Universidade Federal do Rio Grande do Sul

Instituto de Matemática e Estatística

Programa de Pós-Graduação em Matemática

Um processo iterativo para aproximar sub-ações calibradas e exemplos explícitos em Otimização Ergódica

Dissertação de Mestrado

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Dissertação submetida por Hermes Hofmeister Ferreira [1] como requisito parcial para a obtenção do grau de Mestre em Matemática pelo Programa de Pós-Graduação em Matemática do Instituto de Matemática e Estatística da Universidade Federal do Rio Grande do Sul.

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Data da Apresentação: 16/04/2021.

[1]Bolsista do Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq

Resumo

Apresentamos um procedimento iterativo para aproximar numéricamente sub-ações calibradas. O problema principal em Otimização Ergódica consiste de calcular médias maximais ergodicas. Dado um sistema dinâmico $T : X \to X$ e uma função contínua $A : X \to \mathbb{R}$, chamada de potencial, nesta teoria o interesse principal está no valor $m(A) := \sup_{\rho \text{ is a } T\text{-invariant probability}} \int A \, d\rho$ e na probabilidade ρ que atinge este valor, chamada de probabilidade maximizante. Estamos interessados em propriedades de tal ρ. Sub-ações calibradas são funções $V : X \to \mathbb{R}$ tais que $\max_{T(y)=x} [A(y) + V(y)] = m(A) + V(x)$.

O motivo de interesse nas sub-ações é porque os suportes das probabilidades maximizantes de A estão contidos no conjunto $\{x \,|\, V(T(x)) - V(x) - A(x) + m(A) = 0\}$. Uma propriedade importante é a de que se uma probabilidade invariante possui suporte no conjunto acima, ela é maximizante (ver [5]).

Para um potencial Hölder A sempre existe uma sub-ação calibrada. Também é conhecido que genéricamente para um potencial Hölder A a sua probabilidade maximizante é única. Se a probabilidade maximizante é única, então a sub-ação calibada é única a menos de uma constante aditiva.

Nosso procedimento consiste em iterar um operador em uma função inicial, convergindo para um ponto fixo que será uma sub-ação calibrada. Se existir mais de uma sub-ação calibrada o limite depende da condição inicial. A implementação do procedimento é direta e exige pouco poder computacional. O processo também pode ser aplicado na estimação de raio espectral conjunto de matrizes.

Nos restringiremos para $X = S^1 := [0, 1]$ e $A : X \to \mathbb{R}$ contínua. Estamos principalmente interessados no caso $T(x) = 2x (\text{mod } 1)$, mas outras dinâmicas T também são consideradas. Sub-ações calibradas são importantes para obter o valor $m(A)$ e a probabilidade maximizante.

O procedimento proposto aproxima numéricamente sub-ações calibradas e o valor $m(A)$. Com essas aproximações nós podemos adivinhar um sistema de equações que a sub-ação deve satisfazer, a resolução deste sistema fornece a solução explícita para a sub-ação calibrada e o valor $m(A)$. Nós deduzimos o sistema heurísticamente utilizando o gráfico da sub-ação calibrada aproximada.

Abstract

We present an iterative procedure for numerically approximating calibrated sub-actions. The central issue of Ergodic Optimization consists of computing maximal ergodic averages. Given a dynamical system $T : X \to X$, and a continuous function $A : X \to \mathbb{R}$, called the potential, in this theory the main interest is in the value $m(A) := \sup_{\rho \text{ is a } T\text{-invariant probability}} \int A \, d\rho$ and the probability ρ which attains this value, called the maximizing probability. Properties of such probabilities are the main issue here. Calibrated subactions are functions $V : X \to \mathbb{R}$ such that $\max_{T(y)=x} [A(y) + V(y)] = m(A) + V(x)$.

The interest on calibrated subactions is due to the fact that the support of maximizing probabilities for A are contained on the set $\{x \mid V(T(x)) - V(x) - A(x) + m(A) = 0\}$. An important property is: if an invariant probability has support inside the above set, then, this probability is maximizing (see [5]).

For a Hölder potential A a calibrated subaction always exists. It is also known that generically on a Hölder potential A the maximizing probability for A is unique. If the maximizing probability is unique the calibrated subaction is unique up to an additive constant.

Our procedure consists of iterating an operator acting on a given initial function which will converge to a fixed point which will be a subaction. If there exists more than one subaction the limit depends on the initial condition. Its implementation is straightforward and requires little computational power. The iterative procedure can also be applied to the estimation of the joint spectral radius of matrices.

We restrict ourselves to $X = S^1 := [0, 1]$ and $A : X \to \mathbb{R}$ continuous. We are mainly interested in the case $T(x) = 2x(\text{mod } 1)$ but other dynamics T are also considered. Calibrated subactions play an important role in computing the value $m(A)$ and the maximizing probability.

Our proposed iterative procedure approximates numerically calibrated subactions and the value $m(A)$. With these approximations, we can guess a system of equations that the calibrated subaction must satisfy, solving this system yields the explicit expression for the calibrated subaction and the value $m(A)$. We derive the system heuristically by using the graph of the approximated calibrated subaction.

Agradecimentos

Eu expresso minhas profundas gratidões a minha mãe Maria Cristina e minha namorada Elisa. É difícil descrever o quanto são importantes para mim.

Agradeço ao meu sogro Luis Gustavo que me incentivou a cursar matemática quando estava na graduação.

Agradeço ao professor Jairo Krás, um dos primeiros professores do instituto de matemática com o qual eu tive o prazer de ter contato, sempre sendo paciente com minhas dúvidas. Eventualmente Jairo me indicou para trabalhar como bolsista com o professor Artur e esse foi o começo de uma longa jornada.

Agradeço aos meus colegas e amigos matemáticos que me acompanharam desde a graduação, sempre com os quais sempre pude discutir problemas de matemática. Uma das partes mais confortantes de estudar matemática foi poder resolver problemas no quadro juntos.

Em particular agradeço pela amizade de longa história de Bruno Cravo e João Batista. Agradeço a amizade de meu irmão Ramoses.

Agradeço ao meu orientador, Artur O. Lopes que me orientou desde a graduação como bolsista de iniciação científica, sempre acreditando em minha capacidade, compartilhando de seu conhecimento e me guiando. Tenho orgulho de ter produzido trabalhos com Artur.

Agradeço ao CNPQ pelo auxílio financeiro dado durante todo o mestrado.

Contents

Chapter 1

Introduction

In this work, we aim to contribute to the research in Ergodic Optimization. We propose techniques for computing calibrated sub-actions, which are an important tool in Ergodic Optimization. We can extract from them information about maximizing orbits of dynamical systems.

There are two major settings that people analyze questions in Ergodic Optimization: 1) when it is assumed the potential is just continuous, and, 2) when it is assumed some regularity (as Hölder continuity for instance) on the potential. The two cases are conceptually distinct: in the first case, generically, the maximizing probability has support on the all space (see [4] and [16]) and in the second case, generically, the support has support on a periodic orbit (see [6] and [5]). In the first case, generically, subactions are of no help. It is in the second case that subactions are of great help for identifying the support of the maximizing probability. In our work, we introduce a nice tool for identifying, generically, the maximizing probability (see [9]).

In the literature, there are other approaches to computing calibrated subactions, using the involution kernel and techniques from Ergodic Transport. However, obtaining an explicit expression for an involution kernel is not always possible. Our proposed approach allows us to solve explicitly a multitude of examples with great simplicity without requiring to know the involution kernel. Thus, to the best of our knowledge, our proposed techniques are novel.

We begin by stating the definitions which are fundamental to our framework. We also provide motivation to our definitions, so that the reader can better understand how our work relates to other topics.

Given a compact metric space X we consider the associated Borel sigma-algebra $\mathcal{A}$.

Definition 1.0.1. *Given a continuous transformation $T : X \to X$ and a continuous function $A : X \to \mathbb{R}$, a probability μ on $(X, \mathcal{A})$ is said to be T-invariant if for every measurable set $A \in \mathcal{A}$, $\mu(A) = \mu(T^{-1}(A))$*

Definition 1.0.2. *We denote $m(A) := \sup_{\rho \text{ is a} T\text{-invariant probability}} \int A \, d\rho$. If μ is such that $\int A \, d\mu = m(A)$, then we say μ is a maximizing probability for A.*

Under the above assumptions maximizing probabilities for A exist but may not be unique.

Definition 1.0.3. *We say that a function $A : S^1 \to \mathbb{R}$ is Hölder continuous, or simply Hölder if there are nonnegative real constants C and α such that*

$$|A(x) - A(y)| \leq C \, |x - y|^{\alpha}$$

We now give motivation for the interest in this topic within the context of Thermodynamic Formalism - a reader familiar with these concepts can skip to the next definition.

Suppose we have an infinite chain of atoms on the lattice $\mathbb{Z}$, such that in each place of the lattice (the i-th atom) the configuration state could be 0 or 1 (which could play respectively the role of the spin $+$ and $-$). We then identify this chain with the space $\Sigma := \{0, 1\}^{\mathbb{Z}}$, this means that a point $x \in \Sigma$ is of the form $x = (...x_{-2}, x_{-1}|x_0, x_1, ...)$, where x_i is the configuration of the i-th atom.

From the point of view of Equilibrium Statistical Physics is natural to assume that the ergodic properties we should expect from the system should be invariant by translation on the lattice $\mathbb{Z}$. In other words, there is no natural choice for the origin 0 on the lattice $\mathbb{Z}$. In this direction is natural to consider the shift $\sigma : \Sigma \to \Sigma$. That is, $x \to \sigma(x) := (..., x_{-1}, x_0|x_1, x_2, ...)$ and $\sigma^{-1}(x) = (..., x_{-3}, x_{-2}|x_{-1}, x_0, ...)$. This simply represents a translation of a chain of atoms on the lattice. If the configurations of the atoms in this chain are changing according to a stationary probability μ (a probability in thermodynamic equilibrium) we should expect that for each Borel set A on Σ we have the property $\mu(A) = \mu(\sigma^{-1}(A))$. This is because the choice of the origin should not change the likelihood of the observed outcomes. This amounts of saying that μ is an invariant measure with respect to the dynamical system with $X = \Sigma$ and $T = \sigma$. In this case, we describe the state of the system by a T-invariant probability acting on the measurable space, which we will call an equilibrium state of the system.

We can further describe the interaction between the atoms in the chain by a function $A : X \to X$, called the potential of the system. So $A(x)$ could be seen as the energy (or Hamiltonian) of the system in the chain x and could depend on the temperature of the system. For each potential and temperature, one gets an equilibrium probability. In Thermodynamic Formalism, it is natural to consider concepts such as the free energy and the entropy of a given state of the system (an invariant probability on Σ). The thermodynamic equilibrium at a certain temperature is reached on the invariant probability which maximizes free energy.

We will not exhaust this subject here and we refer the interested reader to [3] for more information. For a fixed generic Hölder potential A, it turns out that when the temperature of the system goes to zero, the state that maximizes the free energy converges to the measure that attains the value $m(A) = \sup_{\rho \text{ is } T\text{-invariant probability}} \int A \, d\rho$. That is, the maximizing probability for A.

One interesting aspect is that the computations related to invariant probabilities on $X = \{0,1\}^{\mathbb{Z}}$ can be dealt in the framework of the set $X = \{0,1\}^{\mathbb{N}}$ (see [22]). Note that we can identify points in $\{0,1\}^{\mathbb{N}}$ as the binary expansion of points in the interval $[0,1]$. From this, we can see how our work relates to problems in Thermodynamic Formalism.

The Hölder hypothesis for A represents the fact that the interaction described by the Hamiltonian A decays very fast for two different points x and y far away in Σ.

In Ergodic Optimization we are mainly interested in properties of the invariant probabilities that attain the value $m(A)$. An important result is that when A is Hölder, there is a calibrated subaction (see [6]). Thus, definition 1.0.3 gives a regularity condition to our study. The following definition is crucial to our work.

Definition 1.0.4. *Given a continuous function $A : S^1 \to \mathbb{R}$, a continuous function $V : S^1 \to \mathbb{R}$ that satisfies for any $x \in S^1$, $V(x) = \max_{T(y)=x} [V(y) + A(y) - m(A)]$, is called a calibrated sub-action*

Note that if V is a calibrated subaction for A, then V plus a constant is also a calibrated subaction for A. If the maximizing probability is unique the calibrated subaction is unique up to adding a constant.

For Hölder potentials A there exists Hölder calibrated sub-actions (see [5]). In this work, we provide techniques to compute such calibrated sub-actions, from which one can obtain $m(A)$ and the support of the maximizing probabilities. It follows from the definition that for all $x \in S^1$

$$R(x) := V(T(x)) - V(x) - A(x) + m(A) \geq 0.$$

If a given point $x \in S^1$ belongs to the support of a maximizing measure, then $R(x) = 0$ (see [5]). Moreover, if an invariant probability has support inside the set of x such that $R(x) = 0$, then, this probability is maximizing for A (see [5]). If the potential A is just continuous a calibrated subaction may not exist. Therefore we restrict ourselves to Lipschitz or Hölder potentials.

In this work, we give a mixture of numerical solutions and explicit solutions for the calibrated sub-actions. We use numerical solutions to guide us towards explicit solutions. This procedure has been shown to be quite effective as we will see.

For reproducibility of our analysis, we provide the code used to generate the results in https://github.com/hermes-hf/Explicit_examples_ergodic/. This text derives from our work [9] and [10].

Chapter 2

Approximating calibrated subactions

In this chapter, we introduce the iterative procedure used to approximate calibrated subactions. We first introduce the concepts used and then show that the iterations converge.

2.1 The 1/2-Iterative process

For simplicity we define

Definition 2.1.1. *Denote by* $C[0,1]$ *the set of continuous functions on* $[0,1]$.

We define a building block for our procedure.

Definition 2.1.2. *Denote* $K = K_A : C[0,1] \to C[0,1]$ *the operator such that*

$$K(f)(x) = \max_{T(y)=x} [A(y) + f(y)] - \kappa_f \,.$$

Where $\kappa_f := \max_{z \in S^1} \max_{T(y)=x} [A(y) + f(y)]$.

Observe that if f is a fixed point for K, it must be that it is a calibrated subaction. Unfortunately iterating K does not necessarily lead to a calibrated subaction.

Example 2.1.3. *Consider* $A(x) = -(x - 1/3)^2$ *with* $T(x) = 2x \bmod 1$. *Set* $F_0 = 0$ *and* $F_n = K^n(F_0)$ *to be the iterations of* K. *In this case* F_n *does not converge, see Figure 2.1.*

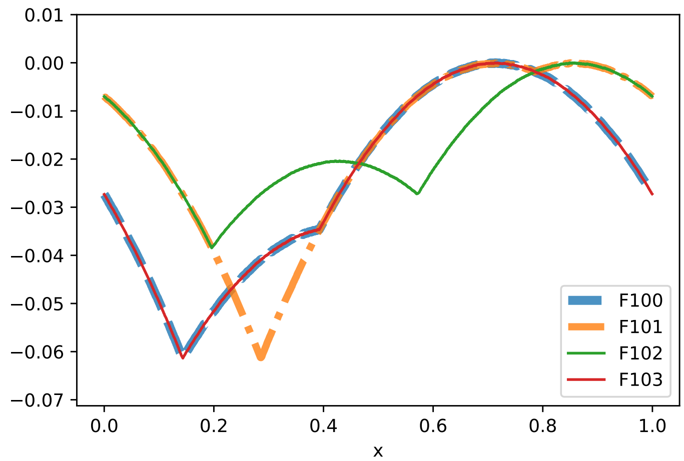

Figure 2.1: With A and T as in Example 2.1.3, iterations of K do not seem to converge. As it can be seen from the figure, $F_{103} \approx F_{100}$, and so the iterations appear to cycle between 3 functions.

To bypass this issue, we construct an averaging scheme by defining

Definition 2.1.4. *Given $A : S^1 \to \mathbb{R}$, consider the operator $G = G_A : C[0,1] \to C[0,1]$ such that, for $f : \mathcal{S}^1 \to \mathbb{R}$,*

$$G(f)(x) = \frac{K(f)(x) + f(x)}{2} - c_f \,,$$

for any $x \in S^1$, where $c_f := \frac{1}{2} \max_{z \in S^1} [K(f)(z) + f(z)]$.

The $\frac{1}{2}$ factor used allows iterations of the operator G to converge as we will show in the following section. This form of averaging for operators has been discussed in [15], [14] and [8].

We first choose the initial function $f_0 := 0$ (the constant zero function), then we compute the iterates $f_n := G^n(f_0)$. If n is sufficiently large, we then expect that

$$f_n(x) + 2 \cdot c_{f_n} \approx \max_{T(y)=x} [A(y) + f_n(y)] \,.$$

So that iterations of G yield estimates to $m(A) \approx 2 \cdot c_{f_n}$ and the calibrated subaction $V \approx f_n$ as well. We then define the iterative procedure consisting of iterating the operator G on the zero function $f_0 = 0$. We will refer to this procedure as the 1/2-Iterative process or procedure.

2.2 Convergence of the iterations

We will discuss the convergence properties of the operator G. This first result shows that K is continuous. We set the sup norm $|f|_\infty := \sup_{x\in[0,1]} |f(x)|$.

Theorem 2.2.1. *The operator K satisfies for $f, g \in C[0,1]$*

$$|K(f) - K(g)|_\infty \leq 2|f-g|_\infty \,.$$

Proof. Let $f, g \in C[0,1]$. We have

$$K(g)(x) = \max_{T(y)=x}[A(y)+f(y)+(g-f)(y)] - \max_{z\in S^1}\max_{T(y)=z}[A(y)+f(y)+(g-f)(y)]$$

Bounding the terms $(g-f)$ by above we obtain $K(g)(x) \leq$

$$\leq \max_{T(y)=x}[A(y)+f(y)] + \max_z[(g-f)(z)] - \max_{z\in S^1}\max_{T(y)=x}[A(y)+f(y)] - \min_z[(g-f)(z)]$$

so that

$$K(g)(x) - K(f)(x) \leq \max_z[(g-f)(z)] - \min_z[(g-f)(z)]$$

and also $K(g)(x) \geq$

$$\geq \max_{T(y)=x}[A(y)+f(y)] + \min_z[(g-f)(z)] - \max_{z\in S^1}\max_{T(y)=x}[A(y)+f(y)] - \max_z[(g-f)(z)]$$

Thus

$$K(g)(x) - K(f)(x) \leq 2|g-f|_\infty$$

Interchanging the roles of f and g we obtain the other inequality, and so

$$|K(f) - K(g)|_\infty \leq 2|f-g|_\infty$$

□

We can strengthen this result by using another norm on $C[0,1]$. To this end, consider the space of functions

Definition 2.2.2. *Define the quotient $\mathcal{C} = C[0,1]/\mathbb{R}$.*

In this space, function classes are defined up to addition, with this we mean that if $f-g$ is constant, then the equivalence class $[f]$ is equal to $[g]$. We equip this space with the norm:

Definition 2.2.3. *Given a function $f \in C[0,1]$, consider its class $[f] \in \mathcal{C}$. Set $|[f]|_{\mathcal{C}} = |f|_{\mathcal{C}} = \min_{\alpha \in \mathbb{R}} |f + \alpha|_\infty$*

Then $\mathcal{C}$ with this norm becomes a Banach space (see [21]).

We state an improvement of the previous setting.

Theorem 2.2.4. *Let $f, g \in C[0,1]$. Then $|K(f) - K(g)|_{\mathcal{C}} \leq |f - g|_{\mathcal{C}}$*

Proof. Let $f, g \in C[0,1]$ and d such that

$$|f - g|_{\mathcal{C}} = |f - g + d|_\infty .$$

Consider $K(f)(x) - K(g)(x) =$

$$= -\kappa_f + \max_i [(A + f) \circ \tau_i(x)] + \kappa_g - \max_i [(A + g) \circ \tau_i(x)] .$$

We can then obtain

$$\begin{aligned} &K(f)(x) - K(g)(x) - \kappa_f + \kappa_g + d = \\ &\qquad \max_i [(A + f - g + g + d) \circ \tau_i(x)] - \max_i [(A + g) \circ \tau_i(x)] . \end{aligned} \tag{2.1}$$

Observe that $-|f - g|_{\mathcal{C}} \leq f(y) - g(y) + d \leq |f - g|_{\mathcal{C}}$ for any $y \in [0,1]$. By monotonicity of the supremum we get

$$\begin{aligned} &-|f - g|_{\mathcal{C}} + \max_i [(A + g) \circ \tau_i(x)] \leq \\ &\qquad \max_i [(A + g + f - g + d) \circ \tau_i(x)] \leq |f - g|_{\mathcal{C}} + \max_i [(A + g) \circ \tau_i(x)] . \end{aligned}$$

Which is equivalent to

$$\begin{aligned} &-|f - g|_{\mathcal{C}} \leq \\ &\qquad \max_i [(A + g + f - g) \circ \tau_i(x)] - \max_i [(A + g) \circ \tau_i(x)] \leq |f - g|_{\mathcal{C}} , \end{aligned}$$

thus

$$\left| \max_i [(A + g + f - g) \circ \tau_i(x)] - \max_i [(A + g) \circ \tau_i(x)] \right| \leq |f - g|_{\mathcal{C}} . \tag{2.2}$$

We assumed $|f - g + d|_\infty = |f - g|_{\mathcal{C}}$. Therefore, using (2.1) and (2.2)

$$|K(f) - K(g) + \kappa_f - \kappa_g + d|_\infty \leq |f - g|_{\mathcal{C}}$$

Recall that

$$|K(f) - K(g)|_{\mathcal{C}} = \\ \min_{k\in\mathbb{R}} |K(f) - K(g) + k|_{\infty} \leq |K(f) - K(g) + (\kappa_f - \kappa_g + d)| \leq |f - g|_{\mathcal{C}} \,. \tag{2.3}$$

This means $|K(f) - K(g)|_{\mathcal{C}} \leq |f - g|_{\mathcal{C}}$ as we wanted to show. □

Theorem 2.2.5 shows that K acting on $\mathcal{C}$ is a weak contraction. It is not a strong contraction - even in the case the subaction is unique - as it was shown in [10]. One can also show that if C_k is the set of Lipschitz functions of same constant k, then $K(C_k) \subseteq C_k$ is compact in $\mathcal{C}$, see [10] Theorem 11. We then borrow results from Theorem 1 in [15] to obtain:

Theorem 2.2.5. *Let $A \in C[0,1]$ be a given Lipschitz or Hölder function. Given $f_0 \in C[0,1]$ we have $\lim_{n\to\infty} G^n(f_0) = V$, where V is a subaction.*

We restrict ourselves to Lipschitz (or Hölder potentials). This is because for a generic continuous potential there might be no subaction.

In the case the maximizing probability is unique, the subaction is unique up to adding a constant. Then, there is a unique class in $\mathcal{C}$ representing a subaction. In this case, independent of the initial condition, the iterative process will converge to such class (which represents a subaction). In the case there exist more than one subaction, given an initial condition, according to Theorem 2.2.5 the iterative process will converge to a subaction, but the limit can depend on the initial condition (see Section 3.9)

For such iterative methods, we can wonder if there exists an exponential convergence rate of convergence. Even though the iterations G^n converge to the desired solution, there are situations where there is no exponential convergence rate. For practical results, we noticed that the method works fine in almost all cases. In our experimental results, we achieved good results with at least 30 iterations.

We will not focus on guaranteeing numerical estimates for the approximate solutions. Instead, we choose to construct the explicit solution from the numerical results. This is possible in many cases as we will see in the following chapter.

Counter example: G may not be a strong contraction (by a factor smaller than 1). We will present an example where $f_0, g_0 \in \mathcal{C}$ but $|G(f_0) - G(g_0)| = 1/2 = |f_0 - g_0|$.

Consider the potential A with the graph given by Figure 2.2. This potential is linear by parts and has the value 0 on the points $1/8, 1/4, 3/4, 7/8$. The value -1 is attained at the points $0, 3/16, 1/2, 13/16, 1$.

Denote $g_0 = 0$ and $f_0 = A$. Then, $|f_0 - g_0| = |f_0 - g_0 + 1/2|_0 = 1/2$. We denote $f_1 = G(f_0)$ and $g_1 = G(g_0)$. The graph of the function $x \to |f_1(x) - g_1(x) + 0.5|$ is described by the bottom rigth picture on Figure 2.2. One can show that $|f_1 - g_1| = |f_1 - g_1 + 1/2|_0 = 1/2$. Therefore, for such potential A the transformation G is not a strong contraction.

$\diamondsuit$

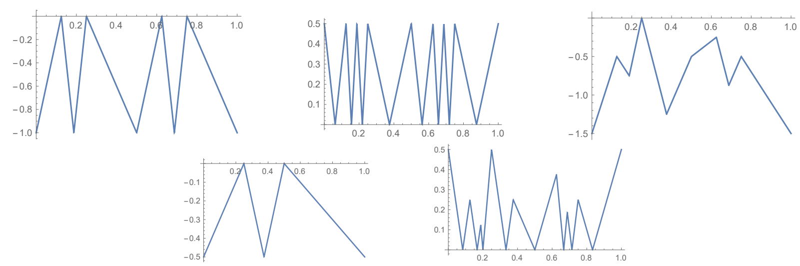

Figure 2.2: On the top: from left to right the graph of $A = f_0$, the graph of $x \to |(f_0(x) - 0) + 0.5|$, the graph of $f_1 = G(f_0)$. On the bottom: from left to right the graph of $g_1 = G(0) = G(g_0)$ and the graph of $x \to |f_1(x) - g_1(x) + 0.5|$. Therefore, G is not a strong contraction because $|f_0 - g_0| = 1/2 = |f_1 - g_1| = |G(f_0) - G(g_0)|$.

Chapter 3

Extracting explicit solutions from the approximations

In this chapter, we introduce heuristical procedures to obtain explicit solutions to the subaction equation

$$\max_{T(y)=x}[A(y)+V(y)] = V(x)+m(A)\,. \tag{3.1}$$

We will be relying on an important fact from [2] : given A and V, if for some constant c

$$V(x) = \max_{T(y)=x}[A(y)+V(y)-c], \tag{3.2}$$

then V is a calibrated subaction and $c = m(A)$. In the figures we will be presenting our numerically approximated subaction is denoted by $\tilde{V}$ in each example (in order in order not to confuse with the exact one V).

3.1 The case $A(x) = -(x-\frac{1}{3})^2$

Consider the potential $A(x) = -(x-\frac{1}{3})^2$. We will present the explicit expression for V in this case (which was not known before). Later we compare the explicit expression with the graph we get via the 1/2-procedure.

We consider in this subsection that $T(x) = 2\,x \pmod 1$ acts on $[0,1]$. Consider also the inverse branches of T given by $\tau_1(x) = \frac{x}{2}$ and $\tau_2(x) = \frac{x+1}{2}$. For the reader familiar with the subject, it is known from [17] that the maximizing probability in this case is Sturmian. Notice that we are in fact computing V such that

$$V(x) = \max_{i=1,2}[A \circ \tau_i(x) + V \circ \tau_i(x) - m(A)] \tag{3.3}$$

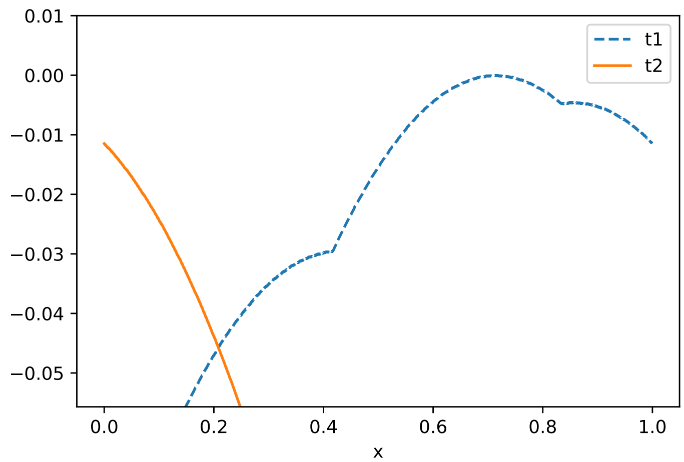

Figure 3.1: Case $A(x) = -(x - \frac{1}{3})^2$ - The blue (dashed) graph describes the values of the approximation of the calibrated subaction V where the 1/2 iterative procedure shows that the maximizing branch was τ_2, by this we mean that Equation 3.3 is satisfied by τ_2 in the blue part of the graph. The orange (straight) graph describes the values of the approximation of V where the 1/2 iterative procedure shows that the maximizing branch was τ_1. The graph for the approximation of V is the supremum of the two curves. We iterate 15 times G obtain this graph.

By looking at Figure 3.1 which we get from the 1/2 iterative procedure, it is natural to assume the existence of V_1, V_2, V_3, V_4, such that

$$V_1(x) + m(A) = V_3 \circ \tau_2(x) + A \circ \tau_2(x),\ V_2(x) + m(A) = V_1 \circ \tau_1(x) + A \circ \tau_1(x),$$

$$V_3(x) + m(A) = V_2 \circ \tau_1(x) + A \circ \tau_1(x)\ ,\ V_4(x) + m(A) = V_3 \circ \tau_1(x) + A \circ \tau_1(x). \tag{3.4}$$

The idea is, for each of the V_i observe which τ_j seems to yield the maximum in the expression $V_j(x) = \max_{i \in \{1,2\}}[A \circ \tau_i(x) + V \circ \tau_i(x) - m(A)]$.

As A is a polynomial of degree two is natural to try to express V on the form $V(x) = \sup\{ V_i(x), i = 1,2,3,4 \} = \sup\{ a_i + b_i x + c_i x^2,\ i = 1,2,3,4 \}$ for some choices of $a_i, b_i, c_i,\ i = 1,2,3,4$. Assuming each $V_i(x) = a_i + b_i x + c_i x^2$ we can convert the four equations (3.4) in a linear system that can be easily solved. From this procedure, we get $m(A) = -2/63$.

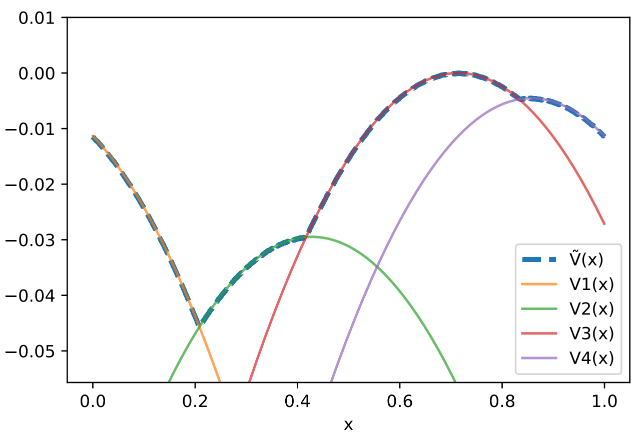

Figure 3.2: Case $A(x) = -(x - \frac{1}{3})^2$ - In blue (dashed) we present the graph of the approximation of the calibrated subaction V via the 1/2 iterative procedure. The picture also show the graphs of the different V_j, $j = 1,2,3,4$.

The function V_1 is a continuation of V_4 when we look these functions V_j as defined on S^1 (periodic). Let us recall that $m(A) := \sup_{\rho\ T\text{-invariant probability}} \int A\, d\rho$

Equation (3.4) suggests that the maximizing probability has support on an orbit of period three (Note that iterating T on $1/7$ yields period 3). Note that $\frac{A(1/7)+A(2/7)+A(4/7)}{3} = -2/63$. This means that the maximizing probability is given by $\mu = \frac{1}{3}(\delta_{1/7} + \delta_{2/7} + \delta_{4/7})$, where δ_x is the Dirac delta at x.

Moreover, we obtain $V_1(x) = \frac{10}{63} - \frac{2x}{21} - \frac{x^2}{3}, V_2(x) = \frac{5}{63} + \frac{2x}{7} - \frac{x^2}{3}, V_3(x) = \frac{10x}{21} - \frac{x^2}{3}$, and $V_4(x) = -\frac{5}{63} + \frac{4x}{7} - \frac{x^2}{3}$. A tedious computation confirms that the V we obtained from $V(x) = \sup\{V_1(x), V_2(x), V_3(x), V_4(x)\}$, is really the calibrated subaction (with maximum value zero) for such A (this can be done by directly checking that V satisfies Eq. 3.1). In Figure 3.2 we compare the graph of the approximated calibrated subaction obtained from the 1/2 iterative procedure (in red) and the exact analytic expression for V

we obtained above (in blue). We have a perfect match. With 15 iterations of the 1/2 iterative procedure, we get a good approximation of V (which was analytically obtained above).

3.2 A procedure to obtain piecewise analytic expressions

In some examples we have to proceed in a different way from the previous one. The general idea is that from observing the numerical approximation of V, we detect it has the form $V(x) = \max\{V_1(x), ..., V_r(x)\}$ where each V_i is analytical. We then determine which τ_i maximizes which V_j, obtaining a system of equations. We then can perform substitutions within this system to obtain the explicit form of one of the V_j. We will look for a way to express such initial V_j via the relation

$$V_j(x) - V_j(\eta(x)) = F(x) - K, \tag{3.5}$$

where F and η are known functions and $K = N\, m(A)$, where N is the period of the maximizing orbit, $j = 1, 2, ..., N$. The function F will be chosen according to convenience in each example. The value K is a fixed variable on the process of trying to find the calibrated subaction. We use the notation $\hat{m}(A) = \frac{K}{N}$ to express the fact that we do not know beforehand the exact value $m(A)$ and in the end we will show that $m(A) = \hat{m}(A)$. We assume $\eta : [0, 1] \to [0, 1]$ is such that

$$\eta^n := \underbrace{\eta \circ \eta \circ ... \circ \eta}_{n-times}$$

satisfies $\lim\limits_{n\to\infty} \eta^n(x) = q$ for some fixed point $q \in [0, 1]$. This indeed will happen in some of the examples we will consider. Note that (3.5) implies

$$V_j \circ \eta(x) - V_j \circ \eta^2(x) = F \circ \eta(x) - K. \tag{3.6}$$

If q is fixed by η we get $F(q) = K$. Therefore, adding (3.5) and (3.6) we get

$$V_j(x) - V_j \circ \eta^2(x) = F(x) + F \circ \eta(x) - 2K. \tag{3.7}$$

We can go on and inductively, obtaining for each n in $\mathbb{N}$,

$$V_j(x) - V_j \circ \eta^n(x) = \sum_{i=0}^{n-1} \left[F \circ \eta^i(x) - \ K\right]. \tag{3.8}$$

If V_j is continuous we get $\lim_{n\to\infty} V_j \circ \eta^n(x) = V_j(q)$. Using the notation $\eta^0(x) = x$ we obtain finally a series (which should be the expression of this V_j we are looking for)

$$V_j(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left[F \circ \eta^i(x) - K\right] - V_j(q). \tag{3.9}$$

We can consider the truncated approximation

$$V_j^{n*}(x) = \sum_{i=0}^{n-1} [F \circ \eta^i(x) - K] - V_j(q). \tag{3.10}$$

Since calibrated subactions added by a constant value are still calibrated subactions, we can assume that $V(q) = 0$. In this way the initial V_j should be given by

$$V_j(x) = \lim_{n\to\infty} V_j^{n*}(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} (F \circ \eta^i(x) - n\,K) = \sum_{i=0}^{\infty} (F \circ \eta^i(x) - K), \tag{3.11}$$

Then, by knowing the expression for a single V_j we are able to obtain the appropriate expression for the other branches V_i. All of this is dependent of smart choices for F and η. In each example we have to show that the above limits V_j, $j = 1, 2, ..., r$, indeed exist. Moreover, we have to show that

$$V(x) = \sup\{V_1(x), V_2(x), V_3(x), ..., V_r(x)\}, \tag{3.12}$$

solves the the subaction equation for A. When F is analytic (if A is analytic this will be the case in most of our examples) the expression (3.11) will provide an analytic expression for V_j, $j = 1, 2..., r$. In this case V will be piecewise analytic. More than that, in most of the cases, there is an analytic dependence of F on the analytic potential A (see Remark 3.2.1). Under appropriate conditions (on absolutely convergence, etc.) this will provide an analytic dependence of the calibrated subaction $V(x)$ for A, in each point x, on the potential A. In the computational procedure to be followed for getting such V_j one does not know in advance the value $m(A)$. When F has Lipschitz constant equal M we get the estimate $|F \circ \eta^i(x) - F(q)| \leq M|\eta^i(x) - q|$. In some of the examples we will get uniform convergence because $\sum_{i=0}^{+\infty} M|\eta^i(x) - q|$ is uniformly bounded. In this way the series defining V_j converges uniformly. We will follow the above reasoning in several examples to be presented next.

Remark 3.2.1. *We point out that if we were considering another potential A close to $\sin^2(2\,\pi\,x)$, then, the reasoning we are going to consider below would apply similarly. Note that F depends nicely on A. In this case (3.11) provides an analytical dependence of V on the nearby potential A.*

3.3 The case $A(x) = \sin^2(2\pi x)$

Consider the periodic function $A(x) = \sin^2(2\pi x)$, $T(x) = 2\,x \bmod(1)$, $\tau_1(x) = \frac{x}{2}$, $\tau_2(x) = \frac{x+1}{2}$. According to page 23 in [7] the maximizing probability μ has support on the periodic orbit of period 2 (the points 1/3 and 2/3). Therefore, we know beforehand that $m(A) = \frac{1}{2}(A(1/3) + A(2/3)) \approx 0.75$.

3.3.1 Overview

In the graphs presented in Figure 3.3 - which were obtained from the 1/2-procedure we call V_2 (blue color) the function we get when the maximizer is τ_2 and V_1 (orange color) the function we get when the maximizer is τ_1. The numerical result we get from the iterative procedure shows the evidence (see Figure 3.3) that the calibrated subaction V should satisfy

$$V(x) = \sup\{\, V_1(x), V_2(x)\,\}. \tag{3.13}$$

We will present an analytic expression for V_2. We will show that

$$V_2(x) = \sum_{i=0}^{+\infty}\left[\ \sin^2\left(\pi\left(\frac{2}{3} + \left(-\frac{1}{2}\right)^i (x - 2/3)\right)\right)\ -\ \sin^2(2\pi/3)\ \right].$$

and

$$V_1(x) = \sum_{i=0}^{+\infty}\left[\ \sin^2\left(\pi\left(\frac{2}{3} - \left(-\frac{1}{2}\right)^i (1/3 - x)\right)\right) - \sin^2(2\pi/3)\ \right].$$

This will finally produce from (3.13) the explicit expression for the subaction V for such A. It is instructive to explain step by step our reasoning. The procedure can be applied to other examples. By observing figure 3.4 we assume that

$$V_2(x) + \hat{m}(A) = V_1(\tau_1(x)) + A(\tau_1(x)), \tag{3.14}$$

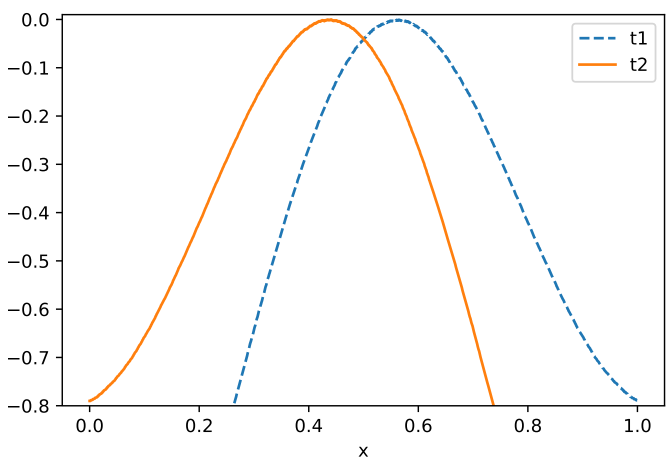

Figure 3.3: Case $\sin^2(2\pi x)$ - From the 1/2 iterative procedure taking $G^{20}(0)$ we get that the approximated subaction V is given by the supremum of the two functions in orange and in blue (dashed). The graph in blue describe the values where the calibrated subaction equation is maximized by τ_2. The graph in orange describes the values where the calibrated subaction equation is maximized by τ_1.

and, also

$$V_1(x) + \hat{m}(A) = V_2(\tau_2(x)) + A(\tau_2(x)). \tag{3.15}$$

Therefore composing (3.15) with $\tau_1(x)$ and substituting in (3.14)

$$V_2(x) - V_2\left(\frac{x}{4} + \frac{1}{2}\right) = A\left(\frac{x}{2}\right) + A\left(\frac{x}{4} + \frac{1}{2}\right) - 2\,\hat{m}(A). \tag{3.16}$$

Taking $\eta(x) = \frac{x}{4} + \frac{1}{2}$ and $K = 2\hat{m}(A)$, note that if $x \in [0,1]$, then $\lim_{n\to+\infty} \eta^n(x) = 2/3$. Define $F(x) = A(\frac{x}{2}) + A(\frac{x}{4} + \frac{1}{2})$, then, by (3.10) $\lim_{n\to+\infty} F(\eta^n(x)) = F(2/3) = A(1/3) + A(2/3) = 2\hat{m}(A)$. We point out that in the present case we already know from [7] that the above $\hat{m}(A) = m(A)$. Note that $F(x) = \sin^2(\pi x) + \sin^2(\pi x/2)$ is analytic. The 1/2 iterative procedure produces the numerical approximation $m(A) \approx 0.75$. We assume that $V(2/3) = 0$. Now

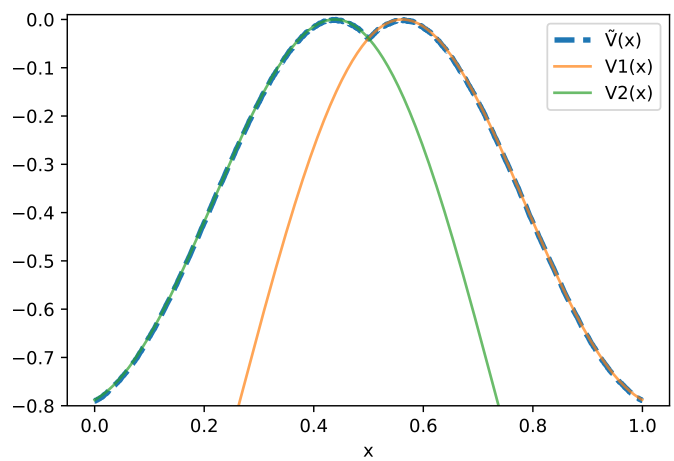

Figure 3.4: Case $\sin^2(2\pi x)$ - The graph in blue (dashed) shows the numerical approximation of the subaction V by $G^{30}(0)$ with a discretization of 2000 points of the form $\frac{n}{2000}$. In green and orange we show the graph of $V_1^{10*}(x)$ and $V_2^{10*}(x)$ (which approximate V_1 e V_2) according to (3.17).

we will express V_2 - using (3.11) - up to adding a constant via truncation

$$V_2^{n*}(x) = \sum_{i=0}^{n-1} [\, F \circ \eta^i(x) - 2m(A)\,]. \tag{3.17}$$

From (3.14), (3.15) and (3.16) we obtain $V_1(x) = V_2(1-x)$ (this is expected, since if $A(x) = A(1-x)$ then the same holds for the calibrated subaction, as shown in [11]). It will be shown in the next subsection that $V_1(x) = V_2(1-x)$ holds. We then define $V_1^{n*}(x) := V_2^{n*}(1-x)$. Figure 3.4 shows that for small values of n one can get a good approximation of the subaction via $V_2^{n*}(x)$, $x \in [0,1]$.

Proposition 3.3.1. *$\lim_{n\to+\infty} V_2^{n*}(x)$, $n \in \mathbb{N}$, given by (3.17), converges uniformly.*

Proof. We get $2\hat{m}(A) = \sin^2(2\pi/3) + \sin^2(\pi/3)$ and

$$|F\circ\eta^i(x) - 2\hat{m}(A)| = \left|(\sin^2(\eta^i(x)\pi) - \sin^2(2\pi/3)) + (\sin^2(\eta^i(x)\pi/2) - \sin^2(\pi/3))\right|$$
$$\leq |\sin^2(\eta^i(x)\pi) - \sin^2(2\pi/3)| + |\sin^2(\eta^i(x)\pi/2) - \sin^2(\pi/3)|.$$

Moreover,

$$\eta^i(x) = 2/3\left(1-\left(\frac{1}{4}\right)^i\right)+\frac{x}{4^i},$$

which means $\eta^i(x)-2/3=\frac{1}{4^i}\left(x-2/3\right)$. We have that $\sin^2$ is Lipschitz in $[0,1]$ for some constant K. Therefore, $|\sin^2(x)-\sin^2(y)|\le K|x-y|$. Then,

$$|\sin^2(\eta^i(x)\pi)-\sin^2(2\pi/3)|\le|\eta^i(x)\pi-2\pi/3|=\frac{K\pi}{4^i}|x-2/3|$$

and $|\sin^2(\eta^i(x)\pi/2)-\sin^2(\pi/3)|\le|\eta^i(x)\pi-\pi/3|\le\frac{K\pi}{2}\frac{1}{4^i}|x-2/3|$. From this

$$|\sum_{i=0}^{+\infty}(F\circ\eta^i(x)-2\hat{m}(A))|\le\sum_{i=0}^{+\infty}|F\circ\eta^i(x)-2\hat{m}(A)|$$

$$\le\sum_{i=0}^{+\infty}\left(\frac{K\pi}{4^i}|x-2/3|+\frac{K\pi}{2}\frac{1}{4^i}|x-2/3|\right)\le K\pi\sum_{i=0}^{+\infty}\frac{1}{4^i}<+\infty.$$

□

Denote $\delta(x)=1-x/2$. It is possible to get from the system (3.16) that $V_1(x)=V_2(1-x)$ and $V_2(x)+m(A)=V_1(x/2)+A(x/2)$ we then obtain $V_2(x)+m(A)=V_2(1-x/2)+A(x/2)$. As $m(A)=A(2/3)$ and $\lim_{n\to+\infty}\delta^n(x)=2/3$, for $x\in[0,1]$ we obtain

$$V_2(x)-V_2(\delta(x))=A(x/2)-A(2/3)\,.$$

From this we get $V_2(x)-V_2(2/3)=\sum_{i=0}^{+\infty}\left(A(\delta^i(x)/2)-A(2/3)\right)$. As $V_2(2/3)=0$, it follows that $V_2(x)=\sum_{i=0}^{+\infty}\left(A(\delta^i(x)/2)-A(2/3)\right)$. Finally, as $\delta^n(x+2/3)=\frac{2}{3}+\left(-\frac{1}{2}\right)^n x$, we obtain the expression

$$V_2(x)=\sum_{i=0}^{+\infty}\left[\sin^2\left(\pi\left(\frac{2}{3}+\left(-\frac{1}{2}\right)^i(x-2/3)\right)\right)-\sin^2(2\pi/3)\right]. \quad (3.18)$$

The corresponding expression for V_1 can be obtained from the equality $V_1(x)=V_2(1-x)$. We will show in the next subsection that $V(x)=\sup\{\,V_1(x),V_2(x)\,\}$ is a calibrated subaction for A. Moreover, we will present a power series expansion around 2/3 for V_2:

$$V_2(x)=\frac{\sin(4\pi/3)}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k(2\pi\left(x-\frac{2}{3}\right))^{2k+1}}{(2k+1)!}\frac{2^{2k+1}}{2^{2k+1}+1}-$$

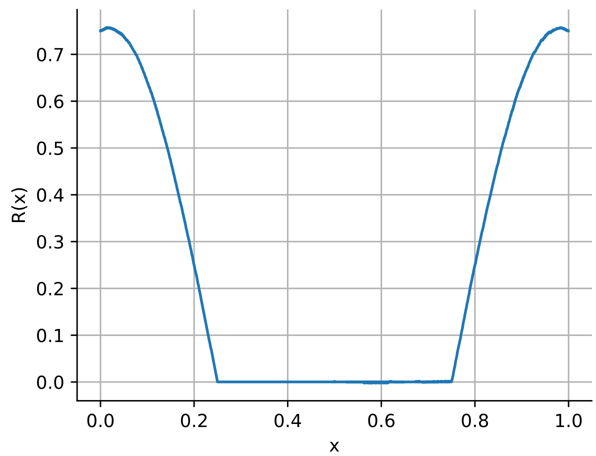

Figure 3.5: case $\sin^2(2\pi x)$ - The graph of R using the approximation of the calibrated subaction. The orbit of period 2 is inside the set $R = 0$.

$$\frac{\cos(4\pi/3)}{2}\sum_{k=1}^{+\infty}\frac{(-1)^k(2\pi\left(x-\frac{2}{3}\right))^{2k}}{(2k)!}\frac{2^{2k}}{2^{2k}-1}. \tag{3.19}$$

As $V_1(x) = V_2(1-x)$ a similar result can be derived for V_1 (which can be expressed in power series around $1/3$). In Figure 3.5 we plot the graph of R we get via the $1/2$ iterative procedure. In Figure 3.6 we compare the obtained power series for V_1 and V_2 with the numerical approximation of V.

3.3.2 Computing the subaction

We want to show that $V(x) = \sup\{V_1(x), V_2(x)\}$ is a calibrated subaction for A.

Lemma 3.3.2. *If* $V_2(x) = \lim_{n\to+\infty} V_2^{n*}(x)$, *then* $V_2(x) = \sum_{i=0}^{N}\left(F\circ\eta^i(x) - 2\hat{m}(A)\right) + \epsilon_N(x)$, *where* $|\epsilon_N(x)| \leq 2\pi\sum_{i=N}^{+\infty}\frac{1}{4^i} = \frac{2\pi}{3\cdot 4^{N-1}} \leq \frac{2}{3\cdot 4^{N-2}}$.

Proof. We just have to use that $\sin^2$ has Lipchitz constant 2 in (3.18). □

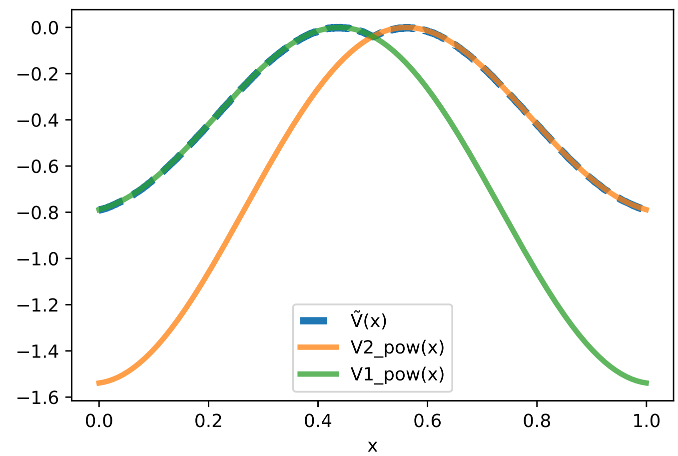

Figure 3.6: Case $\sin^2(2\pi x)$ - The graph in blue (dashed) shows the numerical approximation of the subaction V by $G^{30}(0)$ with a discretization of 2000 points of the form $\frac{n}{2000}$. In green and orange we plot the power series obtained for V_1 and V_2 truncated at the 10th term

We want to show that $V_2 := \lim V_2^{*n}$ indeed satisfies (3.16). It is important to show this because once we define V_2 as this limit, we cannot guarantee it still satisfies (3.16).

Lemma 3.3.3. *If* $V_2(x) = \lim\limits_{n\to+\infty} V_2^{n*}(x)$, *then*

$$V_2(x) = V_2(\eta(x)) + A\left(\frac{x}{2}\right) + A\left(\frac{x}{4} + \frac{1}{2}\right) - 2\,\hat{m}(A).$$

Proof. Denote $H(x) = A\left(\frac{x}{2}\right) + A\left(\frac{x}{4} + \frac{1}{2}\right) - 2\,\hat{m}(A)$. Then, $V_2(x) = \sum\limits_{i=0}^{+\infty} H(\eta^i(x))$ and $V_2(\eta(x)) = \sum_{i=1}^{+\infty} H(\eta^i(x))$. Therefore, $V_2(\eta(x)) = \sum_{i=0}^{+\infty}(H(\eta^i(x)) - H(x)$. From this follows $V_2(\eta(x)) = V_2(x) - H(x)$, and, finally $V_2(x) = V_2(\eta(x)) + A(\frac{x}{2}) + A(\frac{x}{4} + \frac{1}{2}) - 2\hat{m}(A)$. □

Next we want to construct the V_1 which satisfies (3.15) and (3.14).

Lemma 3.3.4. *If* $V_2(x) = \lim\limits_{n\to+\infty} V_2^{n*}(x)$ *and* $\hat{m}(A) = \frac{A(1/3)+A(2/3)}{2}$, *then the function* $V_1(x) = V_2((x+1)/2) + A((x+1)/2) - \hat{m}(A)$ *satisfies* $V_1(x/2) + A(x/2) = V_2(x) + \hat{m}(A)$.

Proof. From the relation between V_1 and V_2 we have $V_2((x+1)/2) + A((x+1)/2) = V_1(x) + \hat{m}(A)$. Taking composition with $\tau_1(x) = x/2$ we get

$$V_1(x/2) + A(x/2) = V_2(x/4 + 1/2) + A(x/4 + 1/2) + A(x/2) - \hat{m}(A)$$

$$= V_2(\eta(x)) + A(x/2) + A(x/4 + 1/2) - \hat{m}(A). \tag{3.20}$$

From Lemma 3.3.3 we obtain $V_2(\eta(x)) - V_2(x) = 2\hat{m}(A) - (A(x/2) + A(x/4 + 1/2))$, therefore, adding and subtrating $V_2(x)$ in (3.20) we have

$$V_1(x/2) + A(x/2) = V_2(\eta(x)) - V_2(x) + V_2(x) + A(x/2) + A(x/4 + 1/2) - \hat{m}(A)$$

$$= 2\hat{m}(A) - (A(x/2) + A(x/4 + 1/2)) + V_2(x) + A(x/2) + A(x/4 + 1/2) - \hat{m}(A).$$

Finally, $V_1(x/2) + A(x/2) = V_2(x) + \hat{m}(A)$. □

Now we can show the simmetry result for V_1.

Lemma 3.3.5. *Defining* $V_2(x) = \lim_{n\to+\infty} V_2^{n*}(x)$ *and* $V_1(x) := V_2((x+1)/2) + A((x+1)/2) - \hat{m}(A)$. *We have the simmetry* $V_1(x) = V_2(1-x)$.

Proof. By Lemma 3.3.4 $V_1(x) - V_1((x+1)/4) = A((x+1)/4) + A(x/2) - 2\hat{m}(A)$. Observe that

$$V_1(x) - V_1((x+1)/4) = \sin^2(\pi x) + \sin^2(\pi x/2 + \pi/2) - 2\hat{m}(A)$$
$$= \sin^2(\pi x) + \sin^2(\pi x/2 + \pi/2) - 2\hat{m}(A)$$
$$= \sin^2(\pi(1-x)) + \sin^2(\pi(1-x)/2) - 2\hat{m}(A) = F(1-x) - 2\hat{m}(A) \tag{3.21}$$

Defining $l(x) = (x+1)/4$, it can be seen that $1 - l(x) = \eta(1-x)$ and therefore $V_1(x) - V_1(l(x)) = F(1 - l(x)) - 2\hat{m}(A) = F(\eta(1-x)) - 2\hat{m}(A)$, so that for natural n, $V_1(x) - V_1 \circ l^n(x) = V_2^{n*}(1-x)$. From $l(1/3) = 1/3$, we obtain $\lim_{n\to\infty} l^n(x) = 1/3$ for $x \in [0,1]$, and so

$$V_1(x) - V_1(1/3) = \lim_{n\to+\infty} V_2^{*n}(1-x)$$

By Lemma 3.3.4 $V_1(1/3) = V_2(2/3) + \hat{m}(A) - A(1/3)$. But $\hat{m}(A) - A(1/3) = 0$ and $V_2(2/3) = 0$. Then $V_1(1/3) = 0$ and finally $V_1(x) = \lim_{n\to\infty} V_2^{*n}(1-x) = V_2(1-x)$. □

We need some differentiability results for V_1 e V_2.

Proposition 3.3.6. *$V_2(x)$ is differentiable in $[0,1]$ and $V_2'(x) = \sum_{i=0}^{+\infty} 2\pi(\eta^i)'(x)\left(\sin\left(\pi\eta^i(x)\right)\cos(\pi\eta^i(x)) + \frac{1}{2}\sin\left(\frac{\pi\eta^i(x)}{2}\right)\cos\left(\frac{\pi\eta^i(x)}{2}\right)\right)$.*

Differentiability follows from uniform convergence.

From this proposition we get $V_2'(x) = \sum_{i=0}^{+\infty} 2\pi\frac{1}{4^i}\left(\sin\left(\pi\eta^i(x)\right)\cos(\pi\eta^i(x)) + \frac{1}{2}\sin\left(\frac{\pi\eta^i(x)}{2}\right)\cos\left(\frac{\pi\eta^i(x)}{2}\right)\right)$.

Lemma 3.3.7. *$V_2'(x) = \varphi_N(x) + \xi_N(x)$, where $|\xi_N(x)| \leq 3\pi\sum_{i=N}^{+\infty}|\frac{1}{4^i}| = \frac{\pi}{4^{N-1}}$,*

$$\varphi_N(x) = \sum_{i=0}^{N} 2\pi\frac{1}{4^i}\left(\sin\left(\pi\eta^i(x)\right)\cos(\pi\eta^i(x)) + \frac{1}{2}\sin\left(\frac{\pi\eta^i(x)}{2}\right)\cos\left(\frac{\pi\eta^i(x)}{2}\right)\right)$$
$$= \sum_{i=0}^{N} 2\pi\frac{1}{4^i}\left(\frac{1}{2}\sin\left(2\pi\eta^i(x)\right) + \frac{1}{4}\sin\left(\pi\eta^i(x)\right)\right). \quad (3.22)$$

We leave the proof for the reader.

I_E denotes the indicator function of the interval E.

Theorem 3.3.8. *Taking $V_2(x) = \lim_{n\to+\infty} V_2^{n*}(x)$ and $V_1(x) = V_2((x+1)/2) + A((x+1)/2) - \hat{m}(A)$, we get that $V(x) = V_1(x)I_{[0,1/2)}(x) + V_2(x)I_{[1/2,1]}(x)$. is a calibrated subaction for A, when $\hat{m}(A) = \frac{A(1/3)+A(2/3)}{2} = m(A)$.*

Proof. We have to show that $\max_{T(y)=x}[A(y) + V(y)] = \max\{V_1(x/2) + A(x/2), V_2((x+1)/2) + A((x+1)/2)\}$. As $V_1(u/2) + A(u/2) = V_2(u) + \hat{m}(A)$, and, $V_1(x) = V_2(1-x)$, then, we have to show that

$$\max_{T(y)=x}[A(y) + V(y)] = \max\{V_2(x) + \hat{m}(A), V_2(1-x) + \hat{m}(A)\} \quad (3.23)$$

We will show first that if $u \in [0,1/2]$, then

$$V_2(u) + \hat{m}(A) \leq V_2(1-u) + \hat{m}(A) = V_1(u) + \hat{m}(A).$$

Denote $\gamma(u) = V_2(u) - V_2(1-u)$. By Lemma 3.3.7 we get

$$\gamma'(u) = V_2'(u) + V_2'(1-u) = \varphi_N(1-u) + \varphi_N(u) + (\xi_N(1-u) + \xi(u))$$

$$\geq \varphi_N(1-u) + \varphi_N(u) - 2\frac{\pi}{4^{N-1}}.$$

Taking $N = 4$ it is possible to see that if $u \in [0.1, 0.9]$ then $\gamma'(u) > 0$. The function γ is monotone increasing from 0.1 to 0.9 and $\gamma(1/2) = 0$. Then γ is negative on the interval $[0.1, 0.5]$. A similar argument can also handle the case $x \in [0, 0.1]$. We use Lemma 3.3.2, the fact that $\gamma(u) = V_2(u) - V_2(1-u)$ and the control of the error $|\epsilon_N(x)|$. Then, finally we get that γ is also negative in $[0, 0.1]$ and is positive for $x \in [0.9, 1]$. From the above we get $\max_{T(y)=u}[A(y) + V(y)] = V_2(1-u) + \hat{m}(A), \quad u \in [0, 1/2]$ and $\max_{T(y)=u}[A(y) + V(y)] = V_2(u) + \hat{m}(A), \quad u \in [0, 1/2]$. Therefore, for all $x \in [0, 1]$ we get $\max_{T(y)=x}[A(y) + V(y)] = V(x) + \hat{m}(A)$ Then, V is a calibrated subaction. □

3.3.3 Expressing V as piecewise power series

Now we will express V_2 in power series. Our final result will be given by expression (3.28). Using the property $\sin^2(x) = \frac{1-\cos(2\pi x)}{2}$, and trigonometric properties for the sum of angles we get from (3.18)

$$V_2(x + 2/3) = \frac{1}{2}\sum_{i=0}^{+\infty}\left[\sin\left(\frac{4\pi}{3}\right)\sin\left(2\pi\left(-\frac{1}{2}\right)^i x\right) - \cos\left(\frac{4\pi}{3}\right)\left[\cos\left(2\pi\left(-\frac{1}{2}\right)^i x\right) - 1\right]\right]. \quad (3.24)$$

Now, define

$$P(x) = \frac{\sin(4\pi/3)}{2}\sum_{i=0}^{+\infty}(\,\sin(2\pi(-1/2)^i x)\; - \,\sin(0)\,)$$

and

$$Q(x) = \frac{-\cos(4\pi/3)}{2}\sum_{i=0}^{+\infty}(\,\cos(2\pi(-1/2)^i\, x\; - \,\cos(0)\,).$$

We will express later V_2 as $V_2(x) = Q(x - 2/3) + P(x - 2/3)$.

Lemma 3.3.9. *P and Q are uniformly convergent in each interval $[-a, a]$.*

Proof. As the function sin is Lipschitz, then, there is a constant C, such that,

$$|\sin(x) - \sin(y)| \leq C|x-y| \leq 2aC,$$

and $\sum_{i=0}^{+\infty} \left|\sin\left(2\pi\left(-\frac{1}{2}\right)^i x\right)\right| \leq \sum_{i=0}^{+\infty} 2\,a\,C\,\left|2\pi\left(-\frac{1}{2}\right)^i\right| \leq +\infty$. For Q we use an analogous argument. ☐

As $\cos(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ one can write Q as

$$Q(x) = \frac{-\cos(4\pi/3)}{2} \sum_{k=1}^{+\infty} \sum_{i=0}^{+\infty} \left(\frac{(-1)^k (2\pi x)^{2k}}{2^{2ik}(2k)!}\right). \tag{3.25}$$

Finally, we get $Q(x) = \frac{-\cos(4\pi/3)}{2} \sum_{k=1}^{+\infty} \frac{(-1)^k (2\pi x)^{2k}}{(2k)!} \frac{2^{2k}}{2^{2k}-1}$ if we exchange the order of summation in (3.25). Proceeding in analogous way we get $P(x) = \frac{\sin(4\pi/3)}{2} \sum_{k=0}^{+\infty} \frac{(-1)^k (2\pi x)^{2k+1}}{(2k+1)!} \frac{2^{2k+1}}{2^{2k+1}+1}$. We need, however, to guarantee we can change the summation order. To show this we will use

Theorem 3.3.10. *Let $f(i,k)$ be a double sequence. Assume that $\sum_{k=1}^{+\infty} |f(i,k)|$ converges for each fixed i and that $\sum_{i=1}^{+\infty}\sum_{k=1}^{+\infty} |f(i,k)|$ converges. Then*

$$\sum_{k=1}^{+\infty}\sum_{i=1}^{+\infty} f(i,k) = \sum_{i=1}^{+\infty}\sum_{k=1}^{+\infty} f(i,k)$$

A proof of this theorem can be found in [1] in page 373.

Proposition 3.3.11. *For a fixed $0 < \varepsilon < 1$, if $x \in [-1+\varepsilon, 1-\varepsilon]$, we can exchange the order in the sum of (3.25) to obtain*

$$Q(x) = \frac{-\cos(4\pi/3)}{2} \sum_{k=1}^{+\infty} \frac{(-1)^k (2\pi x)^{2k}}{(2k)!} \frac{2^{2k}}{2^{2k}-1}.$$

Proof. Note that if $|x| < 1$ there exists a constant M (the coefficients on the power series of cos are decreasing) such that

$$\left|\sum_{k=1}^{+\infty} \frac{(-1)^k (2\pi x)^{2k}}{2^{2ik}(2k)!}\right| \leq \sum_{k=1}^{+\infty} \left|\frac{(2\pi x)^{2k}}{2^{2ik}(2k)!}\right| \leq \frac{1}{2^i} \sum_{k=1}^{+\infty} \left(M x^{2k}\right) =$$

$$\frac{M}{2^i}\left(\frac{x^2}{1-x^2}\right) \leq \frac{M}{2^i}\left(\frac{|1-\varepsilon|^2}{1-|1-\varepsilon|^2}\right).$$

We can exchange the order on the double sum: $\forall x \in [-1+\varepsilon, 1-\varepsilon]$,

$$\sum_{i=0}^{+\infty}\sum_{k=1}^{+\infty}\left|\frac{(-1)^k(2\pi x)^{2k}}{2^{2ik}(2k)!}\right| \leq \sum_{i=0}^{+\infty}\frac{M}{2^i}\left(\frac{x^2}{1-x^2}\right) \leq 2M\left(\frac{|1-\varepsilon|^2}{1-|1-\varepsilon|^2}\right) < +\infty.$$

Note that $(x-2/3) \in [-2/3, 1/3]$. Then,

$$Q(x-2/3) = \frac{-\cos(4\pi/3)}{2}\sum_{k=1}^{+\infty}\frac{(-1)^k(2\pi(x-2/3))^{2k}}{(2k)!}\frac{2^{2k}}{2^{2k}-1}. \tag{3.26}$$

In the same way we get

$$P(x-2/3) = \frac{\sin(4\pi/3)}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k(2\pi(x-2/3))^{2k+1}}{(2k+1)!}\frac{2^{2k+1}}{2^{2k+1}+1}. \tag{3.27}$$

□

As $V_2(x+2/3) = P(x) + Q(x)$, then, $V_2(x) = Q(x-2/3) + P(x-2/3)$. Finally, from (3.26) and (3.27) the power series expression of V_2 around $2/3$ is given by

$$\begin{aligned} V_2(x) = \frac{\sin(4\pi/3)}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k(2\pi\left(x-\frac{2}{3}\right))^{2k+1}}{(2k+1)!}\frac{2^{2k+1}}{2^{2k+1}+1} \\ -\frac{\cos(4\pi/3)}{2}\sum_{k=1}^{+\infty}\frac{(-1)^k(2\pi\left(x-\frac{2}{3}\right))^{2k}}{(2k)!}\frac{2^{2k}}{2^{2k}-1} \end{aligned} \tag{3.28}$$

We can express the power series of V_1 around $1/3$ from $V_1(x) = V_2(1-x)$.

3.4 The case $A(x) = \sin(2\pi x)$

Now we consider the potential $A(x) = \sin(2\pi x)$ and that $T(x) = 2\,x$ (mod 1) acts on $[0,1]$. Consider also the inverse branches of T given by $\tau_1(x) = \frac{x}{2}$ and $\tau_2(x) = \frac{x+1}{2}$. In page 23 in [7] the authors conjectured that in this case the maximizing probability has support on the periodic orbit of period 4 given by $\{1/15, 2/15, 4/15, 8/15\}$. The graph for the subaction V we obtain from the 1/2 iterative procedure for such A is presented in Figure 3.7. Note that in the present case we do not know the value $m(A)$ beforhand. In

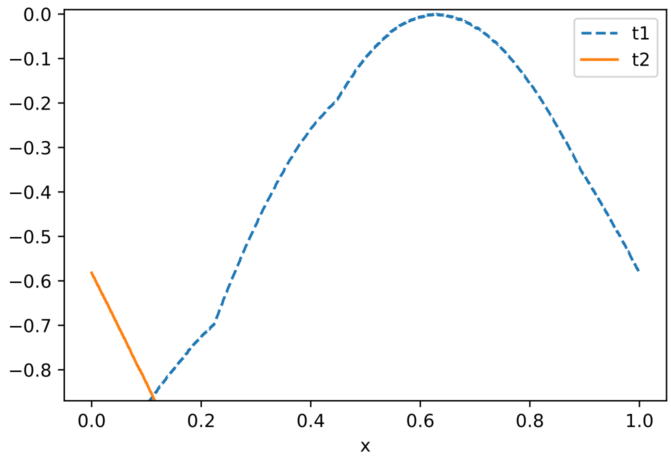

Figure 3.7: Case $\sin(2\pi x)$ - In blue we show the graph of the subaction we get from the 1/2 iterative procedure when the calibrated subaction equation is maximized by the branch τ_1. In orange when it is maximized by the branch τ_2. The graph of the approximation of the calibrated subaction V is the supremum of the blue and orange graphs.

[7] the authors conjectured that $m(A) = \frac{A(1/15)+A(2/15)+A(4/15)+A(8/15)}{4}$. It is possible to show that the conjecture is true. In order to do the computations we consider the $[0,1]$ point of view. From the graph we obtained via the 1/2-procedure it is natural to try to obtain V via the expression $V(x) = \sup\{V_1(x), V_2(x), V_3(x), V_4(x), V_5(x)\}$. Examining the Figure 3.7 we propose the following relations

$$V_5(x) + \hat{m}(A) = V_4(\tau_1(x)) + A(\tau_1(x)), V_4(x) + \hat{m}(A) = V_3(\tau_1(x)) + A(\tau_1(x)),$$

$$V_3(x) + \hat{m}(A) = V_2(\tau_1(x)) + A(\tau_1(x)),\ V_2(x) + \hat{m}(A) = V_1(\tau_1(x) + A(\tau_1(x)),$$
$$\text{and}\ V_1(x) + \hat{m}(A) = V_4(\tau_2(x)) + A(\tau_2(x)).$$

The analysis of this case is similar to the previous one. We will just outline the proof. In order to simplify the analytic expressions on this section (that depends on adding constants) we will write an expression $V_j(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1}(F \circ \eta^i(x) - K) = \sum_{i=0}^{\infty}(F \circ \eta^i(x) - K)$.

With the explicit expression for V_1, from the system above we can also obtain the explicit expressions for V_2, V_3, V_4 and V_5. It will be shown that

$$V_1(x) = \sum_{m=0}^{+\infty} \sum_{j=0}^{3} \left[\sin\left(\frac{\pi}{2^{(j+4m)}} \left(\frac{2^{4(m+1)}-1}{2^4-1} + x \right) \right) - \sin\left(2\pi \frac{2^m}{15} \right) \right]. \tag{3.29}$$

Assuming that the above relations among the V_j are true we get

$$V_1(x) - V \circ \tau_1^3 \circ \tau_2(x) = A \circ \tau_1^3 \circ \tau_2(x) + A \circ \tau_1^2 \circ \tau_2(x) + A \circ \tau_1 \circ \tau_2(x) + A \circ \tau_2(x) - 4\hat{m}(A).$$

Now, we take $\eta(x) = \tau_1^3 \circ \tau_2(x)$, and $F(x) = A \circ \tau_1^3 \circ \tau_2(x) + A \circ \tau_1^2 \circ \tau_2(x) + A \circ \tau_1 \circ \tau_2(x) + A \circ \tau_2(x)$, with $K = 4\,\hat{m}(A)$. Then, we get $\eta(x) = \frac{x}{2^4} + \frac{1}{2^4}$.

Note that if $x \in [0,1]$, then $\lim_{n\to+\infty} \eta^n(x) = \dfrac{1}{15}$. In this way we get numerical evidence that $\hat{m}(A) = \lim_{n\to+\infty} \dfrac{F(\eta^n(x))}{4} = \dfrac{F(1/15)}{4} \approx 0.4841$. This is consistent with the value $m(A) = \frac{A(1/15)+A(2/15)+A(4/15)+A(8/15)}{4} \approx 0.4841$. Using the truncated expression we get $V_1^{n*}(x) = \sum_{i=0}^{n-1} [F(\eta^i(x)) - K]$. Applying the above reasoning in a recursive way we obtain an expression for

$$V_1(x) = \sum_{i=0}^{+\infty} \left[A\left(\frac{\eta^i(x)+1}{2^4} \right) + A\left(\frac{\eta^i(x)+1}{2^3} \right) + A\left(\frac{\eta^i(x)+1}{2^2} \right) + A\left(\frac{\eta^i(x)+1}{2} \right) - 4\hat{m}(A) \right]. \tag{3.30}$$

From this follows (3.29). The function V_2 can be obtained from V_1. The function V_3 from V_2 and so on. One can show that

$$V(x) = \sup\{V_1(x), V_2(x), V_3(x), V_4(x), V_5(x)\}$$

is a calibrated subaction for A and that $\hat{m}(A) = m(A)$.

3.5 Revisiting the case $A(x) = -(x - \frac{1}{3})^2$

Recall that we obtained the expressions for the subaction for A through a linear system, by simply guessing what should be the general expression for each V_j. For reference, we obtained the system of equations

$V_1(x) + m(A) = V_3 \circ \tau_2(x) + A \circ \tau_2(x)$, $V_2(x) + m(A) = V_1 \circ \tau_1(x) + A \circ \tau_1(x)$,

$$V_3(x)+m(A)=V_2\circ\tau_1(x)+A\circ\tau_1(x)\,,\ V_4(x)+m(A)=V_3\circ\tau_1(x)+A\circ\tau_1(x). \tag{3.31}$$

It is possible to obtain the same solution by using the procedure outlined in Section 3.2. We can proceed in the same way as in the last examples by chossing a function F and obtaining the power series for the case $A(x) = -(x-\frac{1}{3})^2$. Taking $F(x) = \frac{-21}{64}\,(x+1/9)^2 + 4/189$, and $\eta(x) = \tau_1\circ\tau_1\circ\tau_2(x)$, we will get

$\lim_{n\to+\infty} V_1^{n*}(x) = \frac{-21}{64}\sum_{i=0}^{+\infty}\left((\eta^i(x)+1/9)^2 - 256/3969\right)$.

One can show that $\eta^i(x+1/7) = \dfrac{1}{7}+\dfrac{x}{8^i}$. Therefore,

$$V_1(x+1/7) = \lim_{n\to+\infty} V_1^{n*}(x+1/7) = \frac{-21}{64}\sum_{i=0}^{+\infty}\left(\left(\frac{16}{63}+\frac{x}{8^i}\right)^2 - \frac{256}{3969}\right). \tag{3.32}$$

After simplification and canceling terms we get $V_1(x) = -\frac{x^2}{3} - \frac{2x}{21} + 1/49$, which shows the same form (up to an additive constant) of the V_1 we obtained before on section 3.1.

3.6 Estimation of the joint spectral radius

In the class of examples we consider in the present section, there is no map acting on $[0,1]$ but there are two naturally defined inverse branches (an iterated function system). The 1/2 iterative procedure will produce useful information.

Consider

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

with

$$\tau_1(x) = \frac{(a_1-b_1)x+b_1}{(a_1+c_1-d_1-b_1)x+b_1+d_1}$$

and

$$\tau_2(x) = \frac{(a_2-b_2)x+b_2}{(a_2+c_2-d_2-b_2)x+b_2+d_2}.$$

Take $I_1=\tau_1([0,1])$, $I_2=\tau_2([0,1])$ and define the potential

$$A(x) = \begin{cases} 1/2\ \left(\log|(\tau_1^{-1})'(x)| + \log(\det(A_1))\right), & x\in I_1, \\ 1/2\ \left(\log|(\tau_1^{-1})'(x)| + \log(\det(A_2))\right), & x\in I_2. \end{cases}$$

In [19] the authors explain how the joint spectral radius can be analyzed from the point of view of Ergodic Optimization. The special space of "invariant probabilities" to be considered on this case is described on Definition 7 of [19]. It follows from results on [19] that the value $e^{m(A)}$, obtained in a similar way as in classical Ergodic Optimization, is equal to the joint spectral radius $\rho(A_1, A_2)$ (under some conditions for A_1, A_2). In this section the main issue is to estimate $m(A)$. We will estimate the value $m(A)$ using the $1/2$ iterative procedure.

3.6.1 First example

Take

$$A_1 = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

In this case the inverse branches are $\tau_1(x) = \frac{x+1}{x+3}$ e $\tau_2(x) = \frac{2}{4-x}$.

The potential is given by

$$A(x) = \begin{cases} 1/2\,(\,\log(|\frac{2}{(x-1)^2}|) + \log(2)\,), & 1/3 \leq x \leq 1/2, \\ 1/2\,\,(\log(|\frac{2}{x^2}|) + \log(2)\,), & 1/2 \leq x \leq 2/3. \end{cases}$$

Observe that $I_1 = [1/3, 1/2]$ and $I_2 = [1/2, 2/3]$. Corollaries 13 and 14 of [19] describe the values of the joint spectral radius $\rho(A_1, t\,A_2)$, for some values of $t > 0$. Looking Figure 3.8 which was obtained from the $1/2$ iterative procedure (showing the possible realizers) we assume that we should take V_1, V_2 (with maximizers, respectively, τ_1 and τ_2) satisfying

$V_2(x) + \hat{m}(A) = V_1(\tau_1(x)) + A(\tau_1(x))$, $V_1(x) + \hat{m}(A) = V_2(\tau_2(x)) + A(\tau_2(x))$.

Finally, we get

$$V_2(x) - V_2 \circ \tau_2 \circ \tau_1(x) = A \circ \tau_2 \circ \tau_1(x) + A \circ \tau_1(x) - 2\hat{m}(A). \tag{3.33}$$

As $q = \frac{1}{2}(\sqrt{17} - 3)$ is the fixed point of $\tau_2 \circ \tau_1$ we obtain $\hat{m}(A) = \frac{A \circ \tau_2 \circ \tau_1(q) + A \circ \tau_1(q)}{2}$ $= \frac{A(\frac{1}{2}(\sqrt{17}-3)) + A(\frac{1}{2}(5-\sqrt{17}))}{2} = \frac{1}{4}\,(2\log(2) + \log(2/(q-1)^2) + \log(2/(q^2))) \approx 1.2702.$

The $1/2$ iterative procedure is able to estimate the joint spectral radius $\rho(A_1, A_2)$. After some computations we will show later that $m(A)$ satisfies $m(A) = \log\left(\frac{1}{2}(3 + \sqrt{17})\right)$, and taking $b = \frac{1}{2}(3 + \sqrt{17})$ we will finally get that $V(x) = \max\{\log(x + b), \log(1 - x + b)\}$ is a subaction. Now we will begin the computations for this case. Taking $F(x) = A \circ \tau_2 \circ \tau_1(x) + A \circ \tau_1(x)$ and

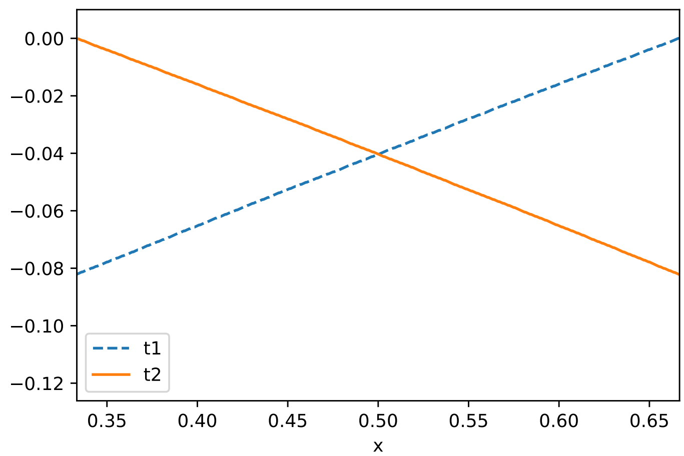

Figure 3.8: The graph in blue indicates where the maximizer is attained by τ_2 and in orange by τ_1.

$\eta(x) = \tau_2 \circ \tau_1(x)$, we get $V_2(x) = \lim_{n \to +\infty} \sum_{i=0}^{n} \left[F \circ \eta^i(x) - \hat{m}(A) \right]$. This means

$$V_2(x) = \log \prod_{i=0}^{+\infty} \frac{11 + 3\eta^i(x)}{11 + (\frac{3}{2}\left(\sqrt{17} - 3\right))}. \tag{3.34}$$

We note that from equation (3.33) we get $V_1(x) = V_2(1-x)$. One can also show that in this case the piecewise analytic expression for the calibrated subaction V can given by $V(x) = \max$ of

$$\left\{ \log \prod_{i=0}^{\infty} \left(\frac{11 + 3(\tau_2 \circ \tau_1)^i(x)}{11 + (\frac{3}{2}\left(\sqrt{17} - 3\right))} \right), \log \prod_{i=0}^{\infty} \left(\frac{11 + 3(\tau_2 \circ \tau_1)^i(1-x)}{11 + (\frac{3}{2}\left(\sqrt{17} - 3\right))} \right) \right\} \tag{3.35}$$

There is a quite strong simplification of all this. Indeed, we get that in this case the subaction V satisfies $V(x) = \max\{V_1(x), V_2(x)\}$, where $V_2(x) = \log\left(h(x)\right)$ for some function h. From the information we get from the 1/2 iterative procedure it seems that h is linear. Assuming that $V_2(x) = \log(x+b)$ we get the system $\log\left(\frac{(b+x)(11+3x)}{b(11+3x)+6+2x}\right) = \log\left((11+3x)e^{-2m(A)}\right)$. This means $e^{-2m(A)} = \frac{b+x}{6+11b+2x+3bx}$. As $m(A)$ satisfies $m(A) = \log\left(\frac{1}{2}(3+\sqrt{17})\right)$, taking derivative on x and using the condition to be equal to zero we get $6 + 11b + 2x + 3bx - (2 + 3b)(b + x) = 0$, that is $6 + 9b - 3b^2 = 0$. Finally, we get $b = \frac{1}{2}(3 + \sqrt{17})$. Note that $b = e^{m(A)}$, therefore we get the candidate for subaction $V(x) = \max\{\log(x + b), \log(1 - x + b)\} = \max\{V_2(x), V_1(x)\}$. It is not difficult to check that this in fact yields a subaction. The next example illustrates how to do such proof, for a more general case.

3.6.2 Second example

We consider a more general case. Given $t > 0$, denote

$$A_1 = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad tA_2 = \begin{pmatrix} 2\,t & 2\,t \\ 1\,t & 2\,t \end{pmatrix}.$$

In this case $\tau_1(x) = \frac{x+1}{x+3}$ and $\tau_2(x) = \frac{2}{4-x}$. As $t > 0$, then

$$A(x,t) = \begin{cases} (1/2)(\log(|\frac{2}{(x-1)^2}|) + \log(2)), & 1/3 \le x \le 1/2, \\ (1/2)(\log(|\frac{2}{x^2}|) + \log(2t^2)), & 1/2 \le x \le 2/3. \end{cases}$$

With different values of t we get different maximal values $m(A)$ and different subactions. Denote by $m(A,t)$ the function which gives the maximal value of $A(x,t)$ (where $e^{m(A,t)}$ is the joint spectral radius $\rho(A_1,\, t\,A_2)$), for each $t > 0$. We are not able to obtain in a rigorous manner the subaction for all cases of $t > 0$. However, we are able to show rigorously that there is an interval $0 \le t \le \frac{4(4+3\sqrt{2})}{18+13\sqrt{2}}$ where the maximal value is constant. Via the 1/2 iterative procedure we will be able to plot (a non rigorous estimation) the maximal value as a function of t (see figures 3.9 and 3.11). The main idea here is to try to take one of the V_i in the form $V_i(x) = \log(x + b)$ (or, $\log(b - x)$). To guess the total number r of V_i, $i = 1, 2..., r$, we use the graph we get from the 1/2 iterative procedure.

We will obtain explicitly that $m(A,t) = \log(2 + \sqrt{2})$, when $0 \le t \le \frac{4(4+3\sqrt{2})}{18+13\sqrt{2}}$. We will elaborate on that. For small values $t \sim 0$, the approximated value $m(A,t)$ indicates that $m(A,t) = \log(2 + \sqrt{2})$. Moreover, it suggests that in order to get the calibrated subaction V we should work with two V_i:

$$V_1(x,t) + \hat{m}(A,t) = A(\tau_2(x),t) + V_2(\tau_2(x),t), \tag{3.36}$$

$$V_2(x,t) + \hat{m}(A,t) = A(\tau_1(x),t) + V_2(\tau_1(x),t). \tag{3.37}$$

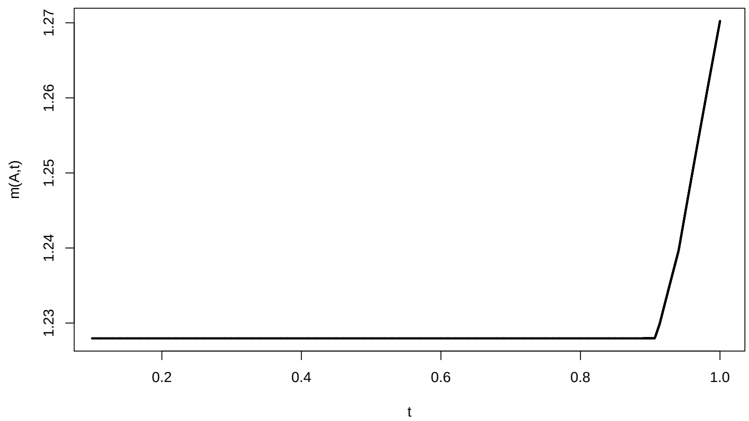

Figure 3.9: Numerical estimation of $m(A,t)$ obtained via the 1/2-procedure.

This system of equations is sugested by Figure 3.10. Then

$$V(x,t) = \max\{V_1(x,t), V_2(x,t)\}$$

is the candidate to be the subaction for $A(x,t)$. As $m(A,t)$ seems to be constant in an interval and $A(\tau_1(x),t) = \log(\frac{2}{1-\tau_1(x)})$ we conclude that V_2 should not depend on t. We assume $V_2(x,t) = \log(x+b)$ and then from last equation we get $b = 1+\sqrt{2}$ and finally $V_2(x,t) = \log(x+1+\sqrt{2})$. It is easy to confirm that $V_2(x,t) + \log(2+\sqrt{2}) = V_2(\tau_1(x),t) + A(\tau_1(x),t)$. Making a substitution in (3.36) we get $V_1(x,t) = \log\left(t(2+\sqrt{2}-\frac{x}{\sqrt{2}})\right)$. Clearly $\hat{m}(A,t) = \log(2+\sqrt{2})$ is a natural candidate to be $m(A,t)$. We will look for the largest interval $[0,t_1]$ such that the subaction V is given by

$$V(x,t) = \max[V_1(x,t), V_2(x,t)]\,. \tag{3.38}$$

We wish to find the largest t such that given $x \in [1/3, 2/3]$ and $i \in \{1,2\}$, for some $j \in \{1,2\}$

$$A(\tau_i(x),t) + V_1(\tau_i(x)) \leq V_j(x,t) + \hat{m}(A,t).$$

That is, the largest t such that

$$\max\left\{\log\left(t(3+x)\left(2+\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{3+x}\right)\right), \log\left(t^2(4-x)\left(2+\sqrt{2}-\frac{\sqrt{2}}{4-x}\right)\right)\right\} \leq$$

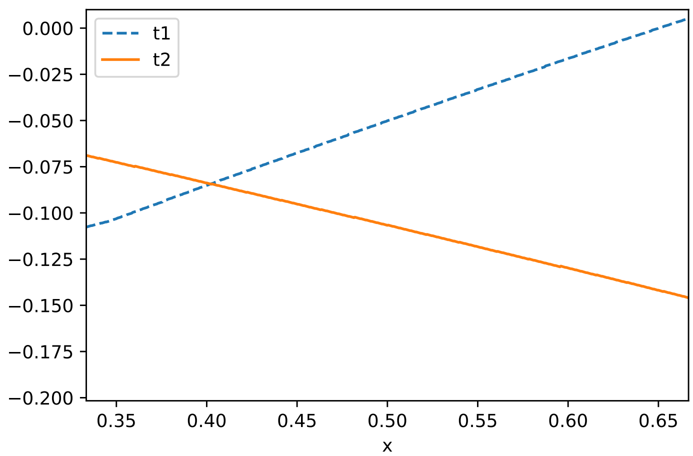

Figure 3.10: Numerical estimation of the subaction obtained via the 1/2-procedure for $t = 0.9$.

$$\leq \max\left\{\log\left(t(2+\sqrt{2}-\tfrac{x}{\sqrt{2}})(2+\sqrt{2})\right), \log\left((1+\sqrt{2}+x))(2+\sqrt{2})\right)\right\}.$$

If $x \in [1/3, \frac{\sqrt{2}}{2+\sqrt{2}}]$ then

$$t(3+x)\left(2+\tfrac{1}{\sqrt{2}}+\tfrac{\sqrt{2}}{3+x}\right) \leq \left(t(2+\sqrt{2}-\tfrac{x}{\sqrt{2}})(2+\sqrt{2})\right).$$

Therefore, in this interval $A(\tau_1(x),t) + V_1(\tau_1(x),t) \leq V(x,t) + \hat{m}(A,t)$. Now, consider $x \in [x(t), 2/3]$, where $x(t)$ is the point such that

$$t(3+x(t))\left(2+\tfrac{1}{\sqrt{2}}+\tfrac{\sqrt{2}}{3+x(t)}\right) = \left((1+\sqrt{2}+x(t)))(2+\sqrt{2})\right).$$ This means that if $x \in [x(t), 2/3]$, then, $A(\tau_1(x),t) + V_1(\tau_1(x),t) \leq V(x,t) + \hat{m}(A,t)$. From this follows that $x(t) \leq \frac{\sqrt{2}}{2+\sqrt{2}} = 0.414214...$, Then, for $x \in [1/3, 2/3]$ we get $A(\tau_1(x),t)+V_1(\tau_1(x),t) \leq V(x,t)+\hat{m}(A,t)$. This condition is satisfied for $t \leq \frac{4(4+3\sqrt{2})}{18+13\sqrt{2}} \approx 0.9061$. It is compatible with the information we get from the 1/2 iterative procedure. Now we will show that $A(\tau_2(x),t) + V_1(\tau_2(x),t) \leq V(x,t) + \hat{m}(A,t)$ for such values of t. Note that if $0 \leq t \leq \frac{(2+\sqrt{2})^2}{8+3\sqrt{2}} \approx 0.952$, then, $t^2(4-x)\left(2+\sqrt{2}-\frac{\sqrt{2}}{4-x}\right) \leq t(2+\sqrt{2}-\frac{x}{\sqrt{2}})(2+\sqrt{2})$. Therefore, $V(x,t)$ given by equation (3.38) is a calibrated subaction with $m(A,t) = \log(2+\sqrt{2})$ if $0 \leq t \leq \frac{4(4+3\sqrt{2})}{18+13\sqrt{2}}$. The final conclusion is that $m(A,t) = \log(2+\sqrt{2})$ for $t \in [0, t_1]$, where $t_1 := \frac{4(4+3\sqrt{2})}{18+13\sqrt{2}} \approx 0.9061$. In Figure 3.11 we show a detailed

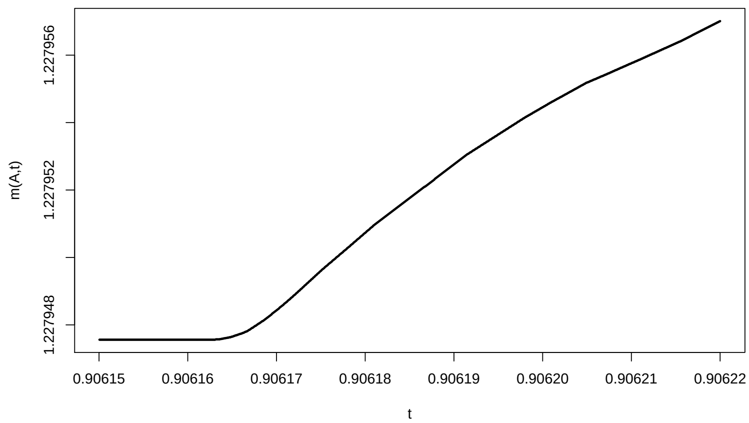

Figure 3.11: Graph of $m(A,t)$ for t around the point t_1 where $m(A,t)$ is not constant anymore.

estimation of the graph of $m(A,t)$ (via the 1/2 iterative procedure) for t close to t_1. It is also possible to determine other intervals contained in $[0,1]$ for t such that we can find explicitly the value $m(A,t)$. We could not, however, obtain joint intervals.

3.7 Minus distance to the Cantor set

Now, we consider the case where $A(x) = -d(x,K)$ where $d(x,K) = \min_{k\in K} |x-k|$ and $K \subset [0,1]$ is the Cantor set. Also, $T(x) = 2\,x \pmod 1$ acts on $[0,1]$ and the inverse branches of T are given by $\tau_1(x) = \frac{x}{2}$ and $\tau_2(x) = \frac{x+1}{2}$. In this section we present pictures we get from the use of the 1/2 iterative procedure and we present some conjectures. We do not provide mathematical proofs. We consider an approximation of the Cantor set via the mesh of points of the form $m = \frac{1}{2} + \sum_{i=1}^{+\infty} a_i \frac{1}{3^i}$ where $a_i \in \{1,-1\}$, and therefore we take $A(x) = -d(x,K) = -\min_{(a_i)\in\{1,-1\}^{\mathbb{N}}} \left|x - \left(\frac{1}{2} + \sum_{i=1}^{+\infty} a_i \frac{1}{3^i}\right)\right|$. It is easy to see that $m(A) = 0$, since $\max A(x) = 0$. Note that $\mu = \frac{1}{2}(\delta_{1/3} + \delta_{2/3})$, δ_0 and δ_1 are all maximizing probabilities. As A is symmetric there is a symmetric subaction. Consider the truncation $A_n(x) = -\min_{(a_i)\in\{1,-1\}^n} \left|x - \left(\frac{1}{2} + \sum_{i=1}^{n} a_i \frac{1}{3^i}\right)\right|$.

The points 0 and 1 are also in the Mather set. We will try to get a subaction via $V_1(x) - V_1(\tau_1(x)) = A(\tau_1(x))$ and $V_2(x) - V_2(\tau_2(x)) = A(\tau_2(x))$.

In this way we get $V_1(x) = \sum_{i=1}^{+\infty} A \circ \tau_1^i(x)$ and $V_2(x) = \sum_{i=1}^{+\infty} A \circ \tau_2^i(x) = V_1(1-x)$. We believe that $V(x) = V_1(x)I_{[0,1/2)} + V_1(1-x)I_{[1/2,1]}$ is a subaction

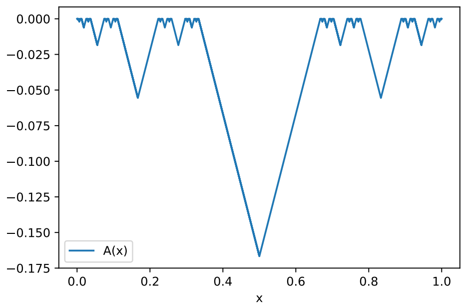

Figure 3.12: *Graph of the truncation* $A_{100}(x)$ *in a discretization of* 10^5 *points*

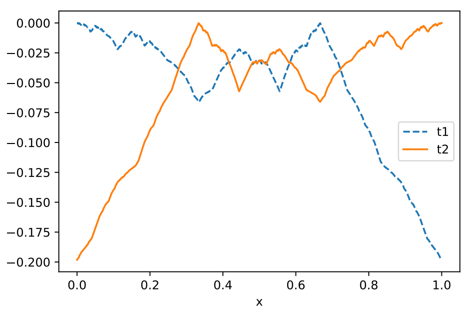

Figure 3.13: Picture obtained using the $1/2$ iterative procedure for $A_{50}(x)$ with a discretization of 10^4 points with 30 iterations. The blue (dashed) graph shows when the maximizer is obtained via τ_1 and the orange graph is for the case when the maximizer is τ_2.

Lemma 3.7.1. *The series* $G(x) = \sum_{i=1}^{+\infty} A(\tau_1^i(x))$ *converges uniformly in the interval* $[0, 1]$.

Proof. Notice that

$$G(x) = \sum_{j=1}^{+\infty} - \min_{(a_i)\in\{1,-1\}^{\mathbb{N}}} \left| x/2^j - \left(\frac{1}{2} + \sum_{i=1}^{+\infty} a_i \frac{1}{3^i} \right) \right|$$

and

$$\min_{(a_j)\in\{1,-1\}^{\mathbb{N}}} \left| x/2^j - \left(\frac{1}{2} + \sum_{i=1}^{+\infty} a_i \frac{1}{3^i} \right) \right| \leq \left| x/2^j \right|.$$

In this way $|G|$ is bounded by a geometric series and therefore we get the claim. □

Conjecture 3.7.2. *Suppose $A(x) = -d(x, K)$ and $T(x) = 2x$ mod(1), then, a subaction is given by $V(x) = G(x)I_{[0,1/2)}(x) + G(1-x)I_{[1/2,1]}(x)$, where $G(x) = \sum_{i=1}^{+\infty} A(\tau_1^i(x))$.*

This subaction is obtained through the two maximizing probabilities which have support in $\{0\}$ and $\{1\}$. Now, we want to try to find another subaction but this time associated to the maximizing probability with support on $\{1/3, 2/3\}$. In this way we will look for solutions of the form $V_2(x) = V_1(\tau_1(x)) + A(\tau_1(x))$ and $V_1(x) = V_2(\tau_2(x)) + A(\tau_2(x))$. As in the previous examples $\eta(x) = \tau_2(\tau_1(x))$ take $V_2(x) = \sum_{i=1}^{+\infty} \left(A(\tau_1(\eta^i(x))) + A(\tau_2(\tau_1(\eta^i(x))))\right)$, and $V_1(1-x) = V_2(x)$. As $\eta(2/3) = 2/3$ one can show that this series is absolutely convergent (similar to the previous Lemma 3.7.1). Define $H(x) = \sum_{i=1}^{+\infty} \left(A(\tau_1(\eta^i(x))) + A(\tau_2(\tau_1(\eta^i(x))))\right)$.

We want to show that $W(x) = H(x)I_{[0,1/2)}(x) + H(1-x)I_{[1/2,1]}(x)$ is a subaction. In the same way as before we want to show that

$\max_{T(y)=x}[A(y) + V(y)] = \max\{H(x), H(1-x)\} = W(x)$.

Conjecture 3.7.3. *The function W given by $W(x) = H(1-x)I_{[0,1/2)}(x) + H(x)I_{[1/2,1]}(x)$, $H(x) = \sum_{i=1}^{+\infty} \left(A(\tau_1(\eta^i(x))) + A(\tau_2(\tau_1(\eta^i(x))))\right)$, is a subaction for A.*

Above we conjectured that W and V were subactions. If this was true, then $\max\{W + C_1, V + C_2\}$ is also a subaction, where $C_1, C_2 \in \mathbb{R}$.

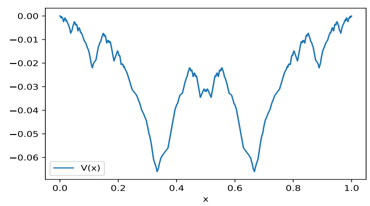

Figure 3.14: Truncation of the subaction V as described in Conjecture 3.7.2, where n=10.

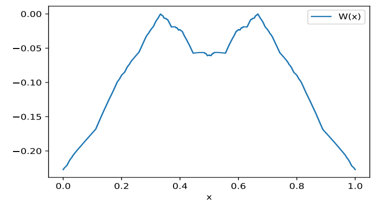

Figure 3.15: Truncation of the subaction W as described in Conjecture 3.7.3, where $n = 10$.

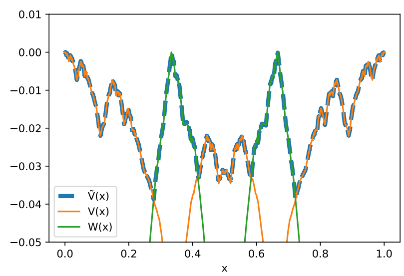

Figure 3.16: Superposition of the two above graphs resulting in a new subaction with the graph in blue (dashed). We can see that the iterative process was in fact computing this superposition.

3.8 A potential equal to its subaction

Set $T(x) = 2\,x \bmod(1)$, with inverse branches $\tau_1(x) = 1/2$, $\tau_2(x) = (x+1)/2$. We will now obtain a non-trivial potential which is equal to its subaction. We make assumptions on A. Set $A = u$ and choose u to be symmetrical

$$u(x) = \begin{cases} f(x), & x < 1/2 \\ f(1-x), & x \geq 1/2 \end{cases} \tag{3.39}$$

where

$$f(x) = \begin{cases} g_1(x), & x < 1/3 \\ g_2(x), & x \geq 1/3 \end{cases}$$

We want to have $\max\limits_{T(y)=x}[A(y)+V(y)] = \max\limits_{T(y)}[2u(y)]$ to be maximized in $[0,1/2]$, by τ_2, and in $[1/2,1]$ by τ_1. This yields the system

$$\begin{array}{c} g_1(x)+m(u)=2g_2(1-\tau_2(x)), \quad g_2(x)+m(u)=2g_1(1-\tau_2(x)) \\ g_1(1-x)+m(u)=2g_2(\tau_1(x)), \quad g_2(1-x)+m(u)=2g_1(\tau_1(x)) \end{array}$$

Two equations above are redundant. Make $\eta(x)=\frac{1+x}{4}$, the system of equations is reduced to

$$g_1(x)+m(u)=2g_2(1-\tau_2(x)), \quad g_2(x)+m(u)=2g_1(1-\tau_2(x))\,. \tag{3.40}$$

We then obtain

$$g_2(1-\tau_2(x))=\frac{g_1(x)+m(u)}{2}=2g_1(\eta(x))-m(u)\,.$$

And then $g_1(x)=4g_1(\eta(x))-3m(u)$ gives $m(u)=g_1(1/3)$. Now suppose that g_1 is differentiable, obtaining $g_1'(x)=4g_1'(\eta(x))\eta'(x)$, then

$$g_1'(x)-g_1'(\eta(x))=0$$

composition with η and substitution gives $g_1'(x)-g_1'(\eta^3(x))=0$. Continuing $g_1'(x)-\lim\limits_{k\to+\infty}g_1'(\eta^k(x))=0$. This means that

$$g_1'(x)=g_1'(1/3)\,.$$

Therefore g_1 and g_2 must be linear. By (3.40)

$$g_1(x)=\alpha\left(x-\frac{1}{3}\right)+\beta, \quad g_2(x)=\alpha\left(\frac{1}{3}-x\right)+\beta$$

With the restrictions

$$\begin{array}{c} g_1(\tau_1(x))\le g_2(1-\tau_2(x)), \quad x\in[0,1/3] \\ g_1(\tau_1(x))\le g_1(1-\tau_2(x)), \quad x\in[1/3,1/2] \end{array}$$

Which means with $\alpha>0$

$$\begin{array}{c} -\alpha/6\le 0, \quad x\in[0,1/3] \\ \alpha(x-1/2)\le 0 \quad x\in[1/3,1/2] \end{array}$$

And this always holds. We conclude that

$$\max_{T(y)=x}[2u(y)]=u(x)+\beta\,.$$

By symetry the same holds in $[1/2,1]$. Finally u is its own subaction with maximizing probability with support in $\{1/3,2/3\}$, $m(u)=\beta$. Figure 3.17 shows the general form of such potential u.

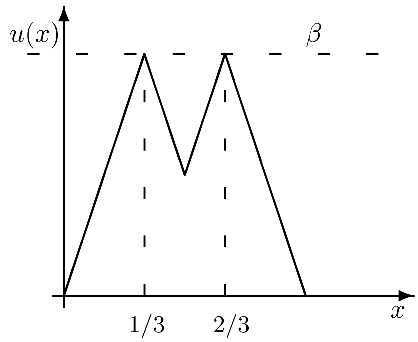

Figure 3.17: *The graph of u given by (3.39)*

3.9 The $1/2$ iterative procedure applied to the case where A has more than one maximizing probability.

The discussion that will be made in this section only addresses questions regarding numerical evidence obtained from the 1/2 iterative procedure. We do not present rigorous proofs in this section. The interest in this section is to understand better the dynamics of the 1/2 iterative procedure on the case there is more than one maximizing probability. In some sense, there are basins of attraction depending on where one begins the iteration of the 1/2 iterative procedure.

Consider the potential $A(x) = -x^2(x-1/3)^2(x-2/3)^2(x-1)^2$ which has maximal value $m(A) = 0$. The ergodic maximizing probabilities are $\mu_1 = \delta_0$, $\mu_2 = \delta_1$ and $\mu_3 = \frac{1}{2}(\delta_{1/3} + \delta_{2/3})$. In this case there exist more than one calibrated subaction (see Theorems 12 and 15 in [11] or Theorem 5 in [13]). One can obtain numerical evidence of the graph of these different calibrated subactions by considering the iteration of G on distinct initial conditions. Taking the initial condition $f_0 = 0$ and iterating G we get the function V which has the graph shown on Figure 3.18. This function $V(x) := G^{30}(0)(x)$ "should be" a calibrated subaction. The graph of the associated function R is displayed on Figure 3.19. Suppose we did not know the maximizing ergodic probabilities. From Figure 3.19 we have numerical evidence that the values of R on the periodic orbits $\{0\}$, $\{1\}$ and $\{1/3, 2/3\}$ are equal to zero (or, ~ 0).

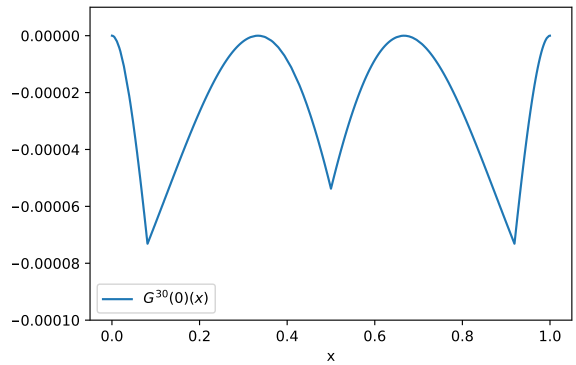

Figure 3.18: The approximated subaction obtained from the initial condition $f_0(x) = 0$.

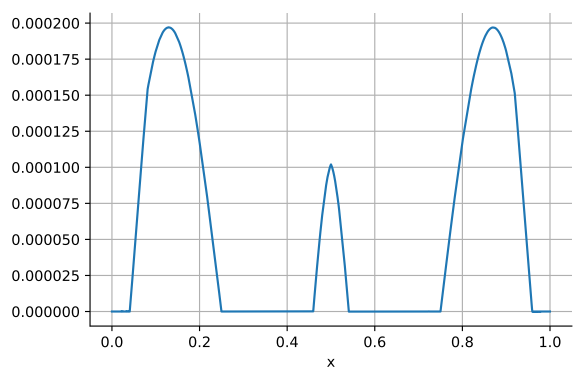

Figure 3.19: The approximated R function obtained from the initial condition $f_0(x) = 0$.

The general idea is: even in the case the maximizing probability is not unique we get numerical evidence about the possible maximizing probabilities. Another initial condition f_0 can be attracted to another calibrated subaction V by iteration of G. Indeed, let $\alpha_{\varepsilon,a} : [0,1] \to \mathbb{R}$ be a piecewise linear bump function defined by

$$\alpha_{\varepsilon,a}(x) = \begin{cases} 0, & 0 \leq x \leq a-\varepsilon \\ -x+(a-\varepsilon), & a-\varepsilon \leq x \leq a \\ x-(a+\varepsilon), & a \leq x \leq a+\varepsilon \\ 0, & a+\varepsilon \leq x \leq 1 \end{cases}$$

where $a \in (0,1)$ and $\varepsilon > 0$ is arbitrary small. We consider two different initial conditions:
a) $A(x) = -x^2(x-1/3)^2(x-2/3)^2(x-1)^2$ and $f_0(x) = \alpha_{0.01,1/5}(x)$: In this case there is a numerical evidence that the high iterates $G^n(f_0)$ converge to the blue (dashed) graph described by Figure 3.20.
b)$A(x) = -x^2(x-1/3)^2(x-2/3)^2(x-1)^2$ and $f_0(x) = \alpha_{0.01,2/3}(x)$: In this case, there is numerical evidence that the high iterates of $G^n(f_0)$ converge to the orange graph described by Figure 3.20. In these two last cases, the graph of the corresponding R (see Figure 3.21) also confirms the numerical evidence that such functions V are calibrated subactions. An interesting future work is to analyze the basin of attraction of each subaction by the iteration G^n.

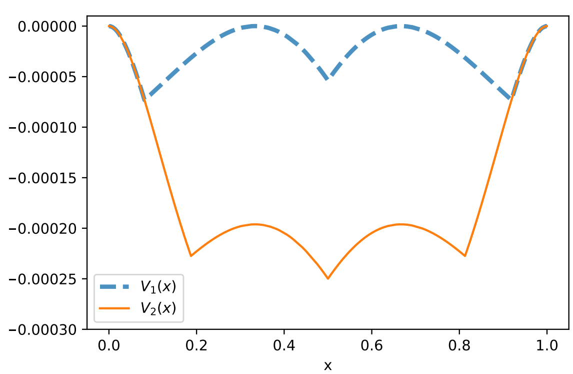

Figure 3.20: The approximated subactions V_1 with initial condition $\alpha_{0.01,1/5}$ and V_2 with initial condition $\alpha_{0.01,2/3}$

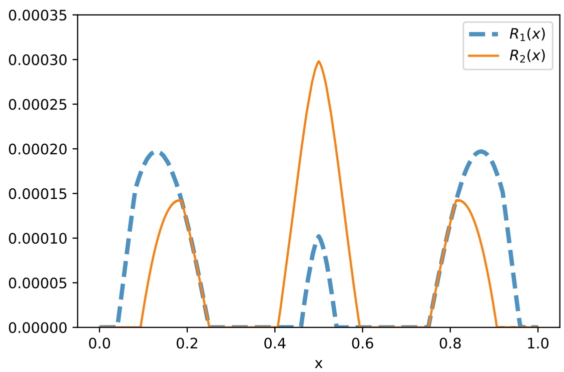

Figure 3.21: The approximated R functions for V_1 and V_2 as in Figure 3.20

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