


Localization and Universality of Eigenvectors in Directed Random Graphs

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Although the spectral properties of random graphs have been a long-standing focus of network theory, the properties of right eigenvectors of directed graphs have so far eluded an exact analytic treatment. We present a general theory for the statistics of the right eigenvector components in directed random graphs with a prescribed degree distribution and with randomly weighted links. We obtain exact analytic expressions for the inverse participation ratio and show that right eigenvectors of directed random graphs with a small average degree are localized. Remarkably, if the fourth moment of the degree distribution is finite, then the critical mean degree of the localization transition is independent of the degree fluctuations, which is different from localization in undirected graphs that is governed by degree fluctuations. We also show that in the high connectivity limit the distribution of the right eigenvector components is solely determined by the degree distribution. For delocalized eigenvectors, we recover in this limit the universal results from standard random matrix theory that are independent of the degree distribution, while for localized eigenvectors the eigenvector distribution depends on the degree distribution.

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Introduction.—Complex systems, such as neural networks [1–3], ecosystems [4], gene regulatory networks [5–7], social networks [8,9], and the World Wide Web [10,11] are described by large, directed graphs. Therefore, there is much interest in understanding how the topology of directed graphs impacts the dynamics of processes and algorithms on them.

Much insight in the dynamical processes on graphs is gained from the spectral properties of the adjacency matrix that represents the network. This is because the linearized dynamics of a complex system in the vicinity of a fixed point is determined by the spectral properties of the adjacency matrix [12,13]. As a consequence, spectral analysis of the adjacency matrix has proven to be important in the study of neural networks [14–18], ecosystems [19–21], gene regulatory networks [22,23], and disease spreading [24–28]. In these systems, the eigenvectors of the adjacency matrix determine the dynamical modes evoked by external perturbations. In addition, right eigenvectors of adjacency matrices of directed graphs are used in algorithms for node centrality [29–31], community detection [32–34], and matrix completion [35].

In disordered systems, eigenvectors localize when the strength of the disorder is large enough [36,37]. Localized eigenvectors occupy a few vertices, whereas delocalized eigenvectors are extended over the whole system. The transition from a delocalized to a localized state leads to a qualitative change in the dynamics of processes and

algorithms. For example, the localization transition implies a metal-insulator phase transition in solid state physics [36,37], a transition from an algorithmic successful to a failure phase in spectral algorithms [31,35,38], and a transition from a regime where the linear dynamics of a large complex system is governed by a finite number of vertices to a regime where the dynamics is governed by a finite fraction of all vertices. In the context of disease spreading, eigenvector localization implies that the fraction of infected vertices is very small right above the epidemic threshold [25].

For undirected random graphs, the localization of eigenvectors of the adjacency matrix has been well studied [25,26,36,37,39–49]. The eigenvector of the largest eigenvalue is localized if the maximal degree of the graph is larger than a certain value. Hence, degree fluctuations are crucial for the localization of eigenvectors in undirected graphs.

For directed random graphs, the statistical properties and the localization of eigenvectors have been studied for one-dimensional chains, such as the Hatano-Nelson model [50–52] and its extensions to biological systems [53,54], and a diluted Ginibre ensemble [55]. However, the localization of eigenvectors in directed random graphs that model complex systems, such as the World Wide Web or neural networks, have not been studied so far.

In this Letter, we make a significant step forward by developing an exact theory for the statistical properties of the right (or left) eigenvectors of directed random graphs

with a prescribed degree distribution and random couplings. We derive exact analytic expressions for the inverse participation ratio and for the critical point of the localization-delocalization transition. Surprisingly, when the moments of the degree distribution are finite, the critical point of the localization-delocalization transition is independent of the degree distribution. Moreover, the right eigenvectors are localized if the degree distribution has a diverging fourth moment. We also show that in the high connectivity limit the statistics of the components of right eigenvectors are only determined by the degree distribution. In this limit, we obtain distinct universality classes that depend on an exponent that quantifies the degree fluctuations.

Model setup.—We consider random matrices \mathbf{A} of dimension $n \times n$ with elements

$$A_{ij} = J_{ij}C_{ij}, \quad i, j \in \{1, 2, \dots, n\}, \quad (1)$$

where $C_{ij} \in \{0, 1\}$ are the entries of the adjacency matrix \mathbf{C} of a simple and directed random graph with a prescribed degree distribution

$$p_{K^{\text{in}}, K^{\text{out}}}(k, \ell) = p_{K^{\text{in}}}(k)p_{K^{\text{out}}}(\ell) \quad (2)$$

of in degrees K^{in} and out degrees K^{out} . We set $C_{ij} = 1$ when there exists a directed link pointing from i to j , such that the out degree (in degree) of the i th node is $K_i^{\text{out}} = \sum_{j=1}^n C_{ij}$ ($K_i^{\text{in}} = \sum_{j=1}^n C_{ji}$). The J_{ij} are real valued, independent, and identically distributed random variables drawn from a distribution $p_J(x)$.

Random graph models with undirected edges and a prescribed degree distribution are surveyed in Ref. [56]. Here we consider their extension to the directed case. Directed random graphs with a prescribed degree distribution [57–62] model the World Wide Web [10,11] and neural networks [1,3,63]. In this model, the in degrees and out degrees are drawn from Eq. (2) subject to the constraint $\sum_{j=1}^n K_j^{\text{in}} = \sum_{j=1}^n K_j^{\text{out}}$, and subsequently nodes are randomly connected according to the given degree sequences. Hence, given a sequence of degrees, random graphs are drawn uniformly from the set of simple and directed graphs. This model provides the ideal setting to explore the influence of network topology on the spectral properties of \mathbf{A} .

In what follows, brackets $\langle \cdot \rangle$ denote the average with respect to the distribution of \mathbf{A} . In particular, we use

$$c = \langle K^{\text{out}} \rangle \quad (3)$$

for the mean out degree, and we denote the variance of a random variable X by $\text{var}(X) = \langle X^2 \rangle - \langle X \rangle^2$.

Spectra of infinitely large matrices \mathbf{A} .—The spectrum of \mathbf{A} has been studied in Refs. [64–67]. For $n \rightarrow \infty$ and $c > 1$, directed random graphs have a giant strongly connected component [68] and the spectral distribution $\rho_{\mathbf{A}}(\lambda) = n^{-1} \sum_{j=1}^n \delta[\lambda - \lambda_j(\mathbf{A})]$ of the eigenvalues $\{\lambda_j(\mathbf{A})\}_{j=1}^n$ is

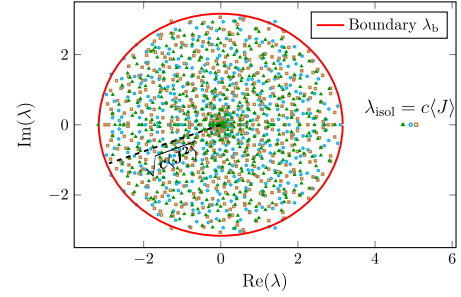


FIG. 1. Eigenvalues of three realizations (circles, triangles, and squares) of the adjacency matrix \mathbf{A} of directed random graphs with $n = 500$ [see Eq. (1)]. The in degrees and out degrees follow a Poisson distribution with average $c = 5$. The weights J_{ij} are drawn from a Gaussian distribution p_J with mean and variance equal to 1.

supported on a disk of radius $|\lambda_b| = \sqrt{c\langle J^2 \rangle}$ centered at the origin of the complex plane. In addition, if

$$c > c_{\text{gap}} = \frac{\langle J^2 \rangle}{\langle J \rangle^2}, \quad (4)$$

then there exists an eigenvalue outlier located at $\lambda_{\text{isol}} = c\langle J \rangle$ that is separated from the boundary λ_b by a finite gap. Figure 1 shows the eigenvalues for an example of a directed random graph, where one clearly identifies the outlier λ_{isol} and the boundary λ_b of $\rho_{\mathbf{A}}(\lambda)$ for $n \rightarrow \infty$.

Distribution of the right eigenvector components.—A right eigenvector $\vec{R}(\lambda)$ associated to an eigenvalue λ of \mathbf{A} satisfies

$$\mathbf{A}\vec{R}(\lambda) = \lambda\vec{R}(\lambda), \quad (5)$$

and the distribution of the entries of $\vec{R}(\lambda)$ reads

$$p_R(r|\lambda) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta[r - R_i(\lambda)]. \quad (6)$$

If λ is an outlier ($\lambda = \lambda_{\text{isol}}$) or λ is located at the boundary of the spectrum ($\lambda = \lambda_b$), then $p_R(r|\lambda)$ fulfills [65–67]

$$p_R(r|\lambda) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) \int \left(\prod_{j=1}^k dx_j d^2 r_j p_J(x_j) p_R(r_j|\lambda) \right) \times \delta \left(r - \frac{1}{\lambda} \sum_{j=1}^k x_j r_j \right), \quad (7)$$

where $d^2 r \equiv d\text{Re}r d\text{Im}r$. Equation (7) is exact for infinitely large and directed random graphs with a prescribed degree distribution, because they are locally treelike. In fact, the solutions of Eq. (7) are well corroborated by direct diagonalizations of large adjacency matrices [65–67]. The analytic results presented below follow from Eq. (7).

Inverse participation ratio.—The localization of $\vec{R}(\lambda)$ can be characterized in terms of the inverse participation ratio (IPR) [44,69,70]

$$\mathcal{I}(\lambda) \equiv \lim_{n \rightarrow \infty} \frac{n \sum_{i=1}^n |R_i(\lambda)|^4}{\left(\sum_{i=1}^n |R_i(\lambda)|^2\right)^2} = \frac{\langle |R(\lambda)|^4 \rangle}{\langle |R(\lambda)|^2 \rangle^2}, \quad (8)$$

where we have used that \mathcal{I} is self-averaging [71]. The IPR is finite if $\vec{R}(\lambda)$ is delocalized, whereas $\mathcal{I}(\lambda)$ diverges if $\vec{R}(\lambda)$ is localized on a finite number of nodes.

From Eq. (7), we derive in the Supplemental Material [71] exact expressions for the IPR when $\lambda = \lambda_{\text{isol}}$ or $\lambda = \lambda_b$. We find that

$$\mathcal{I}(\lambda_b) = \frac{(\gamma + 1)[\langle (K^{\text{out}})^2 \rangle - c]}{c(c - \langle J^4 \rangle / \langle J^2 \rangle^2)}, \quad (9)$$

where $\gamma = 2$ when $\lambda_b \in \mathbb{R}$ and $\gamma = 1$ when $\lambda_b \notin \mathbb{R}$. Analogously, the IPR at $\lambda = \lambda_{\text{isol}}$ reads

$$\begin{aligned} \mathcal{I}(\lambda_{\text{isol}}) = & \frac{3\beta_1 \langle J^2 \rangle^2}{(c^4 \langle J^4 \rangle - c \langle J^4 \rangle)} + \frac{\beta_3 (c^2 \langle J^2 \rangle^2 - c \langle J^2 \rangle)^2}{\beta_1^2 (c^4 \langle J^4 \rangle - c \langle J^4 \rangle)} \\ & + \frac{12\beta_1 \langle J^3 \rangle \langle J^2 \rangle (c^2 \langle J^2 \rangle^2 - c \langle J^2 \rangle)}{(c^4 \langle J^4 \rangle - c \langle J^4 \rangle)(c^3 \langle J^3 \rangle - c \langle J^3 \rangle)} \\ & + \frac{4\beta_2 \langle J^3 \rangle (c^2 \langle J^2 \rangle^2 - c \langle J^2 \rangle)^2}{\beta_1 (c^4 \langle J^4 \rangle - c \langle J^4 \rangle)(c^3 \langle J^3 \rangle - c \langle J^3 \rangle)} \\ & + \frac{6\beta_2 \langle J^2 \rangle (c^2 \langle J^2 \rangle^2 - c \langle J^2 \rangle)}{\beta_1 (c^4 \langle J^4 \rangle - c \langle J^4 \rangle)}, \end{aligned} \quad (10)$$

where

$$\beta_\ell \equiv \sum_{k=\ell+1}^{\infty} p_{K^{\text{out}}}(k) \frac{k!}{(k-\ell-1)!}, \quad \ell = 1, 2, 3. \quad (11)$$

Figure 2 illustrates Eqs. (9) and (10) as a function of c for a Gaussian distribution p_J and three different out degree distributions: Poisson, exponential, and Borel distribution (see Supplemental Material [71]). All moments of these degree distributions are finite and each $p_{K^{\text{out}}}$ is parametrized only by c . Figure 2 shows that the IPR is finite if c is large enough and it diverges for small c , which demonstrates the existence of a delocalization-localization phase transition in directed random graphs.

The localization phase transition.—There are two mechanisms for localization, one governed by fluctuations of J_{ij} , and a second one governed by degree fluctuations.

The first mechanism is illustrated in Fig. 2 and it holds for arbitrary $p_{K^{\text{out}}}$ with a finite fourth moment. In this case, the right eigenvectors associated to $\lambda = \lambda_b$ and $\lambda = \lambda_{\text{isol}}$ are localized when c is smaller than

$$c_b = \frac{\langle J^4 \rangle}{\langle J^2 \rangle^2} \quad \text{and} \quad c_{\text{isol}}^3 = \frac{\langle J^4 \rangle}{\langle J \rangle^4}, \quad (12)$$

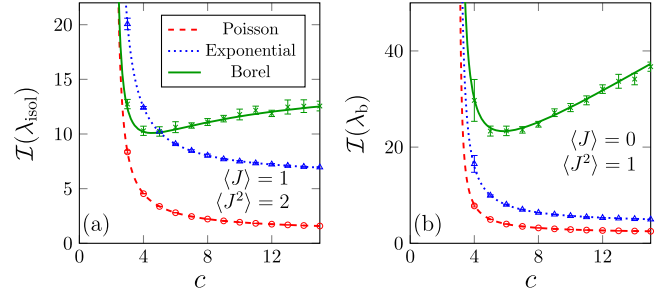


FIG. 2. The IPR $\mathcal{I}(\lambda)$ of right eigenvectors associated with λ_{isol} [panel (a)] and $\lambda_b \notin \mathbb{R}$ [panel (b)]. Equations (9) and (10) (different line styles) are shown as a function of the average degree c for different out degree distributions: Poisson, exponential, and Borel (see Supplemental Material [71]). The weights J_{ij} are drawn from a Gaussian distribution p_J with first and second moments indicated on each panel. The symbols are obtained from the numerical solutions of Eq. (7) using the population dynamics algorithm [43,66], while direct diagonalization results for $\mathcal{I}(\lambda)$ are presented in the Supplemental Material [71]. The error bars are the standard deviation of the IPR for 10 independent runs of population dynamics. The results for the Borel distribution are rescaled as $\mathcal{I}(\lambda_{\text{isol}}) \rightarrow \mathcal{I}(\lambda_{\text{isol}})/c$ in panel (a).

respectively. Thus, the critical points for the localization transitions only depend on the lower moments of p_J and they are independent of $p_{K^{\text{out}}}$. When $p_J(x) = \delta(x-1)$, we obtain $c_b = c_{\text{isol}} = 1$ and the delocalization-localization transition is governed by the percolation transition for the strongly connected component [68]. According to Eq. (10), a localization transition at $c_{\text{isol}}^* = \sqrt{\langle J^3 \rangle / \langle J \rangle^3}$ is, in principle, possible, but we could not find an example of p_J for which $c_{\text{isol}}^* > c_{\text{isol}}$ and $c_{\text{isol}}^* > c_{\text{gap}}$.

Figure 3 shows the phase diagram when p_J is a Gaussian distribution with mean μ and variance σ^2 . In this case, c_{gap} , c_b , and c_{isol} only depend on σ/μ . A few generic properties of eigenvector localization in directed random graphs, which also hold for non-Gaussian p_J , are illustrated in Fig. 3. First, $\vec{R}(\lambda_{\text{isol}})$ is delocalized when $\langle J^2 \rangle^3 > \langle J^4 \rangle \langle J \rangle^2$ because $c_{\text{gap}} > c_{\text{isol}}$. Second, the transition lines fulfill

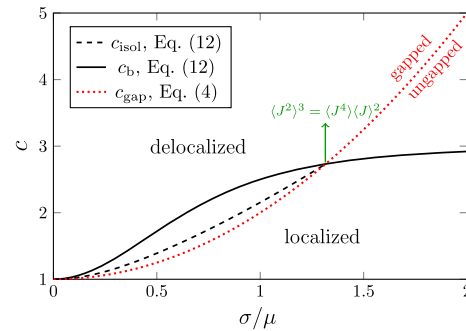


FIG. 3. Phase diagram for the localization of right eigenvectors associated to λ_{isol} and λ_b . The distribution p_J is Gaussian with mean μ and standard deviation σ .

$c_{\text{gap}} < c_{\text{isol}} < c_b$ for $\langle J^2 \rangle^3 < \langle J^4 \rangle \langle J \rangle^2$. Lastly, the critical transitions c_{gap} , c_{isol} , and c_b intersect in a common point because $c_{\text{isol}}^3 = c_b c_{\text{gap}}^2$.

The second mechanism for localization is due to large degree fluctuations. From Eqs. (9) and (10), it follows that $\mathcal{I}(\lambda_b) \rightarrow \infty$ if $\langle (K^{\text{out}})^2 \rangle \rightarrow \infty$ and $\mathcal{I}(\lambda_{\text{isol}}) \rightarrow \infty$ if $\langle (K^{\text{out}})^4 \rangle \rightarrow \infty$, independently of p_J . Hence, localization of $\vec{R}(\lambda_b)$ and $\vec{R}(\lambda_{\text{isol}})$ also occurs in graphs with power-law degree distributions. In the sequel, we show that degree-based localization persists in the high connectivity limit.

Localization and universality in the high connectivity limit.—In Fig. 2, $\mathcal{I}(\lambda)$ flows to different asymptotic values for $c \gg 1$. To explore the localization and universality of eigenvectors in the high connectivity limit $c \rightarrow \infty$, we analyze the moments of the distribution p_R . Since $\langle R(\lambda_{\text{isol}}) \rangle$ is finite, we characterize the limit $c \rightarrow \infty$ of $p_R(r|\lambda_{\text{isol}})$ through the relative variance

$$\mathcal{R}_c = \frac{\text{var}[R(\lambda_{\text{isol}})]}{\langle R(\lambda_{\text{isol}}) \rangle^2}. \quad (13)$$

On the other hand, since $\langle R(\lambda_b) \rangle = 0$, we characterize the limit $c \rightarrow \infty$ of $p_R(r|\lambda_b)$ through the kurtosis

$$\mathcal{K}_c = \frac{\langle (\text{Re}R(\lambda_b))^4 \rangle}{\langle (\text{Re}R(\lambda_b))^2 \rangle^2} = \frac{(4-\gamma)}{2} \mathcal{I}(\lambda_b), \quad (14)$$

where we used the fact that odd moments of $p_R(r|\lambda_b)$ are zero [71]. Setting $c \rightarrow \infty$ in Eqs. (13) and (14), we obtain [71]

$$\mathcal{R}_\infty = \lim_{c \rightarrow \infty} \frac{\text{var}[K^{\text{out}}]}{c^2}, \quad (15)$$

$$\mathcal{K}_\infty = 3 \left(1 + \lim_{c \rightarrow \infty} \frac{\text{var}[K^{\text{out}}]}{c^2} \right), \quad (16)$$

which indicates that the limit $c \rightarrow \infty$ of p_R is determined by the degree distribution. We see that, in general, $p_R(r|\lambda_b)$ and $p_R(r|\lambda_{\text{isol}})$ are not Gaussian in the high connectivity limit.

With the purpose of classifying the universal behavior of p_R for $c \rightarrow \infty$, let us consider degree distributions that satisfy

$$\text{var}[K^{\text{out}}] = Bc^\alpha (c \gg 1), \quad (17)$$

where α and B depend on the specific choice of $p_{K^{\text{out}}}(k)$. Equation (17) holds for different examples of degree distributions, including those in Fig. 2. Plugging this *ansatz* for $\text{var}[K^{\text{out}}]$ in Eqs. (15) and (16), we obtain three universality classes for $\lim_{c \rightarrow \infty} p_R(r|\lambda)$, which are determined by the exponent α that controls the degree fluctuations. The results for the universality classes are summarized in Table I. In terms of \mathcal{R}_∞ and \mathcal{K}_∞ , we find that for $\alpha \leq 2$ the

TABLE I. The relative variance \mathcal{R}_c of $\vec{R}(\lambda_{\text{isol}})$ and the kurtosis \mathcal{K}_c of $\vec{R}(\lambda_b)$ in the high connectivity limit $c \rightarrow \infty$ [see Eqs. (15) and (16)], together with an example of the out degree distribution $p_{K^{\text{out}}}$ in each regime of α [see Eq. (17)].

	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$
\mathcal{R}_∞	0	B	∞
\mathcal{K}_∞	3	$3(1+B)$	∞
Example	Poisson	Exponential	Borel

eigenvectors $\vec{R}(\lambda_b)$ and $\vec{R}(\lambda_{\text{isol}})$ are delocalized in the limit $c \rightarrow \infty$, whereas for $\alpha > 2$ these eigenvectors are localized due to large degree fluctuations.

The eigenvector distributions in the high connectivity limit.—The results in Table I indicate that $p_R(r|\lambda)$ is universal for $c \rightarrow \infty$. Below we present explicit expressions for $p_R(r|\lambda)$ when $c \rightarrow \infty$. Henceforth we set $\langle |R|^2 \rangle = 1$ without losing generality.

The characteristic function of $p_R(r|\lambda)$ is given by [71]

$$g_R(u, v|\lambda) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) e^{k \ln F(u, v|\lambda)}, \quad (18)$$

where

$$F(u, v|\lambda) = \int dx p_J(x) \int d^2 r p_R(r|\lambda) \exp^{-\frac{ux}{2\lambda} + \frac{vx^*r^*}{2\lambda^*}}, \quad (19)$$

and $z = u + iv$. The symbol $(\dots)^*$ denotes complex conjugation. If $\lambda \in \mathbb{R}$, the eigenvector components are real and $F(u, v|\lambda)$ is independent of v .

Setting $\lambda = \lambda_{\text{isol}}$ or $\lambda = \lambda_b$ in Eq. (19), we can expand $F(u, v|\lambda)$ for $c \gg 1$ up to order $O(1/c)$ if $\alpha \leq 2$ (see Table I). This approach does not work for $\alpha > 2$, because the moments of p_R can diverge in this regime. Thus, performing this expansion for $\alpha \leq 2$ and substituting the resulting expression for $F(u, v|\lambda)$ in Eq. (18), we obtain [71]

$$g_R(u, v|\lambda_b) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) \exp \left[-\frac{\gamma k}{4c} (u^2 + (2-\gamma)v^2) \right], \quad (20)$$

$$g_R(u, v|\lambda_{\text{isol}}) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) \exp \left(-\frac{iuk}{c\sqrt{Bc^{\alpha-2} + 1}} \right). \quad (21)$$

Remarkably, the characteristic functions for $c \rightarrow \infty$ are fully specified by $p_{K^{\text{out}}}$ and they are independent of p_J .

For degree distributions where $\lim_{c \rightarrow \infty} \text{var}[K^{\text{out}}]/c^2 = 0$ ($\alpha < 2$), we can set $p_{K^{\text{out}}}(k) = \delta_{k,c}$ in Eqs. (20) and (21), leading to [71]

$$p_R(r|\lambda_b) = \frac{1}{\pi} e^{-|r|^2} (\lambda_b \notin \mathbb{R}), \quad (22)$$

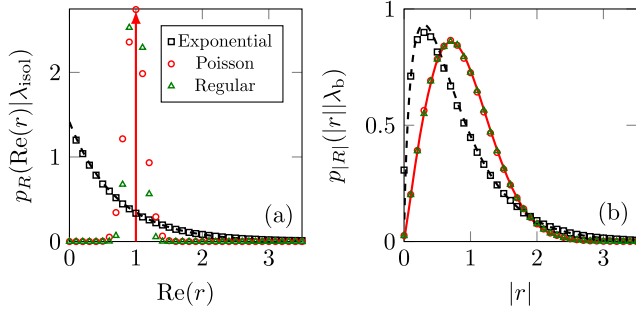


FIG. 4. The high connectivity limit $c \rightarrow \infty$ of the distribution $p_R[\text{Re}(r)|\lambda_{\text{isol}}]$ of the real part of the eigenvector components at λ_{isol} [panel (a)], and of the distribution $p_{|R|}(|r||\lambda_b)$ of the eigenvector components at $\lambda_b \notin \mathbb{R}$ [panel (b)]. The solid red lines and the dashed black lines are, respectively, the analytic results for regular or Poisson and exponential degree distributions [see Eqs. (22)–(25)], while the symbols are numerical solutions of Eq. (7) with $c = 100$. The numerical data for regular or Poisson graphs in panel (a) is a Gaussian distribution with variance of $O(1/c)$, approaching the Dirac delta distribution (vertical arrow) for $c \rightarrow \infty$.

$$p_R(r|\lambda_{\text{isol}}) = \delta[\text{Im}(r)]\delta[\text{Re}(r) - 1]. \quad (23)$$

Equation (22) yields the well-known Porter-Thomas distribution for the eigenvector components of Gaussian random matrices [77,78]. Thus, standard results from random matrix theory are recovered when $\alpha < 2$.

If $p_{K^{\text{out}}}$ is an exponential distribution, where $\alpha = 2$, we obtain in the limit $c \rightarrow \infty$ [71]

$$p_R(r|\lambda_b) = \frac{2}{\pi} K_0(2|r|) \quad (\lambda_b \notin \mathbb{R}), \quad (24)$$

$$p_R(r|\lambda_{\text{isol}}) = \sqrt{2}\delta[\text{Im}(r)]\Theta[\text{Re}(r)]e^{-\sqrt{2}\text{Re}(r)}, \quad (25)$$

where $\Theta(x)$ is the Heaviside step function and $K_0(x)$ is a modified Bessel function of the second kind [79]. Figure 4 illustrates the shape of the distributions p_R given by Eqs. (22)–(25), and compares them with numerical solutions of Eq. (7) for $c = 100$. The derivation of Eqs. (22)–(25) is explained in the Supplemental Material [71].

Conclusions.—We have shed light on the relationship between graph topology and the localization of right eigenvectors in directed random graphs. If the moments of the out degree distribution $p_{K^{\text{out}}}$ are finite, then right eigenvectors at the edge of the spectrum are localized below a critical mean out degree. It is striking that the critical points for the localization transitions are universal, in the sense they only depend on the lower moments of the distribution p_J of the edge weights, regardless of the network topology. Therefore, localization in directed random graphs is fundamentally different from localization in undirected graphs, for which degree fluctuations are

important [25,42–44,46,47,80,81]. Indeed, the eigenvector associated with the largest eigenvalue of the adjacency matrix of an undirected random graph is localized if the maximal degree is large enough [25]. Degree-based localization is also possible for directed random graphs, but then $p_{K^{\text{out}}}$ requires a divergent fourth moment.

In the high connectivity limit, the distribution p_R of the right eigenvector components is determined by the graph topology, independently of p_J . If the out degree fluctuations are small enough, then eigenvectors are delocalized and p_R is given by the same universal distribution as in the case of Gaussian random matrices [77,78]. On the other hand, if the out degree fluctuations are large enough, then eigenvectors are localized and the distribution p_R depends on $p_{K^{\text{out}}}$. More generally, these results indicate that Gaussian random matrix theory describes well the spectral properties of high connectivity graphs only when the degree fluctuations are sufficiently small [82].

For future work, it would be interesting to explore the implications of eigenvector localization for the dynamics of neural networks [53,54] and ecosystems [20,83], to compare the theoretical predictions for the IPR with empirical values in real-world networks [26,84], and to study eigenvector localization of Laplacians of directed graphs [85–87].

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