{k}}\#1_{\mathbb{k}G}) = 0$. \square

Actually, we have the following.

Proposition 2.5.6. *Let A be a partial H -module algebra. Then, $1_A\#1_H$ is a unit in $A\#H$ if and only if A is an H -module algebra.*

Proof. Suppose $1_A\#1_H$ a unit in $A\#H$. Then $a\#h = a(h_1 \cdot 1_A)\#h_2$ and, applying ε_L on the second element of each term of this equality, we have

$$a \triangleleft \varepsilon_L(h) = a(h_1 \cdot 1_A) \triangleleft \varepsilon_L(h_2).$$

Hence,

$$\begin{aligned} a(\varepsilon_L(h) \cdot 1_A) &\stackrel{(2.5.3)}{=} a \triangleleft \varepsilon_L(h) \\ &= a(h_1 \cdot 1_A) \triangleleft \varepsilon_L(h_2) \\ &= a(h_1 \cdot 1_A)(\varepsilon_L(h_2) \cdot 1_A) \\ &= a(h_1 \cdot 1_A)(h_2 S(h_3) \cdot 1_A) \\ &\stackrel{(9)}{=} a(1_1 h \cdot 1_A)(1_2 \cdot 1_A) \\ &\stackrel{(2.1.8)}{=} a(h \cdot 1_A)(1_H \cdot 1_A) \\ &= a(h \cdot 1_A) \end{aligned}$$

and, taking $a = 1_A$ we have $h \cdot 1_A = \varepsilon_L(h) \cdot 1_A$. By Lemma 2.1.6, A is an H -module algebra. The converse is straightforward and standard. \square

We have some more properties for the smash product. The first of them shows that we can see A inside $A\#H$. It will be useful in Chapter 4 to give an A -module structure for some objects.

The second one says we can break a simple element $a\#h$ in two parts, making easier some computations.

Proposition 2.5.7. *Let A be a partial H -module algebra. Then the map*

$$\begin{aligned} \iota : A &\rightarrow A\#H \\ a &\mapsto a\#1_H \end{aligned}$$

is an algebra monomorphism.

Proof. Note that $a\#1_H = a(1_1 \cdot 1_A)\#1_2 \stackrel{(2.1.9)}{=} (1_1 \cdot a)\#1_2 = (1_A\#1_H)(a\#1_H) = a\#1_H$ for all $a \in A$.

Supposing $a\#1_H = b\#1_H$, then $a\#1_H = b\#1_H$ and so $a = b$. It means ι is injective. Moreover, if $a, b \in A$,

$$\begin{aligned} \iota(a)\iota(b) &= (a\#1_H)(b\#1_H) \\ &= a(1_1 \cdot b)\#1_2 \\ &\stackrel{(2.1.9)}{=} (1_1 \cdot ab)\#1_2 \\ &= (1_A\#1_H)(ab\#1_H) \\ &= (ab\#1_H) \\ &= \iota(ab) \end{aligned}$$

then ι is an algebra morphism. □

In other words, the theorem says that we can see A inside of $A\#H$ as $A\#1_H$.

And, for the second one, we have:

Proposition 2.5.8. *Let A be a partial H -module algebra. Then $a\#h = (a\#1_H)(1_A\#h)$ for all $a \in A$ and $h \in H$.*

Proof. In fact,

$$\begin{aligned} (a\#1_H)(1_A\#h) &= a(1_1 \cdot 1_A)\#1_2h \\ &\stackrel{(2.1.9)}{=} (1_1 \cdot a)\#1_2h \\ &= (1_A\#1_H)(a\#h) \\ &= (a\#h). \end{aligned}$$

□

2.6 A Morita context

In the setting of partial actions of Hopf algebras with an invertible antipode there exists a Morita context relating the partial smash product $A\#H$ and the (global) smash product $B\#H$, where B denotes a globalization of A such that the image of A inside B is an ideal of B (cf. [2]). In this section we extend this result to the setting of partial actions of weak Hopf algebras.

First, recall the definition of a Morita context.

Definition 2.6.1. Let A and B be unital rings. A Morita context for A and B is a sextuple $(A, B, M, N, (\cdot), [\cdot])$ where M is a (A, B) -bimodule, N is a (B, A) -bimodule, and $(\cdot): M \otimes_B N \rightarrow A$ and $[\cdot]: N \otimes_A M \rightarrow B$ are homomorphisms of (A, A) -bimodules and (B, B) -bimodules, respectively, such that

$$(i) \quad (m, n)m' = m[n, m'],$$

$$(ii) \quad [n, m]n' = n(m, n'),$$

for all $m, m' \in M$, $n, n' \in N$.

We will first construct a non unitary monomorphism of algebras from $A\#H$ into $B\#H$. For this we need the following lemma.

Lemma 2.6.2. *Let A be a partial H -module algebra and (B, θ) a globalization of A . Then, $S_R^{-1}(h) \triangleright \theta(a) = S_R^{-1}(h) \cdot \theta(a)$, for all $h \in H_L$.*

Proof. Note that $S_R^{-1}(h) \cdot \theta(a) = \theta(1_A)(S_R^{-1}(h) \triangleright \theta(a))$. But $S_R^{-1}(h)$ lies in H_R , thus, by Lemma 2.1.9(i), we have $\theta(1_A)(S_R^{-1}(h) \triangleright \theta(a)) = S_R^{-1}(h) \triangleright (\theta(1_A)\theta(a)) = S_R^{-1}(h) \triangleright \theta(a)$. \square

Proposition 2.6.3. *Let (B, θ) be a globalization of the partial H -module algebra A . Then, there exists a non unitary algebra monomorphism Ψ from $A\#H$ to $B\#H$.*

Proof. Define

$$\begin{aligned} \tilde{\Psi}: A \times H &\rightarrow B \otimes_{H_L} H \\ (a, h) &\mapsto \theta(a) \otimes h. \end{aligned}$$

For all $a \in A, h \in H$ and $z \in H_L$, we have

$$\begin{aligned} \tilde{\Psi}(a, zh) &= \theta(a) \otimes zh \\ &= \theta(a) \triangleleft z \otimes h \\ &= S_R^{-1}(z) \triangleright \theta(a) \otimes h \\ &\stackrel{(2.6.2)}{=} S_R^{-1}(z) \cdot \theta(a) \otimes h \\ &= \theta(S_R^{-1}(z) \cdot a) \otimes h \\ &= \theta(a \triangleleft z) \otimes h \\ &= \tilde{\Psi}(a \triangleleft z, h), \end{aligned}$$

which shows that $\tilde{\Psi}$ is H_L -balanced. Thus, there exists a \mathbb{k} -linear map Ψ from $A \otimes_{H_L} H$ to $B \otimes_{H_L} H$ defined by $\Psi(a \otimes h) = \theta(a) \otimes h$.

It follows from the injectivity of θ and similar calculations that the \mathbb{k} -linear map $\Psi': \theta(A) \otimes_{H_L} H \rightarrow A \otimes_{H_L} H$ given by $\Psi'(\theta(a) \otimes h) = a \otimes h$ is well-defined. Furthermore, Ψ is a monomorphism because $\Psi' \circ \Psi = I_{A \otimes_{H_L} H}$.

It remains to check that $\Psi: A\#H \rightarrow B\#H$ is multiplicative. In fact, for all $a, b \in A$ and $h, g \in H$, we have

$$\begin{aligned}
\Psi((a\#h)(b\#g)) &= \Psi(a(h_1 \cdot b)\#h_2g) \\
&= \theta(a(h_1 \cdot b))\#h_2g \\
&= \theta(a)\theta(h_1 \cdot b)\#h_2g \\
&= \theta(a)(h_1 \cdot \theta(b))\#h_2g \\
&= \theta(a)(h_1 \triangleright \theta(b))\#h_2g \\
&= (\theta(a)\#h)(\theta(b)\#g) \\
&= \Psi(a\#h)\Psi(b\#g).
\end{aligned}$$

□

In the sequel, we will construct the bimodules which will define a Morita context for $A\#H$ and $B\#H$. For this construction we will suppose that $\theta(A)$ is an ideal of B and the antipode S of H is invertible. This assumption on S is necessary because in this construction we will need to make use of Lemma 2.1.11(ii).

Now let $M = \Psi(A\#H)$ and N be the vector space generated by the elements of the form $(h_1 \triangleright \theta(a))\#h_2$, for all $a \in A$ and $h \in H$.

Proposition 2.6.4. *With the above notations and assumptions, M is a right $B\#H$ -module and N is a left $B\#H$ -module, via the multiplication of $B\#H$.*

Proof. Let $\theta(a)\#h \in M$ and $k \triangleright \theta(b)\#g \in B\#H$, so

$$\begin{aligned}
(\theta(a)\#h)(k \triangleright \theta(b)\#g) &= \theta(a)(h_1k \triangleright \theta(b))\#h_2g \\
&= \theta(a)(h_1k \cdot \theta(b))\#h_2g \\
&= \theta(a)(h_1k \cdot b)\#h_2g
\end{aligned}$$

that lies in M .

Let $k \triangleright \theta(a)\#h \in B\#H$ and $g_1 \triangleright \theta(b)\#g_2 \in N$. Then we have

$$\begin{aligned}
(k \triangleright \theta(a)\#h)(g_1 \triangleright \theta(b)\#g_2) &= (k \triangleright \theta(a))(h_1g_1 \triangleright \theta(b))\#h_2g_2 \\
&\stackrel{(2.1.11)}{=} h_2g_2 \triangleright [(S^{-1}(h_1g_1)k \triangleright \theta(a))\theta(b)]\#h_3g_3
\end{aligned}$$

that lies in N because $\theta(A)$ is an ideal of $B = H \triangleright \theta(A)$.

Now, the assertion follows from the associativity of $B\#H$. □

Proposition 2.6.5. *Keeping the same notations and assumptions as above, M is a left $A\#H$ -module and N is a right $A\#H$ -module via the actions*

$$\begin{aligned}
\blacktriangleright: A\#H \otimes M &\rightarrow M \\
a\#h \otimes m &\mapsto \Psi(a\#h)m
\end{aligned}$$

and

$$\begin{aligned}
\blacktriangleleft: N \otimes A\#H &\rightarrow N \\
n \otimes a\#h &\mapsto n\Psi(a\#h)
\end{aligned}$$

respectively, where Ψ is the non unitary monomorphism defined in 7.3. Moreover, M and N are bimodules.

Proof. We need only to ensure that \blacktriangleleft is well-defined. The well-definition of \blacktriangleright as well as the other assertions follow from the fact that $A\#H$ is a subalgebra of $A\#H$ and Ψ is multiplicative.

Given $h_1 \triangleright \theta(a)\#h_2 \in N$ and $a'\#g \in A\#H$, we have

$$\begin{aligned}
(h_1 \triangleright \theta(a)\#h_2) \blacktriangleleft (a'\#g) &= (h_1 \triangleright \theta(a)\#h_2)(\Psi(a'\#g)) \\
&= (h_1 \triangleright \theta(a)\#h_2)(\theta(a'(g_1 \cdot 1_A))\#g_2) \\
&= (h_1 \triangleright \theta(a)\#h_2)(\theta(a')(g_1 \cdot \theta(1_A))\#g_2) \\
&= (h_1 \triangleright \theta(a)\#h_2)(\theta(a')(g_1 \triangleright \theta(1_A))\#g_2) \\
&= (h_1 \triangleright \theta(a))(h_2 \triangleright \theta(a'))(h_3g_1 \triangleright \theta(1_A))\#h_4g_2 \\
&= (h_1 \triangleright \theta(aa'))(h_2g_1 \triangleright \theta(1_A))\#h_3g_2 \\
&\stackrel{(2.1.11)}{=} h_3g_2 \triangleright [(S^{-1}(h_2g_1)h_1 \triangleright \theta(aa'))\theta(1_A)]\#h_4g_3
\end{aligned}$$

that lies in N , because $\theta(A)$ is an ideal of B , which ensures that \blacktriangleleft is also well-defined. \square

Now, we consider the maps $[\cdot, \cdot]: N \otimes_{A\#H} M \rightarrow B\#H$ and $(\cdot, \cdot): M \otimes_{B\#H} N \rightarrow \Psi(A\#H) \simeq A\#H$ given by the multiplication of $B\#H$. Both such maps are well-defined because $M, N \subseteq B\#H$.

Theorem 2.6.6. *($A\#H, B\#H, M, N, (\cdot, \cdot), [\cdot, \cdot]$) is a Morita context. Moreover, the maps $[\cdot, \cdot]$ and (\cdot, \cdot) are both surjective. In particular, if B also has an identity element, then $A\#H$ and $B\#H$ are Morita equivalent.*

Proof. The main assertion follows from Propositions 7.4 and 7.5, and from the associativity of the multiplication of $B\#H$.

For the surjectivity of (\cdot, \cdot) and $[\cdot, \cdot]$ it is enough to show that $MN = \Psi(A\#H)$ and $NM = B\#H$.

In fact, clearly $\Psi(A\#H) \subseteq MN$. Conversely, given $g_1 \triangleright \theta(b)\#g_2 \in N$ and $\theta(a)\#h \in M$ we have

$$\begin{aligned}
(\theta(a)\#h)(g_1 \triangleright \theta(b)\#g_2) &= \theta(a)(h_1g_1 \triangleright \theta(b))\#h_2g_2 \\
&= \theta(a)(h_1g_1 \cdot \theta(b))\#h_2g_2 \\
&= \theta(a)(h_1g_1 \cdot b)\#h_2g_2 \\
&= \theta(a)(h_1g_1 \cdot b1_A)\#h_2g_2 \\
&= \theta(a)(h_1g_1 \cdot b)(h_2g_2 \cdot 1_A)\#h_3g_3 \\
&= \Psi(a(h_1g_1 \cdot b)\#h_2g_2)
\end{aligned}$$

which lies in $\Psi(A\#H)$. Hence, $MN = \Psi(A\#H)$.

Clearly, we have that $NM \subseteq B\#H$. To prove that $B\#H \subseteq NM$ it is enough to check that the equality

$$(h_1 \triangleright \theta(a)\#h_2)(\theta(1_A)\#S(h_3)g) = h \triangleright \theta(a)\#g$$

holds for all $a, b \in A$ and $h, g \in H$. Indeed,

$$\begin{aligned}
(h_1 \triangleright \theta(a) \# h_2)(\theta(1_A) \# S(h_3)g) &= (h_1 \triangleright \theta(a))(h_2 \triangleright \theta(1_A)) \# h_3 S(h_4)g \\
&= h_1 \triangleright \theta(a) \# h_2 S(h_3)g \\
&\stackrel{(1.9)}{=} 1_1 h \triangleright \theta(a) \# 1_2 g \\
&\stackrel{(2.1.7)}{=} 1_1 \triangleright h \triangleright \theta(a) \# 1_2 g \\
&= (h \triangleright \theta(a)) \triangleleft S(1_1) \# 1_2 g \\
&= h \triangleright \theta(a) \# S(1_1) 1_2 g \\
&= h \triangleright \theta(a) \# \varepsilon_R(1_H)g \\
&= h \triangleright \theta(a) \# g.
\end{aligned}$$

Therefore, $B \# H = NM$. The last assertion follows from [41, Theorems 4.1.4 and 4.1.17] \square

In [33], J. García and J. Simón has shown that the identity element hypothesis can be replaced by idempotent ring hypothesis. It can be applied to our case, allowing B to be a non-unital algebra and keeping the categories equivalent.

Chapter 3

Partial coactions of weak Hopf algebras on algebras

In this chapter we explain some points concerning partial coactions of weak Hopf algebras. In Section 3.1 we define a partial H -comodule algebra and show some examples. We also show that for finite dimensional weak Hopf algebras, there is a bijective correspondence between partial H -comodule algebra and partial H^* -module algebra. In Section 3.3 we show that every partial H -comodule algebra has a globalization.

The Section 3.2 is dedicated to the study of the partial coactions on the ground field. We characterize all coactions, as well as apply it to particular weak Hopf algebras. This section is useful to construct new examples of partial coactions.

In Section 3.4 we show that if A is a partial H -comodule algebra, the reduced tensor product $A \otimes H$ is an A -coring.

Along this chapter all coactions will be assumed to be right coactions.

3.1 Partial coactions

We start the section recalling the notion of an H -comodule algebra.

Definition 3.1.1. Let B be an algebra. B is said to be an H -comodule algebra if there exists a linear map $\rho : B \rightarrow B \otimes H$ such that

$$(CA1) \quad (I \otimes \varepsilon)\rho(a) = a;$$

$$(CA2) \quad \rho(ab) = \rho(a)\rho(b);$$

$$(CA3) \quad (\rho \otimes I)\rho = (I \otimes \Delta)\rho.$$

Denote the image of an element $b \in B$ by $\rho(b) = b^0 \otimes b^1$.

We present some trivial examples, which will be used later in the construction of the globalization.

Example 3.1.2. H is an H -comodule algebra via Δ_H .

Example 3.1.3. For any algebra A , the tensor algebra $A \otimes H$ is an H -comodule algebra with coaction given by $\rho = I_A \otimes \Delta_H$.

Note that in the case of Hopf algebras, given any algebra B there exists a coaction given by $\rho(b) = b \otimes 1_H$. This is not a coaction in the case of weak Hopf algebra. Moreover the following statement holds:

Proposition 3.1.4. Let H be a weak Hopf algebra. Then B is an H -module algebra by

$$\begin{aligned} \rho : B &\rightarrow B \otimes H \\ b &\mapsto b \otimes 1_H \end{aligned}$$

if and only if H is a Hopf algebra.

Proof. Supposing B an H -comodule algebra by ρ , then for any $b \in B$ we have $b \otimes 1_1 \otimes 1_2 = (I \otimes \Delta)\rho(b) = (\rho \otimes I)\rho(b) = b \otimes 1_H \otimes 1_H$. Hence $\Delta(1_H) = 1_H \otimes 1_H$ which implies that H is a Hopf algebra.

The converse is straightforward. \square

Remember that if H is a Hopf algebra and A is an H -module algebra, then $\rho(1_B) = 1_B \otimes 1_H$. For weak Hopf algebras this is not true in general. In this case there exists a similar result:

Proposition 3.1.5. Let B be an H -comodule algebra. Then $\rho(1_B) = 1^0 \otimes \varepsilon_L(1^1)$.

Proof.

$$\begin{aligned} 1^0 \otimes \varepsilon_L(1^1) &= 1^0 \otimes 1^1_1 S(1^1_2) \\ &= 1^{00} \otimes 1^{01} S(1^1) \\ &= (1 \ 1^0)^0 \otimes (1 \ 1^0)^1 S(1^1) \\ &= 1'^0 1^{00} \otimes 1'^1 \ 1^{01} S(1^1) \\ &= 1'^0 1^0 \otimes 1'^1 1^1_1 S(1^1_2) \\ &= 1'^0 1^0 \otimes 1'^1 \varepsilon_L(1^1) \\ &\stackrel{(1.11)}{=} 1'^0 1^0 \otimes \varepsilon(1'^1_1 1^1) 1'^1_2 \\ &= 1'^0 1^0 \varepsilon(1'^0 1^1) \otimes 1'^1 \\ &= (1'^0 \ 1)^0 \varepsilon((1'^0 \ 1)^1) \otimes 1'^1 \\ &= 1'^0 \varepsilon(1'^0 1) \otimes 1'^1 \\ &= 1'^0 \otimes 1'^1 \\ &= \rho(1_B). \end{aligned}$$

\square

Then using [17, Proposition 4.10] we obtain some properties:

Theorem 3.1.6. Let B be an H -comodule algebra. The following assertions hold:

(i) $(\rho \otimes I)\rho(1) = (\rho(1) \otimes 1)(1 \otimes \Delta(1));$

(ii) $(\rho \otimes I)\rho(1) = (1 \otimes \Delta(1))(\rho(1) \otimes 1);$

$$(iii) \quad b^0 \otimes \bar{\varepsilon}_R(b^1) = b^1 \otimes 1^1;$$

$$(iv) \quad b^0 \otimes \varepsilon_L(b^1) = 1^0 b \otimes 1^1;$$

$$(v) \quad \rho(1_B) = 1^0 \otimes \varepsilon_L(1^1);$$

$$(vi) \quad \rho(1_B) = 1^0 \otimes \bar{\varepsilon}_R(1^1);$$

$$(vii) \quad \rho(1) \in B \otimes H_L;$$

where $\bar{\varepsilon}_R$ is defined by $\bar{\varepsilon}_R(h) = \varepsilon(h1_1)1_2$.

We introduce now the definition of partial H -comodule algebra.

Definition 3.1.7. Let A be an algebra. Then A is a partial H -comodule algebra if there exists a linear map $\bar{\rho} : A \rightarrow A \otimes H$ such that:

$$(PCA'1) \quad (I \otimes \varepsilon)\bar{\rho}(a) = a;$$

$$(PCA'2) \quad (\bar{\rho} \otimes I)[(a \otimes 1_H)\bar{\rho}(b)] = (\bar{\rho}(a) \otimes 1_H)((I \otimes \Delta)\bar{\rho}(b));$$

moreover it is symmetric if

$$(PCA'3) \quad (\bar{\rho} \otimes I)[\bar{\rho}(b)(a \otimes 1_H)] = ((I \otimes \Delta)\bar{\rho}(b))(\bar{\rho}(a) \otimes 1_H).$$

Note that in the above definition we do not need A to be unital. Moreover, if A is unital, it is clear that the above definition is equivalent to the following:

Definition 3.1.8. Let A be a unital algebra. Then A is a partial H -comodule algebra if there exists a linear map $\bar{\rho} : A \rightarrow A \otimes H$ such that:

$$(PCA1) \quad (I \otimes \varepsilon)\bar{\rho}(a) = a;$$

$$(PCA2) \quad \bar{\rho}(ab) = \bar{\rho}(a)\bar{\rho}(b);$$

$$(PCA3) \quad (\bar{\rho} \otimes I)[\bar{\rho}(a)] = (\bar{\rho}(1_A) \otimes 1_H)((I \otimes \Delta)\bar{\rho}(a));$$

moreover it is symmetric if

$$(PCA4) \quad (\bar{\rho} \otimes I)[\bar{\rho}(a)] = ((I \otimes \Delta)\bar{\rho}(a))(\bar{\rho}(1_A) \otimes 1_H).$$

The first definition is more general than the second one as it does not require A to be unital. However, since we are working with unital algebras, we will use the second one in order to make the proofs more detailed and easier for the reader.

We adopt the notation $\bar{\rho}(a) = a^{\bar{0}} \otimes a^{\bar{1}}$ for the partial coaction.

Remark 3.1.9. *It is straightforward to check that any H -comodule algebra is also a partial H -comodule algebra*

Now we give a characterization which allows us to decide when a partial H -comodule algebra is an H -comodule algebra.

Proposition 3.1.10. *Let A be a partial H -comodule algebra. Then $\rho(1_A) = 1^{\bar{0}} \otimes \varepsilon_L(1^{\bar{1}})$ if and only if A is an H -comodule algebra.*

Proof. In fact, if $\rho(1_A) = 1^{\bar{0}} \otimes \varepsilon_L(1^{\bar{1}})$, then for any $a \in A$ we have

$$\begin{aligned}
(I \otimes \Delta)(\rho)(a) &= a^{\bar{0}} \otimes a^{\bar{1}}_1 \otimes a^{\bar{1}}_2 \\
&= (1_A a)^{\bar{0}} \otimes (1_A a)^{\bar{1}}_1 \otimes (1_A a)^{\bar{1}}_2 \\
&= 1^{\bar{0}} a^{\bar{0}} \otimes 1^{\bar{0}}_1 a^{\bar{1}}_1 \otimes 1^{\bar{0}}_2 a^{\bar{1}}_2 \\
&= 1^{\bar{0}} a^{\bar{0}} \otimes \varepsilon_L(1^{\bar{0}})_1 a^{\bar{1}}_1 \otimes \varepsilon_L(1^{\bar{0}})_2 a^{\bar{1}}_2 \\
&\stackrel{(1.7)}{=} 1^{\bar{0}} a^{\bar{0}} \otimes \varepsilon_L(1^{\bar{0}}) 1_1 a^{\bar{1}}_1 \otimes 1_2 a^{\bar{1}}_2 \\
&= 1^{\bar{0}} a^{\bar{0}} \otimes \varepsilon_L(1^{\bar{0}}) a^{\bar{1}}_1 \otimes a^{\bar{1}}_2 \\
&= 1^{\bar{0}} a^{\bar{0}} \otimes 1^{\bar{0}} a^{\bar{1}}_1 \otimes a^{\bar{1}}_2 \\
&= a^{\bar{0}\bar{0}} \otimes a^{\bar{0}\bar{1}} \otimes a^{\bar{1}} \\
&= (\rho \otimes I)\rho(a)
\end{aligned}$$

hence A is an H -comodule algebra.

The converse is exactly Proposition 3.1.5 □

The following proposition gives us the correspondence between partial module algebras and partial comodule algebras where the weak Hopf algebra is assumed to be finite dimensional.

Proposition 3.1.11. *Let H be a finite dimensional weak Hopf algebra. Then A is a (symmetric) partial H -comodule algebra if and only if A is a (symmetric) partial H^* -module algebra.*

Proof. Supposing A to be a partial H -comodule algebra via $\bar{\rho} : A \rightarrow A \otimes H$, define

$$\begin{aligned}
\cdot : H^* \otimes A &\rightarrow A \\
f \otimes a &\mapsto f \cdot a = a^{\bar{0}} f(a^{\bar{1}})
\end{aligned}$$

then:

(PMA1) Let $a \in A$,

$$\begin{aligned}
1_{H^*} \cdot a &= \varepsilon_H \cdot a \\
&= a^{\bar{0}} \varepsilon_H(a^{\bar{1}}) \\
&\stackrel{(PCA1)}{=} a.
\end{aligned}$$

(PMA2) Let $f \in H^*$ and $a, b \in A$

$$\begin{aligned}
f \cdot ab &= (ab)^{\bar{0}} f((ab)^{\bar{1}}) \\
&\stackrel{(PCA2)}{=} a^{\bar{0}} b^{\bar{0}} f(a^{\bar{1}} b^{\bar{1}}) \\
&= a^{\bar{0}} f_1(a^{\bar{1}}) b^{\bar{0}} f_2(b^{\bar{1}}) \\
&= (f_1 \cdot a)(f_2 \cdot b).
\end{aligned}$$

(PMA3) Let $f, g \in H^*$ and $a \in A$

$$\begin{aligned}
f \cdot (g \cdot a) &= f \cdot (a^{\bar{0}}g(a^{\bar{1}})) \\
&= a^{\bar{0}\bar{0}}f(a^{\bar{0}\bar{1}})g(a^{\bar{1}}) \\
&\stackrel{(\text{PCA3})}{=} 1^{\bar{0}}a^{\bar{0}}f(1^{\bar{1}}a^{\bar{1}}_1)g(a^{\bar{1}}_2) \\
&= 1^{\bar{0}}f_1(1^{\bar{1}})a^{\bar{0}}f_2(a^{\bar{1}}_1)g(a^{\bar{1}}_2) \\
&= 1^{\bar{0}}f_1(1^{\bar{1}})a^{\bar{0}}(f_2 * g)(a^{\bar{1}}) \\
&= (f_1 \cdot 1_A)(f_2 * g \cdot a).
\end{aligned}$$

Thus, A is a partial H^* -module algebra. Moreover, if A is a symmetric partial H -comodule algebra then

(PMA4) Let $f, g \in H^*$ and $a \in A$

$$\begin{aligned}
f \cdot (g \cdot a) &= f \cdot (a^{\bar{0}}g(a^{\bar{1}})) \\
&= a^{\bar{0}\bar{0}}f(a^{\bar{0}\bar{1}})g(a^{\bar{1}}) \\
&\stackrel{(\text{PCA4})}{=} a^{\bar{0}}1^{\bar{0}}f(a^{\bar{1}}_11^{\bar{1}})g(a^{\bar{1}}_2) \\
&= a^{\bar{0}}f_1(a^{\bar{1}}_1)g(a^{\bar{1}}_2)1^{\bar{0}}f_2(1^{\bar{1}}) \\
&= a^{\bar{0}}(f_1 * g)(a^{\bar{1}})1^{\bar{0}}f_2(1^{\bar{1}}) \\
&= (f_1 * g \cdot a)(f_2 \cdot 1_A).
\end{aligned}$$

Hence, A is a symmetric partial H^* -module algebra.

Conversely, if A is a partial H^* -module algebra with action \cdot and $\{h_i\}_{i=1}^n$ is a basis for H , define

$$\begin{aligned}
\bar{\rho} : A &\rightarrow A \otimes H \\
a &\mapsto \sum_{i=1}^n h_i^* \cdot a \otimes h_i
\end{aligned}$$

where $\{h_i^*\}_{i=1}^n$ is the dual basis for H^* , then:

(PCA1) Let $a \in A$

$$\begin{aligned}
(I \otimes \varepsilon_H)\bar{\rho}(a) &= \sum_{i=1}^n \varepsilon_H(h_i)h_i^* \cdot a \\
&= \varepsilon_H \cdot a \\
&= 1_{H^*} \cdot a \\
&\stackrel{(\text{PMA1})}{=} a.
\end{aligned}$$

(PCA2) For all $f \in H^*$ and $a, b \in A$

$$\begin{aligned}
(I \otimes f)(\bar{\rho}(ab)) &= \sum_{i=1}^n f(h_i)h_i^* \cdot ab \\
&= f \cdot ab \\
&\stackrel{(\text{PMA2})}{=} (f_1 \cdot a)(f_2 \cdot b) \\
&= \left(\sum_{i=1}^n f_1(h_i)h_i^* \cdot a\right) \left(\sum_{j=1}^n f_2(h_j)h_j^* \cdot b\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n (h_i^* \cdot a)(h_j^* \cdot b)f(h_i h_j) \\
&= (I \otimes f)(\bar{\rho}(a)\bar{\rho}(b))
\end{aligned}$$

hence $\bar{\rho}(ab) = \bar{\rho}(a)\bar{\rho}(b)$.

(PCA3) For all $f, g \in H^*$ and $a \in A$

$$\begin{aligned}
(I \otimes f \otimes g)((\bar{\rho} \otimes I)\bar{\rho}(a)) &= a^{\bar{0}\bar{0}}f(a^{\bar{0}\bar{1}})g(a^{\bar{1}}) \\
&= f \cdot (g \cdot a) \\
&\stackrel{(\text{PMA3})}{=} (f_1 \cdot 1)(f_2 * g \cdot a) \\
&= 1^{\bar{0}}f_1(1^{\bar{1}})a^{\bar{0}}f_2(a^{\bar{1}}_1)g(a^{\bar{1}}_2) \\
&= 1^{\bar{0}}a^{\bar{0}}f(1^{\bar{1}}a^{\bar{1}}_1)g(a^{\bar{1}}_2) \\
&= (I \otimes f \otimes g)((\bar{\rho}(1) \otimes 1)[(I \otimes \Delta)(\bar{\rho}(a))]),
\end{aligned}$$

then $(\bar{\rho} \otimes I)\bar{\rho}(a) = (\bar{\rho}(1) \otimes 1)[(I \otimes \Delta)(\bar{\rho}(a))]$.

Therefore, A is a partial H -comodule algebra. Moreover, if A is a symmetric partial H^* -module algebra then

(PCA4) For all $f, g \in H^*$ and $a \in A$

$$\begin{aligned}
(I \otimes f \otimes g)((\bar{\rho} \otimes I)\bar{\rho}(a)) &= a^{\bar{0}\bar{0}}f(a^{\bar{0}\bar{1}})g(a^{\bar{1}}) \\
&= f \cdot (a^{\bar{0}}g(a^{\bar{1}})) \\
&= f \cdot (g \cdot a) \\
&\stackrel{(\text{PMA4})}{=} (f_1 * g \cdot a)(f_2 \cdot 1) \\
&= a^{\bar{0}}(f_1 * g)(a^{\bar{1}})1^{\bar{0}}f_2(1^{\bar{1}}) \\
&= a^{\bar{0}}f_1(a^{\bar{1}}_1)g(a^{\bar{1}}_2)1^{\bar{0}}f_2(1^{\bar{1}}) \\
&= a^{\bar{0}}1^{\bar{0}}f(a^{\bar{1}}_11^{\bar{1}})g(a^{\bar{1}}_2) \\
&= (I \otimes f \otimes g)[(I \otimes \Delta)(\bar{\rho}(a))](\bar{\rho}(1) \otimes 1),
\end{aligned}$$

then $(\bar{\rho} \otimes I)\bar{\rho}(a) = [(I \otimes \Delta)(\bar{\rho}(a))](\bar{\rho}(1) \otimes 1)$.

Hence, A is a symmetric partial H -comodule algebra. \square

Example 3.1.12 (Induced partial coaction). *Let B be an H -comodule algebra with coaction ρ and A a unital right ideal of B . Then we can induce*

$$\begin{aligned}
\bar{\rho} : A &\rightarrow A \otimes H \\
a &\mapsto (1_A \otimes 1_H)\rho(a),
\end{aligned}$$

which defines in A a partial coaction.

Example 3.1.13. *Let $H_4 = \mathbb{k} \langle g, x \mid g^2 = 1, x^2 = 0 \text{ and } gx = -xg \rangle$ be the Sweedler's algebra over a field with characteristic different of 2. Take $v = \frac{1}{2}1 + \frac{1}{2}g + rxg$ an element in H_4 , with $r \in \mathbb{k}$.*

For a weak Hopf algebra H , the map

$$\begin{aligned}
\bar{\rho} : H &\rightarrow H \otimes H \otimes H_4 \\
h &\mapsto h_1 \otimes h_2 \otimes v,
\end{aligned}$$

defines in H an structure of a partial $H \otimes H_4$ -comodule algebra.

3.2 Partial coactions on the ground field

In a similar way we did in Section 2.3 for partial actions, we describe and give in this section examples of partial coactions on the ground field.

Note that if we have a partial coaction of H on its ground field \mathbb{k} , then the partial coaction $\bar{\rho} : \mathbb{k} \rightarrow \mathbb{k} \otimes H$ should be \mathbb{k} linear, so it is enough to study who is $\bar{\rho}(1_{\mathbb{k}})$.

We start with the main theorem which describes all partial coactions of H on \mathbb{k} .

Theorem 3.2.1. *Let H be a weak Hopf algebra over a field \mathbb{k} and $\bar{\rho} : \mathbb{k} \rightarrow \mathbb{k} \otimes H$ a \mathbb{k} -linear map such that $\bar{\rho}(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes h$. Then $\bar{\rho}$ defines on \mathbb{k} a partial coaction if and only if the following conditions hold:*

$$(i) \quad \varepsilon(h) = 1_{\mathbb{k}};$$

$$(ii) \quad h = h^2;$$

$$(iii) \quad h \otimes h = hh_1 \otimes h_2.$$

Proof. We will proceed by showing that the properties (PMA1), (PMA2) and (PMA3) are one-to-one related with items (i), (ii) and (iii). In fact,

$$[(PCA1) \Leftrightarrow (i)]:$$

$$\begin{aligned} (I \otimes \varepsilon)\bar{\rho}(1_{\mathbb{k}}) = 1_{\mathbb{k}} &\Leftrightarrow 1_{\mathbb{k}}\varepsilon(h) = 1_{\mathbb{k}} \\ &\Leftrightarrow \varepsilon(h) = 1_{\mathbb{k}}. \end{aligned}$$

$$[(PCA2) \Leftrightarrow (ii)]:$$

$$\begin{aligned} \bar{\rho}(1_{\mathbb{k}}) = \bar{\rho}(1_{\mathbb{k}})\bar{\rho}(1_{\mathbb{k}}) &\Leftrightarrow 1 \otimes h = (1 \otimes h)(1 \otimes h) \\ &\Leftrightarrow h = h^2. \end{aligned}$$

$$[(PCA3) \Leftrightarrow (iii)]:$$

$$\begin{aligned} (\bar{\rho} \otimes I)\bar{\rho}(1_{\mathbb{k}}) = (\bar{\rho}(1_{\mathbb{k}}) \otimes 1_{\mathbb{k}})(I \otimes \Delta)\bar{\rho}(1_{\mathbb{k}}) &\Leftrightarrow 1_{\mathbb{k}} \otimes h \otimes h = (\bar{\rho}(1_{\mathbb{k}}) \otimes 1_{\mathbb{k}})(1_{\mathbb{k}} \otimes h_1 \otimes h_2) \\ &\Leftrightarrow 1_{\mathbb{k}} \otimes h \otimes h = (1_{\mathbb{k}} \otimes h \otimes 1_{\mathbb{k}})(1_{\mathbb{k}} \otimes h_1 \otimes h_2) \\ &\Leftrightarrow 1_{\mathbb{k}} \otimes h \otimes h = 1_{\mathbb{k}} \otimes hh_1 \otimes h_2 \\ &\Leftrightarrow h \otimes h = hh_1 \otimes h_2. \end{aligned}$$

And the proof is complete. □

Note that under the assumption $\varepsilon(h) = 1_{\mathbb{k}}$, the item (iii) implies item (ii) simply applying $I \otimes \varepsilon$ on it. However, we keep the item (ii) because it allow us to proceed exactly as in the proof of the above theorem, i.e., we can show one-to-one relations between them and the partial coactions axioms.

We now describe all coactions of the groupoid algebra $\mathbb{k}G$ and his dual $\mathbb{k}G^*$ on the ground field, for a finite groupoid G . For a better understanding by the reader, we suppose that the characteristic of the field \mathbb{k} is 0 in the next statements. Similar results also work for any field \mathbb{k} .

Proposition 3.2.2. (Partial coactions of $\mathbb{k}G^*$ on \mathbb{k}) Let G be a finite groupoid, $\mathbb{k}G$ the groupoid algebra and $\mathbb{k}G^*$ its dual algebra. Take $\{p_g \mid g \in G\}$ the dual basis of $\mathbb{k}G^*$ and define $\rho(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes x$. Then ρ is a right partial coaction of $\mathbb{k}G^*$ on \mathbb{k} if and only if $x = \sum_{h \in V} p_h$ where V is a group in G .

Proof. Let $x = \sum_{g \in G} \alpha_g p_g$. Note that $x^2 = x \Leftrightarrow \alpha_g = 0$ or $\alpha_g = 1$ for each $g \in G$. So we can suppose $x = \sum_{h \in V} p_h$ for V a subset of G .

Now we have

$$\begin{aligned} \varepsilon_{\mathbb{k}G^*}(x) &= \varepsilon_{\mathbb{k}G^*}\left(\sum_{h \in V} p_h\right) \\ &= \sum_{h \in V} p_h(1_{\mathbb{k}G}) \\ &= \sum_{h \in V} p_h\left(\sum_{e \in G_0} \delta_e\right) \\ &= \sum_{h \in V} \sum_{e \in G_0} p_h(\delta_e) \\ &= \sum_{h \in V} \sum_{e \in G_0} \delta_{e,h}, \end{aligned}$$

where $\delta_{e,h}$ is the Kronecker's delta defined by $\delta_{e,h} = \begin{cases} 1 & \text{if } h = e \\ 0 & \text{otherwise} \end{cases}$.

So, if $e \notin V$ for all $e \in G_0$, $\varepsilon_{\mathbb{k}G^*}(x) = 0$. Therefore, the condition (i) of Theorem 3.2.1 is equivalent to say that there is a unique $e \in V$ such that $e \in G_0$.

Moreover,

$$\begin{aligned} (x \otimes 1_H)\Delta(x) &= \left(\sum_{g \in V} p_g \otimes 1_H\right)\Delta\left(\sum_{h \in V} p_h\right) \\ &= \sum_{g,h \in V} (p_g \otimes 1_H)\Delta(p_h) \\ &= \sum_{g,h \in V} (p_g \otimes 1_H)\left(\sum_{l \in G \exists l^{-1}h} p_l \otimes p_{l^{-1}h}\right) \\ &= \sum_{g,h \in V} (p_g \otimes p_{g^{-1}h}) \\ &= \sum_{g \in V} (p_g \otimes \sum_{h \in V} p_{g^{-1}h}). \end{aligned}$$

But $(\sum_{g \in V} p_g) \otimes (\sum_{h \in V} p_h) = \sum_{g \in V} (p_g \otimes \sum_{h \in V} p_h)$. Since $\{p_g \mid g \in G\}$ is a basis of $\mathbb{k}G^*$, the condition (iii) of Theorem 3.2.1 is equivalent to $\sum_{h \in V} p_h = \sum_{h \in V} p_{g^{-1}h}$ for each $g \in V$. □

For the groupoid algebra $\mathbb{k}G$, it is not so easy to determine all partial coactions on a field as in the last proposition. We should find special idempotents in the groupoid G . Then, we can use Proposition 3.1.11 and apply the result for partial actions of $\mathbb{k}G^*$.

Proposition 3.2.3. (Partial coactions of $\mathbb{k}G$ on \mathbb{k}) Let G be a finite groupoid and $\mathbb{k}G$ the groupoid algebra. If we define $\rho(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes x$, then ρ is a right partial coaction of $\mathbb{k}G$ on \mathbb{k} if and only if $x = \sum_{h \in V} \frac{1}{|V|} \delta_h$ where V is a group in G .

Proof. Suppose first that we have $\rho : \mathbb{k} \rightarrow \mathbb{k} \otimes \mathbb{k}G$ a partial right coaction where $\rho(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes x$ and $x = \sum_{h \in G} \alpha_h h$.

So we can define a left partial action of $\mathbb{k}G^*$ on \mathbb{k} by

$$p_g \cdot 1_{\mathbb{k}} = 1^0 p_g(1^1) = 1_{\mathbb{k}} p_g \left(\sum_{h \in G} \alpha_h h \right) = \alpha_g.$$

In this way we have that

$$\begin{aligned} \cdot : \mathbb{k}G^* \otimes \mathbb{k} &\rightarrow \mathbb{k} \\ p_g \otimes 1_{\mathbb{k}} &\mapsto \alpha_g \end{aligned}$$

is a partial action.

Since it is well known all partial actions of $\mathbb{k}G^*$ on \mathbb{k} (by Proposition 2.3.7), we need to have $V = \{g \in G \mid \alpha_{d(g)} = \alpha_g \neq 0\}$ a subgroup of G and $\alpha_g = \frac{1}{|V|}$ for $g \in V$.

So, if ρ is a partial coaction, we need to have $x = \sum_{h \in V} \frac{1}{|V|} p_h$ where V is a subgroup of G .

Conversely, if we have ρ defined by $\rho(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes \left(\sum_{h \in V} \frac{1}{|V|} p_h \right)$ for V a subgroup of G and $\text{char}(\mathbb{k})$ does not divide $|V|$, we can define

$$\begin{aligned} \cdot : \mathbb{k}G^* \otimes \mathbb{k} &\rightarrow \mathbb{k} \\ p_g \otimes 1_{\mathbb{k}} &\mapsto \alpha_g = \begin{cases} \frac{1}{|V|}, & \text{if } g \in V \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

a partial action.

The above partial action gives origin to a partial coaction $\bar{\rho}$ defined by

$$\bar{\rho}(1_{\mathbb{k}}) = 1_{\mathbb{k}} \otimes \left(\sum_{g \in G} p_g \cdot 1_{\mathbb{k}} g \right) = 1_{\mathbb{k}} \otimes \left(\sum_{h \in V} p_h \cdot 1_{\mathbb{k}} h \right) = \rho(1_{\mathbb{k}}).$$

So $\bar{\rho} = \rho$ and ρ is a right partial coaction. □

3.3 Globalization

In this section we show that every partial H -comodule algebra can be seen as induced partial coactions, as described in Example 3.1.12.

We start defining globalization for partial H -comodule algebras.

Definition 3.3.1. Let $(A, \bar{\rho})$ be a partial H -comodule algebra. An H -comodule algebra (B, ρ) is said to be a globalization for $(A, \bar{\rho})$ if there exists a non unitary algebra monomorphism $\theta : A \rightarrow B$ such that:

(GCA1) $\theta(A)$ is a right ideal of B ;

(GCA2) B is the H -comodule algebra generated by $\theta(A)$;

(GCA3) the partial coaction in $\theta(A)$ is induced by the coaction in B , i.e., the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\theta} & B \\ \bar{\rho} \downarrow & & \downarrow (\theta(1_A) \otimes 1_H) \rho \\ A \otimes H & \xrightarrow{\theta \otimes I} & B \otimes H. \end{array}$$

The reader should understand that by an H -comodule algebra generated by an algebra A we mean the smallest H -comodule algebra containing A .

The next lemma will be useful to construct a globalization for a partial H -comodule algebra. We will denote by ρ the coaction of the partial H -comodule algebra on A and by \cdot_ρ the partial action of H^* on A induced by ρ . Moreover, we denote by $H^* \cdot_\rho A$ the \mathbb{k} -vector space generated by the elements $\{f \cdot_\rho a \mid f \in H^*; a \in A\}$.

Lemma 3.3.2 ([3]Lemma 1). *Let Λ be an H -comodule algebra. If $A \subseteq \Lambda$ is a subalgebra, the subcomodule algebra B generated by A is the subalgebra generated by $H^* \cdot_\rho A$. In other words, the set $S = \{f \cdot_\rho a; a \in A, h \in H\}$ generates B as an algebra.*

The proof of the above lemma for the weak Hopf algebra case does not require a special attention. It is enough to observe that we are only interested in the coalgebra structure of H , which does not change replacing a Hopf algebra by a weak Hopf algebra.

We are going now in direction to show that given a partial module algebra A , then there exists a globalization for it.

First of all recall from Example 3.1.3 that $A \otimes H$ is an H -module algebra with coaction given by $\rho = (I_A \otimes \Delta_H)$. Moreover, if $\bar{\rho}$ is the induced partial coaction on A , then $\bar{\rho}$ is injective by (PCA1). In other words, the map $\bar{\rho} : A \rightarrow A \otimes H$ is an algebra monomorphism.

Using Lemma 3.3.2, take B the comodule algebra generated by $\bar{\rho}(A)$. Moreover, since $\bar{\rho}$ is a partial action, then by (PCA3) we have that $(\bar{\rho} \otimes I_H) \bar{\rho} = (\bar{\rho}(1_A) \otimes 1_H) ((I_A \otimes \Delta_H) \bar{\rho})$.

In other words, for the immersion $\bar{\rho}$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{\rho}} & B \\ \bar{\rho} \downarrow & & \downarrow (\bar{\rho}(1_A) \otimes 1_H) \rho \\ A \otimes H & \xrightarrow{\bar{\rho} \otimes I} & B \otimes H. \end{array}$$

is commutative and B is the comodule algebra generated by $\bar{\rho}(A)$.

It just remains to show the item (GCA1), i.e., we need to show that $\bar{\rho}(A)$ is a right ideal of B . Since $\bar{\rho}$ is an algebra morphism, it is enough to show that $\bar{\rho}(1_A) b \in \bar{\rho}(A)$, for any generator b of B , to obtain $\bar{\rho}(A) B \subseteq \bar{\rho}(A)$.

Note that B is generated by the elements

$$\begin{aligned} \{f \cdot_\rho \bar{\rho}(a) \mid f \in H^*; a \in A\} &= \{f \cdot_\rho (a^{\bar{0}} \otimes a^{\bar{1}}) \mid f \in H^*; a \in A\} \\ &= \{a^{\bar{0}} \otimes a^{\bar{1}} f(a^{\bar{1}}_2) \mid f \in H^*; a \in A\} \end{aligned}$$

and so the generators of B are of the form $b = a^{\bar{0}} \otimes a^{\bar{1}} f(a^{\bar{1}}_2)$.

Thus,

$$\begin{aligned} \bar{\rho}(1_A)(a^{\bar{0}} \otimes a^{\bar{1}} f(a^{\bar{1}}_2)) &= (1^{\bar{0}} \otimes 1^{\bar{1}})(a^{\bar{0}} \otimes a^{\bar{1}} f(a^{\bar{1}}_2)) \\ &= 1^{\bar{0}} a^{\bar{0}} \otimes 1^{\bar{1}} a^{\bar{1}} f(a^{\bar{1}}_2) \\ &\stackrel{(\text{PCA3})}{=} a^{\bar{0}\bar{0}} \otimes a^{\bar{0}\bar{1}} f(a^{\bar{1}}) \\ &= \bar{\rho}(a^{\bar{0}} f(a^{\bar{1}})) \end{aligned}$$

which lies in $\bar{\rho}(A)$. Therefore (GCA1) hold.

So, B with the coaction $\rho = (I_A \otimes \Delta_H)$ is a globalization for the partial module algebra A , called the *standard* globalization.

Therefore, we have the following theorem:

Theorem 3.3.3. *Every partial module algebra has a globalization.* □

Remark 3.3.4. *Note that a simple computation, similar to this one we did above, shows that $\bar{\rho}(A)$ is an ideal of B if and only if A is a symmetric partial module algebra.*

3.4 The reduced tensor product is an A -coring

Let A be a symmetric partial H -comodule algebra. The vector space

$$A \underline{\otimes} H = (A \otimes H) \bar{\rho}(1_A) = \left\{ \sum_{i=1}^n a_i 1^{\bar{0}} \otimes h_i 1^{\bar{1}} \mid a_i \in A; h_i \in H \right\}$$

is called the reduced tensor product of A by H .

We denote by $a \underline{\otimes} h$ the element $a 1^{\bar{0}} \otimes h 1^{\bar{1}}$ in $A \underline{\otimes} H$. Note that $a \underline{\otimes} h = a 1^{\bar{0}} \underline{\otimes} h 1^{\bar{1}}$, thus $A \underline{\otimes} H$ is a subalgebra of $A \otimes H$ with right unit given by $\bar{\rho}(1_A)$. Then we can define left and right actions of A on $A \underline{\otimes} H$ as follows:

$$a \cdot (b \underline{\otimes} h) = (ab) \underline{\otimes} h \quad \text{and} \quad (b \underline{\otimes} h) \cdot a = b a^{\bar{0}} \underline{\otimes} h a^{\bar{1}}$$

for all $a \in A$ and $b \underline{\otimes} h \in A \underline{\otimes} H$.

Note that $A \underline{\otimes} H$ is an A -bimodule via these actions. In fact, $A \underline{\otimes} H$ is clearly a left A -module and since $\bar{\rho}$ is an algebra morphism, $A \underline{\otimes} H$ is a right A -module. The compatibility between these actions is straightforward.

Thus, we can define two maps

$$\begin{aligned} \Delta_{A \underline{\otimes} H}: A \underline{\otimes} H &\rightarrow A \underline{\otimes} H \otimes_A A \underline{\otimes} H \\ a \underline{\otimes} h &\mapsto a \underline{\otimes} h_1 \otimes_A 1_A \underline{\otimes} h_2 \end{aligned}$$

and

$$\begin{aligned}\varepsilon_{A\otimes H}: A\otimes H &\rightarrow A \\ a\otimes h &\mapsto a1^{\bar{0}}\varepsilon_H(h1^{\bar{1}}).\end{aligned}$$

Proposition 3.4.1. *With the above defined maps, $A\otimes H$ becomes an A -coring.*

Proof. First of all, we will show that $\Delta_{A\otimes H}$ is well-defined. In fact, the map

$$\begin{aligned}\tilde{\Delta}: A\otimes H &\rightarrow A\otimes H\otimes_A A\otimes H \\ a\otimes h &\mapsto a\otimes h_1\otimes_A 1_A\otimes h_2\end{aligned}$$

is such that

$$\begin{aligned}\tilde{\Delta}(a\otimes h) &= \tilde{\Delta}(a1^{\bar{0}}\otimes h1^{\bar{1}}) \\ &= a1^{\bar{0}}\otimes h_11^{\bar{1}}_1\otimes_A 1_A\otimes h_21^{\bar{1}}_2 \\ &= a1^{\bar{0}}1^{\bar{0}'}\otimes h_11^{\bar{1}}_11^{\bar{1}'}\otimes_A 1_A\otimes h_21^{\bar{1}}_2 \\ &\stackrel{(\text{PCA4})}{=} a1^{\bar{0}\bar{0}}\otimes h_11^{\bar{0}\bar{1}}\otimes_A 1_A\otimes h_21^{\bar{1}} \\ &= a\otimes h_1\cdot 1^{\bar{0}}\otimes_A 1_A\otimes h_21^{\bar{1}} \\ &= a\otimes h_1\otimes_A 1^{\bar{0}}\cdot 1_A\otimes h_21^{\bar{1}} \\ &= a\otimes h_1\otimes_A 1^{\bar{0}}\otimes h_21^{\bar{1}} \\ &= a\otimes h_1\otimes_A 1_A\otimes h_2 \\ &= \tilde{\Delta}(a\otimes h),\end{aligned}$$

therefore, since $\Delta_{A\otimes H} = \tilde{\Delta}|_{A\otimes H}$, the well-definition follows.

Moreover, $\Delta_{A\otimes H}$ is an A -bimodule map, as follows

$$\begin{aligned}a\cdot\Delta_{A\otimes H}(b\otimes h)\cdot c &= a\cdot b\otimes h_1\otimes_A 1_A\otimes h_2\cdot c \\ &= ab\otimes h_1\otimes_A c^{\bar{0}}\otimes h_2c^{\bar{1}} \\ &= ab\otimes h_1\otimes_A c^{\bar{0}}\cdot 1_A\otimes h_2c^{\bar{1}} \\ &= ab\otimes h_1\cdot c^{\bar{0}}\otimes_A 1_A\otimes h_2c^{\bar{1}} \\ &= abc^{\bar{0}\bar{0}}\otimes h_1c^{\bar{0}\bar{1}}\otimes_A 1_A\otimes h_2c^{\bar{1}} \\ &= abc^{\bar{0}}1^{\bar{0}}\otimes h_1c^{\bar{1}}_11^{\bar{1}}\otimes_A 1_A\otimes h_2c^{\bar{1}}_2 \\ &= abc^{\bar{0}}\otimes h_1c^{\bar{1}}_1\otimes_A 1_A\otimes h_2c^{\bar{1}}_2 \\ &= \Delta_{A\otimes H}(abc^{\bar{0}}\otimes hc^{\bar{1}}) \\ &= \Delta_{A\otimes H}(a\cdot b\otimes h\cdot c).\end{aligned}$$

Now we will check that $\Delta_{A\otimes H}$ is coassociative. In fact,

$$\begin{aligned}(\Delta_{A\otimes H}\otimes I)\Delta_{A\otimes H}(a\otimes h) &= \Delta_{A\otimes H}(a\otimes h_1)\otimes 1_A\otimes h_2 \\ &= a\otimes h_1\otimes 1_A\otimes h_2\otimes 1_A\otimes h_3 \\ &= a\otimes h_1\otimes \Delta_{A\otimes H}(1_A\otimes h_2) \\ &= (I\otimes \Delta_{A\otimes H})(a\otimes h_1\otimes 1_A\otimes h_2) \\ &= (I\otimes \Delta_{A\otimes H})\Delta_{A\otimes H}(a\otimes h).\end{aligned}$$

To show the well-definition of $\varepsilon_{A\otimes H}$, consider

$$\begin{aligned}\tilde{\varepsilon}: A\otimes H &\rightarrow A \\ a\otimes h &\mapsto a\varepsilon_H(h),\end{aligned}$$

which is clearly well-defined. Therefore $\varepsilon_{A \otimes H}$ is also well-defined since $\varepsilon_{A \otimes H} = \tilde{\varepsilon}|_{A \otimes H}$. Moreover,

$$\begin{aligned}
a \cdot \varepsilon_{A \otimes H}(b \otimes h) \cdot c &= ab1^{\bar{0}}c \varepsilon_H(h1^{\bar{1}}) \\
&= ab(1^{\bar{0}}c)^{\bar{0}} \varepsilon_H((1^{\bar{0}}c)^{\bar{1}}) \varepsilon_H(h1^{\bar{1}}) \\
&= ab1^{\bar{0}\bar{0}}c^{\bar{0}} \varepsilon_H(1^{\bar{0}\bar{1}}c^{\bar{1}}) \varepsilon_H(h1^{\bar{1}}) \\
&\stackrel{(\text{PCA4})}{=} ab1^{\bar{0}}1^{\bar{0}'}c^{\bar{0}} \varepsilon_H(1^{\bar{1}}_11^{\bar{1}'}c^{\bar{1}}) \varepsilon_H(h1^{\bar{1}}_2) \\
&= ab1^{\bar{0}}c^{\bar{0}} \varepsilon_H(1^{\bar{1}}_1c^{\bar{1}}) \varepsilon_H(h1^{\bar{1}}_2) \\
&= ab1^{\bar{0}}c^{\bar{0}} \varepsilon_H(h1^{\bar{1}}c^{\bar{1}}) \\
&= abc^{\bar{0}} \varepsilon_H(hc^{\bar{1}}) \\
&= abc^{\bar{0}}1^{\bar{0}} \varepsilon_H(hc^{\bar{1}}1^{\bar{1}}) \\
&= \varepsilon_{A \otimes H}(abc^{\bar{0}} \otimes hc^{\bar{1}}) \\
&= \varepsilon_{A \otimes H}(a \cdot b \otimes h \cdot c).
\end{aligned}$$

It just remains to show the counit property.

$$\begin{aligned}
(I_{A \otimes H} \otimes \varepsilon_{A \otimes H})\Delta_{A \otimes H}(a \otimes h) &= a \otimes h_1 \otimes_A \varepsilon_{A \otimes H}(1_A \otimes h_2) \\
&= a \otimes h_1 \cdot \varepsilon_{A \otimes H}(1_A \otimes h_2) \otimes_A 1_A \\
&= a \otimes h_1 \cdot 1^{\bar{0}} \varepsilon_H(h_2 1^{\bar{1}}) \otimes_A 1_A \\
&= a1^{\bar{0}\bar{0}} \otimes h_1 1^{\bar{0}\bar{1}} \varepsilon_H(h_2 1^{\bar{1}}) \otimes_A 1_A \\
&= a1^{\bar{0}}1^{\bar{0}'} \otimes h_1 1^{\bar{1}}_1 1^{\bar{1}'} \varepsilon_H(h_2 1^{\bar{1}}_2) \otimes_A 1_A \\
&= a1^{\bar{0}} \otimes h_1 1^{\bar{1}}_1 \varepsilon_H(h_2 1^{\bar{1}}_2) \otimes_A 1_A \\
&= a1^{\bar{0}} \otimes h 1^{\bar{1}} \otimes_A 1_A \\
&= a \otimes h \otimes_A 1_A \\
&\simeq a \otimes h
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon_{A \otimes H} \otimes_A I_{A \otimes H})\Delta_{A \otimes H}(a \otimes h) &= (\varepsilon_{A \otimes H} \otimes_A I_{A \otimes H})\Delta_{A \otimes H}(a1^{\bar{0}} \otimes h1^{\bar{1}}) \\
&= \varepsilon_{A \otimes H}(a1^{\bar{0}} \otimes h_1 1^{\bar{1}}_1) \otimes_A 1_A \otimes h_2 1^{\bar{1}}_2 \\
&= 1_A \otimes_A \varepsilon_{A \otimes H}(a1^{\bar{0}} \otimes h_1 1^{\bar{1}}_1) \cdot 1_A \otimes h_2 1^{\bar{1}}_2 \\
&= 1_A \otimes_A a1^{\bar{0}}1^{\bar{0}'} \varepsilon_H(h_1 1^{\bar{1}}_1 1^{\bar{1}'} \otimes h_2 1^{\bar{1}}_2) \\
&= 1_A \otimes_A a1^{\bar{0}}1^{\bar{0}'} \varepsilon_H(h_1 1^{\bar{1}}_2) \varepsilon_H(1^{\bar{1}}_1 1^{\bar{1}'}) \otimes h_2 1^{\bar{1}}_3 \\
&= 1_A \otimes_A a1^{\bar{0}}1^{\bar{0}'} \varepsilon_H(1^{\bar{1}}_1 1^{\bar{1}'}) \otimes h 1^{\bar{1}}_2 \\
&= 1_A \otimes_A a1^{\bar{0}\bar{0}} \varepsilon_H(1^{\bar{0}\bar{1}}) \otimes h 1^{\bar{1}} \\
&= 1_A \otimes_A a1^{\bar{0}} \otimes h 1^{\bar{1}} \\
&= 1_A \otimes_A a \otimes h \\
&\simeq a \otimes h.
\end{aligned}$$

Therefore the reduced tensor product is an A -coring. \square

Chapter 4

Galois extension and Morita theory

In this chapter we will discuss the existence of a Morita context relating the subalgebra of invariants of a partial H -module algebra A and the correspondent partial smash product $A\#H$. We will also show that under some hypothesis, which are trivial in the Hopf algebra case, the Morita context constructed here extends the one described by S. Montgomery in [37].

Using the obtained context, we will develop a Galois theory. The main theorem consist of 3 equivalences, which will be used to define a Galois extension in the partial (co)actions of weak Hopf algebras case.

Finally, we will construct another Morita context, inspired in the coring's case given by S. Caenepeel.

4.1 A Morita context for the subalgebra of invariants

In this section we are interested in a Morita context between the subalgebra of invariants of a partial H -module algebra A and the partial smash product $A\#H$.

4.1.1 A general context

Definition 4.1.1. Let A be a left partial H -module algebra. The set ${}^H A = \{a \in A \mid h \cdot a = (h \cdot 1_A)a \ \forall h \in H\}$ is called the *subalgebra of invariants* of A .

Proposition 4.1.2. Let A be a left partial H -module algebra. Then c lies in ${}^H A$ if and only if $h \cdot (ac) = (h \cdot a)c$, for all $a \in A$ and $h \in H$.

Proof. If $c \in {}^H A$, then for all $a \in A$ and $h \in H$

$$\begin{aligned} h \cdot (ac) &= (h_1 \cdot a)(h_2 \cdot c) \\ &= (h_1 \cdot a)(h_2 \cdot 1_A)c \\ &= (h \cdot a 1_A)c \\ &= (h \cdot a)c. \end{aligned}$$

The converse is immediate taking $a = 1_A$. □

It follows easily by Proposition 4.1.2 that ${}^H A$ is, in fact, a subalgebra of A .

Note that $A\#H$ is clearly a right A -module via the immersion of A into $A\#H$, given in Proposition 2.5.7.

We are now interested in showing that A is a left $A\#H$ module.

Proposition 4.1.3. *Let A be a partial H -module algebra. Then A is a left $A\#H$ -module by*

$$\begin{aligned} \dashv: (A\#H) \otimes_A A &\rightarrow A \\ (a\#h) \dashv b &\mapsto a(h \cdot b). \end{aligned}$$

Proof. Firstly we need to show that \dashv is well-defined. For each $b \in A$, we can define $\tilde{\varphi}_b : A \times H \rightarrow A$ by $\tilde{\varphi}_b(a, h) = a(h \cdot b)$ which is H_L balanced.

In fact, recalling that A is a right H_L -module via $a \triangleleft z = S_R^{-1}(z) \cdot a$ (see Proposition 2.5.1), we have

$$\begin{aligned} \tilde{\varphi}_b(a \triangleleft z, h) &= (a \triangleleft z)(h \cdot b) \\ &= (S_R^{-1}(z) \cdot a)(h \cdot b) \\ &\stackrel{(2.1.9)}{=} a(S_R^{-1}(z) \cdot 1_A)(h \cdot b) \\ &\stackrel{(2.1.8)}{=} a(1_1 S_R^{-1}(z) \cdot 1_A)(1_2 h \cdot b) \\ &\stackrel{(1.28)}{=} a(1_1 \cdot 1_A)(1_2 z h \cdot b) \\ &\stackrel{(2.1.8)}{=} a(1_H \cdot 1_A)(z h \cdot b) \\ &= a(z h \cdot b) \\ &= \tilde{\varphi}_b(a, z h) \end{aligned}$$

for all $a, b \in A$, $z \in H_L$ and $h \in H$. This ensures that the map $\varphi_b : A\#H \rightarrow A$ given by $\varphi_b(a\#h) = a(h \cdot b)$ is well-defined.

So we can define $\phi : (A\#H) \times A \rightarrow A$ by $\phi(a\#h, b) = \varphi_b(a\#h) = a(h \cdot b)$. It is A -balanced:

$$\begin{aligned} \phi(a\#h, bc) &= a(h \cdot bc) \\ &= a(h_1 \cdot b)(h_2 \cdot c) \\ &= \varphi_c(a(h_1 \cdot b)\#h_2) \\ &= \varphi_c((a\#h)(b\#1)) \\ &= \phi((a\#h)(b\#1), c). \end{aligned}$$

Which shows that the map \dashv is well-defined.

It just remains to prove that A is, in fact, a left $A\#H$ module. It is our next step:

For any $a \in A$

$$(1_A\#1_H) \dashv a = 1_A(1_H \cdot a) = a$$

and for any $a, b, c \in A$ and $h, g \in H$,

$$\begin{aligned} (a\#h) \dashv [(b\#g) \dashv c] &= (a\#h) \dashv b(g \cdot c) \\ &= a(\overline{h} \cdot (b(g \cdot c))) \\ &= a(h_1 \cdot b)(h_2 g \cdot c) \\ &= [a(h_1 \cdot b)\#h_2 g] \dashv c \\ &= (a\#h)(b\#g) \dashv c. \end{aligned}$$

And so A is a left $A\#H$ -module, as desired. \square

The next proposition show us an interesting property for the action \rightarrow . It will be useful to show that A is a right $A\#H$ module as well as in Theorem 4.1.25 where we will show an isomorphism concerning \bar{A} .

Proposition 4.1.4. *Let A be a partial H -module algebra. Then for all $h, k \in H$,*

$$\{(1\#h)[(1\#k \rightarrow 1_A)\#1_H]\} \rightarrow 1_A = (1\#h)(1\#k) \rightarrow 1_A.$$

Proof. In fact,

$$\begin{aligned} \{(1\#h)[(1\#k \rightarrow 1_A)\#1_H]\} \rightarrow 1_A &= \{(1\#h)[(k \cdot 1_A)\#1_H]\} \rightarrow 1_A \\ &= [(h_1 \cdot k \cdot 1_A)\#h_2] \rightarrow 1_A \\ &= (h_1 \cdot k \cdot 1_A)(h_2 \cdot 1_A) \\ &= (h \cdot k \cdot 1_A) \\ &= (h_1 \cdot 1_A)(h_2 k \cdot 1_A) \\ &= [(h_1 \cdot 1_A)\#h_2 k] \rightarrow 1_A \\ &= (1\#h)(1\#k) \rightarrow 1_A \end{aligned}$$

as desired. \square

Remark 4.1.5. *Clearly, A is a right ${}^H A$ -module with action given by*

$$\begin{aligned} \triangleleft: A \otimes {}^H A &\rightarrow A \\ a \otimes b &\mapsto ab. \end{aligned}$$

Theorem 4.1.6. *With the maps \rightarrow and \triangleleft , A becomes an $(A\#H, {}^H A)$ -bimodule.*

In fact,

$$\begin{aligned} (a\#h) \rightarrow (b \triangleleft c) &= (a\#h) \rightarrow bc \\ &= a[h \cdot (bc)] \\ &= a[(h \cdot b)c] \\ &= [a(h \cdot b)]c \\ &= (a\#h \rightarrow b) \triangleleft c \end{aligned}$$

for all $a, b \in A$, $h \in H$ and $c \in {}^H A$.

At this point, the savvy readers already noticed that we will use A as one of the modules of the Morita context. Actually, it is usual to use A as both modules in the Morita context. For weak Hopf algebras it is not true in general. So our next step is to construct another module which should be an $({}^H A, A\#H)$ -bimodule.

Proposition 4.1.7. *Define*

$$\begin{aligned} Q &= \{a\#h \mid (1\#g)(a\#h) = [(g \cdot 1_A)\#1](a\#h), \forall g \in H\} \\ &= \{a\#h \mid (g_1 \cdot a)\#g_2 h = (g \cdot 1_A)a\#h, \forall g \in H\}. \end{aligned}$$

Then Q is a right ideal in $A\#H$.

Proof. To show that Q is a right ideal we need to show that given $a\#h \in Q$ and $(b\#g) \in A\#H$, then $(1\#k)[(a\#h)(b\#g)] = ((k \cdot 1)\#1)[(a\#h)(b\#g)]$ for all $k \in H$.

In fact, using the associativity of the partial smash product and the fact that $a\#h \in Q$ we have

$$\begin{aligned} (1\#k)[(a\#h)(b\#g)] &= [(1\#k)(a\#h)](b\#g) \\ &= [((k \cdot 1)\#1)(a\#h)](b\#g) \\ &= ((k \cdot 1)\#1)[(a\#h)(b\#g)] \end{aligned}$$

as desired. \square

Remark 4.1.8. *By the above proposition Q becomes a right $A\#H$ -module with action defined by $(b\#h) \leftarrow (a\#g) = (b\#h)(a\#g)$.*

We can also define

$$\begin{aligned} \triangleright: {}^H A \otimes Q &\rightarrow Q \\ a \otimes (b\#h) &\mapsto (a\#1_H)(b\#h). \end{aligned}$$

To show that Q is a left ${}^H A$ -module with this map, it is enough to check that $a \triangleright (b\#h)$ lies in Q for all $a \in {}^H A$ and $(b\#h) \in Q$.

In fact, let $a \in {}^H A$ and $(b\#g) \in Q$, then for all $h \in H$ we have

$$\begin{aligned} (1\#h)[a \triangleright (b\#g)] &= (1\#h)(a\#1_H)(b\#g) \\ &= ((h_1 \cdot a)\#h_2)(b\#g) \\ &\stackrel{(2.5.8)}{=} ((h_1 \cdot a)\#1_H)(1_A\#h_2)(b\#g) \\ &= ((h_1 \cdot a)\#1_H)((h_2 \cdot 1_A)\#1_H)(b\#g) \\ &\stackrel{(2.5.7)}{=} ((h_1 \cdot a)(h_2 \cdot 1_A)\#1_H)(b\#g) \\ &= ((h \cdot a)\#1_H)(b\#g) \\ &= ((h \cdot 1_A)a\#1_H)(b\#g) \\ &\stackrel{(2.5.7)}{=} ((h \cdot 1_A)\#1_H)(a\#1_H)(b\#g) \\ &= ((h \cdot 1_A)\#1_H)[a \triangleright (b\#g)]. \end{aligned}$$

Note that with these actions, Q becomes an $({}^H A, A\#H)$ -bimodule.

Now we have two rings and two bimodules, as required in the definition of a Morita context. It just remains to define the appropriate maps. It is our next goal.

Define

$$\begin{aligned} \tau: Q \otimes_{A\#H} A &\rightarrow {}^H A \\ (d\#l) \otimes a &\mapsto d(l \cdot a) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mu: A \otimes_{H_A} Q &\rightarrow A\#H \\ a \otimes (d\#l) &\mapsto (a\#1_H)(d\#l). \end{aligned} \tag{4.2}$$

Proposition 4.1.9. *The following properties hold:*

(1) *The map τ defines a morphism of ${}^H A$ -bimodules*

(2) *The map μ defines a morphism of $A \# H$ -bimodules*

Proof. (1) We firstly show that τ is well-defined. In fact, define

$$\begin{aligned}\tilde{\tau}: Q \times A &\rightarrow A \\ (d \# h, a) &\mapsto (d \# h) \rightharpoonup a = d(h \cdot a).\end{aligned}$$

To show that τ is well-defined we just need to check that $\tilde{\tau}$ is $A \# H$ -balanced. Let $d \# l \in Q$, $a \# h \in A \# H$ and $c \in A$ we have that

$$\begin{aligned}\tilde{\tau}((d \# h), (a \# h) \rightharpoonup c) &= (d \# h) \rightharpoonup [(a \# h) \rightharpoonup c] \\ &= (d \# h)(a \# h) \rightharpoonup c \\ &= \tilde{\tau}((d \# h)(a \# h), c) \\ &= \tilde{\tau}((d \# h) \leftarrow (a \# h), c)\end{aligned}$$

and it means that $\tau: Q \otimes_{A \# H} A \rightarrow A$ is well-defined.

Moreover, $\text{Im}(\tau) \subseteq {}^H A$. In fact, let $h \in H$, $d \# l \in Q$ and $a \in A$. Then

$$\begin{aligned}h \cdot \tau((d \# l) \otimes a) &= h \cdot [d(l \cdot a)] \\ &= (h_1 \cdot d)(h_2 l \cdot a) \\ &= [(h_1 \cdot d) \# h_2 l] \rightharpoonup a \\ &\stackrel{(4.1.7)}{=} [(h \cdot 1_A) d \# l] \rightharpoonup a \\ &= (h \cdot 1_A) d(l \cdot a) \\ &= (h \cdot 1_A) \tau((d \# l) \otimes a).\end{aligned}$$

It is straightforward that τ is a morphism of ${}^H A$ -bimodules.

(2) To show that μ is well-defined, take

$$\begin{aligned}\tilde{\mu}: A \times Q &\rightarrow A \# H \\ (a, (d \# l)) &\mapsto (a \# 1_A)(d \# l).\end{aligned}$$

It is enough to show $\tilde{\mu}$ is ${}^H A$ -balanced. Let $a \in A$, $c \in {}^H A$ and $d \# l \in Q$, then

$$\begin{aligned}\tilde{\mu}(ac, d \# l) &= (ac \# 1)(d \# l) \\ &= (a \# 1)(c \# 1)(d \# l) \\ &= \tilde{\mu}(a, c \triangleright (d \# l))\end{aligned}$$

and it follows that μ is well-defined.

Moreover, it remains to show that μ is $A \# H$ -bilinear

In fact, let $(a\#h), (c\#g) \in A\#H$, $b \in A$ and $d\#l \in Q$. Then

$$\begin{aligned}
(a\#h)\mu(b \otimes (d\#l))(c\#g) &= (a\#h)(b\#1)(d\#l)(c\#g) \\
&= (a(h_1 \cdot b)\#h_2)(d\#l)(c\#g) \\
&= (a(h_1 \cdot b)\#1)(1\#h_2)(d\#l)(c\#g) \\
&= (a(h_1 \cdot b)\#1)((h_2 \cdot 1)\#1)(d\#l)(c\#g) \\
&= (a(h_1 \cdot b)(h_2 \cdot 1)\#1)(d\#l)(c\#g) \\
&= (a(h \cdot b)\#1)(d\#l)(c\#g) \\
&= \mu(a(h \cdot b) \otimes (d\#l))(c\#g) \\
&= \mu((a\#h) \rightharpoonup b) \otimes ((d\#l) \leftarrow (c\#g)).
\end{aligned}$$

□

Theorem 4.1.10. *The sextuple $({}^H A, A\#H, A, Q, \tau, \mu)$ is a Morita context.*

Proof. It just remains to prove that

$$\mu(a \otimes (d\#l)) \rightharpoonup c = a \triangleleft \tau((d\#l) \otimes c) \quad (4.3)$$

and

$$\tau(d\#l \otimes b) \triangleright (d'\#l') = (d\#l) \leftarrow \mu(b \otimes d'\#l') \quad (4.4)$$

for all $a, b, c \in A$ and $(d\#l), (d'\#l') \in Q$. In fact,

$$\begin{aligned}
\mu(a \otimes (d\#l)) \rightharpoonup c &= (a\#1)(d\#l) \rightharpoonup c \\
&= (ad\#l) \rightharpoonup c \\
&= ad(l \cdot c) \\
&= a\tau((d\#l) \otimes c) \\
&= a \triangleleft \tau((d\#l) \otimes c)
\end{aligned}$$

and

$$\begin{aligned}
\tau(d\#l \otimes b) \triangleright (d'\#l') &= d(l \cdot b) \triangleright (d'\#l') \\
&= (d(l \cdot b)\#1)(d'\#l') \\
&= (d(l_1 \cdot b)\#1)((l_2 \cdot 1)\#1)(d'\#l') \\
&= (d(l_1 \cdot b)\#1)(1\#l_2)(d'\#l') \\
&= (d(l_1 \cdot b)\#l_2)(d'\#l') \\
&= (d\#l)(b\#1)(d'\#l') \\
&= (d\#l) \leftarrow (b\#1)(d'\#l') \\
&= (d\#l) \leftarrow \mu(b \otimes (d'\#l'))
\end{aligned}$$

and we have the required result. □

4.1.2 Recalling the classical context

In this subsection H will be a finite dimensional weak Hopf algebra.

Definition 4.1.11 ([9, Lemma 3.2]). An element $t \in H$ is a left integral in H if one of the following equivalent conditions hold:

- (1) $h t = \varepsilon_L(h) t$;
- (2) $t_1 \otimes h t_2 = S(h) t_1 \otimes t_2$;

for all $h \in H$. Denote by \int_l^H the vector space of the left integrals in H .

Remark 4.1.12. If H has an invertible antipode, then applying $S^{-1} \otimes I$ in the equality (2) we obtain

$$S^{-1}(t_1) \otimes h t_2 = S^{-1}(t_1) h \otimes t_2. \quad (4.5)$$

Proposition 4.1.13. For all $h \in H$

$$T \in \int_l^{H^*} \Leftrightarrow h_1 T(h_2) = \varepsilon_L(h_1) T(h_2). \quad (4.6)$$

Proof. Supposing T a left integral in H^* , then for any $f \in H^*$,

$$\begin{aligned} f(h_1 T(h_2)) &= f(h_1) T(h_2) \\ &= (f * T)(h) \\ &= (\varepsilon_L^*(f) * T)(h) \\ &= \varepsilon_L^*(f)(h_1) T(h_2) \\ &= f(\varepsilon_L(h_1)) T(h_2) \\ &= f(\varepsilon_L(h_1) T(h_2)) \end{aligned}$$

and so $h_1 T(h_2) = \varepsilon_L(h_1) T(h_2)$.

Conversely, if $h_1 T(h_2) = \varepsilon_L(h_1) T(h_2)$, then for any $f \in H^*$

$$\begin{aligned} (f * T)(h) &= f(h_1) T(h_2) \\ &= f(\varepsilon_L(h_1) T(h_2)) \\ &= f(\varepsilon_L(h_1)) T(h_2) \\ &= \varepsilon_L^*(f)(h_1) T(h_2) \\ &= (\varepsilon_L^*(f) * T)(h) \end{aligned}$$

and it means that T is a left integral in H^* . □

Remark 4.1.14. Recall that H is a left H^* -module with the action given by $h \rightarrow f = f_2(h) f_1$ and also H^* is a left H -module with action given by $f \rightarrow h = h_1 f(h_2)$.

Using the above remark we can define a dual pair of integrals in a weak Hopf algebra.

Definition 4.1.15. Let $t \in \int_l^H$ and $T \in \int_l^{H^*}$ we say that the pair (t, T) is a dual pair of left integrals if $t \rightarrow T = 1_{H^*}$ and $T \rightarrow t = 1_H$.

In [9] the definition of dual pair is simplified in a theorem:

Theorem 4.1.16. [9, Theorem 3.18] *Let $t \in \int_l^H$. If there exists $T \in H^*$ such that $T \rightarrow t = 1_H$, then it is unique and it is a left integral. Moreover, $t \rightarrow T = 1_{H^*}$.*

The above theorem says, in others words, that if $T \rightarrow t = 1_H$, then (t, T) is a dual pair of left integrals.

Remark 4.1.17. *If H is a finite dimensional Hopf algebra, it is well known that the space \int_l^H has dimension 1 over the field \mathbb{k} . Moreover, the map*

$$\begin{aligned} \theta: H^* \otimes \int_l^H &\rightarrow H \\ f \otimes t &\mapsto f \rightarrow t \end{aligned}$$

is an isomorphism of vector spaces.

Then there exists an element $T \in H^$ such that $T \rightarrow t = 1_H$ and by Theorem 4.1.16 the pair (t, T) is a dual pair of left integrals.*

The dimension argument does not works for weak Hopf algebras. As an example, note that for a finite groupoid G , the weak Hopf algebra $\mathbb{k}G$ has the integrals $t_e = \sum_{gg^{-1}=e} \delta_g$ and $t_{e'} = \sum_{hh^{-1}=e'} \delta_h$ and they are linearly independent if $e \neq e'$. Moreover, the dimension of $\int_l^{\mathbb{k}G}$ over \mathbb{k} is the order of G_0 .

In [9], the authors show that the existence of a dual pair is equivalent to the weak Hopf algebra to be a Frobenius algebra. Moreover, in [34], there is an example of weak Hopf algebra which is not a Frobenius algebra. Hence, there exists a weak Hopf algebra which do not have a dual pair of left integrals. However, the integrals can be found in the groupoid algebra, as follows:

Example 4.1.18. *Let G be a finite groupoid. Take $t = \sum_{g \in G} \delta_g$ in $\mathbb{k}G$ and $T = \sum_{e \in G_0} p_e$ in $\mathbb{k}G^*$. Thus (t, T) is a dual pair of left integrals.*

Remark 4.1.19. *Note that, if (t, T) is a dual pair, then*

$$1_1 \otimes 1_2 = t_1 \otimes t_2 T(t_3). \quad (4.7)$$

Proposition 4.1.20 ([9, (3.44)-(3.45) with $x=1$]). *Let (t, T) be a dual pair of left integrals, so we have the following equality*

$$T(S^{-1}(t_1))t_2 = S^{-1}(t_1)T(t_2) = 1_H. \quad (4.8)$$

Proposition 4.1.21. *Let t be a left integral in H and consider*

$$e = 1\#t_2 \otimes 1\#S^{-1}(t_1)$$

in $A\#H \otimes_A A\#H$. Then, for any $a\#h$ in $A\#H$, we have that

$$1\#t_2 \otimes (1\#S^{-1}(t_1))(a\#h) = (a\#h)(1\#t_2) \otimes 1\#S^{-1}(t_1).$$

Proof. Let $a\#h$ in $A\#H$, so

$$\begin{aligned} 1\#t_2 \otimes_A (1\#S^{-1}(t_1))(a\#h) &= 1\#t_2 \otimes_A (1\#S^{-1}(t_1))(a(h_1 \cdot 1)\#h_2) \\ &= 1\#t_3 \otimes_A S^{-1}(t_2) \cdot a(h_1 \cdot 1)\#S^{-1}(t_1)h_2 \\ &= 1\#t_3 \otimes_A [S^{-1}(t_2) \cdot a(h_1 \cdot 1)\#1][1\#S^{-1}(t_1)h_2] \\ &= (1\#t_3)[S^{-1}(t_2) \cdot a(h_1 \cdot 1)\#1] \otimes_A 1\#S^{-1}(t_1)h_2 \\ &= t_3 \cdot S^{-1}(t_2) \cdot a(h_1 \cdot 1)\#t_4 \otimes_A 1\#S^{-1}(t_1)h_2 \\ &\stackrel{(PMA4)}{=} [t_3S^{-1}(t_2) \cdot a(h_1 \cdot 1)](t_4 \cdot 1)\#t_5 \otimes_A 1\#S^{-1}(t_1)h_2 \\ &\stackrel{(1.27)}{=} [1_1 \cdot a(h_1 \cdot 1)](1_2 t_2 \cdot 1)\#t_3 \otimes_A 1\#S^{-1}(t_1)h_2 \\ &\stackrel{(2.1.8)}{=} a(h_1 \cdot 1)(t_2 \cdot 1)\#t_3 \otimes_A 1\#S^{-1}(t_1)h_2 \\ &= [(a(h_1 \cdot 1)\#t_2)(1\#1)] \otimes_A 1\#S^{-1}(t_1)h_2 \\ &= a(h_1 \cdot 1)\#t_2 \otimes_A 1\#S^{-1}(t_1)h_2 \\ &\stackrel{(4.5)}{=} a(h_1 \cdot 1)\#h_2 t_2 \otimes_A 1\#S^{-1}(t_1) \\ &= (a\#h)(1\#t_2) \otimes_A 1\#S^{-1}(t_1) \end{aligned}$$

as desired. □

Now let (t, T) be a dual pair of left integrals and $z \in H_L$, so we have the following equality

$$\begin{aligned} (a\#z_1h_1 \rightarrow 1)T(z_2h_2) &\stackrel{(1.7)}{=} (a\#z_1h_1 \rightarrow 1)T(1_2h_2) \\ &= (a\#zh_1 \rightarrow 1)T(h_2) \\ &= [(a \triangleleft z)\#h_1 \rightarrow 1]T(h_2), \end{aligned}$$

which means that the following map

$$\begin{aligned} \Phi : A\#H &\longrightarrow A \\ a\#h &\longmapsto (a\#h_1 \rightarrow 1)T(h_2), \end{aligned} \tag{4.9}$$

is well-defined because it is H_L -balanced.

Restricting Φ to $A\#H$ we obtain that

$$\begin{aligned} \Phi(a\#h) &= \Phi(a(h_1 \cdot 1)\#h_2) \\ &= [(a(h_1 \cdot 1)\#h_2) \rightarrow 1]T(h_2) \\ &= a(h_1 \cdot 1)(h_2 \cdot 1)T(h_2) \\ &= a(h_1 \cdot 1)T(h_2). \end{aligned}$$

Proposition 4.1.22. *With the above notation, Φ is a map of A -bimodules.*

Proof. It is clear that Φ is left A -linear. Now we will check that it is also right A -linear.

$$\begin{aligned}
\Phi((a\#h)(b\#1)) &= \Phi(a(h_1 \cdot b)\#h_2) \\
&= a(h_1 \cdot b)(h_2 \cdot 1)T(h_3) \\
&= a(h_1 \cdot b)T(h_2) \\
&\stackrel{(4.6)}{=} a(\varepsilon_L(h_1) \cdot b)T(h_2) \\
&\stackrel{(2.1.9)}{=} a(\varepsilon_L(h_1) \cdot 1)bT(h_2) \\
&\stackrel{(4.6)}{=} a(h_1 \cdot 1)bT(h_2) \\
&= \Phi(a\#h)b
\end{aligned}$$

as desired □

Proposition 4.1.23. *The map Φ and the element e above constructed have the following compatibility:*

- (1) $(\Phi \otimes_A I)(e) = 1_A \otimes_A 1\#1 \simeq 1\#1$;
- (2) $(I \otimes_A \Phi)(e) = 1\#1 \otimes_A 1_A \simeq 1\#1$.

Proof. (1)

$$\begin{aligned}
(\Phi \otimes_A I)(e) &= (\Phi \otimes_A I)(1\#t_2 \otimes_A 1\#S^{-1}(t_1)) \\
&= \Phi(1\#t_2) \otimes_A 1\#S^{-1}(t_1) \\
&\simeq [\Phi(1\#t_2)\#1][1\#S^{-1}(t_1)] \\
&= [(t_2 \cdot 1)T(t_3)\#1][1\#S^{-1}(t_1)] \\
&\stackrel{(2.5.8)}{=} (t_2 \cdot 1)T(t_3)\#S^{-1}(t_1) \\
&= (t_2T(t_3) \cdot 1)\#S^{-1}(t_1) \\
&\stackrel{(4.7)}{=} (1_2 \cdot 1)\#S^{-1}(1_1) \\
&\stackrel{(2.5.3)}{=} 1 \triangleleft 1_2\#S^{-1}(1_1) \\
&= 1\#1_2S^{-1}(1_1) \\
&= 1\#1.
\end{aligned}$$

(2)

$$\begin{aligned}
(I \otimes_A \Phi)(e) &= (I \otimes_A \Phi)(1\#t_2 \otimes_A 1\#S^{-1}(t_1)) \\
&= 1\#t_2 \otimes_A \Phi(1\#S^{-1}(t_1)) \\
&\simeq 1\#t_2[\Phi(1\#S^{-1}(t_1))\#1] \\
&= (1\#t_3)[S^{-1}(t_2) \cdot 1\#1]T(S^{-1}(t_1)) \\
&= t_3 \cdot (S^{-1}(t_2) \cdot 1)\#t_4T(S^{-1}(t_1)) \\
&\stackrel{(PMA4)}{=} (t_3S^{-1}(t_2) \cdot 1)(t_4 \cdot 1)\#t_5T(S^{-1}(t_1)) \\
&\stackrel{(1.27)}{=} (1_1 \cdot 1)(1_2t_2 \cdot 1)\#t_3T(S^{-1}(t_1)) \\
&= (t_2 \cdot 1)\#t_3T(S^{-1}(t_1)) \\
&= 1\#t_2T(S^{-1}(t_1)) \\
&\stackrel{(4.8)}{=} 1\#1.
\end{aligned}$$

□

Now, we are able to equip A with a structure of right $A\#H$ -module, via

$$\leftarrow : A \otimes_A A\#H \rightarrow A$$

given by

$$a \leftarrow b\#h = \Phi\{(1\#t_2)[(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1_H](b\#h)\} \quad (4.10)$$

which is clearly well-defined.

Proposition 4.1.24. *With the above definition, A is a right $A\#H$ -module.*

Proof. In fact, for all $a, b, c \in A$ and $g, h \in H$,

$$\begin{aligned} a \leftarrow (1\#1) &= \Phi\{(1\#t_2)[(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1_H](1\#1)\} \\ &= \Phi\{(1\#t_2)[(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1_H]\} \\ &= \Phi\{(1\#t_2)[(S^{-1}(t_1) \cdot 1_A)a\#1_H]\} \\ &= \Phi\{(t_2 \cdot S^{-1}(t_1) \cdot 1_A)(t_3 \cdot a)\#t_4\} \\ &\stackrel{(PMA4)}{=} \Phi\{(t_2S^{-1}(t_1) \cdot 1_A)(t_3 \cdot 1_A)(t_4 \cdot a)\#t_5\} \\ &\stackrel{(PMA2)}{=} \Phi\{(t_2S^{-1}(t_1) \cdot 1_A)(t_3 \cdot a)\#t_4\} \\ &\stackrel{(1.27)}{=} \Phi\{(1_1 \cdot 1_A)(1_2t_1 \cdot a)\#t_2\} \\ &\stackrel{(2.1.8)}{=} \Phi\{(t_1 \cdot a)\#t_2\} \\ &= (t_1 \cdot a)(t_2 \cdot 1_A)T(t_3) \\ &\stackrel{(PMA2)}{=} (t_1 \cdot a)T(t_2) \\ &= t_1T(t_2) \cdot a \\ &= (T \rightarrow t) \cdot a \\ &= 1_H \cdot a \\ &= a \end{aligned}$$

and moreover

$$\begin{aligned} (a \leftarrow b\#h) \leftarrow c\#g &= \\ &= \Phi\{(1\#t_2)[(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1_H](b\#h)\} \leftarrow c\#g \\ &= \Phi\{(1\#t_{2'})[(1\#S^{-1}(t_{1'}) \rightarrow 1_A)\Phi\{(1\#t_2) \\ &\quad [(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1_H](b\#h)\}\#1_H](c\#g)\} \\ &\stackrel{(4.1.22)}{=} \Phi\{(1\#t_{2'})[\Phi\{[(1\#S^{-1}(t_{1'}) \rightarrow 1_A)\#1_H](1\#t_2) \\ &\quad [(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1_H](b\#h)\}\#1_H](c\#g)\} \\ &\stackrel{(4.1.21)}{=} \Phi\{(1\#t_{2'})[\Phi\{(1\#t_2)[((1\#S^{-1}(t_1)) \\ &\quad [(1\#S^{-1}(t_{1'}) \rightarrow 1_A)\#1_H] \rightarrow 1_A)a\#1_H](b\#h)\}\#1_H](c\#g)\} \\ &\stackrel{(4.1.4)}{=} \Phi\{(1\#t_{2'})[\Phi\{(1\#t_2)[((1\#S^{-1}(t_1))(1\#S^{-1}(t_{1'})) \rightarrow 1_A)a\#1_H] \\ &\quad (b\#h)\}\#1_H](c\#g)\} \end{aligned}$$

It just remains to prove that α is $({}^H A, A \# H)$ -bilinear. In fact, let $c \in {}^H A$ and $a \in A$, so

$$\begin{aligned}
\alpha(ca) &= (1 \# t_2)[(1 \# S^{-1}(t_1) \rightarrow 1_A)ca \# 1_H] \\
&= (1 \# t_2)[(S^{-1}(t_1) \cdot 1_A)ca \# 1_H] \\
&\stackrel{(4.1.1)}{=} (1 \# t_2)[(S^{-1}(t_1) \cdot c)a \# 1_H] \\
&= (1 \# t_3)[(S^{-1}(t_2) \cdot c)(S^{-1}(t_1) \cdot 1_A)a \# 1_H] \\
&= (1 \# t_2)[((S^{-1}(t_2) \cdot c) \# S^{-1}(t_1) \rightarrow 1_A)a \# 1_H] \\
&= (1 \# t_2)[((1 \# S^{-1}(t_1))(c \# 1) \rightarrow 1_A)a \# 1_H] \\
&\stackrel{(4.1.21)}{=} (c \# 1)(1 \# t_2)[((1 \# S^{-1}(t_1)) \rightarrow 1_A)a \# 1_H] \\
&= (c \# 1)\alpha(a)
\end{aligned}$$

and this means that α is left ${}^H A$ -linear.

For the right $A \# H$ -linearity we use a dirty trick which consist in apply β to the element and discard it later using its injectivity. Moreover, remember that Q is a right $A \# H$ ideal. If $a \in A$ and $b \# h \in A \# H$, then

$$\begin{aligned}
\beta(\alpha(a \leftarrow b \# h)) &= a \leftarrow b \# h \\
&= \Phi\{(1 \# t_2)[(1 \# S^{-1}(t_1) \rightarrow 1_A)a \# 1_H](b \# h)\} \\
&= \Phi[\alpha(a)(b \# h)] \\
&= \beta[\alpha(a)(b \# h)] \\
&= \beta[\alpha(a) \leftarrow (b \# h)].
\end{aligned}$$

Since β is injective, we have $\alpha(a \leftarrow b \# h) = \alpha(a) \leftarrow (b \# h)$ as desired. \square

The above theorem says that we can see the Morita context from Theorem 4.1.10 switching Q for A .

We will now use the definition of α to make clear who is τ and μ in the Morita context.

Consider $(,) : A \otimes_{A \# H} A \longrightarrow {}^H A$ defined by $(a, b) = \tau(\alpha(a) \otimes b)$ and $[,] : A \otimes_{A \# H} A \longrightarrow A \# H$ defined by $[a, b] = \mu(a \otimes \alpha(b))$. Then

$$\begin{aligned}
[a, b] &= \mu(a \otimes \alpha(b)) \\
&= \mu(a \otimes (1 \# t_2)[(1 \# S^{-1}(t_1) \rightarrow 1_A)b \# 1_H]) \\
&= (a \# 1_H)(1 \# t_2)[(1 \# S^{-1}(t_1) \rightarrow 1_A)b \# 1_H] \\
&= (1 \# t_2)[((1 \# S^{-1}(t_1))(a \# 1_H) \rightarrow 1_A)b \# 1_H] \\
&= (1 \# t_3)[((S^{-1}(t_2) \cdot a) \# S^{-1}(t_1)) \rightarrow 1_A)b \# 1_H] \\
&= (1 \# t_3)[(S^{-1}(t_2) \cdot a)(S^{-1}(t_1) \cdot 1_A)b \# 1_H] \\
&= (1 \# t_2)[(S^{-1}(t_1) \cdot a)b \# 1_H] \\
&= t_2 \cdot ((S^{-1}(t_1) \cdot a)b) \# t_3 \\
&\stackrel{(PMA4)}{=} (t_2 S^{-1}(t_1) \cdot a)(t_3 \cdot b) \# t_4 \\
&\stackrel{(1.27)}{=} (1_1 \cdot a)(1_2 t_1 \cdot b) \# t_2 \\
&\stackrel{(2.1.8)}{=} a(t_1 \cdot b) \# t_2 \\
&= (a \# t)(b \# 1_H) \\
&= (a \# 1)(1 \# t)(b \# 1)
\end{aligned}$$

and

$$\begin{aligned}
(a, b) &= \tau(\alpha(a) \otimes b) \\
&= \tau((1\#t_2)[(1\#S^{-1}(t_1) \rightarrow 1_A)a\#1] \otimes b) \\
&= \tau(t_2 \cdot [(1\#S^{-1}(t_1) \rightarrow 1_A)a]\#t_3 \otimes b) \\
&= (t_2 \cdot [(1\#S^{-1}(t_1) \rightarrow 1_A)a])(t_3 \cdot b) \\
&= t_2 \cdot [(1\#S^{-1}(t_1) \rightarrow 1_A)ab] \\
&= t_2 \cdot [(S^{-1}(t_1) \cdot 1_A)ab] \\
&\stackrel{(PMA4)}{=} (t_2S^{-1}(t_1) \cdot 1_A)(t_3 \cdot ab) \\
&= (S^{-1}(t_1S(t_2)) \cdot 1_A)(t_3 \cdot ab) \\
&= (S^{-1}(S(1_1)) \cdot 1_A)(1_2t \cdot ab) \\
&= (1_1 \cdot 1_A)(1_2t \cdot ab) \\
&\stackrel{(2.1.8)}{=} t \cdot ab.
\end{aligned}$$

So the Morita context $(\overset{H}{A}, A\#H, A, A, (,), [,])$ for partial actions of weak Hopf algebras with a dual pair of integrals extends the one described by S. Montgomery in [37] for actions of Hopf algebras as well as the one described by M. Alves and E. Batista in [3] for partial actions of Hopf algebras.

Now we are able to start thinking about Galois theory for partial actions of weak Hopf algebras.

4.2 Galois theory

Galois theory is largely studied in several contexts. Starting from the classical theory, where the structures are field extensions, several authors gave new approaches to the theory using groups, groupoids, Hopf algebras, weak Hopf algebras and corings (cf. [3, 7, 13–20, 25–27, 30, 37]) usually acting on algebras. In some cases we can also have partial actions on algebras.

All those theories are centered in a main theorem which give equivalent conditions for an extension to be Galois. Our purpose now is to construct such a theorem for partial actions of weak Hopf algebra.

Along this section H will be a finite dimensional weak Hopf algebra, (t, T) will be a dual pair of left integrals and A will be a symmetric left partial H -module algebra and consequently a symmetric right partial H^* -comodule algebra. We will denote by \cdot the partial action and by $\bar{\rho}$ the corresponding partial coaction.

Since T is a left integral in H^* , so $T' = T \circ S^{-1}$ is a right integral in H^* .

Note that H is a right H^* -module algebra with action given by $h \leftarrow f = f(h_1)h_2$.

Moreover, using the fact that (t, T) is a dual pair, one can show that $t \leftarrow T' = 1_H$. In fact,

$$\begin{aligned}
t \leftarrow T' &= t \leftarrow T \circ S^{-1} \\
&= T(S^{-1}(t_1))t_2 \\
&\stackrel{(4.1.20)}{=} 1_H.
\end{aligned}$$

To show our main theorem, we first need to show that the reduced tensor product $A \otimes H^*$ is isomorphic to the partial smash product $A \# H$ as vector spaces. We draw the attention to the fact that the partial smash product was obtained from the tensor product over H_L and no more over the ground field like in the case of Hopf algebras.

To achieve the aim, we should define the following maps:

$$\begin{aligned} \alpha : A \otimes H^* &\rightarrow A \# H \\ a \otimes f &\mapsto a \# (t \leftarrow f) \end{aligned}$$

and

$$\begin{aligned} \beta : A \# H &\rightarrow A \otimes H^* \\ a \# h &\mapsto a(h_1 \cdot 1) \otimes T'_1(S(h_2))T'_2. \end{aligned}$$

Theorem 4.2.1. *With the above notation, α is an isomorphism of vector spaces whose inverse is β .*

Proof. First of all, we need to show that α is well-defined. For it, consider the map

$$\begin{aligned} \tilde{\alpha} : A \otimes H^* &\rightarrow A \# H \\ a \otimes f &\mapsto a \# (t \leftarrow f) \end{aligned}$$

which is clearly well-defined. Now let us see that $\tilde{\alpha}(A \otimes H^*) \subseteq A \# H$. In fact, for all $a \in A$ and $f \in H^*$,

$$\begin{aligned} \tilde{\alpha}(a \otimes f) &= \tilde{\alpha}(a1^{\bar{0}} \otimes f * 1^{\bar{1}}) \\ &= a1^{\bar{0}} \# [t \leftarrow (f * 1^{\bar{1}})] \\ &= a1^{\bar{0}} \# f(t_1)1^{\bar{1}}(t_2)t_3 \\ &= a1^{\bar{0}}1^{\bar{1}}(t_2) \# f(t_1)t_3 \\ &\stackrel{(*)}{=} a(t_2 \cdot 1_A) \# f(t_1)t_3 \\ &= (a \# f(t_1)t_2)(1_A \# 1_H) \\ &= a \# f(t_1)t_2 \\ &= a \# (t \leftarrow f) \in A \# H \end{aligned}$$

where in (*) we used that the partial action and the partial coaction are related to each other by $h \cdot a = a^{\bar{0}}a^{\bar{1}}(h)$.

Note that α can be defined as the restriction $\tilde{\alpha}|_{A \otimes H^*}$.

Next, we will prove that β is well-defined. With this aim in view, we will first define a map

$$\begin{aligned} \tilde{\beta} : A \otimes_{H_L} H &\rightarrow A \otimes H^* \\ a \otimes h &\mapsto a \otimes T'_1(S(h))T'_2. \end{aligned}$$

To show that $\tilde{\beta}$ is well-defined, we just need to show that it is H_L -balanced.

Recall from classical linear algebra that for a finite dimensional vector space the map

$$\begin{aligned} \wedge : H &\rightarrow H^{**} \\ k &\mapsto \hat{k} : H^* \rightarrow \mathbb{k} \\ &\quad f \mapsto f(k). \end{aligned}$$

is an isomorphism.

Let $z \in H_L$ and $h, k \in H$.

$$\begin{aligned}
(I \otimes \hat{k})(\tilde{\beta}(a \otimes zh)) &= (I \otimes \hat{k})(a \underline{\otimes} T'_1(S(zh))T'_2) \\
&= (I \otimes \hat{k})(a1^{\bar{0}} \otimes T'_1(S(zh))T'_2 * 1^{\bar{1}}) \\
&= a1^{\bar{0}}T'_1(S(zh))[T'_2 * 1^{\bar{1}}](k) \\
&= a1^{\bar{0}}T'_1(S(zh))T'_2(k_1)1^{\bar{1}}(k_2) \\
&= a1^{\bar{0}}1^{\bar{1}}(k_2)T'(S(zh)k_1) \\
&= a(k_2 \cdot 1)T'(S(h)S(z)k_1) \\
&= a(1_2k_2 \cdot 1)T'(S(h)S(z)1_1k_1) \\
&\stackrel{(1.29)}{=} a(z1_2k_2 \cdot 1)T'(S(h)1_1k_1) \\
&= a(zk_2 \cdot 1)T'(S(h)k_1) \\
&\stackrel{(2.1.10)}{=} a(z \cdot k_2 \cdot 1)T'(S(h)k_1) \\
&\stackrel{(2.1.10)}{=} a(z \cdot 1)(k_2 \cdot 1)T'(S(h)k_1) \\
&\stackrel{(2.5.3)}{=} (a \triangleleft z)(k_2 \cdot 1)T'(S(h)k_1) \\
&= (a \triangleleft z)1^{\bar{0}}1^{\bar{1}}(k_2)T'(S(h)k_1) \\
&= (a \triangleleft z)1^{\bar{0}}T'_1(S(h))T'_2(k_1)1^{\bar{1}}(k_2) \\
&= (I \otimes \hat{k})[(a \triangleleft z)1^{\bar{0}} \otimes T'_1(S(h))T'_2 * 1^{\bar{1}}] \\
&= (I \otimes \hat{k})[(a \triangleleft z) \underline{\otimes} T'_1(S(h))T'_2] \\
&= (I \otimes \hat{k})(\tilde{\beta}((a \triangleleft z) \otimes h))
\end{aligned}$$

then clearly $\tilde{\beta}$ is well-defined, i.e., it is H_L -balanced. Note that $\beta = \tilde{\beta}|_{A \# H}$ and so it is also well-defined. Now just remains to show that β is the inverse of α . In fact, for all $a \in A$, $k \in H$ and $f \in H^*$,

$$\begin{aligned}
(I \otimes \hat{k})[\beta \circ \alpha(a \underline{\otimes} f)] &= (I \otimes \hat{k})[\beta(a \# (t \leftarrow f))] \\
&= (I \otimes \hat{k})[\beta(a \# f(t_1)t_2)] \\
&= (I \otimes \hat{k})[a(t_2 \cdot 1) \underline{\otimes} f(t_1)T'_1(S(t_3))T'_2] \\
&= (I \otimes \hat{k})[a(t_2 \cdot 1)1^{\bar{0}} \otimes f(t_1)T'_1(S(t_3))T'_2 * 1^{\bar{1}}] \\
&= a(t_2 \cdot 1)1^{\bar{0}}f(t_1)T'_1(S(t_3))T'_2(k_1)1^{\bar{1}}(k_2) \\
&= a(t_2 \cdot 1)1^{\bar{0}}1^{\bar{1}}(k_2)f(t_1)T'(S(t_3)k_1) \\
&= a(t_2 \cdot 1)(k_2 \cdot 1)f(t_1)T'(S(t_3)k_1) \\
&= a(f(t_1)t_2 \cdot 1)(k_2 \cdot 1)T'(S(t_3)k_1) \\
&= a[(t_1 \leftarrow f) \cdot 1](k_2 \cdot 1)T'(S(S^{-1}(k_1)t_2)) \\
&\stackrel{(4.1.11-2)}{=} a[((k_1t_1) \leftarrow f) \cdot 1](k_2 \cdot 1)T'(S(t_2)) \\
&= a[((k_1t_1) \leftarrow f) \cdot 1](k_2 \cdot 1)T(t_2) \\
&= a[((k_1t_1T(t_2)) \leftarrow f) \cdot 1](k_2 \cdot 1) \\
&= a[((k_1(T \rightarrow t)) \leftarrow f) \cdot 1](k_2 \cdot 1) \\
&= a[(k_1 \leftarrow f) \cdot 1](k_2 \cdot 1)
\end{aligned}$$

$$\begin{aligned}
&= a(f(k_1)k_2 \cdot 1)(k_3 \cdot 1) \\
&= a(f(k_1)k_2 \cdot 1) \\
&= a(k_2 \cdot 1)f(k_1) \\
&= a1^{\bar{0}}1^{\bar{1}}(k_2)f(k_1) \\
&= (I \otimes \hat{k})(a1^{\bar{0}} \otimes f * 1^{\bar{1}}) \\
&= (I \otimes \hat{k})(a \underline{\otimes} f)
\end{aligned}$$

and

$$\begin{aligned}
\alpha(\beta(a \underline{\#} h)) &= \alpha(\beta(a(h_1 \cdot 1) \underline{\#} h_2)) \\
&= \alpha[a(h_1 \cdot 1) \underline{\otimes} T'_1(S(h_2))T'_2] \\
&= a(h_1 \cdot 1) \underline{\#} (t \leftarrow T'_1(S(h_2))T'_2) \\
&= a(h_1 \cdot 1) \underline{\#} T'(S(h_2)t_1)t_2 \\
&\stackrel{(4.1.11-2)}{=} a(h_1 \cdot 1) \underline{\#} T'(t_1)h_2t_2 \\
&= a(h_1 \cdot 1) \underline{\#} h_2T'(t_1)t_2 \\
&= a(h_1 \cdot 1) \underline{\#} h_2(t \leftarrow T') \\
&= a(h_1 \cdot 1) \underline{\#} h_2 \\
&= a \underline{\#} h
\end{aligned}$$

Therefore, $A \underline{\otimes} H^* \simeq A \underline{\#} H$ as vector spaces. \square

For a partial H -comodule algebra A , the subalgebra of the partial coinvariants is defined as

$$A^{\underline{\text{co}}H} = \{a \in A \mid \bar{\rho}(a) = (a \otimes 1_H)\rho(1_A)\}.$$

Note that if A is a partial H -module algebra, then it is a partial H^* -comodule algebra and ${}^H A = A^{\underline{\text{co}}H^*}$.

Consider now the following maps:

$$\begin{aligned}
\text{Can} : A \otimes_{A^{\underline{\text{co}}H^*}} A &\rightarrow A \underline{\otimes} H^* \\
a \otimes b &\mapsto a b^{\bar{0}} \otimes b^{\bar{1}}
\end{aligned}$$

and

$$\begin{aligned}
[\cdot, \cdot] : A \otimes_{H_A} A &\rightarrow A \underline{\#} H \\
a \otimes b &\mapsto (a \underline{\#} 1)(1 \underline{\#} t)(b \underline{\#} 1).
\end{aligned}$$

Then:

Lemma 4.2.2. *The image of $[\cdot, \cdot]$ is a left ideal in $A \underline{\#} H$.*

Proof. It is enough to show that $(A \underline{\#} H)(1 \underline{\#} t) \subseteq (A \underline{\#} 1)(1 \underline{\#} t)$.

In fact,

$$\begin{aligned}
(a \underline{\#} h)(1 \underline{\#} t) &= a(h_1 \cdot 1) \underline{\#} h_2 t \\
&= a(h_1 \cdot 1) \underline{\#} \varepsilon_L(h_2) t \\
&= [a(h_1 \cdot 1)] \triangleleft \varepsilon_L(h_2) \underline{\#} t \\
&= a(h_1 \cdot 1)(\varepsilon_L(h_2) \cdot 1) \underline{\#} t \\
&\stackrel{(1.9)}{=} a(1_1 h \cdot 1)(1_2 \cdot 1) \underline{\#} t \\
&\stackrel{(2.1.8)}{=} a(h \cdot 1) \underline{\#} t \\
&= (a(h \cdot 1) \underline{\#} 1)(1 \underline{\#} t).
\end{aligned}$$

□

Now we are able to announce the main theorem of this section:

Theorem 4.2.3. *Let A be a partial H -comodule algebra. The following statements are equivalent:*

- (1) *Can is surjective;*
- (2) *$[\cdot, \cdot]$ is surjective;*
- (3) *There exists $\{x_i, y_i\}_{i=1}^n$ in A such that $\sum_{i=1}^n x_i(h \cdot y_i) = T'(h_1)h_2 \cdot 1_A$.*

Moreover, in this case the maps $[\cdot, \cdot]$ and Can are bijective.

Proof. Note that the following diagram is commutative:

$$\begin{array}{ccc}
 A \otimes_{A \text{co} H^*} A & \xrightarrow{Can} & A \otimes H^* \\
 \searrow [\cdot, \cdot] & \circlearrowleft & \searrow \alpha \\
 & & A \# H
 \end{array}$$

In fact,

$$\begin{aligned}
 \alpha(Can(a \otimes b)) &= \alpha(ab^{\bar{0}} \otimes b^{\bar{1}}) \\
 &= ab^{\bar{0}}(t \leftarrow b^{\bar{1}}) \\
 &= ab^{\bar{0}} \# b^{\bar{1}}(t_1)t_2 \\
 &= ab^{\bar{0}}b^{\bar{1}}(t_1) \# t_2 \\
 &= a(t_1 \cdot b) \# t_2 \\
 &= (a \# t)(b \# 1) \\
 &= (a \# 1)(1 \# t)(b \# 1) \\
 &= [\cdot, \cdot](a \otimes b).
 \end{aligned}$$

Since α is bijective, it is clear that (1) \Leftrightarrow (2).

Moreover, from the usual Morita theory if $[\cdot, \cdot]$ is surjective then it is bijective. Using again that α is bijective we have Can is bijective.

Supposing that (3) holds, then

$$\begin{aligned}
 [\cdot, \cdot](\sum_{i=1}^n x_i \otimes y_i) &= \sum_{i=1}^n (x_i \# 1)(1 \# t)(y_i \# 1) \\
 &= \sum_{i=1}^n x_i(t_1 \cdot y_i) \# t_2 \\
 &= T'(t_1)t_2 \cdot 1_A \# t_3 \\
 &= 1 \# T'(t_1)t_2 \\
 &= 1 \# t \leftarrow T' \\
 &= 1 \# 1
 \end{aligned}$$

and this implies $1\#1 \in \text{Im}([\ , \])$. By Lemma 4.2.2, it is clear that $[\ , \]$ is surjective.

Conversely, if Can is surjective then there exists $\{x_i, y_i\}_{i=1}^n$ such that $\text{Can}(\sum_{i=1}^n x_i \otimes y_i) = 1 \otimes T'$. So,

$$\begin{aligned}
\sum_{i=1}^n x_i(h \cdot y_i) &= \sum_{i=1}^n x_i y_i^{\bar{0}} y_i^{\bar{1}}(h) \\
&= (I \otimes \hat{h})(\sum_{i=1}^n x_i y_i^{\bar{0}} \otimes y_i^{\bar{1}}) \\
&= (I \otimes \hat{h})(\text{Can}(\sum_{i=1}^n x_i \otimes y_i)) \\
&= (I \otimes \hat{h})(I \otimes T') \\
&= (I \otimes \hat{h})(1^{\bar{0}} \otimes T' * 1^{\bar{1}}) \\
&= 1^{\bar{0}} 1^{\bar{1}}(h_2) T'(h_1) \\
&= T'(h_1) h_2 \cdot 1
\end{aligned}$$

and this means that (3) holds. □

The elements in item (3) of the above theorem are called the *Galois coordinates* of A .

4.3 Morita Theory for partial comodule algebras

4.3.1 Some categorical results

We saw in Section 3.4 that $\mathcal{C} = A \otimes H = (A \otimes H) \bar{\rho}(1_A)$ is an A -coring. We will first discuss the relation between comodules over \mathcal{C} , whose categories will be denoted by ${}^{\mathcal{C}}\mathcal{M}$ and the relative partial Hopf module.

Definition 4.3.1. Let M a vector space and A a symmetric partial H -comodule algebra. We say M is a relative partial Hopf module if

(PRHM1) M is a right A -module;

(PRHM2) There exists $\rho_M : M \rightarrow M \otimes H$ such that:

(a) $(I \otimes \varepsilon_H) \rho_M = I_M$;

(b) $(\rho_M \otimes I) \rho_M(m) = m^0 \cdot 1^{\bar{0}} \otimes m^1_1 1^{\bar{1}} \otimes m^1_2$;

(PRHM3) $\rho(m \cdot a) = m^0 \cdot a^{\bar{0}} \otimes m^1 a^{\bar{0}}$.

Definition 4.3.2. Let M and N relative partial Hopf modules. A morphism of relative partial Hopf modules is a linear map $f : M \rightarrow N$ such that

(i) $f(m \cdot a) = f(m) \cdot a$;

(ii) $\rho_N \circ f = (f \otimes I) \circ \rho_M$.

It is clear that $\mathcal{M}_A^H = \{M \mid M \text{ is a relative partial Hopf module}\}$ becomes a category with the morphism given in Definition 4.3.2.

Our next aim is to show that the categories \mathcal{M}^C and \mathcal{M}_A^H are equivalent.

Let M be an object in \mathcal{M}^C , i.e., there is a coaction $\tilde{\rho} : M \rightarrow M \otimes_A \mathcal{C} (= M \otimes_A (A \underline{\otimes} H))$.

Supposing $\tilde{\rho}(m) = \sum_{i=1}^n m_i \otimes_A (a_i \underline{\otimes} h_i)$ we have

$$\begin{aligned} \tilde{\rho}(m) &= \sum_{i=1}^n m_i \otimes_A (a_i \underline{\otimes} h_i) \\ &= \sum_{i=1}^n m_i \otimes_A (a_i 1^{\bar{0}} \otimes h_i 1^{\bar{1}}) \\ &= \sum_{i=1}^n m_i \otimes_A a_i 1^{\bar{0}} \cdot (1_A \otimes h_i 1^{\bar{1}}) \\ &= \sum_{i=1}^n m_i \cdot a_i 1^{\bar{0}} \otimes_A 1_A \otimes h_i 1^{\bar{1}}, \end{aligned}$$

then we can define

$$\begin{aligned} \rho : M &\rightarrow M \otimes H \\ m &\mapsto \sum_{i=1}^n m_i \cdot a_i 1^{\bar{0}} \otimes h_i 1^{\bar{1}}. \end{aligned}$$

Conversely, given $\rho : M \rightarrow M \otimes H$ with $\rho(m) = m^0 \otimes m^1$ we can define

$$\begin{aligned} \tilde{\rho} : M &\rightarrow M \otimes_A \mathcal{C} \\ m &\mapsto m^0 \otimes_A 1_A \underline{\otimes} m^1. \end{aligned}$$

Remark 4.3.3. Using the notation $\rho(m) = m^0 \otimes m^1$ and $\tilde{\rho}(m) = m^{\bar{0}} \otimes m^{\bar{1}}$, ρ and $\tilde{\rho}$ satisfy the following relation:

$$\begin{aligned} m^{\bar{0}} \otimes m^{\bar{1}} &= m^0 \otimes_A (1^{\bar{0}} \underline{\otimes} m^1 1^{\bar{1}}) \\ &= m^0 \otimes_A (1_A \underline{\otimes} m^1) \\ &= m^0 \otimes_A (1^{\bar{0}} \otimes m^1 1^{\bar{1}}). \end{aligned} \tag{4.12}$$

Remark 4.3.4. Note that from the definition of ρ we have $\rho(m) = m^0 \cdot 1^{\bar{0}} \otimes m^1 1^{\bar{1}}$. In fact,

$$\begin{aligned} \tilde{\rho}(m) &= \sum_{i=1}^n m_i \otimes_A (a_i \underline{\otimes} h_i) \\ &= \sum_{i=1}^n m_i \otimes_A (a_i 1^{\bar{0}} \otimes h_i 1^{\bar{1}}) \\ &= \sum_{i=1}^n m_i \otimes_A (a_i 1^{\bar{0}'} 1^{\bar{0}} \otimes h_i 1^{\bar{1}'} 1^{\bar{1}}) \\ &= \sum_{i=1}^n m_i \cdot a_i 1^{\bar{0}'} 1^{\bar{0}} \otimes_A 1_A \otimes h_i 1^{\bar{1}'} 1^{\bar{1}} \\ &= \sum_{i=1}^n m_i \cdot a_i 1^{\bar{0}'} \cdot 1^{\bar{0}} \otimes_A 1_A \otimes h_i 1^{\bar{1}'} 1^{\bar{1}} \\ &= m^0 \cdot 1^{\bar{0}} \otimes_A 1_A \otimes m^1 1^{\bar{1}} \end{aligned}$$

then

$$\begin{aligned} m^0 \otimes_A 1_A \otimes m^1 &= \tilde{\rho}(m) \\ &= m^0 \cdot 1^{\bar{0}} \otimes_A 1_A \otimes m^1 1^{\bar{1}} \end{aligned}$$

and so $\rho(m) = m^0 \cdot 1^{\bar{0}} \otimes m^1 1^{\bar{1}}$.

Proposition 4.3.5. *With the above construction, (M, ρ) is a relative partial Hopf module if and only if $(M, \tilde{\rho})$ lies in \mathcal{M}^C .*

Proof. Note that M is an A -module by construction. Then (PRHM1) is satisfied.

(PRHM2)(a) \Leftrightarrow (CM2)

$$\begin{aligned} (I \otimes_A \varepsilon_C) \tilde{\rho}(m) &\stackrel{(4.12)}{=} (I \otimes_A \varepsilon_C)(m^0 \otimes_A 1 \otimes m^1) \\ &= m^0 \otimes_A \varepsilon_C(1 \otimes m^1) \\ &= m^0 \cdot \varepsilon_C(1 \otimes m^1) \otimes_A 1_A \\ &= m^0 \cdot 1^{\bar{0}} \varepsilon_H(m^1 1^{\bar{1}}) \otimes_A 1_A \\ &\stackrel{(4.3.4)}{=} m^0 \varepsilon_H(m^1) \otimes_A 1_A. \end{aligned}$$

Then $(I \otimes_A \varepsilon_C) \tilde{\rho}(m) = m \otimes_A 1_A$ if and only if $m^0 \varepsilon_H(m^1) = m$.

(PRHM3) \Leftrightarrow (CM1) We have

$$\begin{aligned} \tilde{\rho}(m \cdot a) &= (m \cdot a)^{\bar{0}} \otimes_A (m \cdot a)^{\bar{1}} \\ &\stackrel{(4.12)}{=} (m \cdot a)^0 \otimes_A 1_A \otimes (m \cdot a)^1 \\ &= (m \cdot a)^0 \cdot 1^{\bar{0}} \otimes_A 1_A \otimes (m \cdot a)^1 1^{\bar{1}} \\ &\stackrel{(4.3.4)}{=} (m \cdot a)^0 \otimes_A 1_A \otimes (m \cdot a)^1 \end{aligned}$$

and, on the other hand

$$\begin{aligned} \tilde{\rho}(m) \cdot a &= (m^{\bar{0}} \otimes_A m^{\bar{1}}) \cdot a \\ &\stackrel{(4.12)}{=} m^0 \otimes_A (1_A \otimes m^1) \cdot a \\ &= m^0 \otimes_A (a^{\bar{0}} \otimes m^1 a^{\bar{1}}) \\ &= m^0 \cdot a^{\bar{0}} \otimes_A 1_A \otimes m^1 a^{\bar{1}}, \end{aligned}$$

then $\rho(m \cdot a) = (m \cdot a)^0 \otimes (m \cdot a)^1 = m^0 \cdot a^{\bar{0}} \otimes m^1 a^{\bar{1}}$ if and only if $\tilde{\rho}(m \cdot a) = \tilde{\rho}(m) \cdot a$.

(PRHM2)(b) \Leftrightarrow (CM3) Note that

$$\begin{aligned} (I \otimes_A \Delta_C) \tilde{\rho}(m) &\stackrel{(4.12)}{=} m^0 \otimes_A \Delta_C(1_A \otimes m^1) \\ &= m^0 \otimes_A (1_A \otimes m^1_1) \otimes_A (1_A \otimes m^1_2) \\ &= m^0 \otimes_A (1^{\bar{0}'} \otimes m^1_1 1^{\bar{1}'}) \otimes_A (1^{\bar{0}} \otimes m^1_2 1^{\bar{1}}) \\ &= m^0 \otimes_A (1^{\bar{0}'} 1^{\bar{0}\bar{0}} \otimes m^1_1 1^{\bar{1}' 1^{\bar{0}\bar{1}}}) \otimes_A (1_A \otimes m^1_2 1^{\bar{1}}) \\ &= m^0 \otimes_A (1^{\bar{0}\bar{0}} \otimes m^1_1 1^{\bar{0}\bar{1}}) \otimes_A (1_A \otimes m^1_2 1^{\bar{1}}) \\ &= [m^0 \cdot 1^{\bar{0}\bar{0}} \otimes_A (1_A \otimes m^1_1 1^{\bar{0}\bar{1}})] \otimes_A (1_A \otimes m^1_2 1^{\bar{1}}) \\ &= [m^0 \cdot 1^{\bar{0}'} 1^{\bar{0}} \otimes_A (1_A \otimes m^1_1 1^{\bar{1}' 1^{\bar{1}}})] \otimes_A (1_A \otimes m^1_2 1^{\bar{1}'_2}) \\ &\stackrel{(4.3.4)}{=} [m^0 \cdot 1^{\bar{0}} \otimes_A (1_A \otimes m^1_1 1^{\bar{1}})] \otimes_A (1_A \otimes m^1_2) \end{aligned}$$

and, on the other side

$$\begin{aligned}
(\tilde{\rho} \otimes I)\tilde{\rho}(m) &= \tilde{\rho}(m^0) \otimes_A 1_A \underline{\otimes} m^1 \\
&= m^0 \otimes_A 1_A \underline{\otimes} m^{01} \otimes_A 1_A \underline{\otimes} m^1 \\
&= [m^0 \cdot 1^{\overline{00}} \otimes_A (1_A \otimes m^{01} 1^{\overline{01}})] \otimes_A 1_A \otimes m^1 1^{\overline{1}} \\
&\stackrel{(\text{PRHM3})}{=} [m^0 \otimes_A (1_A \otimes m^{01})] \otimes_A 1_A \otimes m^1
\end{aligned}$$

then $(\rho \otimes I)\rho(m) = m^0 \cdot 1^{\overline{0}} \otimes m^1 1^{\overline{1}} \otimes m^1 2$ if and only if $(I \otimes_A \Delta_C)\tilde{\rho}(m) = (\tilde{\rho} \otimes I)\tilde{\rho}(m)$. \square

Using the above stated we have the functor

$$\begin{aligned}
F : \mathcal{M}^{\mathcal{C}} &\rightarrow \mathcal{M}_A^H \\
(M, \tilde{\rho}) &\mapsto (M, \rho) \\
f &\mapsto f,
\end{aligned}$$

remaining only to prove that if $f : M \rightarrow N$ is a morphism in $\mathcal{M}^{\mathcal{C}}$ then f is a morphism in \mathcal{M}_A^H , and that given $f : M \rightarrow N$ a morphism in \mathcal{M}_A^H then f is a morphism in $\mathcal{M}^{\mathcal{C}}$.

In fact,

$$\begin{aligned}
\tilde{\rho}_N(f(m)) &= f(m)^{\tilde{0}} \otimes_A f(m)^{\tilde{0}} \\
&\stackrel{(4.12)}{=} f(m)^0 \otimes_A 1_A \underline{\otimes} f(m)^1 \\
&= f(m)^0 \cdot 1^{\overline{0}} \otimes_A 1_A \otimes f(m)^1 1^{\overline{1}} \\
&\stackrel{(4.3.4)}{=} f(m)^0 \otimes_A 1_A \otimes f(m)^1
\end{aligned}$$

and on the other hand

$$\begin{aligned}
(f \otimes I)\tilde{\rho}_M(m) &= (f \otimes I)(m^{\tilde{0}} \otimes_A m^{\tilde{1}}) \\
&= (f \otimes I)(m^0 \otimes_A 1_A \underline{\otimes} m^1) \\
&= (f \otimes I)(m^0 \cdot 1^{\overline{0}} \otimes_A 1_A \otimes m^1 1^{\overline{1}}) \\
&= f(m^0 \cdot 1^{\overline{0}}) \otimes_A 1_A \otimes m^1 1^{\overline{1}} \\
&\stackrel{(4.3.4)}{=} f(m^0) \otimes_A 1_A \otimes m^1.
\end{aligned}$$

Then $\rho_N(f(m)) = (f \otimes I)\rho_M(m)$ if and only if $(f \otimes I)\tilde{\rho}_M(m) = \tilde{\rho}_N(f(m))$.

And this completes the proof of the following theorem:

Theorem 4.3.6. *The categories $\mathcal{M}^{\mathcal{C}}$ and \mathcal{M}_A^H are isomorphic.*

4.3.2 Morita theory

Recall now that if $M \in \mathcal{M}^{\mathcal{C}}$ where \mathcal{C} is any coring with a fixed grouplike x , then the corresponding subcomodule of coinvariants is $M^{co\mathcal{C}} = \{m \in M \mid \tilde{\rho}(m) = m \otimes_A x\}$.

On the other side if $M \in \mathcal{M}_A^H$ then $M_A^{coH} = \{m \in M \mid \rho(m) = m \cdot 1^{\overline{0}} \otimes 1^{\overline{1}}\}$.

Then, for the coring $\mathcal{C} = A \underline{\otimes} H$ and the grouplike $\rho(1)$ we have $M_A^{coH} = M^{co\mathcal{C}}$. In fact,

$$\begin{aligned}
M^{co(A\otimes H)} &= \{m \in M \mid \tilde{\rho}(m) = m \otimes_A \rho(1)\} \\
&= \{m \in M \mid m^{\bar{0}} \otimes_A m^{\bar{1}} = m \otimes_A 1^{\bar{0}} \otimes 1^{\bar{1}}\} \\
&= \{m \in M \mid m^0 \otimes_A 1_A \otimes m^1 = m \cdot 1^{\bar{0}} \otimes_A 1_A \otimes 1^{\bar{1}}\} \\
&= \{m \in M \mid m^0 \otimes_A 1^{\bar{0}} \otimes m^1 1^{\bar{1}} = m \cdot 1^{\bar{0}} \otimes_A 1_A \otimes 1^{\bar{1}}\} \\
&= \{m \in M \mid m^0 \cdot 1^{\bar{0}} \otimes_A 1_A \otimes m^1 1^{\bar{1}} = m \cdot 1^{\bar{0}} \otimes_A 1_A \otimes 1^{\bar{1}}\} \\
&= \{m \in M \mid m^0 \cdot 1^{\bar{0}} \otimes m^1 1^{\bar{1}} = m \cdot 1^{\bar{0}} \otimes 1^{\bar{1}}\} \\
&= \{m \in M \mid m^0 \otimes m^1 = m \cdot 1^{\bar{0}} \otimes 1^{\bar{1}}\} \\
&= \{m \in M \mid \rho(m) = m \cdot 1^{\bar{0}} \otimes 1^{\bar{1}}\} \\
&= M_A^{coH}.
\end{aligned}$$

Note that $A \in \mathcal{M}_A^H$ and the action of A on A is given by multiplication. So

$$A^{coH} = A_A^{coH} = A^{co(A\otimes H)}.$$

Consider now the set ${}^*(A\otimes H) = {}_A \text{Hom}(A\otimes H, A)$ with an associative product $\#$ defined by $(\gamma\#\xi)(a\otimes h) = \xi[(a\otimes h_1) \cdot \gamma(1\otimes h_2)]$.

We also have the algebra $\text{Hom}(H, A)$ with a multiplication $\#$ given by $f\#g(h) = f(h_2)^{\bar{0}}g(h_1f(h_2)^{\bar{1}})$.

Note that it is associative. In fact,

$$\begin{aligned}
[(f\#g)\#k](h) &= (f\#g)(h_2)^{\bar{0}}k(h_1(f\#g(h_2)^{\bar{1}})^{\bar{1}}) \\
&= [f(h_3)^{\bar{0}}g(h_2f(h_3)^{\bar{1}})]^{\bar{0}}k(h_1[f(h_3)^{\bar{0}}g(h_2f(h_3)^{\bar{1}})]^{\bar{1}}) \\
&= f(h_3)^{\bar{0}}1^{\bar{0}}g(h_2f(h_3)^{\bar{1}}_2)^{\bar{0}}k(h_1f(h_3)^{\bar{0}}_11^{\bar{1}}g(h_2f(h_3)^{\bar{1}}_2)^{\bar{1}}) \\
&= f(h_3)^{\bar{0}}g(h_2f(h_3)^{\bar{1}}_2)^{\bar{0}}k(h_1f(h_3)^{\bar{0}}_1g(h_2f(h_3)^{\bar{1}}_2)^{\bar{1}}) \\
&= f(h_2)^{\bar{0}}g((h_1f(h_2)^{\bar{1}})_2)^{\bar{0}}k((h_1f(h_2)^{\bar{0}})_1g((h_1f(h_2)^{\bar{1}})_2)^{\bar{1}}) \\
&= f(f_2)^{\bar{0}}(g\#k)(h_1f(h_2)^{\bar{1}}) \\
&= [f\#(g\#k)](h).
\end{aligned}$$

Consider $\underline{\text{Hom}}(H, A) = \{f \in \text{Hom}(H, A) \mid f(h) = 1^{\bar{0}}f(h1^{\bar{1}})\}$ that is a subalgebra of $\text{Hom}(H, A)$. In fact, let $f, g \in \underline{\text{Hom}}(H, A)$, so

$$\begin{aligned}
(f\#g)(h) &= f(h_2)^{\bar{0}}g(h_1f(h_2)^{\bar{1}}) \\
&= [1^{\bar{0}}f(h_21^{\bar{1}})]^{\bar{0}}g(h_1[1^{\bar{0}}f(h_21^{\bar{1}})]^{\bar{1}}) \\
&= 1^{\bar{0}\bar{0}}f(h_21^{\bar{1}})^{\bar{0}}g(h_11^{\bar{0}\bar{1}}f(h_21^{\bar{1}})^{\bar{1}}) \\
&= 1^{\bar{0}}1^{\bar{0}'}f(h_21^{\bar{1}}_2)^{\bar{0}}g(h_11^{\bar{1}}_11^{\bar{0}''}f(h_21^{\bar{1}}_2)^{\bar{1}}) \\
&= 1^{\bar{0}}f(h_21^{\bar{1}}_2)^{\bar{0}}g(h_11^{\bar{1}}_1f(h_21^{\bar{1}}_2)^{\bar{1}}) \\
&= 1^{\bar{0}}f((h1^{\bar{1}})_2)^{\bar{0}}g((h1^{\bar{1}})_1f((h1^{\bar{1}})_2)^{\bar{1}}) \\
&= 1^{\bar{0}}(f\#g)(h1^{\bar{1}}).
\end{aligned}$$

Now given an element $\xi \in {}^*(A\otimes H)$ we can consider the map $\alpha_\xi : H \rightarrow A$ given by $\alpha_\xi(h) = \xi(1\otimes h)$.

Note that α_ξ lies in $\underline{Hom}(H, A)$, in fact

$$\begin{aligned}
1^{\bar{0}}\alpha_\xi(h1^{\bar{1}}) &= 1^{\bar{0}}\xi(1\otimes h1^{\bar{1}}) \\
&= \xi(1^{\bar{0}} \cdot (1\otimes h1^{\bar{1}})) \\
&= \xi(1^{\bar{0}}\otimes h1^{\bar{1}}) \\
&= \xi(1\otimes h) \\
&= \alpha_\xi(h).
\end{aligned}$$

Therefore, we can define the following map

$$\begin{aligned}
\alpha : *(A\otimes H) &\rightarrow \underline{Hom}(H, A) \\
\xi &\mapsto \alpha_\xi.
\end{aligned}$$

Conversely, given a map f in $\underline{Hom}(H, A)$ we can define a map β_f in $*(A\otimes H)$ by $\beta_f(a\otimes h) = a1^{\bar{0}}f(h1^{\bar{1}}) = af(h)$, so we have

$$\begin{aligned}
\beta : \underline{Hom}(H, A) &\rightarrow *(A\otimes H) \\
f &\mapsto \beta_f.
\end{aligned}$$

Proposition 4.3.7. *With the above notation, we have that $\underline{Hom}(H, A)$ isomorphic to $*(A\otimes H)$ as algebras.*

Proof. Consider the maps $\alpha : *(A\otimes H) \rightarrow \underline{Hom}(H, A)$ and $\beta : \underline{Hom}(H, A) \rightarrow *(A\otimes H)$ given above. Let $f \in \underline{Hom}(H, A)$ and $h \in H$, so

$$\begin{aligned}
\alpha(\beta(f))(h) &= \beta(f)(1\otimes h) \\
&= 1_A f(h) \\
&= f(h)
\end{aligned}$$

and for $\xi \in *(A\otimes H)$ and $a\otimes h \in A\otimes H$,

$$\begin{aligned}
\beta(\alpha(\xi))(a\otimes h) &= a\alpha(\xi)(h) \\
&= a\xi(1\otimes h) \\
&= \xi(a \cdot (1\otimes h)) \\
&= \xi(a\otimes h).
\end{aligned}$$

Now, let $\gamma, \xi \in *(A\otimes H)$ and $h \in H$, so

$$\begin{aligned}
\alpha(\gamma\#\xi)(h) &= (\gamma\#\xi)(1\otimes h) \\
&= \xi[(1\otimes h_1) \cdot \gamma(1\otimes h_2)] \\
&= \xi[\gamma(1\otimes h_2)^{\bar{0}}\otimes h_1\gamma(1\otimes h_2)^{\bar{1}}] \\
&= \xi[\gamma(1\otimes h_2)^{\bar{0}} \cdot (1\otimes h_1\gamma(1\otimes h_2)^{\bar{1}})] \\
&= \gamma(1\otimes h_2)^{\bar{0}}\xi[1\otimes h_1\gamma(1\otimes h_2)^{\bar{1}}] \\
&= \alpha(\gamma)(h_2)^{\bar{0}}\alpha(\xi)[h_1\alpha(\gamma)(h_2)^{\bar{1}}] \\
&= [\alpha(\gamma)\#\alpha(\xi)](h).
\end{aligned}$$

Since α is an isomorphism that respects the product, clearly $\alpha(\varepsilon_C) = 1_{\underline{Hom}(H, A)}$, therefore an algebra isomorphism. \square

In [14], Caenepeel constructed the subalgebra $Q = \{q \in {}^* \mathcal{C} \mid c_1 \cdot q(c_2) = q(c) \cdot x, \forall c \in \mathcal{C}\}$ where x is a group-like element in \mathcal{C} . In our case, we have that

$$\begin{aligned} Q &= \{\xi \in {}^*(A \otimes H) \mid (a \otimes h_1) \cdot \xi(1 \otimes h_2) = \xi(a \otimes h) \cdot \rho(1_A)\} \\ &= \{\xi \in {}^*(A \otimes H) \mid a \xi(1 \otimes h_2)^{\bar{0}} \otimes h_1 \xi(1 \otimes h_2)^{\bar{1}} = \xi(a \otimes h) 1^{\bar{0}} \otimes 1^{\bar{1}}\} \\ &= \{\xi \in {}^*(A \otimes H) \mid a \alpha(\xi)(h_2)^{\bar{0}} \otimes h_1 \alpha(\xi)(h_2)^{\bar{1}} = a \alpha(\xi)(h) 1^{\bar{0}} \otimes 1^{\bar{1}}\}. \end{aligned}$$

Now consider

$$\tilde{Q} = \alpha(Q) = \{q \in \underline{Hom}(H, A) \mid q(h_2)^{\bar{0}} \otimes h_1 q(h_2)^{\bar{1}} = q(h) \cdot \rho(1)\}$$

and

$$\begin{aligned} j : A &\rightarrow \underline{Hom}(H, A) \\ a &\mapsto j(a)(h) = [\alpha(\varepsilon_{A \otimes H})(h)]a \\ &= 1^{\bar{0}} a \varepsilon_H(h \ 1^{\bar{1}}) \\ &= a^{\bar{0}} \varepsilon_H(h \ a^{\bar{1}}) \end{aligned}$$

which is clearly well-defined and, moreover, an algebra morphism. In fact, let $a, b \in A$ and $h \in H$, so

$$\begin{aligned} (j(a) \# j(b))(h) &= j(a)(h_2)^{\bar{0}} j(b)[h_1 j(a)(h_2)^{\bar{1}}] \\ &= [a^{\bar{0}} \varepsilon_H(h_2 \ a^{\bar{1}})]^{\bar{0}} j(b)[h_1 [a^{\bar{0}} \varepsilon_H(h_2 \ a^{\bar{1}})]^{\bar{1}}] \\ &= a^{\bar{0}\bar{0}} j(b)[h_1 a^{\bar{0}\bar{1}}] \varepsilon_H(h_2 \ a^{\bar{1}}) \\ &= a^{\bar{0}\bar{0}} b^{\bar{0}} \varepsilon_H(h_1 a^{\bar{0}\bar{1}} b^{\bar{1}}) \varepsilon_H(h_2 \ a^{\bar{1}}) \\ &= a^{\bar{0}} 1^{\bar{0}} b^{\bar{0}} \varepsilon_H(h_1 a^{\bar{1}} 1^{\bar{1}} b^{\bar{1}}) \varepsilon_H(h_2 \ a^{\bar{1}}) \\ &= a^{\bar{0}} b^{\bar{0}} \varepsilon_H((h a^{\bar{1}})_1 b^{\bar{1}}) \varepsilon_H((h \ a^{\bar{1}})_2) \\ &= a^{\bar{0}} b^{\bar{0}} \varepsilon_H(h a^{\bar{1}} b^{\bar{1}}) \\ &= j(ab)(h). \end{aligned}$$

Clearly, $j(1_A) = \alpha(\varepsilon_{A \otimes H}) = 1_{\underline{Hom}(H, A)}$ and, moreover, since $a = j(a)(1_H)$, we have that j is injective.

Note that, for all $q \in \tilde{Q}$, $f \in \underline{Hom}(H, A)$, $h \in H$ and $a \in A$, we have that

$$(q \# j(a))(h) = q(h)a \tag{4.13}$$

$$(j(a) \# f)(h) = a^{\bar{0}} f(h \ a^{\bar{1}}). \tag{4.14}$$

In fact,

$$\begin{aligned} (q \# j(a))(h) &= q(h_2)^{\bar{0}} j(a)(h_1 \ f(h_2)^{\bar{1}}) \\ &\stackrel{\tilde{Q}}{=} q(h) 1^{\bar{0}} j(a)(1^{\bar{1}}) \\ &= q(h) 1^{\bar{0}} \varepsilon_H(1^{\bar{1}}) a \\ &= q(h)a \end{aligned}$$

and

$$\begin{aligned}
(j(a)\#f)(h) &= j(a)(h_2)^{\bar{0}}f(h_1j(a)(h_2)^{\bar{1}}) \\
&= [a^{\bar{0}}\varepsilon_H(h_2a^{\bar{1}})]^{\bar{0}}f(h_1[a^{\bar{0}}\varepsilon_H(h_2a^{\bar{1}})]^{\bar{1}}) \\
&= a^{\bar{0}\bar{0}}f(h_1a^{\bar{0}\bar{1}})\varepsilon_H(h_2a^{\bar{1}}) \\
&= a^{\bar{0}\bar{1}}1^{\bar{0}}f(h_1a^{\bar{1}}1^{\bar{1}})\varepsilon_H(h_2a^{\bar{1}}) \\
&= a^{\bar{0}\bar{1}}1^{\bar{0}}f(ha^{\bar{1}}1^{\bar{1}}) \\
&= a^{\bar{0}}f(ha^{\bar{1}}).
\end{aligned}$$

Proposition 4.3.8. *With the above notation, \tilde{Q} is a $(\underline{Hom}(H, A), A^{coH})$ -bimodule via*

$$f \triangleright q \triangleleft a = f \# q \# j(a). \quad (4.15)$$

Proof. First of all, we need to show that $f \triangleright q \triangleleft a$ lies in \tilde{Q} . In fact, for $f \in \underline{Hom}(H, A)$ and $q \in \tilde{Q}$, we have

$$\begin{aligned}
&(f\#q)(h_2)^{\bar{0}} \otimes h_1 (f\#q)(h_2)^{\bar{1}} \\
&= [f(h_3)^{\bar{0}}q(h_2 f(h_3)^{\bar{1}})]^{\bar{0}} \otimes h_1 [f(h_3)^{\bar{0}}q(h_2 f(h_3)^{\bar{1}})]^{\bar{1}} \\
&= f(h_3)^{\bar{0}\bar{0}}q(h_2 f(h_3)^{\bar{1}})^{\bar{0}} \otimes h_1 f(h_3)^{\bar{0}\bar{1}}q(h_2 f(h_3)^{\bar{1}})^{\bar{1}} \\
&= f(h_3)^{\bar{0}\bar{1}}1^{\bar{0}}q(h_2 f(h_3)^{\bar{1}}_2)^{\bar{0}} \otimes h_1 f(h_3)^{\bar{1}\bar{1}}1^{\bar{1}}q(h_2 f(h_3)^{\bar{1}}_2)^{\bar{1}} \\
&= f(h_2)^{\bar{0}}q((h_1 f(h_2)^{\bar{1}})_2)^{\bar{0}} \otimes (h_1 f(h_2)^{\bar{1}})_1q((h_1 f(h_2)^{\bar{1}})_2)^{\bar{1}} \\
&\stackrel{\tilde{Q}}{=} f(h_2)^{\bar{0}}q(h_1 f(h_2)^{\bar{1}})1^{\bar{0}} \otimes 1^{\bar{1}} \\
&= (f\#q)(h)
\end{aligned}$$

and, for $a \in A^{coH}$,

$$\begin{aligned}
(q \# j(a))(h_2)^{\bar{0}} \otimes h_1(q \# j(a))(h_2)^{\bar{1}} &\stackrel{(4.13)}{=} [q(h_2) a]^{\bar{0}} \otimes h_1[q(h_2) a]^{\bar{1}} \\
&= q(h_2)^{\bar{0}} a^{\bar{0}} \otimes h_1q(h_2)^{\bar{1}} a^{\bar{1}} \\
&\stackrel{\tilde{Q}}{=} q(h)1^{\bar{0}} a^{\bar{0}} \otimes 1^{\bar{1}} a^{\bar{1}} \\
&= q(h) a^{\bar{0}} \otimes a^{\bar{1}} \\
&\stackrel{A^{coH}}{=} q(h) a 1^{\bar{0}} \otimes 1^{\bar{1}} \\
&\stackrel{(4.13)}{=} (q\#j(a))(h) 1^{\bar{0}} \otimes 1^{\bar{1}}.
\end{aligned}$$

Since j is an algebra morphism and $\underline{Hom}(H, A)$ is an associative algebra, then \tilde{Q} is a $(\underline{Hom}(H, A), A^{coH})$ -bimodule. \square

Proposition 4.3.9. *A is an $(A^{coH}, \underline{Hom}(H, A))$ -bimodule via*

$$a \rightarrow b \leftarrow f = ab^{\bar{0}}f(b^{\bar{1}}). \quad (4.16)$$

Proof. Let a be an element in A^{coH} , b in A and f in $\underline{Hom}(H, A)$, then

$$\begin{aligned}
b \leftarrow 1_{\underline{Hom}(H, A)} &= b \leftarrow \alpha(\varepsilon_{A \otimes H}) \\
&= b^{\bar{0}}\alpha(\varepsilon_{A \otimes H})(b^{\bar{1}}) \\
&= b^{\bar{0}}\varepsilon_{A \otimes H}(1 \otimes b^{\bar{1}}) \\
&= \varepsilon_{A \otimes H}(b^{\bar{0}} \otimes b^{\bar{1}}) \\
&= b^{\bar{0}}\varepsilon_H(b^{\bar{1}}) \\
&= b
\end{aligned}$$

and, for $f, g \in \underline{Hom}(H, A)$,

$$\begin{aligned}
(b \leftarrow f) \leftarrow g &= (b^{\bar{0}} f(b^{\bar{1}})) \leftarrow g \\
&= (b^{\bar{0}} f(b^{\bar{1}}))^{\bar{0}} g((b^{\bar{0}} f(b^{\bar{1}}))^{\bar{1}}) \\
&= b^{\bar{0}\bar{0}} f(b^{\bar{1}})^{\bar{0}} g(b^{\bar{0}\bar{1}} f(b^{\bar{1}})^{\bar{1}}) \\
&= b^{\bar{0}} 1^{\bar{0}} f(b^{\bar{1}_2})^{\bar{0}} g(b^{\bar{1}}_1 1^{\bar{1}} f(b^{\bar{1}_2})^{\bar{1}}) \\
&= b^{\bar{0}} f(b^{\bar{1}_2})^{\bar{0}} g(b^{\bar{1}}_1 f(b^{\bar{1}_2})^{\bar{1}}) \\
&= b^{\bar{0}} (f \# g)(b^{\bar{1}}) \\
&= b \leftarrow f \# g.
\end{aligned}$$

Clearly, it is a bimodule, as follows

$$\begin{aligned}
(a \rightarrow b) \leftarrow f &= a b \leftarrow f \\
&= a^{\bar{0}} b^{\bar{0}} f(a^{\bar{1}} b^{\bar{1}}) \\
&\stackrel{A^{coH}}{=} a 1^{\bar{0}} b^{\bar{0}} f(1^{\bar{1}} b^{\bar{1}}) \\
&= a b^{\bar{0}} f(b^{\bar{1}}) \\
&= a \rightarrow (b \leftarrow f).
\end{aligned}$$

□

Now consider the following maps

$$\begin{aligned}
\tau : A \otimes_{\underline{Hom}(H, A)} \tilde{Q} &\rightarrow A^{coH} \\
a \otimes q &\mapsto a^{\bar{0}} q(a^{\bar{1}})
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\mu : \tilde{Q} \otimes_{A^{coH}} A &\rightarrow \underline{Hom}(H, A) \\
q \otimes a &\mapsto q \# j(a).
\end{aligned} \tag{4.18}$$

First of all, we need to check that τ and μ are well-defined. In fact,

$$\begin{aligned}
\bar{\rho}(\tau(a \otimes q)) &= \bar{\rho}(a^{\bar{0}} q(a^{\bar{1}})) \\
&= a^{\bar{0}\bar{0}} q(a^{\bar{1}})^{\bar{0}} \otimes a^{\bar{0}\bar{1}} q(a^{\bar{1}})^{\bar{1}} \\
&= a^{\bar{0}} 1^{\bar{0}} q(a^{\bar{1}_2})^{\bar{0}} \otimes a^{\bar{1}}_1 1^{\bar{1}} q(a^{\bar{1}_2})^{\bar{1}} \\
&= a^{\bar{0}} q(a^{\bar{1}_2})^{\bar{0}} \otimes a^{\bar{1}}_1 q(a^{\bar{1}_2})^{\bar{1}} \\
&= a^{\bar{0}} q(a^{\bar{1}}) 1^{\bar{1}} \otimes 1^{\bar{1}} \\
&= \tau(a \otimes q) 1^{\bar{1}} \otimes 1^{\bar{1}}.
\end{aligned}$$

Therefore $\mathcal{I}m(\tau) \subseteq A^{coH}$. Now let us see that τ and μ are “balanced” maps. Let $f \in \underline{Hom}(H, A)$, then

$$\begin{aligned}
(a \leftarrow f)^{\bar{0}} q((a \leftarrow f)^{\bar{1}}) &= (a^{\bar{0}} f(a^{\bar{1}}))^{\bar{0}} q((a^{\bar{0}} f(a^{\bar{1}}))^{\bar{1}}) \\
&= a^{\bar{0}\bar{0}} f(a^{\bar{1}})^{\bar{0}} q(a^{\bar{0}\bar{1}} f(a^{\bar{1}})^{\bar{1}}) \\
&= a^{\bar{0}} 1^{\bar{0}} f(a^{\bar{1}_2})^{\bar{0}} q(a^{\bar{1}}_1 1^{\bar{1}} f(a^{\bar{1}_2})^{\bar{1}}) \\
&= a^{\bar{0}} f(a^{\bar{1}_2})^{\bar{0}} q(a^{\bar{1}}_1 f(a^{\bar{1}_2})^{\bar{1}}) \\
&= a^{\bar{0}} (f \# q)(a^{\bar{1}})
\end{aligned}$$

and taking $b \in A^{coH}$

$$\begin{aligned} q\#j(b \rightarrow a) &= q\#j(\bar{b} a) \\ &= q\#j(b)\#j(a) \\ &= (q \leftarrow b)\#j(a). \end{aligned}$$

Therefore τ and μ are well-defined.

Proposition 4.3.10. *With the above notation, τ is a (A^{coH}, A^{coH}) -linear map and μ is a $(\underline{Hom}(H, A), \underline{Hom}(H, A))$ -linear map.*

Proof. By the above discussion, τ and μ are well-defined.

For the linearity of τ , suppose $a, a' \in A^{coH}$ and $b \otimes q \in A \otimes_{\underline{Hom}(H, A)} \tilde{Q}$. Then

$$\begin{aligned} \tau(a \rightarrow (b \otimes q) \leftarrow a') &= \tau((a \rightarrow b) \otimes (q \leftarrow a')) \\ &= \tau(ab \otimes q\#j(a')) \\ &= a^{\bar{0}}b^{\bar{0}}(q\#j(a'))(a^{\bar{1}}b^{\bar{1}}) \\ &\stackrel{(4.13)}{=} a^{\bar{0}}b^{\bar{0}}q(a^{\bar{1}}b^{\bar{1}}) a' \\ &\stackrel{A^{coH}}{=} a1^{\bar{0}}b^{\bar{0}}q(1^{\bar{1}}b^{\bar{1}}) a' \\ &= a b^{\bar{0}}q(b^{\bar{1}}) a' \\ &= a \tau(b \otimes q) a'. \end{aligned}$$

Now, given $h \in H$, $f, g \in \underline{Hom}(H, A)$ and $q \otimes a \in \tilde{Q} \otimes_{A^{coH}} A$, we have that

$$\begin{aligned} [f \rightarrow \mu(q \otimes a) \leftarrow g](h) &= [f\#\mu(q \otimes a)\#g](h) \\ &= [f\#q\#j(a)\#g](h) \\ &= (f\#q)(h_2)^{\bar{0}}(j(a)\#g)[h_1 (f\#q)(h_2)^{\bar{1}}] \\ &= (f\#q)(h)1^{\bar{0}}(j(a)\#g)[1^{\bar{1}}] \\ &= (f\#q)(h)1^{\bar{0}}a^{\bar{0}}g(1^{\bar{1}}a^{\bar{1}}) \\ &= (f\#q)(h)a^{\bar{0}}g(a^{\bar{1}}) \\ &= (f\#q)\#j(a^{\bar{0}}g(a^{\bar{1}}))(h) \\ &= (f \rightarrow q)\#j(a \leftarrow g)(h) \\ &= \mu[(f \rightarrow q) \otimes (a \leftarrow g)](h) \\ &= \mu[f \rightarrow (q \otimes a) \leftarrow g](h) \end{aligned}$$

and it shows the desired. □

Theorem 4.3.11. *Let H be a weak Hopf algebra, and A a symmetric partial H -comodule algebra. Then*

$$(A^{coH}, \underline{Hom}(H, A), A, \tilde{Q}, \tau, \mu)$$

defines a Morita context.

Proof. It just remains to prove the relation between τ and μ .

In fact,

$$\begin{aligned}
a \leftarrow \mu(q \otimes b) &= a^{\bar{0}} \mu(q \otimes b)(a^{\bar{1}}) \\
&= a^{\bar{0}} (q \# j(b))(a^{\bar{1}}) \\
&= a^{\bar{0}} q(a^{\bar{1}}) b \\
&= \tau(a \otimes q) \rightarrow b
\end{aligned}$$

and

$$\begin{aligned}
[\mu(q \otimes a) \rightarrow q'](h) &= (q \# j(a) \# q')(h) \\
&= (q \# j(a))(h_2)^{\bar{0}} q'[h_1 (q \# j(a))(h_2)^{\bar{1}}] \\
&= [q(h_2) a]^{\bar{0}} q'[h_1 [q(h_2) a]^{\bar{1}}] \\
&= q(h_2)^{\bar{0}} a^{\bar{0}} q'[h_1 q(h_2)^{\bar{1}} a^{\bar{1}}] \\
&= q(h) 1^{\bar{0}} a^{\bar{0}} q'[1^{\bar{1}} a^{\bar{1}}] \\
&= q(h) a^{\bar{0}} q'[a^{\bar{1}}] \\
&= [q \# j(a^{\bar{0}} q'[a^{\bar{1}}])](h) \\
&= [q \leftarrow (a^{\bar{0}} q'[a^{\bar{1}}])](h) \\
&= [q \leftarrow \tau(a \otimes q')](h).
\end{aligned}$$

□

Now it is clear that Theorem 1.2.7 can be simply applied to the above context, giving rise to some new Galois correspondences.

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