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A comparison of Range Value at Risk forecasting models

Porto Alegre, 2021.

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Este trabalho é dedicado ao meu pai, Luís Carlos Gössling, que nunca duvidou de mim.

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Abstract

Risk forecasting is an important and helpful process for investors, fund managers, traders, and market makers. Choosing an inappropriate risk forecasting model can trigger irreversible losses. In this context, this study aims to evaluate the quality of different models to forecast the Range Value at Risk (RVaR), in both univariate and multivariate analysis, and compare the forecasts to other important risk measures like Value at Risk (VaR) and Expected Shortfall (ES). To assess the performance of both the univariate and multivariate models to RVaR forecasting, we consider an empirical exercise with different asset classes, rolling window estimations, and significance levels. We also evaluate prediction accuracy using Monte Carlo simulations in the univariate analysis, considering different scenarios. We evaluate the empirical forecasts with the score functions of each risk measure. We identified that different models could forecast better different assets, and the GARCH model with Johnson's SU distribution overcoming the other distributions. We observed the RVine and CVine copulas as better models in the multivariate study. Besides that, we noted that the models with Student's t marginal distribution have better performance according to realized loss (score function). We identified that even if a model can forecast RVaR well, that does not imply that the same model will forecast other risk measures well.

Keywords: Risk Forecasting; Risk Measures; Range Value at Risk (RVaR); Elicitability; Monte Carlo simulations.

Contents

1 Introduction

The 1987 stock market crash contributed to a greater interest in risk forecasting tools that considered larger and less likely losses (tail risks). In 1996, the Basel Committee on Banking Supervision published changes regarding the computation of minimum capital requirements for financial institutions (TIAN et al., 2019). From that moment, Value at Risk (VaR) was adopted as the standard risk measure to be calculated by all banks, for each row of their balance sheets. This measure can be understood as the maximum loss for a given period and confidence interval. See (DUFFIE; PAN, 1997) and (JORION, 2000).

Nowadays, the VaR stands as one of the most popular risk measures in the financial industry and risk forecasting literature (see, for example, Kuester et al. (2006)). This happens, as explained in the works Yamai et al. (2002) and Yamai e Yoshiba (2005), primarily due to its conceptual simplicity, computational ease, and ready applicability. However, despite its popularity, it has several problems when used as a market risk measure. Two of its weaknesses are (i) the fact that VaR is not subadditive and hence not a coherent risk measure (ARTZNER et al., 1999)[1] and (ii) VaR is insensitive to potential losses that exceed the quantile of interest. Many alternatives have been proposed in the literature, aiming to overcome the theoretical deficiencies of the VaR. The most prominent one is the Expected Shortfall (ES), which has the advantage of being coherent. It was proposed by Acerbi e Tasche (2002). As stated by Föllmer e Schied (2011) and Föllmer e Weber (2015), ES is the expected loss once this loss overcomes the VaR. Different authors present similar measures to ES, which were named as Tail Conditional Expectation (TailVaR) and Conditional Value at Risk (CVaR), for instance.

For a long period, ES being a coherent risk measure induce a general belief that it is more useful, as a risk measure, than VaR. However, these risk measures are essentially theoretical concepts and, as stated by Righi e Ceretta (2015), an estimation approach is necessary to compute their values, and reliable forecasts become needed. For a comparison of alternative models to forecast VaR and ES, see, for instance, Basu (2011), Zhou e Anderson (2012), Degiannakis et al. (2013), Müller e Righi (2018), and Trucíos et al. (2019). In this context, another theoretical concept that started to be explored is that of elicitability.[2] For short, functionals allowing verification and comparison of competing risk forecasting models are called elicitable. Importantly, VaR is elicitable while ES is not (GNEITING, 2011). However, as stated by Brehmer (2017), non-elicitable functionals that are components of an elicitable functional are called jointly elicitable. Thus ES, although not directly elicitable, can, in practice, be jointly elicited with VaR. See Fissler e Ziegel (2016).

[1] For details regarding coherence, see the section 2.1, or the seminal work Artzner et al. (1999) itself.

[2] A functional is named elicitable when it minimizes the expectation of some score function. For details regarding elicitability, see section 2.1, or the works Gneiting (2011) and Ziegel (2016).

Another important statistical property of risk measures is robustness[3]. Informally, robustness can be understood as the stability of a functional concerning deviations from the theoretical distribution of the data. See (KRÄTSCHMER et al., 2014). Cont et al. (2010) showed that a risk measure cannot be both coherent (in the sense proposed by Artzner et al. (1999)) and robust (in the sense proposed by Hampel (1971)). Therefore, ES is not a robust risk measure. To obtain a robust risk measure that captures losses beyond the quantile of interest, Cont et al. (2010) proposed a risk measure called Range Value at Risk ($RVaR^{\alpha,\beta}$, where α and β are the significance levels). This measure is defined as the average of all VaR^{λ} for λ between α and β, with $0 < \alpha < \beta < 1$ (FISSLER; ZIEGEL, 2019b). $RVaR^{\alpha,\beta}$ is a conditional interquantile expectation (IQE, see Barendse (2017)), given its realization lies in an interval between α and β. Thus, it can be regarded as an interpolation of VaR and ES. Also, RVaR is a robust and, hence, not a coherent risk measure and nests VaR and ES as special cases. Regarding the elicitability of RVaR, Fissler e Ziegel (2019a) shows that the triplet (VaR^{α}, VaR^{β}, $RVaR^{\alpha,\beta}$) is elicitable under weak regulatory conditions, and so RVaR is jointly elicitable. Despite the statistical advantages presented by RVaR, which make it attractive to be used in a practical sense, to the best of our knowledge, there are no studies that evaluate models and/or probability distributions to make their predictions.

In this context, our main objective is to assess the out-of-sample performance of both univariate and multivariate models to to forecast the RVaR of financial time series, using both empirical and simulated data. We considered an autoregressive moving average (ARMA) - Generalized Autoregressive Heteroskedasticity (GARCH) model with different probability distributions of z_t for univariate analyses. Garcia-Jorcano e Novales (2019) results indicate that models are associated with the probability distribution of innovations, hence models play a secondary role. The risk forecasts were evaluated considering both an extensive Monte Carlo experiment as well an empirical financial data from different classes. For the multivariate analyses, we did not use Monte Carlo sinulations, yet we used financial data and the following models: C-Vine and R-Vine copulas with different specifications for the marginal distributions, dynamic conditional correlation (DCC) with normal and Student-t distributions, generalized orthogonal GARCH (GO-GARCH), and historical simulation (HS). For both univariate and multivariate evaluations, we considered different significance levels, rolling estimation windows, and the score functions proposed by Fissler e Ziegel (2019a) to analyze the performance of competing models. Given the direct connection between RVaR, VaR, and ES, we also quantified the predictions of VaR and ES. Our intention with this analysis is to verify how the RVaR predictions behave in comparison to both measures. The lower the realized losses, the better was the model performance.

To the best of our knowledge, this is the first study to evaluate the accuracy of competitive models for RVaR forecasting. Previous studies focused on VaR and ES (for example, Righi

[3] For more regarding robustness, see the section 2.1, or the works Hampel (1971) and Cont et al. (2010)

e Ceretta (2015)), or alternative risk measures as Expectile Value at Risk (EVaR) - proposed by Newey e Powell (1987) and extensively tested in works like Müller e Righi (2018). Also, our study differs from previous works on RVaR because we are using the RVaR loss function proposed by Fissler e Ziegel (2019a) to compare competitive forecasting models.

Another highlight of our study is that we are using Monte Carlo simulations to evaluate the performance of univariate models regarding RVaR forecasting in addition to empirical analysis. Monte Carlo simulation is a computational method that allows an experiment to be replicated enough times to the point that the empirical distribution of certain statistics of interest approximates the underlying probability measure. See (MCCRACKEN, 1955). Our study makes use of an adaptation of the Monte Carlo simulation technique used in Christoffersen e Gonçalves (2004) for estimating one-step-ahead VaR, ES and RVaR forecasts applying univariate and multivariate models. Although Fissler e Ziegel (2019a) illustrates RVaR and Fissler e Ziegel (2019b) evaluate it, these studies are not focused on a model comparison for this measure forecasts. We also forecasted VaR and ES using the same specifications we did for RVaR, to compare its results to their. Thus, our study contributes to the literature that assesses VaR and ES risk predictions. Besides that, our study also contributes to the literature that assesses the effects of misspecified marginals in predicting risk measures. See (FANTAZZINI, 2009).

The remainder of this work is structured as follow: Chapter 2 addresses the background of the financial risk measures presented in this paper (in particular, Range Value at Risk - RVaR), including the models that were used to forecast these risk measures, and the literature that makes comparison between competing models to predict risk measures; Chapter 3 defines methodological aspects regarding this project; Chapter 4 addresses the results we have obtained, with a descriptive analysis of the empirical data; and Chapter 5 adresses to the summary of our main conclusions after this study.

2 Theoretical Background

We have divided this chapter into four sections. In the first section, we recall the most important axioms in the literature of risk measures and provide the formal definition of VaR and ES. In the second, we formally introduce the Range Value-at-Risk. In the third section, we provide the formal definition of the time series models, both scalar- and vector-valued, that we have considered in our empirical application and Monte Carlo study. The fourth and last section provides an overview of studies tackling comparison of forecasting methods in the context of risk measures.

2.1 Preliminaries

We denote by $\mathscr{G} := L^1(\Omega, F, \mathbb{P})$ the space of feasible financial positions. An element $X \in \mathscr{G}$ is interpreted as the random result of some financial asset or portfolio; $X \geq 0$ is a gain, $X \leq 0$ is a loss. The expected value, the cumulative distribution function, and the left quantile of a random variable $X \in \mathscr{G}$ are defined, respectively, as $E[X] := \int_\Omega X d\mathbb{P}$, $F_X(x) := \mathbb{P}(X \leq x)$ for $x \in \mathbb{R}$, and $F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$ for $\alpha \in (0;1)$, respectively.

We begin by defining a generic risk measure and some of its theoretical properties. A **risk measure** is a functional $\rho : \mathscr{G} \to \mathbb{R} \cup \{\infty\}$, which may fulfill the following axioms:

- **Translation Invariance**: for all $X \in \mathscr{G}$, all real valued and constant C, we have $\rho(X+C) = \rho(X) - C$.
- **Monotonicity**: for all X and $Y \in \mathscr{G}$, if $X \leq Y$, then $\rho(Y) \leq \rho(X)$.
- **Positive homogeneity**: for all $X \in \mathscr{G}$ and all $\lambda \geq 0$, we have $\rho(\lambda x) = \lambda \rho(X)$.
- **Subadditivity**: for all X and $Y \in \mathscr{G}$, we have $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
- **Law invariance**: for all X and $Y \in \mathscr{G}$, we that if $F_X = F_Y$, then $\rho(X) = \rho(Y)$.

If a measure of risk respects Translation Invariance and Monotonicity it is called **monetary**. Also, when a risk measure fulfills Translation Invariance, Monotonicity, Positive Homogeneity and Subadditivity, the risk measure is said to be **coherent** in the sense proposed by Artzner et al. (1999). Translation Invariance means that adding or subtracting the sure initial amount C to the initial position and investing it in the reference instrument, simply decreases or increases the risk of the position by C. Monotonicity is the requirement that the risk measure appraises as less risky those positions that pay more in every state of the world, see (KLUGMAN et al., 2012). Positive Homogeneity says that the risk of a portfolio is directly proportional to its size.

Subadditivity indicates that the risk to which a portfolio is exposed cannot be higher than the sum of the individual risk exposure of each one of the positions that compose that portfolio, thus capturing the benefit of diversification. Law invariance ensures that two positions with the same probability function have equal risks.

Two important examples of monetary measures, which fulfill law-invariance, are the Value at Risk and the Expected Shortfall. Given a significance level $\alpha \in (0,1)$, the Value at Risk is defined through (2.1):

$$\mathrm{VaR}^{\alpha}(X) = F_{-X}^{-1}(1-\alpha) \equiv -\inf\left\{x \in \mathbb{R} : F_X^{-1}(x) \geq \alpha\right\} \equiv -F_X^{-1}(\alpha), \quad X \in \mathscr{G} \tag{2.1}$$

Based on this definition, we can note that the VaR^{α} does not consider information beyond the quantile of interest, only the point itself. The Expected Shortfall, in turn, is defined through (2.2):

$$\mathrm{ES}^{\alpha}(X) = -E\left[X|X \leq -F_X^{-1}(\alpha)\right] \equiv \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}^{u}(X)du. \tag{2.2}$$

Where $\alpha \in (0,1)$. When $\alpha = 0$, one defines that $\mathrm{ES}^0(X) = -F_X^{-1}(0) = \mathrm{VaR}^0(X)$.

The first statistical property we will explore is elicitability, in the sense used in Gneiting (2011). This property permits, in particular, the comparison of competitive models in risk forecasting. For details we suggest Gneiting (2011) and Ziegel (2016). In order to define elicitability, first we need to define a scoring function. According to Bellini e Bignozzi (2015), a map $S : \mathbb{R}^2 \to \mathbb{R}_+$ can be called a scoring function if fulfills these properties:

1. $S(x,\mathrm{y}) = 0$ if, and only if, $x = \mathrm{y}$;
2. $y \mapsto S(x,\mathrm{y})$ is non-decreasing for $\mathrm{y} > x$ and non-increasing for $\mathrm{y} < x$, $\forall\, x \in \mathbb{R}$;
3. $S(x,\mathrm{y})$ is continuous in y, $\forall\, x \in \mathbb{R}$.

With the notion of scoring function at hand, a risk measure ρ is called **elicitable** if is law-invariant and there is some scoring function $S : \mathbb{R}^2 \to \mathbb{R}_+$ such that:

$$\rho(X) = -\arg\min_{\mathrm{y}\in\mathbb{R}} \mathbb{E}[S_{\rho}(X,\mathrm{y})],\ \forall\, X \in \mathscr{G}. \tag{2.3}$$

Following Fissler e Ziegel (2019a), the VaR is elicitable under the following score:

$$S_{\mathrm{VaR}^{\alpha}}(x,y) = [(\mathbf{1}_{x<y} - \alpha)y - x\mathbf{1}_{x<y}]. \tag{2.4}$$

The ES, in turn, is not elicitable (FISSLER; ZIEGEL, 2016). However, it is is jointly elicitable with Value at Risk, i.e., $\rho(X) := (\mathrm{VaR}^{\alpha}, \mathrm{ES}^{\alpha})$. The score function of the ES under $\mathbb{R}^3$, considering the adaptation suggested by Gerlach et al. (2017), can be defined by:

$$S_{\mathrm{ES}^{\alpha}}(x,y,z) = y\left(\mathbf{1}_{x<y} - \alpha\right) - x\mathbf{1}_{x<y} + e^{z}\left(z - y + \frac{\mathbf{1}_{x<y}}{\alpha}\left(y-x\right)\right) - e^{z} + 1 - \log\left(1-\alpha\right). \tag{2.5}$$

The second statistical property we investigate is robustness. Intuitively, robustness means that mild misspecification yields only a small variation in the output functional. As stated by Embrechts et al. (2015), the qualitative robustness of a statistical estimator, which is defined in Cont et al. (2010), is equivalent to the continuity, at the point P_X, of the corresponding risk measure at the true distribution, with respect to the weak topology. In a more formal way, let d be a pseudo-metric on $\mathscr{G}$, a risk measure is $\rho : \mathscr{G} \to \mathbb{R} \cup \{\infty\}$ is called d-robust if it is continuous with respect to d. We suggest Cont et al. (2010), Embrechts et al. (2015), and Kiesel et al. (2016) for more details.

VaR does not respect the subadditivity property, hence is not a coherent risk measure. This is one of the main reasons for the criticism over VaR along the years. The ES was initially presented as a suitable (and likely) substitute of VaR, once it respects the four properties needed to be a coherent risk measure. The problem is that coherence is not the only good quality expected from a risk measure, since robustness and elicitability are desirable as well. VaR is a robust risk measure, whereas Cont et al. (2010) showed that if the quantile of the (true) loss distribution is uniquely determined, then the empirical quantile is a robust estimator. The study also explains there is a conflict between the subadditivity property and robustness, indicating that a risk measure cannot be both qualitative robust and coherent, consequently ES is not a robust risk measure. Even though ES has a non robust historical estimator, is possible to remove this drawback by slightly modifying its definition to be simply the average of VaR levels across a range of loss possibilities. That is, modifying the definition of an ES to be the average between two confidence levels, creating the definition of the RVaR, which is important because enjoys similar robustness properties of the VaR, once it is an interquantile expectation.

2.2 Range Value at Risk (RVaR)

Given two significance levels α and β, where $0 \leq \alpha \leq \beta \leq 1$, the Range Value at Risk, $\text{RVaR}^{\alpha,\beta}(X)$, is a conditional interquantile expectation between α and β, formally defined in Equation (2.6):

$$\text{RVaR}^{\alpha,\beta}(X) = \begin{cases} \dfrac{1}{\beta - \alpha} \displaystyle\int_{\alpha}^{\beta} \text{VaR}^{u}(X) du, & if \quad \alpha < \beta; \\ \\ -F_X^{-1}(\alpha), & \text{if} \quad \alpha = \beta. \end{cases} \tag{2.6}$$

As stated in Fissler e Ziegel (2019a), we have the following inequality:

$$\text{VaR}^{\alpha}(X) \geq \text{RVaR}^{\alpha,\beta}(X) \geq \text{VaR}^{\beta}(X). \tag{2.7}$$

This inequality is valid for any F_X when $0 \leq \alpha \leq \beta \leq 1$. For more details regarding RVaR, we suggest Cont et al. (2010) and Fissler e Ziegel (2019a).

Equation (2.6) can also be written in terms of ES, using the equation (2.2), and considering the premise that both $\text{ES}^{\alpha}(X)$ and $\text{ES}^{\beta}(X) \in \mathbb{R}$, as it follows:

$$\text{RVaR}^{\alpha,\beta}(X) = \frac{\beta \text{ES}^{\beta}(X) - \alpha \text{ES}^{\alpha}(X)}{\beta - \alpha}. \tag{2.8}$$

As the VaR, the RVaR does not respect the axiom of subadditivity, but the axioms of monotonicity, translation invariance and positive homogeneity are maintained. Also quantiles are elicitable and, therefore, the VaR is elicitable. Gneiting (2011) showed that that the ES is not elicitable, once, as stated by Gerlach et al. (2017), there is no loss function that is minimized by the true ES series. Unfortunately, the RVaR is not elicitable as well, as showed by Wang e Wei (2020).

In this context, another concept to be explored is the *jointly elicitability*. Following Brehmer (2017), a solution against non-elicitable functionals is the use of multi-dimensional functionals. As stated by the author, the vector-valued functional consisting of the mean and the variance is elicitable, although the variance alone fails to have this property. In short, non-elicitable functionals which are a component of an elicitable functional are called jointly elicitable. For instance, although ES is not elicitable, as proved by Gneiting (2011), was showed by Fissler e Ziegel (2016) that, fortunately, VaR and ES can be used as a family of score functions, which is minimized by the by the true VaR and ES series. In other words, they are strictly consistent scoring functions for (VaR, ES). This idea was applied in Gerlach et al. (2017). Similarly, RVaR is not elicitable, as proved by Wang e Wei (2020), but can be jointly elicitable as a family of score functions. As shown by Fissler e Ziegel (2019a), the triplet (VaR^{α}, VaR^{β}, $\text{RVaR}^{\alpha,\beta}$) is elicitable, once the true VaR^{α}, VaR^{β}, and $\text{RVaR}^{\alpha,\beta}$ series are minimizers of the triplet, hence they are strictly consistent scoring functions for (VaR^{α}, VaR^{β}, $\text{RVaR}^{\alpha,\beta}$). The score function of RVaR under $\mathbb{R}^4$, following the logical of equations 2.5 and 2.4, can be defined by:

$$\begin{aligned} S_{\text{RVaR}^{\alpha,\beta}}(x,y,z,w) = {} & G_1(y)\left(\mathbf{1}_{x<y} - \alpha\right) - G_1(x)\mathbf{1}_{x<y} + G_2(z)\left(\mathbf{1}_{x<z} - \beta\right) - G_2(x)\mathbf{1}_{x<z} \\ & + \phi'(w)\left[w + \frac{1}{\beta - \alpha}\left(S_{\text{VaR}^{\beta}}(x,z) - S_{\text{VaR}^{\alpha}}(x,y)\right)\right] \\ & - \phi(w) + a(x). \end{aligned} \tag{2.9}$$

Gerlach et al. (2017) suggested $G_1(x) = G_2(x) = x$. As for $a(\cdot)$, the requirement of it being a real-valued integrable function remains, so we are using again the choice made by the authors in the ES score function: $a(x) = 1 - \log(1 - \alpha)$. As for $\phi'(\cdot)$, Fissler e Ziegel (2019a) suggested four possible functions to fit it, but, for now, we will use only the first one, defined in these equations:

$$\phi'(w) = (\beta - \alpha)\tanh((\beta - \alpha)w) \tag{2.10}$$

and hence:

$$\phi(w) = \ln(\cosh((\alpha - \beta)w)). \tag{2.11}$$

In that way, our definitive score function for $\text{RVaR}^{\alpha,\beta}$, considering the score functions defined in equations (2.10) and (2.11), is defined in equation (2.12), as it follows:

$$\begin{aligned} S_{\text{RVaR}^{\alpha,\beta}}(x,y,z,w) = {} & y\left(\mathbf{1}_{x<y}-\alpha\right) - x\mathbf{1}_{x<y} + z\left(\mathbf{1}_{x<z}-\beta\right) - x\mathbf{1}_{x<z} \\ & + (\beta-\alpha)\tanh\left((\beta-\alpha)w\right)\left[w + \frac{1}{\beta-\alpha}\left(S_{\text{VaR}^{\beta}}(x,z) - S_{\text{VaR}^{\alpha}}(x,y)\right)\right] \\ & - \log\left(\cosh\left((\alpha-\beta)w\right)\right) + 1 - \log\left(1-\alpha\right). \end{aligned} \tag{2.12}$$

In this context, the RVaR is a market risk measure very interesting to be valued. It is robust, hence has more stability against perturbations of F_X than ES. Also, does not (fully) suffer of VaR shortcomings against extreme losses. Another advantage of RVaR is in not be subject to errors when estimating the expectation for extreme tail values, like ES does.

2.3 Time Series Models

In this section, we recall the definition of the ARMA-GARCH model as well as the multivariate models used to predict risk measures, namely C-Vine and R-Vine copulas, DCC-GARCH, and GO-GARCH.

2.3.1 ARMA-GARCH

A common model to estimate risk measures is the Autoregressive Moving Average-Generalized Autoregressive Conditional Heteroskedasticity (ARMA(p,r)-GARCH(q,s)). See, for instance, Li et al. (2002). A time series $(X_t)_{t\in\mathbb{Z}}$ is called an ARMA(p,r)-GARCH(q,s) process if, for some white noise process $(Z_t)_{t\in\mathbb{Z}}$, some non-negative process $(\sigma_t)_{t\in\mathbb{Z}}$, and some parameters $\phi_i, \theta_i, a_i, b_i$..., the following system of stochastic difference equations holds:

$$\begin{aligned} X_t &= \phi_0 + \sum_{i=1}^{p}\phi_i X_{t-i} + \sum_{j=1}^{r}\theta_j\varepsilon_{t-j} + \varepsilon_t; \\ \varepsilon_t &= \sigma_t Z_t, \qquad Z_t \sim i.i.d.\ F(\theta); \\ \sigma_t^2 &= \omega + \sum_{j=1}^{q} a_j\varepsilon_{t-1}^2 + \sum_{k=1}^{s} b_k\sigma_{t-k}^2, \end{aligned} \tag{2.13}$$

where σ_t^2 is the conditional (at time t) variance. ϕ_0; ϕ_i with $i = 1,\cdots,p$; and θ_j with $j = 1,\cdots,n$ (with p and n being the ARMA model order) are the parameters of the ARMA model. ω; a_j with $j = 1,\cdots,q$; and b_k with $k = 1,\cdots,s$ (with q and s being the GARCH model order) are the parameters of the GARCH model. Also, $\omega > 0$, $a_j \geq 0$, $b_k \geq 0$, and the process $(Z_t)_{t\in\mathbb{Z}}$ follows the F distribution with a vector of parameters θ.

Applying the formulation (2.13), we will forecast conditional (at time t) mean ($\mu_{t+1} = \phi_0 + \sum_{i=1}^{\phi} {}_iX_{t+1-i} + \sum_{j=1}^{n}\theta_j\varepsilon_{t+1-j}$) and standard deviation (σ_{t+1}). Using a parametric approach, we will forecast, in univariate analysis, $\text{VaR}^{\alpha}_{\text{t+1}}$, $\text{ES}^{\alpha}_{\text{t+1}}$, and $\text{RVaR}^{\alpha,\beta}_{\text{t+1}}$ with the following equations:

$$\hat{\text{VaR}}^{\alpha}_{\text{t+1}} = -\mu_{t+1} + \sigma_{t+1}\text{VaR}^{\alpha}(Z_{t+1}); \tag{2.14}$$

$$\hat{\text{ES}}^{\alpha}_{\text{t+1}} = -\mu_{t+1} + \sigma_{t+1}\text{ES}^{\alpha}(Z_{t+1}) \text{ and} \tag{2.15}$$

$$\hat{\text{RVaR}}^{\alpha,\beta}_{\text{t+1}} = -\mu_{t+1} + \sigma_{t+1}\text{RVaR}^{\alpha,\beta}(Z_{t+1}). \tag{2.16}$$

Where VaR^{α}, ES^{α}, and $\text{RVaR}^{\alpha,\beta}$ are defined in equations (2.1), (2.2), (2.6), and (2.8). To derive these formulations, let $X_t = \mu_t + \sigma_t(Z_t)_{t\in\mathbb{Z}}$ be financial return distribution that have a fully parametric location-scale specification based on the expectation, dispersion and random component, where μ_t and σ_t are measurable considering the information available up to $t-1$. Also, $(Z_t)_{t\in\mathbb{Z}}$ represents i.i.d. innovations, which can assume different probability distribution functions. Assume that ρ refers to a risk measure. Remember that our idea here is to represent a monetary loss, so there is a minus sign in front of the risk measure. Also consider the functional form of $X_t = \mu_t + \sigma_t(Z_t)_{t\in\mathbb{Z}}$, where μ_t and σ_t are the mean and conditional standard deviation and $(Z_t)_{t\in\mathbb{Z}}$ is a white noise with an F distribution. Thus, a risk estimate, in a parametric way, is obtained by $-\rho(X_t) = -\rho(\mu_t + \sigma_t(Z_t)_{t\in\mathbb{Z}}) = -\mu_t - \sigma_t\rho((Z_t)_{t\in\mathbb{Z}})$. For more details, see McNeil et al. (2015).

2.3.2 Multivariate Models

In this subsection, we define the multivariate models used in our analysis. We divided this section into two subsections: i) Multivariate GARCH Models, and ii) Copulas.

2.3.2.1 Multivariate GARCH Models

Consider n log-returns series, X_t a vector of these n returns (n x 1) at time t, μ_t the conditional mean at time t, and z_t the error term which follows a white noise process with distribution F_X, at time t. X_t is defined by the formulation (2.17):

$$\begin{aligned} X_t &= \mu_t + \varepsilon_t, \\ \varepsilon_t &= \sqrt{\mathbf{H}_{t,m}}\ z_t \qquad z_t \sim i.i.d.\ F(0,1). \end{aligned} \tag{2.17}$$

Where $t = 1, 2, \cdots, T$; $\mu_t = 0$; $\sqrt{\mathbf{H}_{t,m}}$ is the unique positive-definite matrix of dimensions $n \times n$ such that $(\sqrt{H_{t,m}})^2 = E_{t-1}(X_t X_t')$, specific for each model m. Also, $\mathbf{H}_{t,m} = E[X_t\ X_t^{'}]$ is the covariance matrix of X_t, considering the past information, computed by multivariate GARCH models.

2.3.2.1.1 DCC-GARCH

The DCC-GARCH model, proposed by Engle (2002), allows to jointly model the conditional volatility and correlation. For this model, $\mathbf{H}_t$ is defined by the formulation (2.18):

$$\mathbf{H}_t = \mathbf{D}_t \, \mathbf{R}_t \, \mathbf{D}_t, \qquad \mathbf{D}_t = diag\left\{\sqrt{\sigma_{i,t}}\right\}. \tag{2.18}$$

Where i indexes the different assets in the market, $\mathbf{D}_t$ the diagonal matrix of the conditional standard deviations (computed by the univariate GARCH, formulation (2.13)), and $\mathbf{R}_t$, defined by the equation (2.19) the correlation matrix which contains the conditional correlations coefficients.

$$\mathbf{R}_t = \mathbf{Q}_t^{*-1} \, \mathbf{Q}_t \, \mathbf{Q}_t^{*-1}. \tag{2.19}$$

Where $\mathbf{Q}_t$ is given by equation (2.20):

$$\mathbf{Q}_t = (1 - a + b)\, \overline{\mathbf{Q}} + a\, \varepsilon_{t-1}\, \varepsilon_{t-1}^{'} + b\, \mathbf{Q}_{t-1}. \tag{2.20}$$

Where $\mathbf{Q}_t$ is the conditional covariance matrix of the residuals of the unconditional covariance matrix $\overline{\mathbf{Q}}$. The residuals of $\overline{\mathbf{Q}}$ are obtained by a univariate GARCH model, again by formulation (2.13). Also, a and b are non-negative parameters, which satisfy $a + b < 1$, and finally $\mathbf{Q}_t^*$ represents a diagonal matrix which contains the square root of the diagonal elements of $\mathbf{Q}_t$.

2.3.2.1.2 GO-GARCH

The Generalized Orthogonal - GARCH (GO-GARCH) model, proposed by Weide (2002) is an extension of the O-GARCH (Orthogonal-GARCH) model. This model assumes X_t as a process ruled by a linear combination of non-observed factors z_t, independents, with $\mu = 0$. For this model, $\mathbf{H}_t$ is defined by the equation (2.21):

$$\mathbf{H}_t = \mathbf{W}\, \mathbf{H}_t^2\, \mathbf{W}'. \tag{2.21}$$

Where $\mathbf{H}_t^2$ is the conditional variance matrix at time t, estimated by the univariate GARCH model. The positiveness of $\mathbf{H}_t$ implies the positiveness of $\mathbf{H}_t^2$, assured by the positive estimators of the GARCH model. $\mathbf{W}$ is a non-singular and invertible parametric matrix. Also, $\mathbf{H} = E[X_t\, X_t^{'}] = \mathbf{W}\, \mathbf{W}'$, and $\mathbf{W}$ is defined by the equation (2.22):

$$\mathbf{W}_t = \mathbf{P}\, \sqrt{\Lambda}\, \mathbf{U}. \tag{2.22}$$

Where the orthonormal eigenvectors of $\mathbf{H}$ generate the columns of $\mathbf{P}$, and the eingenvectors of $\mathbf{P}$ compose the diagonal of the matrix Λ. $\mathbf{U}$ is an orthogonal matrix n x n of eingenvectors from $\mathbf{H}$, and $\det \mathbf{U} = 1$. $\mathbf{P}$ and Λ are obtained considering the unconditional information of $\mathbf{H}$ at time t.

2.3.2.2 Copulas

A Copula function C is a n-dimensional distribution function with uniformly distributed marginals. Consider a collection of random variables $X_1,\, X_2,\, \ldots,\, X_n$ with their respectively distribution functions $F_1,\, F_2,\, \ldots,\, F_n$ and their respectively generalized inverse distribution functions $F_1^{-1},\, F_2^{-1},\, \ldots,\, F_n^{-1}$. Applying Sklar (1959) Theorem, F n-dimensional distribution function of C is given by equation (2.23):

$$F(x_1,\, x_2,\, \ldots,\, x_n) = C(F_1(x_1),\, F_2(x_2),\, \ldots,\, F_n(x_n)). \tag{2.23}$$

For every $\mathbf{x} = (x_1,\, x_2,\, \ldots,\, x_n)' \in \mathbb{R}^n$. C is unique if their corresponding distribution functions are continuous. Using equation (2.23), is possible to represent the n-dimensional distribution function C using the marginal distribution functions F_i $(i = 1,\, 2,\, \ldots,\, n)$ by expression (2.24):

$$C(u_1,\, u_2,\, \ldots,\, u_n) = F(F_1^{-1}(u_1),\, F_2^{-1}(u_2),\, \ldots,\, F_n^{-1}(u_n))\,. \tag{2.24}$$

Where $(u_1,\, u_2,\, \ldots,\, u_n) \in [0,1]^n$, and F_i^{-1} represents the generalized inverse function of F_i with $i = 1,\, 2,\, \ldots,\, n$. For more formal details about copulas, see Nelsen (2007).

2.3.2.2.1 Vine Copulas

Vine copulas provide a flexible model to describe multivariate copulas, and are constructed following a waterfall of bivariate copulas. The literature presents three variations for the vine copulas, but we will apply only two of them, namely: R-Vines, and C-Vines. Following Kurowicka e Cooke (2006), the R-Vine density is given by expression (2.25):

$$f(u_1,\, u_2,\, \ldots,\, u_n) = \prod_{k=1}^{n} f_k(u_k) \prod_{i=1}^{n-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)} \left\{ \begin{array}{l} F(u_{j(e)}|\mathbf{u}_{D(e)}); \\ F(u_{j(e)}|\mathbf{u}_{D(e)}). \end{array} \right\}. \tag{2.25}$$

Where E_i, with $i = 1,\, 2,\, \ldots,\, n$, are the edges, $D(e)$ are the nodes shared by the bivariate copulas, and $j(e)$ and $k(e)$ are the non-shared nodes. The nodes $j(e)$ and $k(e)$ are called conditional nodes, whereas $D(e)$ represents the conditioning set, that is the union of the three nodes which are called restrictions set. $\mathbf{u}_{D(e)}$ is a subvector of pseudo-observations. f_k is the density function of each i asset and c is the bivariate copula density, which can be different for each pair of copulas.

The C-Vine density function is defined by the expression (2.26):

$$f(u_1,\, u_2,\, \ldots,\, u_n) = \prod_{k=1}^{n} f_k(u_k) \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,i+j|i+1,\ldots,i+j-1} \left\{ \begin{array}{l} F(u_j|u_1,\ldots,u_{j-1}); \\ F(u_{i+j}|u_1,\ldots,u_{j-1}). \end{array} \right\}. \tag{2.26}$$

The C-Vine trees have only one node, which is connected to $n - j$ edges. According to Joe (1996), the marginal distribution for waterfall constructed process is given by the expression (2.28):

$$F(u_i|\mathbf{u}) = \frac{\partial C_{u_i,u_j}|\mathbf{u}_{-j}\{F(u_i|\mathbf{u}_{-j}), F(u_j|\mathbf{u}_{-j})\}}{\partial F(u_j|\mathbf{u}_{-j})}. \tag{2.27}$$

Where i and $j \in \mathbb{N}$, $i \neq j$, and $C_{u_i,u_j}|\mathbf{u}_{-j}$ represents the dependence structure of u_i and u_j. The C-Vine trees have only one node, which is connected to $n-j$ edges. According to Joe (1996), the marginal distribution for waterfall constructed process is given by the expression (2.28):

$$F(u_i|\mathbf{u}) = \frac{\partial C_{u_i,u_j}|\mathbf{u}_{-j}\{F(u_i|\mathbf{u}_{-j}), F(u_j|\mathbf{u}_{-j})\}}{\partial F(u_j|\mathbf{u}_{-j})}. \tag{2.28}$$

Where i and $j \in \mathbb{N}$, $i \neq j$, and $C_{u_i,u_j}|\mathbf{u}_{-j}$ represents the dependence structure of u_i and u_j.

2.4 Literature Review

In this section, we make a review of some studies that used concurring models comparison for risk measures in the literature. They are organized in ascending chronological order.

Table 1 – Review of the Literature

Study	**Objectives**	**Method**	**Main Conclusions**
Yamai et al. (2002)	1) To provide an overview of studies comparing VaR and ES, to draw practical implications for financial risk management.	1) The paper is mainly theoretical, with many definitions and properties of risk measures.	1) Information given by VaR may mislead rational investors who want to maximize their expected utility. Rational investors employing VaR as a risk measure are likely to construct a perverse position, and that would result in a larger loss in the states beyond the VaR level. 2) Investors can alleviate this problem stated in 1) adopting ES as their conceptual viewpoint, since it also takes into account the loss beyond the VaR level. 3) The effectiveness of ES depends of the estimation stability and the choice of efficient backtesting methods.

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Table 1 – *Continuation of the previous page.*

Study	**Objectives**	**Method**	**Main Conclusions**
Yamai e Yoshiba (2005)	1) To illustrate how the tail risk of VaR can cause serious problems in certain cases, cases in which ES can serve more aptly.	1) The authors used as historical data the exchange rates per one US dollar from 1991/11/01 to 2001/10/29. They used $\alpha =$ 1% for VaR, since the 95% confidence level is not optimal to capt the tail risk, and $\alpha = 5\%$ for both VaR and ES.	1) Rational investors, who maximize their expected utility, may be misled using VaR as a risk measure. They are likely to construct positions with unintended weaknesses, which results in greater losses under conditions beyond the VaR level. 2) VaR is unreliable under market stress. Under extreme assets price fluctuations or an extreme dependence structure of assets, VaR may underestimate risk. 3) Investors or risk managers can solve problems shown in 2) adopting Expected Shortfall, which, by definition, considers account losses beyond the VaR level. 4) The effectiveness of Expected Shortfall, however, depends on the estimation accuracy.
Christoffersen e Gonçalves (2004)	1) The authors want to assess the potential loss of accuracy from estimation error when calculating Var and ES. 2) The authors want to assess their ability to quantify ex-ante the magnitude of this error via the construction of confidence intervals around the VaR and ES measures.	1) GARCH (1,1) for Historical Simulation, Normal Conditional Distribution, Extreme Value Theory, Gram-Charlier and Cornish-Fischer expansions, Filtered Historical Simulation and Bootstrap Algorithms. 2) Define the VaR and ES measures for t+1. 3) The authors use samples of 500 and 1,000 observations. 4) The Monte Carlo Simulation is applied using 5,000 and 100,000 replications. 5) alpha = 1% for VaR and ES.	1) The authors suggest a bootstrap method for calculating confidence intervals around the VaR point estimate. 2) FHS VaR models confidence intervals have the correct coverage, but are quite wide. 3) VaR models based on normal distribution are much narrower. 4) The accuracy of ES forecasts is much lower than the VaR forecasts one.

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Study	**Objectives**	**Method**	**Main Conclusions**
Harmantzis et al. (2006)	1) The paper aims to test, empirically, the performance of different models in measuring VaR and ES in the presence of heavy tails in returns, using historical data.	1) Daily returns of popular indices (S&P500, DAX, CAC, Nikkei, TSE, and FTSE). 2) Currencies (US Dollar, Euro, Yen, Pound and Canadian Dollar). 3) An observation period of 10 years. 4) Modeled with empirical, Gaussian, Generalized Pareto (Peak Over Threshold - POT) of EVT and Stable Paretian distributions (both symmetric and non-symmetric). 5) The authors used $\alpha = 1\%$ and $\alpha = 5\%$ for both VaR and ES. 6) The authors utilize Monte Carlo Simulation with 100,000 random Stable variates.	1) Estimating VaR, models that capture rare events can predict risk more accurately than non-fat-tailed models. 2) Estimating ES, the historical model and the POT method are proved to give more accurate estimations. The Gaussian model underestimates ES, and The Stable Paretian framework overestimates ES.
Kuester et al. (2006)	1) To compare alternative approaches for univariate VaR prediction, and provide some guidance for choosing an appropriate strategy. 2) To extend the EVT framework and introduce a new specification for VaR quantile regressions.	1) Historical simulation (empirical quantiles based on the available past data, fully parametric models: location-scale and Dynamic Feedback (describe the entire distribution of returns), extreme value theory (parametrically models only the tails of the return distribution), and quantile regression approach (directly models a specific quantile rather than the whole return distribution). 2) The estimation is based on daily return data on the NASDAQ Composite Index, from 1971-02-08 to 2001-06-22, yielding a total of 7,681 observations, and multiple windows (1,000, 500 and 250). 3) The authors used $\alpha = 1$, 2.5 and 5%.	1) With only a few exceptions, all methods perform less than what is desirable from a statistical viewpoint. 2) GPD and skewed t distribution perform better than normal.

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Table 1 – *Continuation of the previous page.*

Study	**Objectives**	**Method**	**Main Conclusions**
Basu (2011)	1) To identify wich VaR-ES models are the best suited for stress test.	1) $\alpha = 1\%$ both for VaR and ES. 2) portfolio of long and short of USD 100 and EURO 100 under normal and stressed conditions from 2002 to 2008. 3) Monte Carlo simulations using 50,000 replications. 4) HS, VWHS, Hybrid HS, and EVT-HS. 5) The authors use a stress shock of 5%, 10%, and 15%.	1) The paper suggests that fat-tailed distributions might lead to large capital charges without stress, which do not change much with the size of stress shocks. Hence, with such distributions, the amount of capital might be too high for benign markets (i.e. much of the time), but increasingly inadequate as stress conditions worsen (i.e. when most required).
Zhou e Anderson (2012)	1)To cope with extreme risks, comparing the performances of several commonly used methods in predicting VaR and ES for REITs (Real Estate Investment Trusts - Canada). 2) Comparison of the forecasting performances of alternative models.	1) Three methodological categories were considered: nonparametric, parametric and semiparametric. 2) Six methods were considered: Filtered Historical Simulation (FHS), RiskMetrics, GARCH with normal distribution, GARCH with t distribution, Exponential GARCH with skewed-t distribution and GARCH with Extreme Value Theory. 3) The methods were applied to daily US equity REITs returns from 1997-04-01 through 2009-02-28, with a total of 3,000 observations. 4) To produce the risk forecasts, were used an adopted moving window method to estimate the model parameters. 5) The authors used $\alpha = 0.05$, 1 and 5%.	1) The trio of EGARCH-skewed t, GARCH-t and GARCH-EVT provided the most reliable forecasts among all the methods. 2) GARCH-normal and RiskMetrics were the worst performers and FHS somewhere between then and the conclusion 1).

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Study	**Objectives**	**Method**	**Main Conclusions**
Degiannakis et al. (2013)	1) Test empirically whether the short memory GARCH model is outperformed for forecasting multi-period VaR and ES for longer time horizons (10-day and 20-day) by the long memory FIGARCH model, which accounts for the persistence of financial volatility.	1) The empirical analysis makes use of an adaptation of the Monte Carlo Technique for estimating multiple-step-ahead VaR and ES to the FIGARCH model, which enables comparisons to be made between the forecasting performances of the GARCH and FIGARCH models for: i) 1-step-ahead, ii) 10-step-ahead and iii) 20-step-ahead VaR and ES predictions. 2) The 95 percent VaR and ES forecasting performances of the GARCH and FIGARCH models are tested on daily data across 20 leading stock indices worldwide. 3) The data, which was obtained from Datastream for the period from 1989-01-12 to 2009-02-12, was conditioned to remove any non-trading days. 4)The authors used $\alpha = 5\%$ for both VaR and ES.	1) The long memory FIGARCH model, as compared to the short memory GARCH model, does not appear to improve the VaR and ES forecasting accuracy, meaning the study contributes to the debate on the out-of-sample forecast performance of fractionally integrated models. 2) The estimated parameters of the models present a time-varying characteristic, which can be linked to market dynamics in response to the unexpected news. 3) The estimated parameters of the FIGARCH model exhibit relatively a more time-varying characteristic than those of the GARCH model, inferring evidence that not all of the time-varying characteristics can be due to the news information arrival process of the market arrival.

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Study	**Objectives**	**Method**	**Main Conclusions**
Weiß (2013)	1) The authors try to answer the question of whether the optimality of a parametric copula for estimating VaR and ES is driven by common data characteristics that can in turn be used to identify an optimal parametric copula.	1) COPULA-GARCH model. 2) 1,500 portfolio returns. 3) 1,000 observations and alpha = 1% both for VaR and ES. 4) Monte Carlo simulation using 10,000 replications.	1) The results show that the quality of the DCC model's VaR and ES estimates is positively correlated with the portfolio marginals' volatility, while the opposite is true for the elliptical copulas. 2) For the Archimedean copulas in particular the excess kurtosis of the marginals has a significant positive influence on the quality of the VaR and ES estimates. 3) The results of the empirical study provide no evidence of superiority of parametric copula models over the DCC benchmark. On the contrary, the DCC model outperforms the elliptical, the Archimedean and the mixture copulas in forecasting VaR and ES.
Tolikas (2014)	1) To investigate the limiting distribution of the extreme returns of the NIKKEI225, FTSE100 and S&P500 indices, as well as the indices of some of the largest sectors in Japan, UK and US.	1) Distributions for the extreme financial returns. 2) Identifying the best candidate distributions for the extremes. 3) Parameter estimation using the probability-weighted moment's method. 4) Assessing the goodness of fit of the fitted distributions. 5) Homogeneity test of the extremes. 6) Japanese, UK and US stock markets, NIKKEI225, FTSE100 and S&P500 indices as well as indices for the OIL and GAS, Industrial Financial and Technology sectors. 7) 30 year period, from 1983 to 2012, of observations. 8) The authors used $\alpha = 1$ and $\alpha = 5\%$ for both VaR and ES.	1) The paper provides further support to the usefulness and superiority of the GL (Generalised Logistic) distribution, describing and quantifying downside risk and low probability events, when compared to the GEV and normal distributions in the Japanese, UK, and US stock markets. 2) There are a new set of statistical tools for risk analysis that could be used to better describe and predict the probabilities of large stock price movements.

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Study	**Objectives**	**Method**	**Main Conclusions**
Righi e Ceretta (2015)	1) To verify if there is a pattern regarding the model advantage of the ES estimation, considering the distinct asset classes, estimation windows and significance levels.	1) classes of models: (i) unconditional models, (ii) conditional models and (iii) quantile/expectile regression models. For i), were considered as the possible F distribution functions: Normal, Student's t, Skewed Student's t, Generalized Pareto. In ii), were considered location-scale filters under the normal, Student's t and skewed Student's t specifications for G. 2) Were considered the main factors of market risk: (i) equity, (ii) fixed income, (iii) exchange rates, and (iv) commodities. 3) In (i) the authors collect the data of the daily closing prices of the S&P500, DAX, FTSE100, Nikkei225, Ibovespa and Hang Seng indices. 4) In (ii) the authors considered the 1-Month, 3-Months, 6-Months, 1-Year, 3-Years and 10-Year US Treasury bonds. 5) In (iii) the authors considered the ratio to the US Dollar of Euro, Yen, Pound sterling and Brazilian Real. 6) In (iv), the authors considered WTI crude oil and Gold. 7) All series are from January 2000 to December 2012. 8) The authors used $\alpha = 1$ and 5% for VaR, ES and SD.	1) The rejection rates for a significance level of 0.01 are larger than for 0.05. 2) Historical simulation is not a good choice at the crisis interval, on the other hand, the conditional models Normal-EVT, Skew-Student t and asymmetric CARES can follow the market evolution correctly, but CARES shows very turbulent estimations after the crisis. Also, in the ES estimation CARES and HS lead to discrepant results. 3) The larger the window is, the more conservative tend to be the risk predictions. 4) Table 15 (pg. 44) presents the ES estimation models that have the best performance, given that the VaR violation is correct. 5) The predominance of the conditional models and both GARCH and FHS methods exhibit very good performance in most cases for ES estimation. 6) For all models, skewed Student's t is important.

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Study	**Objectives**	**Method**	**Main Conclusions**
Diaz et al. (2017)	1) To examine the performance of VaR under different distributional models, and considering the financial crisis. 2) Provide useful insight into distributions that appropriately capture tail risk in measuring VaR.	1) Three distributions to appropriately capture tail risk: alpha-stable, g-and-h, and generalized Pareto distributions. 2) two-step procedure: ARMA(1,1) and GARCH(1,1). 3) Five indexes that proxy the portfolio investment of US stocks, European stocks expressed in US dollars, US equity options, foreign exchange rates, and commodities during the 2006/11 to 2015/12 period. 4) The authors used $\alpha =$ 1 and 5%.	1) ARMA(1,1)-GARCH(1,1) is a sound model to capture the volatility process. 2) GPD, alpha-stable, and g-and-h distributions perform well for heavy-tailed data. 3) The g-and-h model is very conservative when the data are not so heavy-tailed, but the GPD and the alpha-stable work well in these cases.
Müller e Righi (2018)	1) To analyze the performance of models for forecasting VaR, ES and Expectile VaR (EVaR) from the multivariate perspective.	Were considered the following approaches: Historical Simulation, Dynamic Conditional Correlation (DCC) - GARCH, Regular Copulas, Vine Copulas and NAC (Nested Archimedean Copulas). The performance was evaluated using 10.000 Monte Carlo simulations, on samples with portfolios of 4 and 16 assets, for the confidence of 1% and 5%. Were computed absolute bias (A. Bias) and Root-Meansquare Error (RMSE), using the R programming language.	1) For Gaussian marginal distribution, Regular and Vine copulas performed better in predicting the risk measures. On the other hand, for data that follows the Student's t distribution, NAC showed the best performance. 2) Copulas method offers better performance than traditional methods, such as HS and DCC - GARCH. 3) HS method tends to overestimate the risk, with positive bias when presenting the worst performance.

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Study	**Objectives**	**Method**	**Main Conclusions**
Tian et al. (2019)	1) To review the recent development in the estimation of the most popular risk measures: VaR, ES, and Expectile regression.	1) Parametric models: EVT (Extreme Value Theory), Quantile Regression technique and regression approach for ES. 2) Nonparametric Models: Nonparametric Approach (kernel smoothing to estimate the conditional distribution function), Extreme Value Theory (EVT) and Quantile Approach. 3) Expectile Models: Parametric Models, Nonparametric Models (A Varying Coefficient Expectile Model) and Semiparametric Models (A Partially Varying Coefficient Expectile Model). 4) Models for Characterizing Tail Dependence: Copula Approach, Network and CoVar Procedure.	1) Expectile has proved to be superior to VaR and ES in some aspects, but more critical analysis of the relative merits and potential estimation procedures accustomed to specific regression techniques are required.
Trucíos (2019)	1) To compare the predictability of the one-step-ahead volatility and VaR of Bitcoin using several volatility models.	1) Volatility models: GARCH models, GARCH-MIDAS model, realized GARCH model, GAS models. 2) The data uses Bitcoin closing prices (in US dollars) traded on Bitstamp from 2011-09-13 to 2017-12-31 (totalizing 2,280 observations). 3) The daily closing prices are considered as the last price traded at each day and were constructed using the tick-by-tick data obtained from bitcoin charts.	1) Robust procedures (proposed by the authors in previous works) outperform non-robust ones when forecasting the volatility and estimating the Value-at-Risk. 2) The paper addresses the presence of outliers and show that a better volatility forecast is obtained using robust procedures, thus suggesting that outliers play a crucial role in the modeling and forecasting of the volatility of Bitcoin.
Trucíos et al. (2019)	1) To propose a methodology based on in copulas and robust volatility models to estimates the Value-at-Risk and Expected Shortfall of cryptocurrencies portfolios.	1) Vine copula and robust univariate volatility models (Robust GARCH model and GAS model. 2) Were examined the daily closing prices (in US dollars) of cryptocurrencies traded from 2015/01/01 to 2019/06/14 (totalizing 1,625 observations). 3) Were considered cryptocurrencies with no missing values, resulting in Bitcoin (BTC), Dash (DASH), DigiByte (DGB), Dogecoin (DOGE), Litecoin (LTC), MaidSafeCoin (MAID) and Vertcoin (VTC).	Vine copulas combined with robust univariate volatility models can be successfully used to estimate the VaR and ES in cryptocurrencies portfolios.

The standard rolling estimation windows are 250, 500, and 1,000 observations, as, for instance, applied in studies like Kuester et al. (2006) and Righi e Ceretta (2015). The most common risk measures comparisons are focused in VaR and ES, as shown by studies like Harmantzis et al. (2006), Zhou e Anderson (2012), Degiannakis et al. (2013), and Righi e Ceretta (2015). For VaR, the standard α is 1%, used, for example, in Yamai e Yoshiba (2005), Kuester et al. (2006), Tolikas (2014), and Diaz et al. (2017). For ES, both $\alpha = 1\%$ and $\alpha = 5\%$ are standard values, as stated in studies like Yamai e Yoshiba (2005), Zhou e Anderson (2012), and Müller e Righi (2018), just to mention a few. There are studies that use $\alpha = 5\%$ for VaR too, but, as noted by Yamai e Yoshiba (2005), the 95% confidence level is not optimal to capt the tail risk of VaR. The most common models are historical simulation and GARCH. Also, as shown by studies like Righi e Ceretta (2015), Degiannakis et al. (2013), and Zhou e Anderson (2012), GARCH have good performance on risk measures estimation proposes. Studies like Zhou e Anderson (2012), Degiannakis et al. (2013), and Diaz et al. (2017) used ARMA(1,1)-GARCH(1,1) combined with multiple distributions to verify the most satisfactory ones. Several studies utilize the Monte Carlo simulation technique adaptations in risk measures forecasting, often following the algorithm proposed by Christoffersen e Gonçalves (2004). As examples of those studies: Harmantzis et al. (2006), Basu (2011) and Weiß (2013).

3 Methodology

In this chapter, we define the main procedures used to achieve the proposed objective. We have divided this chapter into two sections: I) Univariate Analysis, which is divided in i) Numerical Procedures, where we present the GARCH process to generate the returns for the Monte Carlo simulation and the metrics used to assess the risk forecasts, and ii) Empirical Analysis, where we describe the describe the empirical methodological procedures; and II) Multivariate Analysis, where are described the methodological procedures used in the study with multivariate models.

3.1 Univariate Analysis

3.1.1 Numerical Procedures

We used Monte Carlo simulations to evaluate the performance of different methodologies for VaR, ES, and RVaR forecasting. We conducted all the computational implementations using the `R` programming language, see R Core Team (2021), making use of the `fGarch` package (WUERTZ, 2007). We used 1,000 replications, which can provide satisfactory results as observed by Escanciano e Olmo (2011) and Yi et al. (2014). For the sample size, we considered 250, 500, and 1,000 observations[4]. Following Righi e Ceretta (2015) and Müller e Righi (2018), the data generating process utilized to generate the log-returns is an AR(1)-GARCH(1,1) model with parameters $\phi_0 = 0$, $\phi_1 = 1/2$, $\omega = 4 \times 10^{-6}$, $a_1 = 0.1$ and $b_1 = 0.85$.

F_X is the cumulative distribution function of Z_t with a vector of θ parameters. For it, we assumed normal (GARCH$_{norm}$), skewed normal (GARCH$_{snorm}$, considering also the parameter ξ), Student's t (GARCH$_{std}$, considering also the parameter ν), and skewed Student's t (GARCH$_{sstd}$, considering also the parameters ξ and ν) distributions. We used as parameters for skewness (called ξ) and shape (called ν) 0.92 and 8, respectively. The ν parameter value was taken from Righi e Ceretta (2015), once this parametrization represents an equity portfolio return distribution, while the ξ parameter represents a similar value to the one we obtained estimating the S&P 500 returns with the GARCH model. We used the normal distribution because it is often applied in stock market analysis. The other three distributions were used because they consider the heavy-tailed and/or asymmetric behaviour of financial assets. Similar distributions are also used in Diaz et al. (2017). According to the results of Garcia-Jorcano e Novales (2019), the risk forecast performance is associated with the probability distribution of the innovations. Furthermore, the choice of the model playing a secondary role. For this reason, we focused

[4] 250 is the minimum size recommended by the Basel Committee and 500, and 1,000 are the most common values used in the literature. See Kuester et al. (2006), Zhou e Anderson (2012), and Leccadito et al. (2014).

on using the ARMA(1,1)-GARCH(1,1) model with different probability distributions instead of considering different risk forecasting models. Besides, for each generated return series, we forecasted the risk measures using also a ARMA(1,1)-GARCH(1,1) model. The ARMA-GARCH model is defined in equation (2.13). Also, this model order is competitive according the Akaike Information Criterion (AIC). We estimated the risk measures using the equation (2.8), for RVaR, and equations (2.1) and (2.2), respectively, for VaR and ES. We forecasted RVaR using the equation (2.16), and equations (2.14) and (2.15), respectively, for VaR and ES.

Applying a similar technique to the one of Christoffersen e Gonçalves (2004), we generated one-step-ahead forecast for VaR, ES, and RVaR with a ARMA(1,1)-GARCH(1,1) model. On each step, for F_X we assumed Normal (GARCH$_{norm}$), skewed Normal (GARCH$_{snorm}$), Student's t (GARCH$_{std}$), skewed Student's t (GARCH$_{sstd}$), Generalized Error (GARCH$_{ged}$), skewed Generalized Error (GARCH$_{sged}$), and Johnson's SU (GARCH$_{jsu}$) distributions. Regarding significance levels, we used $\alpha = 1\%$, $\alpha = 2.5\%$, and $\alpha = 5\%$ for both $\mathrm{VaR}^{\alpha}_{t+1}$ and $\mathrm{ES}^{\alpha}_{t+1}$, since $\alpha = 1\%$ and $\alpha = 5\%$ are the most used values for the measures in the literature. See, for instance, Tolikas (2014). And $\alpha = 2.5\%$ because the Basel Committee on Banking Supervision recommends[5] it for the Expected Shortfall. For $\mathrm{RVaR}^{\alpha,\beta}_{t+1}$, since it is an interpolation of $\mathrm{VaR}^{\alpha}_{t+1}$ and $\mathrm{ES}^{\alpha}_{t+1}$, we applied the same α used for them, always respecting the condition $0 < \alpha < \beta < 1$, that is $\beta = 2.5\%$ or $\beta = 5\%$ when $\alpha = 1\%$ for both $\mathrm{VaR}^{\alpha}_{t+1}$ and $\mathrm{ES}^{\alpha}_{t+1}$, or $\beta = 5\%$ when $\alpha = 1\%$ for $\mathrm{VaR}^{\alpha}_{t+1}$ and $\alpha = 2.5\%$ for $\mathrm{ES}^{\alpha}_{t+1}$. As to the cases when $0 < \alpha = \beta < 1$, we will obtain $\mathrm{RVaR}^{\alpha,\beta}_{t+1} = \mathrm{VaR}^{\alpha}_{t+1} = \mathrm{VaR}^{\beta}_{t+1}$, resulting with the same confidence levels already chosen for VaR.

Consider ρ_{t+1} and $\hat{\rho}_{t+1}$ respectively the real and the estimated value for one risk measure. To evaluate the performance of the forecasts for all scenarios, we computed for each risk measure, considering each scenario, these metrics:

$$\mathrm{Bias} = E_{\mathrm{MC}}\left[\rho_{t+1} - \hat{\rho}_{t+1}\right]; \tag{3.1}$$

$$\mathrm{R.Bias} = E_{\mathrm{MC}}\left[\frac{\rho_{t+1} - \hat{\rho}_{t+1}}{\rho_{t+1}}\right]; \tag{3.2}$$

$$\mathrm{RMSE} = E_{\mathrm{MC}}\left[\sqrt{(\rho_{t+1} - \hat{\rho}_{t+1})^2}\right]. \tag{3.3}$$

Where $E_{\mathrm{MC}}[X] = \frac{1}{n_{\mathrm{repl}}}\sum_{i=1}^{n_{\mathrm{repl}}} X_i$ represents the Monte Carlo average value of a statistic X. The Bias represents the mean of the differences between the real and the estimated value for some risk measure ρ. The Relative Bias (R. Bias) represents percentage of the bias in relation to the real value of some risk measure ρ. The Root-Mean-Square-Error (RMSE) represents the square root of the squared bias, hence always resulting in a positive value. Besides those metrics, we also computed the realized loss, which is the mean value of scoring functions. The scoring functions of the equations (2.12), (2.4), and (2.5) refers to RVaR, VaR, and ES.

[5] For details regarding the document: Fundamental review of the trading book: A revised market risk framework. See (BCBS) (2013).

The first three errors are commonly used in the literature to compare models performance. The lesser the values computed for the Bias, R. Bias, RMSE, and the Loss Functions, the better is the model in analysis. In general, the closer these metrics values are to 0, the better is the model in analysis.

3.1.2 Empirical Analysis

Like in the Section 3.1.1, we conducted all the computational implementations using the R programming language (R Core Team, 2021), but also making use of the packages `BatchGetSymbols` (PERLIN, 2019), to download the database, and `rugarch` (GHALANOS, 2019), to estimate the model parameters. We used data from five distinct classes of financial assets, which are: 1) index market; 2) fixed income; 3) exchange rates; 4) commodities; and 5) cryptocurrencies. The first four classes were also considered by Righi e Ceretta (2015). We included cryptocurrencies because, to the best of our knowledge, the studies estimating risk measures for cryptocurrencies are limited to VaR and ES. See, for instance, Trucíos (2019) and Trucíos et al. (2019). And also because they have attracted the interest of many investors, practitioners, and researchers since it's creation in 2008. For each class, we refer to two, except for commodities and cryptocurrencies, assets that will be considered. For index market, we collected the data of the daily closing prices of Ibovespa, and S&P 500, which represents, respectively, the American and Brazilian market's daily closing prices. For fixed income, U.S. Treasury Bonds rates for 5, and 10 years yield. For exchange rates, the ratio to the U.S. Dollar of Euro, and of Brazilian Real. For commodities, WisdomTree (WTI) crude oil. For cryptocurrencies, the ratio to the U.S. Dollar of Bitcoin. Differently of the other classes, we used only Bitcoin for cryptocurrencies because it is a very recent market, and only Bitcoin is probably the most successful virtual currency scheme to date. See Katsiampa (2017).

We collected the data from September 2014 to February 2020. We considered the log-returns, which are calculated as $X_t = ln(\frac{P_t}{P_{t-1}})$, with P_t being the closing price on day t, and P_{t-1} the closing price on day t-1, standard procedure in the literature, for all assets, except for fixed income, as the series are already stationary, in accordance with the Dickey-Fuller Test. See Fuller (2009).

We fitted an ARMA(1,1)-GARCH(1,1) model to each of the above time series data and generated the forecasts for the VaR, ES, and RVaR, according to Section 3.1.1. When it comes to the significance levels, we applied the same used in Section 3.1.1. For F_X, we assumed the same probability distribution of the numerical example. For empirical illustration, we also considered the same significance levels of Section 3.1.1. We applied the rolling estimation windows of 250, 500, and 1,000 daily observations, standard numbers of this technique in literature. See Kuester et al. (2006). Moreover, we evaluate the empirical risk forecasts using realized loss obtained by score functions. Again, the lesser the values computed for them, the better is the model performance.

3.2 Multivariate Empirical Analysis

3.2.1 Methodological procedures

The data used is the same used in the univariate empirical analysis, according to subsection 3.1.2, namely the data of the daily closing prices of Ibovespa, and S&P 500, which represents, respectively, the American and Brazilian market's daily closing prices for Index Market. For fixed income, U.S. Treasury Bonds rates for 5, and 10 years yield. For exchange rates, the ratio to the U.S. Dollar of Euro, and of Brazilian Real. For commodities, WisdomTree (WTI) crude oil. For cryptocurrencies, the ratio to the U.S. Dollar of Bitcoin. With this sample, we generated portfolios with 8 and 4 assets. The 8-asset portfolio considered the whole sample, and the 4-asset portfolios considered both 4 assets with a positive correlationship between them (namely S&P 500, Ibovespa, U.S. Treasury Bonds for 5 years yield, and U.S. Treasury Bonds for 10 years yield), and 4 assets with a negative correlationship between them (namely Euro, Real, U.S. Treasury Bonds for 5 years yield, and Bitcoin). Table 2 shows the correlation between each one of the analyzed assets. These portfolios were generated with an equally-weighted approach. The equally-weighted portfolio was considered in the study because it is common in studies that compare models to predict risk measures; see Müller e Righi (2018). Moreover, DeMiguel et al. (2009) point out that the performance out-of-sample of the naive portfolio is competitive concerning optimal portfolios.

The estimation methods used to predict VaR are the traditional HS, Vine copulas, and multivariate GARCH models. For the copula method, at first, we quantify the mean and conditional standard deviation of each univariate series using an AR(1)-GARCH(1,1) model. For RVine and Cvine, we assumed for the marginal distribution these distributions: Normal (RV_{norm} and CV_{norm}), skewed Normal (RV_{snorm} and CV_{snorm}), Student's t (RV_{std} and CV_{std}), skewed Student's t (RV_{sstd} and CV_{sstd}), Generalized Error (RV_{ged} and CV_{ged}), skewed Generalized Error (RV_{sged} and CV_{sged}), and Johnson's SU (RV_{jsu} and CV_{jsu}) distributions. Thus, we can verify the influence of different marginal distributions on RVaR, VaR and ES forecasts. Fantazzini (2009) explains that incorrect specifications in the marginal distribution result in biased estimates for VaR. Based on the adjusted AR(1)-GARCH(1,1) model, we obtain the standardized residuals after forecasting the mean and standard deviation values. Then, we transform them into pseudo-observations $\mathbf{u} \in [0,1]$ by inverting the fitted distribution of each series. This procedure is necessary to predict risk measures using copulas.

Given the marginal distribution and the estimated parameters, we use the following algorithm to predict VaR with copulas. See Aas e Berg (2010), Righi et al. (2013):

1. For each asset, we forecast the mean $\mu_{i,t+1}$ and the conditional standard deviation $\sigma_{i,t+1}$ using the AR(1)-GARCH(1,1) model.

2. We simulate n $u_{i,T}$ samples with size T, and each i represents a asset of our sample, using

RVine and CVine copulas.

3. Given $u_{i,T}$, we generated n $z_{i,T}$ through the inversion of marginal probability, as $z_{i,T} = F^{-1}(u_{i,T})$.

4. For each asset i, we obtain the returns by $r_{i,T} = \mu_{i,t+1} + \sigma_{i,t+1} z_{i,T}$.

5. Based $r_{i,T}$ of each i, we calculate the portfolio returns $W\mathbf{r}_T$, where $W = \{w_1, w_2, \cdots, w_n\}$ is a vector with the weights and $\mathbf{r}_T = \left\{r_{1,T}, r_{2,T}, \cdots, r_{n,T}\right\}$ are the log-returns vector of stocks.

6. Then, we quantified the RVaR, ES and VaR forecasts of the portfolio returns using the Historical Simulation, i.e., the empirical data distribution.

The Vine copulas are constructed from a cascade of bivariate copulas. In our study, we chose bivariate copulas utilizing AIC. The bivariate copulas considered were Gaussian, Student's t, Clayton, Gumbel, Frank, Joe BB1, BB6, BB7, and BB8. For n, we considered 4 and 8 assets; for T, we used the same value of the rolling estimation window, which was 250, 500, and 1,000 observations. As significance levels, we use 1%, 2.5%, and 5%. For W, we used the equally weighted weights, i.e., $\frac{1}{n}$.

In the case of multivariate GARCH models, the difference is that $z_{i,T}$ is obtained employing these models instead of the copula method. For the DCC-GARCH model, we assumed Normal (DCC), and Student's t (aDCC) distributions. For the GO-GARCH model, we performed the estimation of the $\mathbf{W}$ matrix through independent component analysis (ICA). Regarding the HS approach, the returns $r_{i,T}$ of each stock i is the empirical data distribution. To describe the risk forecasts, we quantified the mean and standard deviation, as well the realized losses.

Table 2 – Correlation table between all the assets analyzed, namely S&P 500, Ibovespa, U.S. 5-year Treasury Yield, U.S. 10-year Treasury Yield, Euro, Real, WTI Crude Oil, and Bitcoin.

	Ibovespa	**S&P 500**	**Treasure 5-years**	**Treasure 10-years**	**Real**	**Bitcoin**	**WTI Crude Oil**	**Euro**
Ibovespa	1.0000	0.4099	0.1106	0.1323	-0.0162	0.0319	0.2776	0.0304
S&P 500	0.4099	1.0000	0.2442	0.2692	-0.0171	0.0250	0.2909	0.0066
Treasure 5-years	0.1106	0.2442	1.0000	0.9356	-0.0412	-0.0023	0.1978	-0.0186
Treasure 10-years	0.1323	0.2692	0.9356	1.0000	-0.0342	0.0006	0.2208	-0.0096
Real	-0.0162	-0.0171	-0.0412	-0.0342	1.0000	0.0019	-0.0099	-0.1453
Bitcoin	0.0319	0.0250	-0.0023	0.0006	0.0019	1.0000	-0.0081	-0.0248
WTI Crude Oil	0.2776	0.2909	0.1978	0.2208	-0.0099	-0.0081	1.0000	-0.0043
Euro	0.0304	0.0066	-0.0186	-0.0096	-0.1453	-0.0248	-0.0043	1.0000

4 Results

4.1 Monte Carlo Study

In this subsection we present the results of our Monte Carlo simulation study. The results were divided in Scenarios, each one with different distributions used to generate the data, followed by the error metrics defined in Equations (3.1), (3.2), (3.3), and (2.12), (2.4), and (2.5), namely Bias, R. Bias, RMSE, and RVaR, VaR, and ES score functions. Tables 3, 4, 5, 6, and 7 show, respectively, the results for the forecast average values, bias, relative bias, RMSE, and realized losses of Scenario 1, that is, using a $GARCH_{norm}$ model to generate the data. Tables 8, 9, 10, 11, and 12 show, respectively, the results for the same metrics above, but now of Scenario 2, that is, using a $GARCH_{snorm}$ model to generate the data. Tables 13, 14, 15, 16, and 17 also expose the same metrics, but for the Scenario 3, that is, using a $GARCH_{std}$ model to generate the data. Finally, Tables 18, 19, 20, 21, and 22 refer to the same metrics with a $GARCH_{sstd}$ model generating the data. The results were summarized by Scenario, risk measures, and sample sizes applied. Tables 23 and 24 refer to the results summaries of, respectively, RVaR, VaR, and ES of Scenario 1 - $GARCH_{norm}$. Tables 25 and 26 to the summaries of RVaR, VaR, and ES of Scenario 2 - $GARCH_{snorm}$. Tables 27 and 28 to the summaries of RVaR, VaR, and ES of Scenario 3 - $GARCH_{std}$. Finally, Tables 29 and 30 to the summaries of RVaR, VaR, and ES of Scenario 4 - $GARCH_{sstd}$. We chose as the best model for each sample size in each Scenario those that performed best in two or more metrics. If no no model had the best performance among two or more error metrics, we called the results inconclusive, and if two models had the best performance among two error metrics, we called both the best ones. Tables 31 and 32 illustrate the results for, respectively, RVaR and ES for each Scenario. We also analyzed the risk measures individually, observing the number of occurrences of each model as the best for each risk measure. Table 33 summarises this approach. Also, we understand that if a model occurred as the best among four or more risk measures, it had a great performance.

Initially, observing the RVaR behavior, Tables 3, 8, 13, and 18 show, respectively, the average values of this measure for Scenarios 1, 2, 3, and 4. We can note that the inequality 2.7 is valid for every Scenario. For instance, for Scenario 1 with 500 observations, considering $GARCH_{sged}$ forecasts, we identified for $RVaR^{1\%,5\%}$ an average value equal to of 1.6754, which is higher than $VaR^{5\%}$ value of 1.4485 and, at the same time, lower than $VaR^{1\%}$ value of 2.0265. As noted by Righi e Ceretta (2015), the larger the window estimation is, the more conservative tends to be the risk predictions. That occurs because the larger the window is, lower tends the risk measures to be, thus more conservative.

Observing the Bias behavior, Tables 4, 9, 14, and 19 show, respectively, this metric for Scenarios 1 to 4. Scenario 1 shows a great performance of $GARCH_{jsu}$ and $GARCH_{ged}$

for, respectively, 250 and 1,000 observations (for instance, $VaR^{5\%}$ of 1.0149, and $RVaR^{2.5\%,5\%}$ of 1.4926, respectively). Scenario 2 shows a great performance of $GARCH_{ged}$, $GARCH_{norm}$, and, again, $GARCH_{norm}$ for 250, 500, and 1,000 observations (for instance, $RVaR^{1\%,5\%}$ of 1.6218 and 1.6279, and $VaR^{5\%}$ of 1.3792, respectively). Scenario 3 shows a great performance of $GARCH_{snorm}$, again $GARCH_{snorm}$, and $GARCH_{jsu}$ for, respectively, 250, 500, and 1,000 observations (for instance, $RVaR^{1\%,5\%}$ of 1.6070, $RVaR^{2.5\%,5\%}$ of 1.4493, and $VaR^{5\%}$ of 1.1521, respectively). Scenario 4 shows a great performance of $GARCH_{norm}$, again $GARCH_{norm}$, and $GARCH_{jsu}$ for, respectively, 250, 500, and 1,000 observations (for instance, $RVaR^{1\%,5\%}$ of 1.6591, $VaR^{2.5\%}$ of 1.6512, and $VaR^{5\%}$ of 1.2992, respectively). Lima e Néri (2007) has found great results of $GARCH_{norm}$ for $VaR^{1\%}$, our study too. Based on these results, we can notice that the distribution considered in the data generating process is not the one with the best performance according to the bias. Similar results are found in Telmoudi et al. (2016). According to the authors, this finding is due to the fact that the choice of the distribution of z_t implies an estimation risk, which compromises the estimates.We also observed that all the Bias results are positive, that is a risk underestimation. Although making a multivariate study, Müller e Righi (2018) had similar results with the normal distributions. Also, as stated by the authors, this can be dangerous and lead the institution to bankruptcy, hence the portfolio composition may not be appropriate for the investors' risk tolerance, besides compromising their investment strategies. This finding is possibly a reflection of the fact that GARCH estimates usually underestimate the true value of the parameter, which tends to compromise risk predictions Hwang e Pereira (2006).

Observing the R. Bias behavior, Tables 5, 10, 15, and 20 show, respectively, this metric for Scenarios 1 to 4. One can observe that, except for 250 observation's window of Table 5, with $GARCH_{sstd}$ having the best results according to this metric, in most cases there is no hegemony of any model. Also, we can observe no pattern regarding the estimation windows applied. The R. Bias is useful for comparisons between different risk measures, as the values are standardized by the respective mean values. However, like Bias, this metric can show positive or negative values, hence we have to observe their absolute values for a better analysis. With that in mind, we can observe that the overall lowest absolute values for $RVaR^{1\%,5\%}$, $RVaR^{2.5\%,5\%}$, and $RVaR^{1\%,2.5\%}$ are, respectively, 0.1067 (Scenario 1 - 250 observations, $GARCH_{sstd}$), -0.0007 (Scenario 4 - 250 observations, $GARCH_{std}$), and 0.0631 (Scenario 2 - 1,000 observations, $GARCH_{ged}$). The absolute value of -0.0007 was the lowest between every measure's forecasts. Furthermore, $RVaR^{2.5\%,5\%}$ having the lowest R. Bias overall shows the potential of using this risk measure instead of others for the same quantile, which is very important to our study. Also, it was forecasted with the $GARCH_{std}$ model. We highlighted this point because, Zhou e Anderson (2012) had the same results with $VaR^{1\%}$ and $VaR^{5\%}$ forecasts, and, considering Inequality 2.7, that is perfectly valid for RVaR too, once it is an interpolation of two different quantile VaRs. As for $VaR^{1\%}$ and $VaR^{5\%}$, the lowest absolute values are, respectively, -0.2597 (Scenario 1 - 500 observations, $GARCH_{sstd}$) and -0.0120 (Scenario 1 - 250 observations, $GARCH_{sstd}$). Although Lima e Néri (2007) had the $GARCH_{norm}$ performing with the lowest bias for $VaR^{1\%}$, they had good results with GARCH$_{sstd}$ for the same measure. The value of 47.3641 was the highest between all the risk measure's lowest values forecasts, corresponding to ES$^{1\%}$ (Scenario 1 - 250 observations, GARCH$_{sged}$). We also observed that the the results are well divided between positive and negative values, which is good, as it does not imply an overestimation or underestimation of the risk. However, we also noted a stronger presence of the negative values at the first two Scenarios, that is with GARCH$_{norm}$ and GARCH$_{snorm}$ models to generate the data, indicating a risk overestimation. As stated by Müller e Righi (2018), this can cause higher capital requirements that could be applied more profitably. A trader may also be required to adjust his/her portfolio at an inopportune time. In general, this event is caused by the fact of the GARCH estimators being lower than the parameter values. The differences in relation to the results of the absolute bias can be explained by the fact that the relative bias is standardized by the mean risk value, which can be positive or negative.

Observing the RMSE behavior, Tables 6, 11, 16, and 21 show, respectively, this metric for Scenarios 1 to 4. Scenario 1 shows a great performance of GARCH$_{jsu}$, GARCH$_{sged}$, and GARCH$_{ged}$ among most of the risk measures for, respectively, 250, 500, and 1,000 observations (for instance, respectively VaR$^{5\%}$ of 1.2089, RVaR$^{1\%,5\%}$ of 1.7056, and RVaR$^{2.5\%,5\%}$ of 1.5893). Müller e Righi (2018) also observes that the performance of competing models is similar for predicting VaR, ES and EVaR, according to RMSE. The Bias results for Scenario 1 also showed the same models performing great for 250 and 1,000 observations. Scenario 2 shows a great performance of GARCH$_{ged}$, GARCH$_{norm}$, and GARCH$_{norm}$ for, respectively, 250, 500, and 1,000 observations (for instance, respectively RVaR$^{1\%,5\%}$ of 1.6739, RVaR$^{2.5\%,5\%}$ of 1.5743, and VaR$^{5\%}$ of 1.4564). The Bias result was the same. Scenario 3 shows a great performance of GARCH$_{snorm}$, GARCH$_{snorm}$, and GARCH$_{jsu}$ for, respectively, 250, 500, and 1,000 observations (for instance, respectively RVaR$^{1\%,5\%}$ of 1.6592, ES$^{5\%}$ of 1.7447, and VaR$^{5\%}$ of 1.2684). The Bias result was equal. Finally, Scenario 4 shows a great performance of GARCH$_{norm}$, GARCH$_{jsu}$, and GARCH$_{jsu}$ for, respectively, 250, 500, and 1,000 observations (for instance, respectively RVaR$^{1\%,5\%}$ of 1.7220, VaR$^{5\%}$ of 1.3222 and 1.3757). The Bias result was the same for 250 and 1,000 observations. We can conclude that for both metrics, Bias and RMSE, the results were in most cases the same, indicating a greater solidity of these results.

Finally, the realized loss analysis. Tables 7, 12, 17, and 22 show, respectively, this metric for Scenarios 1, 2, 3, and 4. Scenario 1 shows a great performance of GARCH$_{std}$, GARCH$_{norm}$, and GARCH$_{sged}$ for, respectively, 250, 500, and 1,000 observations. Both Bias and R. Bias had different models with great performance for the same number of observations. Scenario 2 shows a great performance of GARCH$_{snorm}$, GARCH$_{norm}$, and GARCH$_{sged}$ for, respectively, 250, 500, and 1,000 observations. GARCH$_{norm}$ had a great performance in both Bias and R. Bias for 500 observations too, however different for the other observation's sizes. Scenario 3 shows a great performance of GARCH$_{std}$, GARCH$_{ged}$, and GARCH$_{jsu}$ for, respectively, 250, 500, and 1,000 observations. GARCH$_{jsu}$ had a great performance in both Bias and R. Bias for 1,000 observations too, however different for the other observation's sizes. We can note that, with a few exceptions, the Bias and R. Bias results showed themselves quite different from the ones of the realized losses. This result is justified because the previous metrics are applied under the real risk value and the risk forecasts. In contrast, the loss functions are applied under the returns for the out-of-sample period and the risk forecasts.

Observing individually each risk measure, we chose as the best model for each risk measure those that had more occurrences performing as the best models. For example, if GARCH_{ged} performs as the best model for an $\text{ES}^{1\%}$, it counts as an occurrence. Table 33 summarises the risk measures by the number of occurrences. One can observe that GARCH_{jsu} was the best model overall with 96 occurrences, followed by GARCH_{norm} with 93. Observing RVaR behavior, GARCH_{jsu} was the best model forecasting $\text{RVaR}^{1\%,5\%}$ and $\text{RVaR}^{1\%,2.5\%}$. $\text{RVaR}^{2.5\%,5\%}$ was better forecasted by GARCH_{std} with 15 occurrences. Table 33 also shows that GARCH_{jsu} was the best forecasting $\text{RVaR}^{1\%,5\%}$, $\text{RVaR}^{1\%,2.5\%}$, $\text{VaR}^{2.5\%}$, and $\text{VaR}^{5\%}$. Also, GARCH_{norm} was the best forecasting $\text{VaR}^{1\%}$, $\text{ES}^{1\%}$, and $\text{ES}^{2.5\%}$. Unfortunately, we were not able to compare the results with other studies in the literature because there are too few ones exploring RVaR.

Table 3 – Scenario 1 - $GARCH_{norm}$ - risk measures results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR_{t+1}^{\alpha}}$			$\mathbf{ES_{t+1}^{\alpha}}$			$\mathbf{RVaR_{t+1}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	1.9956	1.40135	1.6761	2.2915	1.7657	2.0056	1.6344	1.5258	1.8153
	$GARCH_{snorm}$	2.0239	1.4350	1.7073	2.3167	1.7961	2.0338	1.6659	1.5584	1.8452
	$GARCH_{std}$	2.0394	1.4210	1.7032	2.9789	1.9361	2.3576	1.6754	1.5147	1.9434
	$GARCH_{sstd}$	2.0588	1.4338	1.7185	2.3907	1.8190	2.0758	1.6761	1.5622	1.8658
	$GARCH_{ged}$	2.0086	1.4328	1.6995	2.2937	1.7859	2.0180	1.6589	1.5537	1.8342
	$GARCH_{sged}$	2.0333	1.4495	1.7199	2.3225	1.8075	2.0428	1.6787	1.5721	1.8564
	$GARCH_{jsu}$	2.1407	1.0326	1.4561	2.3311	1.7661	2.0203	1.6249	1.5120	1.8132
500 Observations	$GARCH_{norm}$	2.0359	1.4443	1.7178	2.3301	1.8070	2.0459	1.6763	1.5682	1.8564
	$GARCH_{snorm}$	2.0284	1.4341	1.7089	2.3238	1.7985	2.0383	1.6671	1.5586	1.8480
	$GARCH_{std}$	2.0512	1.4305	1.7146	3.0098	1.9520	2.3795	1.6876	1.5245	1.9594
	$GARCH_{sstd}$	2.0796	1.4572	1.7413	2.4065	1.8404	2.0953	1.6989	1.5855	1.8879
	$GARCH_{ged}$	2.0409	1.4477	1.7222	2.3352	1.8114	2.0507	1.6805	1.5721	1.8610
	$GARCH_{sged}$	2.0265	1.4485	1.7161	2.3129	1.8029	2.0360	1.6754	1.5698	1.8514
	$GARCH_{jsu}$	2.2072	1.0882	1.5159	2.4071	1.8316	2.0905	1.6877	1.5727	1.8794
1000 Observations	$GARCH_{norm}$	2.0373	1.4462	1.7195	2.3312	1.8086	2.0472	1.6780	1.5700	1.8579
	$GARCH_{snorm}$	2.0400	1.4411	1.7180	2.3378	1.8083	2.0501	1.6759	1.5665	1.8582
	$GARCH_{std}$	2.0962	1.4686	1.7563	3.0751	1.9987	2.4338	1.7296	1.5637	2.0062
	$GARCH_{sstd}$	2.0789	1.4538	1.7398	2.4043	1.8384	2.0938	1.6969	1.5830	1.8868
	$GARCH_{ged}$	2.0051	1.4163	1.6887	2.2977	1.7774	2.0150	1.6473	1.5397	1.8265
	$GARCH_{sged}$	2.0644	1.4692	1.7446	2.3598	1.8342	2.0743	1.7027	1.5940	1.8839
	$GARCH_{jsu}$	2.1850	1.0898	1.5088	2.4191	1.8436	2.1027	1.6997	1.5845	1.8918

Table 4 – Scenario 1 - $GARCH_{norm}$ - Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	**Model**	$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$ $\alpha = \mathbf{1\%}$	$\alpha = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}$	$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$ $\alpha = \mathbf{1\%}$	$\alpha = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}$	$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$ $\alpha = \mathbf{1\%}, \beta = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}, \beta = \mathbf{5\%}$	$\alpha = \mathbf{1\%}, \beta = \mathbf{2.5\%}$
250 Observations	$GARCH_{norm}$	**1.9720**	1.3836	1.6557	**2.2645**	1.7444	**1.9819**	1.6143	1.5069	1.7934
	$GARCH_{snorm}$	2.0003	1.4173	1.6868	2.2902	**1.7747**	2.0101	1.6459	1.5394	1.8233
	$GARCH_{std}$	2.0158	1.4033	1.6828	2.9524	1.9148	2.3339	1.6554	1.4957	1.9216
	$GARCH_{sstd}$	2.0352	1.4161	1.6980	2.3641	1.7977	2.0520	1.6560	1.5433	1.8440
	$GARCH_{ged}$	1.9850	1.4151	1.6791	2.2672	1.7645	1.9943	1.6389	1.5348	1.8124
	$GARCH_{sged}$	2.0096	1.4318	1.6994	2.2959	1.7861	2.0191	1.6587	1.5532	1.8346
	$GARCH_{jsu}$	2.1171	**1.0149**	**1.4356**	2.3045	1.7448	1.9966	**1.6049**	**1.4930**	**1.7914**
500 Observations	$GARCH_{norm}$	2.0143	1.4286	1.6994	2.3055	1.7877	2.0241	1.6583	1.5513	1.8366
	$GARCH_{snorm}$	2.0068	1.4185	1.6905	2.2992	**1.7792**	2.0166	**1.6492**	1.5417	**1.8282**
	$GARCH_{std}$	2.0295	1.4149	1.6962	2.9852	1.9327	2.3578	1.6696	**1.5076**	1.9396
	$GARCH_{sstd}$	2.0580	1.4416	1.7229	2.3819	1.8211	2.0736	1.6809	1.5686	1.8681
	$GARCH_{ged}$	2.0193	1.4321	1.7038	2.3106	1.7921	2.0290	1.6625	1.5552	1.8412
	$GARCH_{sged}$	**2.0049**	1.4329	1.6977	**2.2884**	1.7836	**2.0143**	1.6574	1.5529	1.8316
	$GARCH_{jsu}$	2.1856	**1.0726**	**1.4975**	2.3825	1.8123	2.0688	1.6698	1.5558	1.8596
1000 Observations	$GARCH_{norm}$	1.9854	1.4003	1.6709	2.2764	1.7591	1.9953	1.6298	1.5229	1.8079
	$GARCH_{snorm}$	1.9881	1.3952	1.6693	2.2830	1.7587	1.9981	1.6277	1.5194	1.8082
	$GARCH_{std}$	2.0443	1.4227	1.7077	3.0202	1.9492	2.3818	1.6814	1.5166	1.9562
	$GARCH_{sstd}$	2.0271	1.4080	1.6911	2.3494	1.7889	2.0418	1.6487	1.5359	1.8368
	$GARCH_{ged}$	**1.9532**	1.3705	1.6400	**2.2428**	**1.7278**	**1.9630**	**1.5991**	**1.4926**	**1.7765**
	$GARCH_{sged}$	2.0126	1.4234	1.6960	2.3050	1.7846	2.0223	1.6545	1.5469	1.8339
	$GARCH_{jsu}$	2.1332	**1.0440**	**1.4601**	2.3642	1.7941	2.0507	1.6515	1.5374	1.8417

Table 5 – Scenario 1 - $GARCH_{norm}$ - R. Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR^{\alpha}_{t+1}}$			$\mathbf{ES^{\alpha}_{t+1}}$			$\mathbf{RVaR^{\alpha,\beta}_{t+1}}$		
Size	**Model**	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	2.2256	-1.1823	-1.9613	-3.8569	-3.7502	1.7025	-1.2539	3.4792	-5.6123
	$GARCH_{snorm}$	2.2901	1.0841	-1.9726	-4.1290	-3.6923	1.7602	-1.3577	**1.3760**	-5.5525
	$GARCH_{std}$	1.1795	0.5206	-1.7026	-5.1485	-3.4308	1.1699	-1.1100	2.2201	-5.0029
	$GARCH_{sstd}$	1.4011	**-0.0120**	-0.6222	-3.8917	**-2.2253**	**1.0043**	**0.1067**	5.8218	**-3.6780**
	$GARCH_{ged}$	2.9426	-0.0666	-2.2022	-4.0588	-4.1780	2.3363	-1.4847	2.3429	-6.3141
	$GARCH_{sged}$	4.7104	2.2652	-1.7379	**-2.5583**	-4.0357	4.0267	-0.8803	3.8197	-6.5169
	$GARCH_{jsu}$	**-0.5117**	2.1655	**-0.5269**	-3.8070	-2.9010	1.6118	-0.5670	2.9396	-4.6097
500 Observations	$GARCH_{norm}$	-0.7001	-9.3955	0.4247	11.0441	**-7.5552**	-0.8152	1.5546	**-1.6054**	**3.3307**
	$GARCH_{snorm}$	-0.9520	**-3.4192**	0.1275	5.3544	-9.0466	-1.0719	1.5411	-5.3372	3.5112
	$GARCH_{std}$	-0.6752	-5.2810	0.4079	9.9388	-9.9504	-0.7332	1.9043	-4.8819	4.2132
	$GARCH_{sstd}$	**-0.2597**	-5.5252	-1.0000	9.3349	-16.7915	-0.3915	**0.6506**	-5.5913	6.5663
	$GARCH_{ged}$	-0.4194	-7.0204	-0.1555	6.2753	-10.9216	-0.5261	1.2396	-4.6097	4.4513
	$GARCH_{sged}$	-0.7876	-7.0164	**-0.1159**	**3.0853**	-9.9125	-0.8965	1.2320	-4.4036	3.7839
	$GARCH_{jsu}$	-0.6458	-5.6021	0.4960	9.0831	-12.5131	**-0.3086**	1.7216	-3.5355	5.6439
1000 Observations	$GARCH_{norm}$	63.6052	-3.5594	-4.8216	103.9310	1.9733	20.3342	-3.1829	0.7571	-2.7485
	$GARCH_{snorm}$	27.0125	-3.3542	-5.1843	51.8125	5.1598	5.9079	-3.2564	0.6606	**-0.6455**
	$GARCH_{std}$	30.4237	-3.4709	-5.4026	128.4971	2.6516	6.1855	-3.7230	-0.1242	-2.6987
	$GARCH_{sstd}$	36.9020	-2.8593	-6.6213	95.9754	5.5947	8.8118	-4.5176	-1.5262	-1.1914
	$GARCH_{ged}$	29.0220	-2.4121	-3.7675	60.7343	**-0.3944**	4.3116	-2.6838	**-0.0416**	-3.3693
	$GARCH_{sged}$	**23.1736**	-2.7625	-3.5350	**47.3641**	2.2228	**4.2371**	-2.1996	1.1707	-1.5311
	$GARCH_{jsu}$	51.3148	**-2.1604**	**-1.4732**	87.7060	0.6354	7.2746	**-1.7734**	2.0285	-2.2352

Table 6 – Scenario 1 - GARCH$_{norm}$ - RMSE results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\text{VaR}^{\alpha}_{t+1}$			ES^{α}_{t+1}			$\text{RVaR}^{\alpha,\beta}_{t+1}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH$_{norm}$	**1.9989**	1.4725	1.7066	**2.2803**	1.7869	**2.0083**	1.6699	1.5764	1.8319
	GARCH$_{snorm}$	2.0305	1.5012	1.7403	2.3084	1.8204	2.0398	1.7031	1.6075	1.8650
	GARCH$_{std}$	2.0435	1.4872	1.7323	2.9591	1.9474	2.3496	1.7075	1.5660	1.9538
	GARCH$_{sstd}$	2.0589	1.5028	1.7468	2.3758	1.8375	2.0749	1.7092	1.6103	1.8802
	GARCH$_{ged}$	2.0120	1.4903	1.7251	2.2851	1.8040	2.0209	1.6884	1.5948	1.8484
	GARCH$_{sged}$	2.0451	1.5304	1.7628	2.3150	1.8405	2.0539	1.7266	1.6342	1.8845
	GARCH$_{jsu}$	2.2385	**1.2089**	**1.5769**	2.3153	**1.7775**	2.0164	**1.6488**	**1.5485**	**1.8206**
500 Observations	GARCH$_{norm}$	2.0424	1.5115	1.7503	2.3221	1.8306	2.0518	1.7132	1.6182	1.8758
	GARCH$_{snorm}$	2.0391	1.5116	1.7473	2.3190	1.8279	2.0484	1.7100	1.6152	1.8730
	GARCH$_{std}$	2.0550	1.4995	1.7452	2.9871	1.9635	2.3691	1.7211	**1.5784**	1.9700
	GARCH$_{sstd}$	2.0817	1.5201	1.7695	2.3937	1.8595	2.0966	1.7314	1.6307	1.9031
	GARCH$_{ged}$	2.0548	1.5218	1.7606	2.3351	1.8417	2.0640	1.7231	1.6278	1.8874
	GARCH$_{sged}$	**2.0265**	1.5094	1.7417	**2.2989**	**1.8199**	**2.0353**	**1.7056**	1.6131	**1.8643**
	GARCH$_{jsu}$	2.2995	**1.2654**	**1.6348**	2.3982	1.8562	2.0955	1.7276	1.6270	1.8995
1000 Observations	GARCH$_{norm}$	2.0252	1.5017	1.7347	2.3008	1.8142	2.0344	1.6982	1.6055	1.8589
	GARCH$_{snorm}$	2.0327	1.5036	1.7409	2.3144	1.8210	2.0421	1.7041	1.6095	1.8658
	GARCH$_{std}$	2.0826	1.5283	1.7728	3.0289	1.9933	2.4042	1.7495	1.6067	1.9999
	GARCH$_{sstd}$	2.0740	1.5300	1.7695	2.3786	1.8558	2.0878	1.7326	1.6363	1.8990
	GARCH$_{ged}$	**2.0027**	1.4880	1.7173	**2.2755**	**1.7958**	**2.0117**	**1.6814**	**1.5893**	**1.8399**
	GARCH$_{sged}$	2.0597	1.5398	1.7699	2.3377	1.8487	2.0689	1.7337	1.6415	1.8933
	GARCH$_{jsu}$	2.2535	**1.2366**	**1.6035**	2.3715	1.8241	2.0672	1.6936	1.5934	1.8685

Table 7 – Scenario 1 - $GARCH_{norm}$ - Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0435	0.1349	0.0834	1.3339	1.3145	1.3068	0.1782	0.2184	0.1269
	$GARCH_{snorm}$	0.0401	0.1339	0.0826	1.3190	1.2958	1.2992	0.1736	0.2166	0.1227
	$GARCH_{std}$	**0.0254**	**0.1005**	**0.0550**	**0.9621**	**1.0263**	**0.9726**	**0.1257**	**0.1557**	**0.0805**
	$GARCH_{sstd}$	0.0776	0.1914	0.1294	1.9980	1.6340	1.7392	0.2688	0.3210	0.2071
	$GARCH_{ged}$	0.0297	0.1036	0.0590	1.0215	1.0463	1.0081	0.1331	0.1628	0.0887
	$GARCH_{sged}$	0.0418	0.1329	0.0799	1.2444	1.2665	1.2230	0.1745	0.2130	0.1217
	$GARCH_{jsu}$	0.0711	0.1308	0.0945	1.9077	1.3149	1.4454	0.2015	0.2256	0.1656
500 Observations	$GARCH_{norm}$	**0.0329**	0.1240	**0.0717**	1.1303	1.2068	1.1603	**0.1566**	**0.1958**	**0.1046**
	$GARCH_{snorm}$	0.0640	0.1659	0.1093	1.6661	1.4801	1.5282	0.2295	0.2752	0.1733
	$GARCH_{std}$	0.0400	0.1311	0.0784	1.1257	1.2085	**1.1521**	0.1708	0.2096	0.1185
	$GARCH_{sstd}$	0.0370	0.1351	0.0784	**1.1211**	1.2215	1.1650	0.1716	0.2135	0.1154
	$GARCH_{ged}$	0.0502	0.1452	0.0915	1.3645	1.2823	1.2927	0.1952	0.2368	0.1417
	$GARCH_{sged}$	0.0390	**0.1219**	0.0746	1.2879	**1.1988**	1.2072	0.1607	0.1966	0.1136
	$GARCH_{jsu}$	0.1025	0.1924	0.1390	2.7235	1.7480	1.9808	0.2943	0.3318	0.2415
1000 Observations	$GARCH_{norm}$	0.0462	0.1386	0.0860	1.3932	1.2993	1.3094	0.1845	0.2247	0.1322
	$GARCH_{snorm}$	0.0375	0.1267	0.0766	1.2047	1.2215	1.2020	0.1639	0.2033	0.1141
	$GARCH_{std}$	0.0340	0.1234	0.0718	**1.0469**	1.1468	1.0910	0.1571	0.1953	0.1059
	$GARCH_{sstd}$	0.0394	0.1347	0.0801	1.2029	1.2477	1.2122	0.1738	0.2149	0.1196
	$GARCH_{ged}$	0.0351	0.1175	0.0711	1.1658	1.1620	1.1511	0.1524	0.1887	0.1063
	$GARCH_{sged}$	**0.0313**	**0.1169**	**0.0670**	1.0653	**1.1361**	**1.0896**	**0.1478**	**0.1840**	**0.0983**
	$GARCH_{jsu}$	0.0831	0.1626	0.1164	2.0988	1.4751	1.6263	0.2453	0.2794	0.1995

Table 8 – Scenario 2 - GARCH$_{snorm}$ - risk measures results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR^{\alpha}_{t+1}}$			$\mathbf{ES^{\alpha}_{t+1}}$			$\mathbf{RVaR^{\alpha,\beta}_{t+1}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	GARCH$_{norm}$	2.0230	1.4292	1.7038	2.3183	1.7933	2.0330	1.6621	1.5536	1.8428
	GARCH$_{snorm}$	2.1178	1.4810	1.7749	2.4360	1.8715	2.1289	1.7303	1.6140	1.9242
	GARCH$_{std}$	2.0327	1.4100	1.6941	2.9770	1.9282	2.3521	1.6660	1.5043	1.9355
	GARCH$_{sstd}$	2.1647	1.4872	1.7954	2.5254	1.9047	2.1833	1.7495	1.6262	1.9552
	GARCH$_{ged}$	2.0075	1.4328	1.6991	2.2916	1.7851	2.0168	1.6585	1.5535	1.8335
	GARCH$_{sged}$	2.0940	1.4682	1.7574	2.4060	1.8520	2.1048	1.7135	1.5992	1.9040
	GARCH$_{jsu}$	2.8175	1.2337	1.8359	2.5368	1.8788	2.1720	1.7142	1.5855	1.9288
500 Observations	GARCH$_{norm}$	2.0197	1.4283	1.7017	2.3138	1.7909	2.0296	1.6602	1.5522	1.8402
	GARCH$_{snorm}$	2.0980	1.4604	1.7547	2.4165	1.8514	2.1091	1.7101	1.5937	1.9042
	GARCH$_{std}$	2.0789	1.4544	1.7401	3.0419	1.9787	2.4084	1.7129	1.5489	1.9861
	GARCH$_{sstd}$	2.1331	1.4713	1.7732	2.4809	1.8787	2.1498	1.7281	1.6075	1.9291
	GARCH$_{ged}$	2.0218	1.4348	1.7064	2.3131	1.7947	2.0315	1.6651	1.5579	1.8438
	GARCH$_{sged}$	2.0936	1.4665	1.7563	2.4059	1.8510	2.1043	1.7123	1.5978	1.9032
	GARCH$_{jsu}$	3.0873	1.3142	1.9877	2.6204	1.9306	2.2376	1.7582	1.6236	1.9824
1000 Observations	GARCH$_{norm}$	2.0062	1.4127	1.6871	2.3013	1.7766	2.0162	1.6454	1.5370	1.8261
	GARCH$_{snorm}$	2.0768	1.4468	1.7377	2.3914	1.8331	2.0877	1.6936	1.5785	1.8853
	GARCH$_{std}$	2.0589	1.4355	1.7211	3.0279	1.9611	2.3923	1.6944	1.5299	1.9685
	GARCH$_{sstd}$	2.1456	1.4751	1.7812	2.4965	1.8877	2.1621	1.7355	1.6133	1.9391
	GARCH$_{ged}$	2.0213	1.4302	1.7037	2.3147	1.7926	2.0311	1.6621	1.5541	1.8420
	GARCH$_{sged}$	2.0827	1.4510	1.7427	2.3978	1.8383	2.0935	1.6985	1.5831	1.8907
	GARCH$_{jsu}$	3.3439	1.4134	2.1460	2.7055	1.9954	2.3110	1.8179	1.6798	2.0480

Table 9 – Scenario 2 - $GARCH_{snorm}$ - Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		VaR^{α}_{t+1}			ES^{α}_{t+1}			$RVaR^{\alpha,\beta}_{t+1}$		
Size	**Model**	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	1.9825	1.3949	1.6666	2.2747	1.7552	1.9924	1.6254	1.5180	1.8042
	$GARCH_{snorm}$	2.0773	1.4467	1.7377	2.3924	1.8334	2.0883	1.6936	1.5785	1.8855
	$GARCH_{std}$	1.9922	1.3758	**1.6569**	2.9333	1.8901	2.3115	1.6293	**1.4688**	1.8969
	$GARCH_{sstd}$	2.1242	1.4529	1.7582	2.4818	1.8666	2.1427	1.7128	1.5906	1.9166
	$GARCH_{ged}$	**1.9670**	1.3985	1.6620	**2.2480**	**1.7470**	**1.9761**	**1.6218**	1.5180	**1.7949**
	$GARCH_{sged}$	2.0535	1.4339	1.7202	2.3624	1.8139	2.0642	1.6768	1.5636	1.8653
	$GARCH_{jsu}$	2.7770	**1.1994**	1.7987	2.4932	1.8407	2.1314	1.6775	1.5499	1.8902
500 Observations	$GARCH_{norm}$	**1.9835**	1.3984	**1.6690**	2.2744	**1.7572**	**1.9933**	**1.6279**	1.5210	**1.8060**
	$GARCH_{snorm}$	2.0619	1.4306	1.7220	2.3772	1.8177	2.0728	1.6778	1.5625	1.8699
	$GARCH_{std}$	2.0427	1.4246	1.7074	3.0026	1.9450	2.3721	1.6805	**1.5178**	1.9518
	$GARCH_{sstd}$	2.0969	1.4415	1.7404	2.4416	1.8450	2.1135	1.6958	1.5764	1.8948
	$GARCH_{ged}$	1.9856	1.4050	1.6737	**2.2738**	1.7610	1.9952	1.6328	1.5268	1.8095
	$GARCH_{sged}$	2.0575	1.4367	1.7236	2.3665	1.8173	2.0680	1.6800	1.5667	1.8690
	$GARCH_{jsu}$	3.0511	**1.2844**	1.9549	2.5810	1.8969	2.2013	1.7259	1.5925	1.9482
1000 Observations	$GARCH_{norm}$	**1.9663**	**1.3792**	**1.6507**	**2.2583**	**1.7392**	**1.9762**	**1.6094**	1.5022	**1.7882**
	$GARCH_{snorm}$	2.0370	1.4134	1.7013	2.3484	1.7958	2.0478	1.6576	1.5437	1.8474
	$GARCH_{std}$	2.0191	1.4021	1.6847	2.9848	1.9237	2.3523	1.6585	**1.4952**	1.9306
	$GARCH_{sstd}$	2.1058	1.4416	1.7448	2.4535	1.8503	2.1221	1.6995	1.5785	1.9012
	$GARCH_{ged}$	1.9814	1.3968	1.6673	2.2717	1.7553	1.9912	1.6261	1.5194	1.8041
	$GARCH_{sged}$	2.0428	1.4175	1.7063	2.3548	1.8010	2.0536	1.6625	1.5483	1.8528
	$GARCH_{jsu}$	3.3041	1.3800	2.1096	2.6625	1.9580	2.2711	1.7819	1.6450	2.0101

Table 10 – Scenario 2 - $GARCH_{snorm}$ - R. Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR_{t+1}^{\alpha}}$			$\mathbf{ES_{t+1}^{\alpha}}$			$\mathbf{RVaR_{t+1}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	-1.9133	-1.4975	-1.8343	26.9816	-11.1240	**-2.3041**	2.6094	-3.6839	33.7094
	$GARCH_{snorm}$	-2.8711	-1.9168	-1.7301	**19.9907**	-10.6202	-3.3052	3.0219	-4.1254	30.6038
	$GARCH_{std}$	-4.6479	-1.4807	**0.2107**	52.3638	-5.2703	-5.5145	5.3310	-3.7904	11.7115
	$GARCH_{sstd}$	-3.7109	-1.7275	-2.0090	34.5587	-9.7603	-4.1709	1.9300	-3.6691	25.8667
	$GARCH_{ged}$	-2.7211	**-0.7101**	0.8473	20.9168	**-4.3494**	-3.0819	5.5821	-2.9115	13.8385
	$GARCH_{sged}$	-5.3864	-1.1889	-1.1811	32.6594	-4.6794	-5.7859	**1.6085**	**-2.7439**	**6.6597**
	$GARCH_{jsu}$	**-1.6147**	-1.1578	0.8498	26.2275	-5.0177	-2.9761	5.2573	-3.1131	15.5398
500 Observations	$GARCH_{norm}$	-3.3379	-1.3235	6.8029	2.4992	0.2716	-3.5404	-34.9162	-3.1106	-0.6787
	$GARCH_{snorm}$	-3.5213	-1.5666	6.8639	4.8422	0.1808	-3.7314	-34.9639	-3.3182	-0.7863
	$GARCH_{std}$	-4.1157	-2.0325	8.6988	7.1290	0.6593	-4.7130	-45.5822	-4.0484	-0.6428
	$GARCH_{sstd}$	-3.3098	**-1.2837**	**6.3178**	2.5695	0.2421	-3.5231	**-33.5520**	**-2.9269**	-0.6681
	$GARCH_{ged}$	**-2.3115**	-1.7556	11.9921	2.5695	1.9857	**-2.5117**	-53.5713	-4.1408	0.6867
	$GARCH_{sged}$	-2.5941	-2.0378	12.2295	3.1603	1.8463	-2.8124	-50.9874	-4.4254	**0.5230**
	$GARCH_{jsu}$	-6.0839	-1.5983	7.5055	**-0.0973**	**-0.0608**	-4.7373	-36.2000	-3.6032	-1.1405
1000 Observations	$GARCH_{norm}$	4.7696	**-1.2802**	**-1.5461**	12.2105	**21.0984**	2.9130	-0.5098	**-4.9765**	-0.1422
	$GARCH_{snorm}$	3.9602	-2.1662	-2.2596	15.7999	37.0294	2.1844	-0.6816	-7.7964	0.9479
	$GARCH_{std}$	3.1085	-2.2075	-3.3675	15.6941	54.2338	2.2586	-1.7304	-6.6755	2.0789
	$GARCH_{sstd}$	5.4300	-2.8637	-2.2112	10.2403	41.5028	3.2759	**-0.2385**	-10.6398	1.1325
	$GARCH_{ged}$	9.2515	-2.2145	-3.3765	**9.7462**	39.2564	6.0951	-1.9013	-6.4313	**0.0631**
	$GARCH_{sged}$	**1.4071**	-2.0585	-2.1458	29.0917	22.7898	**-0.1675**	-0.8474	-7.0479	-0.8069
	$GARCH_{jsu}$	6.8848	-2.4429	-2.8352	14.8140	47.4911	2.1113	-0.6917	-8.9779	1.8961

Table 11 – Scenario 2 - GARCH_{snorm} - RMSE results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	**Model**	$\mathbf{VaR}_{\mathbf{t+1}}^{\alpha}$			$\mathbf{ES}_{\mathbf{t+1}}^{\alpha}$			$\mathbf{RVaR}_{\mathbf{t+1}}^{\alpha,\beta}$		
		$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	GARCH_{norm}	2.0081	1.4820	1.7169	2.2871	1.7966	2.0175	1.6802	1.5862	1.8414
	GARCH_{snorm}	2.1007	1.5288	1.7846	2.4031	1.8720	2.1111	1.7449	1.6425	1.9205
	GARCH_{std}	2.0170	1.4610	**1.7052**	2.9385	1.9205	2.3257	1.6804	**1.5391**	1.9269
	GARCH_{sstd}	2.1428	1.5311	1.7980	2.4911	1.8980	2.1606	1.7568	1.6482	1.9448
	GARCH_{ged}	**1.9938**	1.4784	1.7100	**2.2655**	**1.7876**	**2.0025**	**1.6739**	1.5815	**1.8316**
	GARCH_{sged}	2.0735	1.5110	1.7642	2.3724	1.8497	2.0836	1.7249	1.6237	1.8971
	GARCH_{jsu}	2.8194	**1.3017**	1.8510	2.4969	1.8608	2.1412	1.7073	1.5901	1.9079
500 Observations	GARCH_{norm}	**2.0059**	1.4647	**1.7097**	2.2887	**1.7917**	**2.0153**	**1.6719**	**1.5743**	**1.8374**
	GARCH_{snorm}	2.0761	1.4894	1.7536	2.3838	1.8431	2.0867	1.7126	1.6066	1.8923
	GARCH_{std}	2.0598	1.4963	1.7468	3.0041	1.9668	2.3798	1.7221	1.5769	1.9733
	GARCH_{sstd}	2.1126	1.5010	1.7735	2.4480	1.8720	2.1286	1.7318	1.6220	1.9193
	GARCH_{ged}	2.0099	1.4846	1.7202	**2.2882**	1.8003	2.0191	1.6835	1.5895	1.8451
	GARCH_{sged}	2.0773	1.5057	1.7612	2.3767	1.8489	2.0874	1.7209	1.6182	1.8977
	GARCH_{jsu}	3.0737	**1.3825**	1.9945	2.5861	1.9201	2.2135	1.7596	1.6379	1.9688
1000 Observations	GARCH_{norm}	**1.9863**	**1.4564**	**1.6910**	**2.2692**	**1.7718**	**1.9958**	**1.6541**	**1.5600**	**1.8173**
	GARCH_{snorm}	2.0559	1.4875	1.7420	2.3562	1.8286	2.0663	1.7023	1.6003	1.8766
	GARCH_{std}	2.0429	1.4840	1.7309	2.9895	1.9522	2.3647	1.7070	1.5632	1.9587
	GARCH_{sstd}	2.1256	1.5230	1.7877	2.4644	1.8840	2.1414	1.7471	1.6406	1.9313
	GARCH_{ged}	2.0077	1.4805	1.7157	2.2866	1.7958	2.0170	1.6789	1.5849	1.8410
	GARCH_{sged}	2.0611	1.4842	1.7419	2.3648	1.8300	2.0714	1.7015	1.5988	1.8792
	GARCH_{jsu}	3.3133	1.4600	2.1282	2.6631	1.9716	2.2755	1.8050	1.6811	2.0215

Table 12 – Scenario 2 - GARCH$_{snorm}$ - Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH$_{norm}$	0.0370	0.1260	0.0745	1.2671	1.2493	1.2352	0.1627	0.2006	0.1116
	GARCH$_{snorm}$	**0.0282**	0.1065	0.0616	**1.0057**	1.1010	**1.0534**	**0.1343**	0.1682	**0.0898**
	GARCH$_{std}$	0.0448	0.1438	0.0886	1.2644	1.3479	1.3058	0.1883	0.2325	0.1334
	GARCH$_{sstd}$	0.0314	**0.1038**	**0.0612**	1.1511	1.0983	1.0900	0.1349	**0.1651**	0.0926
	GARCH$_{ged}$	0.0322	0.1039	0.0644	1.1261	**1.0865**	1.1049	0.1359	0.1685	0.0966
	GARCH$_{sged}$	0.0400	0.1239	0.0769	1.3289	1.2746	1.2870	0.1636	0.2009	0.1169
	GARCH$_{jsu}$	0.0856	0.1926	0.1306	2.2825	1.6240	1.7527	0.2761	0.3233	0.2160
500 Observations	GARCH$_{norm}$	**0.0241**	**0.0962**	**0.0538**	**0.9475**	**1.0223**	**0.9788**	**0.1200**	**0.1501**	**0.0780**
	GARCH$_{snorm}$	0.0334	0.1150	0.0689	1.1195	1.1520	1.1271	0.1481	0.1840	0.1024
	GARCH$_{std}$	0.0532	0.1479	0.0959	1.3106	1.3294	1.3187	0.2009	0.2439	0.1492
	GARCH$_{sstd}$	0.0350	0.1223	0.0706	1.1312	1.1645	1.1211	0.1569	0.1930	0.1056
	GARCH$_{ged}$	0.0388	0.1225	0.0749	1.1944	1.1857	1.1676	0.1610	0.1975	0.1137
	GARCH$_{sged}$	0.0479	0.1362	0.0881	1.3546	1.2906	1.3121	0.1838	0.2244	0.1360
	GARCH$_{jsu}$	0.0425	0.1460	0.0775	1.3057	1.3199	1.2048	0.1854	0.2234	0.1197
1000 Observations	GARCH$_{norm}$	0.0473	0.1335	0.0844	1.4551	1.2855	1.3172	0.1806	0.2180	0.1318
	GARCH$_{snorm}$	0.0598	0.1615	0.1052	1.5337	1.3998	1.4279	0.2209	0.2668	0.1651
	GARCH$_{std}$	0.0523	0.1504	0.0957	1.2468	1.3073	1.2710	0.2025	0.2462	0.1481
	GARCH$_{sstd}$	0.0541	0.1585	0.0990	1.3822	1.3472	1.3306	0.2121	0.2575	0.1530
	GARCH$_{ged}$	0.0421	0.1359	0.0830	1.2501	1.2491	1.2338	0.1777	0.2190	0.1252
	GARCH$_{sged}$	**0.0404**	**0.1276**	**0.0779**	1.2323	**1.2178**	**1.2038**	**0.1676**	**0.2056**	**0.1183**
	GARCH$_{jsu}$	0.0464	0.1574	0.0877	**1.2199**	1.3331	1.2260	0.2001	0.2449	0.1336

Table 13 – Scenario 3 - GARCH$_{std}$ - risk measures results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	**Model**	**VaR$_{t+1}^{\alpha}$** $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	**ES$_{t+1}^{\alpha}$** $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	**RVaR$_{t+1}^{\alpha,\beta}$** $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH$_{norm}$	1.9924	1.4018	1.6749	2.2861	1.7639	2.0024	1.6334	1.5255	1.8132
	GARCH$_{snorm}$	1.9705	1.3933	1.6602	2.2575	1.7472	1.9802	1.6196	1.5142	1.7953
	GARCH$_{std}$	2.1090	1.3599	1.6831	2.9258	1.8964	2.3125	1.6390	1.4803	1.9036
	GARCH$_{sstd}$	2.1230	1.3557	1.6862	2.6432	1.8412	2.1799	1.6407	1.5025	1.8711
	GARCH$_{ged}$	2.1368	1.4140	1.7389	2.5304	1.8591	2.1584	1.6913	1.5598	1.9104
	GARCH$_{sged}$	2.0944	1.3816	1.7020	2.4825	1.8206	2.1157	1.6551	1.5254	1.8711
	GARCH$_{jsu}$	2.3254	1.1523	1.6008	2.3668	1.8019	2.0561	1.6606	1.5477	1.8489
500 Observations	GARCH$_{norm}$	1.9937	1.4094	1.6796	2.2842	1.7677	2.0035	1.6385	1.5318	1.8164
	GARCH$_{snorm}$	1.9577	1.3787	1.6464	2.2457	1.7337	1.9675	1.6057	1.5000	1.7820
	GARCH$_{std}$	2.1893	1.4184	1.7508	3.0147	1.9643	2.3889	1.7017	1.5398	1.9716
	GARCH$_{sstd}$	2.1637	1.3852	1.7205	2.6874	1.8770	2.2198	1.6744	1.5341	1.9081
	GARCH$_{ged}$	2.1860	1.4446	1.7778	2.5895	1.9011	2.2081	1.7290	1.5942	1.9538
	GARCH$_{sged}$	2.1089	1.3830	1.7094	2.5034	1.8299	2.1304	1.6616	1.5295	1.8817
	GARCH$_{jsu}$	2.3449	1.1513	1.6077	2.3304	1.7698	2.0221	1.6297	1.5175	1.8166
1000 Observations	GARCH$_{norm}$	2.0012	1.4175	1.6874	2.2914	1.7754	2.0110	1.6464	1.5398	1.8240
	GARCH$_{snorm}$	2.0262	1.4408	1.7115	2.3173	1.7997	2.0360	1.6703	1.5634	1.8485
	GARCH$_{std}$	2.1631	1.3939	1.7252	2.9751	1.9346	2.3552	1.6745	1.5141	1.9419
	GARCH$_{sstd}$	2.1389	1.3591	1.6948	2.6634	1.8516	2.1950	1.6486	1.5082	1.8827
	GARCH$_{ged}$	2.1550	1.4143	1.7474	2.5574	1.8703	2.1768	1.6986	1.5638	1.9231
	GARCH$_{sged}$	2.1243	1.3943	1.7224	2.5213	1.8437	2.1459	1.6743	1.5415	1.8957
	GARCH$_{jsu}$	2.3548	1.1410	1.6051	2.2962	1.7374	1.9889	1.5977	1.4859	1.7840

Table 14 – Scenario 3 - GARCH$_{std}$ - Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH$_{norm}$	1.9749	1.3920	1.6618	2.2634	1.7493	1.9843	1.6207	1.5143	1.7982
	GARCH$_{snorm}$	**1.9530**	1.3835	1.6471	**2.2348**	**1.7326**	**1.9622**	**1.6070**	1.5030	**1.7804**
	GARCH$_{std}$	2.0915	1.3501	1.6700	2.9031	1.8817	2.2944	1.6264	**1.4691**	1.8886
	GARCH$_{sstd}$	2.1055	1.3459	1.6731	2.6205	1.8266	2.1618	1.6281	1.4913	1.8561
	GARCH$_{ged}$	2.1193	1.4042	1.7258	2.5077	1.8445	2.1404	1.6787	1.5486	1.8955
	GARCH$_{sged}$	2.0769	1.3719	1.6889	2.4598	1.8059	2.0976	1.6424	1.5142	1.8562
	GARCH$_{jsu}$	2.3079	**1.1425**	**1.5877**	2.3441	1.7872	2.0380	1.6480	1.5364	1.8340
500 Observations	GARCH$_{norm}$	1.9368	1.3602	1.6270	2.2221	1.7136	1.9460	1.5864	1.4811	1.7620
	GARCH$_{snorm}$	**1.9008**	1.3294	1.5938	**2.1836**	**1.6796**	**1.9100**	**1.5536**	**1.4493**	**1.7276**
	GARCH$_{std}$	2.1323	1.3691	1.6982	2.9526	1.9102	2.3314	1.6496	1.4890	1.9172
	GARCH$_{sstd}$	2.1067	1.3359	1.6680	2.6253	1.8229	2.1623	1.6223	1.4834	1.8537
	GARCH$_{ged}$	2.1291	1.3954	1.7252	2.5274	1.8470	2.1506	1.6769	1.5435	1.8994
	GARCH$_{sged}$	2.0520	1.3337	1.6568	2.4413	1.7758	2.0729	1.6095	1.4788	1.8272
	GARCH$_{jsu}$	2.2879	**1.1021**	**1.5551**	2.2683	1.7157	1.9646	1.5776	1.4668	1.7622
1000 Observations	GARCH$_{norm}$	**2.0046**	1.4286	1.6952	**2.2897**	1.7816	2.0139	1.6546	1.5494	1.8300
	GARCH$_{snorm}$	2.0297	1.4519	1.7193	2.3156	1.8060	2.0390	1.6786	1.5731	1.8545
	GARCH$_{std}$	2.1666	1.4050	1.7330	2.9734	1.9409	2.3581	1.6827	1.5237	1.9478
	GARCH$_{sstd}$	2.1423	1.3702	1.7026	2.6617	1.8578	2.1979	1.6569	1.5178	1.8887
	GARCH$_{ged}$	2.1585	1.4254	1.7552	2.5557	1.8766	2.1797	1.7068	1.5735	1.9291
	GARCH$_{sged}$	2.1277	1.4054	1.7302	2.5196	1.8500	2.1488	1.6826	1.5512	1.9016
	GARCH$_{jsu}$	2.3583	**1.1521**	**1.6129**	2.2945	**1.7437**	**1.9918**	**1.6060**	**1.4956**	**1.7899**

Table 15 – Scenario 3 - GARCH$_{std}$ - R. Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		VaR^{α}_{t+1}			**ES^{α}_{t+1}**			**$RVaR^{\alpha,\beta}_{t+1}$**		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	GARCH$_{norm}$	-3.7899	3.0262	-5.6524	3.3429	-22.3634	-16.1105	-3.7081	-0.6628	15.6201
	GARCH$_{snorm}$	**-2.3532**	2.6897	-4.1707	2.4518	**-8.0225**	**-8.3766**	-2.6472	-0.3049	5.0712
	GARCH$_{std}$	-3.6629	2.4722	-4.0549	4.4192	-9.3962	-15.0500	-2.5874	**-0.2832**	**3.0900**
	GARCH$_{sstd}$	-4.5029	0.6656	-5.7174	2.1602	-15.0820	-14.1412	-4.1176	-1.7040	7.1949
	GARCH$_{ged}$	-3.8860	2.2556	-5.1569	**-0.0167**	-19.8932	-14.1578	-3.5852	-1.0038	10.3844
	GARCH$_{sged}$	-3.2943	4.5649	**-2.7303**	2.1639	-9.8315	-10.4623	**-1.5830**	0.6082	5.6807
	GARCH$_{jsu}$	-3.6729	**0.1038**	-5.9763	2.1052	-17.6339	-9.3206	-3.9586	-1.6443	8.5648
500 Observations	GARCH$_{norm}$	5.3182	-0.9218	-18.2114	-4.4327	-3.0766	3.7476	-5.3336	-5.5683	21.5851
	GARCH$_{snorm}$	6.3083	-1.4927	-11.1266	**-4.2011**	-19.6278	4.2757	-3.9415	-6.1518	32.0577
	GARCH$_{std}$	5.9115	-3.1131	-10.9261	-6.7931	-3.5363	4.3487	-4.0135	-8.7601	23.8208
	GARCH$_{sstd}$	**4.9500**	**0.5322**	-10.6559	-8.0564	1.7217	3.5828	-3.3999	-11.7511	**17.3450**
	GARCH$_{ged}$	5.5409	-5.4170	**-6.8571**	-10.7163	**-1.0590**	3.7911	**1.6363**	-36.8949	18.1610
	GARCH$_{sged}$	5.7623	-1.4502	-12.3890	-4.4371	-2.3920	4.0299	-5.0550	-5.2776	19.5140
	GARCH$_{jsu}$	5.5232	-1.3637	-15.3756	-4.3543	-4.1207	**2.8305**	-5.7370	**-3.9247**	18.7845
1000 Observations	GARCH$_{norm}$	-10.4924	-0.6385	-3.5818	-1.9622	-0.2999	25.1787	-48.9324	-3.2705	-1.4790
	GARCH$_{snorm}$	**-8.6739**	-0.7337	-0.7553	2.6777	**-0.2976**	**14.2582**	-53.8495	-3.0872	-1.2416
	GARCH$_{std}$	-10.2407	-1.3391	1.2745	3.1479	-0.7086	21.2771	-70.3773	-3.8468	-1.7441
	GARCH$_{sstd}$	-10.9211	-0.5302	**0.2183**	-1.3495	-1.9253	22.5817	-43.0801	-2.5638	-2.7635
	GARCH$_{ged}$	-10.6469	-1.4520	-0.5359	**-0.2756**	-0.9863	28.1870	-59.8276	-3.6933	-2.1148
	GARCH$_{sged}$	-10.2670	**-0.1292**	-1.3168	0.4444	-0.7620	20.9323	**-39.8175**	**-2.2507**	-1.6401
	GARCH$_{jsu}$	-13.0190	-0.4750	-1.8402	-1.2702	0.4266	18.1613	-49.2364	-2.8048	**-0.5644**

Table 16 – Scenario 3 - $GARCH_{std}$ - RMSE results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathbf{VaR}_{t+1}^{\alpha}$ $\alpha=1\%$	$\mathbf{VaR}_{t+1}^{\alpha}$ $\alpha=5\%$	$\mathbf{VaR}_{t+1}^{\alpha}$ $\alpha=2.5\%$	$\mathbf{ES}_{t+1}^{\alpha}$ $\alpha=1\%$	$\mathbf{ES}_{t+1}^{\alpha}$ $\alpha=5\%$	$\mathbf{ES}_{t+1}^{\alpha}$ $\alpha=2.5\%$	$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$ $\alpha=1\%,\beta=5\%$	$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$ $\alpha=2.5\%,\beta=5\%$	$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$ $\alpha=1\%,\beta=2.5\%$
250 Observations	$GARCH_{norm}$	2.0177	1.4943	1.7296	2.2924	1.8085	2.0265	1.6930	1.5993	1.8529
	$GARCH_{snorm}$	**1.9843**	1.4597	1.6962	**2.2555**	**1.7757**	**1.9931**	**1.6592**	1.5650	**1.8206**
	$GARCH_{std}$	2.1241	1.4428	1.7250	2.9157	1.9237	2.3193	1.6849	**1.5451**	1.9302
	$GARCH_{sstd}$	2.1366	1.4333	1.7282	2.6390	1.8717	2.1911	1.6865	1.5613	1.8991
	$GARCH_{ged}$	2.1545	1.4926	1.7821	2.5293	1.8932	2.1747	1.7383	1.6203	1.9413
	$GARCH_{sged}$	2.1053	1.4602	1.7422	2.4741	1.8500	2.1252	1.6998	1.5845	1.8966
	$GARCH_{jsu}$	2.3459	**1.2613**	**1.6494**	2.3533	1.8109	2.0525	1.6820	1.5813	1.8552
500 Observations	$GARCH_{norm}$	1.9658	1.4429	1.6783	2.2399	1.7573	1.9745	1.6416	1.5477	1.8020
	$GARCH_{snorm}$	**1.9493**	1.4342	1.6667	**2.2155**	**1.7447**	**1.9579**	1.6305	1.5379	**1.7885**
	$GARCH_{std}$	2.1586	1.4533	1.7477	2.9642	1.9461	2.3523	1.7031	1.5583	1.9527
	$GARCH_{sstd}$	2.1349	1.4347	1.7240	2.6390	1.8659	2.1884	1.6830	1.5609	1.8945
	$GARCH_{ged}$	2.1560	1.4744	1.7722	2.5436	1.8862	2.1769	1.7273	1.6057	1.9357
	$GARCH_{sged}$	2.0764	1.4187	1.7047	2.4536	1.8152	2.0964	1.6613	1.5443	1.8632
	$GARCH_{jsu}$	2.3180	**1.2288**	**1.6260**	2.2814	1.7568	1.9894	**1.6298**	**1.5296**	1.7997
1000 Observations	$GARCH_{norm}$	**2.0396**	1.5088	1.7499	2.3131	1.8301	2.0485	1.7126	1.6166	1.8752
	$GARCH_{snorm}$	2.0623	1.5232	1.7687	2.3386	1.8502	2.0712	1.7307	1.6328	1.8959
	$GARCH_{std}$	2.1995	1.4951	1.7897	2.9829	1.9841	2.3828	1.7438	1.6006	1.9906
	$GARCH_{sstd}$	2.1744	1.4624	1.7560	2.6792	1.9020	2.2281	1.7140	1.5893	1.9312
	$GARCH_{ged}$	2.1982	1.5145	1.8154	2.5817	1.9290	2.2186	1.7704	1.6475	1.9789
	$GARCH_{sged}$	2.1606	1.4898	1.7839	2.5404	1.8962	2.1808	1.7400	1.6201	1.9451
	$GARCH_{jsu}$	2.3693	**1.2684**	**1.6606**	**2.3082**	**1.7788**	**2.0140**	**1.6518**	**1.5531**	**1.8222**

Table 17 – Scenario 3 - $GARCH_{std}$ - Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathscr{L}_{VaR^\alpha}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathscr{L}_{ES^\alpha}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathscr{L}_{RVaR^{\alpha,\beta}}$ $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	0.0874	0.2102	0.1411	2.1270	1.7517	1.8401	0.2975	0.3515	0.2286
	$GARCH_{snorm}$	0.0374	0.1156	0.0722	1.2336	1.1802	1.1888	0.1529	0.1879	0.1097
	$GARCH_{std}$	**0.0233**	**0.0812**	**0.0487**	**0.9317**	**0.9437**	**0.9362**	**0.1043**	**0.1300**	**0.0721**
	$GARCH_{sstd}$	0.1462	0.3085	0.2216	2.5362	2.1937	2.3077	0.4541	0.5301	0.3678
	$GARCH_{ged}$	0.0566	0.1497	0.0974	1.3991	1.3276	1.3242	0.2061	0.2473	0.1541
	$GARCH_{sged}$	0.0246	0.0995	0.0556	0.9664	1.0707	1.0117	0.1236	0.1552	0.0802
	$GARCH_{jsu}$	0.0469	0.1286	0.0841	1.4203	1.3098	1.3414	0.1747	0.2130	0.1310
500 Observations	$GARCH_{norm}$	0.0567	0.1455	0.0953	1.4846	1.3366	1.3679	0.2019	0.2409	0.1520
	$GARCH_{snorm}$	0.0800	0.1692	0.1183	2.0343	1.5216	1.6543	0.2489	0.2876	0.1983
	$GARCH_{std}$	0.0436	0.1300	0.0816	1.1373	1.2117	1.1733	0.1730	0.2116	0.1252
	$GARCH_{sstd}$	0.0434	0.1411	0.0845	1.1872	1.2747	1.2095	0.1839	0.2256	0.1279
	$GARCH_{ged}$	**0.0369**	**0.1250**	**0.0722**	**1.1042**	**1.1638**	**1.1038**	**0.1613**	**0.1973**	**0.1091**
	$GARCH_{sged}$	0.0530	0.1458	0.0936	1.4303	1.3346	1.3448	0.1983	0.2394	0.1466
	$GARCH_{jsu}$	0.0712	0.1961	0.1293	1.6684	1.6136	1.6323	0.2663	0.3257	0.2004
1000 Observations	$GARCH_{norm}$	0.0529	0.1378	0.0900	1.4293	1.2747	1.3078	0.1904	0.2278	0.1429
	$GARCH_{snorm}$	0.1316	0.2298	0.1753	2.4719	1.7173	1.9231	0.3612	0.4053	0.3070
	$GARCH_{std}$	0.0391	0.1325	0.0809	1.1053	**1.2245**	1.1681	0.1709	0.2134	0.1200
	$GARCH_{sstd}$	0.0412	0.1328	0.0824	1.1849	1.2568	1.2235	0.1735	0.2153	0.1236
	$GARCH_{ged}$	0.0474	0.1383	0.0883	1.2955	1.2632	1.2645	0.1853	0.2267	0.1357
	$GARCH_{sged}$	0.0938	0.2004	0.1418	1.6511	1.4996	1.5314	0.2937	0.3423	0.2356
	$GARCH_{jsu}$	**0.0302**	**0.1281**	**0.0690**	**1.0239**	1.2400	**1.1210**	**0.1571**	**0.1973**	**0.0991**

Table 18 – Scenario 4 - GARCH_{sstd} - risk measures results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH_{norm}	2.0071	1.4254	1.6944	2.2963	1.7821	2.0169	1.6535	1.5473	1.8306
	GARCH_{snorm}	2.0524	1.4241	1.7141	2.3663	1.8094	2.0633	1.6701	1.5554	1.8614
	GARCH_{std}	2.1491	1.3788	1.7101	2.9771	1.9284	2.3523	1.6663	1.5046	1.9357
	GARCH_{sstd}	2.3085	1.4571	1.8228	2.8910	1.9963	2.3733	1.7727	1.6194	2.0281
	GARCH_{ged}	2.1573	1.4224	1.7527	2.5575	1.8750	2.1793	1.7043	1.5707	1.9271
	GARCH_{sged}	2.2035	1.4447	1.7858	2.6160	1.9119	2.2259	1.7359	1.5978	1.9659
	GARCH_{jsu}	2.8599	1.3109	1.9008	2.4669	1.8636	2.1342	1.7128	1.5930	1.9125
500 Observations	GARCH_{norm}	2.0217	1.4327	1.7050	2.3146	1.7938	2.0316	1.6636	1.5561	1.8429
	GARCH_{snorm}	2.0941	1.4694	1.7578	2.4060	1.8525	2.1049	1.7141	1.6000	1.9042
	GARCH_{std}	2.1769	1.4041	1.7369	2.9985	1.9500	2.3738	1.6879	1.5263	1.9573
	GARCH_{sstd}	2.2885	1.4517	1.8110	2.8571	1.9809	2.3506	1.7618	1.6111	2.0129
	GARCH_{ged}	2.1921	1.4435	1.7800	2.5995	1.9044	2.2144	1.7307	1.5945	1.9577
	GARCH_{sged}	2.2382	1.4581	1.8083	2.6640	1.9384	2.2617	1.7571	1.6152	1.9935
	GARCH_{jsu}	2.8020	1.2743	1.8562	2.3683	1.7877	2.0484	1.6425	1.5269	1.8352
1000 Observations	GARCH_{norm}	2.0605	1.4644	1.7400	2.3569	1.8299	2.0705	1.6982	1.5893	1.8796
	GARCH_{snorm}	2.0570	1.4277	1.7182	2.3712	1.8135	2.0679	1.6741	1.5592	1.8657
	GARCH_{std}	2.1806	1.4029	1.7377	2.9992	1.9491	2.3735	1.6866	1.5246	1.9564
	GARCH_{sstd}	2.2757	1.4433	1.8010	2.8375	1.9690	2.3361	1.7519	1.6020	2.0018
	GARCH_{ged}	2.1857	1.4365	1.7732	2.5934	1.8978	2.2080	1.7239	1.5876	1.9510
	GARCH_{sged}	2.2518	1.4637	1.8177	2.6810	1.9489	2.2753	1.7659	1.6225	2.0048
	GARCH_{jsu}	2.8483	1.2955	1.8869	2.3707	1.7913	2.0516	1.6465	1.5310	1.8388

Table 19 – Scenario 4- $GARCH_{sstd}$ - Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathbf{VaR}^{\alpha}_{t+1}$			$\mathbf{ES}^{\alpha}_{t+1}$			$\mathbf{RVaR}^{\alpha,\beta}_{t+1}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	**2,0074**	1,4342	**1.6995**	**2.2909**	**1.7855**	**2.0165**	**1.6591**	1.5544	**1.8337**
	$GARCH_{snorm}$	2.0527	1.4329	1.7192	2.3609	1.8128	2.0630	1.6757	1.5625	1.8644
	$GARCH_{std}$	2.1493	1.3876	1.7152	2.9716	1.9318	2.3519	1.6719	**1.5117**	1.9388
	$GARCH_{sstd}$	2.3088	1.4658	1.8279	2.8855	1.9997	2.3729	1.7783	1.6265	2.0312
	$GARCH_{ged}$	2.1576	1.4312	1.7578	2.5521	1.8784	2.1789	1.7100	1.5778	1.9302
	$GARCH_{sged}$	2.2037	1.4535	1.7909	2.6106	1.9153	2.2256	1.7415	1.6050	1.9689
	$GARCH_{jsu}$	2.8601	**1.3197**	1.9059	2.4614	1.8670	2.1339	1.7184	1.6002	1.9155
500 Observations	$GARCH_{norm}$	**1.9631**	1.3824	**1.6512**	**2.2504**	1.7383	**1.9724**	1.6103	1.5042	1.7871
	$GARCH_{snorm}$	2.0355	1.4192	1.7039	2.3417	1.7969	2.0457	1.6607	1.5481	1.8483
	$GARCH_{std}$	2.1183	1.3538	1.6830	2.9343	1.8945	2.3146	1.6346	**1.4744**	1.9015
	$GARCH_{sstd}$	2.2299	1.4014	1.7572	2.7929	1.9253	2.2914	1.7084	1.5593	1.9570
	$GARCH_{ged}$	2.1335	1.3932	1.7261	2.5352	1.8489	2.1552	1.6773	1.5426	1.9018
	$GARCH_{sged}$	2.1796	1.4078	1.7544	2.5998	1.8829	2.2025	1.7037	1.5633	1.9376
	$GARCH_{jsu}$	2.7434	**1.2240**	1.8023	2.3041	**1.7321**	1.9892	**1.5891**	1.4750	**1.7793**
1000 Observations	$GARCH_{norm}$	2.0559	1.4682	1.7402	**2.3466**	1.8284	2.0653	1.6988	1.5915	1.8777
	$GARCH_{snorm}$	**2.0524**	1.4314	**1.7184**	2.3610	1.8120	2.0627	1.6748	1.5614	1.8638
	$GARCH_{std}$	2.1760	1.4067	1.7378	2.9890	1.9476	2.3683	1.6872	**1.5268**	1.9546
	$GARCH_{sstd}$	2.2711	1.4470	1.8012	2.8273	1.9675	2.3309	1.7526	1.6042	1.9999
	$GARCH_{ged}$	2.1811	1.4402	1.7733	2.5832	1.8963	2.2028	1.7245	1.5897	1.9492
	$GARCH_{sged}$	2.2472	1.4675	1.8179	2.6707	1.9474	2.2701	1.7666	1.6247	2.0030
	$GARCH_{jsu}$	2.8437	**1.2992**	1.8871	2.3605	**1.7898**	**2.0464**	**1.6471**	1.5332	**1.8370**

Table 20 – Scenario 4 - $GARCH_{sstd}$ - R. Bias results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathbf{VaR}_{t+1}^{\alpha}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathbf{ES}_{t+1}^{\alpha}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$ $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	4.5205	17.1511	3.6698	-1.7646	-0.7739	12.7561	8.1657	-2.6869	-1.6191
	$GARCH_{snorm}$	6.4162	17.7736	1.0992	-2.4126	-1.7487	14.1422	3.0166	-0.0087	-2.4907
	$GARCH_{std}$	**3.7772**	14.3644	0.8684	-3.6852	-1.5715	11.0225	2.5140	**-0.0007**	-2.2285
	$GARCH_{sstd}$	6.7498	135.0162	4.9866	**-1.5309**	**0.6198**	22.0684	7.9692	7.2899	**-0.3841**
	$GARCH_{ged}$	6.5914	**11.0814**	1.8268	-2.4811	-0.8405	**9.9434**	4.0163	-0.4281	-1.4274
	$GARCH_{sged}$	8.8369	18.4652	**0.2111**	-2.4451	-3.0770	15.3694	2.0577	-0.3981	-4.0926
	$GARCH_{jsu}$	4.7957	13.1208	-0.2426	-2.4870	-2.3693	10.9019	**1.0096**	-0.0452	-3.0877
500 Observations	$GARCH_{norm}$	**-2.9197**	1.9756	**1.1751**	-1.4631	-55.6327	5.0577	-11.9010	-4.6235	-23.1043
	$GARCH_{snorm}$	-8.8842	2.7600	3.8483	-1.8001	-47.4127	4.2886	**-9.0254**	-3.3185	**0.1498**
	$GARCH_{std}$	-9.8145	2.4525	5.7573	-1.5928	-64.8774	4.7818	-11.2769	-3.8973	1.8350
	$GARCH_{sstd}$	-3.7036	2.8717	1.3735	-1.5638	-87.9778	5.3871	-9.7302	-2.8653	5.4364
	$GARCH_{ged}$	-15.6407	**-0.4657**	2.9701	-2.8951	**-42.3135**	8.0702	-10.6258	-4.1678	-2.8844
	$GARCH_{sged}$	-9.0312	2.4556	3.4042	-0.9732	-57.5996	5.7052	-11.5329	-2.9954	0.7714
	$GARCH_{jsu}$	-9.5709	1.9916	1.3250	**-0.8757**	-54.4308	**2.5416**	-9.3460	**-2.7063**	-5.5108
1000 Observations	$GARCH_{norm}$	**-0.4917**	6.4994	12.2544	**14.9864**	2.6330	**-1.2563**	-8.4010	**5.1781**	2.2318
	$GARCH_{snorm}$	-1.0274	4.3013	**7.4736**	17.3897	1.7661	-1.7577	**-1.1445**	104.4049	1.4472
	$GARCH_{std}$	-2.0372	3.5970	8.5084	25.3761	1.4339	-3.2837	-4.8424	65.7443	0.9880
	$GARCH_{sstd}$	-1.5535	1.0065	8.0667	19.4094	**1.3130**	-2.2957	-4.8299	52.8599	0.9396
	$GARCH_{ged}$	-1.6996	1.0157	10.9523	38.0028	1.9389	-2.7240	-7.3625	115.9788	1.5067
	$GARCH_{sged}$	-1.1165	1.1966	9.8802	18.0726	1.9511	-1.8914	-5.2159	99.3858	1.5684
	$GARCH_{jsu}$	-2.5495	**-0.8469**	9.4859	21.4038	1.1150	-2.7741	-6.4698	87.8039	**0.7582**

Table 21 – Scenario 4 - $GARCH_{sstd}$ - RMSE results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	VaR^{α}_{t+1}			ES^{α}_{t+1}			$RVaR^{\alpha,\beta}_{t+1}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	**2.0473**	1.5253	**1.7583**	**2.3201**	**1.8374**	**2.0561**	**1.7220**	1.6292	**1.8824**
	$GARCH_{snorm}$	2.0953	1.5254	1.7824	2.3911	1.8690	2.1051	1.7424	1.6396	1.9172
	$GARCH_{std}$	2.1820	1.4752	1.7703	2.9861	1.9742	2.3786	1.7303	**1.5849**	1.9808
	$GARCH_{sstd}$	2.3317	1.5564	1.8774	2.8945	2.0375	2.3939	1.8318	1.6945	2.0669
	$GARCH_{ged}$	2.1967	1.5359	1.8246	2.5745	1.9355	2.2171	1.7811	1.6628	1.9833
	$GARCH_{sged}$	2.2344	1.5397	1.8442	2.6292	1.9599	2.2555	1.7985	1.6744	2.0104
	$GARCH_{jsu}$	2.8717	**1.4049**	1.9397	2.4704	1.8932	2.1503	1.7538	1.6449	1.9395
500 Observations	$GARCH_{norm}$	**2.0175**	1.4831	**1.7254**	**2.2931**	1.8062	2.0263	1.6876	1.5907	1.8517
	$GARCH_{snorm}$	2.0820	1.5126	1.7700	2.3763	1.8566	2.0917	1.7300	1.6273	1.9049
	$GARCH_{std}$	2.1534	1.4544	1.7424	2.9543	1.9391	2.3440	1.6981	1.5566	1.9457
	$GARCH_{sstd}$	2.2670	1.4962	1.8164	2.8182	1.9743	2.3267	1.7714	1.6355	2.0042
	$GARCH_{ged}$	2.1833	1.4980	1.7970	2.5727	1.9125	2.2042	1.7517	1.6298	1.9625
	$GARCH_{sged}$	2.2163	1.4994	1.8152	2.6236	1.9347	2.2383	1.7682	1.6389	1.9860
	$GARCH_{jsu}$	2.7548	**1.3222**	1.8380	2.3171	**1.7683**	**2.0109**	**1.6374**	**1.5352**	**1.8122**
1000 Observations	$GARCH_{norm}$	2.1041	1.5604	1.8058	2.3842	1.8881	2.1131	1.7678	1.6701	1.9346
	$GARCH_{snorm}$	**2.0971**	1.5288	**1.7848**	2.3939	1.8709	2.1069	1.7450	1.6427	1.9190
	$GARCH_{std}$	2.2167	1.5070	1.8043	3.0063	1.9989	2.4009	1.7576	1.6127	2.0055
	$GARCH_{sstd}$	2.3121	1.5443	1.8628	2.8557	2.0201	2.3705	1.8176	1.6822	2.0508
	$GARCH_{ged}$	2.2223	1.5433	1.8402	2.6124	1.9536	2.2432	1.7958	1.6749	2.0028
	$GARCH_{sged}$	2.2927	1.5738	1.8897	2.7026	2.0098	2.3146	1.8423	1.7131	2.0614
	$GARCH_{jsu}$	2.8475	**1.3757**	1.9113	**2.3693**	**1.8182**	**2.0630**	**1.6842**	**1.5797**	**1.8629**

Table 22 – Scenario 4 - GARCH_{sstd} - Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Model	$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH_{norm}	0.0441	0.1280	0.0810	1.5389	1.3054	1.3681	0.1720	0.2091	0.1252
	GARCH_{snorm}	0.0721	0.1624	0.1119	2.0199	1.5164	1.6490	0.2343	0.2744	0.1840
	GARCH_{std}	0.1057	0.2143	0.1563	2.1881	1.8756	2.0009	0.3196	0.3708	0.2621
	GARCH_{sstd}	**0.0341**	**0.1242**	**0.0732**	**1.1228**	1.1983	1.1640	**0.1578**	**0.1976**	**0.1073**
	GARCH_{ged}	0.0395	0.1243	0.0734	1.2177	**1.1950**	**1.1637**	0.1640	0.1980	0.1131
	GARCH_{sged}	0.0453	0.1596	0.0980	1.3755	1.4459	1.4199	0.2044	0.2577	0.1433
	GARCH_{jsu}	0.0529	0.1601	0.1018	1.4140	1.3520	1.3497	0.2113	0.2621	0.1546
500 Observations	GARCH_{norm}	**0.0321**	0.1190	0.0694	1.0892	1.1482	1.1148	0.1516	0.1888	0.1017
	GARCH_{snorm}	0.0332	0.1203	0.0705	1.1034	1.1650	1.1256	0.1534	0.1910	0.1037
	GARCH_{std}	0.0332	**0.1034**	**0.0620**	1.0895	**1.0720**	**1.0522**	0.1363	**0.1656**	0.0953
	GARCH_{sstd}	0.0423	0.1333	0.0805	1.1250	1.2046	1.1472	0.1749	0.2139	0.1229
	GARCH_{ged}	0.0292	0.1058	0.0625	**1.0202**	1.0952	1.0594	**0.1352**	0.1687	**0.0919**
	GARCH_{sged}	0.0353	0.1253	0.0723	1.1498	1.2056	1.1537	0.1601	0.1977	0.1076
	GARCH_{jsu}	0.0505	0.1572	0.0937	1.3633	1.4510	1.3706	0.2055	0.2509	0.1440
1000 Observations	GARCH_{norm}	0.05152	0.14255	0.09098	1.44016	1.31316	1.33442	0.19406	0.23375	0.14258
	GARCH_{snorm}	0.04383	0.13301	0.08259	1.29072	1.24931	1.24291	0.17657	0.21572	0.12646
	GARCH_{std}	0.04194	0.13202	0.08030	1.14824	1.22855	1.18067	0.17335	0.21238	0.12225
	GARCH_{sstd}	0.06291	0.16482	0.10768	1.40725	1.40180	1.38526	0.22710	0.27260	0.17060
	GARCH_{ged}	0.05985	0.15432	0.10188	1.50900	1.37868	1.40708	0.21404	0.25643	0.16182
	GARCH_{sged}	**0.04000**	**0.12415**	**0.07668**	**1.14109**	**1.15053**	**1.13091**	**0.16359**	**0.20093**	**0.11670**
	GARCH_{jsu}	0.07854	0.24583	0.15596	1.63983	1.82615	1.75272	0.32214	0.40179	0.23422

Table 23 – Summary of the results of Scenario 1 - $GARCH_{norm}$ by RVaR - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{RVaR_{t+1}^{\alpha,\beta}}$		
Size	**Best Model**	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	**by Bias**	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	by R. Bias	$GARCH_{sstd}$	$GARCH_{snorm}$	$GARCH_{sstd}$
	by RMSE	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	by Loss Functions	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$
500 Observations	**by Bias**	$GARCH_{snorm}$	$GARCH_{std}$	$GARCH_{snorm}$
	by R. Bias	$GARCH_{sstd}$	$GARCH_{norm}$	$GARCH_{norm}$
	by RMSE	$GARCH_{sged}$	$GARCH_{std}$	$GARCH_{sged}$
	by Loss Functions	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
1,000 Observations	**by Bias**	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	by R. Bias	$GARCH_{jsu}$	$GARCH_{ged}$	$GARCH_{snorm}$
	by RMSE	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	by Loss Functions	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$

Table 24 – Summary of the results of Scenario 1 - GARCH$_{norm}$ by VaR and ES - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR^{\alpha}_{t+1}}$			$\mathbf{ES^{\alpha}_{t+1}}$		
Size	**Best Model**	$\alpha = \mathbf{1\%}$	$\alpha = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}$	$\alpha = \mathbf{1\%}$	$\alpha = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}$
	by Bias	GARCH$_{norm}$	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{norm}$	GARCH$_{snorm}$	GARCH$_{norm}$
250	**by R. Bias**	GARCH$_{jsu}$	GARCH$_{sstd}$	GARCH$_{jsu}$	GARCH$_{sged}$	GARCH$_{sstd}$	GARCH$_{sstd}$
Observations	**by RMSE**	GARCH$_{norm}$	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{norm}$	GARCH$_{jsu}$	GARCH$_{norm}$
	by Loss Functions	GARCH$_{std}$	GARCH$_{std}$	GARCH$_{std}$	GARCH$_{std}$	GARCH$_{std}$	GARCH$_{std}$
	by Bias	GARCH$_{sged}$	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{sged}$	GARCH$_{snorm}$	GARCH$_{sged}$
500	**by R. Bias**	GARCH$_{sstd}$	GARCH$_{snorm}$	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{norm}$	GARCH$_{sstd}$
Observations	**by RMSE**	GARCH$_{sged}$	GARCH$_{std}$	GARCH$_{jsu}$	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{sged}$
	by Loss Functions	GARCH$_{norm}$	GARCH$_{sged}$	GARCH$_{norm}$	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{std}$
	by Bias	GARCH$_{ged}$	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{ged}$	GARCH$_{ged}$	GARCH$_{ged}$
1,000	**by R. Bias**	GARCH$_{sged}$	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{sged}$	GARCH$_{ged}$	GARCH$_{sged}$
Observations	**by RMSE**	GARCH$_{sged}$	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{ged}$	GARCH$_{ged}$	GARCH$_{ged}$
	by Loss Functions	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{std}$	GARCH$_{sged}$	GARCH$_{sged}$

Table 25 – Summary of the results of Scenario 2 - $GARCH_{snorm}$ by RVaR - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{RVaR_{t+1}^{\alpha,\beta}}$		
Size	**Best Model**	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
	by Bias	$GARCH_{ged}$	$GARCH_{std}$	$GARCH_{ged}$
250	**by R. Bias**	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$
Observations	**by RMSE**	$GARCH_{ged}$	$GARCH_{std}$	$GARCH_{ged}$
	by Loss Functions	$GARCH_{snorm}$	$GARCH_{sstd}$	$GARCH_{snorm}$
	by Bias	$GARCH_{norm}$	$GARCH_{std}$	$GARCH_{norm}$
500	**by R. Bias**	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{sged}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Loss Functions	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Bias	$GARCH_{norm}$	$GARCH_{std}$	$GARCH_{norm}$
1,000	**by R. Bias**	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{sged}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Loss Functions	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$

Table 26 – Summary of the results of Scenario 2 - $GARCH_{snorm}$ by VaR and ES - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR_{t+1}^{\alpha}}$			$\mathbf{ES_{t+1}^{\alpha}}$		
Size	**Best Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$
	by Bias	$GARCH_{ged}$	$GARCH_{jsu}$	$GARCH_{std}$	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
250	**by R. Bias**	$GARCH_{jsu}$	$GARCH_{ged}$	$GARCH_{std}$	$GARCH_{snorm}$	$GARCH_{ged}$	$GARCH_{norm}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	by Loss Functions	$GARCH_{snorm}$	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{snorm}$	$GARCH_{ged}$	$GARCH_{snorm}$
	by Bias	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{ged}$	$GARCH_{norm}$	$GARCH_{norm}$
500	**by R. Bias**	$GARCH_{ged}$	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{ged}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Loss Functions	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Bias	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
1,000	**by R. Bias**	$GARCH_{sged}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{ged}$	$GARCH_{norm}$	$GARCH_{sged}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Loss Functions	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{jsu}$	$GARCH_{sged}$	$GARCH_{sged}$

Table 27 – Summary of the results of Scenario 3 - GARCH$_{std}$ by RVaR - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$		
Size	**Best Model**	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	**by Bias**	GARCH$_{snorm}$	GARCH$_{std}$	GARCH$_{snorm}$
	by R. Bias	GARCH$_{sged}$	GARCH$_{std}$	GARCH$_{std}$
	by RMSE	GARCH$_{snorm}$	GARCH$_{std}$	GARCH$_{snorm}$
	by Loss Functions	GARCH$_{std}$	GARCH$_{std}$	GARCH$_{std}$
500 Observations	**by Bias**	GARCH$_{snorm}$	GARCH$_{snorm}$	GARCH$_{snorm}$
	by R. Bias	GARCH$_{ged}$	GARCH$_{jsu}$	GARCH$_{sstd}$
	by RMSE	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{snorm}$
	by Loss Functions	GARCH$_{ged}$	GARCH$_{ged}$	GARCH$_{ged}$
1,000 Observations	**by Bias**	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{jsu}$
	by R. Bias	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{jsu}$
	by RMSE	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{jsu}$
	by Loss Functions	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{jsu}$

Table 28 – Summary of the results of Scenario 3 - $GARCH_{std}$ by VaR and ES - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Best Model	VaR^{α}_{t+1} $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	ES^{α}_{t+1} $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$
250 Observations	**by Bias**	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{snorm}$	$GARCH_{snorm}$
	by R. Bias	$GARCH_{snorm}$	$GARCH_{sstd}$	$GARCH_{sged}$	$GARCH_{ged}$	$GARCH_{snorm}$	$GARCH_{snorm}$
	by RMSE	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{snorm}$	$GARCH_{snorm}$
	by Loss Functions	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$
500 Observations	**by Bias**	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{snorm}$	$GARCH_{norm}$
	by R. Bias	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{ged}$	$GARCH_{snorm}$	$GARCH_{ged}$	$GARCH_{jsu}$
	by RMSE	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{snorm}$	$GARCH_{snorm}$
	by Loss Functions	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
1,000 Observations	**by Bias**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	by R. Bias	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{sstd}$	$GARCH_{ged}$	$GARCH_{snorm}$	$GARCH_{snorm}$
	by RMSE	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	by Loss Functions	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$

Table 29 – Summary of the results of Scenario 4 - GARCH$_{sstd}$ by RVaR - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{RVaR_{t+1}^{\alpha,\beta}}$		
Size	**Best Model**	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	**by Bias**	GARCH$_{norm}$	GARCH$_{std}$	GARCH$_{norm}$
	by R. Bias	GARCH$_{jsu}$	GARCH$_{std}$	GARCH$_{sstd}$
	by RMSE	GARCH$_{norm}$	GARCH$_{norm}$	GARCH$_{norm}$
	by Loss Functions	GARCH$_{sstd}$	GARCH$_{sstd}$	GARCH$_{sstd}$
500 Observations	**by Bias**	GARCH$_{jsu}$	GARCH$_{std}$	GARCH$_{jsu}$
	by R. Bias	GARCH$_{snorm}$	GARCH$_{jsu}$	GARCH$_{snorm}$
	by RMSE	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{jsu}$
	by Loss Functions	GARCH$_{ged}$	GARCH$_{std}$	GARCH$_{ged}$
1,000 Observations	**by Bias**	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{jsu}$
	by R. Bias	GARCH$_{snorm}$	GARCH$_{norm}$	GARCH$_{jsu}$
	by RMSE	GARCH$_{jsu}$	GARCH$_{jsu}$	GARCH$_{jsu}$
	by Loss Functions	GARCH$_{sged}$	GARCH$_{sged}$	GARCH$_{sged}$

Table 30 – Summary of the results of Scenario 4 - $GARCH_{sstd}$ by VaR and ES - Bias, R. Bias, RMSE, and Loss Functions results of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR^{\alpha}_{t+1}}$			$\mathbf{ES^{\alpha}_{t+1}}$		
Size	**Best Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$
	by Bias	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
250	**by R. Bias**	$GARCH_{norm}$	$GARCH_{ged}$	$GARCH_{sged}$	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{ged}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	by Loss Functions	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{ged}$	$GARCH_{ged}$
	by Bias	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$
500	**by R. Bias**	$GARCH_{norm}$	$GARCH_{ged}$	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{ged}$	$GARCH_{jsu}$
Observations	**by RMSE**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{std}$	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	by Loss Functions	$GARCH_{norm}$	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{ged}$	$GARCH_{std}$	$GARCH_{std}$
	by Bias	$GARCH_{snorm}$	$GARCH_{std}$	$GARCH_{snorm}$	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$
1,000	**by R. Bias**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{norm}$	$GARCH_{sstd}$	$GARCH_{norm}$
Observations	**by RMSE**	$GARCH_{snorm}$	$GARCH_{std}$	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	by Loss Functions	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$

Table 31 – Summary of the results of all Scenarios by RVaR - of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{RVaR_{t+1}^{\alpha,\beta}}$		
Scenario	**Size**	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
1	**250**	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}
	500	Inconclusive	GARCH_{norm} and GARCH_{std}	GARCH_{norm}
	1,000	GARCH_{ged}	GARCH_{ged}	GARCH_{ged}
2	**250**	GARCH_{ged}	GARCH_{std}	GARCH_{ged}
	500	GARCH_{norm}	GARCH_{norm}	GARCH_{norm}
	1,000	GARCH_{norm}	GARCH_{norm}	GARCH_{norm} and GARCH_{sged}
3	**250**	GARCH_{snorm}	GARCH_{std}	GARCH_{norm} and GARCH_{std}
	500	GARCH_{ged}	GARCH_{jsu}	GARCH_{snorm}
	1,000	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}
4	**250**	GARCH_{norm}	GARCH_{std}	GARCH_{norm} and GARCH_{sstd}
	500	GARCH_{jsu}	GARCH_{std} and GARCH_{jsu}	GARCH_{jsu}
	1,000	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}

Table 32 – Summary of the results of all Scenarios by VaR and ES - of the 1,000 simulations, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

		$\mathbf{VaR^{\alpha}_{t+1}}$			$\mathbf{ES^{\alpha}_{t+1}}$		
Scenario	**Size**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$
1	**250**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{norm}$	Inconclusive	$GARCH_{norm}$
	500	$GARCH_{sged}$	Inconclusive	$GARCH_{jsu}$	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$
	1,000	$GARCH_{sged}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{sged}$
2	**250**	Inconclusive	Inconclusive	$GARCH_{std}$	$GARCH_{snorm}$ and $GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	500	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	1,000	$GARCH_{norm}$ and $GARCH_{sged}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$ and $GARCH_{sged}$
3	**250**	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{snorm}$	$GARCH_{snorm}$
	500	$GARCH_{snorm}$	$GARCH_{jsu}$	$GARCH_{ged}$ and $GARCH_{jsu}$	$GARCH_{snorm}$	$GARCH_{snorm}$ and $GARCH_{ged}$	Inconclusive
	1,000	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
4	**250**	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$	$GARCH_{norm}$ and $GARCH_{sstd}$	$GARCH_{norm}$	$GARCH_{norm}$
	500	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{norm}$ and $GARCH_{std}$	$GARCH_{norm}$ and $GARCH_{sstd}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	1,000	$GARCH_{snorm}$	$GARCH_{std}$	$GARCH_{snorm}$	$GARCH_{norm}$	$GARCH_{jsu}$	$GARCH_{jsu}$

Table 33 – The number of occurrences of each model as the best by risk measure, using a significance level of 1%, 2.5%, and 5%.
*: Indicates the model with more ocurrences applied to a risk measure.
**: Indicates models with no ocurrences applied to a risk measure.

Model	$\mathbf{VaR}_{\mathbf{t+1}}^{\alpha}$			$\mathbf{ES}_{\mathbf{t+1}}^{\alpha}$			$\mathbf{RVaR}_{\mathbf{t+1}}^{\alpha,\beta}$			Total
	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$	
$GARCH_{norm}$	**19***	4	11	**12***	9	**12***	9	8	9	93
$GARCH_{snorm}$	9	1	3	7	8	7	7	2	8	52
$GARCH_{std}$	2	7	7	3	4	4	2	**15***	3	47
$GARCH_{sstd}$	3	6	4	3	3	2	4	3	4	32
$GARCH_{ged}$	5	4	2	11	**11***	8	7	4	7	59
$GARCH_{sged}$	7	4	6	6	5	7	7	5	6	53
$GARCH_{jsu}$	3	**22***	**15***	6	8	8	**12***	11	**11***	96

4.2 Empirical Analysis

4.2.1 Univariate Results

In this subsection, we present the results obtained by both the univariate approach, analysing the risk measures individually for each asset, and their respective realized losses. Table 34 presents the descriptive statistics of the daily returns of Ibovespa, S&P 500, U.S. Treasury Bonds rates for 5 and 10 years, Euro, Real, WTI Crude Oil, and Bitcoin from September 2014 to February 2020.

Table 34 – Descriptive statistics of the daily returns of Ibovespa, S&P 500, U.S. Treasury Bonds rates for 5 and 10 years, Euro, Real, WTI Crude Oil, and Bitcoin from September 2014 to February 2020.

Asset	Mean	Minimum	Maximum	Standard Deviation	Skewness	Kurtosis
Ibovespa	0.0006	-0.0880	0.0660	0.0145	-0.0428	4.8100
S&P 500	0.0003	-0.0442	0.0496	0.0086	-0.5516	6.8417
Treasure 5-years	-0.0003	-0.1752	0.1262	0.0274	-0.1482	6.4303
Treasure 10-years	-0.0003	-0.1324	0.1128	0.0211	0.1011	5.8131
Euro	-0.0001	-0.0254	0.0285	0.0052	0.0594	5.7110
Real	0.0005	-0.0580	0.0752	0.0110	0.3950	7.2634
WTI Crude Oil	-0.0006	-0.1023	0.1118	0.0225	0.2113	5.2247
Bitcoin	0.0023	-0.2547	0.2720	0.0389	0.2082	9.1618

The highest average returns in the period belong to Bitcoin (0.0023) and the lowest to the WTI Crude Oil (-0.0006), however in general the mean of the assets is close to 0, and that was expected. To the Bitcoin also belong the lowest minimum (-0.2547) and the highest maximum (0.2720) returns, consequently the highest volatility (0.0389), as we can see in the Standard Deviation column. Considering the Bitcoin's results, we expected that this asset would have the highest risk measures values overall. On the other hand, although WTI Crude Oil had the lowest average returns, the lowest volatility belongs to Euro (0.0052). S&P 500 also had a volatility lower than the Bitcoin's one (0.0086), probably because of the benefit of diversification. The positive skewness indicates a fatter tail on the right, which means frequent small losses and a few extreme gains are common. We used the R package *moments*[6] Komsta (2015) to estimate the kurtosis. We can see that all assets have the kurtosis superior than the expected for series that follow the normal distribution (kurtosis = 3), indicating that all of them are leptokurtic series, standard behavior of return series. Figure 1 illustrates the returns of each one of the assets on the period from September 2014 to February 2020 and is good illustration of the the high dispersion of the Bitcoin's returns and the low dispersion of the Euro's returns.

[6] This package applies the estimator of Pearson's measure to calculate the kurtosis of a time series, that is, to find the excess of kurtosis of the values, one must subtract 3 from those.

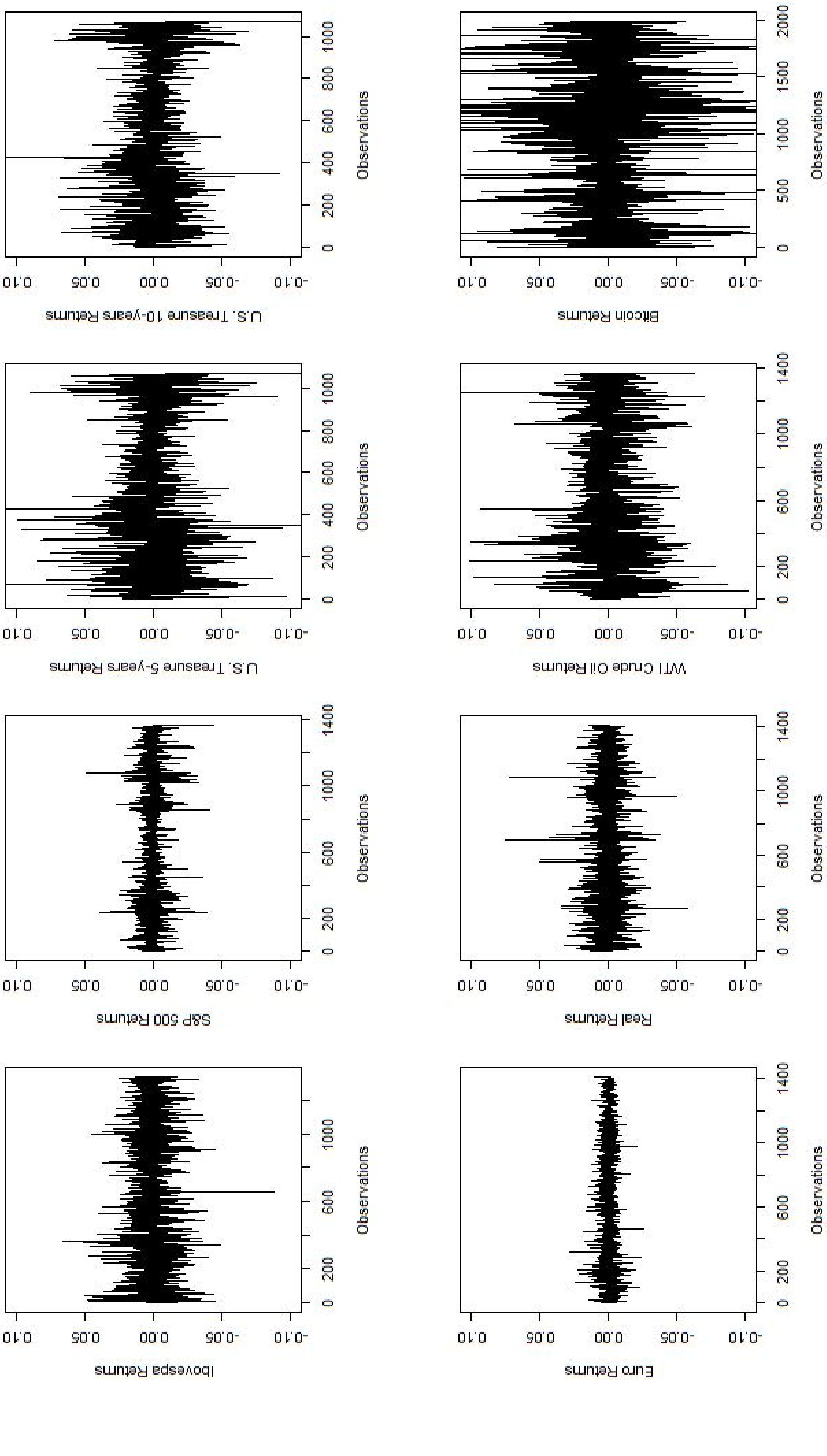

Figure 1 – Daily returns of Ibovespa, S&P 500, U.S. Treasury Bonds rates for 5 and 10 years, Euro, Real, WTI Crude Oil, and Bitcoin from September 2014 to February 2020, totalizing an average of 1662 observations.

Tables 35, 37, 39, 41, 43, 45, 47, and 49 describe the average risk forecasts of, respectively, Ibovespa, S&P 500, the 5 and 10-years Treasury Yield, Euro, Real, WTI Crude Oil, and Bitcoin. We may observe that for all assets the inequality 2.7 is valid. For instance, Bitcoin's $\text{RVaR}^{1\%,5\%}$ with normal distribution of 0.0508, 0.0694, and 0.1059, respectively for 250, 500, and 1,000 observations, smaller than $\text{VaR}^{1\%}$ of 0.0620, 0.0846, and 0.1293, and higher than Bitcoin's $\text{VaR}^{5\%}$ of 0.0436, 0.0596, and 0.0907 for the same rolling estimation window sizes. This result indicates that $\text{RVaR}^{1\%,5\%}$ results in greater protection than $\text{VaR}^{5\%}$ and is less stringent than $\text{VaR}^{1\%}$. It is also worth highlighting about RVaR that we may observe $\text{RVaR}^{2.5\%,5\%}$ has the overall lowest average values for this risk measure. For example, S&P 500 $\text{RVaR}^{2.5\%,5\%}$ with normal distribution of 0.0075, 0.0132, and 0.0158, respectively for 250, 500, and 1,000 observations, smaller than $\text{RVaR}^{1\%,5\%}$ of 0.0081, 0.0142, and 0.0169, which, in turn, is smaller than $\text{RVaR}^{1\%,2.5\%}$ of 0.0090, 0.0158, and 0.0189 for the same rolling window sizes. The $\text{RVaR}^{2.5\%,5\%} < \text{RVaR}^{1\%,5\%} < \text{RVaR}^{1\%,2.5\%}$ inequality occurs over all the results presented in this subsection.

Making a comparison with the descriptive statistics of Table 34, one can observe that the highest average risk measure values for all assets are the Bitcoin's ones, that is $\text{ES}^{1\%}$ of 0.1156, 0.1394, and 0.2276, respectively for 250, 500, and 1,000 observations, all with the GARCH_{sstd} model, corroborating with the statistics of Table 34. Again, the minimum average values among all the risk measures belong to Euro's $\text{VaR}^{5\%}$ of 0.0049, to S&P 500 $\text{VaR}^{5\%}$ of 0.0115, and to Euro's $\text{VaR}^{5\%}$ of 0.0050, respectively for 250, 500, and 1,000 observations, corroborating over again with the values of Table 34. Also, it is important to highlight the RVaR average values compared to the standard deviation ones for each asset. Considering $\text{RVaR}^{2.5\%,5\%}$, we observe that the overall lowest values follow this inequality: S&P 500 (0.0069) - GARCH_{sstd} < Euro (0.0080) - GARCH_{sstd} < Real (0.0169) - GARCH_{sstd} < Ibovespa (0.0190) - GARCH_{std} < 10-years Treasury Yield (0.0244) - GARCH_{sstd} < WTI Crude Oil (0.0250) - GARCH_{std} < 5-years Treasury Yield (0.0254) - GARCH_{std} < Bitcoin (0.0368) - GARCH_{sstd}, whereas Table 34 that, considering the average standard deviation of all assets, we have this inequality: S&P 500 (0.0086) < Ibovespa (0.0145) < 10-years Treasury Yield (0.0211) < WTI Crude Oil (0.0225) < Euro (0.0052) < Real (0.0110) < 5-years Treasury Yield (0.0274) < Bitcoin (0.0389). We highlighted the models which evaluated the lowest values for $\text{RVaR}^{2.5\%,5\%}$ to show GARCH_{sstd} and GARCH_{std} occurred over all assets. Zhou e Anderson (2012) had the same results with $\text{VaR}^{1\%}$ and $\text{VaR}^{5\%}$ forecasts, and, considering Inequality 2.7, that is perfectly valid for RVaR too, as it is an interpolation of two different quantile VaRs.

We divided the realized losses[7] by each class of assets, the rolling window size applied, and the risk measures $\text{VaR}^{1\%}$, $\text{VaR}^{2.5\%}$, $\text{VaR}^{5\%}$, $\text{ES}^{1\%}$, $\text{ES}^{2.5\%}$, $\text{ES}^{5\%}$, $\text{RVaR}^{1\%,5\%}$, $\text{RVaR}^{2.5\%,5\%}$, and $\text{RVaR}^{1\%,2.5\%}$. Tables 36, 38, 40, 42, 44, 46, 48, and 50 show the realized loss of the risk forecasts for, respectively, Ibovespa, S&P 500, the 5 and 10-years Treasury Yield, Euro, Real, WTI Crude Oil, and Bitcoin. Initially, the index markets. Observing Ibovespa, the rolling window

[7] The results were multiplied by 100 for a better analysis.

of 1,000 observations shows a great performance of the GARCH$_{sged}$ model. As for S&P 500, the 250 rolling window had GARCH$_{ged}$ model as the best, while the 500 and 1,000 windows had the GARCH$_{norm}$ as the best model. Zhou e Anderson (2012) and Garcia-Jorcano e Novales (2019) had good results with GARCH$_{sstd}$ for financial returns observing, respectively, VaR$^{1\%}$ and VaR$^{5\%}$, and VaR$^{1\%}$. As for the fixed income, the results for the 5-years Treasury Yield shows that the loss functions of 250 observations had a great performance of the GARCH$_{ged}$ model, although for the 500 and 1,000 rolling windows GARCH$_{norm}$ model had the best results. As for the 10-years Treasury Yield, the 250 rolling window had a great performance of the GARCH$_{ged}$ model, while for 500 and 1,000 observations the GARCH$_{jsu}$ model had the best performance. The third class of assets is exchange, for Euro, the realized loss show a great performance of the GARCH$_{std}$ model for 250 observations, the GARCH$_{sged}$ one for 500 observations, and GARCH$_{ged}$ for 1,000 observations. As for Real the realized loss show a great performance of the GARCH$_{ged}$ for 250 and 1,000 observations. Diaz et al. (2017) had good results with the GARCH$_{norm}$ model for exchange observing VaR$^{5\%}$, although Trucíos et al. (2019) analyzes only cryptocurrencies, they had also great performance of GARCH$_{ged}$ observing VaR$^{2.5\%}$ and VaR$^{5\%}$. The fourth class is the commodities, the WTI Crude Oil realized loss present a great performance of the GARCH$_{sstd}$ model for 250 observations, GARCH$_{std}$ as the best one for 500 observations, and GARCH$_{jsu}$ for 1,000 observations. Diaz et al. (2017) also had good results with the GARCH$_{norm}$ model for commodities observing VaR$^{5\%}$. Finally, analyzing the class of cryptocurrencies, the Bitcoin realized loss present a great performance of the GARCH$_{jsu}$ for 250 observations, while the GARCH$_{snorm}$ model as the best for 1,000 observations. Trucíos et al. (2019) had excellent results applying GARCH$_{ged}$ with the Bitcoin, for VaR$^{2.5\%}$, VaR$^{5\%}$, ES$^{2.5\%}$, and ES$^{5\%}$, and good results with GARCH$_{sstd}$ in general analyzing cryptocurrencies for the same risk measures and significance levels.

We can conclude that different distributions usually forecast better different series. For instance, in the exchange market, GARCH$_{ged}$ performed better for Real, whereas GARCH$_{std}$, GARCH$_{sged}$, and GARCH$_{ged}$ for Euro. Observing Table 51 and 52, we can note that this is valid for all risk measures and significance intervals. Also, for this market, different distributions forecast better in different rolling windows, as one can see in Euro. Furthermore, even at the same class of assets, some models perform better for one asset than for another, as we can see, for instance, on the index market's class, for Ibovespa GARCH$_{sged}$, and GARCH$_{ged}$ performed better, whereas for S&P 500 GARCH$_{ged}$, and GARCH$_{norm}$ performed better. On the other hand, Righi e Ceretta (2013) had GARCH$_{sstd}$ playing an important role on ES stimation with ES$^{1\%}$ and ES$^{5\%}$. Also, the authors observed that commodities are the most sensitive regarding the estimation window because the best model changes conform to distinct past observations and are used to predict the future. Observing Table 52, one can note that, in fact, WTI Crude Oil best models changed quite often with different rolling windows for a same risk measure.

Making an individual analysis of the models behavior compared to the risk measures, Table 51 summarises the best models observing RVaR$^{1\%,5\%}$, RVaR$^{2.5\%,5\%}$, and RVaR$^{1\%,2.5\%}$

realized loss of each asset by rolling window, while Table 52 summarises the same, but for $\text{VaR}^{1\%}$, $\text{VaR}^{2.5\%}$, $\text{VaR}^{5\%}$, $\text{ES}^{1\%}$, $\text{ES}^{2.5\%}$, and $\text{ES}^{5\%}$. Like in the numerical study, we chose as the best model for each risk measure those that had more occurrences performing as the best models. Tables 53 summarises the risk measures by the number of occurrences. One can observe that the GARCH_{jsu} hat 55 occurrences, that is, the model identified more times as the best overall. Besides, GARCH_{jsu} showed a great performance forecasting $\text{VaR}^{1\%}$, $\text{VaR}^{2.5\%}$, $\text{VaR}^{5\%}$, $\text{ES}^{1\%}$, $\text{ES}^{2.5\%}$, and $\text{ES}^{5\%}$. Garcia-Jorcano e Novales (2019) also had good results applying the GARCH_{jsu} model for $\text{VaR}^{1\%}$. The other model that stood out was GARCH_{ged}, with 43 occurrences, the second highest overall. Although having a general good performance, GARCH_{ged} also was the best forecasting $\text{RVaR}^{1\%,5\%}$ and $\text{RVaR}^{2.5\%,5\%}$. Trucíos et al. (2019) also had a great performance of GARCH_{ged} with $\text{VaR}^{2.5\%}$, $\text{VaR}^{5\%}$, $\text{ES}^{2.5\%}$, and $\text{ES}^{5\%}$, although analyzing only cryptocurrencies. Like in the numerical study, we are not able to compare the results with other studies in the literature because there are too few ones exploring RVaR. We have found some models with 0 occurrences as the best, pointed out in Table 53, namely GARCH_{norm} for $\text{VaR}^{5\%}$, and GARCH_{snorm} for $\text{ES}^{1\%}$, $\text{RVaR}^{1\%,5\%}$, and $\text{RVaR}^{1\%,2.5\%}$. Even using only GARCH_{norm}, GARCH_{std}, and GARCH_{sstd}, Zhou e Anderson (2012) also had a poor performance of GARCH_{norm} model with $\text{VaR}^{1\%}$ and $\text{VaR}^{5\%}$. Unlike Garcia-Jorcano e Novales (2019), we had an overall poor performance with the skewed distributions in our empirical analysis.

Table 35 – Risk measures forecasts of Ibovespa using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	**Model**	**VaR$_{t+1}^{\alpha}$** $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	**ES$_{t+1}^{\alpha}$** $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	**RVaR$_{t+1}^{\alpha,\beta}$** $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	GARCH$_{norm}$	0.0284	0.0197	0.0238	0.0328	0.0251	0.0286	0.0232	0.0216	0.0258
	GARCH$_{snorm}$	0.0307	0.0208	0.0254	0.0357	0.0269	0.0309	0.0247	0.0229	0.0277
	GARCH$_{std}$	0.0289	0.0169	0.0219	0.0393	0.0248	0.0307	0.0212	0.0190	0.0249
	GARCH$_{sstd}$	0.0315	0.0184	0.0239	0.0414	0.0268	0.0328	0.0231	0.0208	0.0271
	GARCH$_{ged}$	0.0310	0.0183	0.0238	0.0385	0.0261	0.0315	0.0230	0.0207	0.0269
	GARCH$_{sged}$	0.0331	0.0198	0.0256	0.0408	0.0280	0.0336	0.0248	0.0224	0.0288
	GARCH$_{jsu}$	0.0405	0.0173	0.0261	0.0319	0.0238	0.0274	0.0217	0.0201	0.0244
500 Observations	GARCH$_{norm}$	0.0274	0.0191	0.0229	0.0315	0.0242	0.0275	0.0223	0.0208	0.0249
	GARCH$_{snorm}$	0.0286	0.0197	0.0238	0.0330	0.0252	0.0287	0.0232	0.0216	0.0259
	GARCH$_{std}$	0.0294	0.0183	0.0230	0.0406	0.0260	0.0319	0.0223	0.0200	0.0261
	GARCH$_{sstd}$	0.0304	0.0187	0.0237	0.0385	0.0261	0.0313	0.0230	0.0209	0.0265
	GARCH$_{ged}$	0.0300	0.0191	0.0240	0.0360	0.0258	0.0304	0.0233	0.0213	0.0266
	GARCH$_{sged}$	0.0316	0.0200	0.0252	0.0381	0.0271	0.0320	0.0244	0.0223	0.0279
	GARCH$_{jsu}$	0.0382	0.0174	0.0254	0.0333	0.0250	0.0287	0.0230	0.0213	0.0257
1000 Observations	GARCH$_{norm}$	0.0311	0.0218	0.0261	0.0357	0.0275	0.0312	0.0254	0.0237	0.0283
	GARCH$_{snorm}$	0.0319	0.0222	0.0267	0.0368	0.0282	0.0321	0.0260	0.0242	0.0290
	GARCH$_{std}$	0.0331	0.0209	0.0261	0.0458	0.0294	0.0360	0.0253	0.0228	0.0295
	GARCH$_{sstd}$	0.0335	0.0211	0.0264	0.0419	0.0289	0.0344	0.0257	0.0234	0.0294
	GARCH$_{ged}$	0.0338	0.0215	0.0270	0.0406	0.0291	0.0342	0.0262	0.0240	0.0299
	GARCH$_{sged}$	0.0347	0.0220	0.0277	0.0417	0.0298	0.0351	0.0269	0.0246	0.0307
	GARCH$_{jsu}$	0.0396	0.0186	0.0266	0.0364	0.0275	0.0315	0.0252	0.0234	0.0282

Table 36 – Loss functions results of Ibovespa using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0613	0.1897	0.1187	7.0557	9.0510	7.3071	0.2510	0.3084	0.1800
	$GARCH_{snorm}$	0.0533	**0.1835**	0.1104	6.2658	**8.9216**	6.9712	0.2368	**0.2939**	0.1637
	$GARCH_{std}$	0.0617	0.1992	0.1263	7.0759	9.2449	7.6058	0.2608	0.3255	0.1880
	$GARCH_{sstd}$	0.0527	0.1916	0.1163	6.2039	9.0866	7.2055	0.2443	0.3079	0.1691
	$GARCH_{ged}$	0.0539	0.1919	0.1170	6.3224	9.0946	7.2323	0.2458	0.3089	0.1709
	$GARCH_{sged}$	0.0491	0.1866	**0.1095**	5.8467	8.9844	**6.9329**	**0.2357**	0.2961	0.1586
	$GARCH_{jsu}$	**0.0451**	0.1978	0.1101	**5.4670**	9.2176	6.9628	0.2428	0.3078	**0.1552**
500 Observations	$GARCH_{norm}$	**0.0537**	0.1398	0.0898	**6.3129**	8.0276	6.1520	0.1935	0.2296	**0.1435**
	$GARCH_{snorm}$	0.0542	0.1406	0.0900	6.3536	8.0426	6.1619	0.1947	0.2306	0.1442
	$GARCH_{std}$	0.0541	**0.1382**	0.0894	6.3341	**7.9941**	6.1351	**0.1923**	0.2277	**0.1435**
	$GARCH_{sstd}$	0.0549	0.1389	**0.0887**	6.4082	8.0066	**6.1062**	0.1937	**0.2276**	0.1436
	$GARCH_{ged}$	0.0547	0.1396	0.0902	6.4003	8.0229	6.1684	0.1943	0.2299	0.1450
	$GARCH_{sged}$	0.0558	0.1411	0.0913	6.4968	8.0528	6.2093	0.1968	0.2324	0.1470
	$GARCH_{jsu}$	0.0605	0.1390	0.0913	6.9732	8.0108	6.2109	0.1995	0.2303	0.1518
1000 Observations	$GARCH_{norm}$	0.0440	0.1699	0.0955	5.3607	8.6392	6.3758	0.2139	0.2654	0.1395
	$GARCH_{snorm}$	0.0430	0.1686	0.0947	5.2547	8.6128	6.3440	0.2115	0.2633	0.1377
	$GARCH_{std}$	0.0421	0.1732	0.0948	5.1744	8.7055	6.3446	0.2153	0.2679	0.1369
	$GARCH_{sstd}$	0.0421	0.1719	0.0942	5.1665	8.6793	6.3238	0.2139	0.2661	0.1363
	$GARCH_{ged}$	0.0414	0.1690	0.0930	5.0971	8.6200	6.2770	0.2103	0.2620	0.1344
	$GARCH_{sged}$	**0.0413**	**0.1667**	**0.0920**	**5.0921**	**8.5730**	**6.2351**	**0.2079**	**0.2587**	**0.1333**
	$GARCH_{jsu}$	0.0428	0.1913	0.0941	5.2437	9.0760	6.3181	0.2341	0.2853	0.1369

Table 37 – Risk measures forecasts of S&P 500 using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathbf{VaR}_{t+1}^{\alpha}$			$\mathbf{ES}_{t+1}^{\alpha}$			$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0099	0.0069	0.0083	0.0115	0.0088	0.0100	0.0081	0.0075	0.0090
	$GARCH_{snorm}$	0.0104	0.0072	0.0087	0.0121	0.0092	0.0105	0.0084	0.0078	0.0094
	$GARCH_{std}$	0.0122	0.0067	0.0089	0.0161	0.0103	0.0126	0.0088	0.0079	0.0103
	$GARCH_{sstd}$	0.0110	0.0061	0.0080	0.0158	0.0094	0.0119	0.0078	0.0069	0.0093
	$GARCH_{ged}$	0.0123	0.0072	0.0094	0.0155	0.0104	0.0126	0.0091	0.0081	0.0106
	$GARCH_{sged}$	0.0110	0.0064	0.0084	0.0140	0.0093	0.0113	0.0081	0.0072	0.0095
	$GARCH_{jsu}$	0.0116	0.0056	0.0079	0.0124	0.0093	0.0107	0.0085	0.0079	0.0095
500 Observations	$GARCH_{norm}$	0.0175	0.0120	0.0146	0.0202	0.0154	0.0176	0.0142	0.0132	0.0158
	$GARCH_{snorm}$	0.0196	0.0132	0.0162	0.0229	0.0171	0.0198	0.0157	0.0145	0.0177
	$GARCH_{std}$	0.0209	0.0115	0.0153	0.0276	0.0176	0.0216	0.0151	0.0135	0.0176
	$GARCH_{sstd}$	0.0226	0.0125	0.0166	0.0315	0.0192	0.0241	0.0161	0.0142	0.0191
	$GARCH_{ged}$	0.0214	0.0125	0.0163	0.0268	0.0180	0.0218	0.0158	0.0142	0.0185
	$GARCH_{sged}$	0.0226	0.0133	0.0173	0.0283	0.0191	0.0231	0.0168	0.0151	0.0196
	$GARCH_{jsu}$	0.0272	0.0120	0.0178	0.0223	0.0167	0.0192	0.0153	0.0142	0.0171
1000 Observations	$GARCH_{norm}$	0.0208	0.0144	0.0174	0.0240	0.0184	0.0209	0.0169	0.0158	0.0189
	$GARCH_{snorm}$	0.0228	0.0154	0.0188	0.0265	0.0199	0.0229	0.0183	0.0169	0.0205
	$GARCH_{std}$	0.0251	0.0141	0.0185	0.0331	0.0211	0.0260	0.0181	0.0163	0.0212
	$GARCH_{sstd}$	0.0262	0.0147	0.0193	0.0360	0.0222	0.0277	0.0188	0.0167	0.0222
	$GARCH_{ged}$	0.0251	0.0149	0.0194	0.0312	0.0213	0.0256	0.0188	0.0169	0.0218
	$GARCH_{sged}$	0.0256	0.0152	0.0197	0.0317	0.0216	0.0260	0.0191	0.0172	0.0222
	$GARCH_{jsu}$	0.0309	0.0138	0.0203	0.0259	0.0194	0.0223	0.0178	0.0165	0.0200

Table 38 – Loss functions results of S&P 500 using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0595	0.1218	0.0877	6.9473	7.6681	6.0954	0.1812	0.2095	0.1472
	$GARCH_{snorm}$	0.0557	**0.1188**	0.0841	6.5634	**7.6062**	5.9498	0.1745	0.2030	0.1398
	$GARCH_{std}$	0.0498	0.1267	0.0867	5.9630	7.7683	6.0482	0.1765	0.2134	0.1365
	$GARCH_{sstd}$	0.0519	0.1313	0.0916	6.1727	7.8644	6.2485	0.1832	0.2228	0.1435
	$GARCH_{ged}$	0.0479	0.1193	**0.0804**	5.7814	7.6146	**5.7922**	**0.1672**	**0.1997**	**0.1283**
	$GARCH_{sged}$	0.0526	0.1268	0.0874	6.2539	7.7718	6.0794	0.1794	0.2141	0.1399
	$GARCH_{jsu}$	**0.0464**	0.1373	0.0917	**5.6427**	7.9900	6.2559	0.1837	0.2289	0.1381
500 Observations	$GARCH_{norm}$	**0.0175**	0.0612	**0.0366**	**2.7553**	6.4057	**4.0189**	**0.0787**	**0.0977**	**0.0541**
	$GARCH_{snorm}$	0.0196	0.0664	0.0404	2.9676	6.5138	4.1732	0.0860	0.1068	0.0600
	$GARCH_{std}$	0.0209	**0.0600**	0.0382	3.0950	**6.3818**	4.0882	0.0809	0.0982	0.0592
	$GARCH_{sstd}$	0.0226	0.0640	0.0414	3.2645	6.4667	4.2160	0.0866	0.1054	0.0640
	$GARCH_{ged}$	0.0214	0.0631	0.0408	3.1391	6.4474	4.1906	0.0845	0.1039	0.0622
	$GARCH_{sged}$	0.0226	0.0670	0.0433	3.2620	6.5281	4.2933	0.0896	0.1104	0.0660
	$GARCH_{jsu}$	0.0272	0.0623	0.0445	3.7112	6.4291	4.3384	0.0894	0.1068	0.0717
1000 Observations	$GARCH_{norm}$	**0.0984**	0.3655	**0.2145**	**10.4753**	12.5549	10.9756	**0.4637**	0.5800	**0.3128**
	$GARCH_{snorm}$	0.1001	0.3695	0.2172	10.6365	12.6328	11.0809	0.4694	0.5867	0.3173
	$GARCH_{std}$	0.1030	0.3637	0.2174	10.9490	12.5302	11.1086	0.4664	0.5810	0.3203
	$GARCH_{sstd}$	0.1031	0.3639	0.2177	10.9176	12.5302	11.1075	0.4667	0.5815	0.3207
	$GARCH_{ged}$	0.1008	**0.3631**	0.2165	10.7017	**12.5119**	**11.0584**	**0.4637**	0.5796	0.3173
	$GARCH_{sged}$	0.1016	0.3653	0.2179	10.7760	12.5560	11.1115	0.4667	0.5832	0.3196
	$GARCH_{jsu}$	0.1172	0.3559	0.2192	12.1667	12.3790	11.1558	0.4727	**0.5751**	0.3364

Table 39 – Risk measures forecasts of U.S. 5-year Treasury Yield using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		VaR^{α}_{t+1}			**ES^{α}_{t+1}**			**$RVaR^{\alpha,\beta}_{t+1}$**		
Size	**Model**	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	0.0341	0.0236	0.0284	0.0393	0.0300	0.0343	0.0277	0.0258	0.0309
	$GARCH_{snorm}$	0.0342	0.0236	0.0285	0.0395	0.0301	0.0344	0.0278	0.0259	0.0310
	$GARCH_{std}$	0.0367	0.0233	0.0291	0.0518	0.0330	0.0406	0.0283	0.0254	0.0331
	$GARCH_{sstd}$	0.0368	0.0234	0.0292	0.0453	0.0318	0.0376	0.0284	0.0260	0.0324
	$GARCH_{ged}$	0.0368	0.0240	0.0298	0.0438	0.0319	0.0372	0.0289	0.0266	0.0328
	$GARCH_{sged}$	0.0364	0.0237	0.0294	0.0433	0.0315	0.0368	0.0286	0.0263	0.0325
	$GARCH_{jsu}$	0.0436	0.0199	0.0290	0.0419	0.0312	0.0360	0.0285	0.0264	0.0321
500 Observations	$GARCH_{norm}$	0.0881	0.0600	0.0730	0.1020	0.0772	0.0886	0.0710	0.0659	0.0796
	$GARCH_{snorm}$	0.0952	0.0632	0.0779	0.1113	0.0828	0.0958	0.0757	0.0699	0.0855
	$GARCH_{std}$	0.0990	0.0610	0.0772	0.1372	0.0872	0.1074	0.0747	0.0670	0.0876
	$GARCH_{sstd}$	0.1021	0.0625	0.0794	0.1294	0.0876	0.1052	0.0771	0.0700	0.0890
	$GARCH_{ged}$	0.0967	0.0615	0.0773	0.1159	0.0832	0.0978	0.0750	0.0686	0.0857
	$GARCH_{sged}$	0.1008	0.0636	0.0803	0.1210	0.0865	0.1019	0.0779	0.0711	0.0892
	$GARCH_{jsu}$	0.1307	0.0574	0.0854	0.1097	0.0818	0.0943	0.0748	0.0692	0.0841
1000 Observations	$GARCH_{norm}$	0.0924	0.0641	0.0772	0.1064	0.0815	0.0929	0.0752	0.0701	0.0838
	$GARCH_{snorm}$	0.0944	0.0651	0.0786	0.1090	0.0831	0.0949	0.0766	0.0712	0.0855
	$GARCH_{std}$	0.1004	0.0635	0.0795	0.1401	0.0896	0.1100	0.0770	0.0692	0.0899
	$GARCH_{sstd}$	0.1001	0.0633	0.0792	0.1244	0.0865	0.1026	0.0770	0.0704	0.0881
	$GARCH_{ged}$	0.0980	0.0638	0.0792	0.1163	0.0848	0.0989	0.0770	0.0707	0.0873
	$GARCH_{sged}$	0.0986	0.0641	0.0797	0.1170	0.0853	0.0995	0.0774	0.0711	0.0878
	$GARCH_{jsu}$	0.1172	0.0550	0.0788	0.1105	0.0831	0.0954	0.0762	0.0707	0.0853

Table 40 – Loss functions results of U.S. 5-year Treasury Yield using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.1837	0.4181	0.2903	18.9035	13.6944	14.0847	0.6018	0.7084	0.4740
	$GARCH_{snorm}$	0.1848	0.4202	0.2922	19.0143	13.7369	14.1621	0.6050	0.7124	0.4771
	$GARCH_{std}$	0.1599	0.4200	0.2783	16.4463	13.7142	13.5569	0.5799	0.6983	0.4382
	$GARCH_{sstd}$	0.1602	0.4243	0.2826	16.5409	13.8085	13.7465	0.5845	0.7069	0.4428
	$GARCH_{ged}$	0.1585	**0.4093**	**0.2698**	16.4016	**13.5015**	**13.2451**	**0.5678**	**0.6790**	**0.4283**
	$GARCH_{sged}$	0.1625	0.4175	0.2777	16.7902	13.6708	13.5615	0.5799	0.6951	0.4402
	$GARCH_{jsu}$	**0.1460**	0.4871	0.3085	**15.2161**	15.0898	14.7856	0.6330	0.7956	0.4545
500 Observations	$GARCH_{norm}$	**0.0893**	0.3094	**0.1864**	**9.6341**	11.4502	**9.9104**	**0.3985**	**0.4958**	**0.2757**
	$GARCH_{snorm}$	0.0958	0.3240	0.1983	10.2274	11.7431	10.3642	0.4196	0.5223	0.2941
	$GARCH_{std}$	0.0992	0.3139	0.1965	10.5925	11.5577	10.3225	0.4129	0.5104	0.2958
	$GARCH_{sstd}$	0.1021	0.3206	0.2017	10.8263	11.6864	10.5092	0.4224	0.5223	0.3039
	$GARCH_{ged}$	0.0973	0.3166	0.1969	10.3672	11.6001	10.3142	0.4136	0.5134	0.2941
	$GARCH_{sged}$	0.1008	0.3259	0.2040	10.6894	11.7860	10.5847	0.4264	0.5298	0.3048
	$GARCH_{jsu}$	0.1307	**0.2976**	0.2143	13.3767	**11.2316**	10.9532	0.4277	0.5118	0.3449
1000 Observations	$GARCH_{norm}$	**0.0984**	0.3655	**0.2145**	**10.4753**	12.5549	**10.9756**	**0.4637**	0.5800	**0.3128**
	$GARCH_{snorm}$	0.1001	0.3695	0.2172	10.6365	12.6328	11.0809	0.4694	0.5867	0.3173
	$GARCH_{std}$	0.1030	0.3637	0.2174	10.9490	12.5302	11.1086	0.4664	0.5810	0.3203
	$GARCH_{sstd}$	0.1031	0.3639	0.2177	10.9176	12.5302	11.1075	0.4667	0.5815	0.3207
	$GARCH_{ged}$	0.1008	0.3631	0.2165	10.7017	12.5119	11.0584	**0.4637**	0.5796	0.3173
	$GARCH_{sged}$	0.1016	0.3653	0.2179	10.7760	12.5560	11.1115	0.4667	0.5832	0.3196
	$GARCH_{jsu}$	0.1172	**0.3559**	0.2192	12.1667	**12.3790**	11.1558	0.4727	**0.5751**	0.3364

Table 41 – Risk measures forecasts of U.S. 10-years Treasury Yield using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	Model	$\mathbf{VaR}_{t+1}^{\alpha}$			$\mathbf{ES}_{t+1}^{\alpha}$			$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$		
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	0.0334	0.0233	0.0279	0.0385	0.0295	0.0336	0.0272	0.0254	0.0303
	$GARCH_{snorm}$	0.0321	0.0228	0.0271	0.0368	0.0285	0.0323	0.0264	0.0247	0.0293
	$GARCH_{std}$	0.0351	0.0227	0.0281	0.0495	0.0318	0.0390	0.0274	0.0247	0.0319
	$GARCH_{sstd}$	0.0336	0.0222	0.0272	0.0407	0.0293	0.0342	0.0265	0.0244	0.0299
	$GARCH_{ged}$	0.0355	0.0235	0.0289	0.0419	0.0309	0.0358	0.0281	0.0259	0.0317
	$GARCH_{sged}$	0.0336	0.0226	0.0276	0.0394	0.0293	0.0339	0.0268	0.0248	0.0302
	$GARCH_{jsu}$	0.0358	0.0175	0.0245	0.0400	0.0300	0.0345	0.0276	0.0256	0.0309
500 Observations	$GARCH_{norm}$	0.0728	0.0503	0.0607	0.0841	0.0641	0.0732	0.0592	0.0550	0.0660
	$GARCH_{snorm}$	0.0741	0.0509	0.0616	0.0857	0.0651	0.0745	0.0600	0.0557	0.0671
	$GARCH_{std}$	0.0792	0.0507	0.0631	0.1113	0.0712	0.0874	0.0612	0.0551	0.0715
	$GARCH_{sstd}$	0.0770	0.0495	0.0615	0.0949	0.0668	0.0788	0.0598	0.0548	0.0681
	$GARCH_{ged}$	0.0778	0.0515	0.0634	0.0917	0.0677	0.0785	0.0617	0.0569	0.0696
	$GARCH_{sged}$	0.0772	0.0512	0.0630	0.0911	0.0672	0.0779	0.0612	0.0565	0.0692
	$GARCH_{jsu}$	0.0885	0.0425	0.0601	0.0877	0.0660	0.0758	0.0606	0.0563	0.0678
1000 Observations	$GARCH_{norm}$	0.0746	0.0523	0.0626	0.0857	0.0660	0.0750	0.0611	0.0570	0.0679
	$GARCH_{snorm}$	0.0724	0.0513	0.0611	0.0828	0.0642	0.0727	0.0596	0.0557	0.0660
	$GARCH_{std}$	0.0809	0.0522	0.0647	0.1130	0.0728	0.0891	0.0628	0.0566	0.0731
	$GARCH_{sstd}$	0.0751	0.0493	0.0605	0.0917	0.0655	0.0767	0.0590	0.0543	0.0667
	$GARCH_{ged}$	0.0786	0.0522	0.0641	0.0926	0.0684	0.0793	0.0624	0.0576	0.0704
	$GARCH_{sged}$	0.0737	0.0494	0.0604	0.0867	0.0644	0.0744	0.0588	0.0543	0.0662
	$GARCH_{jsu}$	0.0698	0.0374	0.0499	0.0879	0.0667	0.0763	0.0614	0.0572	0.0685

Table 42 – Loss functions results of U.S. 10-years Treasury Yield using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	GARCH_{norm}	0.0822	0.2391	0.1475	9.0815	10.0553	8.4428	0.3213	0.3866	0.2297
	GARCH_{snorm}	0.0869	0.2440	0.1539	9.5387	10.1574	8.7018	0.3309	0.3980	0.2408
	GARCH_{std}	**0.0768**	0.2463	0.1479	**8.5094**	10.1955	8.4424	0.3231	0.3942	0.2247
	GARCH_{sstd}	0.0792	0.2522	0.1528	8.7576	10.3199	8.6445	0.3314	0.4050	0.2320
	GARCH_{ged}	0.0775	**0.2372**	**0.1437**	8.5935	**10.0125**	**8.2837**	**0.3146**	**0.3809**	**0.2212**
	GARCH_{sged}	0.0805	0.2463	0.1498	8.8912	10.1993	8.5291	0.3267	0.3961	0.2303
	GARCH_{jsu}	0.1103	0.3253	0.2080	11.7906	11.8098	10.8358	0.4355	0.5334	0.3183
500 Observations	GARCH_{norm}	**0.0744**	0.2580	0.1555	**8.2557**	10.4243	8.7246	0.3323	0.4135	**0.2299**
	GARCH_{snorm}	0.0758	0.2608	0.1578	8.3873	10.4821	8.8138	0.3366	0.4186	0.2336
	GARCH_{std}	0.0804	0.2599	0.1613	8.8510	10.4742	8.9663	0.3402	0.4212	0.2417
	GARCH_{sstd}	0.0783	0.2544	0.1572	8.6213	10.3586	8.7970	0.3325	0.4116	0.2355
	GARCH_{ged}	0.0791	0.2635	0.1621	8.6954	10.5374	8.9796	0.3425	0.4255	0.2412
	GARCH_{sged}	0.0787	0.2622	0.1611	8.6579	10.5116	8.9430	0.3408	0.4233	0.2398
	GARCH_{jsu}	0.0885	**0.2239**	**0.1543**	9.5454	**9.7572**	**8.6828**	**0.3121**	**0.3782**	0.2427
1000 Observations	GARCH_{norm}	0.0746	0.2678	0.1577	8.2755	10.6205	8.8099	0.3423	0.4256	0.2324
	GARCH_{snorm}	0.0724	0.2653	0.1548	8.0602	10.5694	8.6959	0.3376	0.4201	0.2271
	GARCH_{std}	0.0809	0.2680	0.1627	8.9017	10.6343	9.0195	0.3487	0.4306	0.2436
	GARCH_{sstd}	0.0751	0.2580	0.1537	8.3242	10.4270	8.6595	0.3329	0.4117	0.2288
	GARCH_{ged}	0.0786	0.2668	0.1611	8.6482	10.6028	8.9413	0.3452	0.4278	0.2396
	GARCH_{sged}	0.0737	0.2570	0.1531	8.1916	10.4056	8.6349	0.3306	0.4101	0.2269
	GARCH_{jsu}	**0.0714**	**0.2390**	**0.1424**	**7.9754**	**10.0610**	**8.2336**	**0.3103**	**0.3814**	**0.2137**

Table 43 – Risk measures forecasts of Euro using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0111	0.0079	0.0094	0.0128	0.0099	0.0112	0.0091	0.0085	0.0101
	$GARCH_{snorm}$	0.0104	0.0075	0.0088	0.0118	0.0092	0.0104	0.0086	0.0081	0.0095
	$GARCH_{std}$	0.0121	0.0078	0.0097	0.0167	0.0109	0.0132	0.0094	0.0085	0.0109
	$GARCH_{sstd}$	0.0109	0.0073	0.0088	0.0133	0.0096	0.0111	0.0086	0.0080	0.0097
	$GARCH_{ged}$	0.0120	0.0080	0.0098	0.0142	0.0105	0.0122	0.0095	0.0088	0.0108
	$GARCH_{sged}$	0.0110	0.0075	0.0091	0.0130	0.0097	0.0111	0.0088	0.0082	0.0099
	$GARCH_{jsu}$	0.0082	0.0049	0.0061	0.0121	0.0094	0.0107	0.0088	0.0082	0.0097
500 Observations	$GARCH_{norm}$	0.0083	0.0059	0.0070	0.0095	0.0074	0.0083	0.0068	0.0064	0.0076
	$GARCH_{snorm}$	0.0084	0.0059	0.0071	0.0097	0.0075	0.0085	0.0069	0.0065	0.0077
	$GARCH_{std}$	0.0086	0.0057	0.0070	0.0121	0.0079	0.0096	0.0068	0.0061	0.0079
	$GARCH_{sstd}$	0.0087	0.0057	0.0070	0.0105	0.0076	0.0088	0.0068	0.0063	0.0077
	$GARCH_{ged}$	0.0087	0.0058	0.0071	0.0102	0.0076	0.0087	0.0069	0.0064	0.0078
	$GARCH_{sged}$	0.0088	0.0059	0.0072	0.0104	0.0077	0.0089	0.0070	0.0065	0.0079
	$GARCH_{jsu}$	0.0103	0.0050	0.0070	0.0097	0.0073	0.0084	0.0068	0.0063	0.0075
1000 Observations	$GARCH_{norm}$	0.0096	0.0068	0.0081	0.0110	0.0085	0.0097	0.0079	0.0074	0.0088
	$GARCH_{snorm}$	0.0096	0.0068	0.0081	0.0109	0.0085	0.0096	0.0079	0.0074	0.0087
	$GARCH_{std}$	0.0108	0.0068	0.0085	0.0147	0.0095	0.0116	0.0082	0.0075	0.0096
	$GARCH_{sstd}$	0.0105	0.0067	0.0083	0.0131	0.0091	0.0108	0.0081	0.0074	0.0092
	$GARCH_{ged}$	0.0106	0.0069	0.0086	0.0126	0.0092	0.0107	0.0083	0.0077	0.0094
	$GARCH_{sged}$	0.0104	0.0068	0.0084	0.0123	0.0090	0.0105	0.0082	0.0075	0.0092
	$GARCH_{jsu}$	0.0110	0.0056	0.0076	0.0115	0.0088	0.0100	0.0081	0.0075	0.0090

Table 44 – Loss functions results of Euro using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	Model	$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathscr{L}_{\mathbf{ES}^{\alpha}}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$ $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	0.0197	0.0585	0.0354	2.9818	6.3509	3.9752	0.0782	0.0939	0.0552
	$GARCH_{snorm}$	0.0201	0.0583	**0.0353**	3.0146	6.3478	**3.9712**	0.0784	0.0936	0.0554
	$GARCH_{std}$	**0.0193**	**0.0580**	0.0354	**2.9361**	**6.3411**	3.9742	**0.0773**	**0.0934**	**0.0547**
	$GARCH_{sstd}$	0.0203	0.0591	0.0358	3.0405	6.3637	3.9922	0.0794	0.0949	0.0562
	$GARCH_{ged}$	**0.0193**	0.0582	0.0356	2.9404	6.3443	3.9813	0.0775	0.0937	0.0549
	$GARCH_{sged}$	0.0201	0.0589	0.0359	3.0136	6.3589	3.9935	0.0789	0.0947	0.0559
	$GARCH_{jsu}$	0.0439	0.0829	0.0593	5.3969	6.8599	4.9434	0.1268	0.1422	0.1031
500 Observations	$GARCH_{norm}$	0.0136	0.0412	0.0250	2.3694	5.9912	3.5506	0.0548	0.0662	0.0385
	$GARCH_{snorm}$	0.0132	0.0409	0.0247	2.3339	5.9856	3.5419	0.0542	0.0657	0.0380
	$GARCH_{std}$	0.0132	0.0416	0.0248	2.3257	5.9994	3.5430	0.0547	0.0664	0.0379
	$GARCH_{sstd}$	0.0130	0.0412	0.0246	2.3074	5.9907	3.5362	0.0541	0.0658	0.0376
	$GARCH_{ged}$	0.0132	0.0414	0.0249	2.3310	5.9960	3.5485	0.0546	0.0663	0.0381
	$GARCH_{sged}$	**0.0128**	**0.0408**	0.0247	**2.2934**	**5.9822**	3.5386	**0.0536**	**0.0654**	0.0375
	$GARCH_{jsu}$	0.0129	0.0435	**0.0241**	2.2972	6.0393	**3.5170**	0.0564	0.0676	**0.0370**
1000 Observations	$GARCH_{norm}$	0.0558	0.1181	0.0846	6.5821	7.5923	5.9709	0.1738	0.2026	0.1403
	$GARCH_{snorm}$	0.0564	0.1183	0.0850	6.6456	7.5975	5.9900	0.1747	0.2033	0.1414
	$GARCH_{std}$	0.0458	0.1173	0.0803	5.5801	7.5760	5.7921	**0.1632**	0.1976	**0.1261**
	$GARCH_{sstd}$	0.0481	0.1191	0.0825	5.8051	7.6130	5.8847	0.1671	0.2016	0.1305
	$GARCH_{ged}$	0.0473	**0.1167**	**0.0801**	5.7343	**7.5630**	**5.7886**	0.1640	**0.1968**	0.1274
	$GARCH_{sged}$	0.0493	0.1181	0.0819	5.9319	7.5934	5.8594	0.1674	0.2000	0.1312
	$GARCH_{jsu}$	**0.0446**	0.1356	0.0903	**5.4636**	7.9571	6.2035	0.1802	0.2259	0.1349

Table 45 – Risk measures forecasts of Real using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	Model	$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$		
		$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0255	0.0182	0.0216	0.0292	0.0227	0.0256	0.0210	0.0197	0.0233
	$GARCH_{snorm}$	0.0221	0.0162	0.0190	0.0249	0.0198	0.0222	0.0185	0.0175	0.0203
	$GARCH_{std}$	0.0271	0.0164	0.0208	0.0357	0.0235	0.0284	0.0204	0.0186	0.0236
	$GARCH_{sstd}$	0.0244	0.0153	0.0190	0.0320	0.0212	0.0255	0.0185	0.0169	0.0212
	$GARCH_{ged}$	0.0282	0.0178	0.0224	0.0343	0.0242	0.0286	0.0217	0.0199	0.0249
	$GARCH_{sged}$	0.0247	0.0161	0.0199	0.0296	0.0214	0.0250	0.0193	0.0178	0.0219
	$GARCH_{jsu}$	0.0212	0.0121	0.0156	0.0275	0.0212	0.0240	0.0196	0.0183	0.0217
500 Observations	$GARCH_{norm}$	0.0219	0.0153	0.0184	0.0252	0.0194	0.0220	0.0179	0.0167	0.0199
	$GARCH_{snorm}$	0.0224	0.0156	0.0187	0.0258	0.0198	0.0225	0.0183	0.0170	0.0203
	$GARCH_{std}$	0.0254	0.0144	0.0188	0.0335	0.0215	0.0263	0.0185	0.0167	0.0216
	$GARCH_{sstd}$	0.0268	0.0149	0.0197	0.0376	0.0228	0.0286	0.0191	0.0170	0.0226
	$GARCH_{ged}$	0.0247	0.0149	0.0192	0.0305	0.0209	0.0251	0.0186	0.0168	0.0215
	$GARCH_{sged}$	0.0250	0.0151	0.0194	0.0310	0.0213	0.0255	0.0188	0.0170	0.0218
	$GARCH_{jsu}$	0.0284	0.0134	0.0191	0.0263	0.0199	0.0228	0.0182	0.0170	0.0204
1000 Observations	$GARCH_{norm}$	0.0249	0.0175	0.0209	0.0285	0.0220	0.0250	0.0204	0.0190	0.0226
	$GARCH_{snorm}$	0.0243	0.0172	0.0205	0.0278	0.0215	0.0244	0.0200	0.0187	0.0221
	$GARCH_{std}$	0.0273	0.0168	0.0212	0.0368	0.0238	0.0290	0.0205	0.0185	0.0239
	$GARCH_{sstd}$	0.0265	0.0164	0.0207	0.0341	0.0229	0.0275	0.0201	0.0183	0.0231
	$GARCH_{ged}$	0.0280	0.0176	0.0222	0.0340	0.0240	0.0284	0.0215	0.0196	0.0247
	$GARCH_{sged}$	0.0272	0.0171	0.0216	0.0330	0.0233	0.0276	0.0209	0.0191	0.0240
	$GARCH_{jsu}$	0.0278	0.0139	0.0192	0.0289	0.0219	0.0251	0.0202	0.0188	0.0225

Table 46 – Loss functions results of Real using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	Model	$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathscr{L}_{\mathbf{ES}^{\alpha}}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$ $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	0.0385	0.1331	0.0777	4.8211	7.8893	5.6708	0.1715	0.2108	0.1161
	$GARCH_{snorm}$	0.0405	0.1309	0.0783	5.0281	7.8459	5.6996	0.1713	0.2092	0.1188
	$GARCH_{std}$	0.0345	0.1325	0.0774	4.4356	7.8803	5.6597	0.1671	0.2099	0.1119
	$GARCH_{sstd}$	0.0375	0.1355	0.0810	4.7308	7.9430	5.8048	0.1730	0.2165	0.1185
	$GARCH_{ged}$	0.0338	**0.1288**	**0.0746**	**4.3589**	**7.8017**	**5.5455**	**0.1625**	**0.2033**	**0.1083**
	$GARCH_{sged}$	**0.0345**	0.1335	0.0778	4.4309	7.8995	5.6783	0.1680	0.2113	0.1123
	$GARCH_{jsu}$	0.0529	0.1542	0.0997	6.2521	8.3285	6.5569	0.2072	0.2539	0.1526
500 Observations	$GARCH_{norm}$	0.0444	0.1242	0.0795	5.4086	7.7077	5.7436	0.1686	0.2036	0.1238
	$GARCH_{snorm}$	0.0420	0.1233	**0.0777**	5.1766	7.6898	**5.6744**	0.1654	0.2011	0.1198
	$GARCH_{std}$	0.0404	0.1254	0.0799	5.0108	7.7316	5.7593	0.1658	0.2053	0.1204
	$GARCH_{sstd}$	**0.0349**	0.1247	0.0781	**4.4669**	7.7155	5.6833	**0.1595**	0.2027	**0.1130**
	$GARCH_{ged}$	0.0425	0.1234	0.0789	5.2144	7.6913	5.7210	0.1659	0.2023	0.1214
	$GARCH_{sged}$	0.0419	**0.1231**	0.0790	5.1614	**7.6846**	5.7222	0.1650	**0.2020**	0.1209
	$GARCH_{jsu}$	0.0368	0.1278	0.0807	4.6564	7.7822	5.7947	0.1645	0.2085	0.1175
1000 Observations	$GARCH_{norm}$	0.0317	0.1180	**0.0673**	4.1544	7.5795	**5.2539**	0.1497	0.1853	0.0989
	$GARCH_{snorm}$	0.0318	0.1182	0.0676	4.1701	7.5832	5.2672	0.1500	0.1858	0.0994
	$GARCH_{std}$	0.0305	0.1194	0.0683	4.0381	7.6091	5.2959	0.1499	0.1877	0.0988
	$GARCH_{sstd}$	**0.0303**	0.1197	0.0685	4.0239	7.6156	5.3051	0.1500	0.1883	0.0989
	$GARCH_{ged}$	0.0309	**0.1179**	0.0674	4.0741	**7.5769**	5.2581	**0.1487**	**0.1852**	**0.0982**
	$GARCH_{sged}$	0.0307	0.1182	0.0675	4.0601	7.5839	5.2645	0.1489	0.1857	**0.0982**
	$GARCH_{jsu}$	**0.0303**	0.1264	0.0702	**4.0160**	7.7542	5.3719	0.1567	0.1966	0.1005

Table 47 – Risk measures forecasts of WTI Crude Oil using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	Model	$\mathrm{VaR}_{t+1}^{\alpha}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathrm{ES}_{t+1}^{\alpha}$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 2.5\%$	$\mathrm{RVaR}_{t+1}^{\alpha,\beta}$ $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
250 Observations	$GARCH_{norm}$	0.0346	0.0242	0.0290	0.0398	0.0306	0.0348	0.0283	0.0264	0.0314
	$GARCH_{snorm}$	0.0369	0.0253	0.0306	0.0427	0.0324	0.0371	0.0298	0.0277	0.0333
	$GARCH_{std}$	0.0352	0.0232	0.0285	0.0505	0.0323	0.0397	0.0278	0.0250	0.0325
	$GARCH_{sstd}$	0.0376	0.0244	0.0302	0.0453	0.0326	0.0381	0.0294	0.0270	0.0334
	$GARCH_{ged}$	0.0362	0.0237	0.0294	0.0429	0.0314	0.0366	0.0285	0.0263	0.0323
	$GARCH_{sged}$	0.0385	0.0251	0.0312	0.0456	0.0333	0.0388	0.0303	0.0278	0.0343
	$GARCH_{jsu}$	0.0566	0.0234	0.0360	0.0432	0.0319	0.0369	0.0291	0.0269	0.0328
500 Observations	$GARCH_{norm}$	0.0473	0.0332	0.0397	0.0543	0.0419	0.0475	0.0388	0.0362	0.0430
	$GARCH_{snorm}$	0.0507	0.0349	0.0422	0.0585	0.0446	0.0509	0.0411	0.0382	0.0459
	$GARCH_{std}$	0.0520	0.0313	0.0399	0.0699	0.0450	0.0550	0.0387	0.0349	0.0451
	$GARCH_{sstd}$	0.0566	0.0338	0.0434	0.0735	0.0484	0.0588	0.0421	0.0380	0.0489
	$GARCH_{ged}$	0.0522	0.0324	0.0411	0.0636	0.0446	0.0530	0.0399	0.0363	0.0459
	$GARCH_{sged}$	0.0559	0.0347	0.0441	0.0679	0.0478	0.0566	0.0427	0.0389	0.0491
	$GARCH_{jsu}$	0.0717	0.0318	0.0470	0.0568	0.0428	0.0491	0.0393	0.0366	0.0440
1000 Observations	$GARCH_{norm}$	0.0469	0.0331	0.0395	0.0538	0.0416	0.0472	0.0385	0.0360	0.0427
	$GARCH_{snorm}$	0.0491	0.0343	0.0411	0.0565	0.0434	0.0494	0.0401	0.0374	0.0446
	$GARCH_{std}$	0.0512	0.0324	0.0404	0.0701	0.0453	0.0553	0.0392	0.0354	0.0455
	$GARCH_{sstd}$	0.0340	0.0427	0.0681	0.0468	0.0558	0.0415	0.0378	0.0476	0.0294
	$GARCH_{ged}$	0.0514	0.0331	0.0413	0.0616	0.0444	0.0520	0.0401	0.0368	0.0456
	$GARCH_{sged}$	0.0540	0.0346	0.0433	0.0647	0.0466	0.0546	0.0420	0.0385	0.0479
	$GARCH_{jsu}$	0.0692	0.0315	0.0458	0.0567	0.0430	0.0491	0.0395	0.0368	0.0441

Table 48 – Loss functions results of WTI Crude Oil using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

Size	Model	$\mathscr{L}_{\textbf{VaR}^\alpha}$ $\alpha = \mathbf{1\%}$	$\alpha = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}$	$\mathscr{L}_{\textbf{ES}^\alpha}$ $\alpha = \mathbf{1\%}$	$\alpha = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}$	$\mathscr{L}_{\textbf{RVaR}^{\alpha,\beta}}$ $\alpha = \mathbf{1\%}, \beta = \mathbf{5\%}$	$\alpha = \mathbf{2.5\%}, \beta = \mathbf{5\%}$	$\alpha = \mathbf{1\%}, \beta = \mathbf{2.5\%}$
250 Observations	GARCH_{norm}	0.0593	0.1979	0.1165	6.8473	9.2159	7.2116	0.2572	0.3144	0.1757
	GARCH_{snorm}	0.0570	0.1924	0.1133	6.6145	9.1006	7.0835	0.2493	0.3057	0.1703
	GARCH_{std}	0.0558	0.1955	0.1138	6.4970	9.1645	7.1003	0.2513	0.3093	0.1696
	GARCH_{sstd}	**0.0535**	**0.1888**	0.1090	**6.2770**	**9.0273**	6.9101	**0.2423**	**0.2978**	**0.1625**
	GARCH_{ged}	0.0564	0.1952	0.1136	6.5587	9.1586	7.0934	0.2515	0.3087	0.1700
	GARCH_{sged}	0.0551	0.1906	0.1108	6.4288	9.0639	6.9824	0.2457	0.3014	0.1659
	GARCH_{jsu}	0.0607	0.1935	**0.1086**	6.9681	9.1233	**6.8958**	0.2541	0.3021	0.1693
500 Observations	GARCH_{norm}	**0.0543**	0.1989	0.1137	**6.3409**	9.2302	7.0932	0.2532	0.3127	**0.1680**
	GARCH_{snorm}	0.0563	0.2009	0.1159	6.5293	9.2704	7.1760	0.2571	0.3168	0.1721
	GARCH_{std}	0.0573	**0.1946**	**0.1133**	6.6407	9.1446	**7.0817**	**0.2518**	**0.3079**	0.1707
	GARCH_{sstd}	0.0601	0.1971	0.1170	6.9008	9.1978	7.2264	0.2571	0.3141	0.1771
	GARCH_{ged}	0.0575	0.1968	0.1149	6.6486	9.1888	7.1413	0.2542	0.3117	0.1724
	GARCH_{sged}	0.0594	0.1986	0.1182	6.8334	9.2247	7.2687	0.2579	0.3167	0.1776
	GARCH_{jsu}	0.0717	0.1938	0.1239	7.9984	**9.1268**	7.4881	0.2653	0.3176	0.1956
1000 Observations	GARCH_{norm}	0.1237	0.3544	0.2183	12.9862	12.3601	11.1822	0.4781	0.5727	0.3421
	GARCH_{snorm}	0.1191	0.3464	0.2099	12.5295	12.1948	10.8446	0.4655	0.5563	0.3290
	GARCH_{std}	0.1150	0.3608	0.2134	12.0707	12.4779	10.9638	0.4758	0.5743	0.3285
	GARCH_{sstd}	0.1097	0.3495	0.2030	11.5817	12.2473	10.5569	0.4591	0.5524	0.3127
	GARCH_{ged}	0.1149	0.3553	0.2093	12.1035	12.3697	10.8120	0.4702	0.5645	0.3242
	GARCH_{sged}	0.1105	**0.3448**	0.2005	11.6649	**12.1553**	10.4633	**0.4552**	**0.5453**	0.3109
	GARCH_{jsu}	**0.0946**	0.3693	**0.1953**	**10.1830**	12.6545	**10.2761**	0.4637	0.5646	**0.2899**

Table 49 – Risk measures forecasts of Bitcoin using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$			$\mathbf{RVaR}^{\alpha,\beta}_{\mathbf{t+1}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	$GARCH_{norm}$	0.0620	0.0436	0.0521	0.0712	0.0549	0.0624	0.0508	0.0475	0.0564
	$GARCH_{snorm}$	0.0619	0.0436	0.0521	0.0710	0.0548	0.0622	0.0508	0.0474	0.0564
	$GARCH_{std}$	0.0669	0.0311	0.0441	0.0992	0.0639	0.0782	0.0551	0.0497	0.0642
	$GARCH_{sstd}$	0.0671	0.0313	0.0443	0.1156	0.0575	0.0782	0.0430	0.0368	0.0534
	$GARCH_{ged}$	0.0814	0.0409	0.0574	0.1112	0.0665	0.0849	0.0553	0.0481	0.0674
	$GARCH_{sged}$	0.0802	0.0404	0.0566	0.1096	0.0656	0.0837	0.0546	0.0475	0.0665
	$GARCH_{jsu}$	0.0830	0.0400	0.0564	0.0787	0.0596	0.0682	0.0548	0.0510	0.0612
500 Observations	$GARCH_{norm}$	0.0846	0.0596	0.0712	0.0970	0.0750	0.0850	0.0694	0.0649	0.0770
	$GARCH_{snorm}$	0.0887	0.0620	0.0743	0.1020	0.0784	0.0892	0.0724	0.0676	0.0806
	$GARCH_{std}$	0.0922	0.0522	0.0682	0.1205	0.0778	0.0951	0.0672	0.0606	0.0781
	$GARCH_{sstd}$	0.0998	0.0558	0.0734	0.1394	0.0849	0.1063	0.0712	0.0634	0.0842
	$GARCH_{ged}$	0.0939	0.0555	0.0722	0.1173	0.0793	0.0958	0.0698	0.0629	0.0814
	$GARCH_{sged}$	0.1004	0.0591	0.0770	0.1256	0.0847	0.1024	0.0745	0.0670	0.0870
	$GARCH_{jsu}$	0.1123	0.0522	0.0751	0.0961	0.0728	0.0833	0.0670	0.0624	0.0747
1000 Observations	$GARCH_{norm}$	0.1293	0.0907	0.1086	0.1485	0.1144	0.1300	0.1059	0.0988	0.1176
	$GARCH_{snorm}$	0.1334	0.0929	0.1116	0.1536	0.1177	0.1341	0.1087	0.1013	0.1211
	$GARCH_{std}$	0.1474	0.0766	0.1036	0.1934	0.1249	0.1526	0.1077	0.0972	0.1253
	$GARCH_{sstd}$	0.1492	0.0773	0.1048	0.2276	0.1269	0.1647	0.1018	0.0891	0.1228
	$GARCH_{ged}$	0.1616	0.0893	0.1198	0.2093	0.1345	0.1662	0.1157	0.1027	0.1374
	$GARCH_{sged}$	0.1604	0.0887	0.1190	0.2079	0.1335	0.1650	0.1149	0.1020	0.1365
	$GARCH_{jsu}$	0.1637	0.0789	0.1113	0.1544	0.1170	0.1338	0.1076	0.1002	0.1201

Table 50 – Loss functions results of Bitcoin using a significance level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observation sizes.

		$\mathscr{L}_{\mathbf{VaR}^{\alpha}}$			$\mathscr{L}_{\mathbf{ES}^{\alpha}}$			$\mathscr{L}_{\mathbf{RVaR}^{\alpha,\beta}}$		
Size	**Model**	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$
250 Observations	GARCH_{norm}	0.1936	0.4013	0.2865	19.6910	13.2977	13.8418	0.5950	0.6879	0.4801
	GARCH_{snorm}	0.1909	0.3995	0.2835	19.4216	13.2590	13.7158	0.5905	0.6830	0.4744
	GARCH_{std}	0.1880	0.4016	0.2851	19.0049	13.3437	13.8033	0.5897	0.6867	0.4731
	GARCH_{sstd}	0.1862	0.3975	**0.2823**	18.7514	13.2507	13.6770	0.5837	0.6798	0.4685
	GARCH_{ged}	0.1893	0.4077	0.2962	19.0608	13.4507	14.1894	0.5970	0.7039	0.4855
	GARCH_{sged}	0.1889	0.4044	0.2954	19.0022	13.3788	14.1512	0.5932	0.6999	0.4843
	GARCH_{jsu}	**0.1782**	**0.3929**	0.2837	**18.1042**	**13.1427**	**13.7131**	**0.5710**	**0.6766**	**0.4619**
500 Observations	GARCH_{norm}	0.2462	0.5929	0.4000	24.3817	17.0681	18.1256	0.8390	0.9929	0.6462
	GARCH_{snorm}	0.2423	**0.5918**	0.3979	23.9928	**17.0413**	18.0292	0.8341	0.9897	0.6402
	GARCH_{std}	0.2305	0.6263	0.4185	22.7229	17.7291	18.7868	0.8565	1.0448	0.6490
	GARCH_{sstd}	0.2149	0.6133	0.4033	21.1969	17.4528	18.1682	0.8280	1.0166	0.6182
	GARCH_{ged}	0.2271	0.6129	0.4025	22.4202	17.4562	18.1735	0.8398	1.0153	0.6296
	GARCH_{sged}	0.2145	0.5986	**0.3903**	21.1859	17.1563	**17.6745**	**0.8128**	**0.9888**	0.6047
	GARCH_{jsu}	**0.1909**	0.6236	0.3973	**19.1550**	17.6910	18.0252	0.8140	1.0209	**0.5881**
1000 Observations	GARCH_{norm}	0.2871	0.6619	0.4490	27.9098	18.3460	19.8409	0.9489	1.1110	0.7361
	GARCH_{snorm}	**0.2851**	0.6618	0.4495	27.6125	18.3259	19.8178	0.9467	1.1113	0.7346
	GARCH_{std}	0.2868	0.6265	**0.4418**	**27.4213**	17.7071	**19.5444**	**0.9128**	**1.0684**	**0.7286**
	GARCH_{sstd}	0.2878	**0.6271**	0.4432	27.4318	17.7220	19.6006	0.9144	1.0702	0.7309
	GARCH_{ged}	0.2901	0.6552	0.4608	27.5248	18.2187	20.1462	0.9447	1.1159	0.7508
	GARCH_{sged}	0.2891	0.6518	0.4594	27.4499	18.1533	20.0989	0.9404	1.1112	0.7486
	GARCH_{jsu}	0.2911	**0.6271**	0.4495	27.8821	**17.6989**	19.7906	0.9175	1.0766	0.7405

Table 51 – Summary of the Loss Functions results of RVaR for Ibovespa, S&P 500, U.S. Treasury Bonds rates for 5 and 10 years, Euro, Real, WTI Crude Oil, and Bitcoin, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observations sizes.

Asset	**Size**	**$RVaR_{t+1}^{\alpha,\beta}$** $\alpha = 1\%, \beta = 5\%$	$\alpha = 2.5\%, \beta = 5\%$	$\alpha = 1\%, \beta = 2.5\%$
Ibovespa	**250 observations**	$GARCH_{sged}$	$GARCH_{snorm}$	$GARCH_{jsu}$
	500 observations	$GARCH_{std}$	$GARCH_{sstd}$	$GARCH_{norm}$ and $GARCH_{std}$
	1,000 observations	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{sged}$
S&P 500	**250 observations**	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	500 observations	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	1,000 observations	$GARCH_{norm}$ and $GARCH_{ged}$	$GARCH_{jsu}$	$GARCH_{norm}$
Treasure 5-years	**250 observations**	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	500 observations	$GARCH_{norm}$	$GARCH_{norm}$	$GARCH_{norm}$
	1,000 observations	$GARCH_{norm}$ and $GARCH_{ged}$	$GARCH_{jsu}$	$GARCH_{norm}$
Treasure 10-years	**250 observations**	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	500 observations	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{norm}$
	1,000 observations	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
Euro	**250 observations**	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$
	500 observations	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{jsu}$
	1,000 observations	$GARCH_{std}$	$GARCH_{ged}$	$GARCH_{std}$
Real	**250 observations**	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$
	500 observations	$GARCH_{sstd}$	$GARCH_{sged}$	$GARCH_{sstd}$
	1,000 observations	$GARCH_{ged}$	$GARCH_{ged}$	$GARCH_{ged}$ and $GARCH_{sged}$
WTI Crude Oil	**250 observations**	$GARCH_{sstd}$	$GARCH_{sstd}$	$GARCH_{sstd}$
	500 observations	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{norm}$
	1,000 observations	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{jsu}$
Bitcoin	**250 observations**	$GARCH_{jsu}$	$GARCH_{jsu}$	$GARCH_{jsu}$
	500 observations	$GARCH_{sged}$	$GARCH_{sged}$	$GARCH_{jsu}$
	1,000 observations	$GARCH_{std}$	$GARCH_{std}$	$GARCH_{std}$

Table 52 – Summary of the Loss Functions results of VaR and ES for Ibovespa, S&P 500, U.S. Treasury Bonds rates for 5 and 10 years, Euro, Real, WTI Crude Oil, and Bitcoin, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 rolling window observations sizes.

Asset	Size	$\mathbf{VaR}^{\alpha}_{\mathbf{t+1}}$ $\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\mathbf{ES}^{\alpha}_{\mathbf{t+1}}$ $\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$
Ibovespa	**250 observations**	GARCH_{jsu}	GARCH_{snorm}	GARCH_{sged}	GARCH_{jsu}	GARCH_{snorm}	GARCH_{sged}
	500 observations	GARCH_{norm}	GARCH_{std}	GARCH_{sstd}	GARCH_{norm}	GARCH_{std}	GARCH_{sstd}
	1,000 observations	GARCH_{sged}	GARCH_{sged}	GARCH_{sged}	GARCH_{sged}	GARCH_{sged}	GARCH_{sged}
S&P 500	**250 observations**	GARCH_{jsu}	GARCH_{snorm}	GARCH_{ged}	GARCH_{jsu}	GARCH_{snorm}	GARCH_{ged}
	500 observations	GARCH_{norm}	GARCH_{std}	GARCH_{norm}	GARCH_{norm}	GARCH_{std}	GARCH_{norm}
	1,000 observations	GARCH_{norm}	GARCH_{ged}	GARCH_{norm}	GARCH_{norm}	GARCH_{ged}	GARCH_{ged}
Treasure 5-years	**250 observations**	GARCH_{jsu}	GARCH_{ged}	GARCH_{jsu}	GARCH_{ged}	GARCH_{ged}	GARCH_{ged}
	500 observations	GARCH_{norm}	GARCH_{jsu}	GARCH_{norm}	GARCH_{norm}	GARCH_{norm}	GARCH_{norm}
	1,000 observations	GARCH_{norm}	GARCH_{jsu}	GARCH_{norm}	GARCH_{norm}	GARCH_{jsu}	GARCH_{norm}
Treasure 10-years	**250 observations**	GARCH_{std}	GARCH_{ged}	GARCH_{ged}	GARCH_{std}	GARCH_{ged}	GARCH_{ged}
	500 observations	GARCH_{norm}	GARCH_{jsu}	GARCH_{jsu}	GARCH_{norm}	GARCH_{jsu}	GARCH_{jsu}
	1,000 observations	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}
Euro	**250 observations**	GARCH_{std} and GARCH_{ged}	GARCH_{std}	GARCH_{snorm}	GARCH_{std}	GARCH_{std}	GARCH_{snorm}
	500 observations	GARCH_{sged}	GARCH_{sged}	GARCH_{jsu}	GARCH_{sged}	GARCH_{sged}	GARCH_{jsu}
	1,000 observations	GARCH_{jsu}	GARCH_{ged}	GARCH_{ged}	GARCH_{jsu}	GARCH_{ged}	GARCH_{ged}
Real	**250 observations**	GARCH_{sged}	GARCH_{ged}	GARCH_{ged}	GARCH_{ged}	GARCH_{ged}	GARCH_{ged}
	500 observations	GARCH_{sstd}	GARCH_{sged}	GARCH_{snorm}	GARCH_{sstd}	GARCH_{sged}	GARCH_{snorm}
	1,000 observations	GARCH_{sstd} and GARCH_{jsu}	GARCH_{ged}	GARCH_{norm}	GARCH_{jsu}	GARCH_{ged}	GARCH_{norm}
WTI Crude Oil	**250 observations**	GARCH_{sstd}	GARCH_{sstd}	GARCH_{jsu}	GARCH_{sstd}	GARCH_{sstd}	GARCH_{jsu}
	500 observations	GARCH_{norm}	GARCH_{std}	GARCH_{std}	GARCH_{norm}	GARCH_{jsu}	GARCH_{std}
	1,000 observations	GARCH_{jsu}	GARCH_{sged}	GARCH_{jsu}	GARCH_{jsu}	GARCH_{sged}	GARCH_{jsu}
Bitcoin	**250 observations**	GARCH_{jsu}	GARCH_{jsu}	GARCH_{sstd}	GARCH_{jsu}	GARCH_{jsu}	GARCH_{jsu}
	500 observations	GARCH_{jsu}	GARCH_{snorm}	GARCH_{sged}	GARCH_{jsu}	GARCH_{snorm}	GARCH_{sged}
	1,000 observations	GARCH_{snorm}	GARCH_{sstd} and GARCH_{jsu}	GARCH_{std}	GARCH_{std}	GARCH_{jsu}	GARCH_{std}

Table 53 – The number of occurrences of each model as the best by risk measure, using a significance level of 1%, 2.5%, and 5%.
*: Indicates the model with more ocurrences applied to a risk measure.
**: Indicates models with no ocurrences applied to a risk measure.

	$\mathbf{VaR}_{t+1}^{\alpha}$			$\mathbf{ES}_{t+1}^{\alpha}$			$\mathbf{RVaR}_{t+1}^{\alpha,\beta}$			
Model	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%$	$\alpha = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{2.5}\%, \beta = \mathbf{5}\%$	$\alpha = \mathbf{1}\%, \beta = \mathbf{2.5}\%$	**Total**
GARCH_{norm}	7	**0****	5	7	1	4	4	2	**7***	37
GARCH_{snorm}	1	3	2	**0****	3	2	**0****	1	**0****	12
GARCH_{std}	2	4	2	3	3	2	5	3	4	28
GARCH_{sstd}	3	2	2	2	1	1	2	2	2	17
GARCH_{ged}	1	**6***	4	2	**6***	**6***	**7***	**6***	5	43
GARCH_{sged}	3	4	3	2	4	3	5	5	2	31
GARCH_{jsu}	**9***	**6***	**6***	**8***	**6***	**6***	3	5	6	55

4.2.2 Multivariate Results

In this subsection, we present the results obtained by the multivariate approach, analyzing the risk measures individually for each portfolio and their respective realized loss. As stated in section 3.2, we divided our assets in three different scenarios. The first one was compound by four assets with a negative or no correlation between them, the second by four assets with a positive correlation, and the third with all analyzed assets. The correlation between the assets is described in Table 2. Tables 54, 55, and 56 expose the average values of all risk measures forecasts for, respectively, scenario 1, 2, and 3, considering the rolling windows of 250, 500, and 1,000 observations. Tables 57, 58, and 59 expose the average values of all risk measures realized losses for, respectively, scenario 1, 2, and 3, considering the rolling estimation windows of 250, 500, and 1,000 observations. We evaluated the performance of the models using the same occurrence counter observed in section 4.1, that is, when a model has the best results over 4 or more risk measures[8] we consider it a great performance. Also, we summarized the results by the total number of occurrences. These summaries are exposed in tables 60 and 61. For a better understanding, we first analyze the average values and, in a second moment, the evaluation of the risk models.

We can observe that the inequality 2.7 is valid for every scenario. For instance, for scenario 1, considering a rolling estimation window of 250 observations and HS forecasts, we have $\text{VaR}^{2.5\%} = 2.4050 > \text{RVaR}^{2.5\%,5\%} = 2.1907 > \text{VaR}^{5\%} = 1.8681$. The multivariate results also confirm that the lowest RVaR estimates are obtained for $\alpha = 2.5\%$ and $\beta = 5\%$, while the highest are found for $\alpha = 1\%$ and $\beta = 5\%$. Thus, as both significance levels assume higher values, the risk estimates are lower and, therefore, the risk estimates are more conservative. We observe that the lowest risk estimates are found for portfolios with 8 assets regarding the different scenarios. This result was expected because a portfolio with 8 assets tends to be better diversified compared to portfolios with 4 assets. For portfolios with 4 assets, we notice that the lowest risk values are observed for portfolios with assets with a negative correlation or close to zero. The negative correlation between assets tends to mitigate the risk associated with the portfolio, decreasing the overall risk of the portfolio. For instance, $\text{RVaR}^{2.5\%,5\%}$ of 1.9835, 2.0792, and 1.5271 for, respectively, scenarios 1, 2, and 3, considering 1,000 rolling estimation window observations and the R-Vine$_{std}$ model.

Considering the same marginal distribution for the risk estimates obtained via copulas, we noticed a similarity of results, regardless of the risk measure. C-Vine and R-Vine copulas describe a multivariate relationship between assets considering a cascade of pair-copulas and the marginal densities. Despite the similarities, both copulas provide a different way of arranging the pair-copulas in a tree structure. See Geidosch e Fischer (2016). For an illustration, observing scenario 2 and 250 observations, all the R-Vine$_{norm}$ values are higher than the C-Vine$_{norm}$ ones

[8] We consider risk measures with different significance levels as different risk measures, for instance, $\text{VaR}^{1\%}$ and $\text{VaR}^{2.5\%}$ count as two different risk measures in the context of occurrences.

considering all the risk measures evaluated, for instance, $RVaR^{2.5\%,5\%}$ of, respectively, 2.1826 and 2.1614. Concerning the multivariate GARCH models, we identified similarities in the mean values of the DCC models, while the GO-GARCH estimates tend to be higher. DCC and GO-GARCH models belong to different subgroups of multivariate GARCH models. GO-GARCH is a model that some heteroskedastic not observable factors drive the returns. At the same time, the DCC is based on estimating the variance and the correlation matrix to estimate the conditional covariance matrix.

Observing scenario realized loss, we observe a superiority of the results obtained via copulas. For the portfolios obtained with all assets and assets that present a positive correlation, it is highlighted that the copulas with marginal Student's t distribution have the best performance. In the univariate empirical analysis, we observed Johnson's SU distribution with good performance. As explained by Müller e Righi (2018), the better performance of the forecasts obtained through copulas may be explained by the fact that they allow the dependence structure to be identified and capture the possible non-linear relationships among variables. Observing Tables 60 and 61, $RVine_{std}$ and $CVine_{std}$ had the best performances with, respectively, 22 and 17 occurrences. Leccadito et al. (2014) also had great results with the marginal Student's t distribution for $VaR^{1\%}$,$VaR^{2.5\%}$, and $VaR^{5\%}$. On the other hand, the Müller e Righi (2018) had good results with the DCC model in their scenario with the high positive association, whereas we did not. That may occur because of the size of their portfolios (16 assets in that case) and our of 4 assets. Furthermore, in the authors' study, simulated data were used, and the correlation was homogeneous, that is, the same among all assets. Our best occurrences of the DCC model are observed in the portfolio with assets that present a positive correlation, emphasizing the predictions obtained with 1000 observations. For bivariate portfolios, Chen e Tu (2013) identified that copulas do not outperform the DCC for predicting VaR and ES. However, it is noteworthy that our study differs in that it uses Vine copulas. Standard multivariate copulas, such as those used in the authors' study lack the requirement of accurately modeling among large numbers of variables.

Observing Tables 60 and 61, we identify that HS perform well in 7 occasions. We note that most occurrences occur in estimated portfolios with 250 observations, with emphasis on the portfolio with 4 assets and close to zero or negative correlation between them. Among the risk measures that presented more occurrences of the HS method with good results, we have ES. These results do not corroborate the findings identified by Müller e Righi (2018). We were able to note a tendency between the models that perform better forecasting VaR and RVaR. In scenario 1, for 250 observations R-Vine_{ged} performed better for $VaR^{2.5\%}$ and $RVaR^{1\%,2.5\%}$, for 500 observations R-Vine_{snorm} performed better for $VaR^{5\%}$, $RVaR^{1\%,5\%}$, and $RVaR^{2.5\%,5\%}$, while C-Vine_{std} performed better for $VaR^{1\%}$ and $RVaR^{1\%,2.5\%}$, and for 1,000 observations aDCC performed better for $VaR^{2.5\%}$ and $RVaR^{1\%,2.5\%}$. Observing scenario 2, for 250 observations R-Vine_{std} performed better for $VaR^{1\%}$ and $RVaR^{1\%,2.5\%}$, for 500 observations R-Vine_{std} performed better for $VaR^{1\%}$, $VaR^{2.5\%}$, and $RVaR^{2.5\%,5\%}$, and for 1,000 observations C-Vine_{std} performed better for $\text{VaR}^{2.5\%}$ and $\text{RVaR}^{1\%,2.5\%}$. Finally, in scenario 3, for 250 observations R-Vine$_{norm}$ performed better for $\text{VaR}^{5\%}$, $\text{RVaR}^{1\%,5\%}$, and $\text{RVaR}^{2.5\%,5\%}$, for 500 observations C-Vine$_{std}$ performed better for $\text{VaR}^{1\%}$, $\text{VaR}^{2.5\%}$, and $\text{RVaR}^{1\%,2.5\%}$, and for 1,000 observations R-Vine$_{std}$ performed better for $\text{VaR}^{1\%}$, $\text{VaR}^{2.5\%}$, and $\text{RVaR}^{15\%,5\%}$. We could not observe the same behavior for ES and RVaR in every cases, for instance in scenario 1 and 250 observations.

We can also note that the performance of the models changes according to the significance level and rolling estimation window considered in the estimation. These results corroborate with the findings from Rossi e Inoue (2012) and Inoue et al. (2017). Observing Table 60, we highlighted some models with no occurrences for any of the rolling estimation windows and significance levels, namely: RVine$_{sstd}$, RVine$_{sged}$, and RVine$_{jsu}$. In our univariate analysis, GARCH$_{sstd}$ and GARCH$_{sged}$ also had poor performances overall. On the other hand, GARCH$_{jsu}$ had the best results. Differently from us, although using GARCH models, Garcia-Jorcano e Novales (2019) had an overall better performance with the skewed distributions than the symmetric versions for Student's t, Generalized Error, and Johnson's SU distributions.

Thus, based on our results, we can conclude that RVaR, VaR, and ES risk predictions tend to be sensitive to model choice. This sensitivity is related to an issue known as model risk. For more details, see Müller e Righi (2020). This risk can compromise decision-making and generate losses or reduce gains since a model that underestimates or overestimates the risk may be being considered for the organization's activities.

Table 54 – Scenario 1 - Forecasts of a portfolio with 4 assets with a negative correlation between them, namely Euro, Real, U.S. 5-year Treasury Yield, and Bitcoin, with multivariate models results for the risk measures, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Measure	Models																	
		RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
250 Obs.	$\mathbf{VaR}_{t+1}^{1\%}$	2.5577	3.8109	2.8184	2.8329	2.5305	2.8031	2.8340	2.5357	3.7784	2.7561	2.7591	2.5496	2.8062	2.7789	3.1323	2.5494	2.5570	2.7403
	$\mathbf{VaR}_{t+1}^{2.5\%}$	2.1768	2.9573	2.2562	2.3191	2.1512	2.2828	2.2711	2.1661	2.9491	2.2254	2.2888	2.1718	2.2940	2.2481	2.4050	2.1804	2.1879	2.3219
	$\mathbf{VaR}_{t+1}^{5\%}$	1.8479	2.3756	1.8258	1.8958	1.8192	1.8698	1.8259	1.8259	2.3679	1.8063	1.8872	1.8292	1.8719	1.8287	1.8681	1.8477	1.8553	1.9654
	$\mathbf{ES}_{t+1}^{1\%}$	2.8888	5.1352	3.5766	3.3488	2.8437	3.3254	3.4250	2.8572	5.0338	3.4494	3.2771	2.8924	3.3100	3.4004	3.6199	2.8683	2.8758	3.1072
	$\mathbf{ES}_{t+1}^{2.5\%}$	2.5707	4.0718	2.9437	2.8754	2.5350	2.8465	2.8952	2.5499	4.0133	2.8667	2.8162	2.5669	2.8462	2.8607	3.1023	2.5625	2.5701	2.7525
	$\mathbf{ES}_{t+1}^{5\%}$	2.3017	3.3960	2.5110	2.5062	2.2694	2.4790	2.4864	2.2832	3.3633	2.4594	2.4665	2.2938	2.4778	2.4650	2.6465	2.2994	2.3070	2.4585
	$\mathbf{RVaR}_{t+1}^{1\%,5\%}$	2.1549	2.9612	2.2445	2.2956	2.1258	2.2674	2.2517	2.1397	2.9457	2.2120	2.2639	2.1441	2.2697	2.2312	2.4032	2.1572	2.1647	2.2964
	$\mathbf{RVaR}_{t+1}^{2.5\%,5\%}$	2.0326	2.7203	2.0783	2.1370	2.0038	2.1116	2.0776	2.0165	2.7133	2.0522	2.1168	2.0206	2.1094	2.0694	2.1907	2.0363	2.0438	2.1645
	$\mathbf{RVaR}_{t+1}^{1\%,2.5\%}$	2.3586	3.3628	2.5217	2.5599	2.3292	2.5272	2.5420	2.3451	3.3330	2.4783	2.5090	2.3499	2.5369	2.5009	2.7573	2.3587	2.3662	2.5161
500 Obs.	$\mathbf{VaR}_{t+1}^{1\%}$	2.6479	3.5532	2.7653	2.8004	2.6654	2.7993	2.7992	2.6930	3.5506	2.7544	2.8241	2.6786	2.7947	2.8247	3.2086	2.6950	2.7018	3.1731
	$\mathbf{VaR}_{t+1}^{2.5\%}$	2.2509	2.7741	2.2132	2.2901	2.2575	2.2801	2.2543	2.2747	2.7845	2.2125	2.3069	2.2578	2.2782	2.2490	2.4949	2.2799	2.2866	2.6103
	$\mathbf{VaR}_{t+1}^{5\%}$	1.8871	2.2147	1.7784	1.8635	1.8889	1.8483	1.8140	1.9033	2.2285	1.7927	1.8756	1.8952	1.8482	1.8160	1.9542	1.9093	1.9161	2.1325
	$\mathbf{ES}_{t+1}^{1\%}$	3.0673	4.7698	3.5751	3.4106	3.0788	3.4035	3.6255	3.1005	4.8496	3.5393	3.4105	3.0956	3.3645	3.6188	3.7514	3.1017	3.1085	3.7972
	$\mathbf{ES}_{t+1}^{2.5\%}$	2.6829	3.7594	2.8920	2.8662	2.6910	2.8623	2.9378	2.7132	3.7914	2.8764	2.8796	2.7029	2.8429	2.9404	3.2424	2.7163	2.7231	3.2284
	$\mathbf{ES}_{t+1}^{5\%}$	2.3828	3.1414	2.4535	2.4805	2.3899	2.4706	2.4964	2.4079	3.1630	2.4489	2.4928	2.3979	2.4620	2.4962	2.7603	2.4128	2.4195	2.8080
	$\mathbf{RVaR}_{t+1}^{1\%,5\%}$	2.2116	2.7343	2.1731	2.2480	2.2176	2.2374	2.2141	2.2348	2.7413	2.1763	2.2634	2.2234	2.2364	2.2156	2.5126	2.2405	2.2473	2.5607
	$\mathbf{RVaR}_{t+1}^{2.5\%,5\%}$	2.0826	2.5234	2.0150	2.0949	2.0887	2.0789	2.0551	2.1026	2.5345	2.0213	2.1060	2.0928	2.0812	2.0520	2.2782	2.1092	2.1160	2.3876
	$\mathbf{RVaR}_{t+1}^{1\%,2.5\%}$	2.4267	3.0858	2.4367	2.5033	2.4325	2.5015	2.4792	2.4551	3.0859	2.4345	2.5256	2.4411	2.4951	2.4881	2.9031	2.4594	2.4662	2.8492
1000 Obs.	$\mathbf{VaR}_{t+1}^{1\%}$	2.5882	3.5368	2.6518	2.7545	2.5721	2.7004	2.7022	2.5941	3.5460	2.6650	2.7669	2.5626	2.7198	2.7101	3.1845	2.6066	2.6247	3.2048
	$\mathbf{VaR}_{t+1}^{2.5\%}$	2.1704	2.7541	2.1101	2.2274	2.1603	2.1891	2.1505	2.1815	2.7538	2.1118	2.2444	2.1663	2.1963	2.1512	2.4807	2.1975	2.2157	2.4266
	$\mathbf{VaR}_{t+1}^{5\%}$	1.8158	2.1978	1.6950	1.8014	1.8030	1.7685	1.7219	1.8228	2.2003	1.7063	1.8116	1.8113	1.7787	1.7325	1.8983	1.8283	1.8464	1.8787
	$\mathbf{ES}_{t+1}^{1\%}$	2.9976	4.8229	3.4361	3.3771	2.9661	3.3090	3.4600	2.9848	4.8188	3.4477	3.3650	2.9537	3.2996	3.4552	3.9606	2.9986	3.0167	4.1877
	$\mathbf{ES}_{t+1}^{2.5\%}$	2.6199	3.7927	2.7875	2.8350	2.5994	2.7769	2.8261	2.6189	3.7940	2.7993	2.8377	2.5939	2.7821	2.8270	3.3364	2.6344	2.6526	3.3423
	$\mathbf{ES}_{t+1}^{5\%}$	2.3017	3.1240	2.3373	2.4183	2.2854	2.3712	2.3732	2.3061	3.1242	2.3465	2.4258	2.2853	2.3782	2.3765	2.7431	2.3188	2.3369	2.7348
	$\mathbf{RVaR}_{t+1}^{1\%,5\%}$	2.1278	2.6993	2.0626	2.1786	2.1152	2.1367	2.1015	2.1365	2.7005	2.0712	2.1910	2.1182	2.1479	2.1069	2.4387	2.1488	2.1669	2.3716
	$\mathbf{RVaR}_{t+1}^{2.5\%,5\%}$	1.9835	2.4553	1.8870	2.0016	1.9714	1.9655	1.9204	1.9933	2.4544	1.8937	2.0140	1.9767	1.9743	1.9261	2.1498	2.0031	2.0212	2.1274
	$\mathbf{RVaR}_{t+1}^{1\%,2.5\%}$	2.3682	3.1060	2.3551	2.4735	2.3550	2.4221	2.4034	2.3750	3.1107	2.3671	2.4862	2.3541	2.4372	2.4081	2.9202	2.3917	2.4098	2.7786

Table 55 – Scenario 2 - Forecasts of a portfolio with 4 assets with a positive correlation between them, namely S&P 500, Ibovespa, U.S. 5-year Treasury Yield, and U.S. 10-year Treasury Yield, with multivariate models results for the risk measures, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Measure	Models																	
		RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
250 Obs.	$\mathbf{VaR}^{1\%}_{t+1}$	2.7267	3.4953	2.9816	2.8816	2.8234	2.9681	3.0060	2.7405	3.4475	2.9472	2.8744	2.8158	2.8887	2.9771	2.7355	2.6764	2.6793	3.9199
	$\mathbf{VaR}^{2.5\%}_{t+1}$	2.3404	2.8068	2.4316	2.3976	2.4190	2.4438	2.4414	2.3365	2.7804	2.4069	2.3988	2.3827	2.4070	2.4296	2.2065	2.2879	2.2908	3.2564
	$\mathbf{VaR}^{5\%}_{t+1}$	1.9829	2.2726	1.9602	1.9775	2.0293	1.9994	1.9858	1.9468	2.2695	1.9429	1.9800	2.0058	1.9856	1.9802	1.7547	1.9264	1.9293	2.7227
	$\mathbf{ES}^{1\%}_{t+1}$	3.0483	4.2872	3.6335	3.3372	3.1803	3.4509	3.5668	3.0811	4.1444	3.5369	3.2971	3.2032	3.3435	3.5317	3.0538	3.0324	3.0353	4.5363
	$\mathbf{ES}^{2.5\%}_{t+1}$	2.7298	3.5924	3.0678	2.9098	2.8426	2.9952	3.0480	2.7482	3.5118	3.0136	2.8903	2.8296	2.9173	3.0263	2.6879	2.6970	2.6999	3.9597
	$\mathbf{ES}^{5\%}_{t+1}$	2.4562	3.0854	2.6482	2.5645	2.5406	2.6234	2.6463	2.4548	3.0378	2.6069	2.5535	2.5245	2.5707	2.6296	2.3456	2.4137	2.4165	3.4920
	$\mathbf{RVaR}^{1\%,5\%}_{t+1}$	2.3082	2.7849	2.4018	2.3713	2.3807	2.4165	2.4162	2.2982	2.7611	2.3744	2.3676	2.3548	2.3775	2.4041	2.1685	2.2590	2.2619	3.2309
	$\mathbf{RVaR}^{2.5\%,5\%}_{t+1}$	2.1826	2.5783	2.2286	2.2191	2.2385	2.2516	2.2446	2.1614	2.5637	2.2002	2.2166	2.2194	2.2242	2.2330	2.0032	2.1303	2.1332	3.0242
	$\mathbf{RVaR}^{1\%,2.5\%}_{t+1}$	2.5175	3.1291	2.6906	2.6249	2.6175	2.6914	2.7022	2.5263	3.0901	2.6647	2.6192	2.5805	2.6331	2.6894	2.4440	2.4735	2.4763	3.5754
500 Obs.	$\mathbf{VaR}^{1\%}_{t+1}$	2.6164	3.2907	2.8414	2.7738	2.7139	2.7887	2.8633	2.6259	3.3088	2.7839	2.7320	2.6715	2.7855	2.8260	2.4025	2.5647	2.5754	3.1299
	$\mathbf{VaR}^{2.5\%}_{t+1}$	2.2125	2.6315	2.2863	2.2798	2.2777	2.3017	2.3057	2.2158	2.6466	2.2565	2.2606	2.2578	2.2947	2.2898	1.9282	2.1716	2.1823	2.5580
	$\mathbf{VaR}^{5\%}_{t+1}$	1.8569	2.1195	1.8474	1.8624	1.8919	1.8823	1.8644	1.8447	2.1338	1.8370	1.8541	1.8807	1.8837	1.8684	1.5594	1.8228	1.8335	2.0825
	$\mathbf{ES}^{1\%}_{t+1}$	3.0059	4.1805	3.5405	3.2983	3.1360	3.3455	3.5616	3.0116	4.1713	3.4713	3.2492	3.1173	3.3183	3.4896	3.0072	2.9454	2.9561	3.7427
	$\mathbf{ES}^{2.5\%}_{t+1}$	2.6356	3.4162	2.9284	2.8150	2.7361	2.8441	2.9488	2.6442	3.4194	2.8749	2.7787	2.7095	2.8313	2.9054	2.4874	2.5845	2.5953	3.1783
	$\mathbf{ES}^{5\%}_{t+1}$	2.3439	2.9072	2.5066	2.4507	2.4178	2.4757	2.5271	2.3439	2.9152	2.4698	2.4256	2.3967	2.4685	2.5017	2.1226	2.2980	2.3087	2.7586
	$\mathbf{RVaR}^{1\%,5\%}_{t+1}$	2.1783	2.5888	2.2481	2.2388	2.2382	2.2582	2.2685	2.1769	2.6012	2.2194	2.2197	2.2166	2.2561	2.2547	1.9015	2.1361	2.1469	2.5126
	$\mathbf{RVaR}^{2.5\%,5\%}_{t+1}$	2.0521	2.3981	2.0847	2.0863	2.0994	2.1072	2.1055	2.0435	2.4109	2.0648	2.0724	2.0839	2.1057	2.0979	1.7578	2.0114	2.0222	2.3389
	$\mathbf{RVaR}^{1\%,2.5\%}_{t+1}$	2.3888	2.9068	2.5203	2.4928	2.4696	2.5098	2.5403	2.3992	2.9182	2.4773	2.4651	2.4376	2.5066	2.5159	2.1408	2.3439	2.3547	2.8021
1000 Obs.	$\mathbf{VaR}^{1\%}_{t+1}$	2.7091	3.5822	2.9469	2.9306	2.7437	2.9031	2.9769	2.7941	3.6286	2.9976	3.0337	2.8274	3.0088	3.0332	3.0879	2.6753	2.6910	3.2348
	$\mathbf{VaR}^{2.5\%}_{t+1}$	2.2808	2.8021	2.3178	2.3656	2.3031	2.3395	2.3646	2.3486	2.8510	2.3727	2.4292	2.3609	2.3997	2.4081	2.5003	2.2633	2.2791	2.6266
	$\mathbf{VaR}^{5\%}_{t+1}$	1.9001	2.2299	1.8616	1.9129	1.9150	1.8892	1.8927	1.9389	2.2703	1.8921	1.9524	1.9439	1.9276	1.9241	1.9620	1.8932	1.9090	2.1282
	$\mathbf{ES}^{1\%}_{t+1}$	3.1217	4.6559	3.7636	3.5750	3.1876	3.5336	3.8132	3.2217	4.6942	3.7872	3.6464	3.2658	3.6429	3.7989	4.3184	3.0768	3.0926	3.8792
	$\mathbf{ES}^{2.5\%}_{t+1}$	2.7383	3.7547	3.0687	3.0085	2.7799	2.9743	3.1121	2.8247	3.7986	3.1116	3.0818	2.8544	3.0660	3.1342	3.3916	2.7054	2.7211	3.2957
	$\mathbf{ES}^{5\%}_{t+1}$	2.4087	3.1248	2.5712	2.5658	2.4402	2.5365	2.6109	2.4787	3.1691	2.6133	2.6281	2.4960	2.6072	2.6395	2.7976	2.3867	2.4024	2.8283
	$\mathbf{RVaR}^{1\%,5\%}_{t+1}$	2.2305	2.7421	2.2730	2.3135	2.2534	2.2872	2.3103	2.2930	2.7878	2.3198	2.3735	2.3036	2.3482	2.3497	2.4174	2.2142	2.2299	2.5656
	$\mathbf{RVaR}^{2.5\%,5\%}_{t+1}$	2.0792	2.4949	2.0736	2.1230	2.1006	2.0986	2.1096	2.1328	2.5395	2.1149	2.1744	2.1376	2.1483	2.1449	2.2036	2.0680	2.0837	2.3609
	$\mathbf{RVaR}^{1\%,2.5\%}_{t+1}$	2.4827	3.1540	2.6054	2.6309	2.5080	2.6013	2.6447	2.5600	3.2016	2.6613	2.7053	2.5801	2.6815	2.6910	2.7737	2.4578	2.4735	2.9067

Table 56 – Scenario 3 - Forecasts of a portfolio full with the 8 assets analyzed, namely S&P 500, Ibovespa, U.S. 5-year Treasury Yield, U.S. 10-year Treasury Yield, Euro, Real, and Bitcoin with multivariate models results for the risk measures, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Measure	Models																	
		RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
250 Obs.	$\mathbf{VaR}_{t+1}^{1\%}$	2.0045	2.8071	2.2157	2.1671	2.0230	2.2151	2.2741	2.0096	2.7464	2.2269	2.1529	2.0492	2.1780	2.2279	2.2049	1.9769	1.9746	1.8140
	$\mathbf{VaR}_{t+1}^{2.5\%}$	1.7203	2.2508	1.7999	1.7865	1.7161	1.8117	1.8311	1.7213	2.2155	1.7829	1.7729	1.7293	1.7892	1.8146	1.7864	1.6869	1.6846	1.5123
	$\mathbf{VaR}_{t+1}^{5\%}$	1.4469	1.8175	1.4513	1.4675	1.4387	1.4866	1.4746	1.4514	1.7898	1.4489	1.4584	1.4544	1.4739	1.4683	1.4113	1.4205	1.4182	1.2773
	$\mathbf{ES}_{t+1}^{1\%}$	2.2544	3.5671	2.7917	2.5236	2.2968	2.6143	2.7538	2.2709	3.4850	2.7202	2.4887	2.3296	2.5281	2.7128	2.5922	2.2299	2.2276	2.0615
	$\mathbf{ES}_{t+1}^{2.5\%}$	2.0149	2.9358	2.3157	2.1882	2.0363	2.2444	2.3249	2.0217	2.8776	2.2828	2.1635	2.0594	2.1930	2.2933	2.2206	1.9848	1.9825	1.8141
	$\mathbf{ES}_{t+1}^{5\%}$	1.8059	2.5005	1.9827	1.9159	1.8156	1.9558	2.0021	1.8110	2.4576	1.9637	1.8990	1.8335	1.9236	1.9803	1.9161	1.7764	1.7741	1.6111
	$\mathbf{RVaR}_{t+1}^{1\%,5\%}$	1.6938	2.2338	1.7804	1.7640	1.6953	1.7912	1.8141	1.6960	2.2008	1.7745	1.7515	1.7095	1.7724	1.7972	1.7471	1.6630	1.6607	1.4985
	$\mathbf{RVaR}_{t+1}^{2.5\%,5\%}$	1.5970	2.0652	1.6497	1.6437	1.5948	1.6673	1.6793	1.6002	2.0377	1.6446	1.6344	1.6077	1.6541	1.6674	1.6116	1.5680	1.5657	1.4081
	$\mathbf{RVaR}_{t+1}^{1\%,2.5\%}$	1.8553	2.5150	1.9984	1.9646	1.8627	1.9977	2.0389	1.8556	2.4726	1.9911	1.9468	1.8792	1.9696	2.0136	1.9728	1.8214	1.8191	1.6491
500 Obs.	$\mathbf{VaR}_{t+1}^{1\%}$	1.9574	2.5834	2.1068	2.0349	2.0106	2.0658	2.1153	1.9728	2.5565	2.0954	2.0682	2.0076	2.0762	2.1226	1.9954	1.9386	1.9395	2.2140
	$\mathbf{VaR}_{t+1}^{2.5\%}$	1.6455	2.0557	1.7030	1.6766	1.6818	1.6919	1.7171	1.6647	2.0522	1.6846	1.6868	1.6902	1.6932	1.7114	1.6633	1.6445	1.6454	1.8328
	$\mathbf{VaR}_{t+1}^{5\%}$	1.3791	1.6429	1.3670	1.3613	1.3973	1.3779	1.3774	1.3831	1.6564	1.3588	1.3759	1.4076	1.3802	1.3832	1.3764	1.3745	1.3754	1.5133
	$\mathbf{ES}_{t+1}^{1\%}$	2.2823	3.3170	2.6563	2.4442	2.3306	2.4717	2.6235	2.2950	3.2698	2.5988	2.4900	2.3386	2.4816	2.6321	2.4776	2.2383	2.2392	2.6280
	$\mathbf{ES}_{t+1}^{2.5\%}$	1.9818	2.6889	2.1844	2.0754	2.0261	2.1010	2.1807	1.9984	2.6639	2.1526	2.1056	2.0320	2.1072	2.1839	2.0780	1.9603	1.9612	2.2484
	$\mathbf{ES}_{t+1}^{5\%}$	1.7525	2.2789	1.8652	1.8027	1.7875	1.8230	1.8706	1.7669	2.2688	1.8441	1.8247	1.7969	1.8271	1.8728	1.8002	1.7399	1.7408	1.9669
	$\mathbf{RVaR}_{t+1}^{1\%,5\%}$	1.6201	2.0194	1.6675	1.6423	1.6517	1.6608	1.6824	1.6348	2.0186	1.6554	1.6584	1.6614	1.6635	1.6830	1.6308	1.6153	1.6162	1.8016
	$\mathbf{RVaR}_{t+1}^{2.5\%,5\%}$	1.5232	1.8690	1.5460	1.5300	1.5488	1.5450	1.5604	1.5353	1.8737	1.5356	1.5438	1.5617	1.5470	1.5617	1.5224	1.5195	1.5204	1.6854
	$\mathbf{RVaR}_{t+1}^{1\%,2.5\%}$	1.7815	2.2702	1.8698	1.8295	1.8231	1.8539	1.8856	1.8006	2.2600	1.8551	1.8493	1.8276	1.8576	1.8852	1.8116	1.7750	1.7759	1.9953
1000 Obs.	$\mathbf{VaR}_{t+1}^{1\%}$	1.9990	2.6668	2.0996	2.1142	2.0018	2.0983	2.1219	2.0127	2.6788	2.1167	2.1212	2.0270	2.1208	2.1320	2.2206	1.9766	1.9875	2.3215
	$\mathbf{VaR}_{t+1}^{2.5\%}$	1.6751	2.1108	1.6873	1.7178	1.6845	1.7055	1.6918	1.6850	2.1184	1.6968	1.7245	1.6971	1.7238	1.7094	1.7721	1.6627	1.6735	1.8560
	$\mathbf{VaR}_{t+1}^{5\%}$	1.3975	1.6878	1.3582	1.3966	1.4030	1.3838	1.3680	1.4017	1.7026	1.3677	1.4023	1.4107	1.3973	1.3816	1.4271	1.3878	1.3987	1.4934
	$\mathbf{ES}_{t+1}^{1\%}$	2.3113	3.4954	2.6473	2.5471	2.3119	2.5297	2.6567	2.3179	3.4402	2.6577	2.5460	2.3426	2.5457	2.6253	3.0247	2.2735	2.2844	2.8576
	$\mathbf{ES}_{t+1}^{2.5\%}$	2.0221	2.8172	2.1850	2.1587	2.0245	2.1451	2.1960	2.0307	2.7987	2.1966	2.1635	2.0471	2.1618	2.1916	2.3882	1.9951	2.0060	2.3829
	$\mathbf{ES}_{t+1}^{5\%}$	1.7746	2.3496	1.8484	1.8526	1.7800	1.8392	1.8574	1.7834	2.3470	1.8585	1.8585	1.7968	1.8556	1.8632	1.9842	1.7569	1.7678	2.0222
	$\mathbf{RVaR}_{t+1}^{1\%,5\%}$	1.6405	2.0632	1.6487	1.6789	1.6471	1.6666	1.6576	1.6498	2.0737	1.6587	1.6866	1.6604	1.6831	1.6727	1.7240	1.6277	1.6386	1.8134
	$\mathbf{RVaR}_{t+1}^{2.5\%,5\%}$	1.5271	1.8820	1.5119	1.5464	1.5356	1.5334	1.5188	1.5362	1.8953	1.5203	1.5536	1.5465	1.5494	1.5349	1.5802	1.5187	1.5296	1.6616
	$\mathbf{RVaR}_{t+1}^{1\%,2.5\%}$	1.8294	2.3651	1.8768	1.8998	1.8329	1.8887	1.8889	1.8391	2.3710	1.8892	1.9084	1.8502	1.9058	1.9025	1.9638	1.8095	1.8204	2.0664

Table 57 – Scenario 1 - Realized losses of a portfolio with 4 assets with a negative correlation between them, namely Euro, Real, U.S. 5-year Treasury Yield, and Bitcoin, with multivariate models results for the risk measures, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Measure	RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
250 Obs.	$\mathbf{VaR_{t+1}^{1\%}}$	0.0509	0.0471	0.0490	0.0429	0.0506	0.0512	0.0476	0.0512	0.0482	0.0447	0.0454	0.0515	0.0516	0.0452	**0.0444**	0.0513	0.0507	0.0604
	$\mathbf{VaR_{t+1}^{2.5\%}}$	0.0902	0.0879	0.0894	**0.0845**	0.0891	0.0879	0.0856	0.0906	0.0929	0.0857	0.0856	0.0915	0.0908	0.0861	0.0884	0.0913	0.0909	0.1001
	$\mathbf{VaR_{t+1}^{5\%}}$	0.1417	0.1456	0.1359	0.1380	0.1397	0.1393	0.1368	0.1400	0.1495	**0.1353**	0.1384	0.1386	0.1405	0.1375	0.1386	0.1427	0.1421	0.1528
	$\mathbf{ES_{t+1}^{1\%}}$	1.2065	1.0813	1.2079	1.0970	1.2446	1.2388	1.1125	1.2284	1.0927	1.1155	1.1463	1.2502	1.2299	1.1952	**1.0521**	1.2412	1.2476	1.5745
	$\mathbf{ES_{t+1}^{2.5\%}}$	1.1612	1.0980	1.1512	1.0998	1.1552	1.1642	1.1082	1.1623	1.1198	1.1122	1.1123	1.1812	1.1660	1.1421	**1.0911**	1.1733	1.1750	1.3597
	$\mathbf{ES_{t+1}^{5\%}}$	1.1669	1.1504	1.1475	1.1311	1.1593	1.1604	1.1356	1.1599	1.1623	**1.1303**	1.1417	1.1662	1.1656	1.1468	1.1308	1.1737	1.1720	1.3008
	$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	0.2123	0.3058	0.2404	0.2167	0.2128	0.2341	0.2308	0.2122	0.3025	0.2270	0.2197	0.2150	0.2317	0.2269	0.2279	0.2124	**0.2115**	0.2414
	$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	0.2434	0.2989	0.2554	0.2421	**0.2414**	0.2520	0.2498	0.2436	0.3009	0.2458	0.2438	0.2455	0.2551	0.2465	0.2566	0.2449	0.2442	0.2702
	$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0.1574	0.1473	0.1515	**0.1469**	0.1557	0.1539	0.1530	0.1568	0.1537	0.1482	0.1498	0.1560	0.1570	0.1518	0.1525	0.1597	0.1583	0.1748
500 Obs.	$\mathbf{VaR_{t+1}^{1\%}}$	0.0462	0.0441	0.0451	0.0456	0.0451	0.0422	0.0462	0.0481	**0.0421**	0.0454	0.0454	0.0469	0.0496	0.0465	0.0429	0.0479	0.0479	0.0485
	$\mathbf{VaR_{t+1}^{2.5\%}}$	0.0885	0.0863	0.0873	0.0877	**0.0849**	0.0851	0.0878	0.0864	0.0881	0.0867	0.0882	0.0890	0.0883	0.0872	0.0879	0.0878	0.0877	0.0924
	$\mathbf{VaR_{t+1}^{5\%}}$	0.1380	0.1429	0.1362	0.1382	0.1374	0.1360	0.1352	0.1373	0.1429	**0.1336**	0.1381	0.1376	0.1371	0.1380	0.1403	0.1387	0.1388	0.1503
	$\mathbf{ES_{t+1}^{1\%}}$	1.0804	**1.0300**	1.0823	1.0647	1.0704	1.0796	1.0877	1.1193	1.0304	1.0740	1.0764	1.0990	1.1092	1.0923	1.0428	1.1018	1.1055	1.1077
	$\mathbf{ES_{t+1}^{2.5\%}}$	1.1036	**1.0707**	1.1042	1.0914	1.0804	1.0883	1.1019	1.1082	1.0771	1.0906	1.1033	1.1106	1.1109	1.1010	1.0870	1.1048	1.1076	1.1238
	$\mathbf{ES_{t+1}^{5\%}}$	1.1276	1.1296	1.1258	1.1250	1.1190	1.1186	1.1201	1.1305	1.1279	**1.1156**	1.1295	1.1276	1.1301	1.1309	1.1337	1.1323	1.1342	1.1712
	$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	0.2100	0.2800	0.2306	0.2226	**0.2082**	0.2162	0.2325	0.2124	0.2820	0.2282	0.2218	0.2133	0.2260	0.2322	0.2316	0.2147	0.2147	0.2450
	$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	0.2418	0.2799	0.2484	0.2475	**0.2367**	0.2415	0.2506	0.2408	0.2811	0.2465	0.2480	0.2444	0.2492	0.2497	0.2605	0.2434	0.2435	0.2695
	$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0.1500	0.1452	0.1525	0.1516	0.1472	0.1470	0.1507	0.1484	**0.1449**	0.1484	0.1514	0.1498	0.1518	0.1533	0.1511	0.1504	0.1503	0.1580
1000 Obs.	$\mathbf{VaR_{t+1}^{1\%}}$	0.0487	0.0455	0.0448	0.0468	0.0492	0.0505	0.0463	0.0477	**0.0430**	0.0497	0.0475	0.0488	0.0500	0.0477	0.0432	0.0459	0.0455	0.0584
	$\mathbf{VaR_{t+1}^{2.5\%}}$	0.0889	0.0898	0.0882	0.0884	0.0874	0.0893	0.0879	0.0877	0.0892	0.0893	0.0875	0.0879	0.0881	0.0880	0.0871	0.0862	**0.0860**	0.1021
	$\mathbf{VaR_{t+1}^{5\%}}$	0.1380	0.1441	0.1371	0.1361	0.1372	0.1375	**0.1345**	0.1372	0.1443	0.1371	0.1367	0.1380	0.1358	0.1370	0.1391	0.1360	0.1359	0.1545
	$\mathbf{ES_{t+1}^{1\%}}$	1.1264	1.0507	1.0837	1.1192	1.1397	1.1464	1.0927	1.1334	**1.0366**	1.1351	1.1140	1.1387	1.1357	1.0995	1.0426	1.0944	1.0940	1.2466
	$\mathbf{ES_{t+1}^{2.5\%}}$	1.1153	1.0902	1.1068	1.1152	1.1144	1.1258	1.1087	1.1129	1.0851	1.1243	1.1092	1.1133	1.1245	1.1078	**1.0828**	1.0917	1.0919	1.2221
	$\mathbf{ES_{t+1}^{5\%}}$	1.1314	1.1379	1.1287	1.1245	1.1285	1.1351	1.1233	1.1284	1.1341	1.1357	1.1269	1.1308	1.1313	1.1291	1.1289	**1.1147**	1.1150	1.2373
	$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	0.2123	0.2856	0.2245	0.2262	0.2111	0.2251	0.2263	0.2096	0.2823	0.2330	0.2257	0.2116	0.2251	0.2278	0.2408	**0.2065**	0.2068	0.2810
	$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	0.2423	0.2877	0.2449	0.2492	0.2393	0.2484	0.2468	0.2398	0.2845	0.2495	0.2477	0.2402	0.2469	0.2476	0.2625	**0.2368**	0.2370	0.2924
	$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0.1541	0.1509	0.1565	0.1521	0.1528	0.1557	0.1524	0.1520	0.1492	0.1575	0.1515	0.1539	0.1533	0.1563	0.1514	0.1488	**0.1478**	0.1819

Table 58 – Scenario 2 - Realized losses of a portfolio with 4 assets with a positive correlation between them, namely S&P 500, Ibovespa, U.S. 5-year Treasury Yield, and U.S. 10-year Treasury Yield, with multivariate models results for the risk measures, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Measure	Models																	
		RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
250 Obs.	$\mathbf{VaR^{1\%}_{t+1}}$	0.0603	**0.0496**	0.0583	0.0562	0.0583	0.0551	0.0581	0.0607	0.0497	0.0558	0.0573	0.0565	0.0557	0.0583	0.0560	0.0600	0.0602	0.1308
	$\mathbf{VaR^{2.5\%}_{t+1}}$	0.1065	0.1019	0.1052	**0.1017**	0.1069	0.1048	0.1059	0.1077	0.1045	0.1044	0.1044	0.1064	0.1063	0.1073	0.1045	0.1043	0.1044	0.1859
	$\mathbf{VaR^{5\%}_{t+1}}$	0.1638	0.1649	0.1625	0.1633	0.1642	0.1654	0.1628	0.1641	0.1675	0.1651	**0.1599**	0.1629	0.1638	0.1655	0.1684	0.1624	0.1625	0.2595
	$\mathbf{ES^{1\%}_{t+1}}$	1.1914	1.0999	1.1558	1.1481	1.1468	1.1161	1.1223	1.1903	**1.0410**	1.1453	1.1820	1.1451	1.1287	1.1602	1.1254	1.1512	1.1508	5.5428
	$\mathbf{ES^{2.5\%}_{t+1}}$	1.1789	1.1244	1.1595	1.1474	1.1658	1.1450	1.1458	1.1793	**1.1059**	1.1410	1.1651	1.1696	1.1551	1.1625	1.1497	1.1473	1.1469	3.0453
	$\mathbf{ES^{5\%}_{t+1}}$	1.2028	1.1749	1.1902	1.1927	1.1937	1.1920	1.1816	1.2030	**1.1697**	1.1883	1.1803	1.1959	1.1945	1.1976	1.2039	1.1822	1.1824	2.2177
	$\mathbf{RVaR^{1\%,5\%}_{t+1}}$	0.2304	0.2688	0.2547	0.2416	0.2373	0.2426	0.2470	0.2364	0.2618	0.2478	0.2392	0.2301	0.2375	0.2522	**0.2263**	0.2293	0.2298	0.4637
	$\mathbf{RVaR^{2.5\%,5\%}_{t+1}}$	0.2749	0.2940	0.2857	0.2743	0.2799	0.2803	0.2832	0.2789	0.2927	0.2795	0.2771	0.2751	0.2792	0.2875	**0.2681**	0.2699	0.2703	0.4860
	$\mathbf{RVaR^{1\%,2.5\%}_{t+1}}$	0.1887	**0.1729**	0.1896	0.1866	0.1882	0.1894	0.1876	0.1923	0.1792	0.1889	0.1837	0.1855	0.1880	0.1926	0.1997	0.1868	0.1869	0.3451
500 Obs.	$\mathbf{VaR^{1\%}_{t+1}}$	0.0625	**0.0488**	0.0530	0.0569	0.0576	0.0545	0.0555	0.0601	0.0531	0.0604	0.0593	0.0602	0.0613	0.0600	0.0636	0.0634	0.0632	0.0579
	$\mathbf{VaR^{2.5\%}_{t+1}}$	0.1058	**0.1012**	0.1046	0.1028	0.1030	0.1027	0.1030	0.1058	0.1042	0.1065	0.1049	0.1052	0.1079	0.1090	0.1163	0.1072	0.1071	0.1061
	$\mathbf{VaR^{5\%}_{t+1}}$	0.1622	0.1616	0.1643	**0.1609**	0.1628	0.1637	0.1628	0.1642	0.1616	0.1630	0.1632	0.1623	0.1641	0.1669	0.1822	0.1651	0.1645	0.1681
	$\mathbf{ES^{1\%}_{t+1}}$	1.2532	**1.0531**	1.1016	1.1615	1.1589	1.1373	1.1219	1.1855	1.0913	1.1871	1.1813	1.1971	1.1570	1.1724	1.1656	1.3141	1.3097	1.1855
	$\mathbf{ES^{2.5\%}_{t+1}}$	1.1920	**1.1020**	1.1432	1.1517	1.1564	1.1473	1.1429	1.1738	1.1305	1.1819	1.1725	1.1799	1.1766	1.1800	1.2125	1.2252	1.2228	1.1688
	$\mathbf{ES^{5\%}_{t+1}}$	1.2090	**1.1615**	1.1894	1.1844	1.1942	1.1932	1.1910	1.2050	1.1771	1.2033	1.2040	1.1981	1.2043	1.2150	1.2669	1.2313	1.2289	1.2052
	$\mathbf{RVaR^{1\%,5\%}_{t+1}}$	0.2372	0.2652	0.2349	0.2387	0.2332	0.2330	0.2431	**0.2308**	0.2712	0.2532	0.2391	0.2363	0.2428	0.2444	0.2337	0.2373	0.2374	0.2614
	$\mathbf{RVaR^{2.5\%,5\%}_{t+1}}$	0.2735	0.2856	0.2713	0.2720	**0.2690**	0.2701	0.2757	0.2702	0.2930	0.2810	0.2744	0.2734	0.2805	0.2832	0.2767	0.2742	0.2743	0.2902
	$\mathbf{RVaR^{1\%,2.5\%}_{t+1}}$	0.1924	**0.1767**	0.1906	0.1881	0.1889	0.1890	0.1904	0.1936	0.1795	0.1947	0.1929	0.1905	0.1944	0.1982	0.2274	0.1980	0.1969	0.1900
1000 Obs.	$\mathbf{VaR^{1\%}_{t+1}}$	0.0586	**0.0446**	0.0563	0.0535	0.0592	0.0586	0.0571	0.0560	0.0476	0.0525	0.0550	0.0547	0.0528	0.0534	0.0556	0.0585	0.0583	0.0570
	$\mathbf{VaR^{2.5\%}_{t+1}}$	0.1023	0.0998	0.1030	0.1033	0.1039	0.1065	0.1056	0.1005	**0.0977**	0.1028	0.1024	0.1015	0.1052	0.1042	0.1082	0.1018	0.1015	0.1079
	$\mathbf{VaR^{5\%}_{t+1}}$	0.1606	0.1610	0.1634	0.1618	0.1603	0.1634	0.1625	**0.1588**	0.1606	0.1621	0.1609	0.1599	0.1631	0.1638	0.1794	0.1604	0.1598	0.1777
	$\mathbf{ES^{1\%}_{t+1}}$	1.1475	**1.0336**	1.0946	1.0786	1.1303	1.1108	1.0756	1.1186	1.0352	1.0561	1.0783	1.0958	1.0641	1.0608	1.0678	1.1547	1.1512	1.1803
	$\mathbf{ES^{2.5\%}_{t+1}}$	1.1438	**1.0951**	1.1326	1.1319	1.1470	1.1540	1.1337	1.1344	1.0893	1.1164	1.1255	1.1288	1.1319	1.1251	1.1312	1.1440	1.1421	1.1919
	$\mathbf{ES^{5\%}_{t+1}}$	1.1803	1.1594	1.1823	1.1775	1.1810	1.1927	1.1814	1.1752	**1.1590**	1.1717	1.1741	1.1747	1.1815	1.1795	1.2204	1.1813	1.1791	1.2479
	$\mathbf{RVaR^{1\%,5\%}_{t+1}}$	0.2347	0.2760	0.2543	0.2424	0.2350	0.2444	0.2546	0.2328	0.2833	0.2425	0.2478	**0.2318**	0.2440	0.2512	0.2777	0.2327	0.2331	0.2665
	$\mathbf{RVaR^{2.5\%,5\%}_{t+1}}$	0.2700	0.2948	0.2799	0.2766	0.2721	0.2818	0.2823	0.2687	0.2971	0.2765	0.2798	0.2695	0.2807	0.2804	0.2955	**0.2679**	0.2680	0.2967
	$\mathbf{RVaR^{1\%,2.5\%}_{t+1}}$	0.1867	0.1713	0.1913	0.1864	0.1860	0.1922	0.1896	0.1815	**0.1685**	0.1870	0.1850	0.1828	0.1891	0.1886	0.2029	0.1864	0.1851	0.1978

Table 59 – Scenario 3 - Realized losses of a portfolio full with the 8 assets analyzed, namely S&P 500, Ibovespa, U.S. 5-year Treasury Yield, U.S. 10-year Treasury Yield, Euro, Real, WTI Crude Oil, and Bitcoin with multivariate models results for the risk measures, using a confidence level of 1%, 2.5%, and 5%; with 250, 500, and 1,000 observations.

Size	Realized Loss	Models																	
		RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
250 Obs.	$\mathbf{VaR_{t+1}^{1\%}}$	0.0372	0.0362	0.0363	0.0384	0.0393	0.0396	0.0380	0.0431	**0.0351**	0.0354	0.0386	0.0377	0.0373	0.0356	0.0376	0.0395	0.0395	0.1045
	$\mathbf{VaR_{t+1}^{2.5\%}}$	0.0708	0.0722	0.0714	0.0726	0.0734	0.0744	0.0725	0.0740	0.0721	0.0704	0.0733	0.0728	**0.0702**	0.0731	0.0715	0.0736	0.0743	0.1381
	$\mathbf{VaR_{t+1}^{5\%}}$	**0.1149**	0.1175	0.1208	0.1184	0.1179	0.1185	0.1151	0.1164	0.1180	0.1163	0.1185	0.1181	0.1190	0.1204	0.1162	0.1185	0.1191	0.1790
	$\mathbf{ES_{t+1}^{1\%}}$	1.1074	**1.0307**	1.0988	1.1381	1.1390	1.1358	1.1368	1.2613	1.0669	1.0909	1.1540	1.1192	1.1338	1.0811	1.0643	1.1456	1.1466	7.1337
	$\mathbf{ES_{t+1}^{2.5\%}}$	1.0794	**1.0512**	1.0756	1.0859	1.0986	1.1068	1.0926	1.1282	1.0631	1.0723	1.1081	1.0862	1.0803	1.0773	1.0599	1.1031	1.1094	3.5296
	$\mathbf{ES_{t+1}^{5\%}}$	1.0857	1.0791	1.1068	1.0999	1.0979	1.1102	1.0889	1.1034	1.0845	1.0897	1.1089	1.0972	1.1030	1.1069	**1.0788**	1.1043	1.1092	2.3145
	$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	**0.1612**	0.2145	0.1817	0.1747	0.1660	0.1793	0.1833	0.1733	0.2129	0.1789	0.1749	0.1635	0.1732	0.1805	0.1757	0.1635	0.1632	0.2920
	$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	**0.1889**	0.2257	0.1995	0.1989	0.1938	0.2022	0.2031	0.1973	0.2228	0.1969	0.1985	0.1923	0.1950	0.2016	0.1978	0.1924	0.1928	0.3168
	$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0.1261	**0.1197**	0.1333	0.1315	0.1319	0.1310	0.1277	0.1316	0.1201	0.1267	0.1319	0.1308	0.1299	0.1330	0.1304	0.1333	0.1345	0.2643
500 Obs.	$\mathbf{VaR_{t+1}^{1\%}}$	0.0409	0.0343	0.0383	0.0407	0.0396	0.0364	0.0376	0.0421	**0.0316**	0.0365	0.0382	0.0409	0.0386	0.0386	0.0398	0.0402	0.0401	0.0404
	$\mathbf{VaR_{t+1}^{2.5\%}}$	0.0736	0.0695	0.0730	0.0746	0.0715	0.0730	0.0711	0.0756	**0.0678**	0.0712	0.0735	0.0751	0.0729	0.0741	0.0734	0.0730	0.0731	0.0756
	$\mathbf{VaR_{t+1}^{5\%}}$	0.1166	**0.1143**	0.1180	0.1181	0.1150	0.1173	0.1146	0.1200	0.1146	0.1177	0.1168	0.1186	0.1188	0.1174	0.1180	0.1187	0.1189	0.1202
	$\mathbf{ES_{t+1}^{1\%}}$	1.1489	1.0146	1.0673	1.1433	1.1582	1.0621	1.0750	1.1699	**1.0017**	1.0456	1.0810	1.1376	1.0988	1.0600	1.0938	1.1397	1.1406	1.1637
	$\mathbf{ES_{t+1}^{2.5\%}}$	1.1018	1.0357	1.0731	1.1066	1.0915	1.0776	1.0629	1.1188	**1.0202**	1.0599	1.0845	1.1060	1.0863	1.0869	1.0805	1.0931	1.0952	1.1212
	$\mathbf{ES_{t+1}^{5\%}}$	1.1007	1.0634	1.0938	1.1096	1.0899	1.0937	1.0789	1.1215	**1.0587**	1.0929	1.0946	1.1081	1.1066	1.0973	1.0941	1.1091	1.1120	1.1133
	$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	0.1660	0.2013	0.1747	0.1741	0.1675	0.1666	0.1750	0.1693	0.1951	0.1722	0.1715	0.1702	0.1708	0.1760	0.1739	0.1649	**0.1646**	0.1826
	$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	0.1925	0.2118	0.1960	0.1971	**0.1913**	0.1931	0.1953	0.1962	0.2057	0.1931	0.1955	0.1973	0.1941	0.1983	0.1952	0.1915	0.1916	0.2060
	$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0.1328	0.1187	0.1334	0.1362	0.1300	0.1325	0.1297	0.1381	**0.1165**	0.1330	0.1329	0.1360	0.1344	0.1335	0.1341	0.1354	0.1356	0.1347
1000 Obs.	$\mathbf{VaR_{t+1}^{1\%}}$	0.0389	**0.0338**	0.0369	0.0360	0.0385	0.0378	0.0375	0.0394	**0.0338**	0.0361	0.0385	0.0379	0.0383	0.0368	0.0378	0.0386	0.0382	0.0397
	$\mathbf{VaR_{t+1}^{2.5\%}}$	0.0712	**0.0681**	0.0720	0.0716	0.0721	0.0722	0.0713	0.0711	0.0687	0.0721	0.0726	0.0721	0.0731	0.0704	0.0715	0.0715	0.0712	0.0725
	$\mathbf{VaR_{t+1}^{5\%}}$	0.1159	0.1137	0.1166	0.1172	0.1151	0.1160	0.1174	0.1143	0.1136	0.1163	0.1169	0.1161	0.1187	0.1168	0.1172	0.1142	0.1141	**0.1127**
	$\mathbf{ES_{t+1}^{1\%}}$	1.1058	**0.9934**	1.0506	1.0478	1.1073	1.0695	1.0594	1.1163	0.9986	1.0319	1.0766	1.0976	1.0853	1.0421	1.0453	1.1028	1.0996	1.0795
	$\mathbf{ES_{t+1}^{2.5\%}}$	1.0724	**1.0146**	1.0609	1.0597	1.0737	1.0656	1.0580	1.0667	1.0218	1.0531	1.0709	1.0727	1.0790	1.0468	1.0530	1.0683	1.0661	1.0590
	$\mathbf{ES_{t+1}^{5\%}}$	1.0853	**1.0520**	1.0879	1.0862	1.0806	1.0828	1.0855	1.0767	1.0541	1.0773	1.0899	1.0850	1.1004	1.0809	1.0836	1.0752	1.0738	1.0607
	$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	0.1659	0.2079	0.1755	0.1707	0.1656	0.1723	0.1769	0.1672	0.2050	0.1734	0.1732	0.1659	0.1752	0.1745	0.1940	**0.1634**	0.1635	0.1912
	$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	0.1903	0.2138	0.1957	0.1935	0.1913	0.1951	0.1954	0.1909	0.2142	0.1949	0.1969	0.1916	0.1981	0.1942	0.2027	0.1893	**0.1891**	0.2074
	$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0.1304	**0.1162**	0.1325	0.1310	0.1296	0.1308	0.1323	0.1285	0.1158	0.1306	0.1322	0.1303	0.1344	0.1310	0.1303	0.1286	0.1277	0.1260

Table 60 – The number of occurrences of each model as the best among all risk measures, using a significance level of 1%, 2.5%, and 5%.
*: Indicates the model with more occurrences among all the risk measures.
**: Indicates models with no occurrences among all the risk measures.

Measure	Models																	
	RV_{norm}	RV_{std}	RV_{sstd}	RV_{ged}	RV_{snorm}	RV_{sged}	RV_{jsu}	CV_{norm}	CV_{std}	CV_{sstd}	CV_{ged}	CV_{snorm}	CV_{sged}	CV_{jsu}	HS	DCC	aDCC	GOGARCH
$\mathbf{VaR_{t+1}^{1\%}}$	0	3	0	0	0	0	0	0	4	0	0	0	0	0	1	0	0	0
$\mathbf{VaR_{t+1}^{2.5\%}}$	0	2	0	2	1	0	0	0	2	0	0	0	1	0	0	0	1	0
$\mathbf{VaR_{t+1}^{5\%}}$	1	1	0	1	0	0	1	1	0	2	1	0	0	0	0	0	0	1
$\mathbf{ES_{t+1}^{1\%}}$	0	5	0	0	0	0	0	0	3	0	0	0	0	0	1	0	0	0
$\mathbf{ES_{t+1}^{2.5\%}}$	0	5	0	0	0	0	0	0	2	0	0	0	0	0	2	0	0	0
$\mathbf{ES_{t+1}^{5\%}}$	0	2	0	0	0	0	0	0	3	2	0	0	0	0	1	1	0	0
$\mathbf{RVaR_{t+1}^{1\%,5\%}}$	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	2	2	0
$\mathbf{RVaR_{t+1}^{2.5\%,5\%}}$	1	0	0	0	4	0	0	0	0	0	0	0	0	0	1	2	1	0
$\mathbf{RVaR_{t+1}^{1\%,2.5\%}}$	0	4	0	1	0	0	0	0	3	0	0	0	0	0	0	0	1	0
Total	3	**22***	**0****	4	6	**0****	1	2	17	4	1	1	1	**0****	7	5	5	1

Table 61 – The number of occurrences of each model as the best by risk measure, using a significance level of 1%, 2.5%, and 5%.
*: Indicates the model with more occurrences for a risk measure.
**: Indicates models with no occurrences for a risk measure.

Models	**Measure**									
	$\mathbf{VaR^{1\%}_{t+1}}$	$\mathbf{VaR^{2.5\%}_{t+1}}$	$\mathbf{VaR^{5\%}_{t+1}}$	$\mathbf{ES^{1\%}_{t+1}}$	$\mathbf{ES^{2.5\%}_{t+1}}$	$\mathbf{ES^{5\%}_{t+1}}$	$\mathbf{RVaR^{1\%,5\%}_{t+1}}$	$\mathbf{RVaR^{2.5\%,5\%}_{t+1}}$	$\mathbf{RVaR^{1\%,2.5\%}_{t+1}}$	**Total**
RV_{norm}	**0****	**0****	1	**0****	**0****	**0****	1	1	**0****	3
RV_{std}	3	**2***	1	**5***	**5***	2	**0****	**0****	**4***	22
RV_{sstd}	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****
RV_{ged}	**0****	**2***	1	**0****	**0****	**0****	**0****	**0****	1	4
RV_{snorm}	**0****	1	**0****	**0****	**0****	**0****	1	**4***	**0****	6
RV_{sged}	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****
RV_{jsu}	**0****	**0****	1	**0****	**0****	**0****	**0****	**0****	**0****	1
CV_{norm}	**0****	**0****	1	**0****	**0****	**0****	1	**0****	**0****	2
CV_{std}	**4***	**2***	**0****	3	2	**3***	**0****	**0****	3	17
CV_{sstd}	**0****	**0****	**2***	**0****	**0****	2	**0****	**0****	**0****	4
CV_{ged}	**0****	**0****	1	**0****	**0****	**0****	**0****	**0****	**0****	1
CV_{snorm}	**0****	**0****	**0****	**0****	**0****	**0****	1	**0****	**0****	1
CV_{sged}	**0****	1	**0****	**0****	**0****	**0****	**0****	**0****	**0****	1
CV_{jsu}	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****	**0****
HS	1	**0****	**0****	1	2	1	1	1	**0****	7
DCC	**0****	**0****	**0****	**0****	**0****	1	**2***	2	**0****	5
aDCC	**0****	1	**0****	**0****	**0****	**0****	**2***	1	1	5
GOGARCH	**0****	**0****	1	**0****	**0****	**0****	**0****	**0****	**0****	1

5 Conclusions

Risk forecasts are an important tool for the investor, portfolio managers, accountants, and so on. By measuring and forecasting the risk, companies can more easily compare risks on many different levels and note declines or improvements in their overall situations or specific problem spots. Not only this, but it also predicts new riskier situations and conflicts based on the previous information. After all, the goal of measurement is to understand the nature of risk and pinpoint problems. Our study's main objective was to extensively forecast RVaR, a relatively new risk measure with theoretical and statistical pleasant properties. We did those forecasts to achieve the viability of RVaR as a market risk measure and compare it with important and traditional measures in the literature, VaR, and ES. We did the univariate procedures applying both empirical and numerical approaches to evaluate its forecasts. As for the multivariate analysis, we observed only the empirical procedures. For both analyses, we used significance levels of 1%, 2.5%, and 5%, samples sizes and rolling window sizes of 250, 500, and 1,000 observations.

Initially, we present some conclusions about the univariate analysis. At the numerical results, we could observe a general underestimation of the risk considering the positive results for the Bias. We had the $RVaR^{1\%,2.5\%}$ having the lowest value overall the risk measures, indicating solid viability of using RVaR instead of VaR or ES. Also, models with Student's t distributions (both skewed and not) were responsible for forecasting the lowest risk measure values overall, indicating another importance of those distributions and corroborating with previous studies in the literature. Also, we should highlight that $ES^{1\%}$ had the highest R. Bias value overall. Thus, compared to the other measures, ES has the highest percentage bias. The RMSE results were able to show a solidity of the previous results once the Bias and RMSE results were almost the same. The realized losses indicated quite different results than the Bias, R. Bias, and RMSE, probably because of their different parameters and error metrics. Finally, we could conclude that Johnson's SU distribution was the best, especially forecasting RVaR. Alike in the numerical study, we had the $RVaR^{1\%,2.5\%}$ having the lowest average values overall. In the empirical analysis, we could conclude that different distributions could forecast better different series for every risk measure, even in the same class of assets, such as the conflicting results between Ibovespa and S&P 500.

In the multivariate study, we note that the composite portfolio with 8 assets is the least risky due to the principle of diversification. Concerning portfolios with 4 assets, as expected, the portfolio with a negative correlation or close to zero is the least risky. Regarding the models evaluated, the C-Vine and R-Vine copulas overachieve the other models. In particular, considering the assets with a positive correlation, the copulas models with Student's t marginal distribution had the best performances. The Historical Simulation model had an overall poor performance, being better in smaller sample sizes and forecasting ES. We also noted a tendency of the same model performing better forecasting VaR and RVaR with the same significance levels. This tendency was not so clear for ES and RVaR.

After all, we conclude that RVaR, VaR, and ES tend to be sensitive to the model choice. Also, these risk measures tend to be sensitive to the analysis applied (univariate or multivariate), samples sizes or rolling estimation window sizes, significance levels, and distributions used, giving space for much more studies in the area. Furthermore, based on our results and given the good statistical properties of RVaR, we suggest that future work explore the use of this measure in financial problems, such as capital determination, for example. About ours, we see that a multivariate Monte Carlo simulation study may be a good achievement as the next steps.

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