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Logic, Ontology, and Arithmetic: A Study of the Development of Bertrand Russell's Mathematical Philosophy from *The Principles of Mathematics* to *Principia Mathematica*

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Tese apresentada ao Programa de Pós- Graduação em Filosofia da Universidade Federal do Rio Grande do Sul (UFRGS) como requisito parcial para a obtenção do Título de Doutor em Filosofia.

Orientador: Prof. Dr. Paulo Francisco Estrella Faria

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Resumo

O presente trabalho tem por objeto de análise o desenvolvimento da Filosofia Matemática de Bertrand Russell desde os *Principles of Mathematics* até - e inlcuindo - a primeira edição de *Principia Mathematica*, tendo como fio condutor as mudanças no pensamento de Russell com respeito a três tópicos interligados, a saber: (1) a concepção de Russell da Lógica enquanto uma ciência (2) os compromissos ontológicos da Lógica e (3) a tese Russelliana de que a Matemática Pura - a Aritmética particular - é nada mais do que um ramo da Lógica. Esses três tópicos interligados formam um fio condutor que seguimos na tese para avaliar qual interpretação fornece o melhor relato da evidência textual disponível em *Principia Mathematica* e nos manuscritos produzidos por Russell no período relevante. A posição geral defendida é que a interpretação de Gregory Landini apresenta argumentos decisivos contra a ortodoxia de comentadores que atribuem à *Principia* uma hierarquia de tipos ramificada de entidades confusamente formulada, e mostramos que os três pontos apontados acima que formam o fio condutor da tese corroboram fortemente a interpretação de Landini. Os resultados que apontam para a conclusão geral de nossa investigação estão apresentados na tese dividida em duas partes. A primeira parte discute o desenvolvimento da lógica de concepção de Russell e do projeto Logicista desde sua gênese e nos *Principles of Mathematics* até *Principia Mathematica*. Esta primeira parte define o contexto para a segunda, que discute a Lógica Russeliana e o Logicismo em sua versão madura apresentada em *Princ*ipia. Mostramos que, ao fim e ao cabo, o a teoria Lógica e a forma da tese Logicista apresentada em *Principia* é o resultado do longo processo iniciado com descoberta da Teoria dos Símbolos Incompletos que levou Russell a gradualmente reduzir os compromissos ontológicos de sua concepção da Lógica enquanto uma ciência, culminando na teoria apresentada na Introdução de *Principia*, na qual ele procura formular uma hierarquia dos tipos que evita o compromisso ontológico com classes, proposições e também com as assim chamadas funções proposicionais e que esse mesmo processo levou Russell a uma concepção da tese de Logicista de acordo com a qual a Matemática é uma ciência cujos compromissos ontológicos não incluem qualquer espécie de *objetos* (no sentido Fregeano) sejam eles particulares concretos ou abstratos.

Palavras-chaves: Bertrand Russell. Filosofia Matemática. Lógica. Fundamentos da Matemática.

Abstract

The present work has as its object of analysis the development of Bertrand Russell's Mathematical Philosophy from the *Principles of Mathematics* up to - and including - the first edition of *Principia Mathematica*, having as a guiding thread the changes in Russell's thought with respect to three interconnected topics, namely: (1) Russell's conception of Logic as a science (2) the ontological commitments of Logic and (3) Russell's thesis that Pure Mathematics - in particular Arithmetic - is nothing more than a branch of Logic. These three interconnected topics form a common thread that we follow in the dissertation to assess which interpretation offers the best account of the available textual evidence in *Principia Mathematica* and in the manuscripts produced by Russell in the relevant period. The general position held is that Gregory Landini's interpretation presents decisive arguments against the orthodoxy of commentators who attribute to *Principia* a confusingly formulated hierarchy of ramfified types of entities, and we show that the three points indicated out above that form the main thread of the thesis strongly corroborate Landini's interpretation. The results that point to the general conclusion of our investigation are stated in the dissertation divided into two parts. The first part discusses the development of Russell's conception of Logic and the of Logicist project from its genesis and in *Principles of Mathematics* up to *Principia Mathematica*. This first part sets the context for a second, which discusses a Russellian Logic and Logicism in its mature version presented in *Principia*. We show that, in the end, the Logic theory and the form of the Logicist thesis presented in *Principia* is the result of a long process that started with the discovery of the theory of Incomplete Symbols which led Russell to reduce the ontological commitments of his conception of Logic as a science, culminating in the theory of types presented in *Principia*'s Introduction, in which Russell seeks to formulate a hierarchy of types that avoids the ontological commitment to classes, propositions and also with so-called propositional functions, and that this same process led Russell to a conception of the Logicist thesis according to Mathematics is a science with no ontological commitments to any kind of *objects* (in the Fregean sense) whether these are conceived as concrete or abstract particulars.

Key words: Bertrand Russell. Mathematical Philosophy. Logic. Foundations of Mathematics.

Contents

1 Introduction

> I think many of us were drawn to our profession by Russell's books. He wrote a spectrum of books for a graduated public, layman to specialist. We were beguiled by the wit and a sense of new-found clarity with respect to central traits of reality. We got memorable first lessons in relativity, elementary particles, infinite numbers, and the foundations of arithmetic. At the same time we were inducted into traditional philosophical problems, such as that of the reality of matter and that of the reality of minds other than our own. For all this emergence of problems the overriding sense of new-found clarity was more than a match. In sophisticated retrospect we have had at points to reassess that clarity, but this was a sophistication that we acquired only *after* we were hooked.[1]

As the subtitle of the present doctoral dissertation unoriginally indicates, this text is about the development of Russell's Mathematical Philosophy from the *Principles of Mathematics* to *Principia Mathematica*. It must be said at the outset, however, that apart from a number of topical points, there is no dramatic novelty in the conclusions reached or in the arguments considered and put forward. The interpretation defended and the main arguments considered are mainly due to Gregory Landini, whose works are cited and discussed less, perhaps, than those of Russell alone[2]. In the chapter on *Principia*'s Logicism an attempt at further defending this interpretation is made by connecting some generally neglected aspects of Russell's 'Regressive Method in Mathematics' and of methodological aspects of his 'Logical Atomism' to the development of his views on the ontology of Logic and Mathematics between 1903-1910. These points, I think, are generally not sufficiently emphasized - in particular by those who disagree with Landini's views - and my discussion may be considered an attempt at a clarificatory contribution to the literature.

Landini has sometimes referred to his interpretation as 'revolutionary' in contrast to what may be called the 'orthodox' interpretation of the development of Russell's views. If I may elaborate the use of this terminology by appealing to the ideas of Thomas Kuhn, the present work is meant at the same time as *normal* and as *revolutionary* scholarship. It is 'normal' scholarship in the sense that it is intended mainly to present and defend an interpretation that has been proposed and developed for some time, so I am aiming mostly at exposition and adjustment of matters of detail. Hence the dearth of novelties or departures from conclusions already reached by other scholars. The work follows, however, an interpretation - or, to maintain the Kuhnian wordplay, a 'paradigm' - that is still being pushed forward by 'revolutionaries' against

1 QUINE, W., 1967, p.657-8. Our emphasis.

2 My doctoral research project as initally submitted aimed at a critical discussion/evaluation of Landini's interpretation and I was attracted to other interpretative approaches - in particlar that of LINSKY, B., 1999 which will also be discussed later on. But I gradually became more and more persuaded that Landini's views are right. There are a few points where I depart slightly from Landini's views, in particular in chapters 3 and 5, but this is indicated in the discussion or in footnotes.

an established orthodoxy that still has to be won over, or converted by persuasion. So, in *this* sense, it is 'revolutionary'.

In fact, still using the Kuhnian terminology, the present work may be seen as a survey of some of the main controversies one finds in the scholarship on Russell's Mathematical Philosophy (*Principia Mathematica*, in particular) and some of the main arguments aimed at showing why the new 'paradigm' must be accepted as common wisdom, at least in light of the currently available evidence. So one of my main goals is to provide a comprehensive exposition and defense of the arguments in favor of Landini's interpretation against what may called the 'orthodoxy'. Whether I have in fact succeeded in this goal is for readers to decide, but if I did succeed then this dissertation has fulfilled its main intended purpose.

Having said that, the purpose of this Introduction is threefold. First, to provide the context for and present what is the 'interpretative orthodoxy' that we aim to argue against. This is done in the first section. Second, to provide the contextualization and motivation for the 'revolutionary interpretation that will be exposed and defended against this 'orthodoxy'. This is done in the second section.[3] Third, to provide a brief summary of the dissertation as a whole and of the main topics/questions that it aims to adress, as well as a brief summary of each chapter and the aims that each of them is supposed to achieve individually.

1.1 The 'Received View' of *Principia Mathematica*

Willard Quine, perhaps the most important philosopher of the second half of 20th century, once justly claimed that Bertrand Russell and Alfred Whitehead's joint work *Principia Mathematica* is "one of the great intellectual monuments of all time"[4]. Similarly, Alfred Tarski - certainly one of the most important logicians of all time - claimed that *Principia* is "undoubtedly the most representative work of modern logic" and that the "influence it has exerted has been no less than epoch-making in the development of logical investigations"[5]. Quine and Tarski are hardly giving excessive praise to *Principia*. The two thousand pages, three volume work is indeed monumental and its importance and influence in the development of Mathematical Logic and the Analytic tradition in Philosophy is so great as to be almost inestimable.

But the fact of the matter is that the monumental character of *Principia* matches (perhaps, only barely) the monumental character of the aims it was suppose to achieve. As is well known, the work was conceived by its authors as an extended defense of the claim that all the main

3 I tried writing these first two sections in such a way that they provide an actual (general and semi-technical) introduction to the issues and literature that are discussed in the body of the text, so that some of it may eventually be useful for someone unfamiliar with some of the complicated historical details and some of the highly specialized literature discussed. This, it must be said, was done at the cost of some repetition with respect to some parts of the text.

4 QUINE, W., 1963, p.14.

5 TARSKI, A., 1941, p.229.

branches of Pure Mathematics can be developed on the basis of Logic alone. As is also well known, this view, which later became inextricable from labels like 'Logicism' and 'logicist project' was originally envisioned by Leibniz in a very rough and programmatic form and then developed systematically and in depth by Frege with respect to Arithmetic and parts of Analysis.

In a broad sense, Frege's work succeeded in carrying out this project to a significant extent, mainly by showing how definitions of 'Zero', 'Number' and 'Sucessor' could be given on the basis of logical and set-theoretical notions and showing that the fundamental principles of Arithmetic could be proved on the basis of a few logical *and* set-theoretical axioms. But as every novice student of Mathematical Philosophy learns, Frege's system had a fundamental flaw with respect to its set theoretical assumptions: they led to contradictions.

Putting things in terms that are instructive, albeit anachronistic[6], Frege's system allowed one (in practice) to assume the following principle that we nowadays call a 'naive' principle of comprehension[7]:

$$(\exists x)(y)(y \,\epsilon\, x \equiv \phi y) \tag{1.1.1}$$

For now, we may read the above as stating 'there is (set) *x* to which every *y* belongs if and only if *y* satisfies the condition ϕ'. The problem, of course, is that the lack of any restrictions as to what sort of conditions determine our 'ϕ' in the above formla allows one to deduce contradictions from it. The most simple of all such contradictions is forthcoming when in place of 'ϕy' we put 'y is not a member of itself', that is, if we instantiate 1.1.13 above as follows:

$$(\exists x)(y)(y \,\epsilon\, x \equiv y \sim \epsilon\, y) \tag{1.1.2}$$

6 Strictly speaking, Frege's system as presented in his *Grundgesetze* (FREGE, G., 1893) differs radically from any modern account of quantification theory and of naive set theory. Frege's logic is best interpreted not as predicate logic as one finds it in modern textbooks but as logic of *terms* (cf. DUARTE, A., 2009 and LANDINI, G., 2012). For an excellent - albeit anachronistic - reconstruction of Frege's system along the lines of modern predicate logic and set theory, cf . HATCHER, W., 1980.

7 The notation used in the present work is mainly that of *Principia Mathematica.* The most basic notation for propositional logic, predicate logic and set theory used throughout the text is the following:
$\supset$, :, :., ::,etc. for punctuation and also for conjunction.
(and) for punctuation.
$\vee$ for (inclusive) disjunction.
$\sim$ for (classical) negation.
$\supset$ for material implication.
$\equiv$ for material equivalence
$(x), (x, y)$, etc... for the universal quantifier.
$(\exists x), (\exists x, y)$, etc... for the existential quantifier.
$\phi x, \psi x, \chi x$, etc... for Russellian 'propositional functions'.
$\supset_{x_1,\ldots,x_n}$ for Peano and Russell's notion of formal implication.
$\equiv_{x_1,\ldots,x_n}$ for Peano and Russell's notion of formal equivalence.
ϵ for the relation of membership to a class.
Other notations will appear in the text with variations in each chapter (in particular in the chapters dealing, respectively, with Russell's *Principles of Mathematics* and *Principia Mathematica*) but the above basic symbols will be used throughout the whole text without variations, except in the appendices containing translations from *Principia Mathematica*.

where, $x \sim \epsilon\, y$ is, of course, $\sim (x \,\epsilon\, y)$. From the above, by instantiating *x* as *z*, it follows at once that:

$$(y)(y \,\epsilon\, z \equiv y \sim \epsilon\, y), \tag{1.1.3}$$

And, thus, by instantiating *y* as *z* above as well, we have:

$$z \,\epsilon\, z \equiv \sim z \,\epsilon\, z \tag{1.1.4}$$

This is the famous paradox that Russell communicated to Frege at the beginning of the 20th century, in what is perhaps the most famous letter in the whole history of both Philosophy and Mathematics[8]. Russell's discovery was groundbreaking because it showed that the assumption about the existence of sets as given by the 'naive' comprehension principle which was tacitally accepted by so many logicians and mathematicians at the time (including Frege and Russell) is contradictory.

To make a quite long story very, *very* short, in the outcome of this event Russell recruited his friend and former teacher Whitehead to try to succeed where Frege had failed and, in fact, to go quite beyond his work[9]. The result, after almost ten years of strenuous work were the three volumes of *Principia Mathematica*, which present their attempt to overcome the contradictions that affected the foundations of Set Theory as they stood at the beginning of the 20th century, and to provide overwhelming evidence that Pure Mathematics is nothing but the development of Mathematical Logic.

Again, the *general* outline of how these objectives were meant to be achieved in *Principia* is well known. In the Introduction to *Principia*'s first edition, which was mainly the work of Russell, the authors put forward the famous 'Theory of Types' as their approach to purge the foundations of Mathematical Logic and Se Theory from contradictions. The gist of that theory is the idea that not all things which can be meaningfully said or expressed about the members of a class can be said of the class itself (and vice versa). In particular, according to the theory of types, one *cannot* legitimately assert either that a class or set is or is not a member of itself[10]. Furthermore, Russell introduced several distinctions among what he called 'orders' of statements involving quantification, distinctions which avoided what he called, following Poincaré, 'vicious circularities'. Concerning the goal of providing evidence for what we nowadays call the 'Logicist thesis', Whitehead and Russell showed - employing their symbolic language which was by and large an enormous improvement over anything which existed at the time with the possible exception of Frege's - how an enormous amount of mathematical definitions and theorems could be grounded on their chosen primitives ideas and propositions.

8 Cf. van HEIJENOORT, J., 1967, pp.124-5.

9 Since Frege never made any claims about the logical status of mathematical laws beyond elementary arithmetic and some parts of the theory of real numbers.

10 Which, of course, was what engendered paradox in Frege's system.

Since *Principia*'s publication (and still to these days), however, it is universally agreed that the goals described above were achieved in *Principia* with radically different degrees of success. Ivor Grattan-Guinnes, for instance, gives us a very representative statement of this almost universal agreement:

> As a technical exercise [*Principia Mathematica*] is a brilliant virtuoso performance, maybe unequalled in the histories of both mathematics and logic; the chain-links of theorems are intricate, the details recorded Peano-style down to the last cross-reference, seemingly always correctly. However, one has to pass beyond the unclear introductory material - an eccentricity these days - before these virtues emerge, especially in the second Volume, which is easily the best of the trio.[11]

Similarly, commenting on the development of Mathematical Logic as conceived by Leibniz in terms of his *Characteristica Universalis* and developed through the works of Peano, Frege, Whitehead and Russell, Gödel famously observed that:

> Frege, in consequence of his painstaking analysis of the proofs, had not gotten beyond the most elementary properties of the series of integers, while Peano had accomplished a big collection of mathematical theorems expressed in the new symbolism, but without proofs. It was only in *Principia Mathematica* that full use was made of the new method for actually deriving large parts of mathematics from a very few logical concepts and axioms. In addition, the young science was enriched by a new instrument, the abstract theory of relations. The calculus of relations had been developed before by Peirce and Schroder, but only with certain restrictions and in too close analogy with the algebra of numbers. In *Principia* not only Cantor's set theory but also ordinary arithmetic and the theory of measurement are treated from this abstract relational standpoint.
>
> It is to be regretted that this first comprehensive and thorough-going presentation of mathematical logic and the derivation of mathematics from it is so greatly lacking in formal precision in the foundations (contained in $*1 - *21$ of *Principia*) that it presents in this respect a considerable step backwards as compared with Frege.[12]

Gödel's remarks, like those of Grattan-Guinness, capture very well what is a quite widespread (almost standard) view about *Principia*. In fact, with respect to its 'foundational' portions - i.e., Russell's *Introduction* which presents the so-called Ramified Theory of Types and the initial sections which treat of elementary portions of Logic and Set Theory - *Principia*'s legacy was incorporated into the canon of Mathematical Logic and Analytic Philosophy with a far greater emphasis on its defects rather than on its merits.

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[11] GRATTAN-GUINNESS, I., 200, p.388.
[12] GÖDEL, K., 1944, pp.119-20.

1.1.1 The Standard Criticisms of *Principia*'s Theory of Types

The main issues in *Principia* which gave rise to this situation are all, in one way or another, related to the presentation of the Theory of Types. As it is presented in *Principia*'s Introduction, the Theory of Types was designed by Russell to dissolve a number of 'contradictions' or 'paradoxes'[13] related to the fundamental notions of Logic and Set Theory. As Russell had done in his paper that made the Theory of Types famous[14], Whitehead and Russell considered a handful of such contradictions, in the following order: (1) the '*Epimenides*', perhaps better known as 'The Cretan Liar'; (2) Russell's own contradiction of the class of all classes which are not members of themselves; (3) the relation version of Russell's paradox; (4) the 'Burali-Forti' contradiction of the greatest ordinal number; (5) the 'Berry Paradox'; (6) the 'least indefinable ordinal' paradox; and finally (7) 'Richard's paradox'.

The general diagnosis which Whitehead and Russell provide concerning these contradictions is that "[...] they all result from a certain kind of vicious circle"[15] that "[...] arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole"[16]. They are thus claiming that a common feature of all paradoxes (1)-(7) indicated above is that they involve statements about sets or totalities such that "[...] if we suppose the set to have a total, it will contain members which presuppose this total"[17]. In their own words "[...] what the theory of types aims at effecting" is to break up such pseudo-totalites about which statements cannot be legitimately made "[...] into smaller sets, each of which is capable of a total"[18], i.e., meaningully considered in a declarative sentence[19].

In order to do this, they incorporate in *Principia*'s logical system what became known as the 'Vicious-circle Principle', which they introduce as follows:

> The Principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves *all* of a collection must not be one of the collection"; or, conversely "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total." We shall call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. Arguments which are condemned by the vicious–circle principle will be called "vicious-circle fallacies"[20].

Principia's attempt to resolve the contradictions by incorporating the Vicious Circle Principle within their formal system has three main components. Above we mentioned two,

[13] Russell used these terms interchangeably.
[14] RUSSELL, B., 1908.
[15] WHITEHEAD & RUSSELL, 1925, p.37 [1910, p.39].
[16] WHITEHEAD & RUSSELL, 1927 [1910], p.37.
[17] WHITEHEAD & RUSSELL, 1925, p.37 [1910, p.39].
[18] WHITEHEAD & RUSSELL, 1925, p.37 [1910, p.39].
[19] And as they observe "by saying that a set has "no total", we mean, primarily, that no significant statement can be maid about "all its members" (WHITEHEAD & RUSSELL, 1927 [1910], p.37.).
[20] WHITEHEAD & RUSSELL, 1925, p.37-8 [1910, p.40].

namely the introduction of type distinctions *and* the introduction of order distinctions which restricted what sort of condition or statement ϕx may be meaningfully formulated within their formalism.

In what is a famously problematic choice of terminology, Whitehead and Russsell called expressions like "ϕx" (which stated the conditions for class membership in comprehension principles like (I.I.I) considered above) 'propositional functions'. Their hierarchy of types of so-called 'propositional functions' restricted the arguments for which such functions had significant values, i.e., so-called 'propositions'. Speaking in terms of *Principia*'s syntax as explained in its Introduction, at the lowest type they had individual variables *x*, *y*, *z*, etc., and a further hierarchy of variables ϕ, ψ, *f*, *g*, etc., whose possible arguments were always restricted to some type. The notation "$\phi\hat{x}$", was meant to distinguish the "function itself "$\phi\hat{x}$" from "the undetermined value "ϕx" "[21] in their exposition[22]. So, for instance, according to *Principia*'s hierarchy of types, if $\phi\hat{x}$ is a propositional function whose possible arguments are individual variables, all of its significant values "ϕa", "ϕb", "ϕc", etc., are propositions - i.e., sentences - which result from substituting an individual constant for "x"; any attempt to substitute anything else for the variable *x* for would result in nonsense, and both "$\phi(\phi\hat{x})$" *and* "$\sim\ \phi(\phi\hat{x})$" , in particular, are ruled out as ungrammatical.

The hierarchy of orders then further splits expressions of the same type according to what sort of quantified variables appears in them in order to avoid another sort of 'vicious circularities'. As they explain:

> [...] the hierarchy which has to be constructed is not so simple as might at first appear. The functions which can take *a* as argument form an illegitimate totality, and themselves require division into a hierarchy of functions. This is easily seen as follows. Let $f(\phi\hat{z}, x)$ be a function of the two variables $\phi\hat{z}$ and *x*. Then if, keeping *x* fixed for the moment, we assert this with all possible values of ϕ, we obtain a proposition:
>
> $$(\phi) \,.\, f(\phi\hat{z}, x).$$
>
> Here, if *x* is a variable, we have a function of *x*; but as this function involves a totality of values of $\phi\hat{z}$, it cannot itself be one of the values included in the totality, by the vicious-circle principle. It follows that the totality of values of $\phi\hat{z}$ concerned in $(\phi) \,.\, f(\phi\hat{z}, x)$ is not the totality of all functions in which *x* can occur as argument, and that there is no such totality as that of all functions in which *x* can occur as argument.[23]

Putting it more generally, and in terms of *Principia*'s syntax, the authors explain that "if the highest order of variable occurring in a function, whether as argument or as apparent variable, is a function of the *n*th order, then the function in which it occurs is of the $n + 1th$

21 WHITEHEAD & RUSSELL, 1927 [1910], p.40.

22 For instance, to indicate when a 'function' - i.e., a non-individual variable - occurred as an argument in some given context, as in "$f(\phi\hat{x})$", "$f(x, \phi\hat{x})$", etc.

23 WHITEHEAD & RUSSELL, 1925, pp.48-9 [1910, p.51].

order"[24]. These distinctions of orders together with those of types are introduced by Whitehead and Russell so that the range of a given variable is always restricted to a given type *and* order and vicious circularities are always blocked gramatically: the distinctions are built up in this way so that the attempt to formulate any of the paradoxes mentioned above results in nonsense[25].

This logical theory whose variables are stratified into types with "superimposed orders"[26], as Quine puts it, is what became known as the *ramified* theory of types. The restrictions of the theory as developed with the Vicious Circle Principle as its guide, however, are quite severe for the purposes of developing classical Mathematics, and Whitehead and Russell were forced to assume a special axiom to attenuate their impact. The assumption in question is their infamous 'Axiom of Reducibility' which asserts that "given any function $\phi\hat{x}$, there is a formally equivalent predicative function"[27]. In symbols, this is stated as follows:

$$(\exists\psi)(x)(\phi x \equiv \psi!x) \tag{1.1.5}$$

As Whitehead and Russell explain in *Principia*'s Introduction, a function is said to be '*predicative*', "when it is of the next order above that of its argument, i.e. of the lowest order compatible with its having that argument"[28]. The import of this axiom is substantial: it allows one, for all intents and purposes, to ignore the superstructure of orders upon types, for it asserts that given any well-formed sentence of *Principia*'s grammar which contains a real (free) variable *x*, there is a function ψ of the lowest order compatible with that of its arguments such that ψ is equivalent to that open sentence for all arguments *x*.

The Axiom of Reducibility played a crucial role in what is actually the third and perhaps most important component of *Principia*'s resolution of the set theoretical contradictions, namely the '*No-class*' theory of classes. This theory is developed in sections $*20$ and $*21$ of *Principia* where Whitehead and Russell attempt to contextually define symbols for classes and relations-in-extension and to recover extensionality principles which suffice for the development of Mathematics without assuming any ontology of sets or classes. This approach was inspired by Russell's famous paraphrase or contextual definition of sentences containing occurrences of definite descriptions, i.e., expressions of the form "the *x* such that ϕx". As is well known, in *On Denoting*[29] - perhaps Russell's most celebrated paper - he proposed an analysis of a sentence like "the present king of France is bald" in terms of an existential generalization asserting that: (i) there is a king of France; (ii) that there is at most one King of France; and, finally, (iii) that he is bald. In

[24] WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56].
[25] Cf. WHITEHEAD & RUSSELL, 1925, p.60-5 [1910, p.63-8].
[26] QUINE, W., 1941, p.25.
[27] WHITEHEAD & RUSSELL, 1925, p.56 [1910, p.58].
[28] WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56].
[29] RUSSELL, B., 1905*a*.

symbols, Russell would later express this in a general manner as follows[30]:

$$\psi(\imath x \phi x) =_{\mathrm{Df}} (\exists y)(\phi y \,.\, (z)(\phi z \supset z = y) \,.\, \psi y] \tag{1.1.6}$$

In the above, the expression "$\imath x \phi x$" stands for the ordinary "the *x* such that ϕx", and the right-side of the definition provides a general way of paraphrasing sentences in which such expressions occur as subjects in a given context $\psi(...)$. So this analysis made it possible to very conveniently (and *rightly*) avoid treating "$\imath x \phi x$" as a singular term, but in fact treat it as what Russell famously called an *incomplete symbol*.

Put roughly, what we find in sections $*20$ and $*21$ of *Principia* is an attempt to extend this method of contextual paraphrase to expressions like "$\hat{x}\phi x$" and "$\hat{x}\hat{y}\psi x$" that are generally thought to stand, respectively, for the set of all things *x* which satisfy some condition ϕ and the class of all couples x and y which satisfy some condition ψ. Still putting things roughly, the main definitions of $*20$ are intended to do this by letting an arbitrary predicative function $\psi!\hat{x}$ that is equivalent to the condition ϕx for all values of x more or less do the job of the set which generally thought to be the referent of such expressions like "$\hat{x}\phi x$". This is done by first letting any occurrence of an expression such as "$\hat{x}\phi x$" within a given condition or context χ be rephrased according to the following contextual definition:

$$f(\hat{z}\phi z) =_{\mathrm{Df}} (\exists \psi)((x) : \phi x \,.\equiv .\, \psi! x : \chi(\psi!\hat{z})) \tag{1.1.7}$$

And then, letting lower-case Greek variables be place-holders for expressions of the form "$\hat{x}\phi x$", Whitehead and Russell emulate quantification over sets by contextually defining "$(\alpha)\chi\alpha$" and "$(\exists\alpha)\chi\alpha$", respectively, as "$(\phi)\chi(\phi!\hat{x})$" and "$(\exists\phi)\chi(\phi!\hat{x})$"[31]. This strategy allowed them at the same time to recover essential extensionality principles like:

$$(x)(\phi x \equiv \psi x) \equiv (\hat{z}\phi x = \hat{z}\psi x) \tag{1.1.8}$$

$$(\alpha)(\beta)((x) \,.\, x \in \alpha \equiv x \in \beta \,.\supset .\, \alpha = \beta) \tag{1.1.9}$$

without the need to assume any ontology of classes *and* to dissolve the set-theoretical contradictions. Russell's paradox, in particular, is blocked directly by the impossibility of meaningfully asserting that a propositional function can be its own argument. For according to the contextual definitions above, any attempt to state anything about a class α always amounts to a statement

30 *Principia* also introduces scope markers in such definitions, which are very important since they also apply to and required by the contextual definitions of class expressions. This will be addressed in chapter 4.

31 Similar definitions are provided by Whitehead and Russell to deal with relations-in-extension in section $*21$. Further distinctions are also added by Whitehead and Russell for dealing with classes of classes as opposed to classes of what Russell called classes of 'individuals'. This is an often neglected and immensely important aspect of *Principia*'s No-Class Theory, as Gregory Landini emphasizes (cf., for instance LANDINI, G., 1998, pp.168-171 and LANDINI, G., 2013b, pp.184-201).

about a predicative function $\phi!\hat{x}$ which is formally equivalent to the defining condition of α; so in particular, the statement that a class α is not a member of itself, i.e., "$\sim \alpha \,\epsilon\, \alpha$" or "$\alpha \sim \epsilon\, \alpha$", must be reduced to a statement that a given function ϕ does not apply to itself, i.e., "$\sim \phi!(\phi!\hat{x})$"; but this, as we already pointed out, is ruled by the restrictions of type as ungrammatical. In order for the No-Class theory to work as intended, however, the Axiom of Reducibility is necessary: without it, there is no guarantee that there is a predicative function which is formally equivalent to the condition which determines the class α in question.

Indeed, the Axiom of Reducibility is also required to recover important results of Classical Mathematics within *Principia*'s formal system, since the restrictions of orders greatly impact the strenght of the system. We may illustrate this with an informal survey of how Whitehead and Russell define the concept of natural number in terms of their notion of the ancestral R_* of a relation *R*. Expressed in *Principia*'s notation, the definition of the notion of the ancestral is given as follows[32]:

$$xR_*y =_{\text{Df}} (\phi)(\phi!x : zRw \,.\, \phi!z \,.\supset_{z,w} .\, \phi!w : \supset .\, \phi!y)$$

In words, this states that x is an *R*-ancestor of *y* if, and only if, *y* has all the '*R*-hereditary' predicative functions (or properties) possessed by *x*. In the presence of the Axiom of Reducibility, the above entails the following:

$$xR_*y \,.\equiv :.\, \psi x : zRw \,.\, \psi z \,.\supset_{z,w} .\, \psi w : \supset .\, \psi y$$

That is, *x* is an ancestor of *y* with respect to *R* if, and only if any *R*-hereditary function (predicative or not) which is true of *x* is also true of *y*.

As is well known, the concept of 'natural number' in *Principia* is defined using this notion and their definition entails the following theorem of mathematical induction:[33]

$$\text{Nc induct}(\gamma) \equiv \phi(0) : \phi\alpha \,.\supset_\alpha .\, \phi(\alpha+1) : \supset .\, \phi\gamma$$

As is also well known, for classical mathematics to be preserved, it is crucial that in the formula above one may legitimately have instead of the so-called 'function' ϕ the predicate '... is a natural number' or a more complex predicate which has this predicate as a subordinate part, as in '*n* is a natural number less than some natural number *m*'. Perhaps the most obvious example is the following version of the principle of induction:

$$\phi(0) \,.\, \text{Nc induct}(\alpha) \,.\, \phi\alpha \,.\supset_\alpha .\, \phi(\alpha+1) : \supset : \text{Nc induct}(\gamma) \,.\supset .\, \phi\gamma$$

The problem is that whenever one attempts to frame such a statement within a 'ramified' syntax, the occurrences of all terms in it must be read as having implicit or explicit indices indicating

32 This is a simplified version of Whitehead and Russell's definition, for *Principia*'s actual definition. *Principia*'s actual presentation of the ancestral will be discussed in chapter 5.

33 As we shall see in chapter 5, this is actually a theorem scheme.

their types/orders which determine what sort of substitutions are allowed in the formula in question. A proof of the principle of strong induction requires just the kind of 'vicious circularity' which the restrictions of order aim at avoiding: it requires that the range of substituends of the so-called 'propositional function' ϕ occurring in the definiens of "Nc induct" include expressions whose order is higher than that of ϕ, since if the definiens of "Nc induct(α)" contains a *bound* variable ϕ whose order is *n*, then "Nc induct(α)" must of the order immediately above that of ϕ. The Axiom of Reducibility cuts right across this difficulty since it asserts that any function whatever is equivalent to a predicative function[34].

It is at this point that *Principia*'s traits which were universally considered problematic and unsatisfactory become quite salient. To begin with, there is the quite problematic use of expressions like "propositional function" and "proposition". In its most strict sense, a 'propositional function' was certainly meant by Whitehead and Russell to be a complex expression containing free variables, schematically symbolized as "ϕx", "ψy", "$\phi(x, y)$", "$\psi(x, y)$", etc[35]; in other words, in the most strict sense in which they employed the notion of a 'propositional funcion' they most likely meant a formal expression ϕ which contained free occurrences of any number of variables *x*, *y*, *z*, etc, and a 'proposition' in its most strict sense, was meant to be just a declarative *sentence* (a statement which can be either true or false). But as Quine time and again observed[36], Whitehead and Russell ambiguously used the expression "propositional function" to mean (at least) two completely different things. Besides employing the label to mean what is best understood as dummy letters or schemata standing for arbitrary open formulas of their formal language, they also used the same label for proper variables of their formal language. To mark this contrast, which was more or less explicitly recognized by the authors, they distinguished this latter use by employing the symbol "!", which distinguished so-called 'predicative' functions from 'non-predicative' ones. So "ϕx" was just a scheme, a mere placeholder for an actual expression containing the variable "*x*" occurring free at any number of places, while "$\phi!$" was an actual expression of their formal language which could occur as an argument of other so-called functions and, crucially, also be bound by quantifiers.The result was that their exposition was plagued by a conspicuously confusing use of the phrase "propositional function", which to this day is a source of both simple misunderstandings and serious interpretative disputes about *Principia*.

Many of these arise at once. For instance, if we follow Quine in distinguishing genuine terms and variables of the formal language from schematic letters, the circumflex notation should be viewed as playing two different roles and, consequently, the symbols "$\phi!\hat{x}$ " and "$\phi\hat{x}$ should be understood as being of radically different kinds: in the first case the role of the circumflex is to indicate a formula where a *genuine term* occurs, as in "$\psi(\phi!\hat{x})$"; while in the second case, the device is used (confusingly) for nominalizing open formulas[37]. This, on its turn, strongly

34 It must be observed, however, that this applies only to extensional contexts.

35 WHITEHEAD & RUSSELL, 1925, pp.38-9 [1910, p.41].

36 Cf., for instance, QUINE, W., 1941, 1963 and 1969, pp.254-256.

37 For instance, in order to distinguish an open formula from the result of substituing another term *a* by a variable

suggests that Whitehead and Russell's lack of care with the use and mention of expressions ended up (unintentionally) blurring the distinction between expressions themselves and what they are be supposed to denote, as Quine vehemently insisted[38].

If we refrain, however, from following Quine's sound advice to distinguish clearly between variables and schematic letters and assume that "propositional function" has a univocal meaning, severe difficulties also arise. To begin with, we seem to find no reasonable explanation whatsoever as to why letters such as "ϕ" *never* appear bound to quantifiers in *Principia* in a formula *without* an accompanying "!"[39].Much more serious, however, the ontological status of so-called propositional functions becomes quite problematic since simply treating them either as entities or as mere expressions may easily - perhaps inevitably - also saddle Whitehead and Russell with a severe lack of care about use and mention and a corresponding ontologically confused use of the phrase "propositional function".

To give a very simple example, Whitehead and Russell repeatedly assert that a function is "an ambiguity"[40]. If they intended "propositional function" to univocally mean some sort of abstract entity, this would be quite misleading, since what may or may not be strictly ambiguous is an expression of some sort. On the other hand, if they intended propositional functions to be mere expressions, they do not make it clear what such expressions stand for - if anything at all - most crucially in the case of the bindable variables accompanied by "!".

In fact, the two most serious interrelated issues involved in the notion of a 'propositional function' concern directly their ontological status, in particular, whether they are anything beyond linguistic expressions. First, their dubious status seriously calls into question whether *Principia* attains any degree of success in its attempt at the ontological elimination of sets or classes. Again, this is a point that has been emphatically pressed by Quine, who urged that the variables "ϕ!", "ψ!", etc., must range over attributes or properties if we are to make any sense of their occurrences bound by quantifiers. This, however, calls into question the very motivation for the contextual elimination of class expressions which is then framed as an outright substitution of classes in favor of attributes on the basis of a confusion between sign and object. As Quine puts it:

> Russell had also a philosophical preference for attributes, and felt that in contextually defining classes on the basis of a theory of attributes he was explaining the obscurer in terms of the clearer. But this feeling was due to his failure to distinguish between propositional functions as predicates, or expressions, and propositional functions as attributes. Failing this, he could easily think that the

in it, as in "$\phi!x \supset \psi!x$" and $\phi!a \supset \phi!a$".

38 Cf., again, any of the following: QUINE, W., 1941, 1963 and 1969, pp.254-256.

39 This is an issue which arises for several interpretations and reconstructions of *Principia*. Cf., for instance, CHURCH, A., 1976; HYLTON, P., 1990, pp.303-9, in particular footnotes 26 and 34, also LINSKY, B., 1999, pp.79-82.

40 WHITEHEAD & RUSSELL, 1910, p.38-9

> notion of an attribute is clearer than that of a class; for that of a predicate is. But that of an attribute is less clear.[41]

Second, there is the question of the plausibility of the 'Vicious Circle Principle' as it appears in *Principia*. Famously, Gödel[42] argued that the principle is admissible only "[...] if the entities involved are constructed by ourselves", that is, if "[...] one takes the constructivistic (or nominalistic) standpoint toward the objects of logic and mathematics"[43]. Following a criticism first put forward by Ramsey[44], Gödel's main point is that if we assume the independent existence of 'propositional functions' and 'propositions' as abstract entities - attributes or properties, for instance - then "[...] there is nothing in the least absurd in the existence of totalities" of such entities that contain "members which can be described only by reference to this totality"[45].

On the heels of Ramsey and Gödel, Quine struck the nerve again, observing that the whole business of order restrictions as imputed by the Vicious Circle Principle were ill-conceived:

> This ramification of type theory is designed for the avoidance of certain contradictions of a quite different sort from [Russell's paradox]. But the treatment is vague, on account of failure to distinguish between expressions and their names. On restoring this distinction one finds that the contradictions against which this part of type theory was directed are no business of logic anyway; they can arise only in discourse that goes beyond pure logic and imports semantic terms such as 'true' or 'designate'. The whole ramification, with the axiom of reducibility, calls simply for amputation.
>
> It is readily seen also on other grounds that this part of type theory was bound to be wholly idle. The axiom of reducibility assures us that from the beginning we could have construed the notations of *Principia* as referreing exclusively to so-called "propositional functions" (predicative *attributes*); but when this is done, the resulting logic is the same as if neither "orders" nor "predicativity" nor "reducibility" had been thought of in the first place. That this simple situation escaped the attention of the authors is attributable, again, to the ambiguity of 'propositional function' and the underlying difficulty over use and mention.[46]

Needless to say, all the points above do afford us with a very bad diagnosis for the 'foundational' portions of *Principia.* The above account, if correct, reveals that the mathematical monument constructed by Whitehead and Russell actually rests on very shaky foundations.

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41 QUINE, W., 1969, p.256-7.

42 GÖDEL, K., 1944, p.127. Famously, Gödel also suggested that there may be several principles instead of one, since Russell at times formulates it in terms of an object being "definable only in terms of" totality, at others is terms of "involving" and also in terms of "presupposing", none of which hold water if attributes are understood as mind-independent entities.

43 GÖDEL, K., 1944, p.128.

44 Cf. RAMSEY, F., 1926, pp.41.

45 GÖDEL, K., 1944, p.128.

46 QUINE, W., 1941, pp.25-6. Cf. also QUINE, W., 1969, pp.254-6.

1.1.2 *Principia*'s Theory of Types and the 'Church Orthodoxy'

Many interpreters of Russell's works as well as authors who were sympathetic to the general approach to the contradictions which is given in *Principia*'s Introduction attempted to reconsider or overcome, in some way or another, the negative assessment of Ramsey, Gödel and Quine[47]. These attempts to present *Principia*'s foundations in a more favourable light can be categorized in two broad (and sometimes overlapping) groups. First, there are attempts to formulate *Principia*'s foundations in a way that is up to modern standards of rigour in order to clear up the problematic and/or confused aspects of the sloppy presentation of the Theory of Types given in the Introduction. Second, there are attempts to consider the views advocated in *Principia*'s Introduction in a more charitable way by considering them in light of Russell's other works in Mathematical Philosophy and by also taking into account the development of his views.

There are many examples of reconstructions of *Principia*'s formal system[48], but that of Church - which at this point may be considered the 'orthodox' reconstruction - is by far the best known. Church manufactured a rigorous formulation of the syntax of ramified type theory which, apart from some (*non*-trivial) departures from *Principia*[49], was intended to capture the resolution of the so-called 'semantic' antinomies or paradoxes by following the restrictions of orders imposed by the Vicious-Circle *Principle*[50]. Crucially, Church's reformulation also assu-

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47 QUINE, W., 1969, p.263. The earliest version of this criticism was introduced by Leon Chwisteck, who endorsed ramified type theory but rejected the Axiom of Reducibility (cf. CHWISTEK, L., 1921, 1922*a*, 1922*b*, 1924, 1925). In his Introduction to the second edition of *Principia*, Russell describes Chwistek's choice of rejecting Reducibility "without adopting any substitute" as "heroic" (WHITEHEAD & RUSSELL, 1927*a*, pp.xiv). For a discussion of Chwisteck's original work on the theory of types see LINSKY, B., 2009; for a discussion relating Chwisteck's views and those of Russell, see LINSKY, B., 2004 and several sections of LINSKY, B., 2011; for a discussion of the influence of *Principia Mathematica* not only on Chwistek but on the whole school of Polish logicians cf. GRATTAN-GUINNESS, I., 2000, pp.489-97 and WOLENSKI, J., 2013. Another important early version of Chwistek's criticism was given by Ramsey, who, of course, rejected both the hierarchy of orders and the Axiom of Reducibility and is generally credited for the concept of a "Simplified" theory of types (cf. RAMSEY, F., 1925). Of course, many - including Quine himself - followed Chwistek and Ramsey in doing this, another important example being Carnap (cf. CARNAP, R., 1929 and 1931) and also Hilbert (cf., HILBERT & ACKERMANN, 1928). Quine's earliest version of this criticism was given in his dissertation, *A System of Logistic* from 1932, later published as QUINE, W., 1934; cf. also QUINE, W., 1936 for similar points. As we already indicated, however, if *Principia*'s language is interpreted as allowing the formulation of non-extensional sentential contexts (as it should), then this criticism is certainly not decisive. Again, the reader is referred to CHURCH, A., 1974 and 1976. As is well known, this charge of redundancy or superfluousness of order distinctions in the presence of the Axiom of Reducibility became standard in the literature (cf., for instance, FITCH, F., 1938; COPI, I., 1950 and 1971; HATCHER, W., 1980, pp.126-7). Through the nineteen-thirties, the simple theory of types as urged by *Principia*'s achieved the status of a standard system for foundation studies and was consolidated in works like GÖDEL, K., 1931, QUINE, W., 1938 and CHURCH, A., 1940 but; this situation changed after the thirties and first-order systems became standard (cf. GOLDFARB, W. 1979, MOORE, G. H., 1988*b*; SHAPIRO, S., 1991, pp.173-202; for a very good study of this historical shift from type-theoretic to first-order approaches in foundation studies, cf. SCHIEMER & RECK, 2013).

48 For instance, COPI, I., 1971; CHIHARA, C., 1973; CHURCH, A., 1976; HATCHER, W., 1968, 1982. General surveys and critical discussions of such reconstructions can be found in HAZEN, A., 1983; LANDINI, G., 1998, pp.267-72 LINSKY, B., 1999, pp.66-72.

49 These departures and the historical *Principia* will be extensively discussed in chapter 4.

50 Cf. CHURCH, A., 1976 for details.

med that the resulting logic - sometimes referred to as 'Russellian Intensional Logic'[51] - requires a realist semantics according to which 'functional' and 'propositional' variables, respectively, range over 'propositional functions' and 'propositions' conceived as *abstracta*. In short: Church faced Quine's criticism of the double role played by the notion of 'propositional function' by biting the bullet and reframing *Principia*'s variables other than individual ones to range over attributes and relations-in-intension stratified into ramified types[52].

Church's reconstruction of *Principia* has exerted a deep and widespread influence in scholarly works on Russell's Mathematical Philosophy. The claim that *Principia*'s so-called 'Ramified' Theory of Types must be interpreted in terms of a realist semantics committed to 'propositional functions' and 'propositions' conceived as entities, in particular, acquired the status of orthodoxy through the influential writings of many important scholars, most notably Peter Hylton and Warren Goldafarb[53]. Apart from minor points of disagreement, what is characteristic of the influential interpreations which these authors put forward is that they are attempts to spell out and elaborate what sort of philosophical motivation lies behind *Principia*'s adoption of a ramified type theory of *entities*[54].

Warren Goldfarb, for instance, captures quite well the general gist of such attempts at making explicit 'Russell's reasons for Ramification' (against Gödel's and Quine's criciticisms), when he writes:

> Gödel and Quine make Russell out to have a general vision of what the existence of abstract entities comes to, and thus to be adopting constructivism as a fundamental stance toward ontology. That does not seem accurate to Russell. Rather, the justification for ramification rests on the particular sorts of entities to which it is applied, namely, propositions and propositional functions. To understand this, we must see more clearly why these entities are central to Russell's logical enterprise and what special features of their structure Russell exploits. The results might have the appearance of constructivism, but Russell's most basic reasons for ramification are not the outgrowth of such a general position; rather, they are far more particular to the nature of the entities he treats.[55]

According to this general line of interpretation, ramification "[...] of a domain of abstract entities is the result of requiring that legitimate *specifications* of such entities be predicative"[56], and such a requirement is justified on the grounds that the entities in question - propositional functions and propositions - are *intensional structured* entities[57].

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51 Cf., for instance, ANDERSON, A., 1989.

52 In fact, Church included axiom-schemes of comprehension for both propositional functions and propositions and distinguished the former from his axioms of reducibility (cf. CHURCH, 1976; and also CHURCH, A., 1956, pp.347-56).

53 Cf. HYLTON, P., 1980, 1990, 2005; GOLDFARB, W., 1989.

54 Cf. also COCCHIARELLA, N., 1980.

55 GOLDFARB., W., 1989, p.26.

56 GOLDFARB., W., 1989, p.24. Our emphasis.

57 See, in particular, GOLDFARB, W., 1989, pp.32-3 and HYLTON, P., 2005, pp.134-6.

Put briefly and roughly, Hylton and Goldfarb argue that the complex structure of the syntax of *Principia*'s theory of types is meant by Russell to reflect to complex structure of the entities which he assumes as fundamental[58]; so against the traditional charges of confusing use and mention and unitentionally assuming a 'constructivistic' stance towards mathematical entities (which would be quite wrong for a Logicist), these authors claimed that 'Russell's reasons for Ramification' are, on the contrary, grounded on a robust realism about propositions and propositional functions. But still, even if such general approach at making sense of *Principia*'s text (in particular the Introduction) is correct, several aspects of the work remain quite baffling and, indeed, many serious interpretative problems remain without a satisfactory answer.

To begin with, the genuine segmentation of the very notion of logical generality which is imputed to Russell in *Principia* by Hylton and Goldfarb is in tension - as they themselves emphasize - with central aspects of Russell's early conception of Logic as elaborated in the *Principles of Mathematics* and many of his subsequent works. For a central logical doctrine which is integral to Russell's early conception of logic was his doctrine of the "unrestricted variable" according to which the range of "the true or formal variable" must always be comprised of "*all* terms", i.e., all *entities* there are[59]. This was also expressed by Russell in terms of the claim that "there is only one kind of being, namely being simpliciter"[60], i.e., the ontological counterpart of the doctrine which Russell encapsulated in terms of the Leibnizan dictum "*Quodlibet ens, est unum*" i.e., "Whatever is, is one"[61]. If taken seriously, this doctrine is plainly incompatible with the genuine segmentation of the notion of logical generality which inevitably accompanies any sort of type-distinctions among entities.

Moreover, as many authors[62] - including Hylton himself[63] - have shown in studies about the development of Russell's views between the *Principles* and *Principia*, it was in order to *preserve this doctrine* that Russell became more and more attracted to eliminativistic approaches to classes; it was also in order to preserve this doctrine that Russell was led at some point to an eliminativistic approach to propositional functions and to propositions as well. And, in fact, *Principia*'s Introduction explicitly advocates an eliminativistic stance towards propositions by adopting Russell's 'multiple-relation' analysis of judgment according to which a proposition "[...] is not a single entity at all"[64]; like expressions for classes, Russell intended expressions like *p*, *q*, *r* etc., to be treated as "incomplete" symbols which are contrasted with individual constants and variables: the latter, unlike incomplete symbols, "[...] do not disappear on analysis"[65].

According to Hylton, however, the implication "[...] that only individuals (neither propo-

58 cf., for instance, HYLTON, P., 2005, pp.134-7.
59 RUSSELL, B., 1937 [1903], p.91, §88
60 RUSSELL, B., 1937 [1903], p.449, §427
61 RUSSELL, B., 1906*a*, p.261.
62 Cf. COCCHIARELLA, N., 1980; HYLTON, P., 1980; LANDINI, G., 1998.
63 Cf., in particular, HYLTON, P., 2005, pp.93-101.
64 WHITEHEAD & RUSSELL, 1925, p.44 [1910, p.46].
65 WHITEHEAD & RUSSELL, 1925, p.44 [1910, p.46].

sitions nor propositional functions) are genuine constituents of propositions, constituents which do not 'disappear on analysis'" can only be explained as "[...] wishful thinking on Russell's part"[66] which displays "unwillingness of Russell to acknowledge the implications of type theory"[67]. Similarly, Goldfarb follows the established common wisdom of Church's reconstruction and claims that the attempt to treat sentences as incomplete symbols is not consistent with the logic of *Principia* and that the multiple-relation analysis of judgment "seems to play no real role in Russell's explanations of his logical system"[68]. The implications of Goldfarb's views are quite like those of Hylton: both are basically claiming that any aspect or passage in *Principia* which may suggest that type distinctions are not meant to apply to genuine entities - as in the case of so-called propositions - is best understood as the result of Russell's failure or "unwillingness" to understand what his own logical system requires in order to make sense or be coherent.

Furthermore, Hylton claims that this same "unwillingness" is also displayed in connection to another fundamental tension which emerges in *Principia* if the theory of types is interpreted in terms of a realistic theory of propositional functions and propositions. Hylton and Goldfarb follow van Heijenoort in attributing to Russell a somewhat naive conception of the "universality of logic"[69]. Goldfarb, for instance, provides a paradigmatic explanation of this interpretation in the following terms:

> Russell took logic to be completely universal. It embodies all-encompassing principles of correct reasoning. Logic is constituted by the most general laws about the logical furniture of the universe: laws to which all reasoning is subject. The logical system provides a universal language; it is the framework inside of which all rational discourse proceeds. For Russell, then, there is no stance outside of logic: anything that can be communicated must lie within it. Thus there is no room for what we would call metatheoretic considerations about logic.[70]

Hylton agrees entirely with that interpretation. He writes:

> Logic, for Russell, is a systematization of reasoning in general, of reasoning as such. If we have a correct systematization, it will comprehend all correct principles of reasoning. Given such a conception of logic there can be no external perspective. *Any* reasoning will, simply in virtue of being reasoning, fall within logic; any proposition that we might wish to advance is subject to the rules of logic. [...] If logic is to be unconditionally and unrestrictedly true in the sense that Russell requires it to be, then it must be universally applicable. This, in turn, implies that statements about logic must themselves fall within the scope of logic, so the notion of a meta-theoretical perspective falls away.[71]

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66 HYLTON, P., 2005, p.106.

67 HYLTON, P., 2005, p.106.

68 GOLDFARB., W., 1989, p.34. That is why Goldfarb takes "[...] the charitable course of ignoring [...]" (idem.) any possible role that the multiple-relation analysis of propositions may play in *Principia*.

69 Cf. van HEIJENOORT, J., 1967.

70 GOLDFARB, W., 1989, p.27.

71 HYLTON, P., 1990, p.203.

Thus according to such interpretation, "intrinsic to Russell's conception of the universality of logic is the denial of the metalinguistic perspective which is essential to the modern conception of logic"[72]. Of course, an important consequence of imputing such a conception of Logic to *Principia* is that it makes the very statement of the theory of types incoherent, in particular if type distinctions are considered as substantial claims about domains of *entities*. As Hylton puts it, "the difficulty here clearly arises from the attempt to state type theory within type theory"[73]: for if propositional functions are entities after all and *Principia*'s *lingua characteristica* is meant to be "all encompassing", then any statement to the effect that two given functions ϕ and ψ are not of the same type would, if true, violate type distinctions, because such statement would presuppose that there is propositional function χ which can be meaningfully applied to two entites ϕ and ψ both when *they are and when they are not of the same type*[74].

Thus, the general situation which we have here is the following. In their efforts to dispel Gödel's and Quine's verdict that the foundations *Principia* are pervaded by basic misunderstandings[75], Church, and following him, Hylton and Goldfarb have argued that *Principia* should be interpreted as advancing an onerous theory of abstract entities that is motivated by a robustly realist, albeit naive, conception of Logic. As Hylton puts it, however, this very conception of Logic gives rise to "[...] a fundamental tension in Russell's philosophy"[76] which can only lead us to conclude that *Principia* is "[...] a technical achievement which is marred by apparent inconsistencies and incoherences"[77].

1.1.3 The 'Axiom' of Infinity

Furthermore, although neither Church, nor Goldfarb nor Hylton explicitly mention this in connection with the issue of propositional functions and propositions, there is a another point that arises here related *Principia*'s so-called 'Axiom of Infinity'.

As is well known, given type restrictions, it is impossible to prove in *Principia*'s system that there are infinitely many objects in any given type. A little more precisely, what *cannot* be shown in any type is that the universal class is non-inductive. In symbols, this may be schematically expressed as follows:

$$V \sim \epsilon \, \text{Cls induct} \tag{1.1.10}$$

Where "Cls induct" stands for the class of all 'inductive' or finite classes. In *Principia*, this class

[72] HYLTON, P., 1990, p.202; cf., also, GOLDFARB., 1989, p.27
[73] HYLTON, P., 2005, p.76.
[74] Cf. HYLTON, P., 1990, pp.316-18; also HYLTON, P., 2005, pp.75-7.
[75] In particular the confusion between the use and mention of expressions.
[76] HYLTON, P., 2005, p.107.
[77] HYLTON, P., 2005, p.107.

in characterized in terms of the following theorem, which expands its more compact definition[78]:

$$\text{Cls induct} = \hat{\alpha}\{(\mu)(\Lambda \,\epsilon\, \mu \wedge (\eta)(y)(\eta \,\epsilon\, \mu \supset \eta \cup \{y\} \,\epsilon\, \mu) \,.\!\supset\!.\, \alpha \,\epsilon\, \mu)\} \tag{1.1.11}$$

Put in terms of cardinal numbers, 1.1.10 is equivalent to the following:

$$\text{Nc(V)} \sim \epsilon \, \text{NC induct} \tag{1.1.12}$$

Which states that the cardinal number of the universal class (of some type[79]) is not a finite cardinal number, i.e., a natural number. Also, in *Principia*'s system there are many statements which are equivalent to the above, including the following:

$$(n)(n \in \text{NC induct} \supset n \neq \Lambda) \tag{1.1.13}$$

$$(n)(n \in \text{NC induct} \supset n \neq n + 1) \tag{1.1.14}$$

$$(m)(n)(n, m \in \text{NC induct} \,.\!\supset\!.\, m + 1 = n + 1 \supset m = n) \tag{1.1.15}$$

$$\text{Nc(NC induct)} \sim \epsilon \, \text{NC induct} \tag{1.1.16}$$

The last three, 1.1.14, 1.1.15 and 1.1.16, in particular cause some alarm: they assert, respectively, that (*a*) no natural number is identical with its immediate successor; (*b*) that no two natural numbers have the same successor and (*c*) that the cardinal number of the class of all natural numbers is not itself a natural number, i.e., that there are infinitely many natural numbers. None of these are provable in *Principia*.

Whitehead and Russell handled this 'difficulty' by introducing 1.1.13 as a hypothesis - *not as a proper axiom of their system* - wherever it was necessary. So instead of proving, say, 1.1.14, they proved only:

$$\text{Infin Ax} \supset (n)(n \,\epsilon\, \text{NC induct} \supset n \neq n + 1) \tag{1.1.17}$$

Where 'Infin Ax' is a label for 1.1.13. Even independently of the issue of the status of 'propositions' and propositional functions, the impossibility of proving 1.1.14 rather than 1.1.17 was universally recognized as attesting *Principia*'s failure to establish the Logicist thesis, since 1.1.13 is actually equivalent to the assumption that there are infinitely many objects of the lowest type, i.e., infinitely many individuals. For instance, William and Martha Kneale's (nearly infuriated) comments on the status of Infin Ax provide a quite representative reaction to its employment in *Principia.* They write:

[78] Cf. $*120\cdot$ and $*120\cdot24$.

[79] For reasons that will be discussed further, it would be more appropriate to speak of the universal class of some *relative* type. Cf. for instance, LANDINI, G., 2016, p.4-5. This issue will be discussed in chapter 5.

> There is something profoundly unsatisfactory about the axiom of infinity. It cannot be described as a truth of logic in any reasonable use of that phrase, and so the introduction of it as a primitive proposition of arithmetic amounts in effect to abandonment of Frege's project of exhibiting arithmetic as a development of logic. Nor is it a sufficient defence to suggest, as Russell has sometimes done, that we may treat as a postulate or hypothesis rather than as an axiom in the old sense. For we want to assert that the series $1/2 + 1/2^2 + 1/2^3 + ...$ converges to 1 as a limit, not that it converges to 1 as a limit if there happens to be an infinity of individuals in the world. But even if we abandon all hope of carrying out Frege's programme in full and say boldly that Russell's axiom is required as an extra-logical premiss for mathematics, how can we justify our acceptance of it? What are the individuals of which Russell speaks, and how can we tell whether there infinitely many of them?[80]

Similar remarks are easily found in the literature about the subject[81] and even authors who are much more sympathetic to Whitehead and Russell's *modus operandi* also agree that *Principia* fails in sustaining the claim that Arithmetic is nothing but the development of Logic[82].

Hylton - like the Kneales and many others - agrees that "Russell's attitude towards the axiom of infinity does [...] threaten the fundamental project of logicism", so much so that according to him "[...] it might be said that *Principia* represents not so much the culmination of Russell's logicism as Russell's abandonment of logicism"[83]. When viewed in light of the highly Platonistic interpretation of type theory which is put forward by Cocchiarella, Golfarb and Hylton himself, we seem to be left with a very negative assessment of the development of Russell's views. Putting matters in a rather blunt way, how could Russell (and Whitehead) have ended up adopting a logical system which at the same time betrayed the philosophical conception of Logic as a science that motivated it (in virtue of the considerations about types made above) *and* also failed to deliver the technical results which were central to the project of establishing Logicism?

As we shall see next, there is an alternative interpretation which provides a more charitable and interesting portray of the Logic and Logicism of *Principia*, that of Gregory Landini.

1.2 Re-assessments of the 'Received View'

1.2.1 Russell's Manuscripts and the 'Substitutional Theory' of Classes and Relations

Before presenting the general outline of Landini's interpretation, it is necessary to provide some context for it, in particular, in order to make clear why several aspects of what we are

80 KNEALE & KNEALE, 1962, p.669.
81 Cf., for instance, HATCHER, W., 1980, pp.123-4; MENDELSON, E., 2010, p.295, footnote; SOAMES, S., 2014, pp.487-8.
82 Cf., for instance, BOOLOS, G., 1998, pp.271-2.
83 HYLTON, P., 2005, p.78. Cf., also, HYLTON, 1990, pp.318-20.

calling 'orthodox' interpretations remained unquestioned for so long.

In the last three quarters of a century[84], we may distinguish three stages of Russellian scholarship as far as Russell's logico-mathematical writings are concerned. The first stage lasted until the end of the nineteen-sixties and was dominated by the negative assessments of Ramsey, Gödel and Quine which we discussed above. In particular, it was a commonplace to assume that the development of Russell's views from the publication of the *Principles* up to the publication of *Principia* was that Russell went through several different dead-ends that were plagued by irresolvable technical difficulties and/or philosophical incoherences, until settling for the type theory of *Principia.* The beginning of the second stage is marked by the acquisition of Russell's library and *Nachlass* by McMaster University, establishing there the Bertrand Russell Archives and the Bertrand Russell Research Centre, leading to the creation of the quarterly newsletter *Russell: the Journal of the Bertrand Russell Archives* - now called *The Journal of Bertrand Russell Studies* and publsihed semiannually - and of the monumental series of *The Collected Papers of Bertrand Russell.*

These developments prompted a complete revolution in our understanding of what happened between the publication of the *Principles of Mathematics* and that of *Principia Mathematica*. Scholars soon found out that the amount of (already fairly voluminous) works that Russell *published* between his two seminal works on Mathematical Philosophy was but the tip of an iceberg of material that included an enormous amount of manuscripts and a vast (and inestimably informative) mathematical correspondence. A little (albeit *very* important) part of this material was made available in the seventies with the publication of some of Russell's so-called '*Essays in Analysis*'[85] and thanks to the pioonering efforts of Ivor Grattan-Guinness[86]. Then, throughout the eighties and nineties, several scholars dedicated an enormous amount of time and effort editing and studying this material, resulting in the publication of many important studies[87] and approximately the first half of the manuscripts Russell produced in the period between the *Principles* and *Principia* in the fourth volume of the *Collected Papers*[88]. And finally, what we may justfiably distinguish as a third stage has recently began with the publication of the long-awaited fifth volume of Russell's *Collected Papers*[89], where we find the published papers and unpublished manuscripts that Russell produced between 1905 and 1908, by far the most crucial period leading to the completion of *Principia Mathematica*.

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84 Taking the publication of Schillpp's volume on Russell (SCHILPP, P. A., 1944) as a (somewhat arbitrary) starting point for detailed Russellian scholarship in general.

85 LACKEY, D., 1973, in particular paper 8 (i.e., RUSSELL, 1906*a*), now published in the *Collected Papers Volume 5* (MOORE, G., 2014), which we'll soon discuss below.

86 An important early survey and appraisal of the content of these manuscripts is given in GRATTAN-GUINNESS, I., 1974. And, of course, there is the classic GRATTAN-GUINNESS, I., 1977 which will be extensively quoted and discussed in the present text.

87 To name just a few: GRATTAN-GUINNESS, I., 1985; COCCHIARELLA, N., 1980; HYLTON, P., 1980; GRIFFIN, N., 1980, 1991; URQUHART, A., 1988; RODRIGUEZ-CONSUEGRA, F., 1989, 1991; GARCIADIEGO, A., 1992; GOLDFARB, W., 1989; LANDINI, G., 1987, 1989, 1991, 1998.

88 URQUHART, A. (ed.), 1994.

89 MOORE, G., 2014.

Now, although there are many, many authors, editors (as well as editorial advisers) and scholarly studies that can be mentioned in detailing how this revolution gradually came about in the use of the Russell Archives and Russell's posthumously published works, four studies deserve particular mention in connection with our topic, namely: Ivor Grattan-Guinness's *Dear Russell, Dear Jourdain*[90], Nino Cocchiarella's *The Development of the Theory of Logical types and the Notion of a Logical Subject in Russell's Early Philosophy*[91], Peter Hylton's *Russell Substitutional Theory*[92] and, lastly and most importantly, Gregory Landini's *Russell's Hidden Substitutional Theory*[93]. The common thread in all these works was that they considered (in increasing amount of detail) a theory that Russell developed between the *Principles* and *Principia*, the 'Substitutional Theory of Classes and Relations'.

The core idea of the Substitutional Theory of Classes and Relations was the same as that of *Principia*'s No-Class Theory[94]: to treat expressions for classes and relations-in-extension as incomplete symbols. The crucial difference, however, is that the Substitutional Theory assumes an ontology of propositions that are treated on par with individuals and functional variables are also explicitly contextually eliminated. Thus, the theory embraces a logical grammar in which all variables are individual variables that have a completely unrestricted range in accordance with Russell's thesis of the univocity of being. The fundamental notions of the theory are those of *an entity occurring as a subject (or term) in a proposition* and of *a proposition q which differs from p by having x occurring in place of term a whenever a occurs as a subject in p*, or, as Russell liked to put it "what *p* becomes when *x* is everywhere substituted for *a* in *p*"[95]. Important details aside, the basic gist of the theory is the employment of what Russell calls a "*matrix*" in place of functional and class expressions[96]. At the basis of the Substitutional calculus, Russell introduces the primitive notation "$p\frac{b}{a}!q$" or "$p/a; x!q$", which stands for the assertion that *q* is exactly like *p* except for containing occurrences of *x* wherever *p* contains occurrences of *a*.; Russell then defined the symbol "$p\frac{x}{a}$" or "$p/a; x$" as "*the q* that is exactly like *p* except for containing occurrences of *x* wherever *p* contains occurrences of *a*"[97], i.e., it is the definite description "$(\imath q)(p\frac{x}{a})$"; he called the component "p/a" of this symbol the *matrix* of the substitution, which is an expression that does not have any meaning by itself, but is in fact an incomplete symbol[98]. Putting things very roughly and briefly, in the Substitutional Theory this symbol played the role of a class expression[99]; Russell's core idea was to emulate within his early quantificational calculus of propositions what effectively amounted to a simple type-theory of classes, classes of

[90] GRATTAN-GUINNESS, I., 1977.
[91] COCCHIARELLA, N., 1980.
[92] HYLTON, P., 1980.
[93] LANDINI, G., 1998.
[94] Presented in sections ∗20 and ∗21 of *Principia*.
[95] Cf., for instance, RUSSELL, B., 1905*a*, p.93; also 1905*b*, p.98,
[96] RUSSELL, B., 1906*a*, p.246.
[97] Cf. RUSSELL, B., 1905*e*, p.93; also 1906*a*, p.246.
[98] This symbol stands for "the result of replacing *a* in *p* by" (cf. RUSSELL, B., 1906*a*, p.246).
[99] Cf. RUSSELL, B., 1905*e*, p.93; and, again, 1906*a*, p.246.

classes, classes of classes of classes, etc., and, similarly, definitions for relations-in-extension[100], thus avoiding the set theoretical contradictions[101] while also allowing the development of a lot of Mathematics on the basis of this calculus *without assuming any distinctions among types of entities* - thus reconciling the doctrine of the unrestricted variable with his logicist goals[102].

Grattan-Guinness's book[103] commented on and made available the correspondence between Russell and the mathematician Philip Jourdain with whom Russell discussed constantly and in great detail the development of his views on Logic between 1905 and 1910. Grattan-Guinness's work called attention for the first time to the major role played by the Substitutional Theory in the development of Russell's views. Cocchiarella's and Hylton's pioneer studies[104] investigated - for the first time in detail - Russell's reasons for adopting and later abandoning the theory, by attempting to reconstruct the tortuous path towards *Principia*'s Theory of Types[105]. Landini's book was the culmination of extensive research[106] in the Russell Archives of the manuscripts that are now published in the fifth volume the *Collected Papers*[107]. Landini made what is to this day the most systematic and detailed reconstruction of the many versions of the Substitutional Theory. Together, these studies showed that there is much more *continuity* in the development of Russell's views towards *Principia* than was previously thought and that any account of this development must consider in detail (and take *seriously*) this important stage of Russell's thought.

However, this (still ongoing) process of piecing together the details of Russell's Substitutional Theory and its role in the development of his views progressed in a gradual way and Landini was the first[108] to challenge basic assumptions that prevented an adequate understanding of Russell's work in this intermediate period. In fact, for quite some time the role played by this theory in the development of Russell's views remained barely understood, in particular because the details of the theory remained, as Landini noted[109], buried and forgotten in manuscripts for almost seventy years, with Russell's published works giving only bare hints as to why his views eventually shifted from the method of Substitution.

Indeed, the Substitution was only discussed in three of Russell's published writings. In

100 Cf., for instance, RUSSELL, B., 1905*e*, pp.93-4; and, again, 1906*a*, p.246; Cf. LANDINI, G., 1998, pp.140-145 and chapter VI for details.

101 Russell's paradox, for instance, is elegantly resolved in the following way: translating into the substitutional theory, "$\alpha \epsilon \alpha$" becomes "$p/a \epsilon p/a$", which is "$p/a; p, a!q$"; this, however, is simply ungrammatical: the language of Substitution requires that only a genuine term like "*q*" or "*a*" may be substituted for another in a given sentence "*p*" and "p/a" is not a term at all. Notice, also, that this restriction is philosophically well motivated: such a grammatical rule is in accordance with Russell's logical doctrine that all and only *terms* (not in the linguistic sense that "*a*" is a (singular) term, but in the sense that *a* is an entity or being) can be logical subjects of propositions.

102 Cf. RUSSELL, B., 1906*a*, p.261.

103 GRATTAN-GUINNES, I., 1977.

104 CHOCCHIARELLA, N., 1980; HYLTON, P. 1980.

105 Other early (but posterior) studies attempting this include LANDINI, G., 1987 and 1989;

106 Cf. also LANDINI, G., 1987 and 1989 for some preliminary presentations of Landini's book.

107 MOORE, G. (ed.), 2014.

108 As we shall see, following a lead of Cocchiarella.

109 LANDINI, G. 1998, p.1.

his 1905 paper *On Some Difficulties of Theory of Transfinite Numbers and Order Types*[110], Russell somewhat reluctantly considered the Substitutional Theory under the label of 'No-Class Theory' and claimed that it could "[...] be accepted as one way of avoiding contradictions, though not necessarily the way"[111]. However, while the article was still in the press he added the following note at the end of it:

> From further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first section of this paper.[112]

The next appearance of the 'Substitutional' or 'No-Class' theory in print was in Russell's paper *Les Paradoxes de la Logique*[113] from 1906, in which he engaged in a polemic started by Henri Poincaré[114] and Louis Couturat[115]. There, again, we find Russell confident in the Substitutional Theory's capacity to purge Set Theory of contradictions, claiming that "[...] there seems reason to hope that the method proposed in [the] article avoids all the contradictions, and at the same time preserves Cantor's results"[116]. This makes clear that at least until September of 1906 Russell was still convinced that the Substitutional Theory could work. However, the next mention to the Substitutional Theory in print came only in 1908 in the seminal paper "*Mathematical Logic as Based on the Theory of Types*"[117], where Russell laconically indicated that the theory can be understood as a philosophically plausible, albeit technically inconvenient, alternative to the hierarchy of so-called 'propositional functions' that he meant to employ *in practice*[118].

But after *Mathematical Logic* Substitution vanished from Russell's works: no explicit mention to the theory appears in any of Russell's subsequent publications in Mathematical Philosophy. In particular, there is not a single explicit mention of it *Principia*, where the label of a 'No-Class Theory' is dissociated from the method and notation of Substitution. Furthermore, none of Russell's published papers provide a satisfactory answer as to why he abandoned the theory. In *On Some Difficulties* Russell mentions that there are *challenges*[119] for the theory, but these did not make his confidence in it waver, as the note added to the paper makes clear. Similarly, in *Les Paradoxes*, Russell tempered his confidence in the theory only in observing that "a

110 RUSSELL, B., 1905*b*.
111 RUSSELL, B., 1905*b*, p.82.
112 RUSSELL, B., 1905*b*, p.89. Russell submitted the paper on 24 November 1905; he read the paper on 14 December and added the note on 5 February 1906 (MOORE, G., 2014, p.62). We will discuss the details of the timeline in Chapter 3.
113 RUSSELL, B., 1906*b*. Russell intended the article to be entitled "*Insolubilia and Their Solution Through Symbolic Logic*", but was persuaded to the contrary by Couturat; the paper was translated by Couturat around the end of June (cf. MOORE, G., 2014, pp.276-7).
114 Cf. POINCARÉ, H., 1905 and 1906.
115 Cf. COUTURAT, L., 1906.
116 RUSSELL, B., 1906*b*, p.296.
117 RUSSELL, B., 1908.
118 Cf. RUSSELL, B., 1908, pp.603-4.
119 Cf. RUSSELL, B., 1905*b*, p.82.

lengthy symbolic development"[120] was still necessary to establish that it was indeed adequate for the development of Mathematics. But beyond that, Russell never published an inkling detailing what led him to abandon Substitution.

This puzzling state of affairs left early interpreters with little to work with in reconstructing Russell's reasons for abandoning his Substitutional or early 'No-Class' theory and made them fail to appreciate its importance. Quine, for instance, viewed the theory as a typical product of use-mention confusion and claimed that the note Russell added to *On Some Difficulties* "[...] was the expression of renewed hope that [Russell] was shortly to abandon"[121].

A fundamental piece of evidence suggesting that this view is mistaken was first published in the early seventies: a paper[122] that Russell read to the London Mathematical Society on April of 1906 which he ended up withdrawing from publication by october of the same year[123]. This paper, entitled "*On the Substitutional Theory of Classes and Relations*" was the closest to a definite presentation of the Substitutional Theory that Russell ever made public[124], but it remained unpublished until 1973, when it came out in the volume *Essays in Analysis*[125]. In this paper Russell voiced again his concern that "the technical development of the principles of mathematics is rendered [...] much more complicated by the Substitutional theory"[126], but also observed that :

> The *only* serious danger, so far as appears, is lest some contradiction should be found to result from the assumption that propositions are entities; but I have not found any such contradiction, and it is very hard to believe that there are no such things as propositions, or to see how, if there were no propositions, any general reasoning would be possible. It would seem, therefore, that the chances of any important lurking fallacy in the method are not great.[127]

This was the first substantial clue as to what led Russell to abandon substitution, for it suggests that he *did find out* that some contradiction follows from the assumption of propositions in the Substitutional calculus.

Thus, we have a first a plausible answer as to what led Russell to withdraw the paper and abandon the Substitutional Theory, namely: *a paradox (or set of paradoxes) which arises from the assumption of propositions within his substitutional calculus.* But which sort of paradox

120 RUSSELL, B., 1906*b*, p.296. Indeed, in *Les Paradoxes* Russell also wrote the following, referring back to *On Some Difficulties*: "In the above-mentioned article, the no-classes theory was merely sketched in the briefest outline, nor did I then know how much of the theory of the transfinite it was possible to express in this language. I have since come to the conclusion that, so far at least as I can yet discover, hardly anything is ruled out except the paradoxes" (RUSSELL, B., 1906*b*, p.287).

121 QUINE, W., 1967, p.151.

122 RUSSELL, B., 1906*a*.

123 Cf. MOORE, G., 2014, pp.236-9.

124 Cf. MOORE, G., 2014, p.173-4.

125 Cf. LACKEY, D., 1973, pp. 165-189.

126 RUSSELL, B., 1906*a*, p.261.

127 RUSSELL, B., 1906*a*, p.261. My emphasis.

or contradiction? Again, Russell never said anything in his published writings that afforded anything close to a definitive answer.

Some further hints were uncovered by Ivor Grattan-Guinness in his research of Russell's correspondence with Jourdain. To begin with, we have a letter of 14 June of 1906, where Russell wrote:

> I feel more and more certain that this theory is right. In order, however, to solve the Epimenides, it is necessary to extend it to general propositions, i.e., to such as $(x).\phi x$ and $(\exists x).\phi x$. This I shall explain in my answer to Poincare's article in the current *Revue de Metaphysique*.[128]

The 'answer to Poincaré' refers to *Les Paradoxes*, and it requires some contextualization. It is in *Les Paradoxes* that Russell acknowledges for the first time that the "paradoxes all spring from some kind of vicious circle"[129] and admits that any account of the foundations of Logic and Mathematics must embody the so-called Vicious Circle principle, which he there states as "whatever involves an apparent variable must not be among the possible values of that variable" and also "less exactly" as "whatever involves *all* must not be one of the *all* which it involves"[130]. Now, concerning the 'vicious circles' involved in the set-theoretical contradictions, Russell claims that they are solved by treating matrices as incomplete symbols, thus blocking problematic substitutions in the way indicated above[131]. But in *Les Paradoxes*, Russell apparently became concerned for the first time with paradoxes of a different sort, like the so-called '*Epimenides*', also known as the 'liar paradox'. A solution of this paradox, Russell explained in the paper, required an extension of the theory of incomplete symbols to statements containing variables bound by quantifiers[132]. The details of this proposal and how it was intended to dissolve paradoxes need not detain us for now[133]. What is relevant is that such claims made by Russell in *Les Paradoxes* together with the letter quoted above may suggest that *it was the liar and related paradoxes that led Russell to withdraw his previous paper on Substitution*. In fact, a similar conclusion may also be extracted from another letter Russell wrote to Jourdain on 10 October of 1906, right after *Les Paradoxes* was published. Responding to Jourdain's praise of the Substitutional Theory, Russell wrote:

> I am glad you feel attracted by the no-classes theory. I am engaged at present in purging it of metaphysical elements as far as possible, with a view to getting the bare residuum on which its success depends. I decided not to publish the

128 GRATTAN-GUINNESS, I., 1977, p.89.
129 RUSSELL, B., 1906*b*, p.278.
130 RUSSELL, B., 1906*b*, p.289.
131 See previous footnote 101. These points will be discussed in more detail in chapter 3.
132 Cf. RUSSELL, B., 1906*b*, p.289-292
133 Russell provides a bare outline of this approach in RUSSELL, B., 1906*b*, pp.288-95. For an attempt at reconstructing this idea in detail cf. LANDINI, G., 1998, pp.216-31.

> paper I read at the London Mathematical [Society] in May; there was much in it that wanted correction, and I preferred to wait till I got things into more final shape.[134]

When viewed in light of Russell's previous letter and of the discussion of the *Epimenides* in *Les Paradoxes*, the 'purging of metaphysical elements' can very reasonably be interpreted as meaning the elimination of propositions as entities[135] - or, at least just *general* propositions. Similarly, Russell's claim that the paper "wanted correction" may suggest that he thought that the version of substitution he put forward in the abandoned paper was susceptible to some version of the *Epimenides.* But the fact of the matter is that these statements are very vague[136] and they provide no detailed indication as to why Russell withdrew the paper much less to why he abandoned substitution.

So commentators were still left in the dark and unable to satisfactorily put this puzzle together. Despite having realized that "[...] the substitutional theory is the missing link between Russell's theory of denoting and *Principia Mathematica*"[137], Grattan-Guinness concurred with Quine and claimed that despite being "[...] an exceptionally ingenious construction even by Russell's standards", the substitutional theory "[...] fell soon after it rose" because it failed "[...] to clarify the status of some of its components as linguistic or abstract objects"[138]. Thus, Grattan-Guinness claimed, the withdrawal of Russell's main paper on the Substitutional Theory ultimately "[...] marked the demise of the theory itself"[139]. Others were led to conclude that paradoxes related to the *Epimenides* were the cause of the Substitutional Theory's demise. Peter Hylton, for instance, who also correctly recognized that "the substitutional theory is an indispensable part of a general understanding of the philosophical context of *Principia Mathematica*"[140] argued that the downfall of the theory was its susceptibility to "paradoxes" that "[...] we would call semantic rather than logical"[141], calling attention, in particular, to a substitutional version of a paradox of propositions[142] that Russell had already considered in the *Appendix B* of the *Principles of Mathematics*[143]. It was such paradoxes, according to Hylton, that led Russell to abandon Substitution

134 GRATTAN-GUINNES, I., 1977, p.93.
135 Cf., for instance, POTTER, M., 2000, p.131.
136 As observed in LANDINI, G., 1998, p.200.
137 GRATTAN-GUINNES, I., 1977, p.94.
138 GRATTAN-GUINNES, I., 1977, p.94. Similarly, Michael Potter claims that the Substitutional Theory was "[...] extremely short-lived, even by Russell's hectic standards of theory revision" (POTTER, M., 2000, p.131).
139 GRATTAN-GUINNES, I., 1977, p.94.
140 HYLTON, P., 1980, p.2.
141 HYLTON, P., 1980, p.23.
142 Cf. HYLTON, P., 1980, p. 24.
143 Cf. RUSSELL, B., 1903, p.527, §500. Russell had already communicated this paradox to Frege in a letter from 29 September of 1902 (cf. FREGE, G., 1980, p.147). The paradox in questions can be described in the following general form: given a class *m* whose elements are all propositions, there is a proposition p_m which asserts that every element of *m* is true; this proposition may or may or may not belong to *m*; let p^*_m be such a proposition p_m which does *not* belong to *m* and let *w* be the class of all such propositions p^*_m; now, let p_w be the proposition which asserts that every element of *w* is true; problem: p_w belongs to w if, and only if it does not belong to *w* - contradiction.

and accept a hierarchy of so-called 'ramified types' in *Mathematical Logic* and *Principia*[144].

Even in light of the evidence available at the time, however, both conclusions, are unsatisfactory. Grattan-Guinness's claim that Substitution was short lived flies in the face of the fact that Russell still considered it as a viable alternative to a theory which embraced (at least *in practice*[145]) a hierarchy of 'propositional functions' at least by June or July of 1907 when he wrote *Mathematical Logic as Based on the Theory of Types*[146]. On the other hand, Hylton's claim that semantic paradoxes brought the theory down is entangled in severe interpretative issues and problematic assumptions. For one thing, there is the following straighforward historical issue: Russell was aware of paradoxes of propositions long before coming up to the Substitutional Theory - are we to say, then, that he simply forgot about them or ignored them when he was confident in the theory[147]? The most notable and important issue, however, concerns a point that Nino Cocchiarella was the first to emphasize, namely, that paradoxes akin to that of *Appendix B* of the *Principles* which arise in the Substitutional Theory should not be viewed as 'semantic' paradoxes akin to the *Epimenides* but as results that demonstrate that there is a conflict between Cantor's 'Power-Class Theorem' and the Substitutional Theory[148]. The conflict - as Cocchiarella first pointed out - and as Landini later extensively investigated - does not depend on the introduction of any semantic notion like 'truth' in the object-language of the Substitutional Theory, but from the possibility of establishing, using the apparatus of the Substituitional Theory which emulates sets of propositions, 1-1 correspondences between propositions and sets of propositions, thus directly violate Cantor's Theorem.

This point first set forth by Cocchiarella is central for understanding Landini's 'revolutionary' interpretation and the development of Russell's views.

1.2.2 Landini's Discovery of the p_o/a_o Paradox and his 'Revolutionary' Interpretation of *Principia*

Following Cocchiarella's hint that the fundamental issue with Russell's Substitutional Theory was a conflict with Cantor's theorem[149], Landini made several important discoveries while investigating the manuscripts and documents on Substitution in the Russell Archives[150]. Perhaps the most striking of them was a letter Russell wrote to the mathematician Ralph Hawtrey on 22 January 1907, where Russell described "*the paradox* which *pilled* the Substitutional

[144] A similar conclusion is drawn by Michael Potter, who also assumes that a version of the Appendix B paradox was responsible for the alleged 'short life' of Substitution (cf. POTTER, M., 2000, p.131-3).
[145] Again, cf. RUSSELL, B., 1908, p.603.
[146] Cf. MOORE, G., 2014, p.585.
[147] Cf. GRATTAN-GUINNESS, I., 2000, p.364.
[148] Cf. COCCHIARELLA, N., 1980, pp.90-1.
[149] Michael Potter is also perceptive on this point, cf. POTTER, M., 2000, p.132.
[150] Early reports of Landini's investigations are given in LANDINI, G., 1987, 1989 and 1991.

Theory"[151]. Like the *Appendix B* paradox, this paradox was *not* semantic but a diagonal construction conflicting with Cantor's power-class theorem. After sketching a derivation of this new contradiction that Landini first dubbed the "p_o/a_o" paradox[152], Russell informed Hawtrey that "in trying to avoid this paradox", he "[...] modified the substitutional theory in various ways, but the paradox always reappeared in more and more complicated forms"[153].

Landini's meticulous studies of the manuscripts on Substitution that Russell produced during 1906 showed that the "p_o/a_o" paradox and its variants were indeed the driving force that led Russell to modify the Substitutional Theory several times[154]. And, indeed, a brief look at the now published manuscripts that Russell produced after May of 1906 show that several formulations of the "p_o/a_o" reappear not only across different texts but more than once in the same manuscript[155].

Now, although the details which Landini provides in reconstructing the many twists and turns that happen in Russell's manuscripts on Substitution are certainly interesting and valuable on their own[156], perhaps what is of greater importance (from a strictly historical point of view) in his studies are the general conclusions to which his reconstructions point to. The main point is that the p_o/a_o paradox which he uncovered is clearly *not* a semantic paradox: it does not rely at all on the use of a truth-predicate applied to propositions or sentences and it does not employ any extra-logical vocabulary for propositional attitudes[157]. In the context of Russell's Substitutional Theory, it arises from assumptions of what may be called the 'pure' Substitutional Calculus of propositions. This, in turn, strongly suggests that Russell was *not* led to 'ramification' and to accept Poincaré's 'Vicious-Circle Principle' in virtue of the semantic paradoxes[158]. Furthermore, it strongly suggests that the ramified theory of types of *Mathematical Logic* (which embraces orders of propositions) is Russell's last unavailing attempt at keeping Substitution *alive*, and thus, that at least by the time he wrote *Mathematical Logic*, Russell still had hopes for the Substitutional Theory as a theory which allowed for the elimination of *both classes and propositional functions* (understood as properties and relations-in-intension stratified into types), a hope that most likely wavered *because* Russell became convinced that the p_o/a_o paradox and its variants could *only* be resolved by introducing hierarchy of *orders* of propositions[159]. In a nutshell, this is what led Landini to formulate a completely novel interpretation of the Theory of Types of *Principia Mathematica*.

[151] RUSSELL, B., 1907*c*, p.125.
[152] Cf. LANDINI, G., 1989 and 1998.
[153] RUSSELL, B., 1907*c*, p.125.
[154] Cf. LANDINI, G., 1998, in particular, pp.206-212, pp.216-220, pp.227-233, pp.235-246, and pp.251-254.
[155] Besides the letter to Hawtrey, i.e., RUSSELL, B., 1907*c*, take, for instance, RUSSELL, B., 1906*d*, p.131, pp.150-1 and p.166; see also RUSSELL, B., 1906*f*, p. 351.
[156] In fact, Landini insists that modifications of Russell's Substitutional Theory may result in a viable alternative for sustaining Logicism (cf. LANDINI, G., 2004).
[157] Cf. LANDINI, G., 1998, p.254.
[158] Cf. LANDINI, G., 1998, pp.213-215 and pp.275-279.
[159] Cf. LANDINI, G., 1998, p.254.

Putting things briefly and roughly, we may summarize Landini's interpretation of *Principia* as follows. According to Landini, the acceptance of orders of propositions understood as *entities* proved philosophically intolerable to Russell, since it went against his most cherished logico-metaphysical doctrine of the unrestricted variable and this reluctance to accept a hierarchy of orders led to the demise of the system of *Mathematical Logic*, burying with it the Substitutional Theory and its ontology of propositions[160]. Russell then embraced the 'multiple-relation analysis of judgement'[161]. Landini views the multiple-relation analysis of propositions as the centerpiece of Russell's attempt to *preserve* the doctrine of the unrestricted variable[162]. Against the orthodox interpretation, he argues that *Principia*'s predicate variables[163] are to be read with *implicit simple type indices* that match their arguments, as in "$\phi^{(o)}x^{o}$", $\phi^{((o),o)}(\psi^{(o)}, x^{o})$, etc[164]. He also claims that the Axiom of Reducibility was meant as an impredicative comprehension axiom for predicate variables[165]. In short, Landini argues that *Principia*'s official grammar[166] was meant to have the same structure of a *simple* (i.e., *not ramified*) theory of types of attributes - just like Russell's original Substitutional Theory[167]. According to Landini, however, in *Principia*'s Introduction Russell meant to offer an *informal* justification for type-distinctions in terms of a nominalistic *substitutional*[168] interpretation of quantified predicate variables[169]. According to him, the notion of 'order' as explained in *Principia*'s Introduction was meant to be justified by a recursive definition of senses of "truth" and "falsehood" applied to *formulas* of *Principia*'s language[170]. At the base of this recursion was the multiple-relation analysis applied to *atomic* sentences (which expressed atomic judgements)[171]. On this interpretation, Russell's hierarchy of senses of "truth" and "falsehood" applying to 'propositions' of different orders was meant to keep track of the complexity of the quantificational structure of formulas[172]. Predicate variables, i.e., 'predicative propositional functions', are then interpreted nominalistically: their occurrences can only be substituted (within a given formula) for *formulas* of the appropriate order and distinguished from schemata standing for open formulas, i.e., 'non-predicative propositional functions'. Also, according to this interpretation a declarative sentence like "*aRb*" functions like an incomplete symbol when flanked by "... is true" - it works like a definite description which purports to denote a fact that may or may not exist[173]. The general gist of this,

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[160] Cf. LANDINI, G., 1998, p.254.
[161] Again, cf. WHITEHEAD & RUSSELL, 1925, p.44 [1910, p.46].
[162] Cf. LANDINI, G., 1998, pp.287-91.
[163] I.e., 'functional' variables "ϕ", "ψ" with the accompanying "!", i.e., those variables that can be bound by quantifiers.
[164] Cf. LANDINI, G., 1998, pp.255-56.
[165] Cf. LANDINI, G., 1998, pp.257.
[166] I.e., that which is assumed and employed in its *numbered propositions*.
[167] Cf. LANDINI, G., 1998, pp.261-67.
[168] In the modern sense one finds, for instance, in KRIPKE, S., 1976, which has nothing to do with the notion of "substitution" of Russell's Substitutional theory.
[169] Cf. LANDINI, G., 1998, pp.279-80.
[170] Cf. LANDINI, G., 1998, pp.281-86.
[171] Cf. LANDINI, G., 1998, pp.287-91.
[172] Cf. LANDINI, G., 1998, pp.283-4.
[173] LANDINI, G., 1998, p.287-91. It must be observed, however, that Landini does not regard the multiple-relation

then, is that given the way Russell envisaged his nominalistic semantics, *only individual* variables are genuine variables ranging over *entities*, in accordance with the doctrine of the univocity of being[174].

If correct, this interpreation dismantles almost all of the traditional objections or difficulties raised by those who hold the 'orthodox' view on the development of Russell's views and *Principia Mathematica*.

To begin with, this interpretation counters Quine's charge that there is a serious use-mention confusion at the core of the No-Class Theory as presented both in *Mathematical Logic* and *Principia*. With respect to *Mathematical Logic*, once one realizes that its type distinctions are grounded on the realist view of propositions of the Substitutional Theory, there is no use-mention confusion[175]. With respect to *Principia*, once we accept that predicate variables (i.e., predicative propositional functions) are interpreted substitutionally and distinguished from 'non-predicative propositional functions', i.e., placeholders for arbitrary open formulas, the charge of serious confusion or deep incoherence can no longer be upheld[176]. The most one can say is that Whitehead and Russell *carelessly* but *deliberately* subsumed two very different ideas under the ambiguous notion of a 'propositional function' or 'propositional functionality'; this, of course, is indeed a serious problem, but it is not rooted in a deep or irredeemable confusion as Quine, for instance, claimed time and again[177,178].

Similarly, the traditional objection against the 'Vicious-circle Principle' and the restrictions of orders is dissolved. If the notion of a 'propositional functions' is understood along the lines indicated by Landini, *Principia*'s Introduction can no longer be seen as putting forward a hierarchy of ramified types of entities and the Vicious-Circle Principle at once loses its problematic status - either as a 'constructivistic' principle grounding type distinctions among mental constructs, ideas, and so on, or as a 'metaphysical' principle grounding type distinctions among mind-independent intensional entities, i.e., propositional functions, propositions, attributes, Fregean concepts or what have you[179]. Indeed, on Landini's interpretation, the 'Vicious-circle Principle' is nothing more than a heuristic principle or guideline that *does not serve as a justification for anything*. Distinctions of type and order are to be justified in terms of Russell's nominalistic (substitutional) semantics for predicate variables[180]. Landini's interpretation is also able to make sense of several aspects of *Principia*'s official syntax and of its presentation of the theory

theory as part of *Principia*'s formal system but only part of an informal semantics that Russell envisaged for justifying type distinctions; this point will be addressed in more detail in chapters 4 and 5.

174 Cf. LANDINI, G., 1998, pp.291-2.

175 Cf. LANDINI, G., 1998, pp.253.

176 Cf. LANDINI, G., 1998, pp.277-278.

177 More recently a criticism almost completely analogous to that of Quine was put forward by Scott Soames (cf. SOAMES, S., 2008).

178 In fact, it must be observed at once that Whitehead and Russell *did* distinguish *predicative* 'propositional functions' that are bindable variables from *non*-predicative ones that on Landini's interpretation are then accordingly treated as schematic letters

179 Cf. LANDINI, G., 1998, pp.275-79.

180 Cf. LANDINI, G., 1998, pp.279-87.

of types that on rival accounts are otherwise puzzling or even incoherent[181]. In this regard, the following particular points are most pressing and salient, namely: (i) *Principia*'s requirement that arguments of a predicate variables must match in both type *and* order[182]; (ii) the abscence of explicit concretion rules for *Principia*'s circumflex as a term-forming operator as well as the abscence of explicit comprehension principles apart from axioms of reducibility[183]; (iii) Whitehead and Russell's explicit practice/rule that so-called 'non-predicative functional variables' are *not* to be bound by quantifiers[184]; (iv) the role played by the notion of scope and the way incomplete symbols are employed by Whitehead and Russell[185] - in particular, the fact their order of elimination is *not* irrelevant, as indicated by Carnap[186] and Gödel[187], for instance.

But perhaps what is most important is that Landini's interpretation is able to make sense of the development of Russell's views from the *Principles* to *Principia* in a much more charitable way than the orthodox interpretation of Church, Hylton, Goldfarb and those who follow them. Landini puts Russell's doctrine of the unrestricted variable and of the univocity of being at the center of his account of this development in a way that provides both an explanation as to why Russell was attracted to the Substitutional version of the No-Class Theory and also why Russell abandoned the Substitutional account in favor of *Principia*'s version of the No-Class Theory. For if in *Principia*'s Introduction Russell intended a justification of type distinctions in terms of a nominalistic semantics, then he was not after all abandoning (deliberately or without acknowledging[188]) his most central logical doctrines but was, in fact, attempting to *preserve* them.

But there is more.

1.2.3 The Axiom of Infinity, Again

Landini's interpretation also provides us with a fresh perspective to evaluate the role Russell (and, to some extent, Whitehead[189]) intended the so-called Axiom of Infinity to play in their attempt to develop Pure Mathematics on a logical basis. To appreciate this we must say a little more about *Principia*'s treatment of classes and cardinal numbers.

As we briefly discussed above, in *Principia* all occurrences of class expressions are

181 Cf. LANDINI, G., 1998, pp.258-67.
182 Cf. LANDINI, G., 1998, pp.267.
183 Cf. LANDINI, G., 1998, pp.265-6.
184 Cf. LANDINI, G., 1998, pp.263-5.
185 Cf. LANDINI, G., 1998, pp.165-171.
186 CARNAP, R., 1947, p.147-50.
187 GÖDEL, K., 1944, p.120.
188 HYLTON, P., 2005, p.106.
189 Landini has recently argued convincingly that Whitehead had a somewhat different 'semantic interpretation' of *Principia* and that he also thought that the assumption that infinitely many universals exist in all types including the lowest could be taken as a *logical* truth, albeit an epistemically inaccessible one (LANDINI, G., forthcoming).

contextually eliminated in terms of the definitions of section $*20$. And as is well known, the cardinal number of a given class α is defined by Russell as the class of all classes β that are similar to α - i.e., the cardinal number of α is the class of all classes β whose elements can be put in a one-one correspondence with the elements of α. In symbols, we may reproduce *Principia*'s definition (in a slightly briefer form[190]) as follows:

$$\mathrm{Nc}‘\alpha =_{\mathrm{Df}} \hat{\beta}(\beta \,\mathrm{sm}\, \alpha) \tag{1.2.1}$$

Where sm stands for the relation of similarity or equinumerosity[191], which for the present purposes may be defined as the relation (in extension) which holds between two classes α and β whenever there is a one-one relation *R* whose domain is α and whose converse domain is β[192]. Thus the symbol "$\hat{\beta}(\beta \,\mathrm{sm}\, \alpha)$" is a class expression, an incomplete symbol whose occurrences must be eliminated in accordance with the definitions of $*20$. That is how *Principia* avoids any commitment to a specific ontology of numbers: by defining the concept of number in terms of the more basic vocabulary of higher-order logic and reducing the ontological commitments of Arithmetic to those of higher-order logic[193].

That is also why Whitehead and Russell need some assumption about the number of entities of the lowest type (i.e., individuals) in order to secure several basic theorems of ordinary Arithmetic. Skipping many (important) technicalities[194] and putting things very roughly, the difficulty is that the existence (i.e., non-emptiness) of a given number *m* (be it finite or not) depends on the existence of some class which has exactly *m* many members. But if at the bottom of the type hierarchy there are exactly *n* individuals, then there can be at most 2^n classes of individuals, 2^{2^n} classes of classes of individuals, and so on; thus, if the number of individuals is finite, then the very series of natural numbers will be finite in all given types, for there will always be some *m* which exceeds the number of elements of the universal class of the particular required type.

Now, the need for a so-called 'axiom' of infinity arises in *Principia* independently of whether the notion of a 'propositional function' is interpreted in realist or nominalist terms: the

190 *Principia* introduces actually introduces the proposition in question as a theorem, since they first introduce "Nc" as a relation-in-extension (cf. $*100$). The proper definition will be discussed and explained in chapter 5.

191 For now we are employing a mix of modern notation and that of *Principia*. Our detailed discussion in chapter 6, however, will fully employ - and explain - *Principia*'s original notation.

192 It must be observed that in *Principia* this definition is framed is such a way that α and β may be classes of what Whitehead and Russell call diferent *relative type* because *R* may be a *heterogeneous* relation with respect to the types of its relata. This is a fundamentally important point because it gives rise to different notions of similarity and consequently different notions of cardinal number (cf. WHITEHEAD & RUSSELL, 1912, pp.4-12). These aspects of *Principia*'s theory of cardinal numbers will be discussed in detail in chapter 5.

193 This, of course, is not to say that Arithmetic remains with no substantial ontological commitments: settling this question involves giving an account of the ontological commitments of higher-order logic. In any event, even if higher-order is understood in terms of a simple type-theory of attributes (i.e., properties and relations-in-intension) this reduction achieves something important: it dispenses with numbers as *objects* in the Fregean sense (i.e., abstract particulars in a more Russellian terminology). For an excelent discussion of this issue cf. KLEMENT, K., 2013. For another thorough discussion of this issue and for a defense of a strong eliminativistic position of numbers, cf. LANDINI, G., 2011.

194 The issues related to the existence of cardinal numbers are discussed in *Principia*'s second volume. The issue is discussed in general in the *Prefatory Statement* and in section A on the basic properties of cardinal numbers; section $*120$ treats finite cardinal numbers. All of this will be discussed in chapter 5.

need for some assumption about the number of individuals springs directly from the way *Principia* emulates a theory of sets using a system of higher-order logic that has the same structure of a simple type theory of attributes[195]. There is no way around this 'difficulty' other than adding some axiom ensuring an infinite (i.e., non-inductive) universal class in some type or by modifying fundamental aspects of *Principia*[196]. From a purely historical point of view, however, the most pressing question for scholars is this: does the use of the 'axiom' in *Principia* compromises *Russell's* claim that Logic and Pure Mathematics are identical?

As is the case with many of the issues we considered so far, 'orthodox' interpreters have sometimes answered by portraying *Principia* in poor lights because crucial aspects of the development of Russell's views were either unknown at the time or simply ignored - in particular with respect to Russell's views on the ontological commitments of Logic.

Logic in the *Principles of Mathematics*, is best understood as the general science of *propositions*[197], a conception of Logic that Russell also took for granted when he was trying to make the Substitutional Theory work. During this period Russell did assume that it was the business of Logic to *prove* a theorem of infinity, and, in fact, one of the virtues of Substitutional Theory was that it allowed Russell, as he put it, to "[...] manufacture $\aleph_0$ entities"[198] from the basic axioms and definitions of the theory. But Russell drastically changed his mind on this point[199].

As Landini first showed in his detailed analyses and reconstructions of the many versions of the Substitutional Theory that appear in Russell's mansucripts, there are even versions of the theory with which Russelll experimented that embraced *orders* of propositions and *were still able to prove an infinity theorem*[200]. This is an immensely important point to notice. In fact, it is so important that it bears spelling out in more detail: Russell's manuscripts show that at some point in 1907 he had formulated a version of the Substitutional Theory that blocked the contradictions - including the p_o/a_o paradox and its variants - *and* allowed a proof of a theorem of infinity, a theory which he almost certainly abandoned because he could not accept orders of propositions understood as *entities* - the same reason why he most likely abandoned the system of *Mathematical Logic*!

In light of the orthodox interpretation, this change of heart on Russell's part becomes almost incomprehensible. If one accepts the view advocated by Church, Hylton, Goldfarb and

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195 That is so whether one starts with a *ramified* structure and adds axioms of reducibility or assumes impredicative comprehension axioms outright.

196 Recently two interesting suggestions have been made in this regard. On the one hand, Landini has proposed an alternative axiom which entails the existence of infintely many natural numbers in a sufficiently high type while dispensing the need for assuming infinitely many entities in the lowest type (LANDINI, G., forthcoming). On the other hand, Landon Elkind has proposed a modification of *Principia*'s type-structure which allows for both infinitely ascending *and* descending types. Elkind then introduces another alternative axiom which allows an ingenious proof from Cantor's power-class theorem that there are infinitely many entities in *any* type (ELKIND, L., forthcoming). Both Landini and Elkind's results employ relations of heterogeneous (simple) types.

197 Cf. RUSSELL, B., 1903, p.1. Cf. also LANDINI, G., 1998, chapter 2 and LANDINI, G., 2010, pp.136-7.

198 RUSSELL, B., 1906*b*, p.238. Cf. also GRATTAN-GUINNESS, I., 1977, p.103.

199 Cf. GRATTAN-GUINNNES, I., 1977, p.105-6.

200 Cf. RUSSELL, B., 1907*e*, pp.515-6; RUSSELL, B., 1906*f*, pp.350-1; also LANDINI, G., 1998, pp.240-6.

many others that *Principia* embraces a ramified hierarchy of propositional functions *and* propositions - and thus deliberately abandons Russell's most central logical doctrine, i.e., that of the unrestricted variable - how can one explain his reasons for not keeping Substitution (*with* orders of propositions) which avoided the need for an axiom of infinity and was (as far as we know) as consistent as *Principia*'s system? It seems that the *only* possible answer that the 'orthodoxy' can provide is that *technically* substituition proved unmanageably complex[201]. But despite its plausibility, this answer goes against established evidence. To begin with, the technical complexity of Substitution did not stop Russell from defending it in *Mathematical Logic*. In fact, Russell knew from the very beginning that from a technical point of view the Substitutional Theory was quite severe[202], but considered writing *Principia* on its basis anyway[203]. Furthermore, we now know that Russell at some point planned to include an appendix in *Principia* explaining how the Substitutional Theory could have served as a foundation for Mathematics[204] - and, in fact, Russell considers this plan in the very manuscript where he claims that "[...] there is much to be said for reviving substitution"[205], precisely on the grounds that it "[...] will allow us to infer *Infin ax*"[206]!

Why then would Russell completely abandon Substitution with orders of propositions in favor of a theory that also betrayed his most central logical doctrines *and* failed to yield desired results? The orthodoxy seems unable to provide a satisfactory answer to this question.

In light of Landini's interpretation, on the other hand, the shift makes sense: according to Landini, Russell's attempt to provide a nominalistic justification for the hierarchy of orders grounded on the multiple relation theory of judgement shows that Russell no longer viewed Logic as the science of propositions, a change that was prompted, first and foremost, by his drive to *preserve* the doctrine of the unrestricted variable and the univocity of being. According to Landini, then, Logic and Pure Mathematics are viewed in *Principia* as one and the same science, namely, the general science of relational structure or of *structure as such*[207] without any commitments to specific ontology of *objects* in the Fregean sense, i.e., Russell's *individuals* or

201 As Russell himself suggests in *Mathematical Logic* (RUSSELL, B., 1908, pp.603).

202 In particular because of its notation.

203 Whitehead, however, was particularly dissatisfied with this idea (cf. LACKEY, D., 1973, pp.130-2), and he was not alone in this (cf. CHURCH, A., 1976*b*, p.702), perhaps with good reason; as Landini himself acknowledges, following (let alone producing) complex proofs formulated in terms of the Substitutional Theory can be a daunting task (cf. LANDINI, G., 1998, p.vi); but be that as it may, it is quite doubtful that this by itself would stop Russell and Whitehead from presenting the Substitutional Theory in *Principia* if they were truly convinced that it was right. Since we are talking about the authors of *Principia Mathematica* one can hardly appeal to prudery about technicalities as the sole excuse for a change of mind.

204 Cf. RUSSELL, B., 1907*e*, p.516. As far as I know, Landini was the first to notice this (cf. LANDINI, G., 1998, p.175). We now also know that Whitehead was favourable the this idea (cf. MOORE, G., 2014, p.lii), despite not being sympthetic to Substitution in general (cf., again, LACKEY, D., 1973, pp.130-2).

205 RUSSELL, B., 1907*e*, p.516.

206 RUSSELL, B., 1907*e*, p.516.

207 Cf. LANDINI, G., 1998, p.294. Cf. also LANDINI, G., 2010, p.95. LANDINI, G., 2011, pp.173-4 and p.187; this idea that Logic and Mathematics are concerned, in general, with *relational structure* appears more or less explicitly, for instance, in RUSSELL, B., 1924, pp.176; RUSSELL, B., 1919, p.59-61 and RUSSELL, B., 1959, pp.99-100.

entities of the lowest type. In this light, the need for an explicit Axiom of Infinity as a *hypothesis* or *conjecture of applied Mathematics is a feature, not a fatal defect* of Russell's approach.

This move would also be in accordance with another trend or pattern in the development of Russell's thought that also seems difficult to reconcile with an interpretation that attributes to (or requires of) him any bloated ontology in *Principia*, namely: his insistence on *the substitution of constructions for postulated entities whenever possible*. This methodological or "heuristic maxim"[208] of substituting "[...] constructions for inferences in Pure Mathematics"[209], as Russell would also later put it, was always a core element of his Logicism and there are *many* instances of it even in Russell's early work - most notably his definition of cardinal numbers as classes of similar classes. As Russell explains in his in the *Philosophy of Logical Atomism* lectures:

> I *always* wish to get on in philosophy with the smallest possible apparatus, partly because it diminishes the risk of error, because it is not necessary to deny the entities you do not assert, and therefore you run less risk of error the fewer entities you assume. The other reason - perhaps a somewhat frivolous one - is that every diminution in the number of entities increases the amount of work for mathematical logic to do in building up things that look like the entities you used to assume.[210]

In fact, Russell claimed that his "philosophical development" since the beginning of the twentieth century "[...] may be broadly described as a gradual retreat from Pythagoras"[211], in the sense that he gradually drifted apart from a view of Mathematics that entailed eliminable commitments to abstract particulars - most notably numbers. As Russell always made clear, the paradoxes led him to further extend this eliminative "retreat" to classes and relations-in-extension, and, as we saw, at some point to propositional functions and even propositions. And, of course, the theory of incomplete symbols is a milestone in this respect - as Russell repeatedly stated[212] - and it was only after Russell discovered the theory that the 'metaphysical purge' really took off[213].

Again, in light of 'orthodox' interpretations it is difficult to reconcile this emphasis on the use of 'Occam's Razor'[214] with the sort of ontology that the Churchian orthodoxy attributes to *Principia*. Indeed, Landini's view that *Principia* embraces a conception of Logic as a science

208 RUSSELL, B., 1924, p.164.
209 RUSSELL, B., 1924, p.166.
210 RUSSELL, B., 1918, p.195. Our emphasis. Of course, as Russell makes clear in this passage, he thought that the application of this methodological maxim was welcome in Philosophy in general. In fact, in the context of this very passage is Russell explaining that he is attracted to Neutral Monism "[...] because it exemplifies Occam's razor".
211 RUSSELL, B., 1959, p.208. Cf. also RUSSELL, B., 1944, p.13-14.
212 Cf., for instance: RUSSELL, B., 1924, pp.165-6; 1944, pp.13-14; 1959, pp.83-5.
213 Indeed, as Kevin Klement points out, in light of what we now know about the development of Russell's views, we see that "the core of Russell's solution to the paradoxes was *not* the theory of types [...] but rather the doctrine of incomplete symbols, the standpoint that words or phrases that apparently stand for such problematic entities as classes or propositions must not be taken at face-value" (KLEMENT, K., 2013, pp.204).
214 RUSSELL, B., 1944, p.13.

of Structure without any commitments to mathematical *objects* in the Fregean sense - thus treating assumptions of infinite classes as *conjectures* that occur as an antecedent clauses in certain existence theorems - is also in accordance with this trend of ontological parsimony.

1.3 Overview of the Goals and Content of Present Work

At this point it should be needless to say that Landini's interpretation is very controversial, in particular since it challenges long-standing assumptions that most interpreters viewed (and which some still view) as unproblematic. Indeed, that is why Landini himself refers to his own interpretation as 'revolutionary' as opposed to 'orthodox'.

In the note preceding this Introduction it was observed that the present dissertation does not present a radically novel interpretation of Russell's works in Mathematical Philosophy, but only aims at slightly expanding and discussing in some detail points already defended by Landini and other authors who follow (at least the most central points of) his interpretation, like Kevin Klement[215] and Graham Stevens[216]. My goal here is to do this by considering three interrelated topics that are of central importance in the development of Russell's views from the *Principles* to *Principia*, namely: his views on the nature of Logic as a science, his views on the ontological commitments of Logic and his thesis that Pure Mathematics (for our purposes, Arithmetic, specifically) is nothing but a development of Logic[217].

As I attempted to indicate in the present Introduction, these three intertwined topics form a unified thread that we can follow in order to assess which interpretation provides the best account of the available textual evidence. The thread, in a nutshell, can be explained as follows. Russell struggled, from the *Principles* onwards, to formulate a logical theory which satisfied the following requirements[218]: (i) it should get rid of the contradictions; (ii) it should keep intact what he thought to be philosphical 'common-sense' assumptions about Logic - most notably his thesis of the unrestricted variable and univocity of being; and finally (iii) it should keep intact as much as possible of the development of Pure Mathematics on the basis of this theory, with the most crucial case being elementary Arithmetic. Russell's whole logico-philosophical itinerary from 1903 to 1910 consisted in attempts to balance these three goals, but as he explained on more than one occasion, tasks (i), (ii) and (iii) were not on a par in terms of priorities. The first one was the truly "imperative" one - in other words, solving the contradictions was *the* main driving task. Reconciling this task with the sort of 'naive' ontological outlook of Russell's early Logicism proved impossible. So emerged Russell's eliminative approaches towards classes, propositional functions and propositions on the heels of his theory of incomplete symbols. This made it

[215] Cf., for instance, KLEMENT, K., 2010, 2013*a*, 2013*b*, 2014 and 2018.
[216] Cf., for instance, STEVENS, G., 2003, 2004 and 2010.
[217] Hence the title of the dissertation, "*Logic*, *Ontology and Arithmetic*".
[218] Cf., for instance, RUSSELL, B., 1959, pp.79-80.

possible to satisfactorily reconcile (i) and (ii), with varying degrees of success with respect to (iii)[219].

According to the 'Orthodoxy', *Principia* failed drastically in reconciling these three tasks, achieving success only with respect to (i), as if Russell had simply given up fullfilling (ii) and (iii). Their conclusion is that the thread ended up in a loose end. This dissertation argues that Landini's interpretation provides a much better approach for untangling the thread.

The dissertation is roughly divided in two parts. The first part discusses the development of Russell's conception logic and of the logicist project from its genesis up to *Principia Mathematica*. This first part sets the context for the second, which discusses Russellian Logic and Logicism in their mature version as presented in *Principia*. Each part consists of two chapters in addition to this Introduction[220]. The **second chapter** is concerned with the Logic and Logicism of the *Principles of Mathematics*. Our starting point is a discussion of what Russell viewed as a fundamental deficiency in Peano's logic: the absence of a treatment of relations. This was, for Russell, the fundamental flaw that prevented Peano from offering nominal definitions where the latter appealed to the so-called definitions by 'abstraction' and is what led Russell to re-discover (and, to some extent reframe) Frege's definition of cardinal numbers as classes of equinumerous classes. We consider this genesis of the Russellian definition of cardinal numbers and its place in the development of Russell's early logicism. Some interpretative issues are also considered and we side with Landini against the interpretation of Rodriguez-Consuegra. Also following to a great extent Landini's interpretation, we explain the conception of logic and the formal system underlying the *Principles of Mathematics*. Our discussion focuses on contrasting Russell's Logic as presented (informally) in the *Principles* with the views of Peano and Frege. Finally, we adress Russell's conception of Logic and Logicism as presented in the *Principles*, emphasizing that the work is tied to a conception of Logic according to which one may prove on a purely logical basis that there are infinite classes.

The **third chapter** is about the development of Russell's views on the ontological comittments of Logic and Mathematics in face of the contradictions. The chapter has two main general goals. First, to articulate the basic problem that permeates Russell's entire logical-philosophical itinerary from 1903 to 1910, namely: attempting to reconcile the absolute conception of logical generality and of the univocity of being of the *Principles* with a technically satisfactory resolu-

[219] If we were to resume even more the already condensed story told in this Introduction, the situation would be the following. The *Principles* failed with respect to (i). Given what little is understood of Russell's attempts at fulfilling this task between the *Principles* and *On Denoting*, it seems that the main difficulties lay in satisfying (ii). Russell's many versions of the Substitutional Theory were sort of a mixed bunch: the original 'simple' substitutional theory failed with respect to (i) since it was vulnerable to the p_o/a_o paradox; the 'no-general-propositions' substituitional theory of *Les Paradoxes* fared somewhat worse: as far as the best conjecture can lead us, the theory could deal with the contradictions in a philosophically sound way, but only at the cost of developing Mathematics adequately - and Russell's attempts at saving Mathematics within it re-introduced the p_o/a_o; then, there were versions of Substitution with orders of propositions that were successful in satisfying (i) and (iii), but failed to satisfy (ii).

[220] Given its extension, the Introduction is being counted as the first chapter.

tion of the contradictions. Second: to discuss in detail why Russell came to think of his theory of incomplete symbols as a means of solving the paradoxes. We begin by explaining the difficulties Russell faced early on in trying resolve the contradiction given his conception of the nature of logic. To this end, we briefly consider the solutions to the contradictions that Russell considered and rejected before the discovery of his theory of incomplete symbols, in particular the type theory of *Appendix B* of the *Principles*. After that, we discuss Russell's Substitutional Theory, taking Landini's work as our main guide. Following Landini's lead, our discussion aims at refuting two groups of interpreters. The first consists of Quine and those who follow him in interpreting Russell's ideas with respect to his theory of incomplete symbols applied to expressions of sets, functions, and so on as irreparably harmed by confusion between use and mention of expressions. The second consists of those who hold what we are calling, following Landini, an 'orthodox' interpretation of the development of Russell's views[221]. The discussion seeks to consider in detail the arguments and the overwhelming evidence[222] against the 'orthodox' interpretation. At the end of the chapter we discuss some issues which arise in connection with Landini's account of why Russell abanoned the Substitutional Theory, noting where there is still place for dispute among 'revolutionaries'. Also, in an appendix to the chapter, we attempt to chronologize the 'rise and fall' of Russell's Substitutional Theory.

Chapter four is about the logic of *Principia Mathematica*. The main task of this chapter is to argue in favor of Landini's interpretation of *Principia*. The bulk of this discussion revolves around *Principia*'s problematic use of the notions of a 'propositional function' and of a 'proposition'. We begin by considering different interpretative approaches for integrating Russell's multiple-relation theory of judgement with *Principia*'s theory of types. We then consider the fundamental technical issues that give rise to interpretative disputes about *Principia*'s hierachy of types and we contrast different approaches to dealing with them. In both cases we contrast the realist interpretations of Nino Cocchiarella and Bernarnd Linsky with the nominalist interpretation of Landini. Our goal is to show that Landini's interpretation has the upper hand in light of the available evidence.

The **fifth and final chapter** is about the logicism of *Principia Mathematica*. This chapter has three objectives. The first is to provide an exposition of *Principia*'s treatment of elementary Arithmetic, in particular the proofs of (the analogues of) Peano's postulates. The second objective is to discuss a variety of problems involved in the use of the notion of 'typically ambiguous cardinal number' in the second volume of *Principia*. As Landini showed, there are several

[221] I.e., authors who hold, for instance, that Russell assimilated Poincaré's 'vicious circle principle' as a means of justifying distinctions between types between entities (cf., for instance, HYLTON, P., 1990, pp.299-300; GOLDFARB, W., 1989, pp.32-4; LINSKY, B., 1999, p.77); or that Russell was led to accept Poincaré's principle and to adopt a theory of types of entities in the 1908 article *Mathematical Logic as Based on the Theory of Types* in virtue of paradoxes we now call "semantic", which he did not distinguish at all from the set-theoretical paradoxes (cf. HYLTON, P., 1980, pp.23-4 and GOLDFARB, W., 1989, p.35-6); or that the he type theory of *Mathematical Logic* is essentially the same as that of *Principia* (cf. CHURCH, A., 1976, p.747; and again, cf. HYLTON, P., 1980, p.1 and GOLDFARB, W., 1989, p.37-8).

[222] First presented by Landini and now availabe in the fifth volume of Russell's *Collected Papers*.

pseudo-theorems in the opening sections of the second volume. We take heed of Landini's detailed study[223], and note the necessary corrections while fulfilling our first goal of discussing the proof of the Peano's postulates. The third objective is to discuss Whitehead and Russell's use the so-called 'Axiom' of Infinity to prove theorems about the existence of cardinal numbers[224], in order to clarify the nature of *Principia*'s logicism. The goal is to argue that there is an internal coherence between the Epistemology and the Metaphysics of Logic underlying the Russellian logicist project as put forward in *Principia*'s own terms[225].

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[223] LANDINI, G., 2016.

[224] Some observations and considerations on the use of the "Multiplicative Axiom" (*Principia*'s version of the Axiom of Choice) are also made.

[225] Before proceeding, it must be emphasized that there are important and relevant topics for discussing some of the themes and issues of the present dissertation that *won't* be addressed (either in detail or not at all) for reasons of space and scope. One such theme that readers might expect to be discussed is the impact of Wittgenstein's ideas and criticism on Russell's views. This expectative can certainly be justified. Wittgenstein was, after all, extremely critical of the theory of types and of the multiple-relation analysis of judgement. While still a student of Russell, Wittgenstein presented a criticism of the multiple-relation theory that famously left the former "paralysed" and contributed in leading Russell to abandon his *Theory of Knowledge* manuscript. Wittgenstein's claim in the *Tractatus* that all and only logical laws are "tautologies" also had an impact on Russell who - at least *apparently* or *superficially* - seems to have accepted this idea, against his fomer view that Logic is a *synthetic* science whose laws have content and are informative. Wittgenstein also played a major role in leading Russell to consider modifications of the theory of types in the Introduction to *Principia*'s second edition. These developments will *not* be discussed in any capacity in the following pages, however, for a very simple reason: the present work is concerned with Russell's works on Mathematical Philosophy leading to and including the *first* edition of *Principia* - the three volumes of which were completed long before Russell even knew of Wittgenstein's existence. And of course, the subject of the impact of Wittgenstein's ideas on Russell's (and vice-versa) involves so many controversial and delicate issues and so vast a literature that it simply could not be addressed in the present dissertation without completely altering its scope.

2 The Logic and Logicism of *The Principles of Mathematics*

> The Nineteenth Century, which prided itself upon the invention of steam and evolution, might have derived a more legitimate title to fame from the discovery of pure mathematics. [...]
>
> What is now required is to give the greatest possible development to mathematical logic, to allow to the full the importance of relations, and then to found upon this secure basis a new philosophical logic, which may hope to borrow some of the exactitude and certainty of its mathematical foundation. If this can be successfully accomplished, there is every reason to hope that the near future will be as great an epoch in pure philosophy as the immediate past has been in the principles of mathematics. Great triumphs inspire great hopes; and pure thought may achieve, within our generation, such results as will place our time, in this respect, on a level with the greatest age of Greece.[1]

2.1 Russell's First Steps Towards Logicism

2.1.1 The Impact of Moore and Peano

The logico-mathematical origins of Russell's Logicism can be traced to the work of 19th century mathematicians on four main trends of investigation in the Foundations of Mathematics. The first was the process of 'rigorization' of Analysis. As is well known, despite the unprecedented scientific progress brought about by the discovery of Calculus by Newton and Leibniz, the discipline lacked a rigorous foundation since fundamental concepts such as *limit* and *continuity* were not clearly understood. Despite the centrality of such notions, they were taken for granted in terms of geometrical notions (e.g., defining a function as continuous if its graph has no 'gaps') or by appeal to intuitions about continuous motion and dubious concepts such as that of infinitely small quantities. This situation changed drastically in the Nineteenth Century when many major figures in the History of Mathematics like Augustin-Louis Cauchy (1789-1857), Bernard Bolzano (1781-1848) and, perhaps, most importantly, Karl Weierstrass (1815-1897) worked on providing a rigorous foundation for Analysis[2].

The second was the culmination of the process of rigorization into what became known

1 RUSSELL, B., 1901*a*, pp.366 and 379.

2 Weierstrass made the most central contribution in this process of rigorization, namely: he showed how to eliminate the appeal to infinitesimals in favor of a rigorously defined notion of a *limit* that relied only upon algebraic properties of real numbers. For an introductory survey of the issues, protagonists and the main works involved in this fundamentally important chapter of the History of Mathematics and Mathematical Philosophy, the reader is referred to GRATTAN-GUINNNES, I. (ed.), 2000*a*; for a discussion which explicitly and thoroughly contextualizes the developments in the foundations of Analysis aiming to discuss the developments of Russell's views, cf. GRATTAN-GUINNESS, I., 2000*b*, pp.14-74.

as the 'arithmetization' of Analysis[3]. Since the efforts of Weierstrass and those before him showed that an adequate definition of *limit* presupposes only the theory of real numbers, the need for a rigorous foundation for this theory became manifest. What Weierstrass showed in effect was that the problem of giving a rigorous, non-intuitive[4] foundation to Analysis was reduced to problem of defining real numbers and proving the existence of limits without any recourse to intuitions about the real line. Of course, the great revolution on the Foundations of Mathematics that occurred in the 19th century was brought about by figures such as Georg Cantor (1945-1918) and Richard Dedekind (1931-1916) who used the theory of sets as a means to do that and in doing so solved a problem that perplexed mathematicians and philosophers since ancient times, namely, that of giving precise definitions of *irrational* numbers[5]. What he and Cantor discovered was that real numbers and operations with them could be defined and their properties proved in a way that dispensed with geometrical intuitions about the real line, assuming only the algebraic properties (and the existence) of the set of rational numbers. Since several means of defining rationals in terms of integers and these in terms of natural numbers were already known at the time, this showed that the theory of natural numbers could provide a solid foundation for Analysis[6].

The third was the development of the theory of cardinal and ordinal numbers by means of set theory, most importantly of a theory of *infinite* numbers. While both Cantor and Dedekind made contributions to the theory of finite cardinals and ordinals, it is in Cantor's work[7] that we find the most important results and generalizations: not only Cantor offered a set theoretical formulation for the arithmetic of finite cardinals and ordinals, but he showed how to extend this theory to infinite numbers in a way that entailed a hierarchy of different *orders* of infinity - his *transfinite numbers*. Cantor's taming of the notion of infinity brought it out of the swamps of philosophical speculation and made it a fruitful mathematical concept. Cantor introduced the notion of one-one correspondence as the standard of measure for sets, exorcising pseudo-problems that haunted Philosophy of Mathematics for centuries[8], introducing a clear-cut distinction between the *power* or *cardinality* of a set, which was what Cantor explained in terms of the notion of one-one correspondence, and its extension, that is the totality of its members or elements[9].

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3 Cf., for instance, KLEIN, F., 1896.

4 To be clear: non-intuitive in the sense of not relying on intuitions about the real line.

5 Perhaps the most famous construction is that of Dedekind, in terms of what nowadays we call "Dedekind Cuts". Cantor also provided a different, albeit equivalent, definition, given in terms of limits of some sequences of rational numbers.

6 The reader is referred again to GRATTAN-GUINESS, I. (ed.), 2000*a* and 2000*b*.

7 His main results are presented in CANTOR, G., 1915, originally published in two parts in 1895 and 1897.

8 One of these was the allegedly 'paradoxical' fact that the natural numbers could be put in a one-one correspondence with the even naturals, for instance. Once one distinguishes, however, between cardinality and extension, there is no paradox: the set of naturals which contains the set of even naturals, is more extensive than the latter, but both have the same cardinality - or *number*. Making a (*merely*) terminological concession to a surpassed tradition, one could say that despite the fact that the set of even numbers is a *part* of greater whole, the set natural numbers, it has the same *number* of terms as that whole.

9 The classical survey of the historical development of Cantor's seminal work on set theory is JOURDAIN, P., 1915; for an even more comprehensive survey of Cantor's life and work cf. DAUBEN, J., 1979. Again, for a discussion of Cantor's works which puts it in the broad context of the developments in the foundations of Analysis, cf., GRATTAN-GUINNESS, I. (ed.), 2000; for a thorough discussion of the impact of Cantor's

Finally, the fourth trend of investigation consisted in the attempts of mathematicians like Augustus DeMorgan (1806-1971), George Boole (1815-1864), Ernst Schröder (1841-1902), Gottlob Frege (1848-1925), Giuseppe Peano[10] (1858-1932) and Alfred North Whitehead (1861-1947) to make Logic a serious mathematical discipline instead of a mere scholastic curiosity[11].

For Russell the first two of these developments showed that the theory of natural numbers could be used as the ultimate foundation upon which Real Analysis rested and indicated that the whole of Pure Mathematics could be *grounded on* or *reduced to* Arithmetic, in the sense that every theorem about higher forms of number (rationals and irrationals) which previously seemed to depend on geometrical intuitions like that of points in a line could be ultimately understood as a theorem about sets of natural numbers[12]. We find this idea clearly articulated in Russell's *Principles of Mathematics* and explicitly formulated as a definitive refutation of Kant's conception of Mathematical knowledge as knowledge grounded on forms of pure intuition:

> [...] during the last thirty or forty years, a new subject, which has added quite immeasurably to theoretical correctness, has been created, which may legitimately be called Arithmetic; for, starting with integers, it succeeds in defining whatever else it requires— rationals, limits, irrationals, continuity, and so on. It results that, for all Algebra and Analysis, it is unnecessary to assume any material beyond the integers, which, as we have seen, can themselves be defined in logical terms. It is this science, far more than non-Euclidean Geometry, that is really fatal to the Kantian theory of *a priori* intuitions as the basis of mathematics. Continuity and irrationals were formerly the strongholds of the school who may be called intuitionists, but these strongholds are theirs no longer. Arithmetic has grown so as to include all that can strictly be called pure in the traditional mathematics.[13]

The *Principles*, of course, was the culmination of Russell's efforts up to 1903 to contribute to the entreprise of expelling the Kantian schools from the strongholds of Mathematics. But, ironically, before 1900, were such philosophical assault to be effected, Russell would not be in the vanguard storming the walls, but defending them. As he vividly describes in *My Philosophical Development*, what led him first to the serious study of Philosophy was his dissatisfaction with the teaching of mathematics to undergraduates at Cambridge:

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works on the development of Russell's views, cf. RODRÍGUEZ-CONSUEGRA, F., 1991 and also GRATTAN-GUINNESS, I., 2000*b*.

10 With Peano we should also include authors who were part of his 'school' of mathematicians like Cesare Burali-Forti (1861-1931), Alessandro Padoa (1868-1937). For thorough and detailed discussions of Peano and his school and their impact on Russell's views, cf. RODRÍGUEZ-CONSUEGRA, F., 1991 and, again, also GRATTAN-GUINNESS, I., 2000*b*.

11 Again, cf., GRATTAN-GUINNESS, I., 2000*b* for a thorough and detailed historical discussion of these developments.

12 The sense and the extent to which Russell did accept arithmetization of all branches of Pure Mathematics and also the extent to which Russell's Logicism requires aritmetization is a controversial subject, however (cf. GANDON, S., 2008; 2012). We provide some clarification on this matter in section 3 of this chapter, footnote 410.

13 RUSSELL, B., 1903, p.157-8 §149.

> The mathematical teaching at Cambridge when I was an undergraduate was definitely bad. Its badness was partly due to the order of merit in the Tripos, which was abolished not long afterwards. The necessity for nice discrimination between the abilities of different examiners led to an emphasis on 'problems' as opposed to 'book work'. The 'proofs' that were offered of mathematical theorems were an insult to the logical intelligence. Indeed, the whole subject of mathematics was presented as a set of clever tricks by which to pile up marks in the Tripos. The effect of all this upon me was to make me think mathematics disgusting. When I had finished my Tripos, I sold all my mathematical books and made a vow that I would never look at a mathematical book again. And so, in my fourth year, I plunged with wholehearted delight into the fantastic world of philosophy.[14]

But back then Cambridge's 'fantastic world' of Philosophy was dominated by the British Idealism of authors like Thomas Hill Green (1836-1882), Francis Herbert Bradley (1846-1924) and John Ellis McTaggart (1866-1925) who were greatly influenced by German Idealism, especially Kant and Hegel. As a young student immersed in this intellectual environment, Russell became convinced that progress in Philosophy could only be achieved through improvements of German Idealism[15]. So it was that, despite his alleged disappointment with Mathematics, he wrote a dissertation on the foundations of Geometry discussing non-euclidean geometries within a broadly Hegelian outlook[16]. The starting point was the recognition that Kant's (supposed) view that Euclidean Geometry was - even in some subjective sense - the *right* geometry was irredeemable incorrect; Russell's goal was to give a transcendental deduction of the axioms that are necessary and sufficient for the development of *any* possible Geometry, Euclidean or not[17].

The main idea of the work was that there were inherent contradictions in the concept of Space and, therefore, within Geometry (in fact, within *every* science) that could only be overcome through a 'dialectical transition' to a new science[18]. In fact, after dealing with Geometry, Russell hoped to give an account of the other mathematical sciences based on this so-called Hegelian dialectic[19]. His ambition was to eventually construct a 'Logic of the Sciences' that would recognize (only) the 'inevitable' contradictions inherent to them. Such a 'logic' was envisaged to enable the 'dialectical transition' from one Science to the next, more abstract one in which the former contradictions would then be (somehow!) resolved. As Hylton points out, this general idea seems to be summed up in Russell's unpublished *Note on the Logic of the Sciences*[20]:

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14 RUSSELL, B., 1959, p.37-8.

15 For a systematic study of this 'idealist apprenticeship', cf. GRIFFIN, N., 1991 and also the first part of HYLTON, P., 1990.

16 The dissertation was presented for examination to George Frederick Stout (1860-1944) and Whitehead in 1895 and published in a revised form (which he had finished by 1896) as *An Essay on The Foundations of Geometry* (RUSSELL, B., 1897). This was Russell's first philosophical book.

17 GRIFFIN, N., 1991, p.129-130.

18 HYLTON, P., 1990, pp.84-89, pp.98-99; GRIFFIN, N., 1991, p.129-130.

19 As Russell later said in recollection regarding *Foundations of Geometry* "there was worse to follow" (RUSSELL, B., 1959, p.40).

20 HYLTON, P., 1990, p.98-9.

> What we have to do, therefore, in a logic of the sciences, is to construct, with the appropriate set of ideas, a world containing no contradictions but those which unavoidably result from the incompleteness of these ideas. Within any science, all contradictions not thus unavoidable are logically condemnable; from the stand point of a general theory of knowledge, the whole science, if taken as a metaphysic [sic], i.e. as independent and self-subsistent knowledge, is condemnable. We have, therefore, first to arrange the postulates of the science so as to leave the minimum of contradictions ; then to supply, to these postulates or ideas, such supplement as will abolish the special contradictions of the science in question, and thus pass outside to a new science, which may then be similarly treated. Thus, e.g., number, the fundamental notion of arithmetic, involves something numerable. [...] Hence geometry, since space is the only directly measurable element in sensation. Geometry, again, involves something which can be located, and something which can move for a position, by definition, cannot move. Hence matter and physics.[21]

In the case of Arithmetic, the alleged inherent contradiction was the fact that the fundamental notion of the science, namely, *number* could be used, as Nicholas Griffin puts it, "[...] both for *counting* and for *measuring*", so the fundamental question which Russell faced was how could the transition be made from countable discrete quantities to *continua*[22]. Young Russell's attempt to treat this problem was given in a paper entitled *On the Relations of Number and Quantity* - which he later claimed to be "unadulterated Hegel"[23]. Russell could also not decide "[...] whether infinite collections have no number or an infinite number" - and in the first draft of *The Principles of Mathematics*, finished around 1898, we still find him reluctant to fully accept Cantor's ideas:

> Cantor has, I think, established a branch of mathematics logically prior to the Calculus and even to irrationals, and has shown how it is presupposed in these. But I cannot persuade myself that his theory solves any of the philosophical difficulties of infinity, or renders the antinomy of infinite number one whit less formidable.[24]

The mentioned difficult - what Russell called the 'Antinomy of Infinite Number' - was still reminiscent of Kant's mathematical antinomies, and ended up being considered "[...] irrelevant, except on the Kantian view that numbers must be schematized in time" in the *Principles*[25]. So up until the turn of the century, Russell's allegiance laid, at least in a broad sense, with the 'Kantian theory' against Weierstrass, Cantor and their followers.

What happened? There are two main factors to be considered, respectively, in the next two sections.

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21 RUSSELL, B.,1959, p.53.

22 GRIFFIN, N., 1991, P.234-5.

23 RUSSELL, B., 1959, p.40. As Russell described it, the basic claim concerning the concept of number was that "extensions beyond the positive integers result from a gradual absorption of the properties of the unit, and give a gradually diminishing information as to the whole" (RUSSELL, B., 1959, p.40). Those interested in the study of logically alien thought can find similar patterns of reasoning in the following: "Quantity, as we saw, has two sources: the exclusive unit, and the identification or equalization of these units. When we look, therefore, at its immediate relation to self, or at the characteristic of selfsameness made explicit by abstraction, quantity is Continuous magnitude; but when we look at the other characteristic, the One implied in it, it is Discrete magnitude." (HEGEL, G.W., ...)

24 RUSSELL, B., 1900a, p.119.

25 RUSSELL, B., 1903, p.355, §337.

2.1.1.1 Moore: The Rejection of Idealism and the Realist Conception of Relations and Propositions

Concerning Philosophical questions in general, the main factor was the influence of Russell's friend and colleague, George Edward Moore (1873-1958). Perhaps the best way to describe the Philosophy put forward by Moore is as "atomist, direct-realist, and radically anti-psychologistic"[26]. As Russell famously recounts in the preface to the first edition of *The Principles*, his views were fundamentally those imposed by Moore's revolt against the idealistic doctrines of Kant, Hegel and his fellow Cambridge colleagues[27]:

> On fundamental questions of philosophy, my position, in all its chief features, is derived from Mr. G. E. Moore. I have accepted from him the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind; also the pluralism which regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities, with relations which are ultimate, and not

[26] GRIFFIN, N., 2013, p.383.

[27] This break with Idealism by Russell and Moore is, for good or bad, frequently considered as the inauguration of Analytic Philosophy (See HYLTON, P., 1990. A dissenting view is proposed by BELL, D., 1999 and, in a sense, by DUMMETT, M., 1973, pp.665-684 and DUMMETT, M., 1994.). The tale - which Russell depicts as a 'revolt' - has been extensively discussed both in favor of its historical accuracy (HYLTON, P., 1990, pp.105-116; HYLTON, P., 2004, pp.1-9.) and against it (GRIFFIIN, N., 1991; RODRÍGUEZ-CONSUEGRA, F., 1991; GRATTAN-GUINNESS, I., 1993.). Russell himself contributed to the narrative of a sudden definitive break in several occasions, for he characterized his conversion to Moore's pluralist realism as an abrupt and radical change. In his *Autobiography*, for instance we find him claiming that "There is one major division in my philosophical work: in the years 1899-1900 I adopted the philosophy of logical atomism and the technique of Peano in mathematical logic. This was so great a revolution as to make my previous work, except such as was purely mathematical, irrelevant to everything that I did later. The change in these years was a revolution; subsequent changes have been of the nature of an evolution" (RUSSELL, 1959, p.11). Although any historical account of Russell's and Moore's 'rebellion' against the idealist tradition as a sudden break - including the one Russell himself sometimes sketches - is indefensible, Russell's retrospective that his change of philosophical perspective was a revolution should be taken *very* seriously and should not be overstated. *There is a radical difference* in philosophical outlook in Russell's writings from, say 1901 owards, and those from his 'idealistic excursus' - what Griffin and Rodríguez-Consuegra showed is that this Revolution was not effected with a sudden and radical break. One indication of this is the fact that Russell was completely dismissive of his early idealistic work. Commenting on the previously discussed paper described as "unadulterd Hegel" he observes that despite Couturat - for whom Russell had much respect - having described the article as a *"petit chef d'oeuvre de dialectique subtile*", Russell simply dismissed the article as being "nothing but unmitigated rubbish". In fact most of Russell's later recollections also display a similar dismissive and aggressive attitude towards Hegelianism, as in *Portraits from Memory*: "My first serious contact with the German learned world consisted in the reading of Kant, whom, while a student, I viewed with awed respect. My teachers told me to feel at least equal respect for Hegel, and I accepted their judgment until I read him. But when I read him I found his remarks in the philosophy of mathematics (which was the part of philosophy that most interested me) both ignorant and stupid." (RUSSELL, 1956, p.20). Such passages fit like a glove what Thomas Kuhn says about scientists who reject their earlier work because they no longer find themselves practicing science in the same paradigm: "Entry into a discoverer's culture often proves acutely uncomfortable, especially for scientists, and sophisticated resistance to such entry ordinarily begins with the discoverer's own retrospects and continues in perpetuity. [...] Systematic distortions of memory, both the discoverer's memory and the memory of many of his contemporaries, are a first manifestation of resistance.". As Paulo Faria - who called my attention to this observation of Kuhn's - puts it, the point is that "[...] a revolutionary scientist is unable to make sense of some of his own earlier work through imposing upon it the new conceptual framework in which he has been working since the heyday of the revolution" (FARIA, P., 2021, pp.61-2). It seems that this is precisely what we witness in Russell's reminiscences.

> reducible to adjectives of their terms or of the whole which these compose. Before learning these views from him, I found myself completely unable to construct any philosophy of arithmetic, whereas their acceptance brought about an immediate liberation from a large number of difficulties which I believe to be otherwise insuperable. The doctrines just mentioned are, in my opinion, quite indispensable to any even tolerably satisfactory philosophy of mathematics, as I hope the following pages will show.[28]

Despite the fact that studies of Russell's unpublished manuscripts which predate the *Principles of Mathematics* established that this break was *gradual*, and thus, that Russell's reminiscence is historically inaccurate[29], the main point which marked a departure from the Idealist tradition is clear. Russell was mainly interested in two theses advocated by the Cambridge idealists: (1) The idea that somehow *features of our thoughts* (be they disguised under the label of *concepts of understanding* or what have you) are constitutive of the things about which we think; and (2) That relations are analyzable in terms of properties or predicates, which for Russell was to say that they were not real, independent entities.

In *Portraits From Memory* he goes into a little more detail about this story, explaining that the denial of second of these theses was the one that most affected his views on Mathematical Philosophy:

> At Cambridge I was indoctrinated with the philosophies of Kant and Hegel, but G. E. Moore and I together came to reject both these philosophies. I think that, although we agreed in our revolt, we had important differences of emphasis. What I think at first chiefly interested Moore was the independence of fact from knowledge and the rejection of the whole Kantian apparatus of *a priori* intuitions and categories, molding experience but not the outer world. I agreed enthusiastically with him in this respect, but I was more concerned than he was with certain purely logical matters. The most important of these, and the one which has dominated all my subsequent philosophy, was what I called 'the doctrine of external relations'. Monists had maintained that a relation between two terms is always, in reality, composed of properties of the two separate terms and of the whole which they compose, or, in ultimate strictness, only of this last. This view seemed to me to make mathematics inexplicable. I came to the conclusion that relatedness does not imply any corresponding complexity in the related terms and is, in general, not equivalent to any property of the whole which they compose.[30]

Russell also stresses this point in *My Philosophical Development*, explaining that "[...] Moore was most concerned with the rejection of idealism" while himself "[...] was most interested in the rejection of *monism*"[31]. Moore's refusal of the idealistic doctrine (1) mentioned above was given in terms of a general conception of judgment based on a - now famous - realist conception of propositions. As Peter Hylton observes, fundamental to Moore's rejection of idealism was the idea that the truth of any judgment should be completely independent of *acts* of judging[32]. Moore and, following him, Russell, understood this idea in the most radical form,

28 RUSSELL, B., 1903, p.xviii.
29 See previous note.
30 RUSSELL, 1959, p.11-12.
31 RUSSELL, B., 1959, p.54.
32 HYLTON, P., 1990, p.109.

claiming that all cognition consists in *direct* relation between a mind and the objects of thought. Perhaps one of the best clear-cut explanation of their views is given by Russell in a letter which he wrote to Frege on 12 December 1904:

> I believe that in spite of all its snowfields Mont Blanc itself is a component part of what is actually asserted in the proposition 'Mont Blanc is more than 4000 meters high'. We do not assert the thought, for this is a private psychological matter: we assert the object of the thought, and this is, to my mind, a certain complex (an objective proposition, one might say) in which Mont Blanc is itself a component part. If we do not admit this, then we get the conclusion that we know nothing at all about Mont Blanc. This is why for me the meaning of a proposition is not the true, but a certain complex which (in the given case) is true.[33]

As Michael Potter puts it, what this "[...] exchange with Frege concerning Mont Blanc demonstrates vividly is that Russell conceived of objects literally as constituents of propositions"[34]; propositions were viewed by Russell as complex *entities* whose constituents are the very things which the proposition is about.

As Russell would later make explicit, the essence of the epistemological doctrine implicit in the *Principles* is the idea that "[...] in every proposition that we can apprehend [...] all the constituents are really entities with which we have immediate *acquaintance*"[35]. Truth and falsity of a proposition was also understood in terms of something (undefinable) which we *recognize* some propositions as having and others as not having. As Russell puts it in the *Principles*, "True and false propositions alike are in some sense entities [...] but when a proposition happens to be true, it has a further quality, over and above that which it shares with false propositions"[36].

Now, from the point of view of the development of Russell's views on Mathematical Philosophy, Russell's reasons for rejection of thesis (2) mentioned above are the most relevant. Russell's main target was Bradley's view that all so-called 'external' relations can be analyzed in terms of "identity and diversity of content", which, themselves are not relations at all but intrinsic properties - or, as Bradley called them, 'internal' relations. To make the point of disagreement as simple as it can be made, take two objects *a* and *b* and some relation *R* which holds between them; are we to say that the correctness of some judgment like *a has R to b* is grounded solely on the monadic predicates which hold of each *relata* or also because of the nature of the relation itself, which is ontologically independent from the terms it relates? The answer which Moore, and following him, Russell, advocated was that 'external' relations are ultimate or non-analyzable constituents of reality, that is, that there are relations that are ontologically independent from their *relata* and not reducible to any intrinsic property or so-called 'internal relation' of these.

33 FREGE, G., 1980, p.169.
34 POTTER, M., 2000, p.121.
35 RUSSELL, B., 1905*a*, p.415.
36 RUSSELL, B., 1903, p.49 §52.

From a logical point of view, this amounted to nothing less than the rejection of the universal analysability of propositions in terms of subject-predicate form, a thesis which Russell thought had as its ultimate and unbearable consequence the disjunction between the two metaphysical views he called 'Monism' and 'Monadism'.

According to the first, the correct mode of analysis of *aRb* is to break it down into the conjunction of two propositions concerning *a* and *b*, that is, a proposition which attributes a property to *a* and another to *b*, while the second holds that *aRb* is to be analyzed as a proposition that attributes a property to a 'whole composed of *a* and *b*'. According to Russell, the first philosopher ever to extract this fundamental consequence of the universal analysis of propositions into subject-predicate form was Leibniz, who, Russell thought, stated the monadistic view with "admirable lucidity"[37] in the following passage:

> The ratio or proportion between two lines L and M may be conceived three several ways; as a ratio of the greater *L* to the lesser *M*; as a ratio of the lesser *M* to the greater *L*; and lastly, as something abstracted from both, that is, as the ratio between *L* and *M*, without considering which is the antecedent, or which the consequent; which the subject, and which the object. [...] In the first way of considering them, *L* the greater is the subject, in the second *M* the lesser is the subject of that accident which philosophers call relation or ratio. But which of them will be the subject, in the third way of considering, them? It cannot be said that both of them, *L* and *M* together, are the subject of such an accident; for if so, we should have an accident in two subjects, with one leg in one, and the other in the other; which is contrary to the notion of accidents. Therefore we must say that this relation, in this third way of considering it, is indeed out of the subjects; but being neither a substance, nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful.[38]

As Russell read him, Leibniz was led, from the logical doctrine that every proposition has the subject-predicate form, to the metaphysical consequence that "[...] relations [...] have only a mental truth" and then to the ultimate conclusion that a "[...] true proposition is one ascribing a predicate to God and to all others who perceive the relation"[39]. The study of Leibniz shaped Russell's understanding of the idealistic doctrines of Bradley, who he thought implicitly accepted the universality of the subject predicate form and due to this was led to the third alternative, the one Leibniz discards, namely that the correct analysis of a relational proposition like *L is greater than M* is not as an assertion about *L* or *M*, but about a "whole composed of them", which is the true subject of the proposition[40].

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37 RUSSELL, B., 1903, p.222 §213.

38 GERHARDT, I.(ed.).,1890, Vol. VII, p.401. The translation is Russell's.

39 RUSSELL, 1900, p.16. At this point in the text, Russell is commenting specifically on the following passage from the *New Essays in Human Understanding*: "The units are separate and the understanding takes them together, however scattered they may be. However, although relations are the work of the understanding they are not baseless and unreal. The primordial understanding is the source of things; and the very reality of all things other than simple substances rests only on the foundation of the perceptions or phenomena of simple substances. " (LEIBNIZ, G., 1765, p.145, §5; this is not Russell's translation).

40 Russell also thought that ultimately, it was this doctrine that led to the idea that every proposition is about the the same unique object, namely, *reality* - hence, the idea of a *monist* metaphysics.

The best way to understand why Russell thought that these views led to 'insuperable' difficulties in the Philosophy of Mathematics is by considering what he called the 'problem of asymmetrical relations'. If we follow the terminology of the *Principles*, *symmetrical* relations are such that xRy always implies yRx; and a relation is *not symmetrical* if for some *x* and *y*, we may have xRy and $\sim yRx$ and an *asymmetrical*[41] *relation* is one such that xRy always implies $\sim yRx$. In symbols:

$$\text{Symmetrical}(R) = (x)(y) \,.\, xRy \,.\supset .\, yRx$$

$$\text{Not} - \text{symmetrical}(R) = \sim \text{Symmetrical}(R)$$

$$\text{Asymmetrical}(R) = (x)(y) : xRy \,.\supset .\sim yRx$$

Relations of the latter kind are fundamental to any attempt to characterize *order* among some given set of terms, and thus, are very important for Mathematics: *greater* or *less than* are the typical examples[42]. Now, at its core, Russell's argument is very simple: both the monadistic as well as the monistic theory claim that relations are unreal (i.e., not ultimate constituents of reality) and this claim is grounded on the logical doctrine that every proposition can be analyzed in terms of a subject and a predicate - so to show that it is impossible to analyze propositions in which *asymmetrical* relations occurs in these terms is to show that the logical basis of monadistic and monistic metaphysics must be rejected. In the case of the monadistic theory, Russell shows that any attempt to analyze "... *has R to z*" as an adjective (where *R* is asymmetric) fails due to the unavoidable complexity of the resulting adjective, which presupposes the relation it was supposed to replace:

> Let *a* and *b* have an asymmetrical relation *R*, so that *aRb* and $b\breve{R}a$[43]. Let the supposed adjectives (which, as we have seen, must each have a reference to the other term) be denoted by β and α respectively. Thus our terms become $a\beta$ and $b\alpha$. α involves a reference to *a*, and β to *b*; and α and β differ, since the relation is asymmetrical. But *a* and *b* have no intrinsic difference corresponding to the relation *R*, and prior to it; or, if they have, the points of difference must themselves have a relation analogous to *R*, so that nothing is gained. Either α or β expresses a difference between *a* and *b*, but one which, since either α or β involves reference to a term other than that whose adjective it is, so far from being prior to *R*, is in fact the relation *R* itself. And since α and β both presuppose *R*, the difference between α and β cannot be used to supply an intrinsic difference between *a* and *b*. Thus we have again a difference without a prior point of difference. This shows that some asymmetrical relations must

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41 The term "*anti*-symmetrical" is sometimes used in textbooks.

42 Of course, strictly speaking, what is required for a relation *R* to generate *order* among the members of some set is the fact that given any two (different) terms *x* and *y* either *xRy* or *yRx* (*R* is connected), the fact that if *xRy* and *yRz*, then *xRz* (*R* is transitive) and the fact no term has *R* to itself (*R* is irreflexive). Transitivity and irreflexivity are sufficient to generate a *partial* order. But any relation that satisfies these conditions must be asymmetric: assume that for some *x* and *y*, we have *xRy.* Assume also *yRx.* Given transitivity, it follows that *xRx*, which contradicts irreflexivity. Thus, if *R* is transitive and irreflexive, it must be asymmetrical.

43 "$\breve{R}$" is the notation borrowed from Schröder for the inverse of *R*, that is, the relation which holds between *y* and *x* when, and only when, *xRy* holds.

> be ultimate, and that at least one such ultimate asymmetrical relation must be a component in any asymmetrical relation that may be suggested.[44]

In the case of the monistic theory, the fundamental problem Russell identifies its the incapacity of the theory to explain what he calls difference of *sense.* If *aRb* is understood as a proposition about a whole composed by *a* and *b*, asymmetrical relations again become a problem. For a sentence like "*a* is greater than *b*" stands for a different proposition than the one which is expressed by "*b* is greater than *a*" or "*a* is lesser than *b*". But if *lesser than* is understood as the converse of *greater than*, then the two propositions have the exact same constituents, and so there must be something that explains the difference between them beyond the mere sum of their parts:

> The proposition "*a* is greater than *b*", we are told, does not really say anything about either *a* or *b*, but about the two together. Denoting the whole which they compose by (*ab*), it says, we will suppose, "(*ab*) contains diversity of magnitude". Now to this statement—neglecting for the present all general arguments—there is a special objection in the case of asymmetry. (*ab*) is symmetrical with regard to *a* and *b*, and thus the property of the whole will be exactly the same in the case where a is greater than *b* as in the case where *b* is greater than *a*. [...] In order to distinguish a whole (*ab*) from a whole (*ba*), as we must do if we are to explain asymmetry, we shall be forced back from the whole to the parts and their relation. For (*ab*) and (*ba*) consist of precisely the same parts, and differ in no respect whatever save the sense of the relation between *a* and *b*. "*a* is greater than *b*" and "*b* is greater than *a*" are propositions containing precisely the same constituents, and giving rise therefore to precisely the same whole; their difference lies solely in the fact that greater is, in the first case, *a* relation of *a* to *b*, in the second, a relation of *b* to *a*.[45]

As Peter Hylton observes the above argument for Russell "[...] is a paradigm case of the interrelation of the philosophical and the technical"[46]: any theory of judgment which purports to be adequate for handling Mathematics must account for the difference between *a is greater than b* and *b is greater than a* (and that which makes them mutually exclusive), i.e., the fact that an asymmetrical relation has a *sense* or a direction. Russell is providing a decisive argument which shows that the monistic analysis cannot explain the difference between propositions in which the same constituents occur with the same relations differing in sense. As Hylton aptly puts it, for Russell the above argument shows that "Mathematics requires the notion of order, which in turn requires irreducible relations"[47]. This is what led Russell to follow Moore in concluding that relations should thus be considered as "ultimate entities", i.e., must be considered *genuine* constituents of propositions that can be analyzed away in terms of predicates or properties.

As we shall see next, this view was decisive in leading Russell to his first major contribution to Mathematical Philosophy.

44 RUSSELL, B., 1903, p.224, §214.
45 RUSSELL, B., 1903, p.225, §215.
46 HYLTON, P., 1990, p.184.
47 HYLTON, P., 1990, p.184.

2.1.1.2 Peano: Mathematical Logic and Axiomatic Method

With respect to the development of Russell's logico-mathematical ideas, *the major influence* on Russell was Peano. Despite the fact that Russell had already read and partially accepted the ideas of Cantor and Dedekind[48], for instance, it was Peano who provided him with a bigger picture of how the works of these authors were truly revolutionary, mainly because Peano showed how their results fitted into a unified conceptual whole tied up by the *Logica Matematica* developed by him and his disciples[49].

As Russell recounts in his *Autobiography*, the most crucial moment in the development of his ideas concerning Mathematical Philosophy was the International Congress of Philosophy:

> The Congress was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. I therefore got him to give me all his works, and as soon as the Congress was over I retired to Fernhurst to study quietly every word written by him and his disciples. It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring anew powerful technique for the work that I had long wanted to do.[50]

As he explains in *My Philosophical Development*, the 'enlightenment' received from Peano came from "two purely technical advances"[51], namely:

1. The distinction between (*a*) the relation of membership which holds between an element *x* and a class α and (*b*) the relation of inclusion between a class β and another class α, which Russell would later symbolise respectively as $x \epsilon \alpha$ and $x \epsilon \beta \supset_x x \epsilon \alpha$ or $\beta \subset \alpha$; and
2. The distinction between the class ιa containing a single entity *a* and *a* itself.[52]

Of course, the notations used here to express these distinctions are also taken from the works of Peano, together with other important distinctions like the one between real (free) and apparent (bound) variables. Important as these distinctions were, marking a break with the part-whole

[48] A detailed discussion of Russsell's gradual acceptance of their works, in which the manuscripts of previous versions of the *Principles* are thoroughly discussed, can be found in RODRÍGUEZ-CONSUEGRA, F., 1991.

[49] For a study of Peano's life and work which also discussed his 'school', see KENNEDY, H., 1980; again, for details relating the works of Peano and his followers to the development of Russell's views, cf. RODRÍGUEZ-CONSUEGRA, F., 1991 and also KENNEDY, J., 1973, 1974.

[50] RUSSELL, B., 1967, p.218.

[51] RUSSELL, B., 1959, p.66.

[52] Russell notes in the same passage that both distinctions were also emphasized by Frege, even before Peano.

conception of a set or class and the acceptance of the superior Cantorian conception of set, however, they barely scratch the surface of what Russell learned from Peano.

Against what was for some time the standard account of what Russell learned from Peano[53], Francisco Rodriguez-Conseugra systematically showed[54] that Russell learned from Peano (and also, almost just as importantly, from his followers) a *method* of analysis of mathematical theories which, by the time of the congress, was presented to Russsell in its most optimized form in Peano's *Formulario mathematico* or *Formulaire de Mathématiques*, in the French version. Inspired by the procedure adopted in Geometry of laying down primitive notions and axioms which embody the fundamental properties of these notions, Peano developed axiomatizations of several branches of pure Mathematics and also of a logical calculus of classes and propositions. Russell recognized Peano's logic and notation as powerful philosophical tools - so much so that at the time Russell saw in Peano's works "[...] the realization of Leibniz's great idea that, if symbolic logic does really contain the essence of deductive reasoning, then all correct deduction must be capable of exhibition as a calculation by its rules"[55]. But Russell also learned from Peano a method of investigating formal theories[56]. What most impressed Russell in Peano's techniques was their wide scope of applications: the method of laying down primitive notions and axioms and then extracting the consequences within a precise calculus was applied by Peano and his school not only to Arithmetic, but also to several branches of Analysis and Geometry and, perhaps most importantly for Russell, even to Logic itself.

The primitive logical notions of the *Formulario* were[57]: 1. *Class* symbolized in some editions by "K" and others by "Cls"[58]; 2. Membership to a class, symbolized by "ϵ"[59]; 3. Material implication, always symbolized by the 'horseshoe', "$\supset$"; 4. Formal implication between two propositions that is, the relation which holds between p and q when "q can be deduced from p, whatever $x, ...z$ are"[60], symbolized as "$p \supset_{x,..,z} q$"; 5. The conjunction or joint affirmation of two propositions p and q, symbolized by "$p \cap q$" or "pq" or yet "$p \,.\, q$"; 6. The notion of definition, which was indicated by "Df" following the formula which defines some string of symbols; 7. The

53 Cf., for instance, KENNEDY, H., 1973. This, account, of course, was standard for good reason: Russell himself suggested it many times, as the passages just quoted make clear.

54 RODRÍGUEZ-CONSUEGRA, F., 1991, pp.91-134.

55 RUSSELL, B., 1901*c*, p.353.

56 See, for instance Russell's little known and even less discussed article *Recent Italian Work on the Foundations of Mathematics* (RUSSELL, B., 1901*c*); some points concerning this article will also be discussed in the present chapter.

57 PEANO, G., 1897, pp.3-4. We shall follow Russell's exposition of Peano's logic in the *Principles* very closely, complementing with details from Peano's own presentation. Russell thought Peano's best expositon of his Symbolic Logic to be the first part of the 1897 edition of the *Formulario*, the reason being that it is the most clear in distinguishing in detail the primitive notions and propositions of Logic. The choice is clearly justified if one looks at the comparative sheer volume of space dedicated to Logic in this edition: in the 1897 edition, Mathematical Logic takes the first seventy pages, while in the 1901 edition this space is almost cut in half.

58 In the 1897 edition he used the letter "K", changing it to "Cls" in the 1901 edition. For the sake of uniformity we employ the latter.

59 Peano used "ε". For the sake of uniformity, we shall employ *Principia*'s counterpart of this symbol.

60 Peano, G., 1897, p.1.

negation of a proposition, symbolized by "$-$"[61]. Peano's notation and methodology drew heavily upon the the algebraic tradition of Boole and Schröder. He followed them in not distinguishing as clearly as he should the operations of the algebra of classes from the those of the propositional calculus, employing, for instance, the same symbol for equality and material equivalence and also the same symbol for class inclusion and material implication, despite the fact that the did drew the distinction[62]. Peano had[63]:

$$\alpha, b \,\epsilon\, \text{Cls} \,.\supset : a \supset b \,.=: x \,\epsilon\, a \,.\supset_x x \,\epsilon\, b \quad \text{Df}$$

which, in fact, as Russell put it, defines "every *a* is *b*" as "*x* is an *a* implies that *x* is a *b*, whatever *x* may be", that is: $x \,\epsilon\, a$ *formally implies* $x \,\epsilon\, b$. Given Peano's definition of formal implication, $x \,\epsilon\, a$ and $x \,\epsilon\, b$ are understood as *propositions containing real (free) variables*, a notion which Russell repudiates in favor of what he calls a *propositional function*[64]. Peano also had an analogous definition for the product (intersection) of two classes[65]

$$\alpha, b \,\epsilon\, \text{Cls} \,.\supset : a \cap b \,.= x \ni (x \,\epsilon\, a \,.\, x \,\epsilon\, b) \,\text{Df}$$

which employed a notion which Russell would later attach much weight to, namely that of *such that*, defined as[66]:

$$a \,\epsilon\, \text{Cls} \,.\supset .\, x \ni (x \,\epsilon\, a) = a \quad \text{Df}$$

This is an attempt to define "the *x*'s such that *so-and so*" or "the *x*'s that satisfiy the condition $x \in a$" as the class *a*. This was probably introduced due to the lack of (and distinction between) a principle of class-abstraction and a principle of extensionality, since the definition seems like a confused amalgam of both. This was later removed in the 1901 edition of *Formulario* after the criticism of Peano's disciple Alessandro Padoa[67], who Russell followed in the *Principles* calling the definition "perfectly worthless" (as we shall see, however, he retained the notion as primitive).

With the primitive notions and definitions described above, Peano assumed the following group of primitive propositions, the formulation of which again, showcase the influence of the algebraic tradition:

$$a \,\epsilon\, \text{Cls} \,.\supset .\, a \supset a$$

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61 Peano used a slightly shorter, bold bar symbol.

62 For more details, cf. GRATTAN-GUINNESS, I., 2000*b*, pp.225-7.

63 PEANO, G., 1897, p.3.

64 This notion had already appeared in a semi-articulate form (cf. GRATTAN-GUINNESS, I., 2000*b*, p.228) in Peano's famous booklet on the principles of Arithmetic, in the following passage: "Let ϕ be a sign or an aggregate of signs such that, if *x* is an object of the class *s*, the expression ϕx denotes a new object; we assume also that equality is defined between the objects ϕx; further, if *x* and *y* are objects of the class *s* and if $x = y$, we assume it is possible to deduce $\phi x = \phi y$. Then the sign ϕ is said to be a function presign in the class *s* [...]" (PEANO, G., 1889, p.91). This passage, as Grattan-Guinness observes is introducing a notion akin to that of Dedekind's 'transformations' (cf. GRATTAN-GUINNESS, 2000*b*, p.228).

65 With this he defined the sum (or union) of two classes was defined as $a, b \,\epsilon\, \text{Cls} \,.\supset .\, a \cup b = -[(-a)(-b)]$ Df.

66 In the 1897 edition of the *Formulario*, Peano used a variable with a bar over it to indicate the use of *such that*, defining: $a \,\epsilon\, \text{Cls} \,.\supset .\, \overline{x}\,\epsilon(x \,\epsilon\, a) = a$ Df. In the 1901 edition this was indicated by an inverted ε. For the sake of convenience and difficulties of typesetting this symbol, we shall employ its modern counterpart $\ni$.

67 RODRÍGUEZ-CONSUEGRA, F., 1991, p.131-4.

$$a, b \,\epsilon\, \text{Cls} \,.\, \supset .\, ab \,\epsilon\, \text{Cls}$$

$$a, b \,\epsilon\, \text{Cls} \,.\, \supset .\, ab \supset a$$

$$a, b \,\epsilon\, \text{Cls} \,.\, \supset .\, ab \supset b$$

$$a, b, c \,\epsilon\, \text{Cls} \,.\, a \supset b \,.\, x \,\epsilon\, a \,.\!\supset\!.\, x \,\epsilon\, b$$

$$a, b, c \,\epsilon\, \text{Cls} \,.\, a \supset b \,.\, b \supset c \,.\, \supset a \supset c$$

$$a, b, c \,\epsilon\, \text{Cls}.\!\supset : a \supset b \,.\, a \supset c \,.\!\supset\!.\, a \supset bc$$

$$a \,\epsilon\, \text{Cls} \,.\, \supset .\, -a \in \text{Cls}$$

$$a \,\epsilon\, \text{Cls} \,.\, \supset .\, -(-a) = a$$

$$a, b, c \,\epsilon\, \text{Cls} \,.\, ab \supset c \,.\, x \,\epsilon\, a \,.\, x - \epsilon\, c \,.\, x \supset x - \epsilon\, b$$

Peano - as everyone else at this time with the exception of Frege[68] - lacked a clear distinction between axioms and rules of inference, with the latter being completely absent from the formulation of his calculus. Despite its imprecision and shortcomings, though, the simple elegant notation and axiomatic formulation of Logic was very effective for cataloging an enormous number laws of propositional logic and the algebra of classes.

But despite the enormous positive impact that Peano's works had on Russell, there emerged a serious point of dispute between the two concerning the fundamental subject of definitions in Mathematics. In the *Formulario* and other works[69], Peano employed three different methods of definition beside the usual nominal, or so-called eliminative ones, namely inductive or recursive definitions, definition by postulates and what Peano called definitions by abstraction. Each of these sorts of definition was employed by Peano in important places in his *Formulario*. In fact, his five primitive propositions for Arithmetic can be understood as an attempt to 'define' the primtive notions of number theory. His now famous axiomatization of Arithmetic assumed as primitives the notions of zero, number and successor, which he symbolized in the *Formulario* as[70]:

$$0 = \ll z\acute{e}ro \gg$$

$$\text{N}_0 = \ll nombre\,(entier,\, positif\, ou\, null) \gg$$

$$a \in \text{N}_0. \supset .a+ = \ll le\, nombre\, qui\, vient\, apr\grave{e}s\, a \gg, \ll le\, successif\, de\, a \gg, \ll a\, plus \gg$$

And had as axioms what we now call the five Peano postulates, more appropriately name the Dedekind-Peano postulates[71]:

$$0 \,\epsilon\, \text{N}_0$$

68 And a little later Russell, who embraced the distinction in the *Principles*, as we shall discuss.

69 An important example is Peano's previous, but better known work *Arithmetices Principia*, partially translated and republished in van Heijenoort's famous collection (van HEIJENOORT, J., 1967, pp.83-97).

70 PEANO, G., 1901, p.41.

71 Cf. PEANO, G., 1901, p.41-43. Dedekind had already made them the basis of Arithmetic in slightly different form in his most famous article (DEDEKIND, R., 1888). Peano himself recognizes his debt to Dedekind in his first exposition of an axiomatic formulation of arithmetic (PEANO, G., 1889). But the fact is that Peano had offered an *explicit axiomatic* treatment of Arithmetic with these axioms, while Dedekind did not (POTTER, M.,

$$a \,\epsilon\, \mathrm{N}_0 \,.\supset .\, a+ \,\epsilon\, \mathrm{N}_0$$

$$a, b \,\epsilon\, \mathrm{N}_0 \,.\, a+1 = b+1 \,.\supset .\, a = b$$

$$a \in \mathrm{N}_0 \,.\supset .\, a+1 \neq 0$$

$$s \,\epsilon\, Cls \,.\, 0 \,\epsilon\, s : x \,\epsilon\, s \,.\supset_x .\, x^+ \,\epsilon\, s : \supset .\, \mathrm{N}_0 \supset s$$

According to Peano, these propositions could legitimately be taken as *defining* the three primitive notions. A very important example of Peano's use of inductive or recursive definition occurs in his treatment of Arithmetic, where he introduces the usual basic equations of addition and multiplication as defining such operations[72]:

$$a \,\epsilon\, \mathrm{N}_0 \,.\supset .\, a+0 = a$$

$$a, b \,\epsilon\, \mathrm{N}_0 \,.\supset .\, a+(b+) = (a+b)+$$

$$a \,.\supset .\, a \times 0 = 0$$

$$a, b \,\epsilon\, \mathrm{N}_0 \,.\supset .\, a \times (b+1) = (a \times b) + a$$

The last kind - definition by abstraction - was employed by Peano at crucial points in his development of Arithmetic, Analysis and Geometry, respectively when defining Cardinal Numbers, Real Numbers and Directions[73]. By a definition by abstraction, Peano meant the procedure of defining the identity of a class of terms (directions of lines, for instance) by means of an equivalence relation (parallelism) that defines and equivalence class. His most famous explanation of definition by abstraction is the following:

> Let *u* be an object; by abstraction, one deduces a new object ϕu. We cannot form an equality
>
> $\phi u =$ known expression,
>
> for ϕu is an object of a nature different from all those that we have considered up to the present. Rather, we define the equality $\phi u = \phi v$ by setting
>
> $h_{u,v} \,.\supset : \phi u = \phi v = p_{u,v}$

2000, pp.81-6). Often forgotten is the fact that Peano's also thought a sixth proposition was necessary, namely $\mathrm{N}_0 \,\epsilon$ Cls. Peano did not have a specific symbol for inequality, but we will use it for convenience. Also, Peano sometimes denoted the successor of *a* with "$a+$" as in the second postulate, and sometimes with "$a+1$", as in the third postulate.

72 Peano, G., 1901, p.40, 51. In this respect, Peano's axiomatic treatment was far inferior than that of Dedekind who not only realized that such procedure presupposed a theorem of recursion, but *proved* the result (again, for details we refer to POTTER, M., 2000, pp.81-6). As we nowadays know, this means that Dedekind was implicitly working within a *second*-order logic. Although Peano, just like Dedekind (and everyone before the twenties) did not distinguish between first and second order quantification (cf. GOLDFARB, W., 1979), we have every reason to suppose that his logic was first order. Thus, Peano should have assumed a theorem of recursion as primitive or added the recursive equations as additional axioms instead of definitions. An illuminating discussion of this last point which connects it with Russell's views can be found in LANDINI, G., 1998, pp.21-26.

73 Peano, .G, 1901 p.70, §32; p.122, §71; p.192-3, §91; for details see RODRÍGUEZ-CONSUEGRA, F., 1987, p.143; 1991, p.124-5; GRATTAN-GUINNESS, I., 2000b, pp.239-41.

> Where $h_{u,v}$ is the hypothesis on the objects *u* and *v*. Thus $\phi u = \phi v$, being the equality defined, means the same as $p_{u,v}$ which is a condition, or relation, between *u* and *v*, having a previously known meaning. This relation must satisfy the three conditions of equality that follow. [...][74]

As Rodríguez-Consuegra notes, although there are three points at which the 'chain of logical definitions' is broken in Peano's *Formulario*[75], the most important case for the purpose of understanding the development of Russell' ideas is the definition of the notion of cardinal number. In the 1901 edition of the *Formulario,* Peano had[76]:

$$a, b \,\epsilon\, \text{Cls} \supset \text{Num}a = \text{Num}b =. \, \exists(b\text{f}a)\text{rcp}$$

He called this a definition "by abstraction of Num$a$" and explained it thus:

> $\ll$ Num$a$ $\gg$ signifie $\ll$ *le nombre (numerus) des a* $\gg$. [...] La définition ·0 est exprimée par les seuls signes de logique. On peut commencer ici l'Arithmétique: nous définirons directemente les signes $> 0\ \text{N}_0 + \times\ \upharpoonright$, sans passer par les idées primitives du §20.
>
> La définition ·0 définit l'égalité $\ll$ Num$a$ = Num$b$ $\gg$, qui subsiste si l'on peut établir une correspondance réciproque entre *a* et *b*.[77]

This is precisely the *definition* which Frege had already considered and rejected in his *Grundlagen der Arithmetik* in 1884[78]; "$\exists(bfa)$rcp" is defined as the existence of a one-one function *f* between two classes *a* and *b*, and hence, what is proposed here is a definition of "the number of the class *a*" via the symmetric, transitive and reflexive relation of similarity or equinumerosity: two classes have the same cardinal number if there exists a one-one function that correlates their members. It is interesting to point out that this section explicitly refers to Cantor's article of 1895 which introduces and works out his concept *power* or *cardinal number* of a class or set [*menge*]. Taking a class as "[...] any collection into a whole M of definite and separate objects *m* of our intuition or our thought"[79], Cantor understood the *cardinal number* or *power* of a class M as something which we obtain by abstracting "[...] from the nature of its various elements *m* and of the order in which they are given"[80], where the *elements m* are the objects which we "separate" in "our intuition or our thought" into a single collection or class M, so that from this double process of abstraction a new, definite object was created. This new

74 KENNEDY, H., 1974, p.397; RODRÍGUEZ-CONSUEGRA, F., 1991, p.124. The "three conditions" that follow are, of course, reflexivity, transitivity and simmetry.

75 RODRÍGUEZ-CONSUEGRA, F., 1991, p.125.

76 PEANO, G., 1901, p.70.

77 PEANO, G., 1901, p.70.

78 FREGE, 1884, p.73-79, §§62-68. To be sure, the passage continues: "Nous n'ecrivons pas une egalité de la forme Num$a$ = (expression composée par les symboles précédents). La P·0 est une Df par abstraction de Num$a$." (Peano, G., 1901, p.70) We shall have occasion to compare briefly Frege's number theory with that of Russell later.

79 CANTOR, G., 1895, p.85.

80 CANTOR, G., 1895, p.86.

object he called the cardinal number of M and he denoted it by the expression "$\overline{\overline{\mathrm{M}}}$". According to Cantor two sets have the same power if "[...] it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only on element of the other"[81] In other words, two sets have the same cardinal number when there is one-one correlation between their elements. When such a condition is fulfilled Cantor called the two sets M and N *equivalent* and abbreviated "M and N are equivalent" by "M∼N". Using this relation of equivalence as a standard of measure for classes, Cantor defined the relations of greater and less and also the operations of addition, multiplication and exponentiation of cardinal numbers without any recourse to the notions of finitude or infinity and could do so in a way that applied *both* to infinite and finite sets. Peano is following Cantor here in not *properly* or strictly defining the notion of *power* or *cardinal number*, but defining them 'by abstraction' via the equivalence relation. Moreover, Peano claimed that it is impossible to (nominally) define the concept of cardinal number using only the logical notions of the *Formulario*.

Now, despite the high regards that Russell had for Peano's formal development of Logic and Arithmetic, he thought Peano's Logic was not sufficiently developed due to the absence of a formal apparatus able to deal with relations. Peano introduced functional and relational symbols as definable (as a rule) in terms of formal implication and membership to a class, as follows:

$$a, b \,\epsilon\, \mathrm{Cls} \,.\, \supset \,:.\, u \,\epsilon\, afb \,. = :\, x \,\epsilon\, a \,.\, \supset_x xu \,\epsilon\, b$$

$$a, b \,\epsilon\, \mathrm{Cls} \,.\, \supset \,:.\, u \,\epsilon\, afb \,. = :\, x \,\epsilon\, a \,.\, \supset_x .\, ux \,\epsilon\, b$$

However, Peano conceived these defintions in a rather vague way, with "f" meant to "[...] représenter par les symboles idéographiques les idées de ≪fonction, correspondence, operátion≫, etc."[82]. Due to the influence of Moore and the recognition of the ultimate reality of relations, this was a fundamental aspect of Peano's Logic that Russell found unacceptable[83].

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[81] CANTOR, G., 1895, p.86-7.

[82] PEANO, G., 1901, p.33.

[83] In Russell's logicist *manifesto*, *Recent Work on The Foundations of Mathematics*, written, as Russell said to Jourdain, "for filthy lucre" and later published as *Mathematics and the Metaphysicians* (RUSSELL, B., 1901*c*?), we find: "The great master of the art of formal reasoning, among the men of our own day, is an Italian, Professor Peano, of the University of Turin. He has reduced the greater part of mathematics (and he or his followers will, in time, have reduced the whole) to strict symbolic form [...] if we wish to learn the whole of Arithmetic, Algebra, the Calculus, and indeed all that is usually called pure mathematics (except Geometry), we must start with a dictionary of three words. One symbol stands for zero, another for number, and a third for next after. What these ideas mean, it is necessary to know if you wish to become an arithmetician. But after symbols have been invented for these three ideas, not another word is required in the whole development. All future symbols are symbolically explained by means of these three. Even these three can be explained by means of the notions of relation and class; but this requires the Logic of Relations, which Professor Peano has never taken up". In the *Principles*, we find: "Peano's logic proceeds by a smooth development. But in one respect it is still defective: it does not recognize as ultimate relational propositions not asserting membership of a class. For this reason, the definitions of a function and of other essentially relational notions are defective. But this defect is easily remedied by applying, in the manner explained above, the principles of the Formulario to the logic of relations" (RUSSELL, B., 1903, p.32 §36).

So in 1901 Russell published his first major work in Symbolic Logic, the paper *On The Logic of Relations with Some Applications to the Theory of Series*[84]. The goal of the paper was to work *within* Peanese Logic in order to extend it with an explicit treatment of relations[85]. This work was the culmination of the influence Moore and Peano had effected on Russell. His goal was to show how the Logic of relations - based on the Moorean conception of these as ontologically independent entities that were genuine constituents of propositions - allowed him to provide nominal definitions in Mathematics, as Russell put it, "[..] wherever definition is possible"[86].

It is at this point that Russell's Logicism was born as a *project*, for it is here that he proposes a nominal definition of cardinal number as an equivalence class, providing the most basic groundwork for establishing that Logic - in his sense - is enough for defining the most basic notions of Mathematics and for proving its most basic propositions. As we shall discuss in the next section, however, there are serious controversies involving the article, concerning both its place in the history of Logic in general, and its philosophical significance within the development of Russell's Mathematical Philosophy. We shall present Russell's main achievement in the article, comparing it with the more definitive claims made in the *Principles* in order to adress the controversy.

2.1.2 Russell's Logic of Relations, the Principle of Abstraction and the Definition of Cardinal Number

The first two sections of Russell's article contain, respectively, his first (published) formal presentation of his logic of relations and a theory of cardinal numbers based on his celebrated definition of cardinals as classes of similar classes.

At the basis of Russell's calculus he introduced the notion of *relation* as primitive in the style of Peano[87]: this meant that under the condition of R belonging to the class of relations Rel, "xRy" meant that "x has the relation R to y". Russell then defined the domain of a relation R as the class of all individuals which stand in the relation R to something and the converse domain of R as the class of all individuals to which something has the relation R. In symbols, Russell

84 RUSSELL, B., 1901*c*. The paper was first published in French as *Sur la logique des relations avec des applications a la théorie des séries* in the journal *Révue de Mathématiques,* edited by Peano and dedicated exclusively for the topics related to the *Formulario.*

85 Rodríguez-Consuegra claims that Peano and his followers had already several (more or less) implicit elements of a theory of relations, and that Russell's claim to have *invented* it should be weakened; for details, see RODRÍGUEZ-CONSUEGRA, F., 1991, pp.72-77; pp.103-105.

86 RUSSELL, B., 1901*b*, p.315.

87 Cf. RUSSELL, B., 1901*c*, pp.315.

put[88]:

$$R \,\epsilon\, \mathrm{Rel} \,.\, \supset .\, \rho = x \ni \{\exists y \ni (xRy)\}\,\mathrm{Df}$$

$$R \,\epsilon\, \mathrm{Rel} \,.\, \supset .\, \breve{\rho} = x \ni \{\exists y \ni (yRx)\}\,\mathrm{Df}$$

Building upon Peano's logic of classes and propositions, his calculus assumed a few additional primitive propositions on the basis of which Russell then developed a logical theory of cardinal numbers. He started by defining the relation of *similarity*, that is, the relation which holds between two classes *u* and *v* whenever there exists a one-one relation *R* such that the domain of *R* is *u* and its converse domain is *v*.

Building upon Peano's notation Russell expressed this as[89]:

$$u, v \,\epsilon\, \mathrm{Cls} \,.\, \supset : u \,\mathrm{sim}\, v \,.=.\, \exists 1 \to 1 \cap R \ni (u \supset \rho \,.\, \breve{\rho} u = v)$$

where many-one, one-many and one-one relations are respectively defined as[90]:

$$\mathrm{Nc} \to 1 = \mathrm{Rel} \cap R \ni \{xRy \,.\, xRz \,.\, \supset_x .\, y = z\}\,\mathrm{Df}$$

$$1 \to \mathrm{Nc} = \mathrm{Rel} \cap R \ni \{yRx \,.\, zRx \,.\, \supset_x .\, y = z\}\,\mathrm{Df}$$

$$1 \to 1 = (\mathrm{Nc} \to 1) \cap (1 \to \mathrm{Nc})\,\mathrm{Df}$$

The crucial point of the section on cardinal numbers is Russell's demonstration of what he called "the principle of abstraction"[91]. This is the key theorem for understanding Russell's reasons for defining cardinal numbers as classes of similar classes. In symbols, Russell expressed it as[92]:

$$R \,\epsilon\, \mathrm{Rel} \,.\, R^2 \supset R \,.\, R = \breve{R} \,.\, \exists R : \supset .\, \exists \mathrm{Nc} \to 1 \cap S \ni (R = S\breve{S})$$

As Russell put it, this asserts that "[...] all relations which are transitive, symmetrical, and non-null can be analyzed as products of a many-one relation and its converse"[93]. In the article, Russell defined R^2 as the relative product of *R* and itself, or RR, so that$R^2 \supset R$ asserts that the relative product of *R* and itself is contained in *R*, which is to say that $xRy \,.\, yRz \,.\, \supset .\, xRz$, that is, that *R* is transitive; $\breve{R}$ is defined as the converse of *R*, so that $R = \breve{R}$ means $xRy \equiv yRx$[94], e.g. *R* is symmetrical; finally, to say $\exists R$ means that *R* is non-null which is to say that its domain and converse domain are non-empty[95]. Thus, it can be unpacked and translated into modern notation as:

$$(R)\left[\left(\begin{array}{c}(x)(y)(z)(xRy \wedge yRz \,.\supset.\, xRz) \wedge \\ (x)(y)(xRy \equiv yRx) \wedge \\ (\exists x)(\exists y)(xRy)\end{array}\right) \supset (\exists S)\left(\begin{array}{c}(x)(y)(z)(xSy \wedge xSz \,.\supset.\, y = z) \wedge \\ (x)(y)(xRy \equiv (\exists \xi)(xS\xi \wedge \xi Sz)\end{array}\right)\right]$$

88 Cf. RUSSELL, B., 1901*c*, pp.315, definition $*1{\cdot}21$ and $*1{\cdot}22$. Recall that ϵ is being used for membership and $\ni$ for *such that* (for instance: "$\rho = x \ni \{\exists y \ni (xRy)\}$" reads "$\rho$ is the class of all *x* such that there is *y* such that *xRy*".

89 Cf. RUSSELL, B., 1901*c*, pp.315, definition $*1{\cdot}1$.

90 Cf. RUSSELL, B., 1901*c*, p.319, definitions $*5{\cdot}1$, $*5{\cdot}2$ and $*5{\cdot}3$, respectively.

91 RUSSELL., B, 1901*c*, p.320.

92 Cf. RUSSELL, B., 1901*c*, pp.320, theorem $*6{\cdot}2$.

93 RUSSELL., B, 1901*c*, p.320.

94 Russell used equality for material equivalence, from this point forward we shall not follow him in this.

95 See propositions $*2{\cdot}2$, $*1{\cdot}72$ and $*1{\cdot}31{\cdot}4$ of section §1, respectively.

This states that every equivalence relation implies the existence of a function[96] S that maps every x and y such that xRy to some ξ in the range of S. According to Russell, this theorem is "[...] presupposed in the definitions by abstraction and it shows that in general these definitions do not give a single individual but a class, since the class of relations S is not in general an element"[97].

As Gregory Landini has urged, one must be careful in order to see the role this principle plays in Russell's definition of cardinal numbers[98]. We may start by making clear how Russell understood the idea of definition by abstraction in the *Principles of Mathematics*, where he treats the subject at lenght. According to Russell, we may take the definition by abstraction of the cardinal number of a class u as an attempt to define the "common property" possessed by every class which is similar to u - namely its *number* or *cardinality*. The reason why this theorem is presupposed in definitions by abstraction is the fact that for such definitions to be legitimate, there must be, for every equivalence relation R, a function δ such that, for every x and y, $\delta(x) = \delta(y)$ if and only if xRy and there must be some unique entity to which x and y are mapped by δ if xRy, which is the entity (e.g., the property) supposedly defined by abstraction. In the case of cardinal numbers, the attempt to define the cardinal number $\text{Nc}(u)$ of a class u by abstraction via the relation of similarity assumes that the function (or many-one relation) Nc is such that $\text{Nc}(u) = \text{Nc}(v)$ if and only if the classes u and v are similar and that u and v are mapped to a unique entity $\text{Nc}(u)$ which is *the* cardinal number of u. The problem, as Russell put it, is that this sort of definition and "[...] generally the process employed in such definitions [...] suffers from an absolutely fatal formal defect: it does not show that only one object satisfies the definition"[99]. Russell continues:

> [...] instead of obtaining one common property of similar classes, which is the number of the classes in question, we obtain a class of such properties, with no means of deciding how many terms this class contains. In order to make this point clear, let us examine what is meant, in the present instance, by a common property. What is meant is, that any class has to a certain entity, its number, a relation which it has to nothing else, but which all similar classes (and no other entities) have to the said number. That is, there is a many-one relation which every class has to its number and to nothing else. Thus, so far as the definition by abstraction can show, any set of entities to each of which some class has a certain many-one relation, and to one and only one of which any given class has this relation, and which are such that all classes similar to a given class have this relation to one and the same entity of the set, appear as the set of numbers, and any entity of this set is the number of some class. If, then, there are many such sets of entities—and it is easy to prove that there are an infinite number of them—every class will have many numbers, and the definition wholly fails to define the number of a class. This argument is perfectly general, and shows that definition by abstraction is never a logically valid process.[100]

[96] Or many-one relation.
[97] RUSSELL., B, 1901*c*, p.320.
[98] LANDINI, G., 1998, p.21. Since his discussion of the point is very clear and instructive, we follow it very closely.
[99] RUSSELL, B., 1903, p.114-15 §§109-110.
[100] RUSSELL, B., 1903, p.114-15 §§109-110.

The attempt to define the property of a class *u* which we call *the number of u* or *the cardinality of u* via the equivalence relation of similarity fails because it does not define a unique entity that can be identified as the number of *u* but a whole class of such properties. For suppose we attempt to define *the number of u* or *the cardinality of* via the equivalence relation of similarity, defining 0 as the number of all classes similar to the empty set, 1 as the number of all singletons, and so on. According to the principle of abstraction, there is a function *S* such that $(x)(y)(x\, sim\, y \equiv (\exists\xi)(xS\xi \wedge yS\xi))$, which is supposed to map the empty set to the number 0, every singleton to the number 1, every couple to the number 2, and so on. But there may be many (infinitely many, in fact) different functions *S*, each of which define different 'numbers'; thus, the method of abstraction does not define a unique property which is *the cardinal number* or *cardinality of u*[101].

Russell's solution in the *Principles of Mathematics* is to define the number of a class as the class of all classes similar to this given class (and, in general, where an equivalence relation holds, to define the 'common property' that arises from it as membership to an equivalence class):

> Membership of this class of classes (considered as a predicate) is a common property of all the similar classes and of no others; moreover every class of the set of similar classes has to the set a relation which it has to nothing else, and which every class has to its own set. Thus the conditions are completely fulfilled by this class of classes, and it has the merit of being determinate when a class is given, and of being different for two classes which are not similar. This, then, is an irreproachable definition of the number of a class in purely logical terms.[102]

Later in the book, he reiterates the point:

> Since similarity is reflexive, transitive and symmetrical, it can be analyzed into the product of a many-one relation and its converse, and indicates at least one common property of similar classes. This property, or, if there be several, a certain one of these properties, we may call the cardinal number of similar classes, and the many-one relation is that of a class to the number of its terms. In order to fix upon one definite entity as the cardinal number of a given class, we decide to identify the number of a class with the whole class of classes similar to the given class. This class, taken as a single entity, has, as the proof of the principle of abstraction shows, all the properties required of a cardinal number. [...] In this way we obtain a definition of the cardinal number of a class.[103]

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101 LANDINI, G., 1998, p.25.
102 RUSSELL, B., 1903, p.115 §§111.
103 RUSSELL, B., 1903, p.305 §§283.

Now, at first sight, the role played by the principle of abstraction in the article is precisely the same it plays in the *Principles*: it allows Russell to (nominally) define the cardinal number of a class *a* as a *class.* Indeed, in the article we find:

> If we wish to define cardinal number by abstraction, we can only define it as a class of classes, of which each has a one-one correspondence with the class "cardinal number" and to which belong every class that has such a correspondence"[104]

It is clear from the passage above that Russell's complain that a definition by abstraction does "[...] not give a single individual but a class[105]" is about the fact that nothing can assure, in general, that the entity defined by abstraction is unique[106].

If we look at the demonstration of the 'Principle of Abstraction' this is further corroborated. The proof is as follows: assume *R* is some non-empty, transitive, symmetric relation; let S be the function that maps *x* to the class $\{z : xRz\}$. In the case of cardinal numbers, let *R* be the relation of similarity; ξ will be the class of all classes similar to some given class x: if we identify this ξ as the number of x it follows immediately that another number μ of a class y is equal to ξ if, and only if x and y are similar: thus, the cardinal number of a class *u* is the class of all classes similar to *u*; to say that a class has the same cardinality of *u* is to say that it belongs to the class of all classes similar to *u*. The uniqueness of the ξ which the process of 'definition by abstraction' assumes is guaranteed when ξ is identified as the converse domain of *R* (this is trivially entailed by the extensionality of classes: if *R* is a relation there is one and only one converse domain of *R*). This, in fact, is surely the point Russell puts forward in the following passage:

> Meanwhile we can always take the class $\breve{\rho}$, which appears in the demonstration of Prop *6.2, as the individual indicated by the definition by abstraction; thus for example the cardinal number of a class *u* will be the class of classes similar to *u*".[107]

So far, so good. Russell seems to have solved a fundamental problem Peano was unable to solve: to define the cardinal number of a class using only logical notions and without relying on definitions by abstraction[108].

Some authors, however, have taken into account the fact that on the one hand, Russell claims in the *Principles* that "[...] definition by abstraction is never a logically valid process"[109]

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[104] RUSSELL., B, 1901*c*, p.321.
[105] RUSSELL., B, 1901*c*, p.320.
[106] See LANDINI, G., 1998, p.25 for a clear demonstration that it is never unique.
[107] RUSSELL., B, 1901*c*, p.320.
[108] This was, for a long time, the received view, again, paradigmatically presented in KENNEDY, J., 1973.
[109] RUSSELL, B., 1903, p.114-15 §§109-110.

while he also claims, in the article, that the principle "[...] is presupposed in the definitions by abstraction"[110], something which may suggest that definitions by abstraction may, after all be legitimate. Some have read him as if one could *justify* the definition of cardinal numbers by abstraction via the principle. Also, the historical record shows that Russell's (re)discovery of the definition of cardinal number as an equivalence class is more complicated than usually thought.

Rodríguez-Consuegra[111] points out that if we compare two of the manuscripts of Russell's 1901 article written in French - a preliminary draft and the final published one - we see that the earlier version does not contain the "Meanwhile..." passage quoted above, in which the nominal definition is suggested[112]. Thus, given that by the time his article was sent to Peano to be published in the *Revue*, Russell had already seen the material Peano had prepared for the 1901 edition of the *Formulario* and the "Meanwhile..." passage was added to the article in proof, Rodríguez-Consuegra further concludes that *Peano* is to be credited with the definition of number as an equivalence class, since, in the 1901 *Formulario* we also find the following passage[113]:

> Etant donée une classe *a*, on peut considérer la classe de classes:
>
> Cls $\cap\, x \ni [\exists(x\mathrm{f}a)\mathrm{rcp}]$;
>
> l'égalité de ces Cls de Cls, calculées sur les classes *a* et *b* importe l'egalité Num$a$ = Num$b$; mais on ne peut pas identifier Num$a$ avec la Cls de Cls considérée, car ces objets ont des propriétés différentes.[114]

There is no doubt that the definition being considered here is the same Russell considers in his 1901 article and that Frege had already proposed in his *Grundlagen* by 1884: the notation "Cls $\cap\, x \ni [\exists(x\mathrm{f}a)\mathrm{rcp}]$" is Peano's equivalent of the class of all classes *x* that are similar to a given class *a* - *precisely* the class that Russell identifies as the number of the class *a*. But there is more: in the same section we find zero defined as the number of the null class, the number one as the class of all singletons, of an infinite class as one which can be one-one correlated with

110 RUSSELL., B, 1901*c*, p.320.

111 Similar considerations of his were antecipated - albeit in much less depth and detail - in Vuillemin's article *Le «platonisme» dans la première philosophie de Russell et le «principe d'abstraction»* (VUILLEMIN, J., 1975).

112 RUSSELL., B, 1901*c*, p.320.

113 RODRÍGUEZ-CONSUEGRA, F., 1987, p.148. Since Burali-Forti was also working with the possibility of turning definitions by abstraction into nominal ones, and Russell certainly knew about this becasuse Burali-Forti discussed precisely this topic at the International Congress, Rodríguez-Consuegra also credits him with the joint priority, claiming that "Burali-Forti completely anticipated the logicist idea" (RODRÍGUEZ-CONSUEGRA, F., 1991, p.34, footnote 1) of defining Peano's primitive notions of Arithmetic. We concur with Landini, however, in that the link between Burali-Forti's views (which involve assuming a theorem of recursion and arithmetical operations as primitive) is "tenuous" at best (LANDINI, G., 1992, p.608).

114 PEANO, G., 1901, p.70. In the *Principles*, Russell claims he fails to see which are these properties Peano thinks cannot be attributed to cardinal numbers if they are understood as classes (RUSSELL, B., 1903, p.115 §111); Peano was probably concerned with unexpected consequences of defining numbers as classes of similar classes like the fact that any pair of entities $\{a, b\}$ belongs, for instance, to the number 2. But from the mathematical point of view these properties are completely irrelevant: they do not alter the formal development of Arithmetic nor its applicability (e.g., as in counting).

a proper subset of itself and the class of finite cardinal numbers as the class of all non-infinite classes. Peano had[115]:

$$0 = \mathrm{Num}\Lambda\,\mathrm{Df}$$

$$1 = \iota\mathrm{Num}`[a \ni (\exists a : x, y \,\epsilon\, a \supset_{x,y} . \, x = y)]\,\mathrm{Df}$$

$$\mathrm{infn} = \mathrm{Num}`\{a \ni [\exists \mathrm{cls} \cap u \ni (u \supset a \,.\, u \neq a \,.\, \mathrm{Num}u = \mathrm{Num}a)]\}\,\mathrm{Df}$$

$$\mathrm{N}_0 = (\mathrm{Num}`\mathrm{cls}) - \mathrm{infn}\,\mathrm{Df}$$

Since we know that Russell saw the material for the 1901 edition of the *Formulario* before it went to the press[116] and that the ‘Meanwhile’ passage was added in print, Rodríguez-Consuegra concludes Russell added it *after* seeing the definition above in the *Formulario*, where it was rejected by Peano on rather flimsy grounds[117].

This suggestion, however, is implausible, for mainly two reasons. First, since we do not have a complete historical record of the exchange between Peano and Russell that allows us to settle the question of the priority of the definition, it is *at least just as plausible* to suppose that Russell suggested the definition to Peano while he was preparing the 1901 edition of the *Formulario*, but Peano rejected it. The second, much less conjectural and more definitive reason is the fact that Russell was, as Nicholas Griffin put it, “[...] remarkably untouchy about matters of priority”[118], which makes Rodríguez-Consuegra’s claim very much implausible, especially since Frege is the indisputably author the one who truly deserves credit for the definition - something Russell always promptly acknowledged. The fact that Russell *never* mentioned Peano’s name in connection with the discovery of the definition makes Rodríguez-Consuegra’s conjecture implausible, despite the fact he showed - *indisputably* - that the story of how (and when) Russell arrived at the definition is more complicated than generally supposed.

Apart from the question of historical priority over the rediscovery of Frege’s definition, Rodríguez-Consuegra’s reading of the role played by the Principle of Abstraction in the philosophical justification of the definition deserves close attention. He points to the puzzling fact that nowhere in the published text there occurs an explicit definition given by a formula of the form “$u \,\epsilon\, \mathrm{Cls} \supset \mathrm{Nc}`u = ...$” which nominally defines the cardinal number of a class *u*. Such a definition would only appear almost a whole year after the 1901 article, in another article for the

[115] PEANO, G., 1901, p.71. Notice that Peano is following Dedekind’s definition of infinite classes here, something that is far from a trivial decision: as Russell, Whitehead and Zermelo would soon find out, to show that this definition coincides with the ‘ordinary’ definition of infinite classes as those that belong to every class s that contains the empty set and $x \cup \{a\}$ whenever x belongs to s requires the Axiom of Choice. Peano’s definition, for instance, would only have its full intended strenght in the presence of the Axiom of (countable/denumerable) Choice, since it would not exclude the possibility that some ordinarily infinite but Dedekind finite cardinal be a natural number. We shall adress this point in more detail later, when discussing *Principia*’s theory of cardinal numbers.

[116] RODRÍGUEZ-CONSUEGRA, F., 1987, p.156-7. This is absolutely indisputable since the references to the *Formulario* in the published article are to the 1901 edition.

[117] PEANO, G., 1901, p.71.

[118] GRIFFIN, N., 1993, p.218

Revue des Mathématiques, entitled "*On the General Theory of Well-Ordered Series*"[119]. There we find the proper definition of the the cardinal number of a class *u* and of the relation of a class to its cardinal number as folloows:

$$u \,\epsilon\, \mathrm{Cls} \supset \mathrm{Nc}‘u = \mathrm{Cls} \cap v \ni (u \,\mathrm{sm}\, v)\ \mathrm{Df}$$

$$\mathrm{Nc} = \mathrm{Cls}‘\mathrm{Cls} \cap w \ni \{\exists \mathrm{Cls} \cap u \ni (v \,\epsilon\, w \,.=.\, u \,\mathrm{sm}\, v)\ \mathrm{Df}$$

Furthermore, Rodríguez-Consuegra points to the fact that Russell had at some point toyed with the idea of introducing the following definitions[120]:

$$\mathrm{sim} = S\breve{S} \,.\, u \,\epsilon\, \mathrm{Cls} \,.\supset.\, \mathrm{Nc} = \breve{\sigma}\ \mathrm{Df}$$

$$\mathrm{sim} = S\breve{S} \,.\, u \,\epsilon\, \mathrm{Cls} \,.\supset.\, \mathrm{Nc}‘u = \iota u_s\ \mathrm{Df}$$

But Russell gave them up, writing in the manuscript margin that "this Df won't do. There may be many such relations as *S*. Nc must be undefinable"[121]. On this basis, together with Russell's apparent ambiguity concerning the legitimacy of definitions by abstraction, Rodríguez-Consuegra claims not only that at some point the principle of abstraction was "thought to *strengthen* the method of abstraction and not to eliminate it"[122] but also that "the section on cardinals [of the published article] was not conceived to give a nominal definition of cardinal numbers in terms of classes of classes"[123]. Thus, Rodríguez-Consuegra ends up claiming that in the paper *On the Logic of Relations*:

> [...] Russell was avoiding offering a definition of number, which can be interpreted, together with the former presentation of the principle of abstraction as a justification of Peano's methods based on abstraction, as an attempt to preserve these methods through a definition by abstraction of cardinal number. In this way the principle of abstraction would permit one all the necessary elements to secure the existence of the entities defined by abstraction [...] and to avoid any attempt to nominally define Nc.[124]

This prompts him to distinguish three distinct 'stages' the principle went through in the development of Russell's ideas from 1901 through 1910. He points out - correctly - that the origin of the principle was the Moorean idea of substituting predicates for relations whenever

119 RUSSELL, B., 1902. Like Russell's previous paper for the journal, this was published in two installments in the *Revue*; it continued Russell's project of incorporating relations within Peano's logic, this time with applications to Cantor's theory of ordinal numbers. A detailed discussion can be found in Gregory Moore's introduction of the article in the Collected Papers third volume (MOORE, G., 1993, pp.384-8).

120 RUSSELL, B., 1901*e*, p.596. See the introduction to Appendix V of Russell's Collected Papers volume 3 (MOORE, G., 1993, pp.589) and the paper by Rodríguez-Consuegra (RODRÍGUEZ-CONSUEGRA, F., 1987, pp.146-150) for details of the manuscript history.

121 As quoted in RODRÍGUEZ-CONSUEGRA, F., 1987, p.149.

122 RODRÍGUEZ-CONSUEGRA, F., 1987, p.149.

123 RODRÍGUEZ-CONSUEGRA, F., 1987, p.148.

124 RODRÍGUEZ-CONSUEGRA, F., 1991, p.160.

possible[125], with the first stage being represented by the (allegedly unsuccessful) paper *On the Logic of Relations*, and the second and third by the *Principles* and *Principia*, respectively. For now we adress his reading of the principle of abstraction in the *Principles*. He takes it to be "[...] the true philosophical axis"[126] of the work, where:

> [...] the function of the principle consists in replacing the supposed common property inferred from the equivalence relation for the class of terms having the given relation to a given term. [...] Thus, the role of the principle is now very different to that which it played in the first writings inspired by Peano, where it seems to be destined to *justify* definitions by abstraction.[127]

Here we shall follow Landini's contention that Rodríguez-Consuegra's interpretation of the role played by the principle of abstraction in the 1901 article is deeply flawed[128].

To see that Landini is correct, we must, as he showed, look at three theorems related to the following definition in Russell's paper[129]:

$$\mathcal{S} = \text{Nc} \to 1 \cap S \ni (\text{sim} = S\breve{S})\,\text{Df}$$

This defines $\mathcal{S}$ as the class of many-one relations *S* such that the relative product *S* and its converse is co-extensive with similarity. This is precisely the class of many-one relations such that $(x)(y)(x\,\text{sim}\,y \equiv (\exists\xi)(xS\xi \wedge \xi Sy))$; using the terminology Russell employs in the *Principles*, this would be the class of functions *S* that yield the (infinitely many) different notions of cardinal numbers which we obtain for each different *S'*s in the attempt to define *cardinality* by abstraction via the relation of similarity[130]. Since the proof of the principle of abstraction assures that ξ can be identified with the converse domain of the relation of similarity[131], Russell can prove[132]:

$$\exists \text{Nc} \to 1 \cap S \ni (\text{sim} = S\breve{S})$$

which asserts that the class $\mathcal{S}$ of many-one relations co-extensive with similarity is non-empty. Using this result and $*6 \cdot 2$, Russell proves the following[133]:

$$S, S' \in \mathcal{S} \,.\supset.\, \breve{\sigma}\,\text{sim}\,\breve{\sigma}'$$

$$S \in \mathcal{S} \,.\, k\,\text{sim}\,\breve{\sigma} \,.\supset.\, \exists S \cap S' \ni (k = \breve{\sigma}')$$

[125] RODRÍGUEZ-CONSUEGRA, F., 1991, p.189.
[126] RODRÍGUEZ-CONSUEGRA, F., 1991, p.161.
[127] RODRÍGUEZ-CONSUEGRA, F., 1991, p.161. Our emphasis.
[128] More generally we disagree with the picture Rodríguez-Consuegra paints of the principle's role in the development of Russell's Mathematical Philosophy. In particular, the principle re-appears in *Principia Mathematica* as a justificatory device for Russell's definition of cardinal number; there are also good reasons to question Rodríguez-Consuegra's reading of the principle in that work cf. footnote 288 of chapter 5.
[129] RUSSELL., B, 1901*c*, p.321, $*1 \cdot 4$.
[130] RUSSELL, B., 1903, p.114-15 §§109-110.
[131] LANDINI, G., 1998, p.26.
[132] RUSSELL., B, 1901*c*, p.321 $*1 \cdot 3$.
[133] RUSSELL., B, 1901*c*, p.321, $*1 \cdot 52$ and $*1 \cdot 54$.

which can be translated as:

$$(S)(R)(S, R \,\epsilon\, \mathcal{S} \,\boldsymbol{.}\!\supset\!\boldsymbol{.}\, \{x : (\exists y)(xSy)\} \;\text{sim}\; \{x : (\exists y)(xRy)\})$$

$$(S)[(S \,\epsilon\, \mathcal{S} \wedge k \approx \{x : (\exists y)(xSy)\}) \supset (\exists R)(R \,\epsilon\, \mathcal{S} \wedge (k = \{x : (\exists y)(xRy)\})]$$

The first theorem asserts that if two relations *S* and *R* are many-one (i.e., functions) such that the relative product of themselves and their converse is co-extensive with similarity, then their converse domains are identical; the second asserts that if the relative product of *S* and its converse is co-extensive with similarity and *k* is similar to the converse domain of *S*, then there is a many-one relation (function) *S'* (or *R*) the converse domain of which is identical with *k*. As Landini puts it, theorems $\cdot 52$ and $\cdot 54$ above show that "[...] Russell provides grounds for claiming his choice of *S* in $\mathcal{S}$ is not on a par with other possible choices"[134], and, in fact, provide the philosophical justification for Russell's definition.

Landini's point is that "[...] the principle is important to Russell because it can be used to demonstrate that the range of any *S* such that $(x)(y)(x \; sim \; y \,\boldsymbol{.}\!\equiv\!\boldsymbol{.}\, x(S|\breve{S})y)$ will have all the formal properties of the cardinal numbers"[135]. Moreover, as Landini observes, when the range of similarity is restricted to *finite* classes, Russell's choice of *S* gives us Frege's "[...] construction of natural numbers as finite cardinals"[136]. This according to Landini is the point made by Russell that when he observes the above theorems "[...] prove that all classes which form the converse domains of different relations of the class $\mathcal{S}$ are similar, and that all classes similar to one of them belong to this class of classes"[137] and thus, that "[...] *the arithmetic of cardinal numbers applies in its entirety to each of these classes*"[138]. Thus, on this basis Russell goes on to offer definitions of zero and successor analogous to the Fregean ones[139]:

$$0_\sigma = \iota \breve{\sigma} \Lambda \;\text{Df}$$

$$x \,\epsilon\, \breve{\sigma} \,\boldsymbol{.}\!\supset\!\boldsymbol{.}\, x + 1_\sigma = \iota \breve{\sigma} \frown y \ni \{uSx \,\boldsymbol{.}\, z \sim \epsilon\, u \,\boldsymbol{.}\!\supset_{u,z}\!\boldsymbol{.}\, u \smile \iota z S y\} \;\text{Df}$$

The proof of the principle of abstraction and the aforementioned comment on it[140] make clear that Russell's idea is to fix the meaning of "cardinality of a class *u*" in terms of the function (i.e., many-one relation) that maps *u* to the converse domain of similarity with respect to *u*. The role of the principle of abstraction in the definition of cardinal number Nc(u) of a class *u* is to assure the existence and uniqueness of such an entity when it is identified with the class of all classes similar to *u*.

134 LANDINI, G., 1998, pp.28.
135 LANDINI, G., 1991, p.609.
136 LANDINI, G., 1991, p.609.
137 RUSSELL, B., 1901*a*, p.322.
138 RUSSELL, B., 1901*a*, p.322. Our emphasis.
139 Cf. RUSSELL, B., 1901*a*, p.322, $*2 \cdot 4$ and p. 324, $*4 \cdot 1$. The definition of successor is derived from the following definitions: Russell also has (RUSSELL, B., 1901*a*, p.322, $*2 \cdot 7$):
$1_\sigma = \iota \breve{\sigma} \cap x \ni (u \in \text{Elm.} \supset_u .uSx)$ Df
$S \in \mathcal{S}. \supset:: m, n \in \breve{\sigma}. \supset .m + n = \iota \breve{\sigma} \cap x \ni \{uSm.vSn.uv = \Lambda. \supset_{u,v} .u \cup vSx\}$ Df
where the second is a general definition of cardinal.
140 RUSSELL, B., 1901*a*, p.322.

This, in turn, allows for a *nominal* definition of cardinal number that *dispenses* with the method of definition by abstraction and which allows the demonstration of the fundamental properties of cardinal numbers. This is the same result that Russell would later announce in the *Principles* as follows:

> Mathematically, a number is nothing but a class of similar classes: this definition allows the deduction of all the usual properties of numbers, whether finite or infinite, and is the only one (so far as I know) which is possible in terms of the fundamental concepts of general logic. But philosophically we may admit that every collection of similar classes has some common predicate applicable to no entities except the classes in question, and if we can find, by inspection, that there is a certain class of such common predicates, of which one and only one applies to each collection of similar classes, then we may, if we see fit, call this particular class of predicates the class of numbers. For my part, I do not know whether there is any such class of predicates, and I do know that, if there be such a class, it is wholly irrelevant to Mathematics. Wherever Mathematics derives a common property from a reflexive, symmetrical and transitive relation, all mathematical purposes of the supposed common property are completely served when it is replaced by the class of terms having the given relation to a given term; and this is precisely the case presented by cardinal numbers. For the future, therefore, I shall adhere to the above definition, since it is at once precise and adequate to all mathematical uses.[141]

Thus what Russell accomplishes here is a solution to a problem Peano had failed to solve adequately. In the 1901 edition of the *Formulario*, Peano recognized that the primitive notions of zero, number and successor admitted an infinite number of different interpretations, all satisfying the Peano postulates; thus he intended to introduce natural numbers as something 'abstracted' from these many different system which share certain structural properties encapsulated in the five primitive propositions. Peano writes:

> Ces Pp, dont nous avons vu la nécessité, sont suffisantes pour déduire toutes les propriétés des nombres qu'on rencontrera dans la suite. Mais il y a une infinité de systémes qui satisfont à toutes les Pp. P.ex. elles sont toutes verifiees si l'on remplace N_0 et 0 par N_1 et 1. Tous les systemes qui satisfont aux Pp sont en correspondance reciproque avec les nombres. Le nombre, N_0, est ce qu'on obtient par abstraction de tous ces systemes; autrement dit, le nombre, N_0, est le systeme qui a toutes les proprietes enoncees par les P primitives, et celles-la seulement.[142]

In the *Principles* Russell argues that this method suffered from two fundamental logical flaws. First, it fails "[...] in indicating any constant meaning for 0, number and succession"[143]; second, it fails "[...] in showing that any constant meaning is possible, since the existence-theorem

[141] RUSSELL, B., 1903, p.116 §111.
[142] PEANO, G., 1901, p.44.
[143] RUSSELL, B., 1903, p.126 §122.

is not proved"[144], that is, it fails in showing that there *are* objects that satisfy the arithmetical postulates. This is a point Russell would emphasize even more clearly *Introduction to Mathematical Philosophy* in terms of applicability:

> Such a procedure is not fallacious; indeed for certain purposes it represents a valuable generalisation. But from two points of view it fails to give an adequate basis for arithmetic. In the first place, it does not enable us to know whether there are any sets of terms verifying Peano's axioms; it does not even give the faintest suggestion of any way of discovering whether there are such sets. In the second place, as already observed, we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a definite meaning, not merely that they should have certain formal properties.[145]

As Russell would later put it in the *Introduction*, what Peano's method accomplishes is a characterization of what Russell calls *progressions*[146]. And, indeed, in 1901 article contains a definition of *progressions* as classes whose ordinal-type is ω[147]:

$$\omega = \text{Cls} \cap u \ni \{\exists 1 \rightarrow 1 \cap R \ni (u \supset \rho \,.\, \breve{\rho} u \supset .\, \exists u \sim \breve{\rho} u : s \,\epsilon\, \text{Cls} \,.\, \exists su \sim \breve{\rho} u \,.\, \breve{\rho}(su) \supset s \,.\, \supset_s .\, u \supset s)\} \text{ Df}$$

Russell then introduces ω_ρ as "the class of progressions of which *R* is the generating relation"[148]. Russell then defines induction generally for any class *u* that is a progression:

$$u \,\epsilon\, \omega \,.\, \supset .\, R \,\epsilon\, \text{Rel}_u \,.=.\, 1 \rightarrow 1 \cap R \ni (u \,\epsilon\, \omega_\rho) \text{ Df}$$

$$\text{Induct} \,.=:.\, u \,\epsilon\, \omega \,.\, R \,\epsilon\, \text{Rel}_u \,.\, \supset : s \,\epsilon\, \text{Cls} \,.\, \exists su \sim \breve{\rho} u \,.\, \breve{\rho}(su) \supset s \,.\, \supset_s .\, u \supset s \text{ Df}$$

And shows that Peano's recursive equations for the basic arithmetical operations for sum and multiplication can be *proved*. We find[149]:

$$a \,\epsilon\, u \,.\, \supset .\, a + 0_u = a$$

$$a, b, x \,\epsilon\, u \,.\, \supset : (x + a) + b = x + (a + b)$$

$$a0_u = 0_u$$

$$a, b, \epsilon\, u \,.\, \supset .\, a(b + 1_u) = ab + a$$

144 RUSSELL, B., 1903, p.126 §122.
145 RUSSELL, B., 1919a, p.10.
146 RUSSELL, B., 1919a, p.9.
147 Cf. RUSSELL, B., 1901, p.325, $*1 \cdot 1$. This, as Russell puts it, "is the class of the classes *u* such that there is a one-one relation *R* such that *u* is contained in the domain of *R*, and that the class of terms to which the different *u*'s have the relation *R* is contained in *u* without being identical with *u*, and which, if *s* is any class whatsoever to which belongs at least one of the terms of *u* to which any *u* does not have the relation *R*, and to which belongs all terms of *u* to which a term of the common portion of *u* and *s* has the relation R, then the class *u* is contained in the class *s*" (RUSSELL, B., 1901, p.325-6).
148 RUSSELL, B., 1901, p.325. Here "ρ" indicates the domain or *R* (cf. RUSSELL, B., 1901, p.315).
149 RUSSELL, B., 1901, p.330.

Thus what sets Russell's account apart from Peano is that he can (i) fix a definite meaning to Peano's primitive notions and (ii) construct an actual class of terms which satisfy Peano's axioms. In a paper published shortly after Russell's 1901 article (which is actually the first collaborative work with Whitehead), Russell re-introduced his definitions of zero, number and successor in the following terms[150]:

$$\mathrm{Nc} = \mathrm{cls}^2 \cap z \ni \{\exists \mathrm{cls} \cap u \ni (z = \mu u)\} \,\mathrm{Df}$$

$$0 = \iota\Lambda \,\mathrm{Df}$$

$$n \,\epsilon\, \mathrm{Nc} \,.\supset.\, n + 1 = \mathrm{cls} \cap u \ni (x \,\epsilon\, u \,.\supset.\, u \sim \iota x \,\epsilon\, n) \,\mathrm{Df}$$

The above state, respectively, that: Nc is the relation between a class *u* and the class of all classes similar to *u*[151]; that zero is the class of classes whose sole member is the empty class; stated informally, the last definition asserts that the immediate successor $n+1$ of a cardinal number *n* is the class of all classes *u* such that if we take an element *x* from *u*, we get a member of *n*.

Using these definitions, Russell then introduced his version of Frege's celebrated definition of finite cardinal numbers - and just as importantly - the principle of induction in their most usual forms[152]:

$$\mathrm{Nc\,fin} = \mathrm{Nc} \cap n \ni \{s \,\epsilon\, \mathrm{Cls} \,.\, 0 \,\epsilon\, s : m \,\epsilon\, \mathrm{Nc} \cap s \,.\supset_m m + 1 \supset s :\supset.\, n \,\epsilon\, s\} \,\mathrm{Df}$$

$$\mathrm{Induct} \,.=:.\, s \,\epsilon\, \mathrm{Cls} \,.\, 0 \,\epsilon\, s \,.\, m \,\epsilon\, \mathrm{Nc} \cap s \,.\supset_m m + 1 \,\epsilon\, s :\supset.\, \mathrm{Nc\,fin} \supset s \,\mathrm{Df}$$

On this basis he proved a version of the Dedekind-Peano postulates, with the exception of the second (which is curiously omitted in the paper). We find[153]:

$$0 \,\epsilon\, \mathrm{Nc\,fin}$$

$$n \,\epsilon\, \mathrm{Nc\,fin} \,.\supset.\, n + 1 \,\epsilon\, \mathrm{Nc\,fin}$$

$$n \,\epsilon\, \mathrm{Nc\,fin} \,.\supset.\, n \sim= n + 1$$

From these Russell showed that no two numbers have the same successor, thus proving that there are infinitely many natural numbers. He defined the immediate predecessor *R* as follows[154]:

$$R = \imath 1 \to 1 \cap R' \cap \ni \{\rho\text{'} = \mathrm{Nc\,fin} : xR'y \,.=.\, x + 1 = y\} \,\mathrm{Df}$$

Then, given the above and induction he showed that this relation is contained in diversity[155], i.e., that if *xRy,* then $x \neq y$, from which it immediately follows that if $a, b \,\epsilon\, \mathrm{Nc\,fin}$, then if $a{+}1 = b{+}1$,

150 WHITEHEAD & RUSSELL, 1902, p.425, defintions, $*1{\cdot}3$, $*1{\cdot}4$ and $*1{\cdot}5$, respectively..
151 To make sense of $*1{\cdot}3$, we need the previous definition "$u \,\epsilon\, \mathrm{cls} \,.\supset.\, \mu u = \mathrm{cls} \cap v \ni (u \,\mathrm{sim}\, v)\}$" (cf. WHITEHEAD & RUSSELL, 1902, p.425, $*1{\cdot}2$).
152 WHITEHEAD & RUSSELL, 1902, p.426, $*1{\cdot}6$ and $*1{\cdot}61$, respectively.
153 WHITEHEAD & RUSSELL, 1902, p.427
154 WHITEHEAD & RUSSELL, 1902, p.427
155 cf. WHITEHEAD & RUSSELL, 1902, p.427-8

then $a = b$. Thus, Russell's definition of cardinal number dispenses the method of definition by abstraction (both in the strict sense of 'defining' number directly in terms of similarity and in the sense of 'abstracting' the structural properties of progressions given by Peano's postulates); the nominal definition of number in terms of number as a class of similar classes gives a definite meaning together with his definition of zero and successor gives a definite meaning to Peano's primitive notions in a way that accounts for their applicability (e.g., counting) and allows for a proof that there is a class of terms that satisfy the Peano postulates[156].

At the heart of Russell's definition of cardinal numbers - be it in the 1901 article or in the *Principles* - is the belief that number is something we attribute to classes and that the relation of similarity is the standard of numerical equality - something he shared with Cantor and, most of all, Frege, who succeeded in finding a nominal definition long before him. This idea is also the cornerstone behind the attempt to define the cardinal number of a class (or concept) by abstraction via the relation of similarity. What Frege - perhaps with greater clarity than anyone ever since - and later Russell realized was that a proper *definition* by abstraction was impossible, for it does not tell *what* numbers are. Frege and Russell's correspondence show that they (unfortunately) talked past each other in many ways, but in one respect the two were in complete agreement, and that is with respect to the idea that "to assign a number involves an assertion about a concept"[157], so to say of *F* and *G* that they have the same number is to say that there is a one-one correspondence between the things that fall under *F* and the things that fall under *G*. Frege rejected, however, a definition by abstraction via the following equivalence which expresses this insight, called now, by default, 'Hume's Principle'[158]:

(HP) $$\text{Nc}(F) = \text{Nc}(G) \equiv (\exists M)[M \,\epsilon\, 1 \rightarrow 1 \,.\, M(F_x, G_x)]$$ [159]

His reason for doing so was the now famous so-called 'Julius Caesar Problem': the above principle does not allow us to determine, in every possible context, the identy conditions for an object indicated by the expression $\text{Nc}(F)$; taking Frege's example, the above equivalence would tell us nothing about the truth-value of the following[160]:

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156 The first interpreter to emphasize this point was Landini, see LANDINI, G., 1998, p.30.

157 FREGE, G., 1950 [1884], §46, §52.

158 This name has been made widespread by George Boolos (BOOLOS, G., 1987) and is now irreversibly standard. Boolos followed Frege himself who quotes the following passage from David Hume's *Treatise*: "When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal" (HUME, D., 1960, p.71 [Book I, Part III, Chapter I]). As Michael Dummett emphatically observes, the name is quite inadequate since "it credits Hume with an idea he probably did not have and certainly did not state" (DUMMETT, M., 1998, p.x). If anyone at all deserves his name attached to the equivalence, it is Cantor. Given its currency, however, we follow the usual terminology.

159 We follow Frege here in the use of expressions of the form "$M_\beta(F_\beta, G_\beta)$". The use of subscripted variables agrees with Frege's in marking which are the argument values for the higher-order function.

160 Frege first raises the Julius Caesar problem for the attempt to define numbers by means of numerically definite quantifiers claiming that "we can never - to take a crude example - decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a

(JC) Julius Caesar $=$ Nc(F)

As Landini suggested in his recent work, Frege's objection *can* be understood as parallel with Russell's argument against definitions by abstraction[161]. The role played by the Principle of Abstraction in the 1901 article was in no way to justify definitions like the one considered and rejected by Frege in *Grundlagen* §§63-68, namely of identity of cardinality by means of the existence of a one-one correspondence between concepts. Rather, the role of the principle was in the phi-

number or is not" (FREGE, G., 1950 [1884], p.68 §56). He later raises the very same problem with the different example for the equivalence *the direction of a is equal to b if and only if a and b are parallel*: "It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis - if I may be forgiven an example which looks nonsensical" (FREGE, G., 1950 [1884], p.78 §67). Frege claims that "for the same reasons" (FREGE, G., 1950 [1884], p.79 §68) the attempt to define *numerical indentity* by means of so-called Hume's Principle is bound to fail. An attempt to show that Hume's Principle gives *the* correct *explanation* if not definition of the concept of number is given in the modern classic *Frege's Conception of Numbers as Objects* (WRIGHT, C., 1983) by Crispin Wright. For a critical discussion of Frege's argument see DUMMETT, M., pp.131-139. For more studies attempting to clarify and further develop Frege's ideas, cf. HALE & WRIGHT, 2001, HECK, R., 2011 and EBERT & ROSSBERG (eds.), 2017.

161 As Landini observes, if we were to express the Principle of Abstraction within Frege's theory of functions, we would have something like the following:

$$\text{Eq}_{x,y}[r(x,y)].(\exists x,y)r(x,y) \supset (\exists k)[(x,y)(k(x,y). = .k(x) = k(y)]$$

Which reads for any first-order non-empty equivalence function (or, for Russell, many-one relation) there is a function *k* which "associates an object with each equivalence group generated by the equivalence function *r*" (LANDINI, G., 2006, p.231) (Observe that the expression "$k(x,y). = .k(x) = k(y)$" asserts an identity between the value of $k(x,y)$ and $k(x) = k(y)$, not a material equivalence. Recall also that identity - like every relation, for that matter - is, for Frege, a function, namely a function of α and β which has as a value *the true* if α and β are identical and *the false* otherwise). This however, is only *one* principle of abstraction one can formulate within Frege's conceptual-notation. Using higher-order functional variables in the style of Frege, Landini points out that we can also have the following second-order counterpart of the above principle (LANDINI, G., 2006, p.232):

$$\text{Eq}_{f,g}[\mathit{\Gamma}_\beta(f_\beta, g_\beta)].(\exists f,g)\mathit{\Gamma}_\beta(f_\beta, g_\beta) \supset (\exists k)[(f,g)(\mathit{\Gamma}_\beta(f_\beta, g_\beta). = .k(\Omega_\beta f_\beta) = k(\Omega_\beta g_\beta)]$$

Where "$\mathit{\Gamma}_\beta(f_\beta, g_\beta)$" is as a scheme for a relation between functions *f* and *g*) If we instantiate this with similarity, we get the following, since similarity is a non-empty equivalence function:

$$(\exists k)[(f,g)(fx \approx_{x,y} fy). = .k(\Omega_\beta f_\beta) = k(\Omega_\beta g_\beta)]$$

Using Frege's $\backslash y$ function wich attached to function *f* gives us the unique y such that fy, we could then define:

$$\text{Nc}(\grave{z}fz) = (\backslash z)(k(\Omega_\beta f_\beta) = z)$$

Landini's point is that the problem here would then be the very same problem Russell raised in the *Principles*: there may be many different *k*'s which give rise to different notions of cardinal number and thus, as with Russell's many different *S*'s in $\mathcal{S}$, Landini continues, "we don't have the genuine cardinal numbers, but rather the notion of "the cardinal numbers with respect to *k*," and "the cardinal numbers with respect to *k**," and so forth", and so he claims that the "[...] the issue underlying the Julius-Caesar Problem, then, is that "definition" by means of Hume's technique amounts to working from ($\text{PA}_{\text{Frege2}}$)". In parallel with the famous result rediscovered by Crispin Wright and baptized as *Frege's Theorem* by George Boolos that an extension of second-order Logic with Hume's Principle is strong enough to derive the Peano postulates, Landini further claims that "The analog of Frege's Theorem [...] is Russell's Theorem – the proof in "On the Logic of Relations" that the range of any function *K* [*S*] satisfying the *Principle of Abstraction* (when the equivalence relation is similarity over attributes exemplified by finitely many entities) forms a progression, and that all progressions are isomorphic". Despite the fact that Landini's reading of Frege's Caesar Problem is very unorthodox and it help us to clarify Russell's position.

losophical justification of Russell's choice of one relation S in $\mathcal{S}$ - namely, similarity - which could formally give a set of terms that satisfy the Peano postulates. This is the first fundamental result Russell obtained in Mathematical Philosophy which found its way into the *Principles of Mathematics*.

2.2 The Logic of the *Principles of Mathematics*

Russell presents the *Principles* as having two main goals. First, to show that "[...] all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts and that all its propositions are deducible from a very small number of fundamental logical principles"[162]. The second is to explain "the fundamental concepts which mathematics accepts as undefinable"[163]. As Russell understood these objectives, the first was a purely mathematical matter that could be established with "all the certainty and precision of which mathematical demonstrations are capable"[164], while the second is a "purely philosophical task"[165].

The work divides into seven parts, namely (I) *The undefinables of Mathematics*; (II) *Number*; (III) *Quantity*; (IV) *Order*; (V) *Infinity and Continuity*; (VI) *Space*; (VII) Matter and Motion. Together, parts II-VII are supposed to accomplish Russell's first main goal of *mathematical* character, namely, that of defining the concepts of pure Mathematics in terms of logical ones and proving its theorems solely from logical premises. Part II offers a programmatic treatment of Cardinal Arithmetic based on Russell's definition of the cardinal number of a class α as the class of all classes similar to α; Russell also develops a treatment of basic arithmetical operations, the distinciton of finite and infinite classes and also ratios and fractions. Part III mainly treats of the notions of continuity and infinitesimals based on Russell's quite complex distinction between the notions of number and quantity. Part IV is mainly concerned with a theory of series and contains the development of a theory of progressions and ordinal numbers with discussions of the works of Dedekind and Cantor. Part V basically builds upon Cantor's work, developing a theory of irrational numbers, limits and continuity of functions and also a theory of transfinite cardinals and ordinals. Part VI is concerned with geometries, with extensive discussions of metric and descriptive ones and the development of distinct conceptions of continuity for these distinct conceptions of spaces. The final part VII is concerned with the foundations of dynamics based on Russell's logicist conception of pure Mathematics and Geometry[166].

Part I, *The undefinables of Mathematics*, is dedicated to Russell's second, philosophical goal - the analysis of the fundamental concepts of Logic upon which the developments of parts

[162] RUSSELL, B., 1903, p.xv.
[163] RUSSELL, B., 1903, p.xv-xvi.
[164] RUSSELL, B., 1903, p.xv.
[165] RUSSELL, B., 1903, p.xv-xvi.
[166] For the history of the composition of the *Principles*, cf. the introduction and the first part of MOORE, 1993.

II-VII should be grounded. The section divides into ten chapters, namely: (I) *Definition of Pure Mathematics*; (II) *Symbolic Logic*; (III) *Implication and Formal Implication*; (IV) *Proper Names, Adjectives and Verbs*; (V) *Denoting*; (VI) *Classes*; (VII) *Propositional Functions*; (VIII) *The Variable*; (IX) *Relations*; (X) *The Contradiction*. The purpose of part I is to enumerate and explain the primitive notions of Mathematical Logic which, in the early chapters, Russell claims to be the following six: [167]: (1) "formal implication"; (2) "implication between propositions not containing variables"; (3) "the relation of a term to a class of which it is a member" (4) "the notion of such that"; (5) the notion of relation; and finally (6) "truth"; these, he also claims, are sufficient to state "all the propositions of symbolic logic". Later portions of the book, however, present a corrected version that actually correspond to the notions discussed in chapters III-IX, namely[168]: (1') *implication*; (2') the relation of a term to a class of which it is a member i.e., *membership*; (3') the notion of *such that*; (4') the notion of *relation*; (5') the notion of a *propositional function*; (6') the notion of *class*; (7') the notion of *denoting*; and finally (8') the notions of *any* or *every* term that are involved in the idea of *variation*[169].

As we discussed, a crucial circumstance which contributed in shifting Russell's interests to Symbolic Logic was his contact with Peano and his school - and the Logic of the *Principles* owes very much to their works, especially Peano's *Formulario*. As we mentioned, Russell viewed Peano's new symbolic logic as being primarily a tool which allowed for the resolution of old (and intractable) philosophical problems because, as Russell saw the matter, it broke free from the shackles of 'traditional' Logic by accounting for methods of deduction that were essential and commonplace in mathematical reasoning. As Russell explains in the opening pages of the *Principles*:

> There was, until very lately, a special difficulty in the principles of mathematics. It seemed plain that mathematics consists of deductions, and yet the orthodox accounts of deduction were largely or wholly inapplicable to existing mathematics. Not only the Aristotelian syllogistic theory, but also the modern doctrines of Symbolic Logic, were either theoretically inadequate to mathematical reasoning, or at any rate required such artificial forms of statement that they could not be practically applied. In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, i.e. the à priori knowledge of space and time. Thanks to the pro-

[167] RUSSELL, B., 1903, p.11 §12.

[168] RUSSELL, B., 1903, p.106 §106.

[169] The analysis of the notion of the variable occupies a large portion of the first part of the work. Indeed, according to Russell himself, it was his puzzlement with respect to this last primitive notion which actually led him from the Philosophy of Physics and Geometry to Symbolic Logic, as Russell explains in the preface: "About six years ago, I began an investigation into the philosophy of Dynamics. I was met by the difficulty that, when a particle is subject to several forces, no one of the component accelerations actually occurs, but only the resultant acceleration, of which they are not parts; this fact rendered illusory such causation of particulars by particulars as is affirmed, at first sight, by the law of gravitation. It appeared also that the difficulty in regard to absolute motion is insoluble on a relational theory of space. From these two questions I was led to a re-examination of the principles of Geometry, thence to the philosophy of continuity and infinity, and thence, with a view to discovering the meaning of the word *any*, to Symbolic Logic." (RUSSELL, B., 1903, p.xvi-xvii).

> gress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable of a final and irrevocable refutation.[170]

In a letter to Jourdain from April 15, 1910, Russell further elaborates this same point, emphasizing why he was also not satisfied with the works of the so-called "algebraic tradition of symbolic logic" and where Peano's own logic required supplementation:

> Until I got hold of Peano, it had never struck me that Symbolic Logic would be any use for the Principles of Mathematics, because I knew the Boolian stuff and found it useless. It was Peano's ϵ, together with the discovery that relations could be fitted into his system, that led me to adopt symbolic logic. I had already discovered that relations which assigned formal properties (transitiveness etc.) are the essential thing in Mathematics, and Moore's philosophy led me to wish to make relations explicit, instead of ϵ and $\subset$. This hangs together with my attack on subject-predicate logic in my book on Leibniz.[171]

The fundamental issue was, of course, the handling of relations, which were an essential component of Russell's break from idealism and the doctrine that every judgment is ultimately analyzable in terms of subject and predicate. Though Russell did partially recognize the merits and importance of the works of the 'algebraic' tradition[172], he still thought that "the subject achieved almost nothing of utility either to philosophy or to other branches of mathematics, until it was transformed by the new methods of Professor Peano"[173].

In a nutshell, the fundamental change brought about by Peano in his analysis of mathematical reasoning was this: he broke with traditional logic by employing his primitive notions of formal implication and of a Cantorian class[174] in analyzing mathematical statements; as we discussed, to Russell this amounted to abandoning the limited subject-predicate analysis of propositions and giving place to an analysis in terms of variables, constants and Russell's new notion of a '*propositional function*'[175]. As we discussed, though Peano himself had not done so in any systematic or detailed way, Russell realized that the framework which was laid down by

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170 RUSSELL, B., 1903, p.4. For similar remarks, cf. RUSSELL, B., 1959, pp.65-6.

171 GRATTAN-GUINNESS, I., 1977, p.133.

172 Russell acknowledges the efforts and importance of George Boole towards the "the recognition of asyllogistic inferences" (RUSSELL, B., 1903, p.10; Russell refers in particular to BOOLE, G., 1854; cf. also BOOLE, G., 1847). Russell also explicitly recognizes (RUSSELL, B., 1903, pp.23-4, footnote*) the origins of the calculus of relations in the works of Augustus De Morgan (cf. De MORGAN, A., 1864, 1847) and its systematic developments in the works of Charles Sanders Peirce (Russell refers specifically to PEIRCE, C. S., 1880; cf. also PEIRCE, C. S., 1870, 1885) and later by Ernst Schröder (cf. SCHRÖDER, E., 1895). Russell recognized Schröder's three volume treatise (i.e., SCHRÖDER, E., 1890, 1891 and 1895) as the "the most complete account of the non-Peanesque methods" (RUSSELL, B., 1903, p.10, footnote *). For a classic survey of the early developments of algebraic logic, cf. LEWIS, C. I., 1918; cf. also GRATTAN-GUINNESS, I., 2000, chapter 2.

173 RUSSELL, B., 1903, p.10.

174 I.e., a class viewed as an *object* as opposed to the part/whole conception which viewed a class as a mere aggregate.

175 As Russell explained in *My Philosophical Development*, Peano's analysis of general propositions, i.e., propositions containing variables, made him realize that "every statement containing the word all involves propositional functions, but does not involve any particular value of these functions" (RUSSELL, B., 1959, p.66).

the Italian mathematician provided a calculus which could be adequately expanded to include a treatment of relations.

The Logic of the *Principles* is by and large Peano's formal system plus Russell's calculus of relations embedded within the realistic Philosophy inherited from Moore[176]. The symbolic logic which is presented in the work rather informally (since the work almost as a rule does not rely on symbolism) was intended to be fully developed in a second volume co-authored with Whitehead, starting from the basic primitives of logic and advancing all the way from Arithmetic through Analysis and into Geometry[177]. So the presentation of logical grammar and of the chain of deductions which is intended to demonstrate Russell's claim that Pure Mathematics is a development of Logic varies from informal to semi-formal. The logical calculus of the *Principles* is divided into three separate branches, namely the calculus of propositions, of classes and relations.

As it will become evident in the brief discussion of the Logic of the *Principles* which follows here, the system of logic that underlies the work looks quite antiquate from the perspective of modern treatments of Logic. This, however, is due to the main logico-philosophical doctrines which Russell holds in the work, not simply because it predates the results and standards of rigor which one finds in modern textbooks.

2.2.1 A Brief Digression: Frege's *Begriffsschrift*

Before discussing the Logic of Russell's *Principles of Mathematics* which is heavily influenced by Peano's *Formulario*, we must make brief digression concerning Frege's works on logic. As is widely known, Frege had developed an analysis and notation for mathematical statements that in many respects was far superior to that which Russell inherited from Peano and which he modified to suit his technical goals and philosophical views. Indeed, by 1879 Frege had published what may rightfully considered the most important work on Logic ever written[178], his *Begriffsschrift* [179].

176 Although, of course, things are not *so* straightforward: Russell's logic corrects a lot of imprecisions in Peano's systems and further adopts principles of comprehension. This will be addressed in detail below.

177 In the preface we find: "The second volume, in which I have had the great good fortune to secure the collaboration of Mr A. N. Whitehead, will be addressed exclusively to mathematicians; it will contain chains of deductions, from the premises of symbolic logic through Arithmetic, finite and infinite, to Geometry, in an order similar to that adopted in the present volume; it will also contain various original developments, in which the method of Professor Peano, as supplemented by the Logic of Relations, has shown itself a powerful instrument of mathematical investigation". (RUSSELL, B., 1903, p.xvi) As is well known, such a volume never appeared in the way they planned.

178 As noted in van HEIJENOORT, J., 1967, p.1. Some, like Quine and Michael Dummett claim that the book actually marks the inauguration of modern Mathematical Logic (cf. QUINE, W., 1960, p.163; DUMMETT, M., 1981, p.xxxv). Perhaps the more historically accurate statement would be that Frege was the first to articulate (or at least to *publish* a work in which he articulates) the modern apparatus of quantificational logic, although the term "quantifier" itself is due to Pierce (cf. CHURCH, A., 288).

179 FREGE, G., 1879. Following an usual convention among commentators of Frege, we shall use " *Begriffsschrift* " with a capital "*B*" to refer to Frege's first booklet on Logic and "*begriffsschrift* " to refer to his conceptual

As is well known, Frege was inquiring (again, long before Russell *and* Peano) whether the concept of *number* should be understood as a *logical concept* and whether the laws of Arithmetic should be understood as *logical principles.* Frege's initial goal, as he described in his first logical booklet, was to analyze the concept of *series* or *sequence* and to prove fundamental theorems about this notion in a way that prevented "anything intuitive" to remain "unnoticed" in a way that kept "the chain of inferences free of gaps"[180]. In this respect, as he explained, natural language became an insurmountable obstacle because "[...] no matter how unweldy the expressions [he] was ready to accept" he "[...] was less and less able, as the relations became more and more complex, to attain the precision that [his] purpose required"[181] - and this is what led him to formulate his "ideography" or, as he more frequently said, his " *Begriffsschrift* ", i.e., "concept-script".

Frege's *Begriffsschrift* contains the first formulation of our modern idea of a formal system: a set of primitive symbols accompanied by rules for forming complex expressions from simpler ones and a set of axioms and rules for manipulating symbols that allowed for a rigorously defined notion of a demonstration or proof. Frege's primitive notions were those of *assertion*[182] (or jugement), *conditionality*[183] (i.e., the material implication), *negation*[184], of *identity*[185], of *functionality*[186], i.e., the notion of *function* (and so also that of an *argument*) and finally, and perhaps most importantly, that of *generaltity*[187], i.e., the universal quantifier.

In Frege's notation, the assertion of what he called a given 'content *A*' and the assertion of not-*A* are respectively symbolized as follows[188]:

$$\vdash\!\!- A$$

$$\vdash\!\!\top\! A$$

In the second formula above, the short vertical stroke is the sign for the primitive notion of negation. Also, the sign "$\vdash\!\!-$" itself is also composed of two distinct symbols; Frege used the expression "—— *A*" to stand for "[...] *a mere combination of ideas*, of which [he] does not state whether he recognizes to be true or not"[189]; the role of the bold vertical bar was to indicate that the

notation. As is well known, Frege modified the original formulation of his *begriffsschrift* throughout the years, most notably in his *Grundgesetze de Arithmetik* (FREGE, G., 1893 and 1903). Some of these changes will be briefly mentioned below. For systematic and in depth discussios of these changes, the reader is referred to the following authoritative studies of the complex development of Frege's ideas: DUMMETT, M., 1967, 1981*a*, 1981*b*, 1991; DUARTE, A., 2009; HECK, R., 2012; LANDINI, G., 2012; cf. also the third part of DEMOPOULOS, W., 1995; a good survey of some issues is given in the Introduction of RICKETTS & POTTER, 2010; for a more recent authoritative collection of essays on Frege's *Grundgesetze*, cf. EBERT & ROSSBERG, 2019.

180 FREGE, G., 1879, p.5.

181 FREGE, G., 1879, p.6.

182 Cf. FREGE, G., 1879, p.11-2.

183 Cf. FREGE, G., 1879, p.13-14.

184 Cf. FREGE, G., 1879, p.17.

185 Cf. FREGE, G., 1879, p.20.

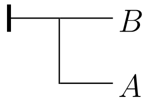
186 Cf. FREGE, G., 1879, p.21-22.

187 Cf. FREGE, G., 1879, p.24.

188 Cf. FREGE, G., 1879, p.11 and 17-18.

189 FREGE, G., 1879, p.11.

content was not merely being considered but actually asserted[190]. Frege's sign for conditionality or material implication gives rise to the famous bi-dimensional aspect of his notation. Frege used the following[191]:



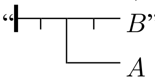
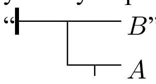
The truth conditions of above are explicitly introduced as excluding the circumstance where *A* can be (truly) affirmed and *B* (truly) denied[192].

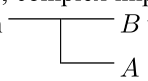
It is in connection with the above notation for material implication that another of Frege's sharpest and most important insights also appear for the first time. As we remarked above,

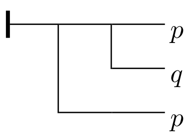
[190] Uncharacteristically, at this point of Frege's explanation of his notation we find some unclear aspects of the exposition of the *Begriffschrift* which give rise to many complex and important interpretative issues about the development of Frege's views. As Frege observes, not every 'content' can be asserted or, as Frege put it, not every content is 'judgeable' or "[...] can become a judgment" (FREGE, G., 1879, p.12). To take an obvious example, one cannot consider the expression "$\vdash x$", where *x* is an individual variable as the expression of a judgment. Frege requires that "[...] whatever follows the content stroke must have a content that can become judgment" (FREGE, G., 1879, p.12). Still, Frege does not provide any clear-cut explanation as to *what* distinguishes a judgeable content from a non-judgeable content. Also, in the *Begriffschrift* Frege introduced the notion of identity in a somewhat confused way. He put forward the following notation (which Russell would later use for the material biconditional in *Principia*):

$$\vdash (A \equiv B)$$

As he first explained it, the above was meant to express that "[...] two names have the same content" (FREGE, G., 1879, p.20) and thus, that it differed from the other symbols like the conditional and negation because "[...] it applies to *names* not to contents" (FREGE, G., 1879, p.20, our emphasis). Both of these imprecisions were addressed by Frege and clarified in his later writings, most notably his papers *On Function and Concept* (FREGE, G., 1891) and *On Sense and Meaning* (FREGE, G., 1892) which introduce what may perhaps be his most famous ideas. In those papers Frege distinguishes a peculiar sort of function which he calls *concepts*. Differently from functions in general, concepts always have as their values for any given argument (or number of arguments) a particular sort of object that Frege called a 'truth-value' (see footnote below); Frege also introduced his famous distinction between the sense of an expression - its mode of presentation - and its *Bedeutung* - its reference or meaning.

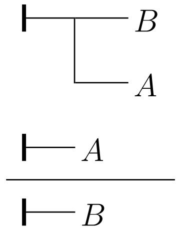
[191] Cf. FREGE, G., 1879, p.13-14. Other propositional connectives are very easily expressible; the expression "", for instance, asserts that *A* is true *and B* is true; while "" asserts that *A* is true *or B* is true.

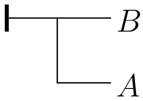
[192] FREGE, G., 1879, p.13-14. Of course, complex implications are formed by nesting implications within others, i.e, putting expressions with the form  within other expressions of the same form. For instance, the Peanesque formula $p \,.\supset .\, q \supset p$ is written as follows:



The above is actually the first numbered proposition in Frege's booklet and one of his axioms ruling the material conditional (cf. FREGE, G., 1879, p.29); the other axioms are (2), (8), (28), (31) and (41) - cf. FREGE, G., 1879, p.31, p.36, p.44, p.45 and p.41, respectively; the last two axioms also rule the behavior of negation. As Alonzo Church observes, this is "[...] the first formulation of the propositional calculus as a logistic system" (CHURCH, A., 1956, p.155-6); the system was later simplified since some axioms are not independent, as showed in TARSKI & ŁUKASIEWICZ, 1930.

Frege's initial goal with his work on Logic was to show that basic propositions concerning the notion of a *sequence* have no intuitive presuppositions and he intended to show this by exhibiting proofs in such a way that all chains of inferences result "free of gaps"[193]. What Frege realized was that in order to truly fulfill this task, it was necessary to not only explicitly articulate within his symbolism the axioms that are needed for proofs, but also the *rules* for manipulation of signs that capture the modes of *inference* that are allowed in the construction of proofs. So in addition to axioms that were assumed as basic logical principles, Frege also introduced a way of representing the inference of a judgment from other judgments, i.e., a mode of inference, in a way that actually provides a *rule* for manipulating expressions of his *begriffsschrift* , i.e, a rule which states what is admissible in the construction of proofs. As his first rule, Frege had[194]:



The above is intended to capture, in terms of a symbolic rule, the fact that, as Frege puts it, "[...] from the two judgments  and $\vdash A$, the new judgment $\vdash B$ follows"[195]. The realization that the statement of such a rule is necessary marks a tremendously important advance towards Frege's goals and for Logic as a whole. As Michael Dummett vividly puts it:

> What Frege wanted was a framework within which all mathematical proofs might be presented and which would offer a guarantee against incorrect argumentation: of proof so set out, it would be possible to be certain that it was not erroneous, or valid only within certain restrictions not made explicit, or dependent upon unstated assumptions. To achieve this purpose, it was necessary to device a symbolic language within which any statement of any given mathematical theory might be framed, as soon as the required additional vocabulary for that theory was specified[196]. This would be, in modern terminology, a formalized language: that is, there would be an effective method of recognizing, for any given collocation of symbols, whether or not it was a formula of that symbolic language. Furthermore, in reference to this language, it was necessary to stipulate formal rules of proof, rules, that is, which would specify in a manner which provided a procedure for effective recognition which sequences of formulas of the language constituted a valid proof. [...] Frege was,

[193] Cf. FREGE, G., 1879, p.5.

[194] Cf. FREGE, G., 1879, p.16. This is the full statement of the rule, but Frege finds this "[...] awkward if long expressions were to take the places of *A* and *B*" (FREGE, G., 1879, p.16) so he uses an abbreviated form.

[195] FREGE, G., 1879, p.15-16.

[196] This is the point which generally used to mark the contrast between Frege's work with those of Boole and Schröder, which Frege famously explained as follows: "My intention was not to represent an abstract logic in formulas, but to express content through written signs in a more precise and clear way than it is possible to do through words. In fact, what I wanted to create was not a mere *calculus ratiocinator*, but a *lingua characterica* in Leibniz's sense" (FREGE, G., 1882, pp.1-2, as quoted in van HEIJENOORT, J., 1967, p.2).

> thus, proposing to take the step from the axiomatization of mathematical theories with which nineteenth-century mathematics had been deeply concerned, to their actual formalization. While the axiomatic method strove to isolate the basic notions of each mathematical theory, in terms of which the other notions of the theory could be defined, and the underlying assumptions, from which all the theorems could be made ultimately to derive, what Frege wanted to do was to subject the process of proof itself to an equally exact analysis.[197]

But Frege went (and had to go) even further in order to achieve the kind of complete analysis which he needed. For as Dummett further observes, "[...] before an analysis of proof is possible, an analysis has first to be given of the structure of the statements which make up the proof"[198] and that's where the notion of *functionality*, i.e., of function and argument comes into play.

The notions of *function* and *argument* and the role they play in Frege's analysis of mathematical judgments (and, in fact, judgments in general) present what is truly revolutionary in Frege's first work. The fundamental gist of Frege's analysis of judgment is that the old dichotomy of subject and predicate is supplanted by the dichotomy of argument and function. Grammatically speaking, Frege thought that statements like "*a* is a prime number" should not be analyzed in terms of a subject and a predicate, but should be decomposed in the same way that arithmetical expressions like "x^2" or "$x + 1$" are - hence the subtitle of his booklet "*a formula language, modeled upon that of arithmetic, for pure thought*"[199]. This idea sprang from a very simple, yet also very deep insight that arose from observing what now seem as very commonplace facts, namely: the fact that functional expressions like "$\sqrt{\ }$" or "$+1$" are incomplete, that is, they have *gaps*; that the use of variables is one possible way of representing these gaps; and that it is by filling such gaps with expressions which have a determinate meaning that the meaning of complex expressions composed of simpler ones is determined, e.g., by filling the gap in "$+1$" to obtain "$0 + 1$" or the gaps in "$\sqrt{\ }$" to obtain "$\sqrt[3]{8}$".

Frege's insight was that even sentences which grammatically appear to have the form of subject and predicate like "2 is a prime number" or "0 is less than 1" can also be analyzed in terms of function and argument. As Dummett observes, what Frege realized was that complex expressions, in particular sentences containing multiple markers of generality are constructed in *stages* and that by employing his analysis of judgments in terms of function and argument instead of subject and predicate, he could formulate a completely perspicuous symbolic device for representing the stages of construction of such sentences, making their internal structure manifest[200].

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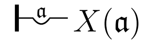
197 DUMMETT, M., 1981, p.1.

198 DUMMETT, M., 1981, p.2. It must be observed that one can accept this point of Dummett's without fully accepting his somewhat anachronistic claim that Frege was interested in formulating a *theory of meaning*.

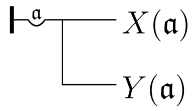
199 FREGE, G., 1879, p.1. It must be observed, though, that the use of lower-case italic letters as variables which indicate the "gaps" of an open sentence does not coincide with that of Frege, who employed different conventions for the use of lower-case italics and also employed Gothic lower-case letters bound by quantifiers to represent generality (cf. FREGE, G., 1879, p.24-6).

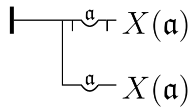
200 Cf. DUMMETT, M., 1981*a*, pp.10-11. For what is surely one the most illuminating discussions of Frege's insights

It was building upon this analysis that in the *Begriffsschrift* Frege was able to present for the first time a clear-cut explanation of the universal quantifier which may be, as Dummett once put it, "[...] the deepest single technical advance ever made in logic"[201]. Frege employed the following expression for representing generality[202]:



Which, as he explains in the *Begriffschrift*, stands for "the judgment that, whatever we may take for its argument, the function is a fact"[203]. So, for instance, the following asserts, respectively, that *everything that is an Y is an X* and that *if everything is a X, then something is an X*:





This simple and elegant notation is the very first formulation of a basic notation for what we now know as 'quantification theory' or 'quantificational logic'.

The above makes it clear how to construct complex statements from simpler ones in stages, in particular those containing generality. If one intends to say, for instance, that every number is odd, one starts from a simple statement like "2 is even", or more shortly, "$E(2)$"; the next stage is to replace "2" by a variable, to obtain a functional expression, i.e., "*x* is odd", or "$E(x)$"; generality comes in asserting that, for every argument which can be assigned for *x*, the resulting statement is true - or as Frege put it in *Begriffschrift* when he did not yet had a notion of a *truth-value* - a statement of a fact.

which led to the 'discovery' of quantification theory - and how truly penetrating and far-sighted Frege was in this regard - cf. DUMMETT, M., 1981*a*, chapters 1 and 2.

201 DUMMETT, M., 1981, p.xxxiii..

202 Thus, to assert that there is an $\mathfrak{a}$ such that $X(\mathfrak{a})$, one writes "$\vdash\!\top\!\smile^{\mathfrak{a}}\!\top X(\mathfrak{a})$".

203 FREGE, G., 1879, p.24. Though in his first presentation of his concept-script Frege does not discuss the issue what a function is or what the values of a function may or may not be (As noted in van HEIJENOORT, J., p.2.), he elucidates his insight in posterior writings by explaining that a predicates like " is a prime number" should be viewed as a particular case of functional expressions that stand for a particular sort of functions that Frege called *concepts*. Functions are opposed to *objects* which are capable of, so to speak, fill the 'gaps' of the function. Shortly after publishing his *Begriffsschrift*, Frege distinguishes between two kinds of functions. On the one hand, there are functions like those of ordinary mathematics that are represented by gappy expressions like " $+1$"; when their gaps are filled, such then stand for a definite object, like, say, the number 1 in case the gap in the expression is filled by the number 0. Concepts, on the other hand, are, for Frege, functions whose values are always another peculiar kind of objects, namely, *truth-values*. So, for Frege, filling the gap of an expression like " is a prime number" may result in an expression which stands either for the value *Truth* or the value *False*, depending on what object is assigned as the argument of the function - as Frege himself puts it "a concept is a function whose value is always a truth-value". Furthermore, the notion of an *object*, for Frege is incapable of definition because "it is too simple to admit of logical analysis" (FREGE, G., 1891, p.147 [18]); similarly the notion function can only be elucidated by observations that either pertain to expressions that represent functions - for instance observing that such expressions 'haves gaps' or 'are incomplete' - or by the use of metaphors - for instance, Frege's favorite one, that a "a function is unsaturated" (cf. FREGE, G., 1891, p.140-1. [7]).

What is truly revolutionary in Frege's analysis and notation is that this construction in stages can be carried on to arbitrary levels of complexity. It is a much more precise (and powerful in terms of expressivenes) tool than anything formulated by Boole or those who build upon his work, including Peano. His notation made it possible to sharply distinguish logical forms of sentences that always eluded traditional (syllogistic) logic like "Everyone loves someone" and "Someone is loved by all". With Frege's symbolism, the two forms can be clearly distinghished as follows:

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!-\!\!\top\!\overset{\mathfrak{b}}{\smile}\!\!\top\, R(\mathfrak{a},\mathfrak{b})$$

$$\vdash\!\!\top\!\overset{\mathfrak{b}}{\smile}\!\!\top\!\!-\!\overset{\mathfrak{a}}{\smile}\!\!-\, R(\mathfrak{a},\mathfrak{b})$$

Indeed, long before Russell, Frege had provided means for handling relations[204] within a logical calculus that allowed every desirable inference[205] to be represented and derived within his system. Frege introduced two additional rules governing expressions containing generality, namely[206]:

$$\frac{\vdash\!\!-\, X(\mathfrak{a})}{\vdash\!\overset{\mathfrak{a}}{\smile}\!\!-\, X(\mathfrak{a})}$$

Which is exactly what we nowadays call the rule of 'universal generalization' and also the following third (and final rule)[207]:

$$\frac{\vdash\!\!\!\begin{array}{l} \top\!\!-\!\!-\, X(\mathfrak{a}) \\ \lfloor\!\!-\!\!-\, A \end{array}}{\vdash\!\!\!\begin{array}{l} \top\!\overset{\mathfrak{a}}{\smile}\!\!-\, X(\mathfrak{a}) \\ \lfloor\!\!-\!\!-\, A \end{array}}$$

under the condition that "A is an expression in which $\mathfrak{a}$ does not occur and if $\mathfrak{a}$ stands only in the argument places of $X(\mathfrak{a})$"[208].

In what is a crucial aspect of Frege's systems, he allowed his quantifiers to bind *functi-*

[204] Relations, for Frege, were a particular case of functions, namely concepts of multiple arguments.

[205] More precisely: every inference that is correct from the point of view of classical Logic.

[206] Cf. FREGE, G., 1879, p.25.

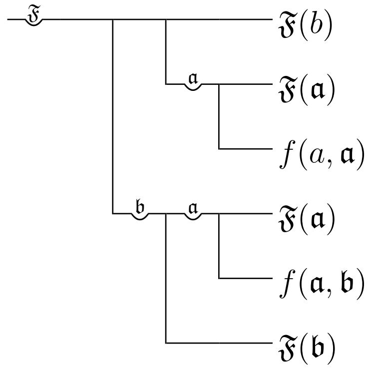
[207] Cf. FREGE, G., 1879, p.26.

[208] FREGE, G., 1879, p.25-26. Frege had only one axiom for quantification theory, namely (FREGE, G., 1879, p.51):

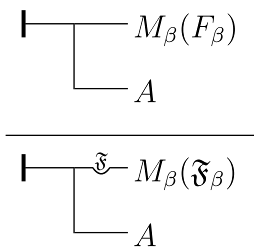
$$\vdash\!\!\!\begin{array}{l} \top\!\!-\!\!-\, X(c) \\ \lfloor\!\overset{\mathfrak{a}}{\smile}\!\!-\, X(\mathfrak{a}) \end{array}$$

which is what we now call the law of universal instantiation.

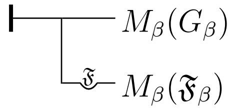
onal variables. In his celebrated definition of the ancestral, for instance, Frege had:



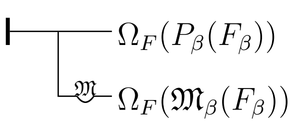
Where the $\mathfrak{F}$ above ranged over *functions* which take objects as arguments, not over objects themselves. Frege's *begriffsschrift* also had provisions for handling expressions for concepts in both predicate and subject positions (or, more approppriately for concepts falling "within" as arguments of higher-order concepts)[209], employing markers for the gaps of functions with subscripted variables, as in '$M_\beta(F_\beta)$'. In fact, as many authors have pointed out, implicit in Frege's *Begriffsschrift* are rules of inference for concepts analogous to those considered above. Following a formulation suggested by Gregory Landini, Frege's rule of generalization for concepts could be expressed as follows:



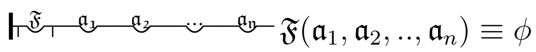
where *F* does not occur free in *A*; similarly, Frege's only axiom for quantification theory[210] has the following analogue for concepts:



These axioms and rules, on their turn, have yet 'higher-order' analogues. The above, for instance has the following analogue for concepts that apply to concepts of second order[211]:



Thus, implicit in Frege's Logic is an ever increasing hierarchy of rules and axioms ruling the corresponding (ever increasing) hierarchy of functions. Indeed, also implicit in Frege's *Begriffsschrift* is a family of axiom schemes of comprehension for functions, as in:



[209] In fact, the quantifiers themselves can be seen as "higher-order concepts" that apply to concepts.
[210] Cf. footnote 208.
[211] That is, concepts that apply to concepts that apply to objects.

where ϕ stands for some judgeable content not containing $\mathfrak{F}$ free. And the same applies to functions like $M_\beta(G_\beta)$, $\Omega_F[P_\beta(F_\beta)]$, and so on[212].

Now, at this point it must be clear that the contrast of Frege's symbolic logic with that of Peano's is quite stark - and, in fact, this contrast was quite extensively discussed by the two authors[213]. For our present purposes, we must consider three fundamental aspects of Frege's notations and analysis of mathematical statements which are much sharper than Peano's - and similar points will apply with to some aspects of the logic of Russell's *Principles of Mathematics*.

First there is Frege's notation and analysis of general judgments. In his correspondence with Peano, Frege praises the former's departure from traditional logic, observing that "[his] designation for generality remedies an essential defect of Boolean logic"; but Frege also surgically notes that Peano's notation "[...] is perhaps less generally applicable than [his]" and that he does "not know whether [Peano] will be able to set sure limits to the scope of generality in all cases"[214]; the latter point, in particular, is important. Peano's notation was not as precise or perspicuous as Frege's. Frege's notation was superior than that of Peano for distinguishing more clearly the scope of quantified variables - in particular in contexts where multiple generality is involved and by allowing relation-symbols[215].

Second, there is Frege's inclusion of rules of inference in his *begriffschrift* and his emphasis that the very notion of deduction or proof in mathematics should be subjected to analysis. From Frege's standard of what counts as a formal deduction or proof - i.e., a sequence of formulas manipulated according to a set of syntatical rules for forming expressions *and* for deducing formulas from axioms or from formulas deducible from the axioms - there is not a single proof in Peano's *Formulario*, since there are no rules of inference *explicitly* articulated. Take, for ins-

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[212] Cf. LANDINI, G., 2012, p.52. As Landini points out, an important change that happens in Frege's *Grundgesetze* is that these implicit axiom schemes will be quite different there. In the *Begriffsschrift* (implicit schematic) comprehension must be framed in terms of an 'identity of content' between an expression like $\overset{\mathfrak{a}}{\smile} F(\mathfrak{a})$ or $\overset{\mathfrak{F}}{\smile} M_\beta(\mathfrak{F}_\beta)$ and some judgeable content *A*; in Frege's *Grundgesetze*, on the other hand, comprehension must be framed in terms of genuine identity, as in (cf. LANDINI, G., 2012, p.57):

$$\vdash \overset{\mathfrak{F}}{\smile}\!\top\!\overset{\mathfrak{a}_1}{\smile}\!-\!\overset{\mathfrak{a}_2}{\smile}\!-\!\overset{..}{\smile}\!-\!\overset{\mathfrak{a}_n}{\smile}\, \mathfrak{F}(\mathfrak{a}_1, \mathfrak{a}_2, .., \mathfrak{a}_n) = \alpha$$

where the schematic α stands for an arbitrary *term*. This is because in the *Grundgesetze* an expression like "$\overset{\mathfrak{x}}{\smile} F(x)$" is assigned a value when asserted (either the true or the false) according to whether all objects *x* fall under *F* or not.

[213] Frege and Peano exchanged a significant number of very illuminating of letters (cf. FREGE, G., 1980, pp.108-129). Peano reviewed the first volume of Frege's *magnum opus*, the *Grundgesetze* (FREGE, G., 1893) in the *Rivista di Mathematica* (cf. PEANO, G., 1895*b*); Frege responded the review with a paper discussing the contrast between his views and those of Peano (cf. FREGE, G., 1897) and also a letter from 29 September 1896 (cf. FREGE, G., 1980, pp. 112-18) which Peano then published in the *Rivista* accompanied by his own counter-reply (cf. PEANO, G., 1896, pp.295-296); for some details about these exchanges, cf. GRATTAN-GUINNESS, I., 2000, pp.247-251.

[214] FREGE, G., 1980, p.109.

[215] Which, again, for Frege, were a particular case of functional expressions.

tance, the following demonstration from Peano's *Principles of Arithmetic*[216]:

$$
\begin{array}{rll}
\multicolumn{3}{l}{Proof:} \\
\mathrm{P1} \,.\supset . & 1 \,\epsilon\, \mathrm{N} & (1) \\
1[a](\mathrm{P6}) \,.\supset . & 1 \,\epsilon\, \mathrm{N} \,.\supset .\, 1+1 \,\epsilon\, \mathrm{N} & (2) \\
(1)\,.\,(2) \,.\supset . & 1+1 \,\epsilon\, \mathrm{N} & (3) \\
\mathrm{P10} \,.\supset . & 2 = 1+1 & (4) \\
(4)\,.\,(3)\,.\,(2, 1+1)[a,b](\mathrm{P4}) \,.\supset . & 2 \,\epsilon\, \mathrm{N} & \text{(Theorem)}
\end{array}
$$

Peano presents the above as a proof in which "[...] we have written explicitly all the steps"; but nowhere in his booklet Peano does introduce an *explicit* rule which justifies step three - indeed, the very law "$a \,.\, a \supset b \,.\supset .\, b$" does not even appear in the initial catalog containing around forty logical propositions[217]. This, of course, does not mean that Peano could not have the rule in his system. Also, it must be observed, this does not mean necessarily that Peano did not recognize - at least implicitly - the distinction between a rule which justifies inference of a proposition from other propositions and an actual axiom or primitive proposition. In fact, the best explanation for such a lacuna in Peano's axiomatization is the confused way in which he characterized his notation "$a \supset b$". In the *Principles of Arithmetic*, in particular, he introduced it as meaning "*a deducitur b*"[218]; in the French version of the *Formulario* he introduced it as "*b on déduit a*"; these explanations suggest that his demonstrations employed an implicit rule but also that his works were affected by a confusion between entailment and material implication; this can explain the absence of an *explicit* rule of detachment: since he explained "$a \supset b$" as "from *a*, one deduces *b*" or "from *a*, *b* can be deduced", he may just as well found unnecessary to introduce an explicit rule (much less a theorem) which stated that from from *a* and $a \supset b$ one may deduce *b*. Similar remarks apply to Peano's notion of formal implication - Peano's constructions of proofs implicitly employed generalization and instantiation in connection with it. Frege, of course, knew better[219] and, given all of his concerns with the rigorous constructions of proofs, he formulated *modus ponens* as *as an explicit symbolic rule* and clearly distinguished between implication and detachment[220]. And even more importantly, Frege also formulated explicit rules for quantification theory.

The points discussed above mark important contrasts in terms of procedures, goals and clarity involved in their attempts at formalizing mathematical statements and show that - in contrast to Frege - Peano was, after all, still tied to some aspects of the Boolean tradition that

216 PEANO, G., 1889, p.94.
217 cf. PEANO, G., 1889, p.87-8.
218 Cf. PEANO, G., 1889, p.
219 And as we shall see, so did Russell.
220 Still, it must be said that it is not clear whether Peano missed the distinction or whether he was just careless or sloppy about it. As we shall see below, Russell himself clearly recognized the distinction in the *Principles of Mathematics*.

prevented substantial progress to be achieved in Logic. As we shall see next, Russell was caught in the middle.[221]

2.2.2 Propositions and Russell's Doctrine of the Unrestricted Variable

At the outset we must recall that the notion of a 'proposition' must be handled with care in the *Principles of Mathematics*[222]. Russell was fully committed to Moore's realist conception of judgment according to which propositions are complex *entities.* As he put it: "[...] a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words"[223]. Thus, when Russell speaks of propositions in the *Principles* he is not talking about sentences, but about complex, mind-independent entities which have as constituents the entities that are the subjects of the proposition. In other words, a 'proposition' in the sense in which Russell employs the notion in *Principles* does not necessarily consists of words, but of *entities*, namely, the *entities which the proposition is about*, i.e., *the logical subjects of the proposition* in question[224]. This is one of the reasons why Russell's calculus of propositions is very different from Peano's and, indeed, from any modern formulation of propositional logic. Of course, the fundamental notions of Russell's logic of propositions are *implication* and *formal implication*, both inherited from Peano and also following Peano, Russell used "$p \supset q$" for material implication. But given his understanding of propositions, he used the symbol "$\supset$" as a name for an actual dyadic (indefinable) relation that holds between *entities*, not expressions.

Russell's calculus of propositions also differed from Peano's (and from modern formulations of propositional logic) in virtue of one of the (if not *the*) most important and central doctrines of the *Principles*, namely Russell's doctrine of the 'univocity of being' - a thesis which we which shall discuss more extensively in the next chapter, but which we must also consider from the outset in order to make sense of the Logic of the *Principles.*

The content of this doctrine of the 'univocity of being' is threefold. The doctrine consists in the following metaphysical claim, as Russell puts it: "[...] there is only one kind of being, namely *being* simpliciter"[225]. This absolute sense of being, Russell elaborates:

> [...] belongs to every conceivable term, to every possible object of thought—in short to everything that can possibly occur in any proposition, true or false, and

[221] Another important point which we may also briefly mention concerns the issue of definitions. Frege voiced to Peano the same concerns that Russell had over definitions: like Russell - but again, before him - Frege criticized Peano's liberal use of non-nominal or non-eliminative definitions, in particular conditional definitions, which formed the great majority of in Peano's *Formulario.*

[222] And, in fact, in all of Russell's works on Logic.

[223] RUSSELL, B., 1903, p.47 §51. Notice that when Russell asserts that "unless it [a proposition] happens to be linguistic", he means something like "unless the [a proposition] happens to be about words or phrases".

[224] Cf. RUSSELL, B., 1903, p.47 §51.

[225] RUSSELL, B., 1903, p.449 §427.

> to all such propositions themselves. Being belongs to whatever can be counted. [...] being is a general attribute of everything, and to mention anything is to show that it is.[226]

This doctrine was accompanied by a logical doctrine and also a 'semantical' thesis, for lack of a better word. The logical doctrine was Russell's doctrine of the 'unlimited' or 'unrestricted variable', i.e., the claim that strictly speaking, there are no *variables*, but only "the true or formal variable" or "*the* variable"[227] whose range is completely unrestricted: as Russell put it, whenever the variable occurs, "the terms dealt with are always *all* terms"[228]. The semantic doctrine is the claim that "[...] it is a characteristic of the terms of a proposition that any one of them may be replaced by any other entity without our ceasing to have a proposition"[229] - that is, for *every* term *b*, whenever there is a proposition *p* whose subject is some term *a*, there is a proposition p^* which differs from *p* solely from the fact that its subject is *b* instead of *a*; in effect, this doctrine boils down to the claim that every term can occur as the subject of every predicate and as a related term of every relation in a way that results in a proposition - i.e., if one can assert meaningfully (no matter if truly or falsely) that a number, say, one, is less than three, one may also assert meaningfully that Alexander the Great is less than three.

In holding this threefold doctrine, Russell significatly reframed Peano's calculus of propositions and its (not explicitly articulated) syntax. Implication for Russell was viewed as a relation that only actually holds between propositions but which can be meaningfully asserted to hold between any entities whatsoever. In other words, for any terms[230] *x* and *y*, one may meaningfully state that *x implies y* or that *y implies x* - even if they are not propositions - but this can only be true if *x* and *y* are propositions[231].

This gives rise to a fundamental feature of the Logic of the *Principles* that may seem strange to the modern reader. One of the hallmarks of modern standard predicate or quantificational logic which was (at least partially[232]) shared by Frege, namely, the employment of a formal language whose variables are 'sorted out' in the following sense: one distinguishes the ranges of propositional, predicate and individual variables in some way or another and imposes

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226 RUSSELL, B., 1903, p.449 §427.
227 RUSSELL, B., 1903, p.91 §88
228 RUSSELL, B., 1903, p.91 §88.
229 RUSSELL, B., 1903, p.45 §48.
230 The notion of *term* is used by Russell in the *Principles* as equivalent to *entity*. We shall discuss this notion at lenght in the next chapter.
231 It is plausible to suppose that Russell thought this to be a self-evident truth, since he offers no justification for it.
232 One sense that Frege's logic is certainly many sorted is that his object variables are sharply distinguished from functional variables. However, with respect to the propositional variables things are more complicated. As Alessandro Duarte has pointed out (unpublished manuscript, available here http://www.alessandroduarte.com.br/?page_id=454 and also here https://www.researchgate.net/publication/313524003_On_a_Problem_concerning_the_Rule_of_Substitution_for_Functions_i it seems that there is a problem with Frege's rule of Substitution in the *Begriffsschrift* which allows substitutions of non-judgeable contents for judgeable ones that result in ill-formed formulas. In case of Frege's later formulations of his *begriffsschrift*, things are also not straightforward given Frege's introduction of truth-values into the system. Thanks for Alessandro Duarte for calling these points to my attention.

grammatical limitations with respect to what sort of terms of the language[233] may or may not be substituted within a given complex expression[234]. In contrast, the variables of Russell's propositional calculus have an absolutely unrestricted range and in his axioms "[...] any conceivable entity may be substituted for any one of our variables without impairing the truth of our proposition"[235]. So, for instance, Russell could not have an axiom or theorem like "$p \supset p$" - for it would result false for any value of "p" that is not a proposition. That is why he actually *defined* a proposition as anything which implies itself[236].

As Russell explains, every primitive proposition or axiom of his calculus of propositions is an implication which has as an antecedent a clause "equivalent to the assertion that the letters which occur in the consequent are propositions"[237] and so "[...] all its propositions have as hypothesis and as consequent the assertion of a material implication"[238]. In a footnote Russell clarifies the status of his primitive propositions by explaining that "[...] the implications denoted by if and then, in these axioms, are formal, while those denoted by implies are material"[239]. Recall that the notion of formal implication was Peano's device for expressing generality, his formal rendition of the universal quantifier. So, for instance, Russell's first axiom of his calculus of propositions, which he informally introduces as asserting that "if *p* implies *q*, then *p* implies *q*; in other words, whatever *p* and *q* may be, "*p* implies *q*" is a proposition"[240] should be translated into symbolic language as something along the following lines:

$$p \supset q. \supset_{p.q} . p \supset q \text{ Pp}$$

where the variables *p*, *q*, etc., have an absolutely unrestricted range. Furthermore, it must be noted that one cannot 'fix' this peculiarity of Russell's Logic by saying that certain variables of the axioms only range over propositions, since the point of having them in this conditional form is to preserve the unrestricted character of the variables and to guarantee that the axioms are true no matter what sort of entities are assigned the variables[241]. This distinguishes Russell's calculus from modern formulations of propositional logic[242]: his axioms are quantified statements that assert something about *everything*, given that the variables have an absolutely unrestricted range.

233 That is, *linguistic terms* in the modern sense, and *not* in Russell's general sense of an *entity*.
234 Of course, the range of substitution of a given category of terms also determines whether such terms can occur significantly in other ways in composing complex expressions other than sentences, for instance, as arguments of certain functional expressions.
235 RUSSELL, B., 1903, p.7 §7.
236 RUSSELL, B., 1903, p.13 §14.
237 RUSSELL, B., 1903, p.13 §14. As we shall extensively discuss later, this is very misleading since, in the *Principles*, propositions are not letters, but are merely indicated by letters.
238 RUSSELL, B., 1903, p.13 §14.
239 RUSSELL, B., 1903, p.16 §18, footnote *.
240 RUSSELL, B., 1903, p.16 §18.
241 That is, in case they are not propositions the antecedents are false and thus the implications are true.
242 In effect, the propositional logic of the *Principles* can be understood as a sort of unique variant of what Alonzo Church calls the '*extended* propositional calculus' which allows propositional variables to be bound by quantifiers. For details, cf. CHURCH, A., 1956, p.151. The first formal treatment of such calculus was given by Russell in his 1906 article *On the Theory of Implication* (RUSSELL, B., 1906*c*); later investigations of such systems were developed by Tarski and Łukasiewicz (cf. TARSKI & ŁUKASIEWICZ, 1930 and CHURCH, A., 1956, p.151-2 for detailed historical information). This means, among other things, that the formal calculus implicit

2.2.3 Quantification, the Notion of a Logical Subject and Denoting

Russell's calculus for Logic is intended to have variables with an absolutely unrestricted range: this applies not only to his calculus of propositions but also to his calculus of classes and relations, and, in fact, to quantification in general. What, then, is the theory of quantifier of the *Principles*?

As many authors have previously noted, the notion of *formal implication* is expected to cover the legwork of the quantifiers in the *Principles of Mathematics*[243]. As we discussed, Peano understood formal implication as a relation that held between two propositions p and q containing real variables $x, ..., z$ whenever "q can be deduced from p, no matter what $x, ..., z$ are"[244]. Differently from Peano, who took this notion as a single primitive notion, Russell realized that the notion is "complex" and "[...] should therefore be separated into its constituents"[245].

The first step of this analysis consisted in noting that to assert a formal implication is to assert all the values (which are propositions) of what Russell called a *propositional function*, a notion which is akin to Frege's notion of a concept and which we'll further discuss in more detail. The relevant point, for now, is that Russell understood a formal implication in a very different way from Peano. For Russell, in the notation "$p \supset_{x,...,z} q$", where p and q contaning real (free) variables, "$p \supset q$" should not be viewed as a sentence expressing a complete proposition, but as an expression that expresses a proposition when all its free variables $x, ..., z$ are assigned determined values or when it is formally asserted that is, when *all* the propositions resulting from assigning a value to $x, ..., z$ are asserted[246]. Thus, Russell analyzed a formal implication as "[...] the assertion of a whole class of material implications"[247]. As Russell explains:

> Given any proposition (not a propositional function), let *a* be one of its terms, and let us call the proposition $\phi(a)$. Then in virtue of the primitive idea of a propositional function, if *x* be any term, we can consider the proposition $\phi(x)$, which arises from the substitution of *x* in place of *a*. We thus arrive at the class of all propositions $\phi(x)$. If all are true, $\phi(x)$ is asserted simply: $\phi(x)$ may then be called a *formal truth*. In a formal implication, $\phi(x)$, *for every value of x*, states an implication, and the assertion of $\phi(x)$ is the assertion of a class of implications, not of a single implication. If $\phi(x)$ is sometimes true, the values of *x* which make it true form a class, which is the class defined by $\phi(x)$: the class is said to exist in this case. If $\phi(x)$ is false for all values of *x*, the class defined by $\phi(x)$ is said not to exist [...][248]

in the *Principles* would not admit a schematic formulation, but would have to have a finite set of axioms and a rule of substitution. This is precisely the route Russell adopted in RUSSELL, B., 1906*c*.

243 And, in fact, as we shall see in the next section of this chapter, Russell defines Pure Mathematics in the *Principles* as the class of all (true) maximally general formal implications.

244 Peano, G., 1897, p.1.

245 RUSSELL, B., 1903, p.16 §18.

246 RUSSELL, B., 1903, p.38-9 §42. Of course, he had not yet fully realized (as Frege had) how to provide an adequate treatment of the notions of *all*, *any*, etc.

247 RUSSELL, B., 1903, p.28 §33.

248 RUSSELL, B., 1903, p.28 §33.

Russell went further, however, seeking an analysis of the very notion of *variation* or *the variable* in the *Principles* which differs drastically with respect to modern approaches and also with respect to Frege's analysis of generality. The contrast with Frege here is illuminating, and we shall use it to conduce our discussion

As we previously observed, following Dummett, we may understand the fundamental insight behind Frege's account of and notation for quantification theory as the idea that all complex expressions are constructed in stages from simpler ones. Such a construction is able to be carried out in several different ways. For example, in a standard language for predicate Logic, we may distinguish two cases of complex expressions which form linguistic units generated in stages from simpler expressions. First we have terms like "$f(x)$", "$f(a)$" and so on, which are formed by putting names or variables as arguments to functional expressions. These are contrasted to basic linguistic units that are formed by placing a singular term or proper name "a" or a variable "x" as an argument to a predicate "F", resulting in an atomic *sentence* like "$F(a)$" or *atomic open formula* "$F(x)$". The first case consists in the most basic kind of complex *terms* and the second the in the most basic kind of *well-formed formulas*[249]. Another different kind of composition comes from prefixing what we call open formulas with quantifiers in order to construct sentences containing logical generality[250]. The essential break with traditional logic that Frege effected consisted in showing that this latter kind of composition was completely different than assigning a predicate to a subject. The internal structure of general statements in Frege's conceptual notation is given by a grammar which has no counterparts for ordinary expressions like "all F's", "some F's" as *subjects* of a sentence.

Although there are important differences between Frege's logic of functions and modern predicate calculi, we may call a language which, in some way or another, embraces such structure a Fregean* Grammar[251]. The Russellian grammar - as put forward in the *Principles* - contrasts markedly with a Fregean* one. The contrast is best explained in terms of the logico-metaphysical distinctions that give rise to it. The most fundamental category of the Russellian grammar is that of a *term* in the sense of the *Principles*, i.e., an entity or being which can occur as a logical subject of a proposition. Among terms Russell distinguishes between *things* and concepts and roughly, the linguistic counterpart of these ontological categories are those of singular terms and predicates; being terms, however, both things *and* concepts can be logical subjects of propositions. In other words, from the perspective of Russellian grammar as presented in the *Principles* what distinguishes "a is red" from "red is a color" is that the former sentence expresses a proposition in which *red* occurs as concept - i.e., not as a the subject of the proposition -

[249] And, of course, iterations or combined iterations of these stages result in expressions of the same kinds but of increasing complexity, as in "$f(f(x)$" or "$F(fa)$", etc.

[250] Of course a different kind of composition of complex expressions in stages comes from taking sentences and iterating what we now call propositional connectives or operators to form new formulas.

[251] Frege's Logic was *not* a predicate calculus, but a logic of *terms*, so such a generic 'Fregean*' cannot strictly be attributed to Frege himself (cf, previous footnote 232 of this chapter and footnote 7 of chapter 3). What we are calling a Fregean* Grammar does, however, incorporate many of Frege's most important and characteristic loigical insights.

while in the latter it does occur as a term, i.e., as the subject of the proposition.

The analysis of quantification of the *Principles* which arises from taking as fundamental the notion of *a term occurring as a logical subject of a proposition* is drastically different from that of Frege. As Hylton points out, Russell's explanation of generality functions primarily in a "non-linguistic level"[252]: the main problem of explaining generality in the *Principles* is not that of providing an adequate symbolic apparatus for the quantifier, but rather that of reaching the ultimate *constituents* of propositions involving generality. Russell referred to this problem as 'The Problem of the Variable'. Upon his return from the Congress of Philosophy by 16 August 1900, Russell explained it to Moore as follows:

> Have you ever considered the meaning of *any*? I find it to be the fundamental problem of mathematical philosophy. E.g. "Any number is less by one than another number." Here *any number* cannot be a new concept, distinct from the particular numbers, for only these fulfil the above proposition. But can any number be an infinite disjunction? And if so, what is the ground for the proposition? The problem is the general one as to what is meant by any member of a defined class. I have tried many theories without success.[253]

Russell was puzzled as to what sort of subject of a proposition would be indicated by "Any number" or "All numbers"[254]. It cannot be the set of all numbers, for the proposition we are considering is not about a certain class, which (so Russell though at the time) is a single entity, but about each member of the class[255]. For, whenever someone asserts "every natural number is either odd or even", this statement is not meant to assert something about the set containing 1, 2, 3, ... nor about the concept of natural number but about the numbers themselves. As Russell puts it, "[...] it is only particular numbers that are odd or even; there is not, in addition to these, another entity, any number, which is either odd or even, and if there were, it is plain that it could not be odd and could not be even"[256].

In the *Principles*, the answer for this conundrum comes with Russell's notion of *denoting*, which is amusingly introduced in the following passage, where Russell writes:

> A concept denotes when, if it occurs in a proposition, the proposition is not about the concept, but about a term connected in a certain peculiar way with

[252] Cf. HYLTON, P., 1990, p.213. This, of course, is not to say that Russell did not recognize the (important) issues of concerning the rules and axioms of quantification theory or that he did not grasped the difference between formulating an adequate formalism and that of providing an analysis of general propositions (in a non-linguistic sense); Russell clearly grasped the difference between a formula or sentence and the proposition that sentence stood for.

[253] MOORE, G., 1993, p.181.

[254] As we shall discuss in some detail in the next chapter, Russell thought that each word of the sentence should stand for a constituent of a proposition.

[255] We are working, for the sake of discussion, with the naive assumption that there could be such a thing as the class of all numbers.

[256] RUSSELL, B., 1903, p.53 §56.

> the concept. If I say "I met a man", the proposition is not about a man: this is a concept which does not walk the streets, but lives in the shadowy limbo of the logic-books. What I met was a thing, not a concept, an actual man with a tailor and a bank-account or a public-house and a drunken wife.[257]

It is important to notice, again, that Russell is not speaking of *denoting* in the usual sense that *words* denote something, but he is talking about a relation which holds between the concept C and a term x when, whenever the concept occurs in a proposition, the logical subject of the proposition is not the concept *C* but the term *x*. As Russell puts it, he is not worried with 'denoting' in the sense that "we denote, when we point to or describe, or employ words as symbols for concepts"[258] but in 'denoting' as "[...] a logical relation between some concepts and some terms, in virtue of which such concepts inherently and logically denote such terms"[259].

Russell understood denoting concepts as derived from what he called "class-concepts"[260]. A class concept for Russell is a concept such that *x is an a* is a propositional function, i.e., when the result of the substitution of a term-name for "*x*" in "*x* is an *a*" results in a sentence that expresses a proposition. So Russell understood all quantifier-like expressions attached to class concept-words as standing for denoting concepts (as in "every number", "any number", "some number", etc.). Russell expressed this, again, with his usual carelessness about use and mention, as follows:

> When a class-concept, preceded by one of the six words *all*, *every*, *any*, *a*, *some*, *the*, *occurs* in a proposition , the proposition is, as a rule, not about the concept formed of the two words together, but about an object quite different from this, in general not a concept at all, but a term or complex of terms.[261]

Fortunately, he explains this more carefully (but still with his characteristic slips):

> Consider again the proposition "I met *a* man". It is quite certain, and is implied by this proposition, that what I met was an unambiguous perfectly definite man: in the technical language which is here adopted, the proposition is expressed by "I met some man". But the actual man whom I met forms no part of the proposition in question, and is not especially denoted by *some man*. Thus the concrete event which happened is not asserted in the proposition. What is asserted is merely that some one of a class of concrete events took place. The

[257] RUSSELL, B., 1903, p.53 §56.

[258] RUSSELL, B., 1903, p.53 §56.

[259] RUSSELL, B., 1903, p.53 §56. Another alternative would be to say that the proposition is about the class of numbers as *many* but according to the doctrine of the univocity of being, there is no such *entity*: whatever is a being or is a possible logical subject of a proposition must be a *single* entity.

[260] RUSSELL, B., 1903, p.74 §73.

[261] RUSSELL, B., 1903, p.64 §65. This passage could be fixed as follows: "When a class-concept [*word*], preceded by one of the six words *all*, *every*, *any*, *a*, *some*, *the*, *occurs* in a ~~*proposition*~~ [sentence], the proposition [indicated by the sentence] is, as a rule, not about the concept [denoted by the denoting-concept-expression] formed of the two words together, but about an object quite different from this, in general not a concept at all, but a term or complex of terms."

> whole human race is involved in my assertion: if any man who ever existed or will exist had not existed or been going to exist, the purport of my proposition would have been different. [262]

Thus, Russell thought that the constituent for which the expression "any number" stands for is a peculiar kind of concept, namely a denoting concept (sometimes also called a denoting complex). The peculiarity of a denoting concept is that despite being a constituent of a proposition like *Any number is less by one than another number*, the proposition is not about the denoting concept but about the totality (as many) of numbers[263]. This peculiarity, in turn, arises from the undefinable nature of the relation of *denoting*.

The very notion of variation or *the variable* or yet "the true formal variable"[264], as Russell also referred to it sometimes, is then identified with the denoting concept *any term*[265]. This analysis of generality amounts to introducing 'the variable' as a genuine constituent of propositions, namely, as Russell puts it, "the object denoted by any term"[266], which has "a kind of individuality"[267].

This individuality, according to Russell accounts for the apparently contradictory fact that although "[...] different variables may occur in a proposition, yet the object denoted by any term, one would suppose, is unique"; once the variable is viewed as this peculiar entity - namely the denoting concept *any term* - this is supposedly resolved because such constituent of a proposition "does not denote, properly speaking, an assemblage of terms, but denotes one term, only not one particular definite term"; thus, Russell concludes:

> [...] *any term* may denote different terms in different places. We may say: any term has some relation to any term; and this is quite a different proposition from: any term has some relation to itself.[268]

Appealing again to the notion of propositional function, Russell concluded:

> The individuality of variables appears to be thus explained. A variable is not *any term* simply, but any term as entering into a propositional function. We may say, if ϕx be a propositional function, that *x* is the term in any proposition of the class of propositions whose type is ϕx. It thus appears that, as regards propositional functions, the notions of class, of denoting, and of *any*, are fundamental, being presupposed in the symbolism employed. With this conclusion,

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[262] RUSSELL, B., 1903, p.62 §62.

[263] In fact, strictly speaking, the proposition is about anything whatsoever, given that *Any number is less by one than another number* is asserting that *for every value of x if x is a number then x is is less by one than another number*. The point is that what is asserted about something in the proposition, the concept of *odd number*, is not being asserted of the denoting concept, but of the things denoted by the concept.

[264] RUSSELL, B., 1903, p.91 §88.

[265] Cf. RUSSELL, B., 1903, p.91 §88.

[266] RUSSELL, B., 1903, p.94 §93.

[267] RUSSELL, B., 1903, p.94 §93.

[268] RUSSELL, B., 1903, p.94 §93.

> the analysis of formal implication, which has been one of the principal problems of Part I, is carried as far as I am able to carry it.[269]

As many authors have pointed out, this account of generality based on the theory of denoting concepts is certainly rich[270] and interesting, but also full of puzzles and difficulties[271]. From a technical standpoint this account does not seem to provide a tenable theory of the quantifier because it fails to provide an adequate account of denoting concepts that do not denote anything and because it fails to account for subtleties of scope of quantifier expressions.

Important details apart, the fundamental subjacent issue here is that Russell still could not let go - as Frege had already done - of ordinary grammar as a guide for analyzing generality (and in fact, for analyzing the structure of propositions in general). As Landini points out, Frege's *Begriffsschfrift*, in contrast to the *Principles*, "[...] allots no logical significance to the fact that 'all men' is the grammatical subject of 'All men are mortal' "[272].

This is another important aspect in which a Fregean* Grammar differs radically from that of Russell's in the *Principles.* A Fregean* Grammar embraces what Michael Dummett called the hierarchy of levels. Embracing this hierarchy of levels amounts to distinguishing sharply the syntatical role which functional symbols and predicates play in contrast to the role of proper names and singular terms. Indeed, if we follow Dummett, we may characterize a Fregean* Grammar, as one in which proper names and singular terms in general are complete linguistic units which may be used (within sentences) to *refer* to *objects*. Similarly, a sentence is a complete linguistic unit that may be used to *assert* something of a particular object or about any number of objects. These complete linguistic units contrast, however, with functional expressions, predicates and also with quantifiers which are *incomplete* expressions. As we mentioned very briefly, Frege elucidated this contrast in terms of the metaphor of 'saturation': a proper name or singular term, as Frege puts it, is *saturated* whereas a functional symbol "$f(...)$" or a predicate or concept symbol "$F(...)$" is *unsaturated.* The point of the metaphor is to convey what is the fundamental trait of a Fregean* Grammar: in order to make a contribution to the truth-conditions or meaning in

269 RUSSELL, B., 1903, p.94 §93.

270 For instance, Russell went to a great deal of effort in order to distinguish the workings of *all*, *every*, *any*, *a*, *some* and especially *the*, all of which he understood as denoting concepts (cf. RUSSELL, B., 1903, p.56 §59).

271 An example appears in the following passage, where Russell writes: "Great difficulties are associated with the null-class, and generally with the idea of *nothing*. It is plain that there is such a concept as nothing, and that in some sense nothing is something. In fact, the proposition "nothing is not nothing" is undoubtedly capable of an interpretation which makes it true [...]" (RUSSELL, B., 1903, p.73 §73). Indeed, nothing is more telling that something is amiss in terms of logical correctness than the attempt to turn "nothing" into a substantive (although, perhaps a case could be made for the attempt to turn "nothing" into a *verb* as a strong contender, but we shall not address the issue). The underlying problem here becomes even more clear in the subsequent explanation of the denoting concept *nothing*. Russell explains: "We may now reconsider the proposition "nothing is not nothing"— a proposition plainly true, and yet, unless carefully handled, a source of apparently hopeless antinomies. *Nothing* is a denoting concept, which denotes nothing. The concept which denotes is of course not nothing, i.e. it is not denoted by itself. The proposition which looks so paradoxical means no more than this: *Nothing*, the denoting concept, is not nothing, i.e. is not what itself denotes. But it by no means follows from this that there is an actual null-class: only the null class-concept and the null concept of a class are to be admitted." (RUSSELL, B., 1903, p.75 §73).

272 LANDINI, G., 1998, p.63.

a well-formed sentence, incomplete expressions must receive some supplementation or become saturated. One such way is to put proper names or singular terms as arguments in them, resulting in atomic sentences in the case of predicates or complex singular terms in the case of functional symbols. Frege's genius was to extend this idea of 'completion' or 'saturation' to the quantifier which he viewed as a 'higher-order' incomplete expression that required a predicate or concept expression in order to become saturated and to express a meaningful sentence. Frege's account of the quantifier and his analysis of generality relied on his modelling predication on functionality. Just like a functional expression like "$f(x)$" is assigned to a value i.e., a specific object according to which value is assigned to "*x*", a sentence like "$F(a)$" is assigned a *truth*-value (also an object) depending on whether the object *a* falls or not under the concept *F*. According to Frege's analysis as put forward in the *Grundgesetze*, the quantifier itself is also seen as an incomplete symbol a kind modelled upon the idea of functionality: the expression "$-\overset{\mathfrak{a}}{\smile}- X(\mathfrak{a})$" is assigned to a truth-value according to what obtains: the True when every object falls under the concept $X(...)$ and to the False otherwise.

Russell's most central doctrines precluded him from accepting Frege's explanation of predication via the notion of functionality. For Frege, the notion of a function was more fundamental than that of a property or a relation-in-intension and, as Landini puts it, for Russell "[...] to have a property or stand in a relation means to occur as a 'term of a proposition' predicating the property or relation" and Russell could simply not accept "[...] the notion that some entities are 'unsaturated' so that variables for them cannot occupy subject positions"[273]; thus he rejected Frege's notion of a function. As Hylton also nicely puts it, "[...] the idea of an entity, or a quasi-entity, which is not self-subsistent but is incomplete, is so alien to Russell's metaphysics at this time that he does not seem able to understand it, even as the idea of another"[274]. The very statement of Frege's doctrine of 'unsaturatedness' was contradictory for Russell as he repeatedly argues throughout the *Principles*[275]: Russell thought that any attempt to assert a sentence like "*F* cannot be a logical subject" turns *F* into a logical subject and this lead him to conclude, as Landini nicely puts it, that "its very statement is self-refuting"[276]. As Hylton rightfully claims, "here, if anywhere, we reach bedrock"[277].

Russell could not but analyze the notion of generality in terms of the fundamental idea of a term occurring as the subject of a proposition. This created a problem for which Russell could simply not provide an adequate solution: that of explaining what sort of subject we have in case of general propositions, in particular formal implications. At the bottom, the problem was that Russell could not accept anything like Frege's doctrine of incomplete or unsaturated entities and this precluded him from understanding quantifiers along the lines of a Fregean* Grammar. Russell had no alternative analysis, however, except one guided by ordinary grammar.

273 LANDINI, G., 1998, p.64.
274 HYLTON, P., 1990, p.220.
275 Cf. for instance, RUSSELL, B., 1903, pp.45-6 §49 or p.510 §483.
276 LANDINI, G., 1998, p.64.
277 HYLTON, P., 1990.

The solution of the *Principles* was given in terms of the notion of denoting concepts, in particular that of *any term*, which he identified with the absolutely unrestricted variable of Logic and Mathematics. A definite solution of the issue had to await Russell's new theory of denoting and of the variable, given in *On Denoting*.

2.2.4 Classes, Propositional Functions and Relations

Russell's calculus of classes, which is build upon the logic of propositions, assumes as primitive, besides the notion of *variable* that is presupposed in Logic as a whole, the three primitive notions of *membership of a term to a class*, of *propositional function* and *such that*, most of which are taken from Peano's logic, with some modifications. As previously noted, one of Russell's debts to Peano is the distinction between *membership* and *containment* together with the recognition of the conceptual priority of the former:

> The insistence on the distinction between ϵ and the relation of whole and part between classes is due to Peano, and is of very great importance to the whole technical development and the whole of the applications to mathematics. In the scholastic doctrine of the syllogism, and in all previous symbolic logic, the two relations are confounded, except in the work of Frege. The distinction is the same as that between the relation of individual to species and that of species to genus, between the relation of Socrates to the class of Greeks and the relation of Greeks to men.[278]

In the *Principles,* Russell adheres to Cantor's conception of a set as a "[...] collection *M* into a whole of definite, well-distinguished objects *m* of our intuition or thought" short of the apparent psychologism involved in it. Put in another formulation that Russell would fully accept, a class or set - in the Cantorion sense - is a "[...] many, which can be thought of as one, i.e., a totality of definite elements that can be combined into a whole by a law"[279]. We find this conception fully articulated in the *Principles* in the following passage:

> A class, we have seen, is neither a predicate nor a class-concept[280], for different predicates and different class-concepts may correspond to the same class. A class also, in one sense at least, is distinct from the whole composed of its terms, for the latter is only and essentially one, while the former, where it has many terms, is, as we shall see later, the very kind of object of which many is to be asserted. The distinction of a class as many from a class as a whole is often made by language: space and points, time and instants, the army and the

[278] RUSSELL, B., 1903, p.20 §20.

[279] CANTOR, G., We are borrowing Boolos's translation (BOOLOS, G., 1971, p.215). Detailed discussions of Russell's gradual acceptance of Cantor's ideas are found in RODRÍGUEZ-CONSUEGRA, F., 1991, GARCIADIEGO, A., 1992 and GRATTAN-GUINNESS, I., 2000b.

[280] This notion is a technical notion in the *Principles*, we explain it below.

> soldiers, the navy and the sailors, the Cabinet and the Cabinet Ministers, all illustrate the distinction.[281]

So called 'propositional functions' were already considered in Russell's aforementioned analysis of formal implication as the assertion of a class of propositions. In place of Peano's notion of a proposition p containing a real variable *x*, Russell has a function ϕx, whose values are propositions. Thus, in a proposition like *Socrates is a man*, if *Socrates* is replaced by a variable, we get *x is a man*, which is a propositional function[282].

As in the case of Russell's theory of generality, the notion of a propositional function plays an explanatory role in Russell's theory of classes, but it is important to observe, in continuity with our previous discussion of the notion of a logical subject and constituents of propositions, that Russell did *not* assume an ontology of propositional functions as he did an ontology of classes. This role is that of indicating what Russell calls a "constancy of form" of certain classes of propositions which Russell assumes as more fundamental than the notion of a *class* itself. This notion is derived, again, from the fundamental concept of a term occurring as a subject in a proposition. In the chapter on the notion of *variable*, he writes:

> When a term occurs as term in a proposition, that term may be replaced by any other while the remaining terms are unchanged. The class of propositions so obtained have what may be called constancy of form, and this constancy of form must be taken as a primitive idea. The notion of a class of propositions of a constant form is more fundamental than the general notion of a class, for the latter can be denned in terms of the former, but not the former in terms of the latter. Taking any term, a certain member of any class of propositions of constant form will contain that term.[283]

As Landini emphatically observes, however, there is no *ontology* of propositional functions in the *Principles*. Russell recognizes this explicitly in the following - often neglected - passage:

> According to the theory of propositional functions here advocated, the ϕ in ϕx is not a separate and distinguishable entity: it lives in the propositions of the form ϕx, and cannot survive analysis.[284]

Russell recognizes that this view may be in tension with the very argument he raises against Frege's notion of functions as incomplete entities which cannot be genuine logical subjects. But the reason for Russell's doubts against the recognition of propositional functions as entities is none other than the threat of contradiction:

281 RUSSELL, B., 1903, p.68 §70.
282 Again, observe that the expressions "occurring in a proposition", "substituted by a variable", etc., have *no linguistic connotations* here.
283 RUSSELL, B., 1903, p.89, §86.
284 RUSSELL, B., 1903, p.88, §85.

> If ϕ were a distinguishable entity, there would be a proposition asserting ϕ of itself, which we may denote by $\phi(\phi)$; there would also be a proposition not-$\phi(\phi)$, denying $\phi(\phi)$. In this proposition we may regard ϕ as variable; we thus obtain a prop- ositional function. The question arises: Can the assertion in this propositional function be asserted of itself? The assertion is non-assertibility of self, hence if it can be asserted of itself, it cannot, and if it cannot, it can. This contradiction is avoided by the recognition that the functional part of a propositional function is not an independent entity.[285]

As we shall also discuss in the next chapter, this difficulty - like those involved in Russell's account of the quantifier and of the nature of classes - will only be resolved with the new account of generality which he comes up in the article *On Denoting*.

The notion of *such that* is, Russell tells us "[...] roughly equivalent to *who* or *which*, and represents the general notion of satisfying a propositional function"[286]. Russell's reluctance to accept *such that* as a definable notion - as Peano did - is in accordance with the idea that there are propositions which are not reducible to subject-predicate form. And this, of course, is due to his acceptance of the reality of relations. As he explains, given a class of propositions of the form *xRb*: "We cannot reduce this [form of][287] proposition to the form "*x* is an *a* "' without using *such that*"[288]. The point here is closely related to Russell's rejection of Leibniz's, and Bradley's theories of judgment: if a proposition is relational, as in *bRa*, any attempt to reduce *R-to-a* to an attribute of *b* or any other object would, in some cases[289], presuppose the complexity involved in what is indicated by the expression "*has R to a*". This point re-appears with the definition of *such that*: in cases where one would want to specify a class of all *x* such that *xRa*, it is pointless, Russell tells us, to define the expression "all *x* such that *xRa*" in terms of membership to a class *a'*, as in "*x* is an *a* "' because "*a'* must be *such that* each of its terms, and no other terms, have the relation *R* to *a*"[290].

The notions of a *propositional function* and *such that* acquire their most basic significance and importance in connection with the notions of *class* and *class-concept*. A *class-concept* is a concept *a* such that "[...] '*x* is an *a*' is a propositional function"; so a class-concept is just a concept that, when asserted of an individual (or term), "asserts that an individual belongs [or not] to a class"[291]. Thus given that "ϕx is a propositional function if, for every value of *x*, ϕx is a proposition, determinate when *x* is given"[292], a class can be conceived as the totality of entities of which some given propositional function ϕx is true. In fact, Russell puts this forward as a possible definition of the notion of class - as he put it, "[..] a class may be defined as all

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285 RUSSELL, B., 1903, p.88, §85.

286 RUSSELL, B., 1903, p.83 §80.

287 This is surely a slip of Russell, since what contains a real variable cannot be a proposition, but a propositional function. Russell is clearly concerned here with the specification of a class of propositions through a propositional function.

288 RUSSELL, B., 1903, p.82 §80.

289 Most notably in propositions which deal with *asymmetrical* relations.

290 RUSSELL, B., 1903, p.82 §80.

291 RUSSELL, B., 1903, p.54 §57.

292 RUSSELL, B., 1903, p.19 §22.

the terms satisfying some propositional function"[293]. But of course, Russell does not accept this definition in virtue of the contradictions inherent to the naive conception of set the most famous of which he discovered himself.

As is well known, however, the suggestion as it stands is nothing but catastrophic, as Russell knew very well. Since Russell embraces Cantor's conception of set, the central tenet of the class theory of the *Principles* is the idea that "[...] classes must in general be regarded as *objects* denoted by concepts, and to this extent the point of view of intension is essential"[294]. By this, Russell means that despite the fact that "logically [...] extension and intension seem to be on a par", since an infinite mind could, in principle, determine an infinite class by enumeration of its members, in our case "Death would cut short our laudable endeavour before it had attained its goal"[295]. As Russell explains "[...] classes as many are the objects denoted by concepts of classes, which are the plurals of class-concepts"[296]. But before his shocking discovery, now referred to as 'Russell's Paradox', he "took it as axiomatic that the class as one is to be found wherever there is a class as many"[297]. This is precisely what engenders what he called *the* Contradiction:

> A class as one may be a term of itself as many. Thus the class of all classes is a class; the class of all the terms that are not men is not a man, and so on. Do all the classes that have this property form a class? If so, is it as one a member of itself as many or not? If it is, then it is one of the classes which, as ones, are not members of themselves as many, and vice-versâ.[298]

Russell clearly identified the source of the problem as the so-called 'naive' assumption that one can speak of any propositional function as determining a class *as one*:

> The reason that a contradiction emerges here is that we have taken it as an axiom that any propositional function containing only one variable is equivalent to asserting membership of a class defined by the propositional function. Either this axiom, or the principle that every class can be taken as one term, is plainly false, and there is no fundamental objection to dropping either. But having dropped the former, the question arises: Which propositional functions define classes which are single terms as well as many, and which do not?[299]

This difficulty is not solved in the *Principles*. Far from a Cantorion Paradise, the set theory of the work is more like Heaven during the Fall of Angels[300]. Nowhere in it we find a definitive answer as to how we can determine which concepts (or propositional functions) define

[293] RUSSELL, B., 1903, p.20 §23.
[294] RUSSELL, B., 1903, p.66 §66.
[295] RUSSELL, B., 1903, p.69 §71
[296] RUSSELL, B., 1903, p.106 §106.
[297] RUSSELL, B., 1903, p.104 §104.
[298] RUSSELL, B., 1903, p.102 §101.
[299] RUSSELL, B., 1903, p.102-3 §102.
[300] Revelation 12 : 7–10.

a class and which ones do not. There is no axiom in the calculus of classes that can be identified as what we nowadays call a *comphrehension principle*: a principle which tells what formulas of a formal language determine a set and thus, what sets *there are*. The reasons why Russell failed to find a solution will be discussed in the next chapter. For now we will review the main *positive* aspects of the *Principles* theory of classes.

Russell assumed two specific primitive propositions for the calculus of classes, namely:

(Comp.*) If *x* belongs to the class of terms satisfying a propositional function ϕx, then ϕx is true.[301]

(Ext.) If ϕx and ψx are equivalent propositions for all values of *x*, then the class of *x*'s such that ϕx is true is identical with the class of *x*'s such that ψx is true[302]

The first of these is clearly Russell's attempt to keep a class as the extension of a propositional function without saying anything as to which propositional functions determine the class: all that is entailed by the axiom is that *if* a class (as one) u is specified by a propositional function ϕx, *then* ϕx is true of elements of *u*, but nothing is said as to what propositional functions specify the class *u* (as one). So strictly speaking there is no actual axiom of comprehension for classes assumed in the work. The second axiom is clearly a version of extensionality: given that classes are the extensions of propositional functions (conceived, again, *as one*, not as many) the axiom asserts that two classes u and v are identical if they are defined by materially equivalent propositional functions. This axiom asserts that classes (*as one*) are identical when they have the same elements. Class-concepts on the other hand can have the same extension and be distinct[303], since they are intensional entities.

Russell also did not identify relations with as classes of ordered pairs (or trios, and so on) which was the approach taken not only by the Algebric tradition[304], but also by Peano. Again, Russell's approach is grounded in his Philosophy of relations:

301 RUSSELL, B., 1903, p.20 §24.
302 RUSSELL, B., 1903, p.20 §24.
303 RUSSELL, B., 1903, p.20 §24.
304 It is well known that Peirce anticipated in many ways the modern treatment of quantification theory and his 'calculus of relatives' is no exception. Peirce's method consisted in introducing relations as the sum of *relatives*, that is, as classes of ordered pairs. As he put it: "A general relative may be conceived as a logical aggregate of a number of such individual relatives. Let *l* denote "lover;" then we may write $l = \Sigma_i \Sigma_j (l)_{ij} (I : J)$ where $(l)_{ij}$ is a numerical coefficient, whose value is 1 in case *I* is a lover of *J*, and 0 in the opposite case, and where the sums are to be taken for all individuals in the universe." (PEIRCE, C., 1883, p.187). It is not surprising that almost halfway through the twentieth century we find Tarski claiming that "though the significance of the theory of relations is universally recognized [...] the calculus of relations, is now in practically the same stage of development as that in which it was forty-five years ago" (TARSKI, A., 1941, p.74), since the *essence* of the modern treatment of a (first-order) relation as the set of *n*-uples of indivduals from a specified domain *D* is already given in the above passage. And, in fact, a fundamental result in model theory, Löwenheim's theorem, which states that if a first-order formula is true in some infinite model, then it is true in a model of cardinality $\aleph_0$, was proved originally within the framework of the Peirce-Schröder calculus of relatives (LÖWENHEIM, L., 1915, p.235, §2, theorem 2); as is well known Lowenheim's result was further investigated and developed by

> Peirce and Schröder have realized the great importance of the subject, but unfortunately their methods, being based, not on Peano, but on the older Symbolic Logic derived (with modifications) from Boole, are so cumbrous and difficult that most of the applications which ought to be made are practically not feasible. In addition to the defects of the old Symbolic Logic, their method suffers technically [...] from the fact that they regard a relation essentially as a class of couples, thus requiring elaborate formulae of summation for dealing with single relations. This view is derived, I think, probably unconsciously, from a philosophical error: it has always been customary to suppose relational propositions less ultimate than class-propositions (or subject-predicate propositions, with which class-propositions are habitually confounded), and this has led to a desire to treat relations as a kind of class.[305]

To be sure, Russell recognized that for the purposes of Mathematics (and its logicist development) which only deals with extensions, nothing is required beyond the extensionality of classes in order to handle a calculus of relations. Russell writes:

> We may replace a relation *R* by the logical sum or product of the class of relations equivalent to *R*, i.e. by the assertion of some or of all such relations; and this is identical with the logical sum or product of the class of relations equivalent to *R'* , if *R'* be equivalent to *R*. Here we use the identity of two classes, which results from the primitive proposition as to identity of classes, to establish the identity of two relations.[306]

As we saw in the discussion of Russell's 1901 article, *Sur la Logique des Relations*, among the most important definitions of the calculus of relations are those of the *domain* of a relation, "the class of terms which have the relation *R* to some term or other", named in the *Principles* as the *referents* with respect to *R* and the *converse-domain*, "the class of terms to which some term has the relation *R*"[307], referred to as the *relata* with respect to *R.* Despite the fact that Russell repudiates the notion of an ordered couple as an entity, these two notions allow him to speak of the *extension* of a relation. This, however applies only to the development of Mathematics, for which only extensions matter. Russell assumes as more fundamental the *intensional* character of relations and so co-extensiveness - in the sense specified - does not entail *identity*:

> The intensional view of relations here advocated leads to the result that two relations may have the same extension without being identical. Two relations *R*, *R*' are said to be equal or equivalent, or to have the same extension, when *xRy* implies and is implied by *xR'y* for all values of *x* and *y*.[308]

Thoralf Skolem (cf. SKOLEM., T., 1920). For a historical study of this often neglected chapter of the history of Logic which is the development of the 'calculus of relatives', cf. BRADY, G., 2000.

305 RUSSELL, B., 1903, p.24 §27.

306 RUSSELL, B., 1903, p.24-25 §28.

307 RUSSELL, B., 1903, p.24 §28.

308 RUSSELL, B., 1903, p.24-5 §28.

From the point of view of philosophical analysis - that is, of determining *what*, after all, a relation *is*, Russell explicitly denies that one may define relations as classes of couples.

The fundamental issue Russell had with the Peirce-Schröder approach was the appeal to the notion of *ordered couple* in the characterization of relations: in a nutshell, Russell thought that there could be no such thing conceived as an *entity*[309]. As we had occasion to briefly consider, relations, for Russell, must be taken, as he put it, *as ultimate*, *non-analysable entities.* Russell's point is quite simple, and again, it takes us back to the rejection of the logical doctrine that every proposition could be analyzed as having the subject-predicate form or as asserting membership to a class: Russell thought that if a relation *R* is identified as a class of couples, the notion of couple should be either treated as a primitive idea or as a class - thus the *sense* or '*direction*' of *R* remains either unexplained or entirely lost. As Russell put it "it is necessary to give sense to the couple, to distinguish the *referent* from the *relatum*: thus a couple becomes essentially distinct from a class of two terms, and must itself be introduced as a primitive idea"[310]. Again, the point is established by considering propositions containing asymmetrical relations which cannot be analyzed in terms of subject-predicate form. Thus, Russell claims that the more correct procedure is "to take an intensional view of relations, and to *identify them rather with class-concepts than with classes*"[311], so that whenever *R* is a relation, *xRy* "express [...] the propositional function "*x* has the relation *R* to *y*""[312].

2.2.5 Rules of Inference and the 'Universality of Logic'

Before closing our discussion of the logic of the *Principles* we must consider how Russell's handles rules of inference in the work.

The issue appears when Russell introduces the fourth axiom of his calculus of propositions, which is, in fact, the rule of *modus ponens*. Russell claims that it "[...] is a principle incapable of formal symbolic statement" and that illustrates "the essential limitations of formalism"[313]. This is a point which marks an important and fruitful break with Peano, since here Russell is acknowledging - somewhat carelessly - two fundamental points:

1. The distinction between axioms and rules of inference;

309 He also thought his treatment was more convenient and more powerful as a purely mathematical *tool*, that is, independently of philosophical considerations (RUSSELL, B., 1903, p.24 §27).

310 RUSSELL, B., 1903, p.99 §98. Of course, around this time the Wiener-Kuratowski definition had not yet been discovered. We'll discuss it in connection with *Principia*'s theory of relations later on.

311 RUSSELL, B., 1903, p.99 §98.

312 RUSSELL, B., 1903, p.24 §27. This requires him to assume a primitive a proposition "to the effect that *xRy* is a proposition for all values of *x* and *y*". (RUSSELL, B., 1903, p.24 §27).

313 RUSSELL, B., 1903, p.16 §18.

2. The indispensability of rules formulated as such - and not merely as another axiom of formal law - for an adequate formulation of a formal system (and the possibility of providing rules in general).

Now, since much has been made of the alleged 'Universality' of Russell's Logic in the *Principles*, the second point is worth close attention.

The status of so called 'Axiom' 4 is further explained in the discussion of Lewis Carroll's 'paradox' as presented in *What the Tortoise said to Achilles*[314]. Russell's resolution of Carroll's puzzle is to distinguish the meaning of the two locutions "*p implies q*" and "*p therefore q*":

> The principles of inference which we accepted lead to the proposition that, if *p* and *q* be propositions, then *p* together with "*p* implies *q*" implies *q*. At first sight, it might be thought that this would enable us to assert *q* provided *p* is true and implies *q*. But the puzzle in question shows that this is not the case, and that, until we have some new principle, we shall only be led into an endless regress of more and more complicated implications, without ever arriving at the assertion of *q*. We need, in fact, the notion of therefore, which is quite different from the notion of implies, and holds between different entities.[315]

We can understand Russell's distinction here as one between *implication* and *entailment*, the difference being that each of these relations hold between different sorts of entities. The different entities in question are what Russell calls *asserted* and *not-asserted* propositions, where the word "assertion" is taken in an (unexplained) "non-psychological sense"[316].

Roughly, Russell's point is the following. Take the following two assertions, where brackets are inserted to distinguish different propositions which occur as constituents (i.e., terms related by the relation of implication)[317]:

(Implies) $\{p$ implies $q\}$ and $\{p\}$ *implies q*.

(Entails) $\{p$ implies $q\}$ and $\{p\}$ *entails q*.

Russell is claiming that the second has subordinate assertions while the first does not, and that this is what differentiates the *assertion of an implication* from an *inference*. The point is that to

[314] CARROLL., L., 1899. Recall that the problem was the following: Not-so-bright Achilles was attempting to deduce B as a conclusion from A implies B and A; a witty tortoise with socratic mannerisms convinces him that in order to obtain B from A implies B and A, he should add a premise to the effect that *if A implies B and A is true, one can deduce B*, leading poor Achilles into an infinite regress.

[315] RUSSELL, B., 1903, p.35 §38.

[316] RUSSELL, B., 1903, p.35 §38.

[317] Observe that there is no use-mention confusion here. Since propositions are not expressions, the use of quotation marks or even Quine's corner quotation marks would give us what we need, which is a mere notational indicator of *individuating* differently *used* (not mentioned) proposition-names or variables. We follow the notation adopted by Landini, who was particularly worried with expressions of the form "$\{p \supset q\} = q$", where dots can become very inconvenient (LANDINI, G., 1998, p.44, in particular footnote 2).

assert (Implies) is to assert that *if* {*p* implies *q*} *and* {*p*} *are true, then q is true*; while to assert (Entails) one must assert *{p implies q} is true and {p} is true, therefore q is true.* The difference, Russell as explains, is that:

> When we say *therefore*, we state a relation which can only hold between asserted propositions, and which thus differs from implication. Wherever *therefore* occurs, the hypothesis may be dropped, and the conclusion asserted by itself. This seems to be the first step in answering Lewis Carroll's puzzle.[318]

Landini claims that the above is "Russell's way of saying that an inference rule is metalinguistic and not itself an axiom among others"[319]. Although there are certainly are hints of such a view in Russell's discussion, it may be considered implausible and anachronistic to fully project this distinction in this clear-cut way into the logic of the *Principles*. This, however is not to say, as several authors have, following an interpretative tradition started by Jean van Heijenoort[320], that the distinction is antithetical to Russell's Logic.

In an excellent paper[321], Ian Proops scrutinized how the general and somewhat vague claim that "Russell's logic is universal" is often brought up in secondary literature as some sort of bundle of claims such as that Russell's logic is a "universally applicable theory"[322]; or that Russell's Logic is a "universal language"[323]; or Russell's Logic consists of "maximally general truths"[324]; or yet that Russell's Logic is "all-encompassing"[325]. More importantly, he sistematically showed how these claims are not always clearly distinguished and are sometimes entangled with others such as that "the range of the quantified variables in the laws of logic is not subject to change or restriction"[326] or that "Logic is not a system of signs that can be disinterpreted and for which alternative interpretations may be investigated"[327]. The point is very relevant, since from these vague characterizations, of course, many specific conclusions are drawn like the fact that "the question of the completeness of a system simply could not arise for him"[328] or still that "there is no room for what we would call metatheoretic considerations about logic"[329]. His systematic critical discussion of the so-called 'Universalist interpretation' presents *definitive* textual evidence, however, to the effect that many of these views cannot be attributed to Russell without

318 RUSSELL, B., 1903, p.35 §38.
319 LANDINI, G., 1998, p.45.
320 van HEIJENOORT, J., 1967*b*.
321 PROOPS, I., 2007.
322 URQUHART, A., 1988, p.83.
323 HYLTON, P., 1990, p.200.
324 RICKETTS, T., 1996, p.59.
325 GOLDFARB, W., 1989, p.27.
326 GOLDFARB. W., 1989, p.27
327 GOLDFARB. W., 1989, p.27
328 HYLTON, P., 1990, p.202.
329 GOLDFARB, W., 1989, p.27.

some serious qualifications or simply cannot be attributed to him at all[330]. The general lesson Ian Proops gives is that this reading attributes "a general blindness on Russell's part to the distinction between logic itself and one of the logical systems designed to capture it"[331]; a distinction which Russell was well aware that could be made, as Landini and Ian Proops showed.

Taking into consideration the particular case of Russell's 'axiom' 4, the inference rule of *Modus Ponens*, how does such 'universalist' reading fits with Russell's discussion? Goldfarb's central claims are that Russell's Logic consists of "all-encompassing principles of correct reasoning" and that "anything that can be communicated must lie within it"[332]. How does this fit with Russell's claim that the notion of *entailment*, generally marked by a "*therefore*" locution "is a principle incapable of formal symbolic statement"[333]? When Russell asserts this, he seems to be, if not plainly denying Goldfarb's idea that for Russell Logic "is the framework inside of which all rational discourse proceeds"[334], at least saying that such a claim must be somewhat weakened. Perhaps a 'universalist' could claim in response that the laws of Logic are not exhausted by what can be expressed by the formalism of Logic; but that goes against some of the claims they accept - for instance, that "Logic, for Russell, was a universal language, a *Lingua characteristica*" that is "universal and all-inclusive"[335]

Is that to say that there is a strict distinction between theory and meta-theory involved in Russell's discussion of *modus ponens* and Carroll's puzzle? No. But it makes it plausible that some embryonic form of the distinction was there mixed with some confused ideas of Russell, despite the claims of van Heijenoort's followers. This indicates that Russell accepted:

1. No formal symbolic expression of a given formal system can capture what is characteristic of *inference* within that formal system, *i.e.*, that which is described by Russell in terms of *therefore*, or what we called *entailment*; and

2. That there are *are propositions which can capture a rule of inference*.

In fact, since Russell was obviously aware of the possibility of formulating different formal systems with different primitive notions axioms and rules, it is absolutely plausible to suppose that Russell thought that "the limitations of formalism" are, in a sense relative to *a given* specific formalism. Unless we are prone to admit an incoherence on Russell's part or, worse yet, to project some *Tractarian* doctrine that somethings can only be shown but not said into the *Principles*, it

330 The first systematic critique of the claim that Russell's conception of Logic precludes the possibility of a meta-theoretical considerations was presented by Landini in his book on Russell's Substitutional Theory (LANDINI, G., 1998, pp.30-41), and, in fact, many of his points are reiterated by Proops, albeit in more detail.

331 PROOPS, I., 2007, p.19.

332 GOLDFARB, W., 1989, p.27.

333 RUSSELL, B., 1903, p.16 §18.

334 GOLDFARB., W., 1989, p.27.

335 HYLTON, P., 1990, p.200.

seems that such general claims like those made by Hylton and Goldfarb should be mitigated in the face of such difficulties[336].

2.2.6 Summary of the Formal System Underlying the *Principles of Mathematics*

In the *Principles* Russell assumes ten axioms for his calculus of propositions, as follows[337]: (1) If *p* implies *q*, then *p* implies *q*; in other words, whatever *p* and *q* may be, "*p* implies *q*" is a proposition; (2) If *p* implies *q*, then *p* implies *p*; in other words, whatever implies anything is a proposition; (3) If *p* implies *q*, then *q* implies *q*; in other words, whatever is implied by anything is a proposition; (4) A true hypothesis in an implication may be dropped, and the consequent asserted; (5) If *p* implies *p* and *q* implies *q*, then *pq* implies *p*; (6) If *p* implies *q* and *q* implies *r*, then *p* implies *r*; (7) If *q* implies *q* and *r* implies *r*, and if p implies that *q* implies *r*, then pq implies *r*; (8) If *p* implies *p* and *q* implies *q*, then, if *pq* implies *r*, then *p* implies that *q* implies *r*; (9) If *p* implies *q* and *p* implies *r*, then *p* implies *qr*; (10) If *p* implies *p* and *q* implies

[336] It is worth noting that Russell's views on inference in the *Principles* (and elsewhere) are tied to his views on assertion, which are somewhat problematic. There is, in particular, a lack of clarity concerning the notion of *assertion* in the "non-psychological" - and unexplained - sense in which Russell employs the notion in the *Principles*. Russell claims that: "It is plain that, if I may be allowed to use the word assertion, in a non-psychological sense, the proposition '*p* implies *q*' asserts an implication, though it does not assert *p* or *q*. The *p* and the *q* which enter into this proposition are not strictly the same as the *p* or the *q* which are separate propositions, at least, if they are true." (RUSSELL, B., 1903, p.35 §38). Naturally, Russell raises the question of what makes a proposition "differ by being actually true from what it would be *as an entity* if it were not true?" (RUSSELL, B., 1903, p.35 §38 our emphasis) but he simply admits he does not have an answer. He explains that "[...] there are grave difficulties in forming a consistent theory on this point, for if assertion in any way changed a proposition, no proposition which can possibly in any context be unasserted could be true, since when asserted it would become a different proposition. But this is plainly false; for in "*p* implies *q*", *p* and *q* are not asserted, and yet they may be true" (RUSSELL, B., 1903, p.35 §38). Yet, as we saw, the distinction is essential to his resolution of Lewis Carroll's paradox. The problem Russell is struggling with here is one Frege had already dealt with - as Russell himself admits (RUSSELL, B., 1903, p.502 §477) - in a much more subtle way, namely that of explaining how the word "true" can be seemingly "devoid of content" yet it "cannot be dispensed with" (FREGE, G., 1915, p.252). Frege's irreproachable solution is to "distinguish the judgment from the thought" so that the judgment is just "the acknowledgment of the truth of a thought" (FREGE, G., 1893, p.9). Frege, of course, introduces a special symbol for this acknowledgment of the truth of a thought, his *judgment-stroke* "|", which, when combined with his horizontal bar, yields the now familiar "⊢", that, when attached to a sentence, expresses the acknowledgment that the thought expressed by it is true. As is well known, Russell later adopts this symbol. Russell took issue with Frege's view. According to to him: "[...] a difficulty arises owing to the apparent fact, which may however be doubted, that an asserted proposition can never be part of another proposition: thus, if this be a fact, where any statement is made about *p* asserted, it is not really about p asserted, but only about the assertion of *p*. This difficulty becomes serious in the case of Frege's one and only principle of inference (Bs. p. 9): "*p* is true and *p* implies *q*; therefore *q* is true". Here it is quite essential that there should be three actual assertions, otherwise the assertion of propositions deduced from asserted premisses would be impossible; yet the three assertions together form one proposition, whose unity is shown by the word *therefore*, without which *q* would not have been deduced, but would have been asserted as a fresh premiss." (RUSSELL, B., 1903, p.504 §478) Perhaps the root of all evil here is this: Russell is convinced that, despite the fact "that assertion does not seem to be a constituent of an asserted proposition", the assertoric force must be "in some sense, contained in an asserted proposition" (RUSSELL, B., 1903, p.504 §478) and this is why Russell is reluctanct to, as he puts it, "divorce assertion from truth, as Frege does" (RUSSELL, B., 1903, p.504 §478).

[337] RUSSELL, B., 1903, p.16 §18.

q, then "'*p* implies *q*' implies *p*" implies *p*. In symbols, the can be put as follows[338]:

(1) $p \supset q \,.\, \supset_{p.q} .\, p \supset q$

(2) $p \supset q \,.\, \supset_{p.q} .\, p \supset p$

(3) $p \supset q \,.\, \supset_{p.q} .\, q \supset q$

(4) -

(5) $p \supset p \,:.\, \supset_{p.q} \,:.\, q \supset q : \supset .\, pq \supset p$

(6) $p \supset q : \supset_{p,q,r} : q \supset r \,.\, \supset .\, p \supset r$

(7) $p \supset q : \supset_{p,q,r} : p \supset r \,.\, \supset .\, p \supset qr$

(8) $q \supset q :: \supset_{p,q,r} :: r \supset r \,:.\, \supset \,:.\, p \supset : q \supset r \,.\, \supset .\, pq \supset r$

(9) $p \supset p :: \supset_{p,q,r} :: q \supset q \,:.\, \supset \,:.\, pq \supset r : \supset : p \,.\, \supset .\, q \supset r$

(10) $p \supset p :: \supset_{p,q} :: q \supset q \,:.\, \supset \,:.\, p \supset q \,.\, \supset p : \supset p$

Given Russell's doctrine of the unrestricted variable, some interpretative issues arise in connection with respect to Russell's definitions of the logical connectives[339], but we may charitably read his definitions as follows:

$$pq = r \supset r \,:.\, \supset_r \,:.\, p \,.\, \supset .\, q \supset_r r : \supset : r$$

$$p \vee q \,. =: p \supset q \,.\, \supset . q$$

$$\sim p \,. =: r \supset r \,.\, \supset_r .\, p \supset r$$

[338] This formulation is borrowed from LANDINI, G., 1998, p.44, except for the use of letters *p*, *q*, *r*, etc., to emphasize that these are *not* special propositional variables that is, that they are unrestricted variables ranging over all terms. Also Landini observes (LANDINI, G., 1998, p.44, footnote 2) modern reconstructions of the Russellian propositional Logic of the *Principles* like those of Church (CHURCH, A., 1984) and Anderson (ANDERSON, A., 1986 and 1989) generally introduce special propositional variables, something that not only goes against the most distinctive feature of the *Principles* Logic, its unrestricted variables, but also demands the introduction of a special notation for propositional identity. Landini has also recently shows that this system requires some modifications in order to solve what he calls the "conjunction problem" (cf. LANDINI, G., 2020).

[339] Cf. BYRD, M., 1989.

None of the interpretative issues with the connectives is very serious[340], except one concerning negation[341]. Be that as it may, what is of real importance is that the above axioms and definitions must be read in accordance with Russell's doctrine of propositions: every propositional variable and expression is a *term*, or as Landini puts it, it is a "[...] grammatical rule that for any wff *A*,

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[340] For instance, Russell defined *conjunction* (logical *product*) as follows: "If *p* implies *p*, then, if *q* implies *q*, *pq* (the logical product of *p* and *q*) means that if *p* implies that *q* implies *r*, then *r* is true. In other words, if *p* and *q* are propositions, then their joint assertion is equivalent to saying that every proposition is true which is such that the first implies that the second implies it." (RUSSELL, B., 1903, p.16 §18). The problem is that his seems to define conjunction only for propositions - having the look of a conditional definition in the sense of Peano. Symbolically, Russell's original definition seems to be the following:

$$p \supset p \,.\supset.\, q \supset q :\supset:\!.\, pq :=: p \,.\supset.\, q \supset_r r :\supset: r$$

But as Michael Byrd observed, this is at odds with Russell's Simplification Axiom - that which asserts "If *p* implies *p* and *q* implies *q*, then *pq* implies *p*" - for since *p* and *q* can be *any* terms, then "the component "*pq*" must have an interpretation when *p* and *q* are not propositions" (BYRD, M., 1989, p.351). Despite the apparent nonsensicality of it, a definition of conjunction should not exclude a conjunction of two terms that are not propositions like *Russell and Whitehead*: if one can speak (albeit always falsely) of Russell *implying* Whitehead, why should the conjunction be considered nonsensical? The above also faces the following problem: even if both *p* and *q* are *true propositions*, *pq* would be false if *r* is not a proposition, for in that case $p \supset q. \supset .r$ would be false! Byrd fixes the slip by adopting the same definition we are attributing to Russell above, since, as Byrd claims, it the simplest solution, since it allows for the conjunction of *non*-propositional terms and keeps *pq* vacuously true for non-propositional *r*'s. Landini observes that Russell's Simplification Axiom avoids the inconvenient question of whether one could derive *p* or *q* from *pq* in case either one of these is a *non*-propositional term, since in order for *pq* to imply *p* or for *pq* to imply *q*, *p* or *q* must be propositions in each case (LANDINI, G., 1998, p.49.). A similar issue arises with disjunction. Russell has: "[...] "*p* or q" is equivalent to "'*p* implies *q*' implies *q*". It is easy to persuade ourselves of this equivalence, by remembering that a false proposition implies every other; for if *p* is false, *p* does imply *q*, and therefore, if "*p* implies *q*" implies *q*, it follows that q is true." (RUSSELL, B., 1903, p.17 §18). Byrd reads this (charitably) as:

$$p \vee q \,.=: p \supset q \,.\supset.\, q$$

Which gives us a non-conditional definition where *p* and *q* can be any terms. As Landini points out, however, Russell's subsequent remark suggests again a restriction to propositions and, again, we can take this as a slip. We have, however, to live with an asymmetry pointed out by Byrd: if *q* is not a proposition then $p \vee q$ is false; but if *q* is a proposition and *p* is not, then $p \vee q$ may be a true proposition (BYRD, M., 1989, p.351).

[341] Russell defines negation as follows: "[...] not-*p* is equivalent to the assertion that *p* implies all propositions, i.e. that "*r* implies *r*" implies "*p* implies *r*" whatever *r* may be. (RUSSELL, B., 1903, p.18 §18)". Once again, Landini thinks that Russell intends a conditional definition since he comments on a footnote that "[...] the principle that false propositions imply all propositions solves Lewis Carroll's logical paradox"(RUSSELL, B., 1903, p.18). Byrd, on the other hand, puts this as:

$$\sim p \,.=: r \supset r \,.\supset_r.\, p \supset r$$

The serious point of dispute, however, is not about this which we may take as a minor slip on Russell's part. As Byrd observes, Russell's definition of negation "depends on assumptions about existence" that require the use of negation as a primitive notion to be stated: the definition presupposes that *there are false* propositions, something which can only be expressed by asserting that there is a proposition *p* such that *p* is false, which is the same as to say that *p* is *not* true or that *p* is *not* the case (BYRD, M., 1989, p.355). Landini claims that if we accept that it is a rule of the grammar of logic that nominalized formulas are singular terms, then the existence of propositions - false or not - would be "a logical assumption" (LANDINI, G., 1998, p.53, footnote 7). There are reasons to question Landini on this point, however. In *The Theory of Implication*, where Russell offers systematic symbolic treatment of propositional logic and quantification theory, he drops the definition of negation of the *Principles* and adopts it as primitive, but he also considers the definition of the *Principles* in the

$\ulcorner\{A\}\urcorner$ is a singular term"[342].

As noted above, Russell did not have a clear account of the notion of class and so is hesitant with respect to the assumption of primitive propositions, in particular a principle of comprehension. Building up on the numbering of the axioms for propositional logic, Russell further introduced: (11) If *x* belongs to the class of terms satisfying a propositional function ϕx, then ϕx is true[343]; and (12) If ϕx and ψx are equivalent propositions for all values of *x*, then the class of *x*'s such that ϕx is true is identical with the class of *x*'s such that ψx is true[344]. Which we may put in symbols as follows:

(11) $\phi x \supset_x x \epsilon \hat{z}\phi z$

(12) $\phi x \equiv_x \psi x : \supset . \hat{z}\phi z = \hat{z}\psi z$

Both of these fail in characterizing an adequate concept of *class*, however, because at no point Russell provides an adequate or precise account of what sort of conditions ϕ are admissible or not.

The Calculus of relations further assumes the following propositions[345]: (13) if *R* is a relation, then *xRy* is a proposition for all values of *x* and *y*; (14) Every relation has a converse, i.e. that, if *R* be any relation, there is a relation *R'* such that *xRy* is equivalent to *yR'x* for all values of *x* and *y*; (15) Between any two terms there is a relation not holding between any two other terms; (16) The negation of a relation is a relation; (17) The logical product of a class of relations (i.e. the assertion of all of them simultaneously) is a relation; (18) The relative product of two relations must be a relation[346]; (19) Material implication is a relation; (20) The relation of a term to a class to which it belongs is a relation. Differently from the calculus of propositions

article. The reason why he drops the older definition, however, is "the fact that it never enables us to know that anything whatever is false" (RUSSELL, B., 1906*a*, pp.60). Similarly, in a letter to Couturat 23 July 1905, Russell explains that "the reason that he abandoned '$p .=: p . \supset . (r) . r$'" was that he realized "one needs the idea of falsehood, and as soon as one defines the negation of a proposition, one can well say that the negation of *p* is true, but not that p is false" (MOORE, G., 2014, p.19). This seems precisely the point raised by Byrd. Landini is clearly right about the logical *status* of the assumption of propositions (regarding both the *Principles* and the *Theory of Implication*). But the above passages shows that despite the fact that the existence of propositions (true or false) is a matter of Logic, the matter which seems to concern Russell here is *how one can construct a logical vocabulary in which this (granted, logical) assumption can be stated*. Perhaps we may say the following: Russell did understand the existence of false propositions as a logical matter but still struggled over what is the correct method of *expressing* that a proposition is false and to *prove* that there are false propositions. Russell seems to have thought later on that the *Principles'*s system is inadequate regarding this latter task. This seems in agreement with both Byrd's point and Landini's view, since the latter emphasizes in several occasions that Russell did distinguish Logic as a science and the formal system that is supposed to capture it (LANDINI, G., 1996, p.556; LANDINI, G., 1998, p.34).

342 LANDINI, G., 1998, p.53, footnote 7.

343 RUSSELL, B., 1903, p.20 §24.

344 RUSSELL, B., 1903, p.20 §24.

345 RUSSELL, B., 1903, p.25 §28.

346 Where the relative product of two relations *R*, *S* is the relation which holds between *x* and *z* whenever there is a term *y* to which *x* has the relation *R* and which has to *z* the relation *S*

and that of classes for which, by the time of the *Principles*, Russell did not yet have a symbolic treatment of his own to offer, his calculus of relations had already been presented in some detail in his 1901 article, which was incorporated practically unaltered in the *Principles*. In the article we find the following axioms ∗1·1, ∗1 · 7, ∗1·18, ∗2·15, ∗1·194·195, ∗2 · 1 · 11 and ∗3·1 which correspond, respectively, to[347]:

(13) $R \,\epsilon\, \mathrm{Rel} \supset : xRy \,.\!=\!.\, x$ has the relation R to y

(14) $R \,\epsilon\, \mathrm{Rel} \,.\supset.\, \exists \mathrm{Rel} \cap R' \ni (xR'y \,.\!=\!.\, yRx)$

(15) $\exists R \cap R \ni (\rho = \iota x \,.\, \breve{\rho} = \iota y)$

(16) $\sim R \,\epsilon\, \mathrm{Rel}$

(17) $K \,\epsilon\, \mathrm{Cls'Rel} \,.\supset.\, \cup{}'K = \iota R \ni \{xRy \,.\!=\, \exists K \cap R' \ni (xR'y)\} :\supset.\, \cup{}'K \,\epsilon\, \mathrm{Rel}$

(18) $R, S \,\epsilon\, \mathrm{Rel} \,.\supset: xRSz \,.\!=\!.\, \exists y \ni (xRy \,.\, yRz) \supset :.\, xRSz \,\epsilon\, \mathrm{Rel}$

(20) $\epsilon \,\epsilon\, \mathrm{Rel}$

The only remaining axiom can be then put as:

(19) $\supset \,\epsilon\, \mathrm{Rel}$

2.3 The Logicism of the *Principles of Mathematics*

2.3.1 Russell's Definition of Pure Mathematics

In the very opening *Principles*, Russell puts forward the following famous - and, admittedly, "somewhat unusual" - definition of Pure Mathematics:

> Pure Mathematics is the class of all propositions of the form "*p* implies *q*", where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: *implication*, *the relation of a term to a class* of which it is a member, the notion of *such that*, the notion of *relation* and such further notions as may be involved in the general notion of propositions of the above form.[348]

[347] RUSSELL, B., 1901*c*, p.316-8.
[348] RUSSELL, B., 1903, p.1 §1.

What is most important in the above passage, of course, is the statement of Russell's logicist thesis: that the propositions of Pure Mathematics are propositions of Logic, with logical propositions characterized as those which have as constituents only logical constants.

As is well known, Russell would later repudiate this definition in the preface to the *Principles*'s second edition on the grounds that there seem to be propositions that satisfy it and "[...] yet may be incapable of logical or mathematical proof or disproof"[349]. His examples include his Multiplicative Axiom, which is perhaps the most famous equivalent of Zermelo's Axiom of Choice and the so-called Axiom of Infinity of *Principia Mathematica*. As Russell explains, his later position is that "[...] the absence of non-logical constants, though a necessary condition for the mathematical character of a proposition, is not a sufficient condition"[350].

Russell also later repudiated the emphasis on *conditional* propositions in the *Principles*'s definition of Pure Mathematics, i.e., the idea that every proposition of Pure Mathematics could be *analyzed* as an implication. As Russell himself observes, he was "[...] originally led to emphasize this form by the consideration of Geometry"[351]. The point in question was precisely the problem which led Russell to write the *Essay on The Foundations of Geometry*, namely how to account for the existence of both Euclidean and non-Euclidean geometries as branches of pure Mathematics. After he abandoned his project of elaborating a 'complete dialectic' of the sciences as conceived in the *Essay*, Russell accepted as completely unproblematic the possibility of formulating different and incompatible geometries based on different conceptions of space: to put it simply, after Russell abandoned the Kantian conception of Geometry as grounded on some form of pure intuition of space, the question of which sort of Geometry is the 'correct' one acquired the status of an *empirical problem.* In his entry for non-Euclidean Geometry for the *Encyclopedia Britannica*, Russell explained the point very clearly:

> [...] there are a number of possible Geometries, each of which may be developed deductively with no appeal to actual facts. But no one of them, *per se*, throws any light on the nature of our space. Thus geometrical reasoning is assimilated to the reasoning of pure mathematics, while the investigation of actual space, on the contrary, is found to resemble all other empirical investigations as to what exists. There is thus a complete divorce between Geometry and the study of actual space. Geometry does not give us certain knowledge as to what exists. That peculiar position which Geometry formerly appeared to occupy, as an *a priori* science giving knowledge of something actual, now appears to have been erroneous. It points out a whole series of possibilities, each of which contains a whole system of connected propositions; but it throws no more light upon the nature of our space than arithmetic throws upon the population of Great Britain.[352]

349 RUSSELL, B., 1937, p.viii.

350 RUSSELL, B., 1903, p.vii. Explaining in some detail how this change came about will be the task of the next two chapters, where we'll adress in some detail Russell's discovery of the Contradiction, the Theory of Definite Descriptions and the several versions of the Theory of Types which he developed over a whole decade.

351 RUSSELL, B., 1937, p.vii.

352 RUSSELL, B., 1902, p.503.

Geometry - be it Euclidean or not - is a branch of Pure Mathematics inasmuch as it consists in implications between an axiom or set of axioms which describe *some consistent* conception of space and some of its logical consequences. The inquiry concerning *which* consistent conception of space is the correct one, or, equivalently, the problem of determining which set of geometrical axioms is the one that describes the properties of *actual space* does not belong to pure Mathematics, but to *applied* Mathematics. This point, Russell later claimed, made him "[...] lay undue stress on implication, which is only one among truth-functions, and no more important than the others"[353].

Another issue with the opening definition of the *Principles* which Russell himself calls attention to at some points is that it is inadequate to say that he takes the propositions of Geometry (and pure Mathematics in general) to assert implications between two *propositions p* and *q*. Since he claims that a distinctive characteristic of logical (and mathematical) propositions is that they contain *variables*, what Russell intends is to define propositions of pure Mathematics as *formal implications.* So the assertion that "Pure Mathematics is the class of all propositions of the form "*p* implies *q*", where *p* and *q* are propositions"[354] should be read as something along the following lines:

(L_{PoM}) "Pure Mathematics is the class of all propositions *r* of the form "*p* implies *q*", where *p* and *q* are *propositional functions* containing any number of variables x, ..., z and *r* asserts that *p* implies *q* is true for all the values of x, ..., z".

Russell corrects this in the preface to the second edition of the *Principles*, observing that "[...] when it is said that '*p* and q are propositions containing one or more variables', it would, of course, be more correct to say that they are propositional functions"[355].

Indeed, this conception of logical propositions was already present in the 1901 popular article on the Foundations of Mathematics which Rodríguez-Consuegra rightly called Russell's logicist *manifesto*. We find it in the following famous passage:

> Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that if one proposition is true, then so is some other proposition. These rules of inference constitute the major part of the principles of formal logic. We then

353 RUSSELL, B., 1937, p.vii.

354 RUSSELL, B., 1903, p.1 §1.

355 RUSSELL, B., 1937, p.vii. He rightfully observes, however, that the slip should "may be excused on the ground that propositional functions had not yet been defined, and were not yet familiar to logicians or mathematicians".

> take any hypothesis that seems amusing, and deduce its consequences. If our hypothesis is about anything, and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.[356]

And, in fact, Russell also fully articulated it in the *Principles*:

> We assert always in mathematics that if a certain assertion *p* is true of any entity *x*, or of any set of entities *x*, *y*, *z*, . . ., then some other assertion *q* is true of those entities; but we do not assert either *p* or *q* separately of our entities. We assert a relation between the assertions *p* and *q*, which I shall call formal implication.[357]
>
> The typical proposition of mathematics is of the form " $\phi(x, y, z, ...)$ implies $\psi(x, y, z, ...)$, whatever values $x, y, z, ...$ may have"; where $\phi(x, y, z, ...)$ and $\psi(x, y, z, ...)$, for every set of values of $x, y, z, ...$, are propositions. It is not asserted that ϕ is always true, nor yet that ψ is always true, but merely that, in all cases, when ϕ is false as much as when ϕ is true, ψ follows from it.[358]

These passages are also relevant in connection with the frequently made claim that for Russell, Logic is the body of "[...] maximally general truths"[359]. To be sure, some version of this view must be attributed to Russell, since he claims, for instance that "[...] so long as any term in our proposition can be turned into a variable, our proposition can be generalized [...] and so long as this is possible, it is the business of mathematics to do it"[360]. But as Ian Proops urges, Russell *should not* be read as defending that "[...] logic counts as a body of maximally general truths because its propositions are in some sense maximally *generalized*"[361].

Rather, the point should be understood in the light of the idea that there can't be any constants beyond logical ones in the propositions of Logic. This is clearly illustrated by one of Russell's favorite examples which also appears in his posterior discussions about the nature of logical truths and logical constants[362]. Take the following:

(Syl) If every man is mortal and Socrates is a man, then Socrates is mortal.

As Russell understood the idea of "[...] turning its terms into variables", in this particular case what we would get is the following[363]:

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356 RUSSELL, B., 1901, p.366.
357 RUSSELL, B., 1903, p.5 §5.
358 RUSSELL, B., 1903, p.6 §6.
359 RICKETTS, T., 1996, p.59. This is also emphasized in GOLDFARB, W., 1989; HYLTON, P., 1990 and many other works.
360 RUSSELL, B., 1903, p.7 §8.
361 PROOPS, I., 2009, p.10.
362 As in the *Theory of Knowledge Manuscript*, for instance RUSSELL, B., 1913 [1984]. These views will be discussed in connection with the views Russell held around the time of *Principia Mathematica*.
363 RUSSELL, B., 1903, p.7 §8.

(Syl*) For all *x*, *y* and *z*, if *x* and *y* are classes such that *z* belongs to *y* whenever it belongs to *x* and *z* belongs to *x*, then *z* belongs to *y*.

Russell thought that this process of generalization gave the "[...] formal essence of a proposition"[364]. The variables here are completely unrestricted and, most importantly, no mention is made concerning any particular class or object: only variables and logical constants occur in it; this is what Russell thought was characterstic of logical truths. As Proops put it, "[...] in urging that it is the business of mathematics to generalize propositions, Russell is urging only that the propositions of pure mathematics should be free of non-logical constants"[365]. Landini had already made a very similar point, commenting that Russell's intention is to characterize the laws of Pure Mathematics as "[...] true universal generalizations all of whose constants are logical constants"[366].

Unfortunately, Russell's definition of Pure Mathematics has also been widely read as a defense of the so called 'If-thenist' position in the Philosophy of Mathematics[367]. The traditional formulation of this view characterizes a proposition of pure mathematics as asserting that "[...] if anything is a model for a certain system of axioms, then it has certain properties"[368]. The problem, of course, is that according to this view, to establish any branch of knowledge as a logical theory all one needs is to specify a vocabulary and set of logical axioms - one can be a Logicist about Chemistry, Sociology or what have you. One can only say that Russell defended this view, however, if one ignores his explicit claim that the logical vocabulary of Pure Mathematics must be a *purely logical* vocabulary. For Russell, however, a proposition of Logic can only have as constituents logical constants: as we shall discuss below, that is the essence of his definition of Pure Mathematics and of the Logicist thesis of the *Principles*.

Russell intended his characterization to encompass all propositions of pure Mathematics including elementary Arithmetic. But as Ian Proops[369] aptly observes, Russell also thought that the propositions of Arithmetic have a different status than those of Geometry: as we saw, he thought that a particular set of axioms for geometry cannot be adopted, much less proved on a logical basis, while he firmly believed his logical theory provided a logical foundation for a proof of the Peano axioms. In fact, as we shall discuss below, what characterizes one of the central traits of logicism of the *Principles* is the idea that the ultimate analysis of a proposition of elementary Arithmetic is not something like "Peano Postulates imply *p*" much less "If so and so is the case, then the Peano postulates are true", for Russell thought that all of Peano's axioms could be proved by purely logical methods.

364 RUSSELL, B., 1903, p.7 §8.
365 PROOPS, I., 2009, p.12.
366 LANDINI,G., 1998, p.16.
367 See, for instance, COFFA, A., 1981.
368 PUTNAM, H., 1975, p.32.
369 PROOPS, I., 2006, p.270.

Before discussing this central point, however, we must discuss whether a substantial distinction can be traced between Logic and Pure Mathematics as sciences in the *Principles.*

2.3.2 The Relation Between Logic and Mathematics in the *Principles*

Grattan-Guinness observes that there seems to be two distinct formulations of the Logicist thesis in Russell's writings and that Russell himself confused them at some points. On the one hand, the thesis can be understood as an inclusion thesis, which asserts that Pure Mathematics is *part* of Logic, that is, that every proposition of Pure Mathematics is a proposition of Logic. In other words, this version of Logicism is the view that all propositions of pure Mathematics are propositions of Logic, with the inverse not holding. On the other hand, Logicism can also be understood as an *identity* thesis which asserts that Pure Mathematics *is* Logic, that is, that Logic and Pure Mathematics are just the same thing. Grattan-Guinness claims that in the opening of the *Principles* "Russell clearly stated logicism [...] as an inclusion thesis" and holds that the second stronger claim is untenable[370].

In the second edition of the *Principles*, however, Russell stated that "[...] the thesis of the following pages, that *mathematics and logic are identical*, is one which I have never since seen any reason to modify"[371]. Grattan-Guinness takes the latter view to be an indefensible position since "logic can be used in many contexts where mathematics is absent, for instance, in an inference like 'I am hungry', and 'if I am hungry, then I will eat'; hence 'I will eat'"[372].

To be sure, there is a sense in which Grattan-Guinness's point is indisputable. Russell would *not* admit that the conjunction of the aforementioned propositions is a logical truth. For they involve notions like hunger and whatever is denoted by the indexical "I". What is a logical truth for Russell in the *Principles* is something like this:

> If p is a proposition and q is a proposition, then, if p implies q and p is true, then q is true.

According to the conception of Logic which Russell holds in the *Principles*, the above statement stands for a logical truth because it expresses a proposition which is a completely general one and which contains only variables and logical constants. Grattan-Guinness's absolutely unquestionable point is that a logical principle like the above may be employed in apparently non-mathematical contexts.

The claim that Logicism must be viewed as an "inclusion" thesis on the face of this, however, does not stand scrutiny. Russell would surely agree that Logic is applied in non-

370 GRATTAN-GUINNESS, I., 2000b, p.317.
371 RUSSELL, B., 1903, p.xxxii.
372 GRATTAN-GUINNESS, I., 2000b, p.317-18.

mathematical contexts. So are mathematical laws: Arithmetic is surely employed when one deduces from the fact that two cows and one sheep are in a field that there are three animals in the field, but again, a statement like "if there two sheep and two cows in a field, then there are four animals in that field" surely does not express a proposition of Mathematics according to the definition of the *Principles*. The relevant point here is this: according to the definition of Pure Mathematics of the *Principles*, a principle of deduction like "If *p* is a proposition and *q* is a proposition, then, if *p* implies *q* and *p*, then *q* " is just as mathematical a statement as "two plus two equals four" because according to the Logicism of the *Principles* both of them express propositions that contain only variables and logical constants. Thus Russell's later statement that Logic and Pure Mathematics are identical can and should be charitably interpreted as the claim that the principles of deduction should be viewed among the principles of Mathematics. In this light, whether Logicism should be viewed as an inclusion or identity thesis actually becomes merely verbal.

This reading fits, for instance, some of the things Russell says in his *Introduction to Mathematical Philosophy* where he states that "[...] logic has become more mathematical and mathematics has become more logical" and that "[...] they differ as boy and man: logic is the youth of mathematics and mathematics is the manhood of logic"[373]. Russell quite clearly makes a very similar point in the *Principles*:

> The distinction of mathematics from logic is very arbitrary, but if a distinction is desired, it may be made as follows. Logic consists of the premisses of mathematics, together with all other propositions which are concerned exclusively with logical constants and with variables but do not fulfill the above definition of mathematics (§1). Mathematics consists of all the consequences of the above premisses which assert formal implications containing variables, together with such of the premisses themselves as have these marks. Thus some of the premisses of mathematics, e.g. the principle of the syllogism, "if *p* implies *q* and *q* implies *r*, then *p* implies *r*", will belong to mathematics, while others, such as "implication is a relation", will belong to logic but not to mathematics. But for the desire to adhere to usage, we might identify mathematics and logic, and define either as the class of propositions containing only variables and logical constants; but respect for tradition leads me rather to adhere to the above distinction, while recognizing that certain propositions belong to both sciences.[374]

The point is that for Russell - in the *Principles* and elsewhere - the attempt to draw a genuine distinction between pure Mathematics and Logic will fail, both because Mathematics can be developed using only logical notions and primitive propositions and because Logic itself can be reasonably understood as a branch of Mathematics. The fact that Logic can also be employed in reasoning about extra-logical notions and propositions does nothing to impugn this claim.

373 RUSSELL, B, 1919a, p.194.

374 RUSSELL, B., 1903, p.9 §10.

This discussion, however, leads us to a substantial issue. The claim that the propositions of Logic are premises of Mathematics has been widely read in association with the claim that there must be a recursively characterization of set of axioms strong enough to derive every proposition of Mathematics. Since Gödel showed that we cannot even formulate a (consistent) axiomatic calculus that entails the consequences of Peano Arithmetic, Russell has been widely read as defending a claim that is ultimately refutable by a technical result. This understanding of Logicism, not only of Russell's version, but also in general, was already present in Carnap's article where the thesis which asserts that Mathematics is part of Logic was dubbed "Logicism"[375]. There, of course, the thesis is explicitly tied to the idea that there must be a *formal* calculus which is strong enough to prove every proposition of Arithmetic, since Carnap understands the thesis as the twofold claim that mathematical concepts are definable in terms of logical concepts and mathematical theorems are provable by means of purely logical methods[376].

Following Landini, however, we can distinguish the two following theses[377]:

(L) There is a consistent, recursively axiomatizable and semantically complete theory of logic with all mathematical truths among its theorems.

(L*) Mathematical truths are logical truths.

Although it is out of question that Russell endorsed both of these theses in the *Principles* (and - arguably - also in *Principia*) the question about their independence is quite another matter. The first is a mathematical conjecture that is demonstrably false; the second can be understood either as a metaphysical claim about the nature of mathematical entities or about the meaning of mathematical statements - or both[378]. In fact, the counterpart of (L*) in the *Principles* is the following:

(RL*) All true propositions of Pure Mathematics are fully general and have only logical constants as constituents.

There remains the question, however, of whether Russell had a clear understanding of the notion of logical truth that did not rely on the notions of provability or deductive closure under the axioms of his calculus for Logic.

Landini, for instance, observes that for Russell "[...] the purpose of deduction of mathematical formulas within the formal calculus for logic is to demonstrate that intuitions which

375 CARNAP, 1983 [1931].
376 CARNAP, 1983 [1931], p.31.
377 LANDINI, G., 2011, p.167-8.
378 Cf. for instance, LANDINI, G., 2011 and KLEMENT, K., 2013.

were thought to be uniquely mathematical are, in fact, logical"[379] - and so he claims that Russell's committment to (L) "[...] is incidental"[380]. Landini's point is that for Russell the role played by rigorous proofs within a logical calculus is to demonstrate the following *epistemic* thesis:

(LI) The only form of intuition required for understanding and recognizing the truth of pure Mathematics is *logical* intuition.

This reading is supported, for instance, by the fact that Russell frames his opposition to Kant not in stating that (pure) Geometry and Arithmetic are analytic but in claiming that *Logic is synthetic:*

> The question of the nature of mathematical reasoning was obscured in Kant's day by several causes. In the first place, Kant never doubted for a moment that the propositions of logic are analytic, whereas he rightly perceived that those of mathematics are synthetic. It has since appeared that logic is just as synthetic as all other kinds of truth;[381]

If Landini is correct - and the passage above about the synthetic character of Logic strongly corroborates his reading - then, the fact that (L) is false becomes less problematic. For in crucial cases over which the truth of (LI) can be disputed - as in the case of the principle of mathematical induction - Russell provides proofs that employ only purely logical methods where non-logical intuition was thought necessary[382].

But still the central question remains unanswered: does Russell provide an account of logical truth independently of deductive closure under his logical axioms? I.e., does Russell provide an account of Logical necessity in the *Principles*? Landini thinks the answer is given by the idea that logical propositions are *fully general* in the sense which we previously discussed, that is: the idea that logical truths are "[...] true universal generalizations all of whose constants are logical constants"[383]. Landini boldly claims that:

> Russell's notion of logical necessity is similar, in one way, to the Tarski semantic notion of "invariant truth in all interpretations." But instead of interpretations of linguistic syntactic structure in different domains, Russell has the notion of propositional structure or "logical form" and the approximation of the original proposition to a fully general proposition which is true. The full generality captures the idea of different interpretations in different domains. Russell's approach, however, reaches a limit with fully general propositions.

379 LANDINI, G., 1998, p.16.
380 LANDINI, G., 1998, p.16.
381 RUSSELL, B., 1903, p.457 §434.
382 LANDINI, G., 1998, p.16.
383 LANDINI,G., 1998, p.16.

> For Russell, full generality and truth has to be sufficient for logical necessity (logical truth).[384]

This account draws on an unpublished article Russell read to the Oxford Philosophical Society on 22 October 1905 entitled *Necessity and Possibility*[385]. There Russell considers the possibility of giving an *anti-Kantian* account of the notion of analytic judgments as those judgments that "[...] are necessary with respect to all their constituents except [...] logical constants"[386]. Here the main contention is still that logical laws have only logical constants and variables as constituents. Landini takes Russell to be defending something like Tarki's conception of logical truth explained in terms of logical structure.

Those who advocate the van Heijenoort interpretation would probably reject this proposal on the grounds that Russell's views were supposedly antithetical to the idea of *varying* interpretations of the non-logical constants (in the linguistic sense). As the argument usually goes, this is because Russell did not understood his Logic as a mere uninterpreted formalism that could be interpreted differently for varying universes of discourses, but rather as a universal language whose universe of discourse - like that of Frege's Logic - is *the* Universe, which "[...] consists of all that there is, and it is fixed"[387]. The latter point is indisputable: Russell's idea that Logic must embrace an unrestricted variable which ranges over every possible *term* requires that the universe of variation be unique and exhaustive. This is the very core of the doctrine of the unrestricted variable, no question about it. However, this does not preclude an attempt to approximate Russell's ideas to those of later logicians as it is frequently claimed[388]. The point to be emphasized is that Russell (and Frege) did not need to speak of different interpretations. In the case of Frege he explicitly introduced his *begriffsschrift* as a basic framework upon which any topic specific science could be developed axiomatically once a specific vocabulary was added to it. Speaking in somewhat anachronistic terms, restrictions of "universe of discourse" were to be drawn by introducing non-logical vocabulary into the object language, not by actually specifying a restricted domain in the metalanguage. The same applies to Russell.

Be that as it may, Russell does explain his conception of analyticity in terms of closure under logical axioms. He first puts forward a purely syntactic account of deducibility:

> The laws of deduction tell us that two propositions having certain relations of form (e.g. that one is the negation of the negation of the other) are such that one of them implies the other. *Thus q is deducible from p if p and q either have one of the relations contemplated by the laws of deduction, or are connected by any (finite) number of intermediaries each having one of these relations to its successor*. This meaning of deducible is purely logical, and covers, I think,

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[384] LANDINI, G., 1998, p.41.
[385] RUSSELL, B., 1905*d*, Collected papers p.507.
[386] RUSSELL, B., 1905*d*, Collected papers p.507.
[387] van HEIJENOORT, J., 1967, p.325.
[388] Again, cf. van HEIJENOORT, J., 1967; RICKETTS, T., 1985; GOLDFARB, W., 1989; HYLTON, P., 1990.

> exactly the cases in which, in practice, we can deduce a proposition *q* from a proposition *p* without assuming either that *p* is false or that q is true.[389]

Russell then defines analytic propositions as those which are *deducible* from the "[...] small number of general logical premises" and concludes that the notion of a proposition that "[...] can be deduced from the laws of logic" gives us "the class of analytic propositions"[390]. He continues:

> A certain large body of propositions, namely (approximately) those constituting formal logic and pure mathematics, all have some very important logical characteristics in common and are all deducible from a small number of general logical premises, among which are included the laws of deduction already spoken of[391]. These general logical premisses fulfill the functions formerly supposed to be fulfilled by the so-called "laws of thought": they may be called the "laws of logic". *From the laws of logic all the propositions of formal logic and pure mathematics will be deducible.* We may, then, usefully *define as analytic those propositions which are deducible from the laws of logic; and this definition is conformable in the spirit, though not in the letter, to the pre-Kantian usage.*[392]

Given the way Russell understands the notion of deducibility, this passage shows that not only he was committed to Logicism in the sense that Gödel showed to be an untenable thesis (namely (L) discussed above), but that he also wavers between accounts of logical necessity that relies upon the idea of deductive closure under some set of axioms.

So, the appeal to the notion of analyticity defined in the paper on logical necessity does not seem to settle the question of whether Russell embraced a definite account of logical necessity independent of the notion of provability or deductive closure as Landini argues. Still, the paper definitely establishes that we should follow Landini in dissociating logicism (RL*) from (L), for there Russell explicitly draws a line between the idea of a proposition *implying* another, in the sense of a relation which holds between propositions (thus, between ontologically independent entities), and a proposition being *deducible* from another, making clear that the former is the more fundamental:

> It is to be observed that, although *deducible from* (as just defined) is a different notion from *implied by*, it cannot be made to replace the notion of *implied by*. For *deducible from* is defined by means of the laws of deduction, and these laws employ the notion of implication. Hence we cannot substitute deducible from

389 RUSSELL, B., 1905*d*, Collected papers p.515. Our emphasis.

390 RUSSELL, B., 1905*d*, Collected papers p.521.

391 The laws in question were the "laws of identity, contradiction and excluded middle" which have been traditionally understood as having some privileged logical status. As Russell observes, these can be obtained as consequences of his primitive proposisitions, so to some extent he thinks it is arbitrary to treat them as having some privileged position as more fundamental than other logical laws. (RUSSELL, 1905x, p.516)

392 RUSSELL, B., 1905*d*, Collected papers p.516.

> for implied by in the laws of deduction, without incurring a vicious circle. The notion of implication thus remains fundamental, and the notion of deducibility is derivative from it.[393]

Also, we cannot be mislead into supposing, from Russell's revised terminology of logical laws as *analytic* - in contrast to where they were viewed as *synthetic* - that Russell changed his views on the nature of Mathematical reasoning. The change is merely terminological. The fundamental issue is, for Russell, the same as it was for Frege: the claim that Logic is a science that *ampliates* or *extends* our knowledge.

Frege framed his disagreement with Kant in terms of the scope of analyticity. He thought that Kant's conception of analytic propositions as founded on the Principle of Contradiction characterized them "too narrowly" and so he claimed his "[...] the division of judgments into analytic and synthetic is not exhaustive"[394]. Russell and Frege are in complete agreement in this. In *The Philosophy of Leibniz*, we find:

> We may argue generally, from the mere statement of the Law of Contradiction, that no proposition can follow from it alone, except the proposition that there is truth, or that some proposition is true. For the law states simply that any proposition must be true or false, but cannot be both. It gives no indication as to the alternative to be chosen, and cannot of itself decide that any proposition is true. It cannot even, of itself, yield the conclusion that such and such a proposition is true or false, for this involves the premise "such and such is a proposition," which does not follow from the law of contradiction. Thus the doctrine of analytic propositions seems wholly mistaken.[395]

The agreement with Frege's standpoint here is complete; so much so that in a footnote, Frege observes that Kant himself anticipated "an inkling" of his view for, Frege claims, "[...] he says that a synthetic proposition can only be seen to be true by the law of contradiction, if another synthetic proposition is presupposed"[396] - and this is precisely the point raised by Russell. Thus, we disagree with Ian Proops, who claims that Frege read Kant more charitably than Russell did[397]. According to him, in the *Principles* "Russell is working with an extremely narrow conception of analyticity that derives from his literal-minded construal of (what is standardly taken to be) Kant's most adequate formulation of an analyticity criterion"[398]. That criterion, of course, is that the judgment "[...] can be cognized sufficiently in accordance with the principle of contradiction"[399]. But Frege's observation shows that he is also attributing to Kant the same notion of analyticity. The change in terminology which Russell effected was nothing more than

393 RUSSELL, B., 1905d, Collected papers p.515.
394 FREGE, G., 1950 [1884], pp.99-100 §87.
395 RUSSELL, B., 1900, p.25-6 §11
396 FREGE, G., 1950 [1884], p.100, footnote 1.
397 PROOPS, I., 2006, p.286.
398 PROOPS, I., 2006, p.286.
399 KANT, I., 1998, p.279, A151/B190.

that: a *mere* change of terminology (perhaps to avoid association with Kant's views on the Philosophy of Mathematics). In fact, as Michael Potter notes, in order for Frege to have claimed that Arithmetic is fully analytic in his sense, "[...] he had to make the further claim that polyadic logic, despite being ampliative, is nevertheless analytic, by which he meant independent of intuition"[400]. In the *Principles*, Russell is simply biting the bullet: he seems to be committed to the idea that there is such a thing as *logical intuition* and these ground both our knowledge of Logic and Mathematics. Both sciences are synthetic because the truth of their propositions is not grounded in the 'Principle of Contradiction' alone.

Landini further elaborates this by noting that the crucial claim is not that polyadic logic is informative, but that what we call higher-order polyadic logic that embraces some form of impredicative comprehension principle is informative. That is, claims that according to Frege and Russell "[...] logic is informative and extends knowledge" because "[...] it embraces comprehension axioms"[401]. For Landini, Frege's logic is informative because it is committed to a hierarchy of levels of functions akin to that of simple type theory. *Mutatis mutandis*, the logic of Russell's *Principles* - just like his intermediate versions of the No-Class theory - is committed to comprehension principles for propositions as abstract entities[402]. It must be said, however, that perhaps the strongest case to be made in favor of this interpretation is grounded in knowledge we *now* possess about the meta-theory of predicate logic. With his completeness theorem for first-order logic, Gödel showed that given some adequate set of axioms, every valid first-order formula is a theorem, or, equivalently, that every consistent set Δ_1 of first-order formulas has a model[403]. When working with second and higher-order logic, we must distinguish between so-called primary and secondary interpretations and models, and if we frame completeness in terms of primary models, then the result does *not* hold for second-order logic. Henkin, showed, however, that for models in general - primary *and* secondary - it can be proved that every consistent set Δ_2 of second-order formulas has a model[404]. And the result can be generalized for the type hierarchy in general. Also, Turing and Church showed, building upon Gödel's work, that the class of theorems of both first-order and higher-order logic cannot be *effectively* cha-

[400] POTTER, M., 2000, p.65. Potter further claims that "The keystone, then, is Frege's claim that even polyadic logic does not depend on intuition" (POTTER, M., 2000, p.65). The basis for his reading is Frege's claim, in connection with his definition of the ancestral of a relation in the *Begriffsschrift*, that "pure thought (regardless of any content given through the senses or even given a priori through an intuition) is able, all by itself, to produce from the content which arises from its own nature judgments which at first glance seem to be possible only on the grounds of some intuition" (FREGE, G., 1879, p.55). Since Frege does not discuss in substantitve detail how we apprehend logical laws, it seems hasty to conclude that he would not concur with Russell in claiming that there can be such a thing as *logical* intuition. Ian Proops conjectures that "Russell arrived at this more inclusive conception of analyticity as a result of reading Frege's *Grundlagen*" (PROOPS, I., 2006, p.287) and it is indeed plausible to suppose that he adopted this terminology to agree with Frege's use rather than with Kant's.

[401] LANDINI, G., 2011, pp.169-170.

[402] And similarly, Russell's mature theory of types *can* be understood in terms of a hierarchy of attributes.

[403] CHURCH, A., 1956, p.233, §44; p.239-245.

[404] This was proved by Leon Henkin in his doctoral dissertation and the proof was later published as HENKIN, L., 1950. A textbook discussion can be found in CHURCH, A., 1956, pp.307-15, ROBBIN, J., 1969, pp.140-1 and also MENDELSON, E., 2015, pp.389-393; a brief presentation can be found in ROGERS, R., 1971, p.92-4.

racterized[405]. This makes a strong case in favor Landini's view, in particular because it ties the idea that Logic is informative with the idea that Logic has - in some way or another - *ontological commitments* which, as we'll see next, was central to Russell's conception of Logic in the *Principles*.

2.3.3 The Existence of Classes and The Epistemology of Logicism in the *Principles*

The fact that Russell *opens* the *Principles* with his definition of Pure Mathematics may be misleading, for some may take Russell to be arbitrarily choosing his definition. This, of course, would be a gross misrepresentation of the work, since the book as a whole is an attempt to justify the definition[406]. Russell's goal with the logical analysis of mathematics is to *reach* the claim that Pure Mathematics and Logic are identical as a conclusion, as he unequivocally explains:

> It will be shown that whatever has, in the past, been regarded as pure mathematics, is included in our definition, and that whatever else is included possesses those marks by which mathematics is commonly though vaguely distinguished from other studies. The definition professes to be, not an arbitrary decision to use a common word in an uncommon signification, but rather a precise analysis of the ideas which, more or less unconsciously, are implied in the ordinary employment of the term. Our method will therefore be one of analysis, and our problem may be called philosophical—in the sense, that is to say, that we seek to pass from the complex to the simple, from the demonstrable to its indemonstrable premisses.[407]

The crucial case for demonstrating Russell's view of the nature of Pure Mathematics is, of course, that of elementary Arithmetic: since Russell accepts the arithmetization of all branches of Pure Mathematics by means of set theoretical methods discovered by him and his predecessors, the most central case for determining whether his conception of Pure Mathematics stands or falls is that of Arithmetic.

In the *Principles*, as elsewhere, Russell reduces this task to that of defining the concept of *natural number*, *zero* and *successor* and demonstrating the Dedekind-Peano postulates on a purely logical basis. Russell explicitly frames his position - again, as elsewhere - in opposition to Peano's procedure of charaterizing these fundamental notions by assuming the Dedekind-Peano postulates as primitives (i.e., by appealing to what Peano and Russell called "definition

[405] It must also be recognized, however, that neither Frege nor Russell caught a whiff of this and that any interpretation which relies on the distinction between first and second-order quantification theory must be held with a grain of salt, since it appeals to our present knowledge of the meta-theory of predicate Logic.

[406] Indeed, despite being the first claim of the book, were the *Principles* to be arranged from assumptions to conclusions, the definition of Pure Mathematics should be the last.

[407] RUSSELL, B., 1903, p.1 §2.

by postulates"). Russell observes that this method just like that of taking the fundamental notions of zero, number and successor as primitive "[...] yields what mathematicians call the existence-theorem, i.e. it assures us that there really are numbers", but also that "[...] leaves it doubtful whether numbers are logical constants or not, and therefore makes Arithmetic [...] a branch of Applied Mathematics"[408].

As we already discussed, Russell avoided Peano's method appealing to his set theoretical construction of cardinal numbers in general as classes of similar classes, showing how starting from this definition he could then develop the basic notions of elementary Arithmetic and prove the Dedekind-Peano postulates[409]. In a nutshell, given Russell's acceptance of the Arithmetization[410] of all branches of pure Mathematics, it is this accomplishement - (I) the definition of Peano's primitive notions together with (II) the proof that there is a set of objects, i.e., natural numbers, which satisfy Peano's axioms - that gives Russell the needed justification for his claim that the propositions of Pure Mathematics are propositions of Logic.

Indeed, in a sense the whole project of *Principles* is tied to the 'logicization' of Arithmetic. In a beautiful passage from the *Principles*'s very last page[411], Russell summarizes how, starting from the theory of natural numbers, he can demonstrate not only the existence of rational, real and complex numbers, but also of Cantor's transfinite numbers and the mathematical entities required by several branches of Pure Geometry and Mathematical Physics:

> The existence-theorems of mathematics - i.e. the proofs that the various classes defined are not null—are almost all obtained from Arithmetic. [...] from the class of the finite cardinal numbers themselves, follows the existence of α_0[*i.e.* $\aleph_0$], the smallest of the infinite cardinal numbers; from the series of fi-

408 RUSSELL, B., 1903, p.126 §122. That is, of course, in light of Russell's own definition of Pure Mathematics which he puts forward in the *Principles*.

409 Of course, a previous - even more basic step - towards this goal was showing that the general concept of *cardinal number* (not specifically natural numbers) could defined in purely logical terms, a task which, as we discussed, was accomplished in Russell's paper of 1901 *On The Logic of Relations*.

410 Sebastian Gandon argues that Russell's Logicism as put forward in both *The Principles of Mathematics* and *Principia Mathematica* does not require arithmetization of all branches of Pure Mathematics in the sense that one must reduce the branches of Mathematics to Arithmetic beforehand in order to establish them as a branches of Logic; on the contrary, according to Gandon, Russell intended Logicism to be established in a "topic-specific" manner (cf. GANDON, S., 2008; 2012). Gandon bases this view on his detailed studies of the often neglected logicist treatment of Geometry that Russell provides in the *Principles* (cf. RUSSELL, B., 1903 part III) and of the theory of quantity presented in part VI of *Principia* (cf. WHITEHEAD & RUSSELL 1927*b*). At any event, concerning much of the development of Mathematics as given in the *Principles* and *Principia* - in particular the theory of transfinite numbers and some important parts of the theory of real numbers - it is on the basis of the most basic existence-theorems of Arithmetic (e.g., that every finite cardinal *n* has a successor $n + 1$ such that $n \neq n + 1$) that Russell establishes the existence theorems for higher forms of number, as he makes clear in the very closing paragraph of the *Principles* (cf. RUSSELL, B., 1903, p.497-8, §474). So at least in a weak sense there is in Russell's thought an unquestionable link between arithmetization and his Logicism, namely: Russell fully accepted that one *can* prove existential theorems concerning all higher forms of number without assuming any sort of specific ontology of numbers beyond that of the finite or inductive cardinals whose theory, in turn, can (at least to some extent) be constructed on a purely logical basis. It is merely this weaker relation between arithmetization and Russell's Logicism that we shall assume in the present work.

411 If we exclude the appendices.

> nite cardinals in order of magnitude follows the existence of ω, the smallest of infinite ordinals. From the definition of the rational numbers and of their order of magnitude follows the existence of η, the type of endless compact denumerable series; thence, from the segments of the series of rationals, the existence of the real numbers, and of θ, the type of continuous series. The terms of the series of well-ordered types are proved to exist from the two facts: (1) that the number of well-ordered types from 0 to α is $\alpha + 1$, (2) that if *u* be a class of well-ordered types having no maximum, the series of all types not greater than every *u* is itself of a type greater than every *u*. From the existence of θ, by the definition of complex numbers (Chapter 44) [*Dimension and Complex Numbers*] , we prove the existence of the class of Euclidean spaces of any number of dimensions; thence, by the process of Chapter 46 [Descriptive Geometry], we prove the existence of the class of projective spaces, and thence, by removing the points outside a closed quadric, we prove the existence of the class of non-Euclidean descriptive (hyperbolic) spaces. By the methods of Chapter 48 [Relation of Metrical to Projective and Descrivitive Geometry], we prove the existence of spaces with various metrical properties. Lastly, by correLating some of the points of a space with all the terms of a continuous series in the ways explained in Chapter 56 [Definition of a Dynamical World] , we prove the existence of the class of dynamical worlds.[412]

But, of course, in order for all of that to work as a vindication of Russell's claim in the beginning of the *Principles*, the following most crucial step of his argument must work:

> The existence of zero is derived from the fact that the null-class is a member of it; the existence of 1 from the fact that zero is a unit-class (for the null-class is its only member). Hence, from the fact that, if *n* be a finite number, *n* + 1 is the number of numbers from 0 to *n* (both inclusive), the existence-theorem follows for all finite numbers.[413]

It is *this* step that is the most central for Russell to conclude that "[...] throughout this process, no entities are employed but such as are definable in terms of the fundamental logical constants" and so, that "[...] the chain of definitions and existence-theorems is complete, and the purely logical nature of mathematics is established throughout"[414].

If intended at establishing the triumph of Logicism, however, the above sketch of the proof that there are infinitely many natural numbers is inseparable from the idea that the existence of infinite classes can be proved from logical principles alone. And indeed, while writing most of the *Principles*, Russell was convinced beyond any reasonable doubt that there is no problem with this view:

> That there are infinite classes is so evident that it will scarcely be denied. Since, however, it is capable of formal proof, it may be as well to prove it.[415]

412 RUSSELL, B., 1903, pp.497-8 §474.
413 RUSSELL, B., 1903, pp.497 §474.
414 RUSSELL, B., 1903, pp.497 §474.
415 RUSSELL, B., 1903, p.357 §339.

Russell accepted several different alleged demonstrations of the existence of infinite classes. One such 'proof' is the following:

> A very simple proof is that suggested in the Parmenides, which is as follows. Let it be granted that there is a number 1. Then 1 is, or has Being, and therefore there is Being. But 1 and Being are two: hence there is a number 2; and so on. Formally, we have proved that 1 is not the number of numbers; we prove that *n* is the number of numbers from 1 to *n*, and that these numbers together with Being form a class which has a new finite number, so that n is not the number of finite numbers. Thus 1 is not the number of finite numbers; and if $n-1$ is not the number of finite numbers, no more is *n*. Hence the finite numbers, by mathematical induction, are all contained in the class of things which are not the number of finite numbers. Since the relation of similarity is reflexive for classes, every class has a number; therefore the class of finite numbers has a number which, not being finite, is infinite.[416]

The above 'proof' which Russell attributes to Plato (in *Parmenides*) is regarded by him as analogous to Frege's proof, sketched in the *Grundlagen* and fully developed in the *Grundgesetze*, that the number of the class of numbers which are less than or equal to *n* is equal to $n+1$:

> A better proof, analogous to the above, is derived from the fact that, if *n* be any finite number, the number of numbers from 0 up to and including *n* is $n+1$, whence it follows that *n* is not the number of numbers.[417]

This is Frege's famous proof, which employed a definition of cardinal number which essentially coincided with that of Russell. The gist of it is to show that each finite cardinal number $n+1$ is the cardinal number of the class of all finite cardinal numbers less than or equal to *n*; thus $n+1$ is shown to be the class that contains the class $\{0, 1, ..., n\}$ as an element (since the cardinal number of a class a is the class of all classes similar to *a*): 1 is the cardinal number of the class $\{0\}$ whose sole member is 0, so $\{0\} \epsilon 1$[418], also 2 is the cardinal number of the class $\{0, 1\}$,so $\{1, 2\} \epsilon 2$, and so on... In this way, starting from 0 and 1 and then proceeding by induction, it can be shown that each finite cardinal is a non-empty set, i.e., that each finite cardinality $n+1$ is instantiated by at least the set $\{0, 1, ..., n\}$.

Russell also considered a famous 'proof' which purports to derive the existence of a Dedekind infinite - or, as Russell called them later - *reflexive* class from the impossibility of establishing a one-one correspondence between things and ideas of things:

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416 RUSSELL, B., 1903, p.357 §339.

417 RUSSELL, B., 1903, p.357 §339. Surely, Russell would not endorse such a comparison after the *Principles*. Frege's proof is a legitimate piece of Mathematics which was unfortunately carried out within an inconsistent theory; the attempt to show that the number 2 exists by treatint 'being' as *something* is just a blatant example of the kind of nonsense which can be produced without a proper understanding of the existential quantifier.

418 Incidentally, since $\{\Lambda\} = 0$ by definition, 0 itself is also a member of 1.

> Again, it may be proved directly, by the correlation of whole and part, that the number of propositions or concepts is infinite. For of every term or concept there is an idea, different from that of which it is the idea, but again a term or concept. On the other hand, not every term or concept is an idea. There are tables, and ideas of tables; numbers, and ideas of numbers; and so on. Thus there is a one-one relation between terms and ideas, but ideas are only some among terms. Hence there is an infinite number of terms and of ideas.[419]

As Russell acknowledges in a footnote, this proof can be traced to the works of Bolzano[420] and was famously put forward by Dedekind in his seminal paper on natural numbers[421]. Russell's remarks make it clear that he was convinced of the soundness of such demonstrations.

But of course, proofs of the kind considered above were formulated by Bolzano, Cantor, Dedekind, Frege and Russell by employing the notion of class in a disastrously liberalized manner. All the works mentioned by Russell - most explicitly, importantly and famously, those of Frege - assumed either formally or informally some form of naive comprehension for sets, much like Russell did in the *Principles.* And as Russell found out around 1901 the laws governing the 'unrestricted' conception of set employed by Cantor and his followers led to contradictions which cast a shadow of doubt on the logico-mathematical results and set-theoretical definitions that ground Russell's claim that the laws of Pure Mathematics are logical truths as put forward in the *Principles*. This calls for a discussion of how Russell understood what grounds his belief that his chosen primitive propositions are *true*. And again, here it becomes somewhat more doubtful that Russell had a tenable account or criterion for determining what should or should count as a logical truth.

Russell speaks loosely in terms of varying degrees of self-evidence[422], but does not develop his views on the epistemology of Logic and Mathematics in any detail. There is, however, an implicit general epistemology of Logic and Mathematics behind the *Principles*. As Peter Hylton puts it, despite the fact that the doctrines of the *Principles* "say little or nothing" about the nature of mathematical knowledge (and knowledge in general), "[...] a view of knowledge is none the less implicit in those doctrines"[423].

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419 RUSSELL, B., 1903, p.357 §339.

420 BOLZANO, B., 1851.

421 DEDEKIND, R., 1888, p.64. There, we find the following proof of theorem 66 which asserts that "there exist infinite systems"; Following Boolos's translation, we have: "The world of my thoughts, i. e., the totality *S* of all things that can be objects of my thought, is infinite. For if *s* signifies an element of *S*, then the thought *s'*, that *s* can be object of my thought, is itself an element of *S*. If *s'* is regarded as the image $\phi(s)$ of the element *s*, then the mapping ϕ determined thereby has the property that its image *S'* is a part of *S*; and indeed *S'* is a proper part of *S*, because there are elements in *S* (e.g., my own ego), which are different from every such thought *s'* and are therefore not contained in *S'*. Finally, it is clear that if *a* and *b* are different elements of *S*, then their images *a'*, *b'*, are also different, so that the mapping is distinct (similar). Consequently, S is infinite, q.e.d. (BOOLOS, G., 1998, pp.202-3; DEDEKIND, R., 1888, p.64). Both Dedekind and Russell refer to this result as originally proved by Bolzano. Rightfully, George Boolos refers to Dedekind's 'proof' as "[...] one the strangest pieces of argumentation in the history of logic" (BOOLOS, G., 1998, p.202) - although the proof Russell draws from the *Parmenides* mentioned above is a *very* strong contender.

422 RUSSELL, B., 1903, p.17 §18.

423 HYLTON, P., 1990, p.111.

The cornerstone of that view, of course, is Russell's notion of *acquaintance.* This notion is introduced in order to account for the sort of direct or immediate relation that is required by the pluralistic realism Russell inherited from Moore. Thus, in the Preface of the *Principles* we find the following striking passage:

> The discussion of undefinables—which forms the chief part of philosophical logic—is the endeavour to see clearly, and to make others see clearly, the entities concerned, in order that the mind may have that kind of acquaintance with them which it has with redness or the taste of a pineapple. Where, as in the present case, the undefinables are obtained primarily as the necessary residue in a process of analysis, it is often easier to know that there must be such entities than actually to perceive them; there is a process analogous to that which resulted in the discovery of Neptune, with the difference that the final stage—the search with a mental telescope for the entity which has been inferred—is often the most difficult part of the undertaking.[424]

This paragraph gives a rough - and somewhat fantastic - picture of how Russell understood the process of arriving at the primitive notions of Logic. This picture indicates that the *epistemological* grounds for accepting the axioms for the calculus of Logic is our capacity to apprehend with clarity and distinction - to borrow a Cartesian mode of expression - the most basic entities of Logic and Mathematics, as if these could be perceived through some 'mental telescope'; or, if we are to take the pineapple metaphor seriously - as it should - through some sort of logical palate.

Fortunately, Russell had already put forward - albeit in a very rough and schematic form - elements of a much more interesting epistemology for Logicism. In 1901 Russell wrote a paper that never saw print in his lifetime entitled *Recent Italian Work on The Foundations of Mathematics*[425], where he discussed and praised in detail the merits of Peano's symbolic logic and methodology. Concerning what he called Peano's 'logistic method', Russell claims that:

> Its aims is [sic] to discover the necessary and sufficient premisses of the various branches of mathematics, and to deduce results (mostly known already) by a rigid formalism which leaves no opening for the sinister influence of obviousness. Thus the interest of the work lies (I) in the discovery of the premisses and (2) in the absolute correcteness of the deduction. Both these points are (or rather should be) of great interest to the philosopher.[426]

Anyone familiar with Russell's posterior discussions of his 'regressive method' of discovering premises in mathematics (in which mathematical logic plays a prominent role) should recognize here the roots of Russell's later, more sophisticated ideas[427]. It is very plausible to

424 RUSSELL, B., 1903, p.xv.
425 RUSSELL, B., 1901*c*.
426 RUSSELL, B., 1901*c*, p.352.
427 RUSSELL, B., 1907; RUSSELL, B., 1919a; RUSSELL, B., 1924; WHITEHEAD & RUSSELL, 1910; These views will be discussed in detail in chapter five.

suppose then, that by the time of the *Principles* Russell had realized that one way to justify the choice of axioms of Logic was in in virtue of their *consequences*. So we have reason to believe that Russell would endorse the idea that propositions of Pure Mathematics - which we have, *prima-facie*, every reason to believe - can be derived from the axioms of Logic lends these very axioms epistemic support. This point is important, because there is a fly in the ointment in the attempt of justifying the choice of primitive notions and axioms through the postulation of some sort of direct relation like acquaintance: Russell's Paradox. The discovery of the contradiction showed that the notion of class could not be taken for granted as transparent notion; as Quine once observed, the discovery of the contradictions shows that the most basic and at first sight self-evident 'intuitions' about the notion of set are, in fact, "bankrupt"[428]. But given the centrality of the notion of class in the Logicist development of Mathematics in the *Principles*, this means that Russell's entire epistemology of Logic and Mathematics is jeopardized.

Indeed, it is no coincidence that in later writings Russell emphasizes the regressive method as a way of not only discovering but *justifying* the premises of Mathematics: once apparently self-evident principles and highly abstract concepts of Logic - in the sense in which Russell understood this science - are shown to lead to contradiction, self-evidence loses what little justificatory weight it had to begin with. Russell was, of course, the first to be shocked by his discovery precisely because of this. Concerning the calculus of classes, Russell gave the following warning in the preface of the *Principles*:

> In the case of classes, I must confess, I have failed to *perceive* any concept fulfilling the conditions requisite for the notion of class. And the contradiction discussed in Chapter x. proves that something is amiss, but what this is I have hitherto failed to discover.[429]

The analogy with perceptual experience cannot be overstated here: Russell is claiming that he had failed to become *acquainted* with any sort of *entity* which satisfies the conditions which pure Mathematics requires of classes. This is Russell's first acknowledgment in print of the damage done by Russell's discovery to his conception of classes conceived as *entities.* But this puts the very conception of Logic which Russell held in the *Principles* in check.

One of the central tasks of the next chapter is to discuss in more detail how this very difficulty led Russell to reformulate the Logic of the *Principles* and eventually to retreat from a conception of Logicism which embraced the existence of classes as a logical matter.

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428 QUINE, W., 1969, p.x.

429 RUSSELL, B., 1903, p.xv. Our emphasis.

3 Russell's Ontological Development

> I finished this first draft of The Principles of Mathematics on the last day of the nineteenth century i.e. December 31, 1900. The months since the previous July had been an intellectual honeymoon such as I have never experienced before or since. Every day I found myself understanding something that I had not understood on the previous day. I thought all difficulties were solved and all problems were at an end. But the honeymoon could not last, and early in the following year intellectual sorrow descended upon me in full measure.[1]

3.1 The Collapse of Russell's Early Logicism

3.1.1 The Contradiction and the Notion of a Logical Subject in the *Principles*

As we discussed in the previous chapter, one of main goals Russell set out in the *Principles* was to "[...] the endeavor to see clearly, and to make others see clearly" the entities with which Mathematical Logic is concerned, so that "the mind may have that kind of acquaintance with them which it has with redness or the taste of a pineapple"[2]. Unfortunately, in the course of this endeavor Russell made a bitter discovery.

The tale of this discovery is familiar: Cantor famously proved that there cannot be a greatest cardinal number and on the face of such result Russell was led to question if this could be reconciled with the existence of the class of all *things* or *terms* or *entities*, the existence of which, at the time, he thought plausible, if not inevitable to be admitted by any correct logical theory. The question of whether phrases like "the greatest cardinal number" or "the class of all entities" stood for genuine logical subjects, in turn, led Russell to consider other sorts of problematic cases. As Russell vividly recounts this in *My Philosophical Development*:

> I thought, in my innocence, that the number of all the things there are in the world must be the greatest possible number, and I applied his proof to this number to see what would happen. This process led me to the consideration of a very peculiar class. Thinking along the lines which had hitherto seemed adequate, it seemed to me that a class sometimes is, and sometimes is not, a member of itself. The application of Cantor's argument led me to consider the classes that are not members of themselves; and these, it seemed, must form a class. I asked myself whether this class is a member of itself or not.[3]

The outcome of this is now recounted in almost every textbook of Mathematical Logic and Set Theory as a tale of legend. By June 16 1902, almost a year after the discovery, Russell

1 RUSSELL, B., 1959, p.73.
2 RUSSELL, B., 1903, p.xv.
3 RUSSELL, B., 1959, pp.75-6. Compare also RUSSELL, B., 1903, p.101/§100.

wrote a letter to Frege in which he reports "[...] a difficulty only on one point"[4], namely with Frege's Basic Law V. In Frege's *Grundgesetze* this law appears as follows[5]:

$$\vdash (\grave{z}fz = \grave{z}gz) = \overset{\mathfrak{a}}{\smile} f(\mathfrak{a}) = g(\mathfrak{a})$$

Frege described the content of this law in terms of the "[...] transformation of the generalization of an equality into an equality of course of values"[6]. Courses of values were Frege's extensions of *functions*, of which first-order concepts were treated as a particular case that took objects as arguments and had truth-values as values. For our purposes, Frege's Law V can be stated as:

$$(F)(G)(\hat{z}Fz = \hat{z}Gz \equiv (x)(Fx \equiv Gx))$$

The above can be read as: for every concept *F* and every concept *G*, the extension of *F* is identical to the extension of *G* if, and only if for every object *x*, *x* falls under the concept *F* if, and only if *x* falls under *G*[7]. Framed in these terms, this axiom entails that every concept *F* has a unique object associated with it, namely, its extension: the set of all things that fall under *F*. This is easily demonstrated: instantiate Basic Law V as $\hat{z}Fz = \hat{z}Fz \equiv (x)(Fx \equiv Fx)$; since the right side of the bi-conditional is a logical truth, we have $\hat{z}Fz = \hat{z}Fz$; by existential generalization we have $(\exists x)(x = \hat{z}Fz)$ and by universal second-order generalization we have $(F)(\exists x)(x = \hat{z}Fz)$; it must be observed that one does not *need* Law V to show this; alternatively, since in Frege's system "$\hat{z}Fz$" is a genuine term, $\hat{z}Fz = \hat{z}Fz$ holds and one gets $(F)(\exists x)(x = \hat{z}Fz)$ from the laws of predicate logic alone[8].

The *one* difficulty which Russell found was the result of instantiating *F* as the concept "[...] not a member of its own extension"[9], the extension of *that concept*, however, is a member of itself if and only if it does not belong to itself. Since Frege's Law was his fundamental axiom assuring the existence of value-ranges, Russell took that this as a general difficulty of characterizing correctly comprehension or abstraction axioms for classes. As we already remarked, the naive or intuitive assumption that Russell made regarding classes was that the following principle had unrestricted validity regarding the condition or concept F:

$$(F)(\exists y)(x) : x \,\epsilon\, y \,.\equiv.\, Fx$$

Russell took as unproblematic that if classes are extensions of concepts or so-called propositional functions, i.e, functions whose values are propositions, the class $\{x : Fx\}$ must contain as

4 In: FREGE, G. 1980, p.131.

5 FREGE, G., 1893, p.61, §47. It must be observed that in Frege's original system the conflict with Cantor's power-class theorem comes from the 'implication' from left to right, i.e., Law Vb (cf. LANDINI, G., 2017, in particular pp.11-12), something one misses when on Frege's notations are translated to the notation of modern predicate logic. Thanks for Landini for calling my attention to this point.

6 FREGE, G., 1980, p.132.

7 This, of course, is not true to Frege's own formulation, since Frege's Logic is best interpreted as a Logic of terms, not as a predicate calculus in the modern sense. For details, cf. DUARTE, A., 2009 and LANDINI, G., 2012.

8 Thanks to Gregory Landini for calling my attention to this point.

9 This was, in fact, Frege's *correction*: Russell spoke of a concept not applying to itself, but this violates Frege's hierarchy of levels (cf. FREGE, G., 1980, pp.132-3).

members all, and only those objects *x* for which the propositional function *Fx* has as a value a true proposition. The contradiction was forthcoming when it was supposed that the propositional function Fx would result in a significant proposition for every possible value of *x* - even if *x* was the the class $\{x : Fx\}$ itself; this let the way open for the possibility of the following line of reasoning: Let F be the condition $x \notin x$, that is, *x is not a member of x*, then, there is a set $w = \hat{x}(x \sim \epsilon\, x)$; it immediately follows that $x \,\epsilon\, w \,.\equiv_x . \sim (x \,\epsilon\, x)$, and thus $w \,\epsilon\, w \,.\equiv . \sim (w \,\epsilon\, w)$. This is the difficulty which became widely known as 'Russell's Paradox'.

Frege responded to Russell's discovery on June 22 "[...] surprised [...] beyond words and [...] thunderstruck, because it rocked the ground on which [he] meant to build Arithmetic"[10]. By 1903, Russell had published his discovery within his *Principles of Mathematics* as an open problem to which he had, at best, a sketch of solution and Frege discussed it in and Appendix added in print to the second volume of his Basic Laws of arithmetic, where we witness the last breath of his logicist project[11].

Mathematicians and logicians - including Russell himself - were by no means unaware of difficulties similar to Russell's paradox at the time it was discovered. By 1897, one of Peano's followers, Cesare Burali-Forti[12], had established the following: Let *O* be the class of all ordinal numbers and assume that it can be well-ordered[13]; it follows that *O* has an ordinal number - call it Ω; presumably Ω is the greatest ordinal number, since it is the number of the series of *all*

[10] FREGE, G. 1980, p.132. The letter was published for the first time in van HEIJENOORT, J., 1967, p.124-5.

[11] It is there that we find Frege's haunting words: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished" (FREGE, G., 1997 [1903], p.279 [253]) which constrast drastically with his enthusiastic remarks in the first volume of *Basic Laws*: "[...] I will only be able to accept a refutation if someone shows, by actually doing it, that a better more durable edifice can be erected on different fundamental convictions, or if someone proves to me that my principles lead to obviously false conclusions. But no one will succeed in this." (FREGE, G., 1997 [1893], p.207-8 [XXVI]). Frege attempted to solve Russell's contradiction in the Appendix, proposing a mitigated version of (V), namely:
(V') $\hat{z}Fz = \hat{z}Gz \,.\equiv\colon (x)(x \neq \hat{z}Fz \,.\, x \neq \hat{z}Fz \colon\supset\colon Fx \equiv Gx)$
Although Frege recognized that this modification would certainly demand revision of several proofs, he claimed that "we need scarcely fear that this sill raise essential difficulties for the course of the proofs" (FREGE, G., 1997 [1903], p.289 [265]). But he ended up giving up the thesis that arithmetic can be grounded in Logic and concluded that the truths of number theory are grounded on Geometry: "[...] arithmetic cannot be based on sense perception; [...] So an *a priori* mode of cognition must be involved here. But this cognition does not have to flow from purely logical principles, as I originally assumed. There is the further possibility that it has a geometrical source. [...] Only on this view does mathematics present itself as completely homogeneous in nature." (FREGE, G., 1925, pp.276-7). There is no conclusive textual evidence as to what led Frege to abandon his amended system, but some later developments led to several conjectures. Sobocinski published a proof of Lesniewski (from 1938) that the amended version of Basic Law V is also inconsistent (SOBOCINSKI, B., 1949), while Quine and Geach generalized his result (QUINE, W.O., 1955; GEACH, P., 1956). Dummett claimed that even if Frege did not know of the inconsistency of (V'), he must have known that several arithmetical theorems could not be proved in the revised system, including the theorem that asserts the infinity of the natural number-series (DUMMETT, M., 1967, p.225). Recently, Landini claimed that, although the reconstructions of Quine and Geach do not make historical justice to Frege's formal system, the best conjecture is that Frege himself realized the amended system to be inconsistent (LANDINI, G., 2006, pp.18-19). And, of course, the works of George Boolos, Crispin Wright, Richard Heck and Bob Hale show that there is a viable to revive Frege's Logicism from his works (WRIGHT, C., 1983; BOOLOS, G., 1987; 1992; HECK, R., 1993; WRIGHT & HALE, 2001.).

[12] van HEIJENOORT, J., 1967, p.104-5.

[13] A set *M* is well-ordered by a relation *R* if and only if every non-empty subset of *M* has a minimal element with respect to *R*.

ordinals; but since it is known that the class all ordinals up to some ordinal can be well-ordered, it follows that ordinal number Ω' of all ordinals *O* including Ω should be such that $\Omega' > \Omega$. Similarly, Cantor himself discovered arount 1899 the following: Let $\overline{\overline{X}}$ be the cardinal number of a set X, let $\leq$ and $<$ have their usual meaning, let $\wp(X)$ be the power set of X and, finally, let U be the set of all sets; since U is the set of all sets, it follows that $\wp(U)$ is a subset of U, and thus, that $\wp(U) \leq U$; but Cantor's power-class theorem states that for every set *X*, $X < \wp(X)$. But the fact of the matter is that neither Burali-Forti nor Cantor nor anyone else thought of these results *as paradoxes* until Russell stated his discovery *as a contradiction* in the *Principles of Mathematics*[14]. Russell himself would later observe that he thought "[...] that there was some unimportant error in the reasoning"[15] with Burali-Forti's result[16].

Leaving aside the details of how Russell came to actually recognize that he discovered a genuine contradiction in the foundations of Set Theory[17], it is clear that the fundamental question which was brought up by the discovery was this: how should the ontology of classes be handled once the general naive assumption that every propositional function defines a class is shown to be contradictory? Resolving this problem was, of course, nothing less than a monumental task since it was basically the search for a satisfactory logical theory which could (*a*) serve as a foundation for Mathematics and (*b*) solve the contradictions while (*c*) being well motivated from a philosophical point of view.

The problem was particularly pressing in the *Principles of Mathematics* given Russell's radical response to the ontological question about what *is* or has being. For, in the *Principles*, Russell was committed to the doctrine of the unrestricted variable and of the univocity of being, which, as we previously discussed, is encapsulated in the claims that "[...] there is only one kind of being, namely being *simpliciter*"[18] and that "*Being* is what belongs to every conceivable term, to every possible object of thought - in short to everything that can possibly occur in any proposition, true or false, and to all such propositions themselves"[19]. As we briefly discussed,

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14 Burali-Forti was explicitly stating his result as a *proof by reductio* that there cannnot be a greatest ordinal number; the same was the case with Cantor, but dealing with cardinals. See MOORE & GARCIADIEGO, 1981; GARCIADIEGO, A., 1992.

15 RUSSELL, B., 1957, p.77.

16 By 1905, however, Russell had found a common source in all of these set-theoretical paradoxes. In *On Some Difficulties*, Russell makes the following claim: "Burali-Forti's contradiction is a particular case of the following: 'Given a property ϕ and a function *f*, such that, if ϕ belongs to all the members of *u*, $f`u$ always exists, has the property $\phi$, and is not a member of $u$; then the supposition that there is a class *w* of all terms having the property $\phi$ and that $f`w$ exists leads to the conclusion that $f`w$ both has and has not the property ϕ.' This generalization is important, because it covers all the contradictions that have hitherto emerged in this subject." (RUSSELL, B., 1905*b*, p.71). This, as Landini observes, suggests that here Russell is sketching a the following theorem or result about what sort of classes should be admissible or not (LANDINI, G., 2013*a*, pp.182-3):

$(f)\,[((x)(x \,\epsilon\, u \,.\supset \phi x : \supset : (\exists y) \,.\, f`u = y : f`u {\sim}\epsilon : \phi(f`u)) \,.\, {\sim}(\exists w)((x) : x \,\epsilon\, w \,.\equiv .\, \phi x]$

In case of Russell's own paradox, as he explain "we put '*x* is not a member of *x*' for $\phi!x$, and *u* itself for $f`u$. In this case, owing to the fact that $f`u$ is *u* itself, we have only one possibility: namely that '*x* is not a member of *x*' is non- predicative" (RUSSELL, B., 1905*b*, p.71-2), where by "non-predicative" Russell means that it does not determine a class.

17 For these, cf. MOORE, G., 1988 and GARCIADIEGO, A., 1992.

18 RUSSELL, B., 1903, p.449 §427.

19 RUSSELL, B., 1903, p.449 §427.

"*Term*" is Russell's technical expression for *entity* in the most general sense. In the *Principles*, the notion is introduced in the following famous passage:

> Whatever may be an object of thought, or may occur in any true proposition, or can be counted as one, I call a term. This the, is the widest word in the philosophical vocabulary. I shall use as synonymous with it the words unit, individual, entity. The first two emphasize the fact that every term is one, while the third is derived from the fact that every term has being, i.e is in some sense. A man, a moment, a number, a class, a relation, a chimera, or anything else can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false. [20]

Russell's introduction of the notion brings about several important points about his ontology around the time of the publication of the *Principles*. For one thing, it is clear that in introducing the notion, Russell is introducing an *ontological* category into his philosophical vocabulary, not a merely linguistic one: when he asserts that men, moments, numbers and classes are terms, he is not interested in categorizing the sortal concept-words "man", "moment", etc as falling under the category of *term*, but is making a claim about ontology, namely that men, moments, relations and even chimeras are *entities*, or have *being*. This is made clearer by the following subsequent passage:

> A term is, in fact, possessed of all the properties commonly assigned to substances or substantives. Every term, to begin with, is a logical subject: it is, for example, the subject of the proposition that itself is one.[21]

It also shows that in the *Principles* Russell was committed to a distinction between being and *existence*. Everything that is an entity has *being*, but "[...] existence, on the contrary, is the prerogative of only some amongst beings", since, as Russell puts it, "[...] to exist is to have a specific relation to existence—a *relation*, by the way, which existence itself does not have"[22]. Differently from *existence* which, one can deny of some entities, *being* (allegedly) cannot:

> [...] this distinction is essential, if we are ever to deny the existence of anything. For what does not exist must be something, or it would be meaningless to deny its existence; and hence we need the concept of being, as that which belongs even to the non-existent".[23]

This point of fundamental importance is echoed when Russell asserts that "[....] to deny such and such a thing is a term must always be false"[24]. And the reason why "[...] it must always

[20] RUSSELL, B., 1903, p.43 §47.
[21] RUSSELL, B., 1903, p.44 §47.
[22] RUSSELL, B., 1903, p.449 §427.
[23] RUSSELL, B., 1903, p.450 §427.
[24] RUSSELL, B., 1903, p.43 §47.

be false" is this: in order for a sentence like "so-and-so does not have *being*" to be meaningful, there must be *something* for which the phrase "the so-and-so" stands; but to say that *so-and-so* is *something* is to say it has *being*, so it must be a term.

As we also discussed, Russell was led to this conclusion because the radical metaphysical thesis of the univocity of being came hand in hand in the *Principles* with the idea that in a sentence, every word must stand for a constituent of the proposition expressed by the sentence[25]:

> I shall speak of the terms of a proposition as those terms, however numerous, which occur in a proposition and may be regarded as subjects about which the proposition is.[26]
>
> [...] it must be admitted, I think, that every word occurring in a sentence must have some meaning: a perfectly meaningless sound could not be employed in the more or less fixed way in which language employs words. The correctness of our philosophical analysis of a proposition may therefore be usefully checked by the exercise of assigning the meaning of each word in the sentence expressing the proposition.[27]
>
> Words all have meaning, in the simple sense that they are symbols which stand for something other than temselves. But a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words.[28]

According to such passages, propositions are complex entities which contain the *terms* about which the proposition is (and the proposition itself is a term) and sentences are complex *linguistic* items which denote the complex entities that are propositions. The complexity of a sentence reflects the complexity of the proposition it denotes in such a way that to every expression in the sentence there will be a *term* to which the expression refers that is a constituent of the proposition that the whole sentence denotes.

However, there are further taxonomic distinctions that must be observed, which is not something particularly easy to do given Russell's characteristic use-mention carelessness. Russell claims that words, in general, denote *terms*; proper names denote *things*, while all other words denote *concepts*; among concepts, he distinguishes those that are denoted by adjectives, which he calls *class-concepts* and those that are denoted by verbs, which are *relations*[29]. Rus-

[25] Recalling that a *proposition* is understood as a sort of entity which has also entities as constituents, not necessarily linguistic

[26] RUSSELL, B., 1903, p.45 §48.

[27] RUSSELL, B., 1903, p.42 §46.

[28] RUSSELL, B., 1903, p.47 §51. Famously, Russell made similar considerations when discussing with Frege whether "Mont Blanc with its snowfields" is or is not part of the thought that "Mont Blanc is more than 4000 metres high" in their correspondence. (Frege to Russell November 13 1904. In: FREGE, G., 1980, p.163). As Russell saw the matter the "[...] Mont Blanc itself is a component part of what is actually asserted in the proposition 'Mont Blanc is more than 4000 metres high.' We do not assert the thought, for this is a private psychological matter: we assert the object of the thought, and this is, to my mind, a certain complex (an objective proposition, one might say) in which Mont Blanc is itself a component part. If we do not admit this, then we get the conclusion that we know nothing at all about Mont Blanc." (Russell to Frege 12 December 1904. In: FREGE, G., 1980, p.169)

[29] RUSSELL, B., 1903, p.44 §48.

sell's distinctions are not clear enough mainly due to his selection of terminology. On the one hand, when he makes a distinction among *terms* between *things* and *concepts*, it is clear that he *cannot* be making a linguistic distinction: it is a distinction among sorts of entities. On the other hand, the introduction of the further distinction among concepts is not that clear cut, for when he speaks of those concepts indicated by adjectives and those indicated by verbs he asserts that "[...] the former kind will often be called predicates or class-concepts; the latter are always or almost always relations"[30]. Here, one must not be misled by Russell's use of "predicate" as equivalent to "class-concept"; "*attribute*" would be a more appropriate choice. For when speaking of relations as a sort of concept, it is clear what the point is: a relation is a sort of entity (term) generally indicated in sentences by the occurrence of verb-words, and the same must be the case for ajdective-words.

If we take as examples (*p*) "Brutus killed Caesar" and (*q*) "Caesar is human", the first sentence as a whole indicates a complex entity which consists of Brutus in a certain relation to Caesar, namely that of being the agent of his death; this relation which is a sort of concept (therefore a sort of *term*) is indicated by the verb "killed", and the things (terms) which are the *relata* of the relation are denoted by the proper names "Brutus" and "Caesar". Similarly the second indicates a complex which consists in Caesar and the concept *humanity*. Russell introduces his taxonomy in order to introduce a further distinction between a term occurring *as a term* and a term occurring *as a concept*[31]. Take as examples the two sentences (*p*) "Socrates is human" and (*p*') "Humanity belongs to Socrates", both of which Russell thought to be names of propositions (in the ontological sense). According to Russell, despite being equivalent[32], these propositions are distinct in the sense that in (*p*) "[...] the notion expressed by human occurs in a different way from that in which it occurs when it is called humanity, the difference being that in the latter case, but not in the former, the proposition is about this notion"[33]. Russell's point is that in (*p*) the *logical subject* of the proposition is *Socrates* whereas in (*p*') the logical subject is *humanity.* As he further explains "[...] the terms of a proposition" are "[...] those terms, however numerous, which occur in a proposition and may be regarded as subjects about which the proposition is"[34]

As Nino Cocchiarella showed long ago[35], the notion of a logical subject is central for understanding the tension which the contradiction generates within the philosophical framework of the *Principles*[36]. The main reason for this is that the notion is the key for understanding Russell's *unrestricted* conception of variation held in the work. It is Russell's notion of a logical

30 RUSSELL, B., 1903, p.44 §48.
31 COCCHIARELLA, N., 1980, pp.72-3.
32 RUSSELL, B., 1903, p.45 §48. Presumably, when he speaks of equivalence, he is talking about *material* equivalence.
33 RUSSELL, B., 1903, p.45 §48.
34 RUSSELL, B., 1903, p.45 §48.
35 COCCHIARELLA, N., 1980.
36 COCCHIARELLA, N., 1980, pp.72-7.

subject that entails what Cocchiarella dubbed the thesis of "univocity of being"[37]. In the *Principles*, Russell summarizes this thesis with the claim that "[...] there is only one kind of being, namely being *simpliciter*"[38]. As we discussed in the previous chapter, the most important consequence of this thesis is that general statements, that is, statements involving *all* and *any* - i.e., statements involving *variables* - have an *absolutely general* character. As Russell explains, in a statement like "Every *x* is so-and-so" we can make "[...] our *x* always an unrestricted variable", in the sense that "[...] we can speak of the variable, which is conceptually identical in Logic, Arithmetic, Geometry, and all other formal subjects"[39], that is: "[....] the terms dealt with are always *all* terms"[40], so there is no *intrinsic restrictions* upon the range of the variable. According to Russell, one can only 'restrict' a variable by explicitly stating antecedents in a formal implication, as in "$(x)(\phi x \supset \psi x)$", but this can only be understood as an artificial 'restriction' or as no restriction at all for this in fact asserts of *every term x* that if *x* is a ϕ it is also a ψ. This is Russell's famous thesis of the *unrestricted variable*, the fundamental logical doctrine of the *Principles of Mathematics*.

Concerning the terms of a proposition, that is, its logical subjects, Russell asserts that "It is a characteristic [of them] that any one of them may be replaced by any other entity without our ceasing to have a proposition."[41]. This means that a term can only be considered a logical subject of a proposition if it occurs *as a term* and not as a concept and points to a fundamental difference between things and concepts with respect to them as possible logical subjects of propositions: a thing can only occur *as a term*, that is, as a logical subject. If "*is a man*" is substituted for "Socrates", we get the sentence: (*p**) "Socrates is a Socrates" which is nonsensical, for two reasons. First, there is nothing *occurring as a term* corresponding to "is a man" in (*p**) to be substituted in the proposition for Socrates, since the only expression which denotes a term occurring *as a term* in (*p**) is "Socrates". Second, *things* like Socrates (not to be confused with the name "Socrates") can only occur in a propositions *as terms*, that is, *as logical subjects*. A concept, however, can occur as a logical subject but also as a *concept* (or, in Russell's more misleading terminology, as a *predicate*). If we take the proposition (*p*) and substitute Socrates for a (or, better yet, *the*) variable, we obtain what Russell calls a propositional function or propositional form, namely *x is a man*. Given that the concept *is a man* is a term, there must be a proposition which results from taking this concept itself as a value *x*. But this proposition is not expressed by the sentence (*p***) "Is a man is a man" which is also nonsense, but by (*p****) Mankind is a man. And in this proposition, the concept does not occur *as a* concept, but *as a term*. As Russell further explains:

> Predicates, then, are concepts, other than verbs, which occur in propositions having only one term or subject. *Socrates is a thing, because Socrates can*

[37] COCCHIARELLA, N., 1980, p.72.
[38] RUSSELL, B., 1903, p.449 §427.
[39] RUSSELL, B., 1903, p.45 §48.
[40] RUSSELL, B., 1903, p.45 §48. Our emphasis.
[41] RUSSELL, B., 1903, p.45 §48.

> *never occur otherwise than as a term in a proposition: Socrates is not capable of that curious twofold use which is involved in human and humanity.*[42]

Thus, up to this point, what the doctrine of the unrestricted variable states once put in the context of Russell's realist theory of propositions and of his views on the nature of terms can be summarized as follows:

1. Although propositions have many terms as their *constituents*, not every term that occurs in a proposition occurs *as a term*, that is, as a logical subject.
2. The logical subject of a proposition expressed by a sentence that contains an adjective is the entity denoted by the grammatical subject to which the adjective is attributed and in this proposition the term denoted by the adjective occurs as a concept, not as a subject.
3. The logical subjects of a proposition expressed by a sentence that contains a verb are the relata of the relation denoted by the verb and in this proposition the relation occurs as a concept not as a subject.

Now, Russell must answer a question that complicates this: given this "curious twofold use" of predicates [concepts indicated by adjectives] which allows them to occur *as terms,* that is, as logical subjects, and *as concepts*, does this mean that they occur as different sorts of entities? To put it more specifically, given the two propositions (*p*) and (*p*') of our previous example, we have the occurrence of two different terms occurring in each, in one case a concept and in the other a thing?[43]

These questions are particularly pressing in connection with class-concepts like x *is a class that does not belong to itself* and they cast a shadow on the whole philosophical outlook of the *Principles*, for they threaten Russell's most fundamental theses: that whatever can be mentioned *is an entity* or a has *being*; that whatever is *is a term*; and that "[...] any one of them may be replaced by any other entity without our ceasing to have a proposition"[44]. As we shall see, Russell's philosophical development (regarding Mathematical Philosophy) from 1903 through 1910 can be seen as an attempt to reconcile these views with a solution of the contradiction.

3.1.2 Russell's Failed Early Attempts at Solving the Contradiction

Russell's first "tentative" attempt at a solution of the contradiction was his initial formulation of the theory of types presented in the *Appendix B* of the *Principles* and in his correspondence with Frege. The basic idea of that version of the theory was that "every propositional

[42] RUSSELL, B., 1903, p.45 §48. Our emphasis.
[43] RUSSELL, B., 1903, p.45 §48.
[44] RUSSELL, B., 1903, p.45 §48.

function $\phi(x)$ [...] has, in addition to its range of truth, a range of significance, i.e. a range within which *x* must lie if $\phi(x)$ is to be a proposition at all, whether true or false[45]."

Thus, the cornerstone of that theory is that there are values of *x* for which "$\phi(x)$" is meaningless. In this theory 'types' are defined as ranges of significance of propositional functions forming a *hierarchy*:

> A *term* or *individual* is any object which is not a range. This is the lowest type of object. If such an object - say a certain point in space - occurs in a proposition, any other individual may *always* be substituted without loss of significance. [...] The next type consists of ranges or classes of individuals. [...] The next type after classes of individuals consists of classes of classes of individuals[46].

So Russell's schematic response to the contradiction in the *Appendix B* of *Principles* was to somehow weaken the doctrine of the unrestricted variable, admitting that variables could be restricted when quantification over propositional functions and/or classes could lead to contradictions. This however, threatened Frege's and Russell's early conception of Logic as a whole - as Frege himself pointed out to Russell in a letter from 29 June 1902 correspondence:

> [...] if you admit a sign for the extension of a concept (a class) as a meaningful proper name and hence recognize a class as an object, then the class itself must either fall under the concept or not; *tertium non datur.* [47]

Russell's correspondence with Frege shows that the former had envisaged a way out the difficulty by denying that classes such as the class of all classes that do not belong to themselves were genuine *entities*, via his distinction of a class *as one vs* class *as many.* On 10 July 1902, Russell wrote:

> Concerning the contradiction, I did not express myself clearly enough. I believe that classes cannot always be admitted as proper names. A class consisting of more than one object is in the first place not *one* object, but many[48].

Frege saw that this also would not do, pointing out that this was to ground Arithmetic (for Russell, Pure Mathematics in general) on a theory of aggregates or systems (what today we would call *fusions*[49]), thus failing to provide Arithmetic with *objects.* On 28 July 1902 Frege replied the following:

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45 RUSSELL, B., 1903, p.523 §497.
46 RUSSELL, B., 1903, p.523 §497.
47 FREGE, G. 1980, p.135.
48 FREGE, G. 1980, p.137.
49 As Michael Potter puts it "[...] a fusion is conceived of as being nothing over and above its members" a *plurality*, "whereas a set is thought of as distinct from that plurality" (POTTER, M., 1999, p.98).

> It seems to me that you want to admit only systems and not classes. I myself was reluctant to recognize ranges of values and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is: How do we apprehend logical objects? And I have found no other answer to it than this: We apprehend them as extensions of concepts, or more generally, as ranges of values of functions. I have always been aware that there are difficulties connected with this, and your discovery of the contradiction has added to them; but what other way is there?[50]

Frege had good reasons to see no way around this difficulty - reasons that were partially shared by Russell (at that time). Despite their famous and severe divergences on topics of Philosophy of Language, Frege and Russell could not admit that something other than an *object* (Frege's term) or *individual* (Russell's) could be the logical subject of a singular proposition - and if set theory was to serve as a genuine foundation for Mathematics, classes should be thus taken as objects and class expressions as singular terms. Frege made the point with his usual lucidity in a letter from 28 July 1902:

> If a class name is not meaningless, then, in my opinion, it means an object. In saying something about a manifold or set, we treat it as an object. A class name can appear as the subject of a singular proposition and therefore has the character of a proper name, e.g., 'the class of prime numbers comprises infinitely many objects'[51].

As long as Russell shared Frege's view that every class-expression should be treated as a proper name there was no way around this difficulty: either this or the unrestricted variable had to go.

So it was that shortly after the publication of the *Principles*, Russell told Jourdain on 19 January 1906 he thought "[...] had solved the whole thing by denying classes altogether" while keeping propositional functions as entities[52]. In fact, he wrote to Frege by 24 May 1903, just days after publishing the book[53], asserting that he "[...] discovered that classes are entirely superfluous. Your designation $\grave{\varepsilon}\phi(\varepsilon)$ can be used for ϕ itself, and $x \frown \grave{\varepsilon}\phi(\varepsilon)$ for $\phi(x)$"[54]. But, again, at the time this was another a dead-end.

It took Frege almost a year and a half to answer Russell's letter and by the time he did, Russell had already given up the strategy. As Frege pointed out to Russell, the short lived solution was spurious since the the letters "ϕ", "ψ", etc., were "[...] no longer used as function letters but as object letters"[55]. Frege's sound criticism relied on his conception of functions as 'unsaturated' entities, as he explained in a letter to to Russell from 13 November 1904:

50 FREGE, G. 1980, p.140-1.
51 FREGE, G. 1980, p.140.
52 GRATTAN-GUINNESS, I., 1977, p78.
53 GRATTAN-GUINNESS, I., 2000b, p.328-9.
54 FREGE, G. 1980, p.159.
55 FREGE, G. 1980, p.162. Russell seems to have partially accepted Frege's contention, for in his reply from 12 December 1904 he asserts: "I have known already for about a year that my attempt to make classes entirely dis-

> To use a function sign in isolation is to contradict the nature of a function, which consists in its unsaturatedness. For this is how a function differs from an object. This is also why function names must differ essentially from proper names, the difference being that they can carry with them at least one empty place - an argument place. And these argument places must always be preserved in a function name and be recognizable as such; otherwise the function name becomes a meaningless proper name. The same must hold for function letters, at least wherever they are to be replaceable by function names.[56]

Russell's reply shows how deep their disagreement on these matters was. He referred back to the criticism he had already made in the *Appendix B* of the *Principles* to the consequences Frege extracted from his distinction between functions and objects, pointing specifically to paragraph §483, in which he made the two following points[57]: 1. If one tries to claim that *it is not the concept*, *but the name that is involved*, one is already making the concept a logical subject; 2. It is always legitimate to ask what entity a name names and if there is a positive answer (that is, if it is a meaningful name), then, in the case of the symbol for a concept, the symbol must name the concept, which is then a possible logical subject of a proposition.

Frege would accept neither of these points, for regarding both of them he would claim that whenever we try to make a *concept* the subject of a possible judgment, we are not asserting something about *a concept*, but about an object (hence his notorious assertion that "the concept horse is not a concept"[58]). It is puzzling, then, that before making this point Russell asserts in the same letter from 12 December 1904 that he "[...] knew, for about a year that [his] attempt to make classes entirely dispensable was a failure, for essentially the same reasons as you give"[59]. Probably by "essentially the same reasons" he means the fact the his strategy could not avoid the use of function symbols as names of objects, for as Russell later explained in the letter from 19 January 1906 to Jourdain about his work on this period, he "treated ϕ as entity" and this was precisely the reason the strategy "brought back the contradiction"[60]. Given that, according to his unrestricted conception of variation, if a functional variable "ϕ" is treated as an entity variable, one can define a function W as follows:

$$(W) \qquad W(\phi) = \sim \phi(\phi) \text{ Df}$$

pensable was a failure, for essentially the same reasons as you give." However, he still refrained from accepting Frege's views on functions: "But it is not yet clear to me that it is never permissible to use a function letter in isolation. On this point I do not quite share your view, for reasons you will find in my book, sects 480ff, especially sect.483. My present belief is roughly as follows. In the case of a particular function, e.g., $(\xi - 1) \cdot (\xi + 1)$,what arises through the mere omission of ξ can certainly not be regarded as an object. But I believe that if we use the notation ϕx, the letter ϕ must designate something that remains the same when *y* is substituted for *x*. This something is, I believe, what is designated by $\phi\xi$" (FREGE, G. 1980, p.166-7).

56 FREGE, G. 1980, p.160-1.

57 RUSSELL, B., 1903, p.504 §483.

58 FREGE, G., 1892*b*, pp.192-3.

59 FREGE, G. 1980, p.160-1.

60 GRATTAN-GUINNESS, I., 1977, p78.

Which is a function that applies to every function that does not apply to itself. Then, we have[61]:

$$(W') \qquad (\phi)(W(\phi) \,.\!\equiv\!.\! \sim \phi(\phi))$$

The question that can obviously be introduced then is this: Does this *W* apply to itself or not? In exactly the same way the paradox of the class of all classes emerged, we have:

$$(W'') \qquad W(W) \,.\!\equiv\!.\! \sim W(W)$$

Thus, the same problem remained and, as Russell put, he "[...] gained nothing by rejecting classes"[62].

In reminiscence, Russell regarded his work along the period of 1903-1904 as being completely unsuccessful toward the goal of resolving the contradiction, with a breakthrough coming with his discovery of the theory of descriptions[63] first presented in *On Denoting*. In what follows we discuss why that was the case.

3.2 The 'Rise' of Russell's Substitutional Theory of Classes and Relations

3.2.1 Russell's Reception of Frege and the Renewed Theory of the Variable

As noted in the previous chapter, the Logic of the *Principles* had significant improvements with respect to Peano's, but in many respects it also kept some of its shortcomings[64]. The most notable of these are salient in Russell's account of quantification: as in the case of Peano's works, there is no explanation of generality in the *Principles* as clear and precise as that of Frege

[61] GRATTAN-GUINNESS, I., 1977, p78.

[62] GRATTAN-GUINNESS, I., 1977, p78. What is even more puzzling however, is the fact that Russell himself antecipated this outcome already in the *Principles of Mathematics*! For there, he had already claimed that "the ϕ in ϕx is not a separate and distinguishable entity: it lives in the propositions of the form ϕx, and cannot survive analysis", precisely for the reasons discussed above: "If ϕ were a distinguishable entity, there would be a proposition asserting ϕ of itself, which we may denote by $\phi(\phi)$; there would also be a proposition $\sim \phi(\phi)$, denying $\phi(\phi)$. In this proposition we may regard ϕ as variable; we thus obtain a propositional function. The question arises: Can the assertion in this propositional function be asserted of itself? The assertion is non-assertibility of self, hence if it can be asserted of itself, it cannot, and if it cannot, it can. This contradiction is avoided by the recognition that the functional part of a propositional function is not an independent entity. As the contradiction in question is closely analogous to the other, concerning predicates not predicable of themselves, we may hope that a similar solution will apply there also."(RUSSELL, B., 1903, p.88 §85).

[63] RUSSELL, B., 1944, p.224; RUSSELL, B., 1959, p.79; RUSSELL, B., 1967, p.229.

[64] Most notably, the problematic theory of denoting concepts. Indeed, Landini argues that the Substitutional Theory began in the *Principles* itself, but it could not work together with the denoting concepts, so it could only take off with the theory of incomplete symbols in place (cf. LANDINI, G., 1998, pp.89-93).

and none of Russell's works dealing with formal deductions produced up to the publication of the *Principles* provide a notation that is precise (and also flexible) enough to adequately express scopes of quantified variables in the way that Frege had done flawlessly - albeit unknowingly - long before.

This situation changed, however, shortly after the publication of the *Principles*. The extent of Russell's breakthrough is fully displayed in the paper *The Theory of Implication*[65], his first (published) formal development of the elementary portions of Mathematical Logic (i.e., propositional logic and quantification theory) and also the first to exhibit the acceptance of Frege's account of the quantifier[66]. The paper shows Russell modifying both his accounts of propositional connectives and of quantifiers as given in the *Principles*. Russell assumed *both* implication and negation as primitive and he simplified his definitions by changing the meaning of "$p \supset q$" to "if p is true, then q is true"[67]. This, as Landini points out[68], marks a separation of the formal treatment of quantification theory from that of the calculus of propositions. This separation greatly simplified the definitions and primitive propositions. Instead of the somewhat archaic calculus of the *Principles* discussed in the previous chapter, we find the following very familiar axioms and definitions in the 1906 system, where both implication and negation are now assumed as primitive notions[69]:

$$\vdash . \, p \supset p \; \mathrm{Pp}$$

$$\vdash : p \, . \supset . \, q \supset p \; \mathrm{Pp}$$

$$\vdash : p \supset q \, . \supset : q \supset r \, . \supset . \, p \supset r \; \mathrm{Pp}$$

$$\vdash :. \, p \, . \supset . \, q \supset r : \supset : q \, . \supset . \, p \supset r \; \mathrm{Pp}$$

$$\vdash : \sim \sim p \, . \supset . \, p \; \mathrm{Pp}$$

$$\vdash : p \supset \; \sim p \, . \supset . \sim p \; \mathrm{Pp}$$

$$\vdash : p \supset \; \sim p \, . \supset . \, q \supset \; \sim p \; \mathrm{Pp}$$

$$p \, . \, q \, . = . \sim (p \supset \sim q) \; \mathrm{Df}$$

$$p \vee q \, . = . \sim p \supset q \; \mathrm{Df}$$

$$p \equiv q \, . = . \, p \supset q \, . \, q \supset p \; \mathrm{Df}$$

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65 RUSSELL, B., 1906 The paper was finshed writing the paper around July of 1905, but it was published only by April of 1906 (cf. MOORE, G., 2014, p.14).

66 Short, of course, of Frege's doctrine of functions as incomplete entities.

67 RUSSELL, B., 1906, p.24.

68 LANDINI, G., 1998*a*, p.53. Though, as Landini also observes, Russell may have still thought that a calculus like that of the *Principles* was "still viable". His choice in the 1906 was most likely a matter of simplicity in exposition.

69 RUSSELL, B., 1906*c*, p.29-30, propositions ∗2·5, ∗2·6, ∗2·7 , ∗2·8, ∗2·9, ∗2·91, ∗2·92 , ∗4·1, ∗4·11 and ∗4·12. *Modus Ponens* was again put forward as a 'primitive proposition' that cannot be stated in symbols, namely, "[...] anything implied by a true proposition is true" (RUSSELL, B., 1906*c*, p.27.).

Russell used a notation which we can schematically represent by "$(A \between x_1, ..., x_n)$" as standing for propositional functions - i.e., a function whose values are propositions[70] - of *n* many arguments. That is, he employed "$(A \between x)$", "$(B \between y)$", "$(C \between x, y)$", etc., as open formulas, i.e., sentences with real (free) variables. Crucially, employing this notion he introduced the notation "$(x).(A \between x)$" for the universal quantifier standing for the now *primitive* idea "$(A \between x)$ is true for all values of *x*"[71]. Thus, in what was a major step forward with respect to Peano and his own earlier works, Russell could then distinguish clearly the different scope of an apparent (bound variable can have) using dots. This is clearly displayed in the primitive propositions which he assumes[72]:

$$\vdash : (x) \,.\, (A \between x) \,.\, \supset \,.\, (A \between y) \text{ Pp}$$

What is true of any is true of all. Pp

$$\vdash :. \, (x) \,.\, p \supset (A \between x) \,.\, \supset : p \,.\, \supset \,.\, (x) \,.\, (A \between x) \text{ Pp}$$

Indeed, all of the above are taken directly from Frege's *Begriffschrift*: $*7{\cdot}1$ is just the law of universal instantiation, i.e., proposition number 58 of the *Begriffschrift*[73] while $*7{\cdot}11$ and $*7{\cdot}2$ correspond to Frege's rules for quantification theory[74], with the second transformed by Russell into a law or 'primitive proposition'[75].

This approximation to Frege was made possible because Russell gave up his previous goal of giving an account or analysis of the constituents of general propositions - in particular 'the variable' - in terms of other notions, most notably that of *denoting*[76]. In fact, the change was prompted by Russell's abandonment of the *Principles's* theory of denoting concepts. The

[70] RUSSELL, B., 1906*c*, p.26. Since Russell understood propositional variables as genuine variables which ranged over entities (namely, propositions), he also viewed formulas like "$p \supset q$" as propositional functions that become sentences once the variables "*p*", "*q*" are replace by actual sentences or are bound by quantifiers.

[71] RUSSELL, B., 1906*c*, p.52.

[72] RUSSELL, B., 1906*c*, p.54-55, propositions $*7{\cdot}1$ and $*7{\cdot}1$.

[73] Cf. FREGE, G., 1879, p.51.

[74] Cf. FREGE, 1879, pp.25-6.

[75] As with the rule of *modus* ponens, the peculiar status of 'proposition' $*7{\cdot}11$ is also explicitly recognized. Russell writes: "This proposition is the converse of $*7{\cdot}1$. It cannot be symbolized without inventing a new symbol for the purpose, which seems not worth while, as this symbol would not be afterwards required. It might be supposed at first that what we mean could be expresed by

$$\vdash : (C \between y) \,.\, \supset \,.\, (x) \,.\, (C \between x)$$

But this would mean: "$(C \between y)$ implies that $(C \between x)$ is true for all values of *x*, where *y* may be anything", which is not what we mean, and is not in general true. What we mean is: "If $(C \between y)$ is true whatever *y* may be, then $(C \between x)$ is true for all values of *x*". When $(C \between y)$ is asserted, we have:

$$\vdash : (C \between y) \,.\, \supset \vdash . \, (x) \,.\, (C \between x)$$

but we cannot adopt this as our formula, because we must only put the assertion sign before what really is true, and here $(C \between y)$ is to be only a hypothesis, which may or may not be true." (RUSSELL, B., 1906*c*, p.54-5).

[76] It must be observed - and this is a point to be addressed below - that right after abandoning the theory of denoting concepts in *On Denoting* Russell adopts an account of propositions which abandons general propositions, so that particular problem of explaining their constituents vanishes. This, of course, does not mean that Russell abandons the use of formulas or sentences which contain quantifiers and variables; as we shall see, variables as part of the formal apparatus and the notation of the quantifier, in fact, are kept as primitive notions.

transition was completed by the time Russell published[77] his most famous and perhaps most important technical paper, *On Denoting*[78], on October of 1905. There Russell presents his new view of denoting which completely overhauls the theory of the *Principles*:

> I take the notion of the *variable* fundamental; I use "$C(x)$" to mean a proposition [in fact, as Russell indicates a footnote, "more exactly, a propositional function"] in which *x* is a constituent, where *x*, the variable, is essentially and wholly undetermined. Then we can consider the two notions "$C(x)$ is always true" and "$C(x)$ is sometimes true". Then *everything* and *nothing* and *something* (which are the most primitive of denoting *phrases*) are to be interpreted as follows:
>
> *C*(everything) means "$C(x)$ is always true";
>
> *C*(nothing) means " '$C(x)$ is false' is always true";
>
> *C*(something) means "It is false that '$C(x)$ is false' is always true";
>
> Here the notion "$C(x)$ is always true" is taken as ultimate and indefinable and the others are defined by means of it. *Everything*, *nothing*, and *something*, are not assumed to have any meaning in isolation, but a meaning is assigned to every proposition in which they occur. This is the principle of the theory of denoting I wish to advocate: that denoting phrases never have any meaning in themselves, but that every proposition in whose verbal expression they occur has a meaning. The difficulties concerning denoting are, I believe, all the result of a wrong analysis of propositions whose verbal expressions contain denoting phrases.[79]

Against the analysis of the *Principles*, Russell is explicitly denying here that phrases like "*all x*" or "some *x*" must stand for genuine constituents of propositions. In fact, Russell is also flat out refusing an important doctrine which in the *Principles* came hand in hand with the doctrine of the unrestricted variable, namely, that every part of a significant sentence must stand for a constituent of the proposition that the sentence in question expresses.

As is well known, however, this abandonment of denoting concepts is surrounded by interpretative problems that are almost intractable. Famously, Russell puts forward in *On Denoting* an argument against the theory of denoting concepts of the *Principles* that became known in secondary literature as "the Gray Elegy argument"[80]. This argument, as Russell puts it, aims at showing that "the relation of the meaning to the denotation involves certain rather curious difficulties" which suffice to "prove that the theory which leads to such difficulties must be wrong"[81]. It is reasonable - perhaps uncontroversial - to say, however, that the presentation of the argument itself as given in *On Denoting* may very well be the most controversial and problematic set of paragraphs in Russell's logico-philosophical works. The argument seems plagued -

77 Most likely the paper was written between July 3 and August 3 of 1905 (URQUHART, A., 1994, p.li).

78 RUSSELL, B., 1905*a*.

79 RUSSELL, B., 1905*a*, p.416.

80 Russell claims that the argument is directed at what he takes to be Frege's views on the matter, but this, it seems, can only be the case if Russell distorted Frege's views beyond recognition. As Peter Geach pointed out long ago, the target of the argument are Russell's own previous views on the matter (cf GEACH, P., 1959) which he *quite mistakenly* thought to be essentially the same as Frege's. This seems to be one of the very few consensual points about the "Gray Elegy argument".

81 RUSSELL, B., 1905, p.421.

even more than usual - by Russell's customary lack of care with use and mention, to the point that many authors - including, for instance, Alonzo Church - have argued that there is nothing but nonsense to be availed in the argument[82]. Furthermore, as some interpreters have observed, Russell's unpublished manuscripts dealing with the issue of denoting produced between the publication of the *Principles* and *On Denoting* are so dauntingly complex and obscure that they seem to just raise more questions instead of providing helpful answers. Finally - although this may be considered an 'external' issue, the literature which discusses the purpose and content of the "Gray Elegy argument" is at this point so massive and abstruse that any attempt at a concise and clear-cut discussion of it seems doomed from the start[83]. As Gregory Landini colorfully but truly notes, the argument thus seems like "[...] a siren song which lives up to the *Odyssey* in bringing to ruin all who hope to probe its pages" and "[...] sooner or later the interpreter is seduced by one or another of the songs and, impelled by its blissful soporific, dashed on the rocky crags"[84].

So here we shall take heed of Landini's advice of "[...] navigating past the sirens by being firmly lashed to the mast of the *Principles*"[85]. The key for understanding the core of the argument, as Landini convincingly argues[86], is to pay attention to how the theory of denoting concepts threatens Russell's central doctrine of the unrestricted variable. The central point to be observed is this: the doctrine of the unrestricted variable requires that if there is such an entity as a denoting concept *C*, then *C* must be capable of being a logical subject, i.e., occur as a *term* in a proposition; but if *C* is a *concept* in the sense of the *Principles*, in order for *C* to be a logical subject of a proposition *p* it must have what Russell refers to in *Principles* as the "[...] curious twofold use"[87] of concepts, that is, it must be capable of occurring *as a concept* and *as a term* in a proposition. In a nutshell, Landini argues that this is the main issue to which Russell failed to find a satisfactory account: the possibility of a denoting concept occurring as a term or entity in a proposition. This is a straightforward and serious problem that in all likelihood led to the demise of the theory of denoting concepts and for our purposes it is the main relevant issue underlying the tortuous argument.

What then, of the variable? Isn't it viewed as an entity by Russell and if so can it be considered the last standing denoting concept? Some authors have argued in favor of this view. Peter Hylton, for instance suggests that "[...] Russell eliminates denoting only by assuming the crucial case of it - i.e. that he does not eliminate denoting but simply reduces it to the one

[82] Cf. CHURCH, A., 1943, p.302.
[83] For a sample of the controversies surrounding the "Gray Elegy Argument" of *On Denoting*, cf: GEACH, P., 1958; BLACKBURN & CODE, 1978; GEACH, P., 1978; HYLTON, P., 1990, pp. 237-275; PAKALUK, M., 1993; WAHL, R., 1993; KREMER, M., 1994; MAKIN, G., 1995; LANDINI, G., 1998*a*; DEMOPOULOS, D., 1999; POTTER, M., 2000, pp.122-8; LEVINE, J., 2004; KAPLAN, D., 2005.
[84] LANDINI, G., 1998*b*, p.43-44.
[85] LANDINI, G., 1998*b*, p.44.
[86] LANDINI, G., 1998*b*, p.59-67.
[87] RUSSELL, B., 1903, p.45, §48.

case of the variable"[88]; Hylton also thinks that the notion must survive because it is required to make sense of propositional functions, specifically the following idea that later appears explicitly articulated in *Principia* in the following terms:

> When we speak of "ϕx", where x is not specified, we mean one value of the function, but not a definite one. We may express this by saying that "ϕx" ambiguously denotes ϕa, ϕb, ϕc, etc., where ϕa, ϕb, ϕc, etc. are the various values of 'ϕx'. When we say that "ϕx" ambiguously denotes ϕa, ϕb, ϕc, etc., we mean that "ϕx" means one of the objects ϕa, ϕb, ϕc, etc., but an undetermined one.[89]

The question of the nature of propositional functions in *Principia* notwithstanding, the claim that the variable is kept as a denoting concept which is the characteristic constituent of propositional functions cannot be attributed to Russell at this point in time (i.e., between *On Denoting* and *Principia*).

Already in the *Principles* where the variable was explicitly treated as a denoting concept, Russell refused to acknowledge propositional functions as (complex) *terms*. Russell unequivocally voices this view in the - often ignored - passage which we already had occasion to discuss where he claims that "[...] the ϕ in ϕx is not a separate and distinguishable entity: it lives in the propositions of the form ϕx, and cannot survive analysis"[90]; the reason for this apparent exception to Russell's view that "[...] anything that can be mentioned"[91] must be a logical subject (i.e., *a term*), is nothing less than the threat of Russell's paradox framed in terms of predication. As Russell explains:

> I am highly doubtful whether such a view does not lead to a contradiction, but it appears to be forced upon us, and it has the merit of enabling us to avoid a contradiction arising from the opposite view. If ϕ were a distinguishable entity, there would be a proposition asserting ϕ of itself, which we may denote by $\phi(\phi)$; there would also be a proposition not-$\phi(\phi)$, denying $\phi(\phi)$. In this proposition we may regard ϕ as variable; we thus obtain a propositional function. The question arises: Can the assertion in this propositional function be asserted of itself? The assertion is non-assertibility of self, hence if it can be asserted of itself, it cannot, and if it cannot, it can. This contradiction is avoided by the recognition that the functional part of a propositional function is not an independent entity. As the contradiction in question is closely analogous to the other, concerning predicates not predicable of themselves, we may hope that a similar solution will apply there also.[92]

As we just discussed, however, no such solution is given in the *Principles*. But by the time Russell published *On Denoting* he did have a solution given in terms of his Substitutional

88 HYLTON, P., 1990, p.254-5.
89 WHITEHEAD & RUSSELL, 1925, p.39 [1910, p.41].
90 RUSSELL, B., 1903, p.88, §85.
91 RUSSELL, B., 1903, p.43, §47.
92 RUSSELL, B., 1903, p.88, §85.

Theory of Classes and Relations. As we shall discuss in detail below, the Substitutional Theory dispensed *completely* with propositional functions as entities of *any sort*. So the idea that Russell had to appeal to the notion of a propositional function as an entity in order to account for the variable in *On Denoting* simply cannot be right.

Still, Russell was still committed to general propositions understood as entities, so the problem of explaining what sort of constituent characterized general propositions - i.e., variables - remained. But the fact of the matter is that Russell seems to have simply left this problem unsolved. In a now famous letter that Moore wrote to Russell on 23 October 1905, the former raised the issue as follows:

> You say 'all the constituents of propositions we apprehend are entities with which we have immediate acquaintance.' Have we, then, immediate acquaintance with the variable?[93]

On 25 October 1905, Russell replied as follows:

> I only profess to reduce the problem of denoting to the problem of the variable. This latter is horribly difficult, and there seem to be equally strong objections to all the views I have been able to think of.[94]

This reply strongly corroborates Landini's view that by the time Russell had published *On Denoting* and adopted the Substitutional Theory he had "[...] abandoned all hope of giving a *metaphysical* theory of the use of single letters as variables"[95] *On Denoting* really departs with denoting concepts. What remains seems to be the following. The notion of the variable as a constituent of general propositions is kept but without any explanatory role. Variables as expressions are introduced as part of the primitive symbolic vocabulary of Russell's new logical calculus. And, as we shall discuss next, the notion of a 'propositional function' is explained away just like the notions of classes and relations-in-extension by Russell's new method of Substitution.

3.2.2 The Theory of Descriptions, the Genesis of the Substitutional Theory and the Received View of Russell's Ontological Development

Besides a new analysis of generality, i.e., a new theory of the variable, in *On Denoting* Russell also introduces what may very well be his most famous and celebrated contribution to

[93] URQUHART, A., 1994, p.xxxv.
[94] URQUHART, A., 1994, p.xxxv.
[95] LANDINI, G., 1998*b*, p.72.

Mathematical Philosophy and Philosophical Logic: his new theory of definite descriptions. *On Denoting* presents Russell's new analysis of sentences containing ocurrences of definite descriptions like "the present King of France" or "the center of mass of the solar system" informally. In Russell's manuscripts written in the period surrounding the publication of *On Denoting*, however, we already find many formal statements of Russell's contextual definition, whose canonical form can be put as follows[96]:

$$\psi\{(\imath x)(\phi x)\} \,.{=}{:}\, (\exists y) : \phi z \,.\equiv_z .\, z = y : \psi y \,\mathrm{Df} \tag{3.2.1}$$

This defines the truth-conditions of any context ψ in which an expression like "the *x* which is a ϕ", i.e., "$(\imath x)(\phi x)$" occurs as a subject in terms of a sentence which asserts that there is one, and only one *y* such that ψy.

Apart from the 'Gray Elegy argument' against denoting concepts, in *On Denoting* Russell advocated his analysis mainly on the grounds that it was capable of dissolving a variety of logico-philosophical 'puzzles' that are nowadays taught and discussed in almost every introductory course or text to the Philosophy of Language[97]. As Russell stated in his later publications, however, at the time he thought that the chief merits of his new theory of denoting were connected to the resolution of the contradictions and the theory of types, somewhat to the bafflement of his readers. In *My Philosophical Development*, for instance, he writes, concerning his work on solving the contradictions, that:

> Throughout 1903 and 1904, my work was almost wholly devoted to this matter, but without any vestige of success. My first success was the theory of descriptions, in the spring of 1905 [...] This was, apparently, not connected with the contradictions, but in time an unsuspected connection emerged. In the end, it became entirely clear to me that some form of the doctrine of types is essential.[98]

A similar remark is is also made by Russell in his *Autobiography*, where we find the following passage which leaves the link between Russell's work on the contradictions and the theory of descriptions even more enigmatic:

> In 1905 things began to improve. Alys and I decided to live near Oxford, and built ourselves a house In Bagley Wood. (At that time there was no other house there.) We went to live there in the spring of 1905, and very shortly after we had moved in I discovered my Theory of Descriptions, which was the first step

96 Again, this definition requires scope markers to properly work and to be properly understood. This will be addressed in chapter 4.

97 At the time Russell had to convince the then editor of *Mind*, G.F. Stout, that the article deserved to be printed since, according to Russell, Stout thought the views presented expressed in the article "presposterous" to the point that he "begged [Russell] not to demand its publication as it stood" (Cf. RUSSELL, B., 1959, p.83 and URQUHART, A., 1994, p.414).

98 RUSSELL, B., 1959, p.79.

> towards overcoming the difficulties which had baffled me for so long. [...] In 1906 I discovered the Theory of Types. After this it only remained to write the book out.[99]

For quite some time, the above remarks connecting the discovery of the theory of descriptions with the resolution of the contradictions and the theory of types left readers baffled and puzzled and remained without a good explanation in secondary literature.

One reason is this: the first appearance of the Theory of Types was in the *Appendix B* of the *Principles* but the next one is in Russell's famous paper *Mathematical Logic as Based in the Theory of Types* which appeared only halfway through 1908 (and *Principia*'s first volume only by the end of 1910)[100]. In *My Mental Development* Russell had also already claimed explicitly that the theory of descriptions led him at the time to take class expressions as incomplete symbols:

> What was of importance in this theory was the discovery that, in analyzing a significant sentence, one must not assume that each separate word or phrase has significance on its own account. "The golden mountain" can be part of a significant sentence, but is not significant in isolation. It soon appeared that class-symbols could be treated like descriptions, i.e., as non significant parts of significant sentences. This made it possible to see, in a general way, how a solution of the contradictions might be possible.[101]

This throws some light on the issue. As we saw, at the time of the *Principles*, Russell thought that every descriptive phrases like "The class of all classes" or "The class of all classes which do not belong to themselves" had to denote something so that sentences in which they occur would not be disregarded as nonsensical. In *On Denoting* Russell abandons this view, since there he adheres to the view that such linguistic expressions do not express a meaning in isolation, but only in the context of complete sentences. Such expressions Russell called 'incomplete symbols'. But Russell does not go into any detail on the relation of his theory of denoting with the solution of the contradiction, beyond the loose comparison of class expressions with names of fictional entities. So still, the connection here with the Theory of Types is tenuous at best.

The only well-known published text which gives a hint of what was going on with Russell's views on the contradictions between 1905-08 is the article *On Some Difficulties in the Theory of Transfinite Numbers and Order Types*. There, Russell discusses very briefly three approaches which he considered up to 1905 for solving the contradictions. He called them the "Zig-Zag Theory", "The Limitation of Size Theory" and the "No-Class theory". Each of these theories was an attempt to avoid the contradiction by somehow mitigating existential assumptions about classes while at the same time *preserving* the unrestricted variable which was so

99 RUSSELL, B., 1967, p.229.

100 In this regard, as Gregory Moore obsrves, Russell's remark that "it only remained to write the book out" is "grossly misleading" (MOORE, G., 2014, pp.lxx).

101 RUSSELL, B., 1944, p.224.

central to the general conception of Logic (and Mathematics) presented in the *Principles.* According to the Zig-Zag theory, "propositional functions determine classes when they are fairly simple, and only fail to do so when they are complicated and recondite"[102] and the basic idea of the Limitation of Size Theory was to impose "[...] a certain limit of size which no class can reach" in order to show that "any supposed class which reaches or surpasses this limit is an improper class, i.e., is a non-entity"[103]. Russell worked on both theories from 1903 to 1905, but in the end he was not satisfied with either of them. In the case of the zig-zag theory, the main problem was that it was not clear at all where the line between legitimate and illegitimate classes was to be drawn for any reason other than avoiding the contradictions. As he put it:

> [...] in attempting to set up axioms for this theory, I have found no guiding principle except the avoidance of contradictions; and this, by itself, is a very insufficient principle, since it leaves us always exposed to the risk that further deductions will elicit contradictions.[104]

In the case of the limitation of size theory, the problem was that it was unclear whether the theory would be adequate for reconstructing Classical Mathematics, and so sufficient for the purposes of Logicism:

> A great difficulty of this theory is that it does not tell us how far up the series of ordinals it is legitimate to go. It might happen that ω was already illegitimate: in that case all proper classes would be finite.[105]

Finally, there was No-Class Theory also called the Substitutional Theory of Classes and Relations. This was the most radical attempt to resolve the problem: according to it "classes and relations are banished altogether"[106].

As Ivor Grattan-Guiness once remarked, this No-Class sketched in the 1905 article and later developed in writings from 1906 and 1907 as the Substitutional Theory of classes and relations "is the missing link between Russell's theory of denoting and *Principia Mathematica*"[107]. In the only published work in which Russell discusses this theory in detail, he asserts that:

> Technically, the theory of types as suggested in Appendix B [of Principles] differs little from the no-classes theory. The only thing that induced me at that time to retain classes was the technical difficulty of stating the propositions

[102] RUSSELL, B., 1905*b*, p.74.
[103] RUSSELL, B., 1905*b*, p.74.
[104] RUSSELL, B., 1905*b*, p.75.
[105] RUSSELL, B., 1905*b*, p.80.
[106] RUSSELL, B., 1905*b*, p.80.
[107] GRATTAN-GUINNESS, I., 1977, p.94.

> of elementary arithmetic without them—a difficulty which *then* seemed to me insuperable.[108]

In his *Dear Russell, Dear Jourdain*[109] Grattan-Guinness made available Russell's correspondence with Jourdain in which the Substitutional Theory is discussed in detail (also providing commentary informed by Russell's then unpublished manuscripts). There we find conclusive evidence for the connection between Russell's famous theory of incomplete symbols and Russell's acceptance of some form of what he would later call the 'doctrine' of types[110].

Unfortunately, several factors contributed in making this missing link either unknown, ignored or misunderstood for quite a long time. First of all, Russell himself was not only quite dismissive of it in reminiscence[111], but he also never mentioned it explicitly in any published work after 1908. Second, almost *all* the work he produced on this theory remained unpublished for several decades. Between 1905 and 1907 Russell produced hundreds of folios dealing with the No-Class theory. But only one paper dealing at some level of detail with that theory was published in Russell's lifetime. The paper in question was originally entitled *On Insolubilia and Their Solution Through Symbolic Logic*; it ended up being published in French by September 1906 as *Les Paradoxes de la Logique* in the *Revue de méthaphysique et de morale* in connection with a polemic with Poincaré[112]. But this work was not published in English until 1973 in the collection *Essays in Analysis*, thus remaining unavailable and neglected. More importantly, Russell's main paper dealing with the theory, *On The Substitutional Theory of Classes and Relations* which never saw print in Russell's lifetime. Russell sent it to the London Mathematical Society by 24 April of 1906 and read it by 10 May. The paper was accepted for publication, but Russell had second thoughts about it[113].

These circumstances and Russell's cryptic historical remarks contributed to make widespread a largely inadequate picture of the development of his views, which was put forward in its most pristine and paradigmatic form in Quine's *Russell's Ontological Development*, which is worth quoting in length:

> Seeing Russell's perplexities over classes, we can understand his gratification

108 RUSSELL, B. 1906, p.193, our emphasis. The paper was only amde widely available in 1973 when it was republished in enlgish for the first time. See previous footnote.

109 GRATTAN-GUINNESS, I., 1977.

110 RUSSELL, B., 1959, p.79.

111 In a letter to Lucy Martin Donnelly from April 22 1906, Russell wrote: "My work goes ahead at a tremendous pace, and I get intense delight from it" (RUSSELL, B., 1967, p.280), however in his Autobiography written more than sixty years later he added a footnote unfairly asserting that "It turned out to be all nonsense" (RUSSELL, B., 1967, p.280; Ironically, in the same letter he claimed that "I feel better able than anyone else to judge what my work is worth").

112 MOORE, G., 2014, pp.xiviii-xlix. Details will be discussed below.

113 He reported to Jourdain by 10 october: "I decided not to publish the paper I read at the London Mathematical [Society] in May; there was much in it that wanted correction, and I preferred to wait till I got things into more final shape" (GRATTAN-GUINNESS, I., 1977, p78). This paper was also published for the first time only in 1973 in the collection *Essays in Analysis*.

> at accommodating classes under a theory of incomplete symbols. But the paradoxes, which were the most significant of these perplexities, were not solved by his theory of incomplete symbols; they were solved, or parried, by his theory of types. One is therefore startled when Russell declares in "*My Mental Development*" that his expedient of incomplete symbols "made it possible to see, in a general way, how a solution of the contradictions might be possible."[114] If the paradoxes had invested only classes and not class-concepts, then Russell's elimination of classes would indeed have eliminated the paradoxes and there would have been no call for the theory of types. But the paradoxes apply likewise, as Russell knew, to class-concepts, or propositional functions. And thus it was that the theory of types, in this its first full version of 1908, was developed expressly and primarily for propositional functions and then transmitted to classes only through the contextual definitions.
>
> The startling statement that I quoted can be accounted for. It is linked to the preference that Russell was evincing, by 1908, for the phrase 'propositional function' over 'class-concept'. Both phrases were current in *Principles of Mathematics*; mostly the phrase 'propositional function' was visibly meant to refer to notational forms, namely open sentences, while concepts were emphatically not notational. But after laying waste Meinong 's realm of being in 1905, Russell trusted concepts less and favored the more nominalistic tone of the phrase 'propositional function', which bore the double burden. If we try to be as casual about the difference between use and mention as Russell was fifty and sixty years ago, we can see how he might feel that, whereas a theory of types of real classes would be ontological, his theory of types of propositional functions had a notational cast. Insofar, his withdrawal of classes would be felt as part of his solution of the paradoxes. This feeling could linger to 1943, when he wrote "*My Mental Development*," even if its basis had lapsed.[115]

In order to correctly appreciate the proper role the theory of descriptions played on the development of his views up to *Principia Mathematica*, we must take into consideration the work on logical theories Russell produced in the period of 1905-1906 towards a solution of the contradiction.

3.2.3 The Development of Russell's Substitutional Theory in Russell's Early Manuscripts

As already noted, Russell would later state that his desiderata for an adequate logical theory of classes, ordinary functions and propositional functions and relations-in-extension were the following[116]:

1. It should make the contradictions disappear;
2. It should leave intact as much of mathematics as possible;

[114] RUSSELL, B., 1944, p.14. This is the passage quoted above.
[115] QUINE, W., 1966, pp.660-1.
[116] RUSSELL, B., 1959, p.79-80.

3. It should be well-motivated philosophically, or, as Russell put it, it should "[...] appeal to what may be called 'logical common sense'"[117].

As we already discussed in the Introduction, from 1905-1907, the Substitutional Theory was Russell's favored candidate theory for solving the contradictions, developing classical Mathematics and preserving the unrestricted variable thus satisfying the three requisites.

The main idea involved in that theory was to emulate a theory of classes and propositional functions using his recently discovered theory of incomplete symbols. The first public appearance of the theory was in the paper *On Some Difficulties in the Theory of Transfinite Numbers and Order types*, which was finished by 24 November of 1905[118]. The paper was mainly concerned with objections raised by the mathematician Ernest Hobson[119] against the Cantorian theory of cardinals and ordinal numbers, some of them connected to the contradictions and the Axiom of Choice[120]. There Russell gave only a bare outline of how his new symbolic theory[121]:

> Instead of a function $\phi!x$, where the notation inevitably suggests the existence of something denoted by "ϕ", we proceed as follows: Let *p* be any proposition, and *a* a constituent of *p*. (We may say that *a* is a constituent of *p* if *a* is mentioned in stating *p*). Then let "$p\frac{x}{a}$" denote what *p* becomes when *x* is substituted for *a* in the place or places where *a* occurs in *p*. For different values of *x* this will give us what we have been accustomed to call different values of a propositional function. In place of ϕ we have now two variables, *p* and *a*: in respect to the different values of $p\frac{x}{a}$, we may call *p* the prototype and *a* the origin or *initial subject*. (For a may be taken as being, in a generalized sense, the subject of *p*.) Consider now such a statement as "$p\frac{x}{a}$ is true for all values of *x*". Let *b* be an entity which is not a constituent of *p*, and put $q = p\frac{b}{a}$; then "$q\frac{x}{b}$ is true for all values of *x* is equivalent to "$p\frac{x}{a}$ is true for all values of *x*". Thus, subject to certain reservation, the statement "$p\frac{x}{a}$ is true for all values of *x*" is independent of the initial subject *a*, and thus may be said to depend only upon the *form* of *p*. Statements of this sort replace what would otherwise be statements having propositional functions for their arguments. For example, instead of "ϕ is a unit function" (i.e., "There is only one, and only one, *x* for which $\phi!x$ is true"), we shall have "There is an entity *b* such that $p\frac{x}{a}$ is true when, and only when, *x* is identical with *b*". [122]

Unfortunately, *On Some Difficulties* does not give any real indication of how the theory may be used as a foundation for a logical calculus strong enough for the development of Mathematics and capable of resolving the contradictions. In particular, concerning the treatment of

[117] RUSSELL, B., 1959, p.80.
[118] MOORE, G., 2014, p.62.
[119] HOBSON, E., 1905.
[120] For details, see MOORE, G., 2014, pp.62-5.
[121] As we mentioned, Russell considered in that paper not only the No-class theory, but also the 'Zig-zag' theory and the 'limitation of size' theory.
[122] RUSSELL, B., 1905*b*, p.81.

classes, Russell only observes that the expression "[...] the values of *x* for which $p\frac{x}{a}$", can be used to proxy a class-expression without assuming that "[...] these values collectively form a single entity which is the class composed of them"[123]. There is no indication, however, of how the theory of relations-in-extension can be developed, and most importantly, there is no clear answer to the contradictions besides the outright elimination of classes.

That is why scholars were somewhat puzzled for some time by the fact that in the beginning of 1906 Russell was convinced that the Substitutional Theory was the correct alternative, for he added a note to the paper by 5 February 1906 claiming that "[...] from further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first three sections of this paper"[124]. Quine, for instance, conjectured that the theory was short-lived precisely because it could not afford an adequate way to paraphrase discourse about classes (and presumably relations-in-extension). Famously, Quine wrote:

> The central idea of the no-classes theory was that, instead of speaking of the class of all the objects that fulfill some given sentence, one might speak of the sentence itself and substitutions within it. Now discourse about specified classes lends itself well enough to paraphrase in terms of sentences and substitution, but when we talk rather of classes in general, as values of quantifiable variables, it is not evident how to continue such paraphrase. Russell had already acknowledged that the no-class theory might prove inadequate to much classical mathematics; and his more sanguine postscript of February was the expression of renewed hope that he was shortly to abandon. For he soon turned back to his theory of types and proceeded to develop it in detail. The result, published in 1908, is the paper reproduced below [*Mathematical Logic as Based on the Theory of Types*].[125]

Several issues emerge here. First, Russell's notion of 'substitution' in a *proposition* has no linguistic connotation whatsoever. At this time Russell still retained his ontology of propositions conceived as complex *entities* and when he speaks of a constituent being 'substituted by a variable' (or vice-versa) in a proposition, the use of "substitution" is somewhat metaphorical. Strictly speaking what can substituted (in the sense that *we* substitute) is a variable-letter, say "*x*" for a constant or name letter, say "*a*" in a given formal expression (an open or closed *sentence*, say "ϕx"). This sense of substitution was, of course, important in setting up the *formal grammar* of his new symbolic theory but the fundamental sense of 'substitution' is that of an *entity a* occurring in place of another entity *x* in a given proposition *p* (which is also an entity), thus resulting in another proposition *q* (also an entity). This is because the notion of a *subject a* of a proposition *p* here is that of the *Principles*, namely *a* occurring *as a term* in the complex of

123 RUSSELL, B., 1905*b*, p.82.
124 RUSSELL, B., 1905*b*, p.89. The sections in question discussed the contradictions and the existence theorems of the theory of cardinal and ordinal (transfinite) numbers.
125 QUINE , W., 1967, pp.150-1.

entities p[126]. The notion of a logical subject of a proposition understood as complex entity was still the touchstone of Russell's Logic.

Furthermore, Russell's correspondence and work notes of the period show that Quine's historical account is quite inadequate. By 13 January 1906 Russell wrote to Jourdain presenting him the new theory of descriptions, explaining that a descriptive phrase like "$(\imath x)(\phi x)$" is not a name of some kind of "non-entity" but "[...] merely a phrase which fails to describe anything"[127]. Indeed, by 19 January 1906 Russell had provided a general outline of how expressions for classes and relations could be proxied by incomplete symbols like "$p\frac{x}{a}$" briefly mentioned in the 1905 paper, which he now called "matrices":

> Classes, relations, etc. are now for me all of them what I call *matrices*, of which the nature (omitting subtleties) is as follows: Let $p/a{}^{\text{;}}x$[128] represent what p becomes when x is substituted for a in those places (if any) where a occurs in p. Similarly $p/(a,b){}^{\text{;}}(x,y)$ substitutes x, y for a, b, etc. Then p/a or $p/(a,b)$ or $p/(a,b,c)$ or p/a is a class, $p/(a,b)$ a *dual* relation, etc. [...]
>
> One may use single letters for classes and relations, but if one does, these letters are only significant in certain contexts. E.g. $p/a{}^{\text{;}}x$ requires that x should be an entity; thus $p/a{}^{\text{;}}(p/a)$ is meaningless; thus $u \epsilon u$ is meaningless.[129]

This already provides a clear explanation of how Russell started to envision a dissolution of his paradox through the elimination of classes: if an expression like "p/a" is an incomplete symbol which does not denote a single entity and does not have meaning in isolation, it cannot be substituted by the name of an entity in a given sentence; "p/a" is not a single entity, but a pair of entities; thus, it does not denote any constituent of the proposition p in question and any attempt to formulate a sentence by substituting this expression for the name "x" of the variable or a name constant "a" would result in nonsense. It also gives a hint of how relations-in-extension can be emulated by matrices with two or more argument places.

On 14 March 1906 Russell developed this line of thought explaining that a class expression α "[...] is nothing but a typographical abbreviation for something of the form q/b"[130] i.e, a *matrix* of some kind. He also (roughly) defines a class α as the totality of entities x such that, given a proposition p and an entity a, if we substitute x for a in p, we obtain a true proposition, making the parallel with the theory of descriptions even more clear. By 21 March 1906 this view was completely consolidated, for Russell wrote yet another letter to Jourdain detailing how his paradox is dissolved by the Substitutional Theory and addressing the notation for relations-in-extension:

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[126] LANDINI, G., 1998*a*, p.98. Recall that a variable, or as Russell put in the *Principles*, *the* variable, is also a non-linguistic entitty.

[127] GRATTAN-GUINNESS, I., 1977, p.70.

[128] This notation "$p/a{}^{\text{;}}x$" is merely a more convenient way of writing "$p\frac{a}{x}$".

[129] GRATTAN-GUINNESS, I., 1977, p.75-6.

[130] GRATTAN-GUINNESS, I., 1977, p.77.

> My view at present is that every class, great and small, is merely a symbolic abbreviation; as such, they are all legitimate if they have any significant definition; but such forms as $\hat{x}(x \sim\in x)$ are not significant; because one must have $x \in p/a$ or $x \sim\in p/a$ for significance, where we write:
>
> $x \,\epsilon\, p/a \,.\!=.\, p\dfrac{x}{a}$ Df.
>
> Thus, $x \,\epsilon\, x$ or $x \sim \epsilon\, x$ is nonsense. Cls is a class of classes, and it is just as legitimate, in its proper context, as any other *matrix*. (I call a thing a *matrix* when it consists of a prototype and one or more initial subjects, so that it requires one or more arguments supplied to it to be significant. Thus in $p\dfrac{x}{a}$, p/a is the matrix; in $p\dfrac{x,y}{a,b}$, $p/(a,b)$ is the matrix. These replace classes and relations)[131]

Concerning the resolution of the paradoxes, these letters make it very clear why Russell was convinced of the adequacy of the No-class theory: by treating functional expressions like "ϕx", "fx", etc, class expressions like "$\hat{x}\phi x$" and expressions for relations in extension like $\hat{x}\hat{y}\phi(x,y)$ as incomplete symbols, problematic class expressions like "$\hat{x}(x \sim \epsilon\, x)$", "$\sim \phi(\phi)$", etc were ruled as ungrammatical, dissolving the problem. This afforded a resolution of such difficulties which at the time Russell thought could preserve his unrestricted conception of variation and univocity of being, for the theory allowed only type variable which ranges over all entities, namely individuals and propositions.

In this regard, the letter from 14 March 1906 is of particular importance because here Russell introduces the notion *type* of a matrix in his new solution of the Burali-Forti paradox:

> *W* has now become wholly undefinable, for this reason, that if we call a *type* such a thing as Cls^2, Cls^3, etc., the existence-theorems for segments of *W* require us to go further and further up the hierarchy of types, and no type will give the whole series, because a higher type would always give more. Now, a series is meaningless unless all its terms are of the same type; and there is no such thing as Cls^∞ or Cls^ω. Hence there is no such thing as the whole series of ordinals. This is the solution of the Burali-Forti.[132]

Disregarding the technical details of the approach for now, this restores the core idea of *Appendix B* from the *Principles* in a way that for Russell is philosophically acceptable: he no longer has a hierarchy of classes *as entities*, but a hierarchy of *matrices*, which are incomplete symbols. The Burali-Forti is thus also ruled out as ungrammatical without doing violence to what Russell thought to be 'logical common sense'.

Several of Russell's manuscripts from the end of 1906 through the whole year of 1906 and at least half of 1907 show that he was quite convinced that the theory was also adequate for the development of a great deal of Mathematics. The earliest detailed record of the formal details of the Substitutional Theory is a letter from Russell to Hardy written in 15 December of

131 GRATTAN-GUINNESS, I., 1977, p.84.
132 GRATTAN-GUINNESS, I., 1977, p.77.

1905[133]. The earliest text where Russell systematically develops the formal details of the theory is the core manuscript *On Substitution* from 22 December 1905[134]. Between December of 1905 and April of 1906, Russell worked incessantly on the Substitutional Theory, producing many interesting manuscripts that record the (sometimes tortuous and complicated) development of the theory. This period of intense work culminated in a paper which Russell sent to the *London Mathematical Society* by 24 April 1906[135]. This paper, entitled *On The Substitutional Theory of Classes and Relations* is the single most important work dealing with the logic of substitution, for several reasons. First, it was the closest thing next to a finished formal development of the theory which Russell ever produced. Second, it was the only work dealing with the formal details of the theory which he made public (he read it to *London Mathematical Society* by 10 May 1906[136]); in fact, Russell even submitted the manuscript to publication, but later withdrew it, for reasons which we'll discuss below. Moreover, the paper is important because it establishes once and for all the real connection between the theory of incomplete symbols and the theory of types.

As put forward in *On The Substitutional Theory of Classes and Relations*, the basic idea of the Substitutional Theory is to abolish propositional functions, classes and relations and extensions as genuine entities (and possible values of genuine variables). Roughly, this amounts to the following: instead of taking "ϕx", "xRy", or "$x \in \alpha$" as the most basic sentential components of the formal grammar of logic, the language adopts the following as a primitive notion[137]:

$$p\frac{x}{a}!q \text{ (Primitive Idea)}$$

which is defined as meaning "*q* results from *p* by substituting *x* for *a* in all those places (if any) where *a* occurs in *p*". In terms of this notion Russell then defines[138]:

$$a\, ex\, p \,.{=}{:}\, (x) \,.\, p\frac{x}{a}!p \text{ or "}a\text{ does not occur in }p\text{" Df}$$

$$\sim a\, ex\, p = \text{"}a\text{ occurs in }p\text{" Df}$$

$$x = y \,.{=}.\, x\frac{y}{x}!x \text{ Df}$$

Thus, the grammar of the theory is articulated in terms of *occurrence as a term in a proposition* instead of *occurring as argument in a function.* As we mentioned above, the equivalent of a functional expression like "ϕx" is the following[139]:

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133 MOORE, G., 2014, p.90.
134 MOORE, G., 2014, p.90.
135 MOORE, G., 2014, p.236.
136 MOORE, G., 2014, p.236.
137 RUSSELL, B., 1906*a*, p.245.
138 RUSSELL, B., 1906*a*, p.245.
139 RUSSELL, B., 1906*a*, p.246.

$$p\frac{x}{a} \mathbin{.} =\colon (\imath q) \mathbin{.} p\frac{x}{a}!q \text{ Df}$$

which contextually defines the symbol "$p\frac{x}{a}$" as "the q which satisfies $p\frac{x}{a}!q$" - an incomplete symbol, which, given the definition of "$\psi\{(\iota x)(\phi x)\}$", is equivalent to:

$$p\frac{x}{a} \mathbin{.} =\colon (\exists q) \colon p\frac{x}{a}!r \mathbin{.} \equiv_r \mathbin{.} q = r \colon p\frac{x}{a}!q$$

Thus, the theory adopts only one style of entity variable that ranges over individuals and propositions (understood as entities) alike and emulates the notation of a functional calculus using the incomplete symbols of the form "$p\frac{x}{a}$". There is no genuine variable ranging over propositional functions. As indicated in *On Some Difficulties*, using the incomplete symbols "$p\frac{x}{a}$" , which Russell called *matrices*, one can, instead of saying that a *function* is true for all its values as in "$(x).\phi x$", assert that a certain type of substitution always yield a true proposition, for instance in the case where "$(x).\, p\frac{x}{a}!p$" is to mean that every substitution of *x* for *a* in *p* is true. Using only quantification over a *single domain* of propositions *and* individuals, this allowed for the simulation of a theory of propositional functions without actually using functional variables. Only propositions and individuals were taken as entities and so, as legitimate values of genuine variables.

Classes could then be 'explained away' using this very procedure. Ordinarily classes and relations-in-extension are viewed as the extension of a propositional function. In the Substitutional Theory membership to classes and relation-in-extension can be defined in terms of matrices and quantification over class-variables can be dispensed in favor of quantification over individuals and propositions. The basic idea of the theory, in Russell's own words, is that the "[...] shadowy symbol p/a represents a *class*" and "[...] $p/(a, b)$ will represent a dyadic relation" and so on[140]. The general idea is to put:

$x \,\epsilon\, p/a = p\frac{x}{a}$ Df

$x, y \,\epsilon\, p/(a, b) = p\frac{(x,y)}{(a,b)}$ Df

And so on...

Which are to mean that " '*x* is a member of the class *p/a*' is to be interpreted as 'the result of replacing *a* in *p* by *x* is true' "[141], etc. This avoids commitment to an ontology of classes (and relations in extension) given that in the Substitutional Theory the notation of a theory of classes is proxied by the notation of matrices. As Russell puts it, although matrices occur in meaningful

140 RUSSELL, B., 1906*a*, p.246.
141 RUSSELL, B., 1906*a*, p.246.

sentences "[...] it is not a part which means anything in isolation" for the matrix "[...] is not the name of an entity, but is a mere part of symbols which are the names of entities"[142].

Now, as Landini notes, it must be observed that Russell's manuscripts show that he envisaged more than one approach for developing a proxy for the calculus of classes within the Substitutional Theory and that the alternative provided in *On the Substitutional Theory of Classes and Relations* seems to suffer from fatal flaws[143]. But be that as it may, we can assess with a fair degree of certainty that Russell recognized that the Substitutional Theory had the three fundamental merits he required of a satisfactory logical theory independently of determining which route for recovering set theory Russell would give preference to.

First the theory was able to avoid the set-theoretical contradictions. Russell's paradox, for instance was avoided due to the fact that any attempt to assert that a matrix is a member of itself results in an improper nonsensical expression. To assert that "$\alpha \epsilon \alpha$" where "α" is any given class expression would be something like "$p/a; p, a!q$" which is meaningless; "p/a" is an incomplete symbol that does not name a singular entity that can be substituted for a term in a proposition. As Landini puts it, Russell's paradox was simply discarded as violation of the grammar of logic - in practice, the grammar of the Substitutional Theory is that of the simple theory of types, though there are no types of *entities*[144].

Second, the theory had important virtues from a technical point of view, the most important being that it allowed proofs of several existential theorems, in particular the 'existence' of infinite classes. As Russell explained in another paper from 1906 which we'll discuss shortly, the existence of an infinite class could be given along the following lines:

> Given one entity *a*, we have the proposition $a = a$; and by the axiom at the beginning of the last paragraph ["that given any proposition *p*, there is at least one entity *u* which is not explicitly mentioned in stating *p*"[145]] there is an entity *u* such that *u* is not mentioned in '$a = a$' . This entity is not *a*, since *a* is mentioned in '$a = a$'. Hence there are at least two entities. There will be similarly an entity not mentioned in '$a = u$', which must be neither *a* nor *u*. We can in this way show that, if *n* is any finite number, there are more than *n* entities, and by taking propositions into account, we can manufacture $\aleph_0$ entities. E.g. put
>
> $p_0 \,.\!=\!.\, a = u$ [Df], $\quad p_{n+1} \,.\!=\!.\, p_n = u$ [Df]

142 RUSSELL, B., 1906*a*, p.247. Russell further clarifies this statement as follows: "[...] when we say that a matrix is not an entity, we mean that a matrix is a set of symbols, or a phrase which by itself has no meaning at all, but by the addition of other symbols or words becomes part of a symbol which has meaning, i.e. is the name of something." (RUSSELL, B., 1906*a*, p.247).

143 Cf. LANDINI, G., 1998, pp.149-151.

144 LANDINI, G., 1998*a*. In particular, Landini's rigorous presentation of the theory shows that a translation function from the Substitutional Theory for a simple type theory of attributes can be formulated (LANDINI, G., 1998*a*, pp.140-5), thus showing that the Substitutional Theory has the expressive power of simple type theory without its ontological commitments to classes or attributes (in extension).

145 RUSSELL, B., 1906*b*, p.288.

> it is not hard to prove that the successive *p*'s are all different, and that there are therefore at least $\aleph_0$ entities. Hence, the cardinals up to and including $\aleph_0$ exist, and the ordinals finite and of the second class exist.[146]

The above proof sketch shows how, assuming an ontology of propositions, it follows that 'there is' a class of cardinality $\aleph_0$ whose elements are genuine entities. As we commented in the preceding chapter, proving the existence of an infinity of entities from Logical assumptions alone became a pressing issue since the proofs considered by Russell in the *Principles of Mathematics* were all compromised by the inconsistency of the naive conception of set. The Substitutional Theory seemed to afford the most important existential theorems for Arithmetic without any special assumptions about the existence of classes or individuals (though, of course, it assumed an ontology or propositions as structured entities).

Third, the theory attended to what Russell thought, at the time, to be logical common sense, for it avoided the contradictions without doing any damage to Russell's absolute conception of logical generality (his thesis of the unrestricted variable) and his univocal notion of *being* or *logical subject*. The theory, as he put in the end of the main paper dealing with it "[...] affords what at least seems to be a complete solution of all the hoary difficulties about the one and the many; for, while allowing that there are many entities, it adheres with drastic pedantry to the old maxim that 'whatever is, is one' "[147].

3.3 The 'Fall' of Russell's Substitutional Theory of Classes and Relations

3.3.1 Propositional Paradoxes, the Substitutional Theory and the 'Orthodox' View

So far, our discussion of the Substitutional Theory did not address any really controversial topic. As we saw, Quine's old view of Russell's ontological development was debunked long ago by the material made available by Douglas Lackey[148] and Grattan-Guinness[149] and the archival research started by these authors was then carried to new depths by Landini[150] and Moore[151] whose scholarly and editorial work establish the logico-mathematical potential of the Substitutional Theory[152]. But there remain several interpretative issues and unanswered questions about

146 RUSSELL, B., 1906*b*, p.288.
147 RUSSELL, B., 1906*a*, p.261
148 LACKEY, D., 1973.
149 GRATTAN-GUINNESS, I., 1977.
150 Cf. LANDINI, G., 1987, 1989 and 1998.
151 Cf. MOORE, G. (ed.), 2014.
152 There were, of course, many other important contributors to this (still ongoing!) process of reception of Russell's unpublished manuscripts, correspondence and even of some of neglected aspects of his published writings. A small but comprehensive sample was provided in our introduction.

the theory. As is well known, Russell ended up abandoning the substitutional approach in favor of *Principia*'s so called ramified theory of types: the main interpretative difficulties are related to the question of what exactly led Russell to abandon the Substitutional Theory, and many conflicting accounts were given over the years in secondary literature.

We know for certain that by 10 May 1906, when Russell read *On The Substitutional Theory* for the *London Mathematical Society*, the theory was still alive and well. For as we just mentioned, he ends the paper observing that the theory resolves the contradictions in a way that is technically and philosophically satisfactory[153]. However, by 10 October 1906 Russell wrote to Jourdain with cryptic misgivings about the theory and reporting his decision to withdraw *On the Substitutional Theory* from publication:

> I am glad you feel attracted by the no-classes theory. I am engaged at present in purging it of metaphysical elements as far as possible, with a view to getting the bare residuum on which its success depends. [...] I decided not to publish the paper I read at the London Mathematical [Society] in May; there was much in it that wanted correction, and I preferred to wait till I got things into more final shape.[154]

At once this raises several important questions. What led Russell to change his mind about his views as presented in *On the Substitutional Theory*? Did Russell give up the Substitutional Theory together with the paper? And in the long run, what, exactly, led to the abandonment of the Substitutional Theory?

Some clues for answering these questions are provided in the correspondence with Jourdain and also in the abandoned paper, where Russell tempered his confidence in the Substitutional Theory with the following observation:

> The only serious danger, so far as appears, is lest some contradiction should be found to result from the assumption that propositions are entities; but I have not found any such contradiction, and it is very hard to believe that there are no such things as propositions[155].

This indicates that Russell was not completely sure of the theory's consistency at the time he sent the paper, having reservations about an ontology of propositions. And, in fact, as we shall briefly discuss in detail, Russell did find that the danger of contradiction from the assumption of propositions was very real.

We also know for certain that by 1 June of 1907, Russell was convinced that an ontology of propositions should be handled with care due to "[...] the paradox of the liar and its analogs",

153 RUSSELL, B., 1906*a*, p.261.
154 GRATTAN-GUINNESS, I., 1977, p.93.
155 RUSSELL, B., 1906*a*, p.261.

for he wrote to Jourdain explaining he no longer accepted the manufacturing of an infinite set in terms of the method discussed in our previous section:

> Consideration of the paradox of the liar and its analogs has led me to be chary of treating propositions as entities. I therefore no longer regard as valid the proof of $\aleph_0$ entities. [...] I now think that existence-theorems beyond the finite require a definite assumption that the number of entities is not finite.[156]

We also know that difficulties related to "Epimenides" led Russell to apparently reconsider the theory of types *applied to propositions* also, for by 10 September Russell wrote the following to Jourdain:

> I incline at present to the doctrine of types, much as it appears in appears in Appendix B of my book [*The Principles*]. To this I add that propositions and functions can never be *apparent* variables, so that statements about all of them are meaningless.[157]

It is at this point that many controversies become salient leading to differing accounts in secondary literature. Grattan-Guinness, for instance, claims that "[...] the withdrawal [of *On the Substitutional Theory*] marked the demise of the theory itself"[158]. But we have plenty of evidence to conclude that Russell was far from done with the Substitutional Theory.

In early May of 1906 Henri Poincaré published an article in *Revue de Méthaphysique et de Morale*[159] criticizing in detail some of Russell's writings, in particular *On Some Difficulties*. The article was in fact a sequel to a previous paper in the same journal[160] which extensively criticized Russell's and other mathematicians's use of logic and Cantorian set theory to provide a foundation for Mathematics[161]. By the end of 1905 Russell was already considering a reply to Poincaré (first paper), since in 19 December of 1905 he wrote the following to Couturat[162]:

> The solution which, I believe, I have found to the contradictions is not yet precise enough that I can publish it. First of all, it requires a new branch of logistic - the theory of substitutions - which must replace functions, classes, and relations. This theory is rather complicated from the technical viewpoint

156 GRATTAN-GUINNESS, I., 1977, p.105.
157 GRATTAN-GUINNESS, I., 1977, p.91.
158 GRATTAN-GUINNESS, I., 1977, p.94.
159 POINCARÉ, H., 1906.
160 POINCARÉ, H., 1906.
161 As is well known, the disagreement between Russell and Poincaré was *very deep*: it amounted to nothing less than the clash between a Platonic Logicist and a Kantian Constructivist and there was hardly a (philosophical) point about which they could concur. A good survey of the issues involved in the disagreement can be found in GOLDFARB, W., 1988. A good critical discussion of Poincaré's views on the Philosophy of Mathematics can be found in CHIHARA, C., 1973, pp.139-173.
162 Who also was about to publish a reply. See COUTURAT, L., 1906.

> (though not from the philosophical viewpoint), and I will need time to develop it. I intend to send a purely mathematical memoir to the *American Journal of Mathematics* as soon as possible. Then I will be able to publish right away an article showing that the contradictions have disappeared. This article will avoid, so far as possible, technical difficulties, and will bring out as much as possible the philosophical scope of the new theory. If you think that it would be good to publish it in the *Revue de Metaphysique*, I will gladly send it there. But I will probably need a year to assure myself that the new theory is solid.[163]

In 15 May 1906 Russell wrote to Couturat about the rebuttal to *Poincaré*, explaining that he must clarify his resolution of the contradictions before responding. The same letter also makes it clear that Russell was still convinced of the soundness of the Substitutional Theory, but he observes that somehow the theory of incomplete symbols must be extended to propositions in order for it to work:

> I still believe that my solution to the contradictions (the no-classes theory) is good, but it seems to me that it must be extended to propositions, i.e. that they, like classes and relations, cannot be put in the place of ordinary entities. [...]
>
> I will follow your advice in replying to Monsieur Poincaré. For this reason, I will not respond immediately, since I would like to reduce to order what I have to say about the solution to the contradictions.[164]

As Gregory Moore observes, this is one of the first indications that Russell "no longer regarded propositions as entities"[165] because of contradictions which resulted from an unrestricted assumption of these as entities. But there are several interpretative puzzles involved here.

Russell had already discussed contradictions connected to an ontology of propositions long before abandoning the Substitutional Theory. In the *Appendix B* of the *Principles*, he provided a definitive argument establishing the incompatibility of an unrestricted (i.e., 'untyped') ontology of propositions with Cantor's power-class theorem:

> If *m* be a class of propositions, the proposition "every *m* is true" may or may not be itself an *m*. But there is a one-one relation of this proposition to *m*: if *n* be different from *m*, "every *n* is true" is not the same proposition as "every *m* is true". Consider now the whole class of propositions of the form "every *m* is true", and having the property of not being members of their respective *m*'s. Let this class be *w*, and let *p* be the proposition "every *w* is true". If *p* is a *w*, it must possess the defining property of *w*; but this property demands that *p* should not be a *w*. On the other hand, if *p* be not a *w*, then *p* does possess the defining property of *w*, and therefore is a *w*.[166]

163 As quoted in MOORE, G., 2014, pp.xiviii-xlix. The purely mathematical paper could very well be *On the Substitutional of Clases and Relations*, which Russell later withdrew from publication.

164 MOORE, G., 2014, p.274-5.

165 MOORE, G., 2014, p.lii;

166 RUSSELL, B., 1937 [1903], p.527 §500.

This paradox, generally referred to as the *Appendix B paradox*, was first considered by Russell in his correspondence with Frege in early 1902. On 29 February Russell wrote to Frege formulating it as a difficulty which his theory of types (that of *Appendix B*) could not resolve[167]. Russell set out the paradox formally as follows:

> Let me now put |' for 'is identical with'. We then have:
>
> $$w = r \ni (\exists m \ni \{r|'(p \,\epsilon\, m \,.\supset_p .\, p).r \sim \epsilon\, m\}) \,.\supset\, ::: (q \,\epsilon\, w \,.\supset_q .\, q) \,\epsilon\, w. \supset .$$
> $$\exists m \ni ((q \,\epsilon\, w \,.\supset_q .\, q)|'(p \,\epsilon\, m \,.\supset_p .\, p).(q \,\epsilon\, w \,.\supset_q .\, q) \sim \epsilon\, m) \quad (1)$$
> $$(q \,\epsilon\, w \,.\supset_q .\, q)|'(p \,\epsilon\, m \,.\supset_p .\, p) \,.\supset .\, w|'m \quad (2)$$
> $$(1) \,.\, (2) \,.\supset :\, (q \,\epsilon\, w \,.\supset_q .\, q) \,\epsilon\, w \,.\supset .\, (q \,\epsilon\, w \,.\supset_q .\, q) \sim \epsilon\, w \quad (3)$$
>
> Similarly: $(q \,\epsilon\, w \,.\supset_q .\, q) \sim \epsilon\, w \,.\supset .\, (q \,\epsilon\, w \,.\supset_q .\, q) \,\epsilon\, w.$
>
> Hence the contradiction.[168]

Notice that *nowhere* above there occurs any use of a truth-predicate. Put in a more approachable notation and employing brackets "[...]" for forming propositional terms out of formulas, the difficulty arises if we consider a class of propositions *w* defined as follows[169]:

$$w = \{p : (\exists m)(p = [(q)(q \,\epsilon\, m \supset q)] \wedge p \sim \epsilon\, m])\}\text{Df}$$

Given such a class w, we can define the proposition *r* which asserts that all members of *w* are true, that is:

$$r = [(p)(p \,\epsilon\, w \supset p)]\ \text{Df}$$

But by their respective definitions, it follows that r is a member of w if and only if it is *not* a member of *w*, since $r \,\epsilon\, w$ implies $p \,\epsilon\, w \,.\supset_p .\, p : r \sim \epsilon\, w$, and since, by definition, r asserts that $p \,\epsilon\, w \,.\supset_p .\, p$, assuming $r \sim \epsilon\, w$ entails precisely that $r \,\epsilon\, w$. Contradiction.

Another propositional paradox which Russell had also considered earlier than 1906 was the famous 'liar paradox'. This ancient paradox is usually presented in terms of a self-referencing statement that purports to assert of itself that it is false, as in "This statement is false": this is true, if and only if it is false. Russell usually refers to this paradox as the '*Epimenides*' because of another well known different version of it which is the "Paradox of Cretan": it arises when the Cretan Epimenides asserts that all Cretans are liars. The manuscript *On Fundamentals* from June 1905 contains one of Russell's first serious discussions of the liar[170], where he considers treating it as an analogue of his own paradox[171].

[167] FREGE, G., 1980, p.147.
[168] FREGE, G., 1980, p.156.
[169] In this formula we use the brackets [...] to form a propositional term.
[170] URQUHART, A., 1994, p.359. Russell probably considered this paradox for the first time in lost manuscripts from the first half of 1905, since *On Fundamentals* opens with a discussion of the liar paradox under the heading of "Summary of preceding" which according to Urquhart refers to a lost set of notes.
[171] RUSSELL, B., 1905*c*, p.360. The discussion of the liar begins with the heading "summary of the preceding", which indicates that the discussion is a continuation of a manuscript that is no longer extant (cf. URQUHART, A. (ed.)., 1994, p.359).

But if Russell was aware of such difficulties well before submitting *On the Substitutional Theory*, why did he seriously considered abandoning propositions only halfway through 1906? Several authors have argued that the withdrawal of the paper and the fall of the Substitutional Theory were caused by the *Appendix B* paradox and/or the liar paradox. Several important questions are tied to the problem of determining what sort of contradiction led Russell to abandon the Substitutional Theory and the question of what specific form of these contradictions led Russell to abandon the theory was a matter of controversy in the literature for quite some time because of this.

To begin with, the liar or *Epimenides* is the sort of paradox which we nowadays call semantic: its formulation involves the use of truth-predicates or intensional vocabulary like those of propositional attitudes; as we shall see in more detail below, the *Appendix B* paradox, on the other hand, is a purely logical paradox: it involves nothing less than a diagonal argument which shows that under certain assumptions about propositions, one may construct a one-one correspondence between entities and classes of entities, thus a direct violation of Cantor's power-class theorem[172].

Some authors, however, claim that Russell took these paradoxes like the *Appendix B* to be of the same kind as the *Liar* and its variants. Hylton, for instance, has argued that it was a version of the *Appendix B* paradox that eventually led Russell to the so-called 'ramification' of types. As Hylton points out, some matrices p/a have only propositions as members, i.e., some matrices emulate classes of propositions, and each of these matrices can be associated with a proposition which asserts that every member of p/a is true[173], i.e, $(x) : p/a; x \,.\supset.\, x$. Now, let:

$$p = (\exists r, c) : [a = (x) : r/c; x \,.\supset.\, x] : \sim (r/c; a) \text{ Df}$$

From this it follows that[174]:

$$p/a; [p/a; x \supset x] \,.\equiv.\!\sim (p/a; [p/a; x \supset x])$$

The problem here is exactly analogous to that of the Appendix B paradox: start from the notion of a proposition *q* which asserts that all members of a given class *m* of propositions are true i.e., $(x) : r/c; x \,.\supset.\, x$; take all such propositions which assert this but that are such that they do not belong to *m*, i.e., $\sim (r/c; a)$; this forms the class *w* of the Appendix B; now, does the proposition *p* which asserts that all such propositions are true belong to this class *w* or not? From each answer the opposite follows.

172 Cocchiarella was one of the first authors to emphasize this point in COCCHIARELLA, 1980, p.89-90.
173 HYLTON, P., 1980, p.24.
174 Cf. HYLTON, P., 1980, p.24 and also LANDINI, G., 1998, pp.202-3.

In his own formulation, however, Hylton employs truth-predicates as part of the grammar of the Substitutional Theory, something he takes as unavoidable: he claims that "we cannot, for example, define membership in a matrix except by saying something equivalent to: '*b* is a member of *p/a*' is to mean 'the *q* such that $p/a; b!q$ is true' "[175].

Hylton's basic argument in favor of this claim is that Russell cannot distinguish between the use of a sentence as the name of a proposition and as an assertion of one without making "[...] an essential, if tacit, use of the notion of truth"[176]. This leads him to view the above paradox as kindred to what we nowadays call *semantic* paradoxes and assumes that the so-called ramification was introduced as a general solution to such difficulties including the *Epimenides*[177]. Apart from some important aspects which we'll address in detail below, Goldfarb follows the general outline of Hylton's interpretation, claiming that "Russell did not separate the theory of orders, which embodies the ramification of propositional functions, from the theory of types"[178], by which he means that Russell did *not* separate semantic paradoxes from logical ones. From this, there emerges a historical account of how Russell's views developed which we called the 'Orthodox View' in the Introduction. We can summarize it as follows:

> **The Orthodox View.** Russell was attracted to the Substitutional Theory because it seemed to preserve the unrestricted variable. Consideration of paradoxes like the *Appendix B* (understood as semantic in nature) and the *Epimenides* led him to introduce type distinctions among propositions in some form or another[179]. This move, however, undermined the basic feature which motivated the Substitutional Theory of classes and relations, its capacity to preserve the unrestricted variable. Thus, from late 1906 or early 1907 onward, Russell adopted ramified type theory on the heels of Poincaré's idea that impredicative definitions involve vicious circularities. And the adoption of this theory involved assuming an ontology of propositions and propositional functions understood as *entities* and not merely symbolic conveniences, by means of which he emulated a simple type theory of classes (employing the axiom of Reducibility). This theory was supposedly presented in its definitive form in Russell's 1908 paper *Mathematical Logic as Based on the Theory of Types*, which introduces essentially the same system as that of *Principia Mathematica*. Thus, the unrestricted variable fell with the so-called simple Substitutional Theory.

The above account, even when developed in sophisticated ways (as in Hylton and Goldfarb's writings) leaves several questions without a satisfactory answer. For instance, if Russell was

175 HYLTON, P., 1980, p.24.
176 HYLTON, P., 1980, p.24.
177 HYLTON, P., 1980, p.25.
178 GODLFARB, W., 1989, p.24.
179 This is a point of disagreement between Hylton and Goldfarb. See HYLTON, P., 1980; GOLDFARB, W., 1989.

aware of difficulties involved in an ontology of propositions as early as 1902, how could it take so long for him to realize that they affect the Substitutional Theory also? And if *Mathematical Logic* marks the definitive abandonment of the Substitutional Theory why did Russell assert in that paper that a hierarchy of functions "[...] of various orders may be obtained from propositions of various orders by the method of substitution"[180], with the former being preferable only for being "more convenient" in practice?

We shall see next that the Orthodox View does not conform to the current established knowledge we now posses about the development of Russell's views. As Landini observes[181], any reasonable interpretation of the development of Russell's views throughout 1906-1908 must satisfactorily account for the views and changes which appear in the following published works (one almost published):

1. *On the Substitutional Theory of Classes and relations*[182] finisehd in April 1906 and read to the London Mathematical Society 10 may.
2. *On Insolubilia and Their Solution Through Symbolic Logic*[183], finished around July 1906 and published as *Les Paradoxes de la Logique* in September.
3. *Mathematical Logic as Based on the Theory of Types*[184], finished around June of 1907 and published by July of 1908.

We now also have available many of Russell's unpublished manuscripts and work notes of the relevant period. Among these, the most important for our purposes are the following:

1. The manuscript *On Substitution* dated April/May 1905[185].
2. The manuscript *The Paradox of the Liar* dated September 1906[186].
3. The manuscript *Logic in Which Propositions are not Entities*, also dated September of 1906[187].
4. The manuscript *On Types* dated January 1907[188].

180 RUSSELL, B., 1908, p.603
181 LANDINI, G., 2015, p.168-9.
182 RUSSELL, B., 1906*a*.
183 RUSSELL, B., 1906*b*.
184 RUSSELL, B., 1908.
185 RUSSELL, B., 1905*f*.
186 RUSSELL, B., 1906*f*.
187 RUSSELL, B., 1906*e*.
188 RUSSELL, B., 1907*e*.

5. The set of notes *Fundamentals* dated January 1907[189].

6. A letter Russell wrote to Ralph Hawtrey on 22 January 1907[190].

In the next sections we attempt a discussion of the most relevant points of the above papers and documents related to the abandonment of the Substitutional Theory. Needless to say that given our purposes and the centrality and depth of Landini's investigations of these issues, our discussion will be largely driven by them. Our discussion of the unpublished manuscripts will also take into account some recent issues raised by Jolen Galaugher[191] about Landini's views.

3.3.2 The Substitutional Theory and *Insolubilia*: the p_o/a_o Paradox

On 16 June of 1906[192] Russell sent his rebutal of Poincaré's article to Couturat, so that he could translate it to French. The paper was originally entitled *Insolubilia and Their Solution by Symbolic Logic* and by Couturat's request it was published under the title "*Les Paradoxes de la Logique*" in september of 1906 in the *Revue*[193]. Russell's response focused mainly on *Poincaré's* claims concerning the contradictions and their impact in principles of Logic and Set theory. Famously, the mathematician accused Russell (and all followers of Cantor) of indulging in the assumption of illegitimate totalities[194]. And as is also well known one of Poincaré's main contentions was that the contradictions found at the foundations of logic involved 'vicious circles', or 'vicious circularities'. As Russell received Poincaré's criticism, the latter was (informally, of course) arguing that there are two kinds of propositional functions: those that determine a class and those that do not; the former he called 'predicative', the latter 'impredicative'. The criteria proposed by Poincaré for distinguishing them was the sort of quantification involved in their definition. If the definition of a function ϕ involved quantification over a range of values that contained ϕ, then the definition was impredicative, otherwise it was predicative. Poincaré was, of course, urging that in order to avoid contradictions one should avoid impredicative definitions, i.e, those that involve a "vicious circularity". In Russell's hands, this became the guiding principle: "Whatever involves an apparent variable must not be among the possible values of that variable"[195]. Thus, Russell recognized that there was "[...] an element of truth in Poincaré's objections", namely that "[...] whatever in any way concerns all of any or some (undetermined) of the members of a class must not be itself one of the members of a class"[196].

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189 RUSSELL, B., 1907*h*.
190 RUSSELL, B., 1907*c*. Cf. LINKSY, B., 2003 for a photocopy for Russell's original handwriting.
191 GALAUGHER, J., 2013.
192 MOORE, G., 2014, p.274-5.
193 LACKEY, D., 1973, p.190; MOORE, G., 2014, p.273.
194 POINCARÉ, H., 1906, p.316-17.
195 RUSSELL, B., 1906*b*, p.284.
196 RUSSELL, B., 1906*b*.

A long interpretative tradition which includes, but is not restricted to those authors that hold what we called the Orthodox view, has understood this concession of Russell to the famous French mathematician as marking the incorporation of the principle as playing some *justificatory* role for type distinctions of a given domain of *entities*[197]. Hylton, for instance, claims that Russell thought the principle "[...] could justify type distinctions for propositional functions [conceived as intensional entities]"[198] in both *Mathematical Logic* and *Principia Mathematica*, while Goldfarb speaks of the principle as sufficient to "yield ramification"[199], characterizing ramification in *general* as "the result of requiring that legitimate specifications of such *entities* be predicative"[200]. Also, since Poincaré's main examples of contradictions which involve vicious-circularity are semantic in nature, e.g., the *Liar* paradox, Russell's adoption of Poincaré's principle has been often understood as marking his failure to recognize the important distinction later explicitly urged by Frank Ramsey[201] between those paradoxes that can be avoided by the simple theory of types (i.e, the logical paradoxes) and those that require the so-called ramified theory (i.e., the semantic ones).

It is often neglected, however, that Russell accepted Poincaré's principle *only as a negative guideline* - not as a positive principle which *solved* the contradictions. In *Insolubilia* Russell makes this crystal clear:

> It is important to observe that the vicious-circle principle is not itself the solution of vicious circle paradoxes, but merely the result which a theory must yield if it is to afford a solution of them. It is necessary, that is to say, to construct a theory of expressions containing apparent [bound] variables which will yield the vicious circle principle as an outcome. It is for this reason that we need a reconstruction of logical first principles, and cannot rest content with the mere fact that paradoxes are due to vicious circles.[202]

In fact, Russell was still occupied, after accepting the Vicious-Circle Principle, with the search for a genuine *solution* to the contradictions which avoided such vicious-circularity while being well motivated from a philosophical point of view. And at this point Russell still thought that any satisfactory solution should preserve the unrestricted variable and the "old maxim that whatever is is one".

In *Insolubilia*, Russell framed this task around the difficulties involved in dealing with *general* propositions, for assuming that the variables of a logical calculus are completely unrestricted, general propositions becomes problematic: assume for instance that *p* is general, i.e.,

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197 This view can be traced back at least to Gödel's famous paper on Russell's Mathematical Philosopy (GÖDEL, K., 1944), though some hints of it are also present in Carnap's seminal paper on Logicism (CARNAP, R., 1931).

198 HYLTON, P., 2004, p.114. Hylton takes the vicious circle principle to be grounded upon the alleged *metaphysical* doctrine according to which "a propositional function presupposes its values"; still, Hylton assumes that the principle embodies some sort of substantial thesis about the nature of mathematical entities.

199 GOLDFARB, W., 1989, p.37.

200 GOLDFARB, W., 1989, p.24. Our emhpasis.

201 RAMSEY, F., 1925.

202 RUSSELL, B., 1906*b*, p.289.

that it contains an apparent variable *x*; if *p* is an entity, it itself must be a possible value of the very variable *x* which occurs in it, and this leads to a vicious circularity of the kind Russell wanted to avoid. The general idea Russell inclined to at the time was to treat general propositions as incomplete symbols in order to reconcile an ontology of propositions with the unrestricted variable:

> To reconcile the unrestricted range of the variable with the vicious-circle principle, which might seem impossible at first sight, we have to construct a theory in which every expression which contains an apparent variable (i..e,which contains such words as all, any, some, the,) is shown to be a mere *façon de parler*.[203]

How did Russell envision such a solution? Well, the idea which he sketched involved introducing a hierarchy of the senses of *truth* applied to *statements*, with each sense of truth tracking the number of apparent (bound) variables that occur in the statements in question (a view which, as we'll see in the next chapter, also appears in a modified form in *Principia*, as Landini showed[204]). Taking as his example the *Epimenides*, Russell explained that a truth-predicate which applies to a statement like 'There is a proposition *p* which I am affirming and which is false' must have a different meaning from one that applies to the proposition *p* itself. His rough proposal in the paper is that statements or sentences must be distinguished between those that are not general and express (and denote) actual propositions (which are entities) and those that do not, with such distinctions being tracked by the number of apparent variables a statement contains:

> If we want to state what is equivalent to 'I am making a false statement containing *n* apparent variables', we must say something like: 'There is a propositional function $\phi(x_1, x_2, \ldots, x_n)$ such that I assert that $\phi(x_1, x_2, \ldots, x_n)$ is true for any values of $x_1, x_2, \ldots, x_n$ and this is in fact false'. This statement contains $n + 1$ apparent variables, namely $\phi(x_1, x_2, \ldots, x_n)$ and ϕ. Hence, it does not apply to itself.[205]

Taking as an example the law of excluded middle, i.e., "every proposition is either true or false", Russell asserts:

> In the restricted sense which we have given to proposition, the law of excluded middle is not a proposition, since it contains an apparent variable. It is a true statement, but *true* here has a different meaning, namely that all the propositions which the statement ambiguously denotes are true (in the previous sense). As applied to statements the meaning of the word *true* varies as the number

[203] RUSSELL, B., 1906*b*, p.292.
[204] LANDINI, G., 1998*a*, p.281-5.
[205] RUSSELL, B., 1906*b*, p.291.

> of apparent variables in the statements varies. The broad result for the sake of which the above theory is adopted is this: If ϕx is true for all values of *x*, it does not follow that ϕx is true of the statement that ϕx is true for all values of *x*. Thus, the vicious-circle paradoxes which would result if this did follow are avoided.[206]

The general idea considered here is to eliminate general propositions from the ontology of the substitiutional theory; Russell's idea is to take what he calls general statements - i.e., formulas containining apparent variables, which are, of course kept as part of the language of the Substitutional Theory - and explain their truth and falsehood in terms of truth and falsehood of genuine (non-general) propositions. Russell's idea here is that only what he called 'elementary statements', that is, statements that do not contain apparent variables denote propositions; general statements are understood as incomplete symbols that assert that some class of elementary propositions is true or false (with this class treated as a pseudo entity emulated by a matrix like p/a, of course). This, in a sense, creates a hierarchy of types of general propositions and explains Russell's letter of 10 September to Jourdain where he claims to "[...] incline at present to the doctrine of types, much as it appears in appears in *Appendix B* of [*The Principles*]" with the difference that "propositions and functions can never be *apparent* variables, so that statements about all of them are meaningless"[207].

But interpretations of *Insolubilia*, like that of Warren Goldfarb[208], which take it as introducing a form of ramification of entities are plainly inadequate. The hierarchy of types is a hierarchy of *statements*, of incomplete symbols, and Russell makes it absolutely clear that the point of treating them as such is to preserve the unrestricted variable which ranges over a single unique domain, namely that of *all* entities. There are no types of entities in *Insolubilia*. A letter to Jourdain from 14 July 1906 also makes it clear that the solution to the problem of vicious circularity could be attained by treating them as pseudo-entities, like classes and propositional functions:

> [...] the no-classes theory shows that we can employ the symbol $x(\phi x)$ without ever assuming that this symbol in isolation means anything. I feel more and more certain that this theory is right. In order, however, to solve the *Epimenides*, it is necessary to extend it to general propositions, i.e.,to such as $(x).\phi x$ and $(\exists x).\phi x$. This I shall explain in my answer to Poincare's article in the current *Revue de Metaphysique*.[209]

Now, this letter and some points of the discussion in *Insolubilia* strongly indicate that Russell was in fact preoccupied with the liar paradox and its cognates, and thus, that he followed Poincaré in not distinguishing logical paradoxes from semantic ones. Was this the case?

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[206] RUSSELL, B., 1906*b*, p.292.
[207] GRATTAN-GUINNESS, I., 1977, p.91.
[208] GOLDFARB, W., 1989, p.24.
[209] GRATTAN-GUINNESS, I., 1977, p78.

As Landini has extensively argued, the evidence pointing to a negative answer, on the other hand, is quite strong. As we pointed out in the Introduction, in his archival research on the manuscripts and documents Russell left concerning the Substitutional Theory, Landini made a series of game-changing discoveries for understanding the abandonment of the Subsitutional Theory. To begin with, Landini found a letter Russell wrote to the mathematician Ralph Hawtrey on 22 January 1907, where he communicates a "paradox which pilled the Substitutional Theory". The letter reads as follows[210]:

> Dear Hawtrey, - I forgot to send you the paradox which pilled the substitution-theory. Here it is. Put
>
> $$p_0 \,.{=}{:}\, (\exists p, a) : a_o \,.{=}.\, p\frac{b}{a}!q : \sim \left(p\frac{a_o}{a}\right)$$
>
> [where "$p\frac{b}{a}!q$" means "becomes q by substituting b for a"]. Then:
>
> $$p_o \frac{p_o\frac{b}{a_o}!q}{a_o} \;.{=}{:}\; (\exists p,a) : p_o\frac{b}{a_o}!q \;.{=}.\; p\frac{b}{a}!q : \sim \left(p\frac{p_o\frac{b}{a_o}!q}{a}\right)$$
>
> Hence
>
> $$p_o\frac{p_o\frac{b}{a}!q}{a_o} \;.\supset{:.}\; (\exists p,a): p_o\frac{b}{a_o}!q = p\frac{b}{a}!q : p_o\frac{p_o\frac{b}{a_o}!q}{a_o}!q \,.\sim \left(p\frac{p_o\frac{b}{a_o}!q}{a}!q\right) \quad (1)$$
>
> $$\sim\left(p_o\frac{p_o\frac{b}{a_o}!q}{a_o}\right) \;.\supset{:.}\; p_o\frac{b}{a_o}!q \;.{=}.\; p\frac{b}{a}!q \;:\supset_{p,a}.\; \left(p\frac{p_o\frac{b}{a_o}!q}{a}!q\right) \;{:.}\supset{:.}\; p_o\frac{p_o\frac{b}{a_o}!q}{a_o} \quad (2)$$
>
> $$(1).(2)\supset{:.}\; p_o\frac{p_o\frac{b}{a_o}!q}{a_o}!q \;{:.}\; (\exists p,a): p_o\frac{b}{a_o}!q = p\frac{b}{a}!q : \sim\left(p\frac{p_o\frac{b}{a_o}!q}{a}\right) \quad (3)$$
>
> But if $p_o\frac{b}{a_o}!q$ is the same proposition as $p\frac{b}{a}!q$, it seems plain we must have $p = p_o \,.\, a = a_o$, whence $p\frac{p_o\frac{b}{a_o}!q}{a} \,.\equiv.\, p_o\frac{p_o\frac{b}{a_o}!q}{a_o}$.
>
> Thus it is impossible that $p\frac{p_o\frac{b}{a_o}!q}{a}$ should be false while $p_o\frac{p_o\frac{b}{a_o}!q}{a_o}$ is true, which, by (3) is shown to be involved.

[210] The letter was discovered by Landini in the Russell archives (LANDINI, G., 1989). The first published (complete) *transcription* of the letter was by Grattan-Guinness in his *The Search for Mathematical Roots* (GRATTAN-GUINNESS, I., 2000, pp.279-80.). Bernard Linsky pointed out that there are some errors in the original letter (as well as some corrections which are probably due to Hawtrey) and published a new corrected transcription together with Russell's handwritten original and a formal reconstruction within a modern natural deduction system (LINSKY, B., 2003, pp.151-160). There is also a type-set version in the recently published *Collected Papers 5* (MOORE, G., 2014, p.125), but unfortunately the original plate was not published with it. We follow Linsky's corrected transcription, but not his reconstruction of the proof.

> In Trying to fix this paradox, I have modified the substitution-theory in various ways, but the paradox always reappeared in more and more complicated forms.
>
> Yours ever, Bertrand Russell.
>
> 20 Bagley Wood. Jan. 22, 1907[211]

Despite the cumbersome notation and the complex substitutions involved the version of the paradox which appears in the letter, the problem posed there is quite simple. Line (1) introduces by definition a proposition p_0 from which a problematic matrix p_0/a_0 is obtainable. The problem, as Landini puts it, is that there is "[...] a fundamental tension between the Substitutional Theory and the diagonal method used by Cantor to generate his power-class theorem"[212]. Put generally, the issue arises from the fact that one may prove within the substitititional theory the following:

$$p_0 = (\exists p, a) : a_0 = f(p, a) : \sim \left(p\frac{a_0}{a}\right)$$

Where f is a one-one function from propositions into classes of propositions[213]. As Landini observed and Russell's manuscripts confirmed, there are many ways to prove the existence of such a function f within the simple Substitutional Theory and Russell was well aware of them. One of them, that developed by Hylton, is completely analogous to the *Appendix B*. Now, as we mentioned, Hylton thinks the formulation of the paradox requires the use of truth-predicates given Russell's carelessness with use and mention. This is certainly not the case.

The only thing that is required for distinguishing an asserted proposition from a nominalized one is an explicit device for nominalizing formulas, i.e., a term-forming operator, something with which Russell was surely familiar[214]. Again, let us use brackets [and] for this, with "*p*" being an asserted sentence and "$[p]$", $[p \supset q]$", "$[(x) \,.\, p\frac{x}{a}!q]$", etc being terms of the language of the Substitutional Theory that can be flanked by the identity sign. Thus, to generate the *Appendix B* paradox in the Substitutional Theory, we let[215]:

$$f(p, a) = \left[(r)(p\frac{r}{a} \,.\, \supset_r .\, r)\right]$$

From which we obtain:

$$[p_o] = \left[(\exists p, a)\left(a_o = \left[(r)(p\frac{r}{a} \,.\, \supset_r .\, r\right] : \sim \left(p\frac{a_o}{a}\right)\right)\right]$$

Following Galaugher we may call this the *Appendix B* version of the p_0/a_0 paradox, since it is the exact same contradiction formulated in terms of the language of substitution. The function f employed in the letter Russell sent to Hawtrey which Landini originally called p_0/a_0 arises by letting:

$$f(p, a) = \left[p\frac{b}{a}!q\right]$$

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211 RUSSELL, B., 1907*c*, p.125.

212 LANDINI, G., 2010, p.147.

213 LANDINI, G., 2010, p.147; GRATTAN-GUINNESS, I., 2000, p.363; LANDINI, G., 2014, p.170.

214 LANDINI, G., 1998*a*, p.43-4.

215 We follow Galaugher (GALAUGHER, J., 2013, p.15) in uniformly using the notation "$p\frac{x}{a}$" which Russell employs in his manuscripts more instead of "$p/a; x$".

From which it follows that:

$$[p_o] = \left[(\exists p, a).\left(a_o = \left[p_o \frac{b}{a}!q\right] : \sim \left(p \frac{a_o}{a}\right)\right)\right]$$

From whence, by the substitution $p_o \frac{p_o \frac{b}{a_o}!q}{a_o}$ we obtain:

$$(\exists p, a)\left(\left[p_o \frac{b}{a_o}!q\right] = \left[p \frac{b}{a}!q\right] : \sim \left(p \frac{p_o \frac{b}{a_o}!q}{a}\right)\right)$$

And, again, from a series of elementary steps and assuming $p_o \frac{b}{a_o}!q = p \frac{b}{a}!q : \supset : p = p_o \,.\, a = a_o$ as giving identity conditions for propositions, we have[216]:

$$(\exists p, a)\left(\left[p_o \frac{b}{a_o}!q\right] = \left[p \frac{b}{a}!q\right] : \left(p \frac{p_o \frac{b}{a_o}!q}{a}\right)\right) . \equiv . \sim (\exists p, a)\left(\left[p_o \frac{b}{a_o}!q\right] = \left[p \frac{b}{a}!q\right] : \left(p \frac{p_o \frac{b}{a_o}!q}{a}\right)\right)$$

Another simpler way given by Landini is to just let[217]:

$$f(p, a) = [a \supset p]$$

By the axiom of identity of propositions, we have:

$$[p_o \supset a_o] = [p \supset a] : \supset :. p_o = p : a_o = a$$

We let the pair p_o/a_o be such that [218]:

$$(x)\left(p_o \frac{x}{a_o} \equiv \left((\exists p, a)\left(x = [p \supset a] . \sim \left(p \frac{x}{a}\right)\right)\right)\right)$$

Then, if we put forward the following substitution:

$$\left[p_o \frac{p_o \supset a_o}{a_o}\right] = \left[(\exists p, a)\left([p_o \supset a_o] = [p \supset a] : \sim \left(p \frac{p_o \supset a_o}{a}\right)\right)\right]$$

We obtain:

$$p_o \frac{p_o \supset a_o}{a_o} \equiv \sim \left(p_o \frac{p_o \supset a_o}{a_o}\right)$$

Assuming an unrestricted domain of propositions over which we let the individual variables range, these paradoxes spring directly from the axioms of the Substitutional Theory which assert that any two propositions are identical if and only if they have the same constituents. From these assumptions one can show within Russell's original Substitutional calculus that any one

216 We follow closely the formulation of GALAUGHER, J., 2013, p.14.
217 LANDINI, G., 2014, p.171.
218 LANDINI, G., 2010, p.147.

of the above functions f is one-one. As already noted in the Introduction, the importance of Landini's re-discovery of this family of paradoxes in Russell's manuscripts, in particular that presented in the letter to Hawtrey cannot be overstated, not only because the letter brought to light a completely new paradox unique to the Substitutional Theory, but because it strongly indicates that it was this very paradox that eventually led to Russell to abandon the theory. In the letter Russell reports to Hawtrey that "[...] in trying to avoid this paradox, [he] modified the Substitutional Theory in various ways, but the paradox always reappeared in more and more complicated forms"[219].

Indeed, as we already briefly discussed in the Introduction, on the basis of this letter and of his meticulous study of Russell's manuscripts[220] on the Substitutional Theory, Landini has put forward a radical challenge to the Orthodox view and the interpretative tradition which takes Russell to have been led to the ramified theory of types by the semantic paradoxes. Landini has provided persuasive arguments and evidence to show that semantic paradoxes like the *Epimenides* were *not* the driving force towards the abandonment of the Substitutional Theory, for as he observes "like the *PoM Appendix B* this paradox is based on the diagonal method employed in Cantor's power-class theorem"[221]. We have, in fact solid evidence showing that Russell knew of this paradox months before publishing *Insolubilia/Les Paradoxes*. In the manuscript *On Substitution*[222] dated April/May of 1906, for instance, we find the following:

> $$p_o \,.{=}{:}\, (\exists p, a) : a_o = (p\frac{b}{a}!q) : p\frac{a_o}{a}!r \,.\supset_r .\sim r$$
>
> [...]
>
> $$.{=}{:}\, (\exists p, a) : p_o\frac{b}{a_o}!q \,.{=}.\, p\frac{b}{a}!q :\sim (p/a); (p_o\frac{b}{a_o}!r)$$
>
> The only way to avoid *oddities* is to decree a doctrine of types as regards *propositions*, so that in $p\frac{b}{a}!q$, b and a be of lower type than p and q, so that the substitution of $(p\frac{b}{a}!q)$ for a is illegitimate.[223]

Of Russell's manuscripts dealing with the Substitutional Theory, *On Substitution* is certainly among the most difficult ones, with daunting twists and turns and blind alleys. This is, of course, to be expected, since as Landini observes, these notes are nothing more than dialogues Russell was having with himself[224]. It is clear, however, that the main issue with which Russell was struggling was the reconciliation of the "oddities" involving the family of paradoxes p_o/a_o. Several different approaches are considered and discarded over and over again. Russell considered for instance distinguishing between propositional and non-propositional substitutions:

[219] RUSSELL, B., 1907*c*, p.125.
[220] Now published in the fifth volume of the *Collected Papers of Bertrand Russell* (Cf. MOORE, G. (ed)., 2014).
[221] LANDINI, G., 1998a, p.204.
[222] RUSSELL, B., 1906*d*.
[223] RUSSELL, B., 1906*d*, p.131, folio 7.
[224] LANDINI, G., 1998a, p.207.

> All we need is such a modification as will permit
>
> $p\frac{b}{a}!q \,.=.\, p_o\frac{b}{a_o}!q : p/a \neq p_o/a_o$
>
> Again we might notationally distinguish propositions and entities, and have two sorts of substitution, propositional and entity. The notational distinction may, to begin with, be as hitherto, by a difference of letters. Thus p, q, r, s will be propositions. The different sorts of substitutions may be indicated in the same way: $\frac{q}{p}$ is a propositional substitution, $\frac{x}{a}$ an entity substitution. $\frac{q}{a}$ and $\frac{x}{p}$ are to be meaningless.[225]

But as Landini showed well before the publication of the manuscripts, Russell recognized that the approach was hopeless for several reasons[226]. In fact, time and again Russell returned to type distinctions as the best approach, but found it unbearable to put aside the unrestricted variable and to reconcile type-distinctions with propositions as entities. A typical passage which labors this point in some detail is the following:

> What is essential is not the denial that propositions, functions, classes, relations, etc. are entities, but the denial that they can be substituted in the sort of functions which we have called entity-functions. But it will be difficult to define these functions, or to settle what among entities are the sort we have hitherto considered as entities *sans phrase*. Thus it seems simpler to stick to the view that propositions etc. are not entities.
>
> It seems as if propositions *must* be entities. They may, however, be entities of different kind from others, and such that $\phi(p)$ and $\phi(x)$ are never both significant.
>
> But if propositions are entities, how are we to avoid classes of propositions? And if we once allow them , we can't escape such propositions as $\wedge`u \in u$, if *u* is a class of propositions[227]. And if we allow this, the liar becomes insoluble; unless we introduce a hierarchy of propositions, which is intolerable.[228]

Thus, as Landini puts it, by the time *Insolubilia* was published Russell's "[...] philosophical objections to orders of propositions had won out"[229]. But none of the main manuscripts dealing with substitution from April and May of 1906, namely *On Substitution*[230] and *Logic in Which Propositions are not Entities*[231] consider the approach which Russell ended up defending in *Insolubilia*[232].

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225 RUSSELL, B., 1906*d*, p.164, folio 94.

226 Roughly, the issue was that *propositional substitutions* which this approach permitted allowed the paradox to be formulated. Cf. RUSSELL, B., 1906*d*, pp.164-5, in particular folio 96 and also LANDINI, G., 1998a, p.206-212 for details.

227 Recall that $\wedge`u$ is the notation Russell employed for the logical product of all elements of a class *u* of propositions.

228 RUSSELL, B., 1906*d*, p.185, folio 140.

229 LANDINI, G., 1998a, p.213.

230 RUSSELL, B., 1906*d*.

231 RUSSELL, B., 1906*e*.

232 RUSSELL, B., 1906*b*.

The approach of *Insolubilia* is considered in the manuscript *The Paradox of the Liar*[233] dated September 1906, where we find it as one possible approach:

> Another possible basis for a hierarchy of propositions is the following. There are such things as propositions, but no propositions contain apparent [bound] variables. Statements contain apparent variables, but are merely ambiguous: they assert one of a number of propositions, without deciding which. Thus statements containing 'all propositions' do not themselves state propositions. This suffices to avoid the liar and kindred paradoxes.[234]

We also find it in the following summary of approaches to the contradictions:

> I. In order to avoid the paradox of the liar, it is necessary to have a hierarchy of propositions (i.e. of what one would naturally call propositions), so that a proposition can never be about all propositions, but only about all at one grade in the hierarchy; and a proposition about all propositions of a certain grade must be of the next grade. So far as the liar is concerned, there are various ways in which such a hierarchy might be constructed.
>
> II. The hierarchy which seems both inherently the most plausible and the best designed to obviate paradoxes is arranged according to the number of apparent variables in a statement. A statement containing no apparent variables affirms a definite proposition; a statement containing apparent variables affirms an ambiguous proposition, i.e. any one of the propositions which result from assigning values to the variables. Hence a statement about all propositions does not state a new proposition.[235]

However, in the same manuscript we find Russell considering several ways of forming a hierarchy of propositions which he tried to avoid in *Insolubilia* by eliminating general propositions[236]. What happened?

3.3.3 From *Insolubilia* to *Mathematical Logic as Based on the Theory of Types*

In *Insolubilia* Russell did not pursue in detail the question of how a viable logical theory could be developed by not assuming general propositions. Although he ended his reply to Poincaré with confidence, stating that "[...] there seems reason to hope that the method proposed in this article avoids all the contradictions, and at the same time preserves Cantor's results"[237] he also recognized that this was not established beyond reasonable doubt: a demonstration of this would require "[...] a lengthy symbolic development" and "[...] a long and patient labor of analysis and reconstruction will probably be necessary before the principles of mathematics can be

[233] RUSSELL, B., RUSSELL, B., 1906*f*.
[234] RUSSELL, B., 1906*f*, p.328, folio 23.
[235] RUSSELL, B., 1906*f*, p.332-3, folio 30.
[236] For instance, RUSSELL, B., 1906*f*, p.352, folio 72.
[237] RUSSELL, B., 1906*b*, p.296.

stated in the absolutely best form"[238]. Well, it seems that in the process of carrying out this task, Russell realized that adopting either a hierarchy of statements as in *Insolubilia* or of genuine propositions as he tentatively considers in the *Paradox of the Liar* requires some mitigating axiom for classical Mathematics to be developed. In the same summary quoted above he explains:

> III. In order to be able to deal with things like mathematical induction where a variable function appears, it is essential to have some means of inferring that certain formulae, if they hold for all functions below a certain grade, hold for all functions absolutely. For this purpose we need an axiom to assure us that any propositional function containing a real variable *x* and any number of apparent variables is formally equivalent to some function containing less than some given (constant) number of apparent variables.[239]

The difficulty here is the essential impredicativity involved in the principle of mathematical induction and some parts of Analysis[240] and the required axiom is akin to the Axiom of Reducibility formulated in terms of the Substitutional Theory without general propositions. Indeed, Russell even considered this informally in *Insolubilia*, where explains that:

> [...] unless this restriction[241] is mitigated by an axiom, it will render most of the usual uses of induction fallacious; and in other ways it will destroy many pieces of ordinary mathematical reasoning.[242]

Such mitigating axiom is considered by Russell in *Insolubilia* as one which asserts of any general statement that there is an elementary one equivalent to it, as he puts it: "[...] every statement containing *x* and an apparent variable is *equivalent*, for all values of *x*, to some statement ϕx containing no apparent variable"[243].

Landini conjectures that Russell must have realized that the theory of *Insolubilia* allowed the introduction of p_0/a_0 paradoxes once such mitigating axioms were added to the theory and that this was the sole cause of the abandonment of the Substitutional Theory of that paper[244]. Unfortunately, there is no extant manuscript with a formal treatment of *Insolubilia*'s system to

[238] RUSSELL, B., 1906*b*, p.296.
[239] RUSSELL, B., 1906*f*, folio 30.
[240] As is well known, the restrictions of the hierarchy of orders are quite severe: they don't block only the contradictions, but also portions of ordinary Mathematics. Among the casualties are, for instance, the definitions of mathematical induction and the completeness property of the reals in terms of the existence of least upper bounds. To remedy this Russell would later introduce a family of *Axioms of Reducibility*, which assert that given any non-predicative monadic propositional function *f*, there is a predicative function $\phi!\hat{x}$ co-extensive with *f*, and so on for every *n*-ary non predicative propositional function.
[241] I.e., that "a statement involving an apparent function-variable must not be of the form ϕx even when it contains *x*" (RUSSELL, B., 1906*b*, p.211).
[242] RUSSELL, B., 1906*b*, p.211.
[243] RUSSELL, B., 1906*b*, p.212.
[244] Cf. LANDINI, G., 1998a, pp.227-231; also LANDINI, G., 2010, pp.155-8.

confirm this, so Landini's claim relies completely on his own reconstruction which does establish his conclusion. Landini shows, in fact, that his reconstructed system eliminates general propositions and solves the Liar, the Appendix *B* and all variants of the p_0/a_0[245]. But the system requires the introduction of mitigating axioms of the form of the following scheme[246]:

> $(\exists p, a)(x)(\exists q)\left(p\frac{x}{a}!q : q \equiv A\right)$ where *p*, *a* are not free in *A*, and *A* is any *wff* (general or otherwise) containing *x* free.

This does not allow the return of the *liar* or the *Appendix B*, but it reintroduces some forms of the p_0/a_0, in particular the p_0/a_0 of the Hawtrey letter[247]. So, in sum, without mitigating axioms, the system of *Insolubilia* could not recover Arithmetic and with them, it was vulnerable to some forms of the p_0/a_0 paradoxes.

Landini's conjecture is corroborated by the fact that in September of 1906 Russell was considering a hierarchy of *orders* of propositions (not mere statements). At some point *The Paradox of the Liar* he considers the view that "[...] there are propositions, though not in the same sense in which there are individuals"[248], and puts particular weight to the fact that the Hawtrey variant of p_0/a_0 can be dissolved in this way:

> A second-order proposition is one in which either "all values" or "any value" of *p* occurs, or a complex $p\frac{x}{a}!q$ occurs. I think the latter alone is a sufficient: all second-order propositions that arise contain $p\frac{x}{a}!q$.
>
> We shall need a notation, say p^2, for any second-order proposition. Then we have $p^2\frac{x}{a}!q^2$ and also $p^2\frac{p}{q}!q^2$: both are significant. The former substitution will affect only origin and argument in $p\frac{x}{a}!q$, the latter will affect only prototype and resultant. Both these substitutions are third-order propositions. Thus $p\frac{\alpha}{\beta}!q$ is always of higher order than any of its constituents; this disposes of the fallacy which led to the abandonment of substitution before, i.e.
>
> $$p_o \,.=: (\exists p, a) : a_o \,.=.\, p\frac{x}{a}!q :\sim p\frac{a_o}{a} :. \supset .\, p_o/a_o; p_o\frac{b}{a_o}!q :\sim p_o/a_o; p_o\frac{b}{a_o}!q$$
>
> for here we substitute for a_o the proposition $p_o\frac{b}{a_o}!q$, which is necessarily of higher grade than a_o.[249]

[245] See LANDINI., 1998, p.216-227

[246] Cf. LANDINI, G., 1998a, p.229-30 for details.

[247] LANDINI, G., 1998a, pp.227-33 for a full demonstration. Roughly, this is due to a fundamental difference between some variants of p_0/a_0 and the *Liar* and *Appendix B*: the former does *not* require an identity between $f(a,p)$ and a general proposition in order to generate a one-one function between propositions and classes of propositions (LANDINI, G., 1998a, p. 230).

[248] RUSSELL, B., 1906*f*, p.352, folio 72.

[249] RUSSELL, B., 1906*f*, p.352, folio 72.

And, in, fact, Landini's conjecture together with the available evidence also provides a good solution to interpretative issues which puzzled scholars for quite some time about Russell's next main work, *Mathematical Logic as Based on the Theory of Types*.

Guided again, by the Vicious-Circle Principle, Russell introduced types in that work as "[...] the range of significance of a propositional function, i.e. as the collection of arguments for which the said function has values"[250]. There Russell employs the functional notation which would later appear in *Principia*: he employs single letter variables for (propositional) functional variables instead of matrices and he makes it clear that, *in practice*, the theory which is put forward is a hierarchy of propositional functions from which a derived hierarchy of propositions can be obtained:

> A function whose argument is an individual and whose value is always a first-order proposition will be called a first-order function. A function involving a first-order function or proposition as apparent variable will be called a second-order function, and so on. A function of one variable which is of the order next above that of its argument will be called a predicative function; the same name will be given to a function of several variables if there is one among these variables in respect of which the function becomes predicative when values are assigned to all the other variables. Then the type of a function is determined by the type of its values and the number and type of its arguments.
>
> The hierarchy of functions may be further explained as follows. A first-order function of an individual *x* will be denoted by $\phi!x$ (the letters ψ, χ, θ, F, G will also be used for functions). No first-order function con- tains a function as apparent variable; hence such functions form a well-defined totality, and the ϕ in $\phi!x$ can be turned into an apparent variable. Any proposition in which ϕ appears as apparent variable, and there is no apparent variable of higher type than ϕ, is a second-order proposition. If such a proposition contains an individual *x*, it is not a predicative function of *x*; but if it contains a first-order function ϕ, it is a predicative function of ϕ, and will be written $f!(\psi!\hat{z})$. Then *f* is a second- order predicative function; the possible values of *f* again form a well-defined totality, and we can turn f into an apparent variable. We can thus define third-order predicative functions, which will be such as have third-order propositions for their values and second-order predicative functions for their arguments. And in this way we can proceed indefinitely. A precisely similar development applies to functions of several variables.[251]

On the surface, the hierarchy introduced here can be illustrated by some fairly familiar examples and ideas. Take the following statements as examples, where "Joe" is a proper name, where Dx stands for *x is a dead parrot*, Rx stands for *x is resting* and $Q(\phi\hat{x})$ stands for $\phi\hat{x}$ *is a quality of a dead parrot.* Take the following statements: "D(Joe)", "R(Joe)" , "$(x) : Dx \,.\supset .\, Rx : \supset .\, R$(Joe)" and "$(\phi) : Q(\phi\hat{x}) \,.\supset .\, \phi$(Joe)". The first two statements contain (only) functions which take individuals as arguments and each statement is what Russell calls an *elementary* proposition: they have no apparent (bound variables), individual or otherwise. The third statement also contains only functions which take individuals as arguments, but it is

[250] RUSSELL, B., 1908, p.601.
[251] RUSSELL, B., 1908, pp.603-4.

possible, by the substitution of the proper name "Joe" by a free variable z, to obtain a function which contains bound individual variables, namely:

$$(x) : Dx \,.\supset.\, Rx : \supset .\, R(\hat{z})$$

This function is like the first two in that it also only takes individuals as arguments, but it contains bound variables - so the propositions which are its values are not elementary. Now, the fourth function is also true of individuals, but by turning "Joe" into a free variable we obtain the following function, which we may call F:

$$(\phi) : Q(\phi\hat{x}) \,.\supset.\, \phi(\hat{y})$$

This function, like the third, contains an apparent variable, but it is an apparent variable which ranges over functions that take individuals as arguments. According to the distinctions introduced, all the four functions we considered that are involved in the statements above are of the same type, for they take individuals as arguments, and not functions, functions of functions and so on. Thus, according to *type* distinctions alone, we could have any of the above functions as possible values of the same variable ψ. To take our example involving the deceased parrot Joe, the function $F\hat{y}$, i.e., $(\phi) : Q(\phi\hat{x}) \,.\supset.\, \phi(\hat{y})$, involves quantification over a totality of functions which contain, for instance, the function $R\hat{x}$ (*i.e. x is resting*); so if $R\hat{x}$ and $F\hat{y}$ were of the same type, we would have a "vicious circle". Thus, besides distinguishing types of functions only in terms of what sort of arguments it may take (individuals, functions of individuals, functions of functions of individuals, and so on), Russell also distinguish types of functions by taking into account what sort of quantifiers occur in them, thus introducing what he calls the *order* of a function and the propositions which are their values, with the same applying, of course, to functional and propositional variables. Put roughly, a function like $F\hat{y}$ must be of a higher order than the order of any bound variable which occurs in it. A predicative function is a function ϕ of order $n+1$ which takes arguments of order *n*; thus, in the above, $D(x)$ is predicative, while $(\phi) : Q(\phi\hat{x}). \supset .\phi(y)$ is not. This gives us a hierarchy of functions from which a hierarchy of propositions can be obtained: a proposition *p* is always of order *n*, where *n* is the highest order among its apparent variables.

Mathematical Logic also contains the exact same contextual elimination of class expressions presented in *Principia*'s section $*20$, and relations in extension given in $*21$, namely[252]:

$$f\{\hat{z}(\psi z)\} \,.=: (\exists\phi) : \phi!x \,.\equiv_x.\, \psi x : f\{\hat{z}(\phi z)\} \text{ Df}$$

$$f\{\hat{x}\hat{y}(\psi(x,y)\} \,.=: (\exists\phi) : \phi!(x,y) \,.\equiv_{x,y}.\, \psi(x,y) : f\{\hat{x}\hat{y}(\phi(x,y)\} \text{ Df}$$

which Russell describes as guaranteeing the possibility of extracting an extensional function from any intensional one[253]; Russell also introduces for the first time in print his Axioms of

[252] Cf. RUSSELL, B., 1908, p.613.
[253] RUSSELL, B., 1908, p.x; MARSH, C., 1956, pp.89; van HEIJENOORT, J., 1967, p.172-3.

Reducibility[254]:

$$\vdash :. (\exists f) : \phi x \,.\equiv_x .\, f!x$$

$$\vdash :. (\exists f) : \phi(x, y) \,.\equiv_{x,y} .\, f!(x, y)$$

which "states that, given any function ϕx, there is a predicative function $f!x$ such that $f!x$ is always equivalent to ϕx"[255]. Thus on the surface everything looks as if Russell had assumed a hierarchy of types of propositional functions as primitive entities.

Several interpreters took *Mathematical Logic* as the abandonment of the Substitutional Theory, for one reason or another. First, there are close similarities between the formulation of the type theory of *Mathematical Logic* and the final theory of types of 1910 put forward in *Principia* - the notation, in particular, is exactly the same. Second, after the publication of *Mathematical Logic* Russell *never mentioned* the Substitutional Theory again, not in *Principia* nor in any of the works surrounding its publication. An interpretation which takes *Mathematical Logic* as putting forward a theory of types/orders of propositional functions faces some serious difficulties, however. First, Russell initially introduces his hierarchy in terms of individuals and propositions of any finite arbitrary order:

> Elementary propositions together with such as contain only individuals as apparent variables we will call first-order propositions. These form the second logical type.
>
> We have thus a new totality, that of first-order propositions. We can thus form new propositions in which first-order propositions occur as apparent variables. These we will call second-order propositions; these form the third logical type. [...]
>
> The above process can be continued indefinitely. The $n+1$*th* logical type will consist of propositions of order *n*, which will be such as contain propositions of order $n-1$, but of no higher order, as apparent variables. The types so obtained are mutually exclusive, and thus no reflexive fallacies are possible so long as we remember that an apparent variable must always be confined within some one type.[256]

Second, and much more serious, Russell explicitly asserts that functional variables are a mere *convenience* that may be eliminated by the use of matrices of the Substitutional Theory:

> In practice, a hierarchy of functions is more convenient than one of propositions. Functions of various orders may be obtained from propositions of various orders by the method of substitution. If *p* is a proposition, and *a* a constituent of p, let "$p\frac{x}{a}$" denote the proposition which results from substituting *x* for *a* wherever *a* occurs in *p*. Thenp/a,which we will call a matrix, may take the

254 RUSSELL, B., 1908, pp.611..

255 RUSSELL, B., 1908, pp.611..

256 RUSSELL, B., 1908, pp.602-3.

> place of a function; its value for the argument x is $p\frac{x}{a}$, and its value for the argument a is p. Similarly, if "$p\frac{(a,b)}{(x,y)}$" denotes the result of first substituting for a and then substituting y for b, we may use the double matrix $p/(a,b)$ to represent a double function. In this way we can avoid apparent variables other than individuals and propositions of various orders. [...]
>
> Although it is possible to replace functions by matrices, and although this procedure introduces a certain simplicity into the explanation of types, it is technically inconvenient. Technically, it is convenient to replace the prototype p by ϕa, and to replace $p\frac{x}{a}$ by ϕx; thus where, if matrices were being employed, p and a would appear as apparent variables, we now have ϕ as our apparent variable. In order that ϕ may be legitimate as an apparent variable, it is necessary that its values should be confined to propositions of some one type. Hence we proceed as follows. [257]

It is only *after* this clarification that Russell introduces the hierarchy of functions. As Landini insists, these passages strongly indicate that Russell took the philosophical foundations of the theory presented in *Mathematical Logic* to be the Substitutional Theory and that Russell probably adopted functional variables given the fact that working with the Substitutional Theory's notation is a daunting task[258]. Landini claims that the type theory of *Mathematical Logic* is Russell's last effort to put the Substitutional Theory to work and that the paper "[...] embraces the hierarchy of orders of propositions that *Les paradoxes* [*i.e.*, *Insolubilia*] sought so desperately to avoid."[259]

The manuscripts and the documented evidence strongly suggest that Landini is right. In a manuscript entitled *On Types*[260] - which we have conclusive evidence to believe to have been written *after The Paradox of the Liar*[261] - we find Russell still considering that "[...] there is much to be said for *reviving* substitution"[262]. There Russell considers a *typed* version of the Substitutional Theory which introduces a hierarchy that is very similar to the hierarchy of propositions which starts with individuals and ascends to propositions of several orders that Russell considers in the beginning of *Mathematical Logic*:

> Assume for moment the substitutional view of functions.
>
> Thus p/a will stand for a function, p being a proposition, called the *prototype*, and a a constituent of p, called the *initial subject*.
>
> The first type consists of individuals.
>
> The second type consists of such propositions as contain no apparent variables other than individuals. Any proposition of the second type will be denoted by p^1 or q^1 or r^1 or s^1, and called a *first-order proposition*; p^1/a or $p^1(a,b)$, or etc. will be called a first-order function. [sic]

[257] RUSSELL, B., 1908, pp.603. We modified Russell notation, putting the more readable "$p\frac{x}{a}$" in place of "$p/a;x$", etc.
[258] LANDINI, G., 1998a, p.235.
[259] LANDINI, G., 1998a, p.236.
[260] RUSSELL, B., 1907*e*.
[261] RUSSELL, B., 1906*f*.
[262] RUSSELL, B., 1907*c*, p.516, folio 3. Our emphasis.

> The third type consists of such propositions as contain no apparent variables except first-order propositions and individuals. [Thus the second type is included in it.) Any proposition of the third type will be called a second-order proposition and denoted by p^2 or q^2 or r^2 or s^2. We can thence derive functions of various kinds: p^2/a, $p^2/(p^1, a)$, p^2/p^1, $p^2/(a, b)$, $p^2/(p^1, q^1)$ and so on. Such functions may be jointly called second-order functions, though they are of various types. Similarly for higher types.
>
> Thus the order of a function depends upon that of its prototype, not upon that of its argument. Owing, however, to the reducibility-axiom, a function can, in most contexts, be replaced by one of the order immediately succeeding that of its argument, or (in the case of a function of several variables) of the argument of highest type among its arguments. In the above treatment, the only possible apparent variables are a, p^1, p^2, p^3, ...[263]

Russell claims, however, that "[...] for technical reasons, it is simpler to use functions rather than substitution", and that the formulas of the substitutional language "[...] can be easily translated into functional language"[264], and then goes on to describe a functional hierarchy that fits closely (although not exactly[265]) that of *Mathematical Logic*:

> The first type consists of individuals.
>
> The second type consists of first-order propositions, as before; first-order functions, which may be regarded instead as forming the second type, are such as have first-order propositions for their values.
>
> Second-order propositions will now be such as contain no apparent variables except individuals and first-order functions. Second-order functions will be such as have second-order propositions for their values. Etc.[266]

To recover classical Mathematics, Russell then considers the introduction of Substitutional Axioms of Reducibility as in:

$$\vdash :. (\exists p^1) : p^n/a \equiv p^1/a$$

Which asserts that every matrix of any arbitrary order is equivalent to some matrix of the first-order.

[263] RUSSELL, B., 1907*e*, folio 1.

[264] RUSSELL, B., 1907*e*, p.515, folio 2.

[265] For a detailed discussion and reconstructions of various approaches embracing orders of propositions with which Russell experimented, cf. LANDINI, G., 1998*a*, chapter 9. The most pressing issue concerned how the order of $p\frac{x}{a}!q$ should be determined, in particular whether elementary propositions (i.e., those expressed by sentences which do not contain apparent variables) should be of the same order of individuals or not. The issue here is quite important because it impacts whether substitution can recover a theorem of infinity or not. Landini meticulously compares the many different approaches Russell explores in the manuscripts *On Substitution*, *The Paradox of the Liar* and *On Types*. Landini notes that it is not at all clear what route Russell intended to follow in the Substitutional Theory underlying *Mathematical Logic*, but one difference is clear: in *Mathematical Logic* Russell refrains from letting elementary propositions in the same type of individuals, thus blocking a proof of infinity (cf. RUSSELL, B., 1908, p. 620), although his reasons are not clear (cf. LANDINI, G., 1998a, p.239-40).

[266] RUSSELL, B., 1907*e*, p.515, folio 1.

Unfortunately, however, we do not know the specific date of *On Types*: we know that it was written after *The Paradox of the Liar*, for Russell quotes the latter in it[267]; we also know that although *Mathematical Logic* was *published* by the end July 1908, Russell had the manuscript sent to the *American Journal of Mathematics* fourteen months earlier, but they stalled the publication[268]. As we mentioned before, in early January of 1907 Russell wrote to Hawtrey mentioning the paradox which "pilled" the Substitutional Theory. But as Landini observes "[...] to pill is to despoil, but not necessarily to kill"[269] and at some point after that Russell was led to reconsider substitution with orders of propositions.

What prompted this?

The manuscript *On Types* provides a lot of information. It indicates that Russell's confidence in the Substitutional Theory had clearly not wavered - at lost not completely, for he even considers putting it as an appendix to an unspecified work - perhaps in *Mathematical Logic* but most likely *Principia* itself:

> If this form of substitution turns out feasible, perhaps it should be put into an Appendix. It is philosophically simpler than functions, but technically vastly more complicated.[270]

Furthermore, as Landini showed in detail by rigorously reconstructing and developing the Substitutional Theory of *On Types*[271], Russell had good reasons to be confident in the theory: differently from the theory of *Insolubilia* which became inconsistent in the presence of mitigating/Reducibility axioms, the Substitutional Theory of *On Types* can block p_o/a_o paradoxes and what is more it can recover a theorem of infinity![272] The manuscript also gives us important clues as to why Russell eventually gave up the theory. Russell observes that "[...] *philosophically* all we need to assume is the hierarchy of propositions, together with individuals" but also that he does *not* "[...] think the complication is worth while"[273]; according to Russell, this is because "[...] for technical reasons, it is simpler to use functions rather than substitution"[274]; but employing functions seems to undermine the very point of putting up with the complications of substitution:

267 MOORE, G., 2014, p.496.
268 MOORE, G., 2014, p.585.
269 LANDINI, G., 1998a, p.234.
270 RUSSELL, B., 1907*e*, p.516, folio 5.
271 Cf. LANDINI, G., 1998a, pp.240-251.
272 Cf. LANDINI, G., 1998a, pp.251-4. The same cannot be said about the Substitutional Theory of *Mathematical Logic* which requires some assumption about the number of individuals to prove that there are infinite classes (cf. RUSSELL, B., 1908, p.620). As to why that is the case, cf. the previous footnote 265 of the present chapter.
273 RUSSELL, B., 1907*e*, p.518, folio 7*a*.
274 RUSSELL, B., 1907*e*, p.515, folio 2.

> The point of view of substitution is perhaps unnecessarily complicated, seeing that ϕx is needed in any case, and so is $\phi(x, y)$. The only thing we save is $\phi!x$. The necessity for ϕx makes the philosophical gain less than it would be. If ϕx could be avoided, substitution would be worth adopting.[275]

So the state of affairs in *On Types* seems to be this: on the one hand it shows that Russell was still attracted to the some virtues of (typed) Substitution, namely (i) its avoidance of an *ontology* of functions, since it embraces a hierarchy of individuals and propositions (and nothing else) and (ii) its capacity to, in principle[276], recover an infinity theorem. The problem is that the approach remains excessively complicated and much less convenient than the plan of employing with a hierarchy of functions - at least in *practice*.

All of this fits very well with Landini's conjecture about the demise of *Insolubilia and* his interpretation of *Mathematical Logic.* It shows that the most plausible answer to the question of why Russell revived Substitution in *Mathematical Logic* was the reluctance to accept an ontology of propositional functions.

In another important and very difficult manuscript entitled *Fundamentals*[277] from 1907, Russell was considering the difficulties involved in abandoning proposition and adopting some form of contextual elimination of classes. There we find a possible explanation of the difficulties which lead him back to substitution and an ontology of propositions[278]:

> Note that the no-classes theory is in essence abandoned by distinguishing between ϕx and $x \epsilon \hat{z}(\phi z)$. For this requires that $\hat{z}(\phi z)$ should be a constituent of $x \epsilon \hat{z}(\phi z)$, and therefore that $\hat{z}(\phi z)$ should be something. This difficulty seems inherent in the no-classes theory, since functions must be allowed as apparent variables. A value of an apparent variable must be something, and thus the no-classes theory won't work. It worked while we had propositions, because then they became apparent variables where a variable matrix was wanted. But if propositions are not to be apparent variables, functions must be, and therefore functions must be admitted. But then they may as well be classes.[279]

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275 RUSSELL, B., 1907*e*, p.517, folio 5.

276 Again, cf. the previous footnote 265.

277 RUSSELL, B., 1907*h*. This paper should not be confused with the manuscript *On Fundamentals* from 1905 published the *fourth* volume of Russell's collected papers (URQUHART, A., 1994, p.359).

278 Initially, he considers what he calls axioms of classes and relations-in-extension in terms of what we would call abstraction or concretion principles, namely (RUSSELL, B., 1907*h*, p.545, folio 1):

$\vdash : \phi x \,.\equiv_x .\, x \,\epsilon\, \hat{z}(\phi z)$ [Axiom of Classes]

$\vdash : \phi(x, y) \,.\equiv_{x,y} .\, x\{\hat{x}\hat{y}\phi(x, y)\}y$ [Axiom of Relations]

But he ends up considering this plan "useless" (RUSSELL, B., 1907*h*, folio 12) precisely because of the difficulties involved in employing functions as apparent variables. In the same manuscripts Russell again reconsiders this approach, but instead of assuming the above abstraction principles as primitive, he assumes the Axiom of Reducibility and puts forward the contextual definition that ends up in *Mathematical Logic*, i.e $x \,\epsilon\, \hat{z}(\phi!z) \,.=.\, \phi!x$ and $f\{\hat{z}(\phi z)\} \,.=: (\exists\psi) : \phi z \,.\equiv_z .\, \psi!z : f\{\hat{z}(\psi!z)\}$ as Df's (RUSSELL, B., 1907*h*. p.563, folio 51).

279 RUSSELL, B., 1907*h*. p.542, folio 6.

Although Russell goes back and forth through different approaches to resolving this tension, even considering reviving classes as entities at some point[280], we find him at the end of the manuscript reconsidering the functional hierarchy, but explaining it in the exact way he does in *Mathematical Logic*, i.e., going back back to propositions:

> First draft must be revived to the extent of re-introducing individuals as an absolute type. Then *elementary propositions* are those which assign predicates to individuals, or assert relations among two or more individuals. Then the *first type of functions* consists of all definable by elementary functions, elementary functions being such as have elementary propositions for values. *Second type of functions* consists of all definable by means of individuals and first type, and having functions as arguments; and so on.[281]

Thus, the manuscripts from early 1907 indicate that the best explanation for the revival of he Substitutional Theory was this: Russell concluded that the elimination of classes required either functions or propositions as the residual of analysis; but Russell found an ontology of functions intolerable, which precluded him to accept functions as apparent variables; this led him back to propositions, only stratified to a hierarchy of *orders*. Assuming this is correct and that Landini's interpretation of *Insolubilia* is right, this would explain the passage on the Substitutional Theory present in *Mathematical Logic.* This is also further corroborated by a letter wrote to Russell on 16 June of 1907, i.e., exactly within the period when Russell sent the manuscript of *Mathematical Logic* to the *American Journal*: there we find Whitehead *agreeing* with Russell that "[...] the substitution theory is the proper explanatory starting point"[282].

Why, then, did they abandoned the Substitutional Theory?

Well, the initial motivation for adopting the theory was that it seemed to *preserve* the unrestricted variable. The many versions of p_o/a_o showed, however, that the simple Substitutional Theory put forward in *On the* Substitutional *Theory of Classes and Relations* required emendation. The result was the Substitutional Theory without general propositions put forward in *Insolubilia*: this theory could adhere "with drastic pedantry to the old maxim that "whatever is, is one"", and in accord with its original motivation. But the cost was too high: Classical Mathematics could not be developed within it without mitigating axioms assuring the existence of elementary propositions equivalent to any general one; but if Landini is right, Russell eventually realized that such axioms re-introduced some versions of p_o/a_o. The introductions of orders of propositions blocked these contradictions, but at the cost of the unrestricted variable. This reading is further corroborated by a letter Whitehead wrote to Russell on 7 October 1906:

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280 Cf. RUSSELL, B., 1906*f*, pp.331-2, folios 30-32.
281 RUSSELL, B., 1907*h*. p.567, folio 62.
282 MOORE, G., 2014, p.lxxxvii. As with most of Russell's letters to Whitehead, the one to which he latter was responding is lost.

> The nastiness which you wanted to avoid is the Frege bugbear of propositional functions becoming unmeaning when certain terms are substituted. According to the doctrine of types we have got to put up with this. - Thus certain things (such as functions) which can be named and talked about won't do as arguments in some propositional functions. The result is that we have to use the *restricted* variable. The doctrine of substitution was on stronger ground here; for it did without the function entities, and simply brought impasse a typographical device for pretending that we were talking of one entity when we were really talking of two. Hence if you want the unrestricted variable, the doctrine of substitution is the true solution. - But then this doctrine won't work, will it?[283]

If in its main points the interpretation endorsed here is correct, then the reason the theory would not work was that, as both Russell and Whitehead came to see the matter, there was no longer any reason to adopt the theory. Since type distinctions among propositions conceived as entities went against the very motivation for adopting the theory, namely, that it preserved the unrestricted variable, it remained without any philosophical motivation. Also from a technical and, especially, notational, point of view the theory was a nightmare as Whitehead time and again complained to Russell. All of this indicates that this was the theory's undoing: it ended up being technically inconvenient and lacking a philosophical motivation.

Some questions, however, remain unanswered.

3.3.4 The Paradoxes and the Demise of *Insolubilia*

Almost every turn in Russell's views on Substitution which we discussed so far seem accounted by Landini's interpretation on the basis of abundant textual evidence. The following claims, in particular, are definitely established:

- The p_o/a_o paradox was the main force driving Russell to modify the Substitutional Theory. It was, in particular, this paradox that led Russell to withdraw the paper *On the Substitutional Theory of Classes and Relations* from publication and to pursue theories that eliminated propositions or embraced orders.
- The Vicious-Circle Principle was not introduced by Russell with any sort of *justificatory* role.
- The purpose of the system behind *Insolubilia* was to reconcile the unrestricted variable with Cantorian Set Theory by employing a theory of incomplete symbols to propositional functions, classes *and* general propositions.

[283] MOORE, G., 2014, p.lxxi.

- By June/July of 1907 Russell had abandoned the version of Substitution of *Insolubilia* in favor of a version embracing orders of propositions - which appeared in *Mathematical Logic* cloaked under the convenient functional notation that was later used in *Principia.*

- Substitution was not only alive by January of 1907 when Russell wrote his letter to Hawtrey reporting the p_o/a_o paradox, but it survived (or was revived!) in many forms embracing orders of propositions at least halfway through 1907.

There are, however, some claims made by Landini concerning the development of the Substitutional Theory which cannot be endorsed on the basis of hard textual evidence alone. There are three interrelated claims, in particular, for which we seem to not have enough available textual evidence to settle, which concern (a) the abandonment of the Substitutional Theory of the first half of 1906, i.e., that of the paper *On The Substitutional Theory of Classes and Relations* and (b) the transition from the later version of Substitution without general propositions of *Insolubilia* to the versions of Substitution with orders of propositions that Russell put forward in *Mathematical Logic* and elsewhere[284]. The claims in question are the following:

- The p_0/a_0 is the "*sole* cause" of *On The Substitutional Theory* "[...] being withdrawn and the cause of all the subsequent experimental alterations to the Substitutional Theory found in the manuscripts"[285].

- Russell must have realized that the introduction of mitigating axioms in the system of *Insolubilia* revived the p_0/a_0, which in turn led him to abandon the approach of the paper and to accept orders of propositions.

- The semantic paradoxes played *no role* in leading Russell to abandon the paper *On The Substitutional Theory* or making any of the subsequent modifications.

The three claims are problematic (to varying degrees) with respect to textual evidence, but for somewhat different reasons. The conjecture about mitigating axioms is problematic because, as Landini himself observes, no manuscript where Russell discusses the inconsistency of the system of *Insolubilia* amended by mitigating axioms is extant[286]. In fact, Landini further speculates that such manuscript or group of manuscripts could be in the possession of Whitehead and were thus destroyed with most of his *Nachlass*[287]. Still, the problem is that we simply cannot pinpoint *exactly* when or why Russell became convinced that *Insolubilia* was a failure given the available evidence, despite the plausibility of Landini's theory. The claim that the p_0/a_0 is "*sole* cause" of

[284] For instance in the manuscripts *Paradox of the Liar* (RUSSELL, B., 1906*f*) and *On Types* (RUSSELL, B., 1907*e*) and *Fundamentals* (RUSSELL, B., 1907*h*).
[285] LANDINI, G., 2015, p.170. Our emphasis.
[286] Cf. LANDINI, G., 2015, p.175-6.
[287] Cf. LANDINI, G., 2015, p.173-4.

Russell's modifications of the Substitutional Theory is the least problematic, but problematic nonetheless given all the information we now have available. The timing of Russell's withdrawal of the paper and the dates of the early manuscripts dealing with the p_0/a_0, in particular, make it plausible. Russell read the paper *On the Substitutional Theory* on April of 1906 to the London Mathematical Society and *On Substitution* - where the p_0/a_0 appears for the first time - is dated April/May of 1906, which indicates that he discovered it right after composing and reading the paper. The problem lies with the dates of the remaining manuscripts. The manuscript *The Paradox of the Liar*, in particular, is dated September 1906. This, as we shall discuss in some detail below, casts doubts about his views on the withdrawal of *On the Substitutional Theory*, which happened on October and also seriously calls Landini's theory about the abandoning of *Insolubilia* into question. Then there is Landini's claim about the semantic paradoxes. The problem here (also given the points raised above) is the fact that Russell repeatedly mentions the Epimenides and related paradoxes as if they were genuine *logical* issues both in manuscripts and published writings. Why would he do that, if the real issue were paradoxes of a completely different nature?

In fact, as Jolen Galaughuer has observed in a relatively recent paper[288], Russell seems to repeatedly raise the *Epimenides* as a fundamental issue at several occasions - including *On Insolubilia* and unpublished manuscripts. This, Galaugher points out, calls for an answer to three questions: (a) If Landini is right in that *Insolubilia* was really concerned with the p_o/a_o, "[...] why did Russell advertise '*Insolubilia*' as a solution to the *Epimenides*?" and (b) if Russell "[...] was dissatisfied with the solution, as his correspondence suggests, why did he go on to publish it?"[289]; this, in turn, leads Galaugher to speculate - given that Russell could not come to accept a hierarchy of types, either of functions or propositions - "[...] why did substitution reappear with orders [in *Mathematical Logic*] if he had rejected a hierarchy of orders as intolerable?[290]

Galaugher points out that some historical documents and specific dates established in the fifth volume of Russell's *Collected Papers* seem in tension with Landini's claims. She notes that Russell wrote a letter to Jourdain on 4 of July 1906 manifesting his doubts about his solution to the Epimenides:

> Many thanks for your 'Topsy-Turvy Fairy Tales'. I was amused to see Epimenides the liar reappearing in your last fairy-tale. Have you any views as to how he is to be dealt with? I have some views, but I am not very well satisfied with them.[291]

Since we also know that Russell had already sent the manuscript of *Insolubilia* for Cou-

[288] GALAUGHER, J., 2013.
[289] GALAUGHER, J., 2013, p.5.
[290] GALAUGHER, J., 2013, p.5.
[291] GRATTAN-GUINNESS, I., 1977, p.91. In virtue of a clear typo Grattan-Guinness accidentally dates the letter 4 July 1905. The 'Topsy-Turvy Fairy Tales' was a little booklet published anonymously by Jourdain.

turat to translate halfway through June of 1906[292], this strongly indicates, as Galaugher points out[293], that the solution with which Russell was not satisfied is most likely that of *Insolubilia*. Furthermore the *Paradox of the Liar* - where Russell gives up the solution of *Insolubilia* and explores orders of propositions[294] - is dated September of 1906[295], which indicates that by the time the paper was published Russell had already abandoned his plan of solving the paradoxes by eliminating general propositions.

Galaugher also draws attention to and further develops a point made by Graham Stevens in his excellent review[296] of Landini's book on the Substitutional Theory. Stevens claimed in his review that there is a case to be made in favor of the view that there are structural similarities between the Epimenides and some variants of the p_o/a_o paradox. Stevens has in mind the following substitution which appears in Landini's formal reconstruction of the version of the p_o/a_o that appears in the letter to Hawtrey:

$$(\exists p, a)\left(a \,\text{in}\, p \,.\wedge.\, (z)\left(p\frac{z}{a}!\left[(\exists r, c)\left(z = \left[(y)\left(r\frac{y}{c} \supset y\right)\right] \wedge \sim \left(r\frac{z}{c}\right)\right)\right]\right)\right)$$

Stevens somewhat cryptically claims that "[...] the Epimenides is almost a mirror image, yielding a proposition asserting the falsity of all propositions contained in its correlated class and yet apparently also a member of that class"[297]. His point is that in case of the Epimenides, "[...] in place of the resultant of the above substitution, we will have"[298]:

$$(\exists r, c)\left(z = \left[(y)\left(r\frac{y}{c}\right) \supset\sim y\right] \wedge \left(r\frac{z}{c}\right) \wedge \sim \left(r\frac{z}{c}\right)\right)$$

Stevens claims that the "[...] similarity in structure is evident and, surely, unlikely to have escaped Russell's notice" and it shows that "[...] Russell was interested in the Epimenides and variants of the Liar because he thought they shared important features with the other paradoxes of propositions"[299]. Stevens's claim here seems twofold. One the one hand, he seems to be suggesting, against Landini, that Russell viewed the Epimenides as a paradox whose structure is similar to that of the p_o/a_o, i.e., that Russell must have at some point formulated the Epimenides as a diagonal paradox which violates Cantor's theorem. On the other hand, Stevens claims that this "[...] should make us wary of thinking that Russell clearly perceived the differences between the logical and semantic paradoxes and of the distinction between logical and semantic considerations generally"[300], since he is not suggesting that "that the Epimenides is not a semantic paradox", but only that such formulation of the Epimenides "[...] shows, as the substitutional paradox and *Appendix B* paradox do, that the assumption of propositions as logical objects turned out to introduce problems just as severe as (and remarkably similar to) those introduced by

[292] Cf. MOORE, G., 2014, p.274-5.
[293] Cf. GALAUGHER, J., 2013, p.8.
[294] Cf., for instance, RUSSELL, B., 1906*f*, p.359-365, in particular folio 99.
[295] Cf. MOORE, G., 2014, p.317.
[296] STEVENS, G., 2004.
[297] STEVENS, G., 2004, p.175.
[298] STEVENS, G., 2004, p.175.
[299] STEVENS, G., 2004, p.175-6.
[300] STEVENS, G., 2004, p.175-6.

the assumption of classes"[301] The most central claim here is the first: that Russell construed the Epimenides as a genuine logical paradox akin to the p_o/a_o. The question then, is: does this claim stand scrutiny? And if it does, does that mean that it was the Epimenides, after all, that Russell was attacking in *Insolubilia* and which led him to the modifications of Substitution?

Galaugher argues in favor of a positive answer to these questions, with some compelling points. She argues that "what the manuscripts reveal is that Russell entertained the notion that, when the Epimenides was stripped of such psychological features as 'assertion', or 'belief', what remained was a logical paradox with a form akin to that of the p_o/a_o paradox"[302]. Her argument draws mainly on the manuscript *On Substituition* from April/May of 1906[303]. There, we find Russell struggling with the liar paradox and vindicating Galaugher's claim: Russell explicitly claims that "the liar must be put in a purely logical form to see what it comes to"[304]. Russell puts forward the following as characterizing the liar[305]:

$$fp = \left[(\exists q, b)\left(p = (s)\left[q\frac{s}{b} \supset\ \sim s\right]\right) \wedge \sim \left(q\frac{p}{b}\right)\right]$$

If the above or something like it was in fact is what Russell intended by a "logical form of the liar", this seems to vindicate Stevens's and Galaugher's claim that Russell did formulate a substitutional paradox which he regarded as yielding "[...] the purely logical form of the Epimenides,"[306] as Galaugher puts it[307].

The next question, then, is: was it *this* paradox that concerned Russell in *Insolubilia* and that led him to modify his views on Substitution on more than one occasion? Again, Galaugher argues that it is. On her view, in *Insolubilia* Russell "[...] intended to solve the 'purely logical' Epimenides, the Appendix B version of the p_o/a_o paradox which shared its form, and the familiar Hawtrey version of the p_o/a_o paradox"[308] by dispensing with general propositions. And although Galaugher recognizes that in both *On the Substitutional Theory of Classes and Rela-*

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301 STEVENS, G., 2004, p.175.
302 GALAUGHER, J., 2013, p.16.
303 RUSSELL, B., 1906*d*.
304 Cf. RUSSELL, B., 1906*d*, p.159, folio 82.
305 This exact formulation is actually due to Galaugher (cf. GALAUGHER, J., 2013, p.17). The exact formulation which Russell employed in the when claiming that "the liar must be put in a purely logical form to see what it comes to" was the following (RUSSELL, B., 1906*d*, p.159, folio 82):

$$fp \,.\!=.\, (\exists q, b) : p = \sim \vee\text{‘}q/b \,.\! \sim q\frac{p}{b}$$

which unpacked means:

$$fp \,.\!=\!:.\, (\exists q, b) :. p \,.\!=\!: q\frac{s}{b} \,.\supset_s .\!\sim s \,.\!=\!:\sim q\frac{p}{b}$$

Russell then applies this function to $\sim \vee\text{‘}f\hat{p}$, thus obtaining (Cf. RUSSELL, B., 1906*d*, p.159, folio 82.):

$$f(\sim \vee\text{‘}f\hat{p}) \,.\!=\!:.\, (\exists q, b) :. (fp_o)\frac{p}{p_o} \,.\supset_p .\!\sim p : = : q\frac{s}{b} \,.\supset_s .\!\sim s :.\!\sim q\frac{\sim \vee\text{‘}f\hat{p}}{b}$$

As Galaugher notes, Russell was working with the above formulation assuming the distinction between propositional and individual (or entity) substitutions (cf.GALAUGHER, J., 2013, p.16.), but when the above is re-framed in terms of the original or simple Substititional Theory, it coincides with Galaugher's formulation.
306 GALAUGHER, J., 2013, p.17.
307 As we shall see below, however, it is not completely clear whether this is the case.
308 GALAUGHER, J., 2013, p.19.

tions and *Insolubilia* Russell "[...] gave semantic paradoxes of naming and defining a separate treatment"[309], on her view:

> Russell was concerned with the Epimenides paradox in "Insolubilia" precisely because it was an analogue of the Appendix B version of the p_o/a_o paradox and, on his own construction of it, "purely logical", so that its solution could be extended to the other propositional paradoxes which afflicted his theory.[310]

As noted above, this is indeed corroborated by some known dates, in particular the letter Russell wrote to Jourdain on 4 July 1906, where he manifested his dissatisfaction with his current solution to the Epimenides. This, as Galaugher notes, goes against Landini's account, according to which Russell was committed to the system of *Insolubilia* at least up to September of 1906 and perhaps even as far as January of 1907.

This, however, as Galaugher notes, leads to a somewhat pressing question: if Russell was dissatisfied with the solution of *Insolubilia* to the Epimenides by July, how did he go on to publish it in September? Galaugher's answer is that Russell's "[...] dissatisfaction was not technical but philosophical"[311]. Galaugher notes that in *On Substitution* Russell had raised philosophical concerns about the view put forward in *Insolublia* according to which to assert a general statement like "$(x).\phi x$" is not to assert a single proposition i.e., ϕx for all values of *x*[312]. As Galaugher notes, Russell's preoccupation is not strictly technical for he observes that "[...] the chief merit of this method is that it is technically feasible, and avoids contradictions"[313] but "[...] *as philosophy*, [he is] dissatisfied with the view that 'all men are mortal' is an ambiguous assertion of the mortality of this or that man"[314]; the problem, Russell continues, "[...] is that, in $(x).\phi x$, we don't primarily have the *values* of ϕx, but we have primarily $\phi \hat{x}$; and our proposition is really about $\phi \hat{x}$ rather than about any of its values"[315]. The fundamental issue here, again, was that of imposing limitations on the variable, which *Insolubilia* avoided by treating general statements as incomplete symbols. Galaugher points out, however, that Russell "[...] vacillated more than once during this period [i.e., from April to September of 1906] between a substitutional theory and a theory which placed type indices on predicate variables"[316], which, again, suggests that his dissatisfaction stated in the letter from 4 July referred to *Insolubilia*. And in fact, as Galaugher suggests, this does corroborate her claim that "[...] there is much in the manuscripts to suggest that Russell regarded '*Insolubilia*' as programmatic"[317], above all from a *philosophical* point

309 GALAUGHER, J., 2013, p.19.
310 GALAUGHER, J., 2013, p.19.
311 GALAUGHER, J., 2013, p.22.
312 Cf. RUSSELL, B., 1906*d*, p.229, folio 250.
313 Cf. RUSSELL, B., 1906*d*, p.229, folio 250.
314 Cf. RUSSELL, B., 1906*d*, p.229, folio 251.
315 Cf. RUSSELL, B., 1906*d*, p.229, folio 251.
316 GALAUGHER, J., 2013, p.24. Cf., for instance, RUSSELL, B., 1906*f*, p.364, folio 99 or the whole manuscript *On the Functional Theory of Propositions, Classes and Relations* (RUSSELL, B., 1906*i*)
317 GALAUGHER, J., 2013, p.24.

of view. Galaugher conjectures, then, that "Russell's inability to discover any philosophically satisfactory substitution- theoretic alternative to the solution set forth in '*Insolubilia*' led him to reject substitution in the late fall of 1906 or early winter of 1907"[318] and that *Mathematical Logic* "resurrected" substitution only to avoid quantification over propositional function variables"[319] - a point which is indeed corroborated by the manuscripts - in particular *On Types* - and also by Whitehead and Russell's correspondence, where they agree to adopt a language of functions but taking substitution as "[...] the proper explanatory starting point"[320].

So, what should we conclude from all of this? It seems that, in light of the currently available evidence, Galaugher has established the following points :

- Russell did consider the Epimenides or liar paradox in its "purely logical form" together with variants of the p_o/a_o around April-May of 1906 as genuine issues for the Substitutional Theory.
- So it is very likely that the Epimenides as considered in *On Substitution* played a role in the withdrawal of the paper read to the London Mathematical Society.
- It is also very likely that the Epimenides as considered in *On Substitution* was one of Russell's concerns in *Insolubilia.*
- The dissatisfaction which Russell mentions in the letter to Jourdain from 4 July refers to *Insolubilia* and it is most likely philosophically motivated.
- Even if Landini's conjecture about mitigating axioms is correct, Russell had *philosophical* misgivings about the solution of the 'pure form Epimenides' and the variants of the p_o/a_o which *Insolubilia* puts forward well before January of 1907, when he reports to Hawtrey that the p_o/a_o "pilled" the Substitutional Theory.

It seems also, however, that none of the above points present any fundamental difficulties for Landini's interpretation.

To be sure, the above shows that Landini's claim that the p_o/a_o was the "sole cause" of Russell's withdrawing the paper read to the London Mathematical Society *may* be partially mistaken, because it is plausible that philosophical concerns also played a role in the abandonment of substitution. The same, however, cannot be said of Landini's conjecture about mitigating axioms, which is a more substantial aspect of his theory: like the *Appendix B* paradox, the "purely logical form" of the Epimenides requires identity between a singular term and a general proposition: thus it is *not revived by Insolubilia amended by mitigating axioms*. Thus: *if* Landini is

318 GALAUGHER, J., 2013, p.26.
319 GALAUGHER, J., 2013, p.27.
320 MOORE, G., 2014, p.lxxxvii.

correct in conjecturing that Russell discovered that the system of *Insolubilia* was inconsistent experimenting with mitigating axioms, then Russell would have to have noticed that the Epimenides did not survive, and in that case, Landini's claim that p_o/a_o was the "sole cause" of all subsequent modifications of Substitution from *Insolubilia* up to *Mathematical Logic* still stands. Indeed, this very claim is strongly corroborated by substantial textual evidence prior to January of 1907. As noted before, when commenting on the introduction of orders of propositions in the *The Paradox of the Liar*, Russell wrote:

> This disposes of the fallacy which led to the abandonment of substitution before, i.e.
>
> $p_o \,.{=}{:}\, (\exists p, a) : a_o \,.{=}.\, p\frac{b}{a}!q :\sim p\frac{a_o}{a} :. \supset : p_o/a_o ; p_o\frac{b}{a_o}!q :\sim p_o/a_o ; p_o\frac{b}{a_o}!q$[321]

And here he is dealing with the version of p_o/a_o paradox of the letter to Hawtrey. This makes it quite clear that it was *this* version which was causing the real trouble with Substitution.

It must also be said that one cannot attach much weight to the fact that in *Insolubilia* Russell is concerned with the *Epimenides* and does not even mention the p_o/a_o paradox. One likely reason for this is that Russell was meeting his audience halfway: a proper formulation of the many variants of the p_o/a_o paradox in *Insolubilia* - in particular that which appears in the letter to Hawtrey - would require a heavy dose of formalism which Russell (understandably) may have wanted to spare his readers of, since he was writing for a general philosophical and mathematical audience. Thus, one possible explanation as to why Russell chose the *Epimenides* to explain his resolution of the propositional paradoxes in *Insolubilia* (and elsewhere in his published writings) is that he wanted a straightforward simply explicable problem that dispensed with the subtle technicalities of the Substitutional Theory. Indeed, this would also explain why Russell also does not formulate the *Epimenides* in its "purely logical form" in the paper.

Now, about the fact that Russell raised philosophical concerns over the approach of *Insolubilia*. Again, all that this does establish is that Landini's claim that the he p_0/a_0 is the "*sole* cause" of *On The Substitutional Theory* "[...] being withdrawn and the cause of all the subsequent experimental alterations to the Substitutional Theory found in the manuscripts"[322] *may* be wrong. But here we seem to have two possibilities. The first and most plausible one is that philosophical concerns about the approach of *Insolubilia* cropped up on top of the discovery that the p_0/a_0 survived the introduction of mitigating axioms. The second is that the abandonment of *Insolubilia* was motivated by *purely philosophical* reasons, which is highly unlikely. Russell was always quite transparent in claiming that *all* approaches to solving the contradictions faced - quite often severe - philosophical difficulties; these were the rule, not the exception. And, in fact, as we already discussed, textual evidence repeatedly appears in the manuscripts and documents suggesting the contrary conclusion: the difficulties which *ultimately* led to the demise of

[321] RUSSELL, B., 1906*f*, p.352, folio 74.
[322] LANDINI, G., 2015, p.170. Our emphasis.

Substitution were technical. For instance, in *The Paradox of The Liar* Russell claims that "[...] on the whole, it looks as if substitution could be employed usefully in the philosophy, but was technically rather tiresome"[323] and that "[...] the chief point of substitution was to avoid functions as apparent variables"[324] which he realized was not doable after all. So we still have very strong reasons to believe that the decisive reasons that led Russell to abandon the Substitutional Theory - and, as we shall see in the next section, also the *ontology of propositions* - were technical i.e., related to the contradictions.

Finally, the most crucial point of Landini's interpretation that also seems to be preserved is that strictly semantic paradoxes - like the Liar as ordinarily formulated - had no role to play in shaping Russell's views. To be sure, Russell repeatedly raises "the liar", the "*Epimenides*" or "the liar and related paradoxes" or "the *Epimenides* and kindred puzzles" when referring to his reasons for embracing distinctions of orders. Apart from an already considered possible (conjectural) explanation for that[325], we have a solid reasons to downplay this fact. The main point which has been established by Landini, following the initial hint of Nino Cocchiarella, is this: the p_o/a_o, which Russell constructed again and again in the manuscripts as a diagonal paradox which violates Cantor's Power-Class theorem and which does not rely on any kind of semantic or epistemic vocabulary was driving the modifications; this claim stands even if we accept Steven's and Galaugher's point that Russell also construed the *Epimenides* as a paradox with such a similar structure, in fact, they further corroborate it, showing that in all likelihood Russell mentioned the liar so frequently because he viewed it as a paradox that, when stripped out of its semantic and/or psychological coatings can be understood as a logical paradox. This is corroborated by Russell's claim in *The Paradox of the Liar* that "[...] so far as appears, the introduction of psychological considerations serves no purpose in solving the paradox of the liar" and that "[...] it remains to seek out some logical theory by which the vicious self-reference may be avoided"[326], which was, in fact, the main task of that manuscript.

3.3.5 The Paradoxes and the Demise of Propositions

As is well known, just before the publication of *Principia*, Russell adopts his famous Multiple Relation Theory of Judgment which marks, once and for all, the abandonment of his theory of propositions conceived as *singular* logical subjects. Before proceeding to discussing the solution which *Principia Mathematica* provides for the difficulties which baffled Russell for so long, we need to address why this change came about. We'll also do well to review what this change amounted to in terms of Russell's general philosophical views.

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[323] RUSSELL, B., 1906*f*, p.352, folio 74; similarly cf. also RUSSELL, B., 1906*f*, p.359, folio 86.

[324] RUSSELL, B., 1906*f*, p.367, folio 105.

[325] I.e., that Russell was meeting his audience halfway and avoiding the complications of the complex substitutions involved in the many variants of the p_o/a_o paradox.

[326] RUSSELL, B., 1906*f*, p.322, folio 6

We already observed that after the adoption of Moore's pluralism, Russell thought that "[...] a proposition, unless it happens to be linguistic, does not itself contain words", but "contains the *entities* indicated by words"[327], that is, he became committed to the thesis that propositions are complex entities that have other entities as constituents. This view came along with a picture of *understanding* or apprehension of propositions that was tied to Russell's well known notion of *acquaintance*:

> In every proposition that we can apprehend (i.e., not only in those whose truth or falsehood we can judge of, but in all that we can think about), all the constituents are really entities with which we have immediate acquaintance.[328]

As Russell later explained in *Knowledge by Acquaintance and Knowledge by Description* what he meant by this is the following:

> I say that I am acquainted with an object when I have a direct cognitive relation to that object, i.e. when I am directly aware of the object itself. When I speak of a cognitive relation here, I do not mean the sort of relation which constitutes judgment, but the sort which constitutes presentation. [...] That is, to say that *S* has acquaintance with *O* is essentially the same thing as to say that *O* is presented to *S*.[329]

This sort of "direct cognitive relation" can evidently only be maintained with genuine constituents of propositions, that is, those entities that are denoted by genuine proper names as opposed to those that are denoted by incomplete symbols (like definite descriptions). Thus, up until the adoption of the multiple-relation analysis of judgment Russell was committed to a view of judgment that can be summarized in the following terms: (i) judgment is a dyadic relation between a subject *S* and a proposition *p*; (ii) in order to judge a proposition *p* as true or false one must be acquainted with the proposition *p*; (iii) to be acquainted with a proposition *p* is to be acquainted with the constituents of *p*. Russell also accepted as unproblematic the view that "[...] some propositions are true and some false, just as some roses are red and some white"[330], that is, that both sorts of propositions have *bein*g, i.e., are genuine entities. As Russell asserts in *On The Nature of Truth*, if judgment is understood as a dyadic relation between a subject and a proposition, the difference between true and false propositions should then be taken as undefinable, with true propositions being identified with *facts*, that is, propositions which have this undefinable *property* of being true[331]. Thus, Russell was also committed to: (iv) truth and

327 RUSSELL, B., 1937 [1903], p.47 §51. our emphasis.
328 RUSSELL, B., 1905*a*, p.427;.
329 RUSSELL, B., 1911*a*, p.148; RUSSELL, B., 1917, p.209-210.
330 RUSSELL, B., 1904*a*, p.473.
331 RUSSELL, B., 1907, p.451. This article was partially reprinted in *Philosophical Essays* but the embrionary discussions of the multiple-relation theory of judgement were removed by Russell due to the fact he was not satisfied with them anymore by of the publication of the collection.

falsehood are qualities which belong to non-mental, non-linguistic complexes (propositions) and (v) that there are two sorts of propositions: *facts*, which are *true* propositions, and *fictions*, which are *false* propositions, both being sorts of genuine entities.

Russell started to change his mind regarding this cluster of views on propositions and judgment[332] around the time he discovered the p_o/a_o paradox, the same period when he started to consider abandoning propositions and adopting the multiple-relation analysis. The main appearances in print of the multiple-relation analysis before *Principia* occurred in the following texts:

1. *On the Nature of Truth*, which Russell finished by the end of 1906 and published in the *Proceedings of the Aristotelian Society* in early 1907.
2. *On the Nature of Truth and Falsehood*, which Russell finished early in 1909 and published only in his *Philosophical Essays* in 1910.

In a nutshell[333], the view that Russell considers in these papers is the idea that there is no single logical subject that corresponds to a judgment, i.e., that there is not a single complex entity which is the object of judgment, i.e., a proposition, but that judgment is a relation between a subject *S* and the many constituents $C_1, ..., C_n$ about which the subject *S* is making the judgment[334].

In *On the Nature of Truth.* Russell considers this view only tentatively that "[...] a belief cannot be validly treated as a single thing"[335], i.e., that a judgment *not* as "[...] one idea with a complex object" but as consisting "[...] of several related ideas"[336]. In that paper he was inclined to adopt this view, but was very cautious, claiming that "[...] the difficulties in its way are formidable, and may turn out to be insuperable"[337]. There were two difficulties Russell thought to be involved. The first was a vague skepticism regarding the possibility of giving an adequate explanation of the notion of correspondence involved between a belief and a fact[338], while the second was involved in an argument in favor of the opposite view, i.e., the view according to which there are objective falsehoods[339]. Russell considered the idea that a judgment points or refers to a complex *only if* the complex is a fact, which amounts to treating false sentences as

332 "Judgement" and "belief" are interchangeable in this context, since what matters is that a proposition is the *object* of the relation. See RUSSELL, B., 1910*c*, p.117, footonote 1.

333 Again, the details of the multiple-relation theory and its role in the Introduction of *Principia* will be discussed in the next chapter.

334 I.e., for instance, if *S* judges that a is red, the constituents of such judgment are *a* and *redness*, while the complex fact that arises from *S* making these judgment consists in *S*, *a* and *redness*; the point is that there is no longer a single object of judgment here when *S* makes such judgment, but several, each of which *S* must be acquainted with in order to make the judgment.

335 RUSSELL, B., 1907*a*, p.451.

336 RUSSELL, B., 1907*a*, p.452.

337 RUSSELL, B., 1907*a*, p.453.

338 Cf. RUSSELL, B., 1907, p.452-3.

339 Cf. RUSSELL, B., 1907*a*, p.453.

incomplete symbols. This, as Russell explained was "an extension of the principle applied in [...] "*On Denoting*" [...] where it is pointed out that such propositions as "the King of France is bald" contain no constituent corresponding to the phrase "the King of France"."[340]. And, in fact, Russell refers to *Les Paradoxes/Insolubilia* in a footnote of the article, specifically to the dissolution of the liar paradox by not treating "a belief as a single thing".

The reference to *Insolubilia* suggests that Russell envisaged a treatment of false propositions analogous to that which he sketched for general propositions. However, there is not even a sketch of such analysis of false propositions as being expressed by incomplete symbols in *On The Nature of Truth*, probably due to the fact that Russell had no idea yet how to analyze the notion of correspondence with a fact that characterized true propositions and was absent in false ones. So both the problems of correspondence and of objective falsehoods remained unsolved in the article.

In *On The Nature of Truth And Falsehood*[341], Russell provides a solution for the first difficulty and a dissolution of the second. Regarding the problem of objective falsehoods, Russell ended up abandoning the view that *any sort of judgment* consists in maintaining a relation to a single object, thus abandoning the view that in the case of *true* judgments, they denoted complexes which could be identified with propositions:

> We cannot maintain this view with regard to true judgments while rejecting it with regard to false ones, for that would make an intrinsic difference between true and false judgments, and enable us (what is obviously impossible) to discover the truth or falsehood of a judgment merely by examining the intrinsic nature of the judgment. Thus we must turn to the theory that no judgment consists in a relation to a single object. The way out of the difficulty consists in maintaining that, whether we judge truly or whether we judge falsely, there is no one thing that we are judging. When we judge that Charles I died on the scaffold, we have before us, not one object, but several objects, namely, Charles I and dying and the scaffold. Similarly, when we judge that Charles I died in his bed, we have before us the objects Charles I, dying, and his bed. These objects are not fictions: they are just as good as the objects of the true judgment. We therefore escape the necessity of admitting objective falsehoods, or of admitting that in judging falsely we have nothing before the mind. Thus in this view judgment is a relation of the mind to several other terms: when these other terms have *inter se* a 'corresponding' relation, the judgment is true; when not, it is false.[342]

Thus, again Russell ended up following the most radical route suggested by the theory of incomplete symbols: *every sentence* ended up being viewed as such and this led Russell to a solution of the problem of correspondence in the following terms:

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340 RUSSELL, B., 1907*a*, p.451.
341 RUSSELL, B., 1910*a*.
342 RUSSELL, B., 1910*b*, p.120;

> [...] the judgment that two terms have a certain relation *R* is a relation of the mind to the two terms and the relation *R* with the appropriate sense: the 'corresponding' complex consists of the two terms related by the relation *R* with the same sense. The judgment is true when there is such a complex, and false when there is not. The same account, *mutatis mutandis*, will apply to any other judgment. This gives the definition of truth and falsehood.[343]

Russell introduces the sense of a relation as distinguishing what sort of complex corresponds to distinct judgments like 'aRb' and 'bRa' and correspondence to an existing complex is introduced as a definite criteria for the truth of judgment: a judgment is true when there is a complex which corresponds to what it asserts that exists: for instance, a complex consisting of *a bearing the relation R to b* in the case of 'aRb'; the absence of such a complex characterizes falsehood. Here there are no longer objective truths or falsehoods understood as mind-independent complex entities which are the objects of judgments; by a stretch of terminology what can be called an 'objective truth' is a *fact* but strictly speaking this is now simply wrong: what is true or false is a judgment depending on whether it *corresponds* to an existing complex (i.e., a fact). Declarative sentences, on their turn, are true or false in the derivative sense that they may express true or false judgments and propositions as they were conceived in *The Principles of Mathematics* and in the era of the Substitutional Theory become extinct[344].

Now, in the previous sections we discussed what may have ultimately led Russell to abandon the Substitutional Theory of Classes and relations and the conclusion we reached is that Landini's central claim is right: the major factor was the p_o/a_o paradox, in particular the variants which did not involve general statements flanked by identity (like that of the letter to Hawtrey) - this paradox is the main (even if not the sole) thing leading Russell to shape and re-shape the Substitutional Theory in the manuscripts and that ultimately it was Russell's failure to reconcile (a consistent version of) the Substitutional Theory with the doctrine of the unrestricted variable that led to the theory's demise. A question which we did not address, however, is whether these difficulties also played a major role in leading Russell to abandon the ontology of propositions and to adopt of the multiple-relation analysis of judgment. Landini argues in favor of a positive answer, claiming that "[...] the reason Russell abandoned propositions in 1910 was because of the paradoxes of propositions and his desire to avoid introducing a hierarchy of orders and a calculus for logic with restricted variables"[345]. Ian Proops has claimed in a relatively recent paper that "[...] that this idea lacks compelling textual support"[346].

Although Proops acknowledges that "[...] it is true that certain paradoxes - and especially the Liar - were clearly operative in prompting Russell's initial experiments with the MRTJ", he adds that "[...] something further is required to explain his shift from his earlier (1906) indecisive

343 RUSSELL, B., 1910*b*, p.123-4.
344 In fact, as Landini very aptly puts it, "[...] facts and Russellian propositions are like men and dinosaurs: they never coexisted" (LANDINI, G., 2011, p.253).
345 LANDINI, G., 1996, p.323. Cf. also LANDINI, G., 1998a, p.275; 2010, p.160-1.
346 PROOPS, I., 2011, p.190.

openness toward this theory to his later (1909) firm endorsement of it"[347]. Proops recognizes that propositional paradoxes led Russell to accept eliminativism about *general* propositions in *Insolubilia* and also to accept a ramified version of the Substitutional Theory in *Mathematical Logic*. Also, although Proops accepts that the typed version of Substitution which appears in *Mathematical Logic* may have been abandoned because it amounts to abandoning the doctrine of the unrestricted variable which "[...] was arguably one of the original rationales for the substitutional theory"[348], he claims that:

> What is clear is that existing accounts of this matter become conjectural at this point. The textual record simply fails to supply concrete indications of Russell's reasons for becoming dissatisfied with the ramified version of the substitutional theory.[349]

The availability of Russell's unpublished manuscripts provides overwhelming evidence against Proops's claim and confirms that in all likelihood Russell abandoned the Substitutional Theory with orders of propositions because he could not accept orders and let go of the doctrine of the unrestricted variable, which in turn led to the abandonment of the ontology of propositions. In fact, in our previous discussion of the development of Russell's Substitutional theory we considered *solid* textual evidence in favor of this.

So before closing this chapter we may review our previous discussion and also take note of some further points that appear in Russell's manuscripts which we did not address and which also strongly indicate that the ontology of propositions was also abandoned because of propositional paradoxes. Let us begin by reviewing what we may consider as established conclusions so far, leaving what may correctly be called 'conjectural' claims aside. We may summarize them as follows:

- First, there is no place for doubt that the "analogues of the liar" to which Russell refers concern the paradox that Landini dubbed the 'p_o/a_o' and its many variants and most likely, as Stevens suggested and Galaugher confirmed, also the paradox which Russell referred to as 'purely logical liar'[350].

347 PROOPS, I., 2011, p.190. Furthermore, Proops also claims that Russell's "[...] move from the apparent agnosticism about propositions of the introduction to *Principia* to the eliminativism of 'On the Nature of Truth and Falsehood" (1910)' demands explanation" (PROOPS, I., 2011, p.190-1). In the next chapter we shall address the status of propositions in *Principia* and we argue, following Landini, that *Principia* fully embraces eliminativism about propositions.

348 PROOPS, I., 2011, p.195.

349 PROOPS, I., 2011, pp.195. However, Proops mistakenly takes the paradox responsible for the withdrawal of *On the Substitutional Theory of Classes and Relations* to be the substitutional version of the *Appendix B*. Also, Proops notes that the referees of the paper made several critical comments and he assumes that "in the light of those criticisms, Russell immediately withdrew that article"(PROOPS, I., 2011, p.192) The centrality of the p_o/a_o and its constant re-appearances in the manuscripts from April and May alone show that these views are wrong.

350 Depending, of course, on whether one accepts Stevens's and Galaugher's arguments

- It was Russell's dissatisfaction with the solutions which he sketched to these paradoxes between April and September of 1906[351] that led him to incorporate orders of propositions within the Substitutional Theory.

- We also *know* that Russell became dissatisfied with orders of propositions for at least two important reasons. First and foremost we know that Russell became dissatisfied with orders because it went against the core *philosophical* motivation for Substitution, namely, that it preserved the unrestricted variable. Furthermore as Russell repeatedly observed, the technical core motivation for the Substitutional Theory was that it avoided functional variables. The price of this, however, proved too high and as *Mathematical Logic* and the correspondence with Whitehead confirmed[352], Russell became convinced that *technically* Substitution was not viable also.

The above consists in quite non-speculative, or as Proops puts it, 'concrete' indications of what made Russell dissatisfied with the Substitutional Theory, indications that can be extracted from extensive and solid textual evidence. In fact, what our discussion indicated above is that the only matters which remain subject of speculation concern details, namely: *the exact month and the exact reason when Russell abandoned the system of Insolubilia in favor of orders of propositions* - in particular whether Russell abandoned it because he discovered, as Landini claims, that mitigating axioms re-introduced paradoxes, or whether philosophical reasons played a role as well, as Galaugher claims. Our conclusion above amounted to an agnostic pluralism: that both conjectures are compatible and that both are most likely (at least partially) right, i.e., that despite the fact that no extant manuscripts confirm this, Russell must have at some point realized that *Insolubilia* supplied by mitigating axioms revived the p_o/a_o and that philosophical concerns that may have been building up around *Insolubilia* also cropped on top of this. Speculation concerns these points *only*, however. The claim that Russell gave up the so-called 'ramified' version of Substitution because he was *technically* and *philosophically* dissatisfied with it - for several different reasons - is established by the currently available textual evidence

Then there is issue of the ontology of propositions. Here, again, we have explicit textual evidence connecting the abandonment of the ontology of propositions with concerns with the paradoxes. Russell himself says in *On The Nature of Truth* that there are difficulties with the view that propositions are entities and that "[...] *the chief* of these difficulties is derived from paradoxes analogous to that of the liar"[353]. Russell also explicitly claims in other sources from the same period that propositional paradoxes led him to reject the ontology of propositions. In his correspondence with Jourdain he notes in a letter from 1 June 1907 that "[...] consideration of the paradox of the liar and its analogs has led [him] to be chary of treating propositions as entities"[354];

[351] I.e., between the withdrawal of *On the Substitutional Theory of Classes and Relations* from publication and the publication of *Insolubilia*.
[352] Again, cf. the letter Whitehead wrote to Russell on 7 October 1906 (MOORE, G., 2014, p.lxxi).
[353] RUSSELL, B., 1907*a*, p.451. Our emphasis.
[354] GRATTAN-GUINNESS, I., 1977, p.104.

similarly, doubts about propositions as entities appear almost as a rule scattered throughout the manuscripts of 1906 through 1908 tied to a discussion of paradoxes. In the manuscript *Logic in Which Propositions are not Entities* from September of 1906, for instance, we find the following striking passage under the heading of "*General considerations*":

> Roughly speaking, the view that propositions are not entities amounts to this, that the predicates that can be significantly asserted of propositions are different from those that can be asserted of entities. "The Law of Contradiction is fond of cream cheese" is to be as inadmissible as "The number 1 is fond of cream cheese." I can't help thinking this would solve some problems as to the nature of truth; also the *Epimenides* and kindred puzzles. All significant propositions about propositions, on this view, will really be propositions about entities; just as propositions about classes are. A proposition about a proposition, if it can't be reduced to the form of a proposition about entities, is to be meaningless.[355]

The very motivation of the manuscript is to abolish propositions in order to deal with the contradictions. In the manuscripts we confirm again what Russell says in *On the Nature of Truth*, that it is precisely in the context of dealing with the contradictions that he starts to consider abandoning an ontology of false propositions. In the *On Substitution* manuscript, for instance, he considers the view according to which "there is no such thing as $\sim p$, but only the denial of *p*" and "we may assert or deny *p*, but there is no such thing as not-*p*"[356]. Then there is also the manuscript *The Paradox of the Liar*, where Russell repeatedly raises the elimination of propositions in various different approaches as a means of dealing with the liar and 'kindred puzzles'[357].

All of this shows, against Proops's claim about lack of evidence, that the multiple-relation theory of judgment conceived as an eliminative approach to propositions emerged from the ashes of the Substitutional Theory after it was "pilled" by the p_o/a_o paradox and revived by the distinctions of orders, only to be abandoned again, this time for good.

What we shall discuss next, following Landini's arguments, is that this eliminative approach was also adopted in *Principia Mathematica*, which was conceived by Russell to embrace a No-Propositions, No-Classes and No-Propositonal Functions ontology for the calculus of Logic.

3.4 Chapter Appendix: timeline of the rise and fall of Russell's Substitutional Theory

What follows is a chronology of the relevant events which elapsed during the rise and fall of Russell's Substitutional Theory as cataloged by Douglas Lackey, Ivor Grattan-Guinness

355 RUSSELL, B., 1906*e*, p.265, folio 13.
356 RUSSELL, B., 1906*d*, p.150-1, folio 62.
357 Cf., for instance, RUSSELL, B., 1906*f*, p.336, folio 40; p.337, folio 43.

and Gregory Moore.

-1905

October *On Denoting* is published on *Mind*.[358]

24-10-05 Russell submits *On Some Difficulties in the Theory of Transfinite Numbers and Order Types* [Paper 1, V.5][359] to the London Mathematical Society.[360]

05-12-05 Whitehead writes to Russell and refers to paradoxes within the Substitutional Theory as "oddities"[361]

14-12-05 *On Some Difficulties* is read to the London Mathematical Society.[362]

19-12-05 Russell writes to Couturat about the Substitutional Theory, reporting his plan to send "a purely mathematical memoir to the American Journal of Mathematics" and another which "will avoid, so far as possible, technical difficulties, and will bring out as much as possible the philosophical scope of the new theory" which he considers sending to the *Revue de méthaphysique et de morale* as *Les paradoxes de La Logique*[363].

-1906

January Jourdain writes to Russell requesting more published work on his new approach presented in *On Some Difficulties*.

13-01-06 Russell writes to Jourdain presenting his recent breakthrough: the theory of definite descriptions. Russell explains that he is hesitant of publishing his views because of "errors which [he] keep[s] on discovering"[364]

05-02-06 Russell adds a note to *On Some Difficulties* in the press, stating that the no-class theory is the definite solution to the paradoxes.[365]

21-02-06 Whitehead writes to Russell criticizing the "excessive formalism" of the Substitutional Theory[366].

22-02-06 Whitehead writes another letter complaining about the Substitutional Theory's handling of class-expressions[367].

358 URQUHART, A., 1994, p.li; URQUHART, A., 1994, p.414.
359 RUSSELL, B., 1905b.
360 LACKEY, D., 1973, pp.135; MOORE, G., 2014, p.62.
361 MOORE, G., 2014, p.lxxiii; MOORE, G., 2014, p.105.
362 LACKEY, D., 1973, pp.135; MOORE, G., 2014, p.62.
363 MOORE, G., 2014, pp.xiviii-xlix.
364 GRATTAN-GUINNESS, I., 1977, p.70; GRATTAN-GUINNESS, I., 1977, p.74.
365 MOORE, G., 2014, p.62.
366 LACKEY, D., 1973, p.131.
367 LACKEY, D., 1973, p.131; MOORE, G., 2014, p.xlix.

March Couturat publishes *Pour La Logistique*[368], his reply to Poincaré's first paper[369] on Logic and Logicism, in the *Revue de méthaphysique et de morale*.

08-03-06 Russell writes to Couturat praising his response to Poincaré and reporting his own intention of writing a response in the *Revue.*[370]

24-04-06 Russell submits *On the Substitutional Theory of Classes and Relations* [Paper 6b, V.5] to the London Mathematical Society.[371]

27-04-06 Augustus Edward Love, secretary of London Mathematical Society at the time, writes to the mathematician Alfred Bray Kempe requesting him to be a referee of Russell's paper.[372]

April -May Russell finishes a draft of *Logic in Which Propositions are not Entities* [Paper 7, V5] and sends it to Whitehead.

May Poincaré's second paper[373] on Logic and Logicism is published in the *Revue.*

05-05-06 Whitehead writes back to Russell about the manuscript, pointing out difficulties with false propositions. Couturat writes to Russell about Poincaré's second paper on Logic and Logicism[374].

07-05-06 Kempe writes back to Love with an evaluation of Russell's paper.[375]

April-May Russell works on two different manuscripts the final form of which cannot be dated especifically, namely: *On Substitution* [Paper, 5a, V5] and *On the Functional Theory of Propositions, Classes and Relations* [Paper 8, V5].

09-05-06 Whitehead writes back to Russell mentioning a new "function-type" theory and suggesting emendations for publishing the paper *On the Substitutional Theory.*

10-05-06 The paper *On the Substitutional Theory* is read to the London Mathematical Society.[376]

15-05-06 Russell replies to Couturat about the rebuttal to *Poincaré*, explaining that before responding he wants to clarify his answer to the contradictions; he observes in particular the need of extending the theory of incomplete symbols to propositions[377].

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368 COUTURAT, L., 1906.
369 POINCARÉ, H., 1906*a*.
370 MOORE, G., 2014, p.273.
371 LACKEY, D., 1973, p.165;
372 MOORE, G., 2014, p.236-7.
373 POINCARÉ, H., 1906*b*.
374 MOORE, G., 2014, p.262-3.
375 Cf. MOORE, G., 2014, p.236.
376 LACKEY, D., 1973, p.165; MOORE, G., 2014, p.236.
377 MOORE, G., 2014, p.Iii; MOORE, G., 2014, p.274-5.

15-06-06 Russell writes to Couturat promising to send his reply to Poincaré the next day so that he can start working in the translation to French. The reply in question is the paper *The Paradoxes of Logic* [Paper 9, V5], also known as *On 'Insolubilia' and Their Solution by Symbolic Logic.*[378]

23-06-06 Russell writes to Couturat reporting his intention to make minor corrections in the proof pages of *Insolubilia.*[379]

25-06-06 Couturat sends Russell the translation of *Insolubilia* before sending it to the press, so that he can make corrections. Couturat asks that Russell put more emphasis on the actual infinite and Mathematical Induction and to change the title.[380]

02-07-06 Russell replies to Couturat, refusing to add the emphasis on the points requested by the latter, but agrees to change the title; Russell does not require any modification for the proof pages except for the insertion of a minor observation in a footnote.[381]

04-07-06 Russell writes to Jourdain stating that he is dissatisfied with his current solution to the Epimenides[382].

10-09-06 Russell writes to Jourdain stating he is inclined "at present to the doctrine of types, much as it appears in Appendix B of my book"[383].

14-09-06 *Insolubilia* is published in the *Revue.*[384]

07-10-06 Whitehead writes to Russell stating the Substitutional Theory won't work[385].

10-10-06 Jourdain writes a note to Russell praising the No-class Theory.[386]

12-10-06 Love writes to Russell reporting the acceptance of the paper *On the Substitutional Theory* for publication in the *Proceedings of* London Mathematical Society; he also sent Russell Kemp's observations[387].

13-10-06 Russell replies to Love and states his intention of withdrawing the paper from publication.[388]

October Russell withdraws the paper *On the Substitutional Theory* from publication.[389]

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378 MOORE, G., 2014, p.274-5.
379 MOORE, G., 2014, p.275-6
380 MOORE, G., 2014, p.276.
381 MOORE, G., 2014, p.275.
382 Cf. GRATTAN-GUINNESS, I., 1977, p.91.
383 GRATTAN-GUINNESS-I., 1977, p.91-2; MOORE, G., 2014, p.lxxi-lxxii.
384 LACKEY, D., 1973, p.190; MOORE, G., 2014, p.273.
385 MOORE, G., 2014, p.lxxi.
386 GRATTAN-GUINNESS, 1977, p.93.
387 MOORE, G., 2014, p.238.
388 MOORE, G., 2014, p.238-9.
389 MOORE, G., 2014, p.239.

14-10-06 Love writes to Russell lamenting the withdrawal of the paper *On the Substitutional Theory.*[390]

22-10-06 Russell replies to Jourdain's note from October 10 reporting that the paper *On the Substitutional Theory* was withdrawn and explains that the paper and the theory require corrections at several points, in particular the need of "purging it of metaphysical elements".[391]

-1907

22-01-07 Russell sends the letter to Hawtrey with the p_0/a_0 that "pilled" the Substitutional Theory[392].

June-July Russell sends the manuscript of *Mathematical Logic as Based in the Theory of Types* [Paper 22, V5] to the *American Journal of Mathematics.*[393]

16-07-07 Whitehead writes a very long letter to Russell reviewing the general outline of his approach (by that point) and presenting presenting specific doubts.[394]

1908

06-01-08 Whitehead writes another letter about difficulties with the theory of types.

02-07-08 Russell writes to Jourdain reporting that *Mathematical Logic* is to appear in the *American Journal of Mathematics* and that the journal stalled the publication of his manuscript for a year.

30-07-08 *Mathematical Logic* is published in the *American Journal of Mathematics*.

03-09-08 Hawtrey writes to Russell thanking him for sending the paper and stating his reservations about its content, in particular the Axiom of Reduciblity.

05-07-08 Russell replies to Hawtrey and reports that the *American Journal* kept his manuscript for fourteen months. He agrees with Hawtrey about the problematic status of the Axiom of Reducibility.[395]

1909

26-05-09 Whitehead writes to Russell about the theory of types, inquiring about difficulties[396].

[390] MOORE, G., 2014, p.239.
[391] GRATTAN-GUINNESS, 1977, p.93.
[392] LANDINI, G., 1998a, p.234-5; MOORE, G., 2014, p.105.
[393] MOORE, G., 2014, p.585.
[394] MOORE, G., 2014, p.lxxvi-lxxvii. As Moore observes, Russell's answer is lost, like most of Whitehead's *Nachlass*
[395] MOORE, G., 2014, p.585-6.
[396] MOORE, G., 2014, p. 587.

May-September Russell and Hawtrey begin an exchange of letters discussing the technical details of what will become the last draft of *Principia Mathematica.*[397]

19-10-09 Whitehead and Russell deliver the final manuscript of *Principia Mathematica* to the Cambridge University Press.[398]

[397] MOORE, G., 2014, p.xc-xci..
[398] MOORE, G., 2014, p.xc-xci..

4 Propositional Functions and the Logic of *Principia Mathematica*

My Philosophical Development, since the early years of the present century, may be broadly described as a gradual retreat from Pythagoras.[1]

4.1 Introduction

Some features of *Principia*'s presentation of its formal system - like its axiomatic formulation of propositional and quantificational logic in ∗1 and ∗10 - will probably look and feel fairly familiar to anyone who has studied the subject from modern textbooks; but others - like the presentation of type theory of the Introduction and of quantification theory in ∗9 - will certainly seem strange or archaic. But without question, what is most baffling to the modern reader of *Principia* is the lack of any detailed explanation of the grammatical rules of the formal system, i.e., of its syntax. And, indeed, Gödel once aptly observed that in *Principia* "what is missing, above all, is a precise statement of the syntax of the formalism"[2].

This lack of a precise statement of syntactical rules of formation of expressions is, of course, most problematic with respect to distinctions of types. As we just discussed, the absence of an explicit explanation of the formalism was also responsible for interpretative issues in the paper *Mathematical Logic as Based on the Theory of Types*. As we saw, that paper embraces *in practice* a notation of functional variables with implicit order indices. On the surface the resolution of the contradictions in *Principia* is the same given in *Mathematical Logic as Based on the Theory of Types*. As in the case of Russell's paper, the introduction of type distinctions in

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1 RUSSELL, B., 1959, p.208.

2 GÖDEL, K., 1944, p.120. Gödel's main complaint concerns the absence of an explicit rule of subsitution for terms within arbitrary formulas, in particular when substituted symbols have complex *definiens*, as in the case of incomplete symbols. As Gödel points out, there are cases in which the order of elimination or substitution of defined expressions is *not* irrelevant - Gödel's famous example is "$\phi!\hat{u} = \hat{u}\{\phi!u\}$. Indeed, the lack of care with syntatic matters is by far *Principia*'s most serious defect, in particular since it is this shortcoming that generates many of its complicated interpretative issues including much of the fuss involving the notion of a 'propositional function'. As we discussed in the very introduction of the present work, the main issue with the use of this notion in *Principia* is that it seems to be ambiguous: sometimes the expression "propositional function" is used as an alternate label for a scheme, a dummy letter which stands for some arbitrary open sentence, while at other times the phrase must mean an actual object-language variable which can be bound by quantifiers. As we also had occasion to discuss, Quine argued again and again that such dual role played by the idea is ultimately grounded on a fundamental confusion between sign and symbol on the part of Russell. This issue will be adressed when we discuss different interpretations of *Principia*'s syntax (section 4.4 below). The early attempts to fix the issues are in HILBERT & ACKERMANN, 1928; CARNAP, R., 1934; QUINE, W., 1934. Church points out that none of the early attempts to formulate an adequate rule of substitution were completely correct (CHURCH, A., 1956, p.289), and that the first fully adequate statement of the rule was given in HILBERT & BERNAYS, 1934. For further details cf. CHURCH, A., 1956, p.289-90, in particular footnotes 456, 457 and 461.

Principia is presented as a means to embody the Vicious Circle Principle[3]. Also, like *Mathematical Logic*, *Principia* also embraces the contextual elimination of expressions for classes and relations-in-extensions, and, in fact, the definitions which were already introduced in the paper are kept intact in *Principia*'s sections ∗20 and ∗21. Thus, like the Substitutional Theory, the type theory of *Principia* is not committed to classes; also, similarly to the Type Theory of *Appendix B* of the *Principles*, Russell's contradiction is solved by imposing restrictions upon the ranges of significance of propositional functions[4] and contradictions akin to the *Liar* are again dismissed by employing a hierarchy of so-called orders of functions and propositions[5].

There are many claims made in *Principia*'s Introduction, however that strongly suggest that Russell was putting forward a *radically* different logical theory than that of his previous paper. As we saw, the hierarchy of functions in *Mathematical Logic* was a mere device of *notational convenience*[6], for there Russell explicitly claims that the underlying logical theory of *Mathematical Logic* is the Substitutional Theory, which is a realist theory of propositions that eliminates the use of functional variables contextually. The Substitutional Theory, however, is not even mentioned in *Principia* and the work's Introduction explicitly endorses Russell's new 'multiple-relation' analysis of judgment which handles declarative sentences as incomplete symbols. In the section *Definition and Systematic Ambiguity of Truth and Falsehood* of the *Introduction*, we find:

> [...] a "proposition," in the sense in which a proposition is supposed to be *the* object of a judgment, is a false abstraction, because judgment has several objects, not one.[7]
>
> Owing to the plurality of the objects of a single judgment, it follows that what we call a "proposition" (in the sense in which it is to be distinguished from the phrase expressing it) is not a single entity at all. That is to say, the phrase which expresses a proposition is what we call an "incomplete" symbol; it does not have meaning in itself, but requires some supplementation in order to acquire a complete meaning.[8]

And concerning so-called 'propositional functions', we find the claim that:

> A [propositional] function, in fact, is not a definite object, which could be or not be a man; *it is a mere ambiguity awaiting determination*, and in order that it may occur significantly it must receive the necessary determination, which it obviously does not receive if it is merely substituted for something determinate in a proposition.[9]

3 WHITEHEAD & RUSSELL, 1925, pp.37-8 [1910, pp.39-40].
4 Cf. WHITEHEAD & RUSSELL, 1925, pp.62-3 [1910, pp.65-6].
5 Cf. Cf. WHITEHEAD & RUSSELL, 1925, pp.63-4 [1910, pp.66-7].
6 Cf. RUSSELL, B., 1908, p.603 and our discussion in the previous chapter.
7 Cf. WHITEHEAD & RUSSELL, 1925, p.44 [1910, p.46].
8 WHITEHEAD & RUSSELL, 1925, p.44 [1910, p.46].
9 WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.50].

Such claims strongly suggest that in *Principia*'s Introduction Russell took a further eliminativistic step with respect to *Mathematical Logic* and adopted a no-class, no-propositional-function and no-proposition logical theory. One gets exactly such impression from some of the very early passages of *Principia* where the most basic part its logical vocabulary is introduced:

> We will use such letters as a, b, c, x, y, z, w, to denote objects which are neither propositions nor functions. Such objects we shall call individuals. Such objects will be constituents of propositions or functions, and will be genuine constituents, in the sense that they do not disappear on analysis, as (for example) classes do, or phrases of the form 'the so and so'.[10]

Again, the above quite strongly suggests that classes, propositions and so-called 'propositional functions' unlike 'individuals' are *not* part of the furniture of universe according to *Principia*, but are, as Russell sometimes said, "logical fictions". But of course, to this day the interpretation of such passages is quite controversial and as we already indicated in the introduction to the present work, Russell's eliminativistic hopes were (and, in fact, still are) often met with incredulity.

As we had occasion to remark, Quine emphatically claimed that Russell failed to properly grasp the distinction between schematic letters and predicate variables and that any hopes of ontological elimination which Russell may had about *Principia* sprang from such confusion[11]. Perhaps the most famous of these bracing sermons on use and mention is given in Quine's introduction to *Mathematical Logic* where his remarks apply to *Principia* as well:

> Is Russell assigning types to his objects or to his notations? The confusion persists as he proceeds to defined "*n*-th orders propositions". His lowest type comprises individuals; his next comprises what he calls first-order propositions; and so on up. These propositions, unlike individuals, are evidently notation; at any rate they can contain variables. Yet, like the individuals, they have type and they figure as values of quantified variables. [...] From his hierarchy of Types of propositions, Russell derives a hierarchy of functions. He speaks here of substitution, in a way that suggests that his functions also are notation in character; they seem simply to be open sentences, sentences with free variables. Still, he assigns them types and lets them be values of quantified variables. [...] Failure to distinguish thus between open sentences on the one hand and attributes and relations on the other had grave consequences for this paper and equally for *Principia Mathematica*, for which this paper sets the style.[12]

For other authors, like Church, who insisted that *Principia* should be seen as presenting a system of *intensional* logic, the claim that propositions are incomplete *symbols* was viewed as

10 WHITEHEAD & RUSSELL, 1925, p.51 [1910, pp.53-4].

11 QUINE, W., 1941, 1963, 1967 and 1969, pp.254-256. To be sure, the way in which the very passage above is phrased raises the obvious issue: are the constituents of propositions and functions letters or the individuals they denote? Since Russell speaks of *classes* (*not class-expressions*) as well as definite descriptions - which are *phrases* - as "disappearing upon analysis", it is far from clear which sense is intended here (as elsewhere).

12 QUINE, W.V., 1967, p.151.

incompatible with the ramified type theory, since they argued that the background logic required by such a theory is a fully realist theory of propositions and propositional functions. In Church's seminal paper on ramified type theory, for instance, we find:

> Many passages in [*Mathematical logic as based on the theory of types*] and [*Principia Mathematica*] may be understood as saying or as having the consequence that the values of the propositional variables are sentences. But a coherent semantics of Russell's formalized language can hardly be provided on this basis (notice in particular that, since sentences are also substituted for propositional variables, it would be necessary to take sentences as names of sentences). And since the passages in question seem to involve confusions of use and mention or kindred confusions that may be merely careless, it is not certain that they are to be regarded as precise statements of a semantics.[13]

Discussing Russell's justification for the type-hierarchy of *Principia*, Warren Goldfarb claims that "the substantive constraints on the complex structures of propositions and propositional functions that comprise ramification rely on the details of the theory of these *entities* that Russell adopts"[14]. His reading completely accepts Church's diagnosis, and echoing the latter's words, Goldfarb asserts:

> It does not appear that this view [the multiple-relation theory of judgment] is consistent with the logic of *Principia* (see Church, "*Comparison*," p. 748). Luckily, the view seems to play no real role in Russell's explanations of his logical system. In what follows, I shall take the charitable course of ignoring it."[15]

But although diagnoses of Quine and Church (and, of course, their followers) come from two different philosophical outlooks[16], there is a similar interpretative presupposition underlying their views, namely, the assumption that the system of *Mathematical Logic* and *Principia* are identical apart from some minor details[17]. This seems to be what led Quine to charge the system of *Mathematical Logic* as affected by a serious confusion between use and mention and it is what led Church to claim that *Principia* requires a realist theory of propositions and propositional functions.

This assumption, however, is very contentious and so are many the consequences drawn from it[18]. As we discussed in the previous chapter, Quine's criticism that type-distinctions are

13 CHURCH, A., 1976a, p.748, footnote 4.

14 GOLDFARB, W., 1989, p.34, our emhpasis.

15 GOLDFARB, W., 1989, p.34.

16 One may recall that for Quine "intensions are creatures of darkness" to be joyfully "exorcised" (QUINE, W., 1956, p.180).

17 Cf., for instance, QUINE, W., 1967; CHURCH, A., 1976a, p747; HYLTON, P., 1990, p.282; GOLDFARB, W., 1989, p.37. More recently Moore seems to make this assumption in his introduction and commentaries to the fifth volume of Russell's collected papers (cf., for instance, MOORE, G., 2014, p.lxxvii and also p.585).

18 Incidentally, Goldfarb writes: "In *Principia Mathematica* it is by and large this theory of 1908 that is put to work" (GOLDFARB, W., 1989, p.37)

rooted in a confusion of use and mention is moot with respect to *Mathematical Logic*[19]: the logical theory of that paper is the Substitutional Theory, which is a realist theory of propositions; the reason why the hierarchy of that work starts with individuals and ascends to propositions is that these are the *entities* to which Russell's logical theory was committed[20]. Church also seems not to realize that the background logical theory of *Mathematical Logic* is different from that of *Principia* and draws a conclusion which is, in a sense, opposite to that of Quine: Church does realize that *Mathematical Logic* did assume a realist theory of propositions, but by assuming that those systems are essentially the same he concludes that *Principia* should also assume such a realist theory of propositions with an accompanying realist hierarchy of propositional functions. But *Mathematical Logic* still endorsed the ontology of propositions and the Substitutional Theory and there is plenty of evidence that by the time he was done with *Principia* Russell had abandoned both (which were, indeed, inseparable)[21]. Whatever logical theory Russell intended to serve as a foundation for *Principia* it was *not* the system of *Mathematical Logic*.

This fact and its importance have eluded many interpreters of *Principia*[22]. Indeed, one of the greatest interpretative puzzles about the development of Russell's Mathematical Philosophy was that of determining the specific reason that led Russell to abandon the Substitutional Theory, which is apparently not even *mentioned* in *Principia*. In our previous chapter we saw that this puzzle was solved by Landini, who found in the Russell Archives the now famous letter Russell wrote to Hawtrey reporting the paradox that "pilled" the Substitutional Theory; this paradox, which Landini baptized the "p_0/a_0" paradox, together with its many variants, led Russell to introduce *orders of propositions* into the Substitutional Theory[23]. The final product was the system of *Mathematical Logic*, which is a ramified version of the Substitutional Theory with axioms of Reducibility. Russell abandoned this theory because it failed to preserve the most important feature of the Substitutional Theory, namely, its completely unrestricted variables. Landini's work caused - or, at least it should have caused - nothing short of a revolution for interpreters of Russell's works adjacent to *Mathematical Logic*, especially *Principia*. For one thing, it showed, once and for all, something that cannot be overstated: the system of *Principia*

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19 This is not to say that Russell's works are not infested with use-mention sloppines, they surely are. But this is not as devastating as Quines thinks. Indeed, if Landini is correct - and we intend to argue that he is - then Russell in many ways had views that were much similar to Quine's regarding the status of functional variables.

20 Individual variables, as we previously discussed, were also at some point considered by Russell as *entities* which occurred as genuine constituents of propositions, but this view stopped playing any explanatory role in his account of quantification and nature of propositions.

21 As we discussed, the evidence that by the time of *Principia Mathematica* Russell was no longer commited to a realist ontology of propositions like the one from the time of *Principles of Mathematics* and the era of the Substitutional Theory is simply overwhelming, both in manuscripts and published writings.

22 Goldfarb, for instance, misses the first point completely, writing the following: "In "Mathematical Logic," Russell's view is quite different from that exhibited in the Substitutional Theory. Propositional functions are now taken as legitimate entities; with their reappearance, the realistic view of propositions as complexes also returns" (GOLDFARB, W., 1989, p.36). Hylton's work (HYLTON, 1980), like those of Grattan-Guinness (GRATTAN-GUINNESS, I., 1977) and Nino Cocchiarella (COCCHIARELLA, N., 1980), was pioneer in noticing the importance of the Substitutional Theory, but lacking the evidence we now have at our disposal, they did not fully realize what led Russell to abandon it.

23 This was first reported in LANDINI, G., 1989 and later developed in LANDINI, G., 1998a.

is *not* the same as that presented in *Mathematical Logic*[24].

But be that as it may, one thing should be indisputable: the intent of the authors of *Principia* is that no sentence denotes a single entity; so every expression of the form: "So-and-so is the case" must be treated as an incomplete symbol. Thus, there is no ontology of propositions in *Principia* if these are understood as entities in the same sense that Russell meant in the *Principles* and the era of Substitutional Theory[25]. This should be a point of agreement for anyone who wants to make sense of *Principia*'s Logic in its entirety. And indeed this is a point that has been widely regarded as unproblematic by scholars.

What scholars cannot agree on is how one should answer the following questions: Granted that propositions are not entities in the same sense that individuals, is there still a sense admitted by Whitehead and Russell in which a proposition can be said to be an entity distinct from the phrase (sentence) which expresses it? And if yes, what sense is that? Put in more precise terms, should we take Whitehead and Russell's words literally when they assert that "propositions are incomplete symbols", "propositions contain variables", and so forth, or should we take these assertions as expressing a different view according to which propositions and propositional functions constitute a different sort of entity distinct from (but constituted of) individuals? This is *the* fundamental point of dispute between Russellian scholars concerning *Principia Mathematica*.

Hylton, for instance, emphatically observes that Russell and Whitehead are quite careful in their choice of words here: what they say is that *the phrase which expresses a proposition* is an incomplete symbol *not* that *propositions are* incomplete symbols[26], thus claiming that there is still an ontology of propositions and so-called 'propositional functions' - as entities distinguished from predicate variables and schematic letters - in *Principia*. Hylton's position, which is followed by several authors like Goldfarb and Bernard Linsky, is that "propositional functions are complex, structured entities"[27], and so are propositions:

> Propositions, on Russell's account, are complex structured entities. Two propositions which have some part of their structure in common are both values of one and the same propositional function, which also shares that structure (one might almost say,which is that structure).[28]

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[24] This point was also noticed in by Nino Cocchiarella (COCHIARELLA, N., 1980), although he did not fully draw the consequences of this fact as Landini did.

[25] As Peter Hylton aptly observes, it is certain that in using the notion of "proposition" and *denying that there are* propositions, Russell "[...] is using the word in precisely the same sense as in his earlier works"(HYLTON, P., 1990, p.342), and so "[...] what is he denying is that there are objective non-linguistic and non-mental entities" which "contain the things they are about"(HYLTON, P., 1990, p.342).

[26] HYLTON, P., 1990, p.342. Similar remarks are made by Bernard Linsky, who claims that "[...] the phrase that *expresses* a proposition, he says, is an incomplete symbol or expression, but the proposition is not an incomplete symbol or expression" (LINSKY, B., 1999, p.125).

[27] HYLTON, P., 2004, pp.134.

[28] HYLTON, P., 2005, p.133.

Landini, on the other hand, claims that "[...] if there are no propositions in the ontological sense, then individuals cannot be their constituents"[29], and concludes that there is no ontology of propositions and propositional functions in *Principia*. Thus, he claims that Russell was not committed to a ramified type theory of entities and that the hierarchy of types is concerned with the significance of the expressions for the language of logic and Mathematics, not with their ontology. Also, according to Landini's reading, *Principia*'s formulation of type theory was meant to preserve the doctrine of the unrestricted variable, given that it was not committed to any kind of 'typed' metaphysics and only allowed genuine (referential) variables to range over individuals.

How should such dispute be settled? Well, we must investigate how these interpretations fit within (i) the metaphysical views which Russell held at the time of *Principia*'s publication and (ii) the technical exposition of *Principia*'s theory of types. We already had occasion to discuss a long interpretative tradition that reads most of Russell's exposition as blatant examples of use-mention confusion; we also saw that there are some authors who think that aspects of *Principia*'s logic and ontology that Russell clearly thought to be of central importance should be charitably ignored. Since much has been written to show that these approaches are somewhat misguided, they won't be addressed here anymore. We will address two main sorts of interpretation:

1. Realistic intepretations of so-called propositional functions that take seriously *Principia*'s multiple-relation theory of judgment and its accompanying thesis that sentences are incomplete symbols.

2. 'Nominalistc'[30] interpretations of *Principia*'s according to which 'functional' or predicate variables must be interpreted in terms of a substitutional semantics (in the modern sense)[31].

Our concern here is with what role the multiple-relation plays in the formulation and/or justification of *Principia*'s theory of types[32]. On this regard, the first detailed attempt at an explanation was given by Nino Cocchiarella. We shall consider his reading, and the more recent proposal of Bernard Linsky, who, in a sense, offers a variant of Cocchiarella's interpretation. Concerning the

29 LANDINI, G., 1998a, p.262.

30 It must be observed that "nominalism" and "nominalistic" are used here only for the lack of a better word - perhaps "deflationary" would be more adequate - for there is simply no debate about Russell's commitment to a realist view of universals by the time of *Principia* (and any other time of his life, for that matter). The interpreation is nominalistic only with respect to *Principia*'s higher-order variables.

31 Note that expressions "substitution", "substitutional", etc., have nothing to do with "substitutions" within Russell's Substitutional Theory of classes and Relations.

32 This excludes many influential realist interpretations, like that of Peter Hylton (for instance, HYLTON, P., 1980; 1990), who is rightly regarded as an important interpreter - but most of his work focuses on the question of how Russell's multiple relation theory can account for the kind of direct relation of *acquaintance* with the objects of thought which Russell's realism requires (in particular HYLTON, P., 1990) and he shares Church's skepticism about Russell's commitment to sentences as incomplete symbols.

second kind of interpretation, the first author to claim that *Principia*'s so-called ramified theory of types should be interpreted in terms of a substitutional semantics for its functional variables was Kripke in his famous paper on substitutional quantification[33]. There, Kripke explicitly observes that this is a way "[...] of casting doubt on Quine's repeated assertion that Russell's ramified theory of types provides no significant ontological gain over ordinary set theory"[34]. Later, Mark Sainsury[35] took up Kripke's suggestion as a way to reconcile the "apparent contradiction" involved in Russell's commitment, on the one hand, to the view that logical constructions are *fictions*, and to the view that *there are* classes, relations-in-extension, etc. But neither Kripke nor Sainsbury developed this interpretation in any detail. Landini was the first to develop a detailed interpretation on this basis[36]; he was later followed by Kevin Klement[37] and Graham Stevens[38]. In the end, we will side with Landini: his interpretation affords the best means to make sense of the text of *Principia* and the many twists and turns in the development of Russell's Mathematical Philosophy[39].

Before continuing, some cautionary observations concerning the phrases "proposition", "propositional function" and other similar expressions must be made. In a sense this chapter is a concession to tradition, i.e., to what we are calling the 'orthodoxy', since we will consider many issues and interpretations that arise from the supposition that "propositional function" has a univocal use in *Principia.* Our conclusion, however, will be that the question "what is a propositional function in *Principia*" is badly framed, for it assumes that there was a *unique* sense intended by the authors for the expression "propositional function". Following Landini, we will conclude that the assumption must be rejected. We will not follow Quine, however, in claiming that Russell was just confused about matters of use and mention, that being the reason why "propositional function" does not have an univocal meaning. Whitehead and Russell were not confused about "propositional functions", they were careless. And though it must be conceded that the line between carelessness and confusion is not sharp and that there is certainly room for disagreement about where one ends and the other begins, our task will be to consider here interpretations of *Principia* that attempt to make sense of "propositional functions" without concluding that Whitehead and Russell's exposition was the product of any confusion of a fundamental kind like that imputed to them by Quine. Incidentally, this chapter will also be a concession to tradition because we will use the expression "propositional function" as if it could be a univocally referring term - but this will be done just for the sake of discussing

33 KRIPKE, S, 1976, p.368. It is fair to say that it is only after the publication of that paper that attempts to formulate substitutional semantics for formal theories started to be really taken seriously.

34 KRIPKE, 1976, p.368. Kripke is, in fact, a little more emphatic: he claims that this "is just *one* way of casting doubt on Quine's repeated assertion [...]".

35 SAINSBURY, M, 1980, pp.31-3.

36 LANDINI, G., 1998a.

37 KLEMENT, K., 2010; 2013; 2014; 2018.

38 STEVENS, G., 2003; 2004.

39 The focus of our discussion in the present chapter will be Landini's work. Possible departures from that can be made from his interpretation while still maintaining a nominalistic interpretation of so-called propositional functions will be occasionally noted in footnotes or brief commentaries.

different interpretations of the expression “propositional function”.

4.2 *Principia*’s Multiple-Relation Theory of judgment

In *Principia*, Russell and Whitehead sketch the following metaphysical picture of the furniture of the universe:

> The universe consists of objects having various qualities and standing in various relations. Some of the objects which occur in the universe are complex. When an object is complex, it consists of interrelated parts. Let us consider a complex object composed of two parts *a* and *b*, standing to each other in the relation *R.* The complex object “*a*-in-the-relation-*R*-to-*b*” may be capable of being *perceived*; when perceived, it is perceived as one object. Attention may show that it is complex; we then *judge* that *a* and *b* stand in the relation *R*. Such a judgment, being derived from perception by mere attention, may be called a “judgment of perception.” This judgment of perception, considered as an actual occurrence, is a relation of four terms, namely *a* and *b* and *R* and the percipient.[40]

What is said here is straightforward, albeit very sketchy. They are claiming that objects in general, that is, genuine entities, are divided into two fundamental sorts: those that are complex and those that are simple, with complex objects ultimately being facts, i.e, simple objects having qualities or standing in relations with other objects. Of course, implicit in the above is a distinction between between *universals* and *particulars*, which Russell discussed extensively in many writings published almost concurrently with *Principia*. In *Knowledge by Acquaintance and Knowledge by Description*, for instance, we find:

> Among particulars I include all existents, and all complexes of which one or more constituents are existents, such as this-before-that, this-above-that, the-yellowness-of-this. Among universals I include all objects of which no particular is a constituent. Thus the disjunction “universal-particular” is exhaustive. We might also call it the disjunction “abstract-concrete”.[41]

As Russell further explains in *On The Relation of Universals to Particulars*, particulars “[...] enter into complexes only as the subjects of predicates or the terms of relations and, if they belong to the world of which we have experience, exist in time, and cannot occupy more than one place at one time in the space to which they belong”[42], while universals “[...] can occur as predicates or relations in complexes, do not exist in time, and have no relation to one

40 WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.45].
41 RUSSELL, B., 1911*a*, pp.150; RUSSELL, B., 1917, p.213-4.
42 RUSSELL, B., 1912*a*, pp.182; MARSH, C., 1956, p.124.

place which they may not simultaneously have to another"[43]. Thus, the metaphysical picture to which *Principia*'s Introduction adheres to is the following: there are complex and simple objects; simple objects may be either particulars or universals; complex objects are articulated complexes which consists of simple objects (either particulars or universals) having properties or bearing relations (with these being universals).

Now, to see how the above metaphysical picture of the universe fits with the multiple relation theory of judgment, we must consider how Whitehead and Russell intend to analyze the meaning of sentences like "*aRb*" on the basis of this theory. Whitehead and Russell write:

> [...] we may define *truth*, where such judgments are concerned, as consisting in the fact that there is a complex *corresponding* to the discursive thought which is the judgment. That is, when we judge "*a* has the relation *R* to *b*", our judgment is said to be *true*, when there is a complex "*a*-in-the-relation-*R*-to-*b*", and is said to be *false* when this is not the case. This is a definition of truth and falsehood in relation to judgments of this kind.[44]

The above passage requires some care. Notice that Whitehead and Russell are not directly defining truth for elementary *sentences*, but for elementary *judgments*, which are not linguistic items but complexes that consist in a judging mind and the objects of judgment. What is defined above as true or false is not a sentence but a certain *complex of entities.* This complex will be true when there is another corresponding complex composed of the same entities occurring in the judgment complex and interrelated in the same way they are in the judgment complex, and it will be false when there is no such complex. When *m* judges that *aRb*, there is a judging complex *m-J-a-R-b*; if there is a complex *a-R-b* which corresponds to it, then the judging complex is true, otherwise it is false. As Whitehead and Russell further explain:

> It will be seen that, according to the above account, a judgment does not have a single object, namely the proposition, but has several interrelated objects. That is to say, the relation which constitutes judgment is not a relation of two terms, namely, the judging mind and the proposition, but it is a relation of several terms, namely the mind and what are called constituents of the proposition. [...] When a judgment occurs, there is a certain complex entity, composed of the mind and the various objects of judgment. When the judgment is *true*, in the case of the kinds of judgments we have been considering, there is a corresponding complex of *objects*, of the judgment alone. Falsehood, in regard to our present class of judgments, consists in the absence of a corresponding complex composed of the objects alone.[45]

Following Cocchiarella[46] and Landini, we take Whitehead and Russell to be defining truth regarding the elementary or atomic case "as the correspondence between a belief complex

[43] RUSSELL, B., 1912*a*, pp.182; MARSH, C., 1956, p.124.
[44] WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.46].
[45] WHITEHEAD & RUSSELL, 1925, p.43-4 [1910, p.46].
[46] COCCHIARELLA, N., 1980, p.103-4.

and another complex composed of the objects of the belief complex"[47]. We take it that this must be the case since according to the multiple-relation theory, judging is a relation of various terms, which includes the subject that judges or holds the belief, so there must be a complex of entities which includes the terms about which the belief or judgment is, the mind that judges or holds the belief and the relation of judging or believing. Since Russell and Whitehead (sometimes) took the care to state that it is *the phrase which expresses a proposition* that is an incomplete symbol, it cannot be the phrase which expresses the proposition which is true, but a particular occurrence of judgment/belief, which is a complex of entities. Thus truth and falsehood are not primarily defined for *sentences*: Whitehead and Russell's definition is primarily a definition of the truth of judgment occurrences. And, of course, we say *primarily* because one can speak of the truth of a sentence occurrence, but only in a derivative sense: a true sentence-occurrence is one that expresses a true judgment-occurrence.

It is on the basis of this analysis of the meaning of declarative sentences that Whitehead and Russel claim in the Introduction to *Principia* that (a) "a 'proposition' in the sense in which a proposition is supposed to be *the* object of a judgment, is a false abstraction" and (b) that "a "proposition" (in the sense in which it is to be distinguished from the phrase expressing it) is not a single entity at all"; and finally (c) that "the phrase which expresses a proposition is what we call an 'incomplete' symbol"[48]. A sentence like "*aRb*" is not the name of a single entity which is the object of judgment, because judgment always involves a complex (composed of the judging mind and the many objects which the judgment is about) and this complex may or may not correspond to a fact; in the most simple case which the authors are considering above, a judgment always has as constituents some quality *P* or relation *R* and individuals $a, a_1, ..., a_n$, so that "*Pa*" just asserts that the individual *a* has the quality *P* and "$R(a_1, ..., a_n)$" asserts that the individuals $a_1, ..., a_n$ have the relation *R*. The authors put the point as follows:

> A proposition is not a single entity, but a relation of several; hence a statement in which a proposition appears as subject will only be significant if it can be reduced to a statement about the terms which appear in the proposition. A proposition, like such phrases as "the so-and-so", where grammatically it appears as subject, must be broken up into its constituents if we are to find the true subject or subjects.[49]

To make sense of this, it is crucial to realize that truth and falsehood are primarily applicable to judgments, not to sentences; as Landini puts it, *Principia*'s truth-predicates are meant to "flank statements and *not* names of statements"[50], because "when a genuinely atomic statement is flanked by "is true" it is to be viewed as a sort of name—viz., a disguised definite descrip-

47 LANDINI, G., 1998a, pp.287-8.
48 WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.46].
49 WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.51].
50 LANDINI, G., 1998a, p.287. Our emphasis.

tion"[51]. Of course, the details of how this idea is to be interpreted are a matter of controversy, and they will be addressed below.

For now, we must review how the above account gives rise to *Principia*'s hierarchy of senses of truth and falsehood which, in fact, generates their hierarchy of orders of propositions, i.e., declarative sentences. The general idea which they introduce here is that the senses of "truth" which apply to expressions of the form $(x) \,.\, \phi x$ or $(x) : x\, is\, a\, man \,.\, \phi x$, "[...] is not the same as that in which ϕx is or may be true"[52]. Judgments of the form *aRb*, *Pa*, etc., are what Whitehead and Russell call *elementary judgments* and these, if true, are always so because there is a *single* complex which corresponds to them. The truth or falsehood of g*eneral* judgments, on the other hand, is understood by Whitehead and Russell very differently:

> [...] take now such a proposition as "all men are mortal". Here, the judgment does not correspond to *one* complex, but to many, namely "Socrates is mortal", "Plato is mortal", "Aristotle is mortal", etc. [...] For the purposes of illustration, "Socrates is mortal" is here treated as an elementary judgment, though in fact it is not one, as will be explained later. [...] Our judgment that all men are mortal collects together a number of elementary judgments. [...] We must admit, therefore, as a radically new kind of judgment, such general assertions as "all men are mortal". We assert that, given that *x* is human, it is always mortal. That is, we assert "*x* is mortal" of *every x* which is human. Thus we are able to judge (whether truly or falsely) that *all* the objects which have some assigned property also have some other assigned property; That is, given any propositional functions $\phi\hat{x}$ and $\psi\hat{x}$, there is a judgment asserting ψx with every *x* for which we have ϕx. Such judgments we will call *general judgments*.[53]

More generally, the point is put by them as follows:

> Let us take any function $\phi\hat{x}$, and let ϕa be one of its values. Let us call the sort of truth which is applicable to ϕa, "*first truth*."[54] [...] Consider now the proposition $(x) \,.\, \phi x$. If this has truth of the sort appropriate to it, that will mean that every value ϕx has "*first-truth*." Thus if we call the sort of truth that is appropriate to $(x) \,.\, \phi x$ "*second-truth*," we may define "$\{(x) \,.\, \phi x\}$ has second truth" as meaning "every value for $\phi\hat{x}$ has first truth," i.e. "$(x) \,.\, (\phi x$ has first truth)." Similarly, if we denote by "$(\exists x) \,.\, \phi x$" the proposition "ϕx sometimes," i.e. as we may less accurately express it, "ϕx with some value of *x*," we find that $(\exists x) \,.\, \phi x$ has second truth if there is an *x* with which ϕx has first truth; thus we may define "$\{(\exists x) \,.\, \phi x\}$ has second truth" as meaning "some value for $\phi\hat{x}$ has first truth," i.e. "$(\exists x) \,.\, (\phi x$ has first truth)." Similar remarks apply to falsehood.[55]

51 LANDINI, G., 1998a, p.287.

52 WHITEHEAD & RUSSELL, 1925, p.46 [1910, p.49].

53 WHITEHEAD & RUSSELL, 1925, pp.44-5 [1910, p.47].

54 As they remark, this should be taken with the following disclaimer, that "This is not to assume that this would be first truth in another context: it is merely to indicate that it is firs sort of truth in our context."(WHITEHEAD & RUSSELL, 1925 [1910], pp.42).

55 WHITEHEAD & RUSSELL, 1925, p.42 [1910, pp.44-5].

That is, for Whitehead and Russell any general 'proposition' should be analyzed as asserting that a given propositional function $\phi\hat{x}$ is true for all or some of its values[56]. Taking as an example "$(x) : x\, is\, a\, man \;.\supset.\; x\, is\, mortal$", Whitehead and Russell analyze this as meaning something like: "The propositional function $\hat{x}$ *is a man* $\supset$ $\hat{x}$ *is mortal* has a true proposition as a value for every argument *x*". More generally, to say that "$(x) : \phi x \,.\supset.\, \psi x$", or "$(x) \,.\, \phi x$", or "$(x) \,.\!\sim \psi x$", etc., have truth of the *n* + 1 kind, is to say that 'every value' - as Whitehead and Russell put it - of $\phi\hat{x} \,.\supset.\, \psi\hat{x}$, $\phi\hat{x}$, or $\sim \psi\hat{x}$ has truth of the appropriate *n* kind.

What remains to be seen is how expressions like "$\phi\hat{x}$" fit within this view. Do they stand for entities which enter in complexes as genuine constituents? Furthermore, what happens when we extend the above definition of truth and falsehood to general propositions that contain bound functional variables ϕ, ψ, etc? As we shall see, Cocchiarella, Bernard Linsky, and Landini have very different answers to these questions - all worth considering in detail. Since Linksy's answer can be viewed as a modification of Cocchiarella's and Landini's is formulated in direct opposition to Cocchiarella, we must start with the latter.

4.2.1 Cocchiarella's Reconstruction of *Principia*'s Multiple Relation Analysis of judgment

Cocchiarella was the first author to make a serious effort to fit *Principia*'s Multiple Relation theory within the Theory of Types. He is also one of the first to recognize that the system of *Principia* is not the same of *Mathematical Logic*, the contrast being, as he puts it, that in *Principia* "propositions are no longer to be considered as single entities, and propositional functions (of different orders) are no longer to be considered as non-entities"[57]. As we mentioned above, Cocchiarella, like Landini, takes truth and falsity as primarily a property of particular judgment occurrences:

> [...] instead of taking propositions of different orders as being single entities that are themselves objectively true or false, Russell now assumes that judgments as particular occurrences, or statements as potential judgments, are the primary vehicles of a hierarchically ordered system of truth and falsehood.[58]

He claims that Russell and Whitehead intended to analyze an atomic or elementary judgment as having the form:

(NC-eJ) $\quad J_{n+2}(m, R_n(\hat{x}_1, ..., \hat{x}_n), a_1, \ldots, a_n)$

[56] WHITEHEAD & RUSSELL, 1925, p.42 [1910, pp.44-5].
[57] COCCHIARELLA, N., 1980, p.101.
[58] COCCHIARELLA, N., 1980, p.104.

Where *m* is some mind which judges/or believes that $R_n(a_1, \ldots, a_n)$ is true, $a_1, \ldots, a_n$ are any objects, $R_n(\hat{x}_1, ..., \hat{x}_n)$ is a propositional function which is to be *identified with a universal*[59] and $a_1, \ldots, a_n$ are any objects (particulars or universals) of appropriate type relative to $R_n(\hat{x}_1, ..., \hat{x}_n)$.

An elementary/atomic proposition then, is simply the judgment which is expressed by a sentence of the form like (NC-eJ), that is, a complex that may or may not correspond to a single fact[60]. Thus, he takes that truth can be defined in the case of elementary (atomic) *judgments* as follows[61]:

(C-eT) $J_{n+2}(m, R_n(\hat{x}_1, ..., \hat{x}_n), a_1, \ldots, a_n)$ has atomic truth iff there is complex (fact) corresponding to *m*'s judging with respect to $a_1, \ldots, a_n$ and R_n, for some mind *m*.

Although Cocchiarella does not go into an explicit explanation as to why this amounts to treating atomic sentences as incomplete symbols and the corresponding 'proposition' as "a false abstraction", we may easily reconstruct a good reason for such claim on the basis of his reading: an expression like "*aRb*" purports to denote a complex; but this complex may or may not exist; in order for it to express something definite, then, this expression must enter as a subordinate clause as part of an expression that actually expresses a judgment that stands for a judging complex, as in *Othello-believing-that-Desdemona-loves-Cassius*. Indeed, Cocchiarella writes that:

> [...] it is not the assertoric occurrence of a propositional phrase which is an "incomplete symbol" but only the occurrence of a nominalized propositional phrase, i.e., one occurring as a grammatical subject (or object) of a declarative sentence, and especially one occurring as the nominal subject (or direct object) of a sentence expressing a propositional attitude.[62]

Now, the central aspect of Cocchiarella's interpretation is that propositional functions are to be identified with universals, i.e. qualities/attributes and relations-in-intension, and thus be included as constituents of judgments[63]:

> As a particular occurrence, moreover, a judgment is a complex of entities consisting of a mind and the objects, including the propositional functions (now construed as properties and relations in intension) that would otherwise be taken as the constituents of the proposition being judged.[64]

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59 COCCHIARELLA, N., 1980, p.102.
60 COCCHIARELLA, N., 1980, p.103.
61 COCCHIARELLA, N., 1980, p.102. We are somewhat adapting the authors terminology so that it fits better with our subsequent discussion of Landini's interpretation.
62 COCCHIARELLA, N., 1980, p.101.
63 COCCHIARELLA, N., 1980, p.102, 108.
64 COCCHIARELLA, N., 1980, p.104.

Cocchiarella argues that since propositions conceived as entities are out of the picture, the hierarchy of orders of judgments must be characterized in terms of the orders of propositional functions:

> The order of the judgment's truth or falsehood is then to be identified with the maximum order of the propositional functions occurring in that judgment.[65]

Thus, according to Cocchiarella's interpretation, Whitehead and Russell's analysis of "*m* judges that all men are mortal" is to be understood along the following lines:

$$J_4(m, (x)(\hat{\phi}!x \supset \hat{\psi}!x), \hat{x} \textit{ is a man}, \hat{x} \textit{ is mortal})$$

Where $(x)(\hat{\phi}!x \supset \hat{\psi}!x)$ is "[...] the second-order relation of material implication between two predicative propositional functions of individuals"[66], and since such function is of second order, the proposition is also of second order. For a third-order case of truth and falsehood containing quantification over functions which take functions as arguments, he takes the analysis to be given as something like:

$$J_4'(m, (\phi)(\hat{\chi}!(\phi!\hat{z}) \,.\supset.\, \phi!\hat{x}), f!(\hat{\phi}!\hat{z}), \text{Napoleon})$$

Where "$(\phi)(\hat{\chi}!(\phi!\hat{z}).\ \supset\ .\phi!\hat{x})$" is meant to stand for "a third-order relation between second-order properties of predicative properties of individuals on the one hand and individuals on the other"[67].

The general idea is that the hierarchy of senses of "truth" and "falsehood" which applies primarily to judgment occurrences is a by-product of type distinctions applied to propositional functions understood as entities: if a judgment contains as (a genuine) constituent a propositional function of order *n*, then the judgment itself must be of at least that order *n* and, in fact, any judgment will be of the same order as that of the function of highest order which occurs in it[68].

Thus, to sum up: Cocchiarella takes truth to be primarily defined for belief-complexes (or judging complexes) that may or may not correspond to facts; he assumes (as we shall see, incorrectly) that propositional functions must enter as constituents in both facts and belief complexes; and thus, he concludes that a hierarchy of senses of truth and falsehood arises from a hierarchy of judgments or judgement complexes, with the order of a judgment being given by the propositional function of the highest order which occurs in it.

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65 COCCHIARELLA, N., 1980, p.104.
66 COCCHIARELLA, N., 1980, p.104.
67 COCCHIARELLA, N., 1980, p.105.
68 COCCHIARELLA, N., 1980, p.104-5.

4.2.2 Landini's Reconstruction of *Principia*'s Multiple Relation Analysis of judgment

Landini, like Cocchiarella, thinks that the multiple-relation analysis of judgment must be taken seriously. But there are several points of disagreement between Landini's reading and Cocchiarella's. The fundamental points which Landini (rightly) disputes are the following:

> 1. Propositional functions are universals which can enter into complexes as constituents.
>
> 2. The hierarchy of senses of truth and falsehood is derived from a hierarchy of judgment-complexes/judgment-occurrences, and the order of judgment complex is determined by the propositional functions of the highest order which occur in it.

Unlike Cocchiarella, Landini does not read the passage cited above which introduces the different senses of truth and falsehood[69] as introducing a hierarchy of judgments - rather he takes Russell and Whitehead to be recursively characterizing truth and falsehood for '*propositions*' understood as declarative *sentences* using the multiple-relation theory applied to elementary propositions as the base of the recursion. As Landini puts it:

> The hierarchy of senses of 'truth' and 'falsehood' in *Principia* is applied to what Whitehead and Russell call 'propositions'. It is a hierarchy of orders of propositions which is thereby generated, not a hierarchy of judgment-complexes. Of course, the "propositions" here are not the objective truths and falsehoods Russell formerly endorsed. They must be declarative sentences or statements capable of truth or falsehood.[70]

Now, this does not mean that Landini disagrees that *primarily* truth and falsehood must be applied to judgments (specifically atomic ones); on the contrary, he explicitly recognizes that this is the correct interpretation:

> [...] the foundational notions of 'truth' and 'falsehood' are defined in terms of correspondence between judgment (or belief) and fact in accordance with the multiple-relation theory. That is, when an atomic declarative sentence is asserted there occurs an 'elementary judgment' insofar as there is a mental complex involved with the assertion. An elementary judgment is then said to be 'true' when there is a corresponding complex and 'false' when there is no corresponding complex. The truth conditions for an atomic proposition such as '*a* has the relation *R* to *b*' consist in the fact that there is a complex corresponding

[69] I.e., WHITEHEAD & RUSSELL, 1925, pp.41-2 [1910, pp.44-5].
[70] LANDINI, G., 1991, p.42. Cf. also LANDINI, G., 1998a, p.p.283-4.

> to the judgment-complex which is involved with the assertion of the atomic proposition.[71]

Landini accepts that truth applied to an atomic sentence is derived from truth applied to judgment-complexes, for he is aware that what corresponds or fails to correspond to a fact is a judgment complex, not a declarative sentence.

However, Landini interprets Russell's claim that the multiple-relation theory of judgment "[...] is an extension of the principle applied in [...] "*On Denoting*" [...]" and the accompanying claim that propositions are incomplete symbols[72] more radically than Cocchiarella. He suggests that when the expression "*a* bears *R* to *b*" or "*aRb*" (not its name) is flanked by ". . . is true", the expression is in fact a definite description in disguise[73], to be analyzed as something along the following lines[74]:

(L-eJ) "The complex (fact) corresponding to *m*'s judgment with respect to *a* and *R* and *b*, for some mind *m*".

Thus, Landini claims that the correct interpretation is to take the sentence "*aRb*", when flanked by "... is true" or "... is false" to be functioning as incomplete incomplete symbol which purports to denote the *fact m*-J-*o*-*aRb* for some *m* which judges/believes it. He gives the following general characterization of atomic judgments where *R* is a relation, $a_1, \ldots, a_n$ are objects (universals or particulars), *m* is a mind or subject of belief/judgment and $[R_n(a_1, \ldots, a_n)]$ is the complex wich consists in the terms $a_1, \ldots, a_n$ related by the *n*-ary relation *R*[75]:

(L-eT) $[R_n(a_1, \ldots, a_n)]$ has atomic/elementary truth if and only if there is a fact corresponding to *m*'s judging/believing with respect to $a_1, \ldots, a_n$ and R_n, for some mind *m*.

Since Whitehead and Russell claim that a general judgment "collects together a number of elementary judgments"[76], without ever stating what the constituents of such judgment are, only claiming that they are of a "radically new kind"[77], Landini assumes that the only restraint on the analysis of general judgments is that they must accord with the recursive definition which

[71] LANDINI, G., 1998a, p.
[72] RUSSELL, B., 1907*a*, p.453, footnote 16.
[73] LANDINI, G., 1998a, pp.288.
[74] LANDINI, G., 1998a, pp.288.
[75] LANDINI, G., 1998a, pp.289.
[76] WHITEHEAD & RUSSELL, 1925, p.45 [1910, p.47].
[77] WHITEHEAD & RUSSELL, 1925, p.45 [1910, p.47].

has the multiple relation theory as its base case[78]. He takes it that a general judgment is to be analyzed as asserting the existence of *multiple* elementary complexes, and thus he analyses the truth conditions of the judgment that all men are mortal as follows[79]:

(L-FoT) [(*x*)(Man(*x*) ⊃ Mortal(*x*))] has second truth if and only if for every *x*, if there is a fact corresponding to *m*'s judging with respect to *x* and Humanity, for some mind *m*, then there is a fact corresponding to *m*'s judging with respect to *x* and Mortality, for some mind *m*.

Here, however, it is not a second sort of truth based on a second notion of correspondence that is being introduced: a general judgment only differs from an atomic one because if true, it may correspond to multiple complexes instead of a single one. But all judgment complexes are on a par for Landini: their constituents are all particulars and universals and there are no type-distinctions among them. Thus, strictly speaking, there is only one sense in which truth can be applied to judgments, namely: in terms of correspondence between a judgment complex and facts (either a single one or several).

As we mentioned and shall address in detail below, Landini claims that when Whitehead and Russell distinguish a whole hierarchy of senses of truth and falsehood, they are using truth-predicates in a secondary way that applies to propositions understood as *declarative sentences* and this hierarchy is recursively generated with the multiple relation theory applied to elementary judgments as the base case[80]: all senses of truth and falsehood for sentences/propositions, including those of 'higher-order', are derived from the one legitimate sense of truth applied to judgment complexes. This recursive characterization of senses of "truth" is meant to track what sort of quantifiers occur in a sentence: according to Landini, "truth" applied to sentences containing apparent (bound) variables does *not* apply to belief complexes because these must be uniformly true in the exact same sense (that they correspond to facts). Thus, Landini claims:

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78 He offers two possible alternative analyses of (*m*) which adequately fit within this. The two alternative analyses he offers are the following (LANDINI, G., 1998a, pp.288-90):

(*m*-analysis-L1) [$(x)(x$ *is a man* $\supset x$ *is mortal*)] is true iff for every *x*, if there is a fact corresponding to *m*'s *understanding* with respect to *x* and Humanity, for some mind *m*, then there is a fact corresponding to *m*'s understanding with respect to *x* and Mortality, for some mind *m*.

(*m*-analysis-L2) [$(x)(x$ *is a man* $\supset x$ *is mortal*)] is true iff for some *m*, the general belief complex consisting of *m* related to Humanity and Mortality is true.

He takes 'supposing' or 'understanding' as more "appropriate multiple relations for the general recursive definition" since "to define "truth" as correspondence between a mental state and a fact one must assume at least one mind, but surely not one in a believing relation and not one for every entity. 'Understanding', on the other hand, turns the trick so long as it is not viewed as an occurrent state" (LANDINI, G., 1998a, p.288). For the sake of uniformity we shall discuss general judgments in terms of belief, to conform with Cocchiarella's and facilitate comparison; but one should be aware of the issue Landini points out.

79 LANDINI, G., 1998a, p.288.

80 LANDINI, G., 1998a, p.286.

> [...] Cocchiarella has things wrong side up. He takes a top-down approach which makes the order of a propositional function occurring in a belief-complex as fundamental and holds that this determines the nature of "truth" (correspondence). But Whitehead and Russell would appear to be taking a bottom-up approach. The relation of 'correspondence' is fundamental. It is the nature of the correspondence involved that establishes the sense of 'truth' and 'falsehood' and, accordingly, the orders of propositions.[81]

How then, does Landini's interpretation fit Whitehead and Russell's use of "propositional function"? Well, as we briefly mentioned, Landini's main contention is that there are at least two completely distinct uses of the term "propositional function" in *Principia*'s Introduction. In one sense, "propositional function" means a predicate variable that can be bound by (*substitutional*) quantifiers. In another sense, "propositional function" means a *schematic letter* which may be substituted by well-formed *formulas*[82]. But to fully appreciate his reasons for this, we must discuss the issues involved in *Principia*'s arguments in favor of types and orders in detail.

For now, we may sum up Landini's interpretation of the multiple relation analysis of judgment as follows: against Cocchiarella, Landini claims that there is no hierarchy of complexes; the hierarchy of senses of truth and falsehood presented in *Principia* introduces a hierarchy of truth-predicates that applies to sentences; but this recursively generated hierarchy of truth and falsehood predicates is derived from the primary sense of truth and falsehood which applies to belief complexes and is founded on the fundamental notion of correspondence between judgment complexes and facts (either a single one, or a multiplicity of them). According to this reading, only individuals (either universals or particulars) forming a single non-typed domain enter into complexes as genuine constituents: so-called propositional functions are just expressions.

4.3 The Hierarchy of Types and the Axiom of Reducibility

From a technical point of view, the core of *Principia*'s type theory is expressed by the authors in the following terms: "any expression containing an apparent variable must not be in the range of that variable, *i.e.* must belong to a different type"[83]. There are two presentations of the hierarchy of types in *Principia* which attempt to explain how this idea avoids the contradictions - one in the second chapter of the Introduction and another in $*12$, within the body of the work, where the Axiom of Reducibility is officially introduced. If we look at the expositions in the two portions of the work, there are two main definitions: a recursive characterization of the *order* of a propositional function and a definition of *predicative* propositional functions. Concerning the notion of *order*, in the Introduction and $*12$, respectively, we find the following characterization of first-order functions:

81 LANDINI, G., 1991, p.43.
82 LANDINI, G., 1998a, p.265.
83 WHITEHEAD & RUSSELL, 1910, p.168.

> We will denote any first-order function by "$\phi!\hat{x}$" and any value for such function by "$\phi!x$". Thus, "$\phi!x$" stand for any value of any function which involves no variables except individuals.[84]
>
> A function of the first order is one which involves no variables except individuals, whether as apparent variables or as arguments.[85]

Both these definitions are clearly equivalent and give the base clause of their definition. The recursive clause of their definition of the order of a function is then presented in the Introduction and $*12$, respectively, as follows:

> If the highest order of [sic] variable occurring in a function, whether as argument or as apparent variable, is a function of the *n*th order, then the function in which it occurs is of the $n+1$th order.[86]
>
> A function of the $(n+1)th$ order is one which has at least one argument or apparent variable of order *n*, and contains no argument or apparent variable which is not either an individual or a first-order function or a second-order function or a function of order *n*.[87]

These are also clearly equivalent, and thus, both the Introduction and $*12$ present the same hierarchy of types in terms of the notion of the *order* a function.

Another important notion which is introduced at this point and then reconsidered at $*12$ is that of a *matrix*, i.e., a function which contains no apparent variables[88]. Their key idea is that "all possible propositions and functions are obtainable from matrices by the process of turning the arguments to the matrices into apparent variables"[89]. So the hierarchy of types begins with "matrices [...] whose values are of the forms ϕx, $\psi(x, y)$, $\chi(x, y, z)$, *i.e.*, where the arguments, however many there may be, are all individuals"[90].

This passage may seem again to introduce yet another instance of use-mention confusion. But if we take the authors to be describing the hierarchy of functions in terms of their formal expressions[91], their exposition can be reviewed as follows. They start with matrices which contain only real individual variables[92], that is, functions ϕ, ψ, χ which "contain no apparent variables and have no arguments except individuals, [sic] do not presuppose any totality of functions"[93].

84 WHITEHEAD & RUSSELL, 1925, p.51 [1910, p.54].
85 WHITEHEAD & RUSSELL, 1925, p.167 [1910, p.175].
86 WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56].
87 WHITEHEAD & RUSSELL, 1925, p.167 [1910, p.175].
88 WHITEHEAD & RUSSELL, 1925, pp.50-1 [1910, p.53].
89 WHITEHEAD & RUSSELL, 1925, p.51 [1910, p.53].
90 WHITEHEAD & RUSSELL, 1925, p.51 [1910, p.54].
91 It is worthwhile to emphasize, yet again, that since Whitehead and Russell speak of functions and matrices as containing and/or involving *variables*, it is clear that there is a legitimate sense in which these can be said to be *expressions* which are stratified into a hierarchy, and that this in itself does not mean that propositional functions must be, after all, *just* expressions.
92 Since there are no constants in *Principia*, we shall not bother to include them in our discussion, but it would be evidently easy to do so.
93 WHITEHEAD & RUSSELL, 1925, p.51 [1910, p.54].

Examples of such (first-order) matrices are:

$$\phi x, \psi(x,y), \phi x \vee \psi(x,y), \chi(x,y,z), \ etc...$$

Next, by the process of turning real individual variables into apparent variables, i.e., binding them to quantifiers, they form the totality of *first*-order *functions*, which are either first-order matrices or are obtained from matrices by turning real individual variables into apparent variables. Examples of such matrices are the following:

$$(\exists y) \,.\, \psi(x,y), (x) \,.\, \psi(x,y), (x,y) \,.\, \chi(x,y,z), etc...$$

Now, it is at this point that they introduce the shriek notation (as in "$\phi!\hat{x}$", "$\psi!(\hat{x},\hat{y})$", etc.):

> We shall denote any first-order function by "$\phi!\hat{x}$" and any value for such function by "$\phi!x$". Thus "$\phi!x$" stands for any value for any function which involves no variables except individuals.[94]

Using "$\phi!\hat{x}$", "$\psi!(\hat{x},\hat{y})$", etc., for first-order functional variables, they form the totality of *second*-order *matrices*, that is, the totality of all functions which contain only individual and first-order functional variables, but no apparent variables[95]. Examples of such matrices are:

$$f(\phi!\hat{z}), f(\phi!\hat{z}, \psi!\hat{z}), f(\phi!\hat{z}, x), \ etc...$$

Again, turning real individual variables and real first-order functional variables into apparent variables, they thus form the totality of second-order *functions*, which are either second-order matrices or are obtained from second-order matrices by quantifying over individual or first-order functional variables, as in:

$$(\phi) \,.\, f(\phi!\hat{z}), (\exists\psi) \,.\, f(\phi!\hat{z}, \psi!\hat{z}), (x) \,.\, f(\phi!\hat{z}, x), etc...$$

Second-order functional variables are then introduced, as in $f!(\hat{\phi}!\hat{z})$, $g(\hat{\phi}!\hat{z}, x)$, etc., which can be used to define the totality of third-order matrices and functions, and so on, "indefinitely"[96]. Put generally, a matrix of order *n* is such that it contains only *real* individual variables or *real* variables of *order less than n*, while a function of order *n* is such that it contains apparent individual variables or apparent individual variables of order less than *n*.

Crucially, then, they introduce the following definition of a *predicative* function:

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94 WHITEHEAD & RUSSELL, 1925, p.51 [1910, p.54]. As Bernard Linsky observes (LINSKY, B., 1999, p.79.), this *can* be confusing, since the shriek notation is employed to characterize predicative functions in general.

95 WHITEHEAD & RUSSELL, 1925, p.52 [1910, p.55].

96 WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56]. It is at this point that Russell and Whithead reject the possibility of having transfinite types which Russell had previously considered in *Appendix B* of the *Principles*, and which Gödel would later consider: "We do not arrive at functions of an infinite order, because the number of arguments and of apparent variables in a function must be finite, and therefore every function must be of a finite order. Since the orders of functions are only defined step by step, there can be no process of "preceding to a limit", and functions of an infinite order cannot occur" (WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56]).

> We will define a function of one variable as predicative when it is of the next order above that of its argument, i.e., of the lowest order compatible with its having that argument. If a function has several arguments, and the highest order of function occurring among the arguments is the *n*th, we call the function predicative if it is of the (*n* + l)th order, i.e., again of the lowest order compatible with its having the arguments it has. A function of several arguments is predicative if there is one of its arguments such that, when the other arguments have values assigned to them, we obtain a predicative function of the one undetermined argument.[97]

It is at this points that fundamental questions about the structure of their hierarchy begin to emerge. Right after the above definition, we find the claim that "all possible functions in the above hierarchy can be obtained by means of predicative functions and apparent variables"[98] and that:

> [...] speaking generally, a non-predicative function of the *n*th order is obtained from a predicative function of the *n*th order by turning all the arguments of the $n - 1$th order into apparent variables. (Other arguments also may be turn into apparent variables.) Thus we need not introduce as variables any functions except predicative functions.[99]

Thus, they claim in section $*12$:

> We shall use 'ϕx', or '$f(\chi!\hat{x})$' or etc. to express functions (ϕ or f) whose order, relatively to its argument, is not given. Such a function cannot be made into an apparent variable, unless we suppose its order previously fixed. As the only purpose of the notation is to avoid the necessity of fixing the order, such a function will not be used as an apparent variable; the only functions which will be so used will be predicative functions, because, as we have just seen, this restriction involves no loss of generality.[100]

This imposes an important restriction upon their symbolism: *there are no non-predicative variables* in *Principia*. But if we follow the Introduction's definition of predicative function, then all the possible values of a functional variable are such that its order "[...] is of the next order above that of its argument, i.e., of the lowest order compatible with its having that argument"[101]. As we'll see below when discussing Church's reconstruction of ramified logic, this *can* be understood as a severe (and unwelcome) restriction on the range of functional variables or, as Landini argues, a fundamental aspect of the justification of type distinctions. The main issue here is

97 WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56]
98 WHITEHEAD & RUSSELL, 1910, p.56.
99 WHITEHEAD & RUSSELL, 1925, p.53-4 [1910, p.56] Furthermore, they claim, in accordance with their custom of not using functions with more than two argument-places, that "to obtain any function of one variable *x*", they "need not go beyond predicative functions of *two* variables", since, generally, "if $\phi!\hat{u}$ is a predicative function of a sufficiently high order, any assigned non-predicative function of *x* will be of one of the two forms $(\phi).F!(\phi!\hat{u}, x)$, $(\exists\phi).F!(\phi!\hat{u}, x)$", where F is a predicative function of $\phi!\hat{u}$ and *x*" (WHITEHEAD & RUSSELL, 1910, p.56).
100 WHITEHEAD & RUSSELL, 1925, p.165 [1910, p.173]
101 WHITEHEAD & RUSSELL, 1925, p.53 [1910, p.56]

that the restriction to predicative variables apparently excludes some non-predicative functions from their reach. One recurrent example from the secondary literature is the following[102]: let "$G(x)$" mean "x is a great general", let "$B(x)$" mean "x is brave" and let "Jones" and "Smith" be the name of two individuals; now assume:

1. $(x) : G!(x) \,.\supset .\, B!(x) : B!(Smith)$
2. $(x) : G!(x) \,.\supset .\, B!(x) : B!(Jones)$

From these and higher-order universal instantiation, it follows, from 1 and 2[103]:

3. $(\exists\phi) :. (x) : G!(x) \,.\supset .\, \phi!x : \phi!(Smith)$
4. $(\exists\phi) :. (x) : G!(x) \,.\supset .\, \phi!x : \phi!(Jones)$

These are obtained by the process Whitehead and Russell allow, that is, by turning real predicative functional variables into apparent ones (and, of course, turning constants into variables, which they allow, despite the fact that *Principia* does not have any constants). But with the restriction to predicative variables, the following could not be obtained from 3 and 4:

5. $(\exists\psi) : \psi(Smith) \,.\, \psi(Jones)$

This is because the function ψ of *y* would have to be introduced as something like:

$$\psi\hat{y} =_{\mathrm{Df}} (\exists\phi) :. (x) : G(x) \,.\supset .\, \phi!x : \phi!(\hat{y})$$

This is a non-predicative function not obtainable by generalization on predicative variables. Are we to understand such restrictions as intended by Whitehead and Russell or not?

This point actually leads us to another fundamental question about *Principia*'s system. Most modern formulations of type-theory introduce, at some point, a notational device to turn open formulas into (higher-order) terms that can be substituted for variables of the appropriate kind, or occur as arguments of functions (of higher types). Perhaps the most well-known example of such device is the λ operator introduced by Alonzo Church in his Calculus of λ-conversion.

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[102] Peter Hylton discusses it in his HYLTON, P., 1990, p.309-9, footnote 34 and attributes its formulation to Thomas Ricketts; it is also discussed in LINSKY, B., 1999, p.80-1.

[103] Notice here that if one applies $*12{\cdot}1$ - ignoring the requirement that *Principia* must admit only 'predicative' comprehension for its predicate variables $\phi!$,$\psi!$, etc., this problem vanishes. That is, indeed, one of the reasons why Landini argues that $*12{\cdot}1$ must be viewed as *Principia*'s sole comprehension principle and why there are one only 'predicative functional variables' $\phi!$,$\psi!$, etc., bound to quantifiers in the work..

Such systems generally introduce, along with appropriate axioms[104], an operator such as λ that binds variables of open formulas to turn them into terms. If "Fx", for instance is a formula containing *x* free, then "$\lambda x(Fx)$" or "$[\lambda x : Fx]$" or something along these lines will be a name for a new term[105] (with the same applying for functional or relational expressions of any number of arguments as in "$[\lambda x \lambda y : R(x, y)]$", etc.). And according to the rules of such systems for forming new terms from others, formulas of arbitrary complexity can be turned into terms via the λ device (as in "$[\lambda x \lambda y : (R)(R(x, y). \supset .Fx \supset Fy)]$", or "$[\lambda x \lambda y \lambda R : R(x, y). \supset .Fx \supset Fy]$" for instance). The question, then, is: should *Principia*'s circumflex be understood as a term-forming operator which attached to any open formula yields a (higher-order) term, much like Church's λ operator?

Well, what Whitehead and Russell say in the Introduction regarding this point is the following:

> When we wish to speak of the propositional function corresponding to "*x* is hurt", we shall write "$\hat{x}$ is hurt". Thus, "$\hat{x}$ is hurt" is the propositional function and "*x* is hurt" is an ambiguous value of that function. [...] More generally, ϕx is an ambiguous value of the propositional function $\phi \hat{x}$, and when a definite signification *a* is substituted for *x*, ϕa is an unambiguous value of $\phi \hat{x}$.[106]

This has been almost universally read, in one way or another, as the introduction a term-forming operator. Peter Hylton, for instance claims that "the circumflex notation [...] is used as an abstraction operator", in the following way: "[...] if "*Fx*" is an open sentence, then "$F\hat{x}$" is the name of an entity, a propositional function"[107]. Echoing Hylton's words, Warren Goldfarb puts this as follows:

> Russell uses the circumflex as an abstraction operator: if *Fx* is an open sentence, then $F\hat{x}$ expresses the propositional function that, for each argument *a*, yields as value the proposition expressed by *Fa*.[108]

This is turned into a criticism of *Principia* since, if the circumflex is to be understood as term-forming operator, then some axiom of comprehension is required for the system, but as the story goes "Russell never comes close to formulating this or to having axioms of this form"[109]. In searching for a charitable interpretation, Goldfarb takes *Principia* to have implicit concretion or abstraction principles embedded in the rules of generalization and instantiation

104 Of such axioms, the more characteristic are those determined by axiom schemata such as $(x)([\lambda z : \mathbf{A}]x \equiv \mathbf{A}[z/x])$, $(x, y)([\lambda z \lambda w : \mathbf{A}]xy \equiv \mathbf{A}[z, w/x, y])$, etc., which are generally called principles of λ-conversion together with comprehension principles for concepts, two-place relations and so on.

105 And what sort of thing this term denotes, of course depends on how the formal language is interpreted.

106 WHITEHEAD & RUSSELL, 1925, p.15 [1910, p.15]

107 HYLTON, P., 1980, p.27-8.

108 GOLDFARB, W., 1989, p.39, footnote 20.

109 GOLDFARB, W., 1989, p.32.

for higher-order variables[110]; Hylton, on the other hand, suggests that the absence of a clear account of circumflexion is due to Russell's "unwillingness [...] to acknowledge the implications of type theory", since the operator "is never acknowledged as a piece of basic notation"[111]. Indeed, Hylton is quite correct in claiming that Russell and Whitehead seem to have *not* intended the circumflex to be part of their official grammar. In fact, they practically say so:

> We have found it convenient and possible - except in the explanatory portions - to keep the explicit use of symbols of the type "$\phi\hat{z}$" either as constants [*e.g.*, $\hat{x} = a$] or as real variables, almost entirely out of this work.[112]

The problem is that not only the authors continue to claim - in the body of the work - that the notation $\phi\hat{x}$ "[...] means the function itself, as opposed to an ambiguous value of the function"[113], but much more seriously, the circumflex appears in their numbered propositions - crucially in the contextual definitions of $*20$ and $*21$ which we'll discuss next. This point is absolutely relevant in connection with the previous one, about the restriction of variables to predicative values. To go back to the Ricketts-Hylton example, if we take an open formula like "$(\exists\phi) :. (x) : G(x) . \supset . \phi!x : \phi!(y)$" and interpret the circumflex as a term-forming operator, then "$(\exists\phi) :. (x) : G(x) . \supset . \phi!x : \phi!(\hat{y})$" should be considered a term; but were that the case, then *Principia*'s formulation of type theory is incoherent: if the above function, call it $\psi\hat{y}$, for brevity, is a genuine term, then it should be substituable for variables in universal instantiations, for instance, or turned into a variable by universal generalization; but this is impossible if they only allow predicative variables[114]. So either the circumflex is not a term-forming operator or the predicative restriction is not really intended; or the notion of "predicative function" just means a predicate variable as Landini claims.

But the trouble with the predicativity constraint is not over yet. In $*12$, Whitehead and Russell officially introduce their problematic Axiom of Reducibility which asserts that "every function of one variable is equivalent, for all its values, to some predicative function of the same argument"[115] and that "every function of two variables to a function of those variables"[116]. In symbols, they put these as:

$$*12\cdot1 \vdash :. (\exists f) : \phi x . \equiv_x . f!x \quad \text{Pp}$$

$$*12\cdot11 \vdash :. (\exists f) : \phi(x,y) . \equiv_{x,y} . f!(x,y) \quad \text{Pp}$$

[110] GOLDFARB, W., 1989, p.32.
[111] HYLTON, P., 1980, p.27-8.
[112] WHITEHEAD & RUSSELL, 1925, p.19 [1910, p.20]
[113] WHITEHEAD & RUSSELL, 1925, p.125 [1910, p.132]
[114] This point is emphasized by Landini, who, as far as I know, first called attention to it in LANDINI, G., 1998a, pp.265-6. More on this point below.
[115] WHITEHEAD & RUSSELL, 1925, p.166 [1910, p.174]
[116] WHITEHEAD & RUSSELL, 1925, pp.166-7 [1910, p.174] The authors observe that the same assumption could be made for functions of three or more variables, as in $(\exists f) : \phi(x,y,z) . f!(x,y,z)$, etc., but they refrain from assuming them because "they are not indispensable".

As we saw, the *order* of a function is characterized in the same way in ∗12 and the Introduction. But now Whitehead and Russell seem to give conflicting statements concerning the structure of the hierarchy, since the key notion of a predicative function is re-introduced in ∗12 as follows:

> A function is said to be predicative when it is a matrix. It will be observed that, in a hierarchy in which all the variables are individuals or matrices, a matrix is the same thing as an elementary function.
>
> '*Matrix*' or '*predicative function*' is a primitive idea.
>
> The fact that a function is predicative is indicated, as above, by a note of exclamation after the function letter.
>
> The variables occurring in the present work, from this point onward will all be either individuals or matrices of some order in the above hierarchy.[117]

The problem is that this seems to characterize a narrower notion of predicative function, and thus a different hierarchy than that of the Introduction. In fact, the authors are the first to note that they provide different explanations of the hierarchy in different portions of the work, and explicitly endorse the version given in ∗12:

> The explanation of the hierarchy of types in the introduction differs slightly from that given in ∗12 of the body of the work. The latter explanation is stricter and is that which is assumed throughout the rest of the book.[118]

This difference led several authors - perhaps the most prominent example being Alonzo Church - to argue that there are two senses of "predicativity" at work in *Principia* - one much more restrictive than the other[119]. As Charles Chihara explains, the difference seems to be that "in ∗12, we are told that predicative functions are matrices, that is not all first-order propositional functions would be predicative", while "in the introduction predicative functions of individuals are those that take individuals as arguments and that contain no bound variables that range over propositional functions"[120]. But if the claim that the latter, more restrictive notion of predicativity is the one used throughout the work, what are we to make of the exposition in the Introduction?

Thus, to sum up: there are three fundamental interpretative issues about *Principia*'s type hierarchy. The first concerns the restriction to predicative variables: according to Whitehead and Russell, the only functional variables admitted to occur bound to quantifiers are predicative; but this is a problematic restriction if the circumflex is viewed as an abstraction or term forming operator, so is this restriction really intended? The second problem, then, is just this: should the

117 WHITEHEAD & RUSSELL, 1925, p.164 [1910, p.172]
118 WHITEHEAD & RUSSELL, 1925, p.vii [1910, p.viii]
119 CHURCH, A., 1976a, p.747-8.
120 CHIHARA, C., 1973, p.22.

circumflex be interpreted as a term-forming operator? The third concerns the notion of *predicative propositional function*: are there really two distinct accounts of this important notion, one in the Introduction and another on $*12$? And if there are, which one is that intended by the authors?

4.3.1 The Church-Linsky Reconstruction of the Hierarchy of Types

Church formulated what is generally considered the standard, or orthodox, reconstruction of the ramified theory of types of *Principia Mathematica*. The basic feature of Church's version of ramified type theory is that variables are assigned ramified types (*r*-types for short) according to the domain over which they range: individual variables range over individuals, while functional variables of a certain *r-type* range over propositional functions (understood as *abstracta*) of matching *r-type*.

Church also "take[s] propositions as values of the propositional variables on the ground that this is what is clearly demanded by the background and purpose of Russell's logic"[121] despite the authors explicit claim "that what [they] call a 'proposition' (in the sense in which this is distinguished from the phrase expressing it) is not a single entity at all"[122].

The *r-types* of variables and constants are recursively defined as follows[123]: (i) There is an *r*-type *i* to which every individual variable (or constant) belongs; (ii) If $\ulcorner t_1, \ldots, t_m \urcorner$ are *r*-types and $m \geq 0$, then there is an *r*-type $\ulcorner (t_1, \ldots, t_m)/n \urcorner$ of *level n* where $n \geq 1$, to which every functional variable (or constant) of level *n* belongs; (iii) There is an *r*-type $\ulcorner ()/n \urcorner$ (or $\ulcorner 0/n \urcorner$, for short) to which every propositional variable (or constant) of level *n* belongs; (iv) There are no other *r*-types. Church defines an *r*-type $\ulcorner (t_1, \ldots, t_m)/k \urcorner$ as *directly lower than* the *r-type* $\ulcorner (v_1, \ldots, v_m)/n \urcorner$ if $\ulcorner t_1 = v_1 \urcorner, \ldots, \ulcorner t_m = v_m \urcorner$ and $k < n$; and an *r*-type *v* is *directly higher* than t when t is directly lower than *v*[124]. Also, the *order of a variable or constant* is recursively defined as follows[125]: (i) An individual variable (or constant) is of order 0; (ii) The order of a variable (or constant) of *r*-type $\ulcorner (t_1, \ldots, t_m)/n \urcorner$ is *N*+*n* where N is the greatest of the orders of $\ulcorner t_1, \ldots, t_m \urcorner$ and if $m = 0$, then $N = 0$. Finally, a variable of *r*-type $\ulcorner (t_1, \ldots, t_m)/k \urcorner$ is said to be *predicative* if and only if its level $k = 1$. As the reader can surely attest, there is some terminological divergence between Church's treatment and that of *Principia*, the main one being that Church's notion of *level* somewhat substitutes Whitehead and Russell's notion of *order*[126]. As Landini observes, if levels *k* are kept equal to 1, then "Church's notation is just a variant

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[121] CHURCH, A., 1976a, p.748, footnote 4.
[122] WHITEHEAD & RUSSELL, 1925, p.44 [1910, p.46]
[123] CHURCH, A., 1976a, p.748.
[124] CHURCH, A., 1976a, p.748.
[125] CHURCH, A., 1976a, p.748.
[126] See CHURCH, A., 1976a, p.747, footnote 2. Perhaps the best guide for the maze of terminological distinctions one finds in the literature on Ramified Type Theory is given by Allen Hazen (HAZEN, A., 1983, p.397, footnote 7). One must be careful with Hazen's historical remarks on *Principia*, however, for they are informed by Church's interpretation.

of the notation of simple-types"[127]. Thus, Church's hierarchy (with *levels* being equivalent to Whitehead and Russell's *orders*) follows the account of predicativity given in the Introduction, rather than that of $*12$.

In terms of its grammar, what mainly characterizes Church's formulation of type theory is that an atomic formula $\phi(x_1, ..., x_n)$ is well formed if and only if ϕ is a functional variable or constant of some *r-type* $\ulcorner(t_1, \ldots, t_n)/k\urcorner$, with $k > 0$ and $x_1, ..., x_n$ are such that each respective *r-type* v_j of x_j for $j \leq n$ is such that v_j is either equal to, or directly lower than, the *r-type* t_j, that is: $\ulcorner\phi^{(t_1,...,t_m)/n}(\alpha_1^{t_1}, ..., \alpha_m^{t_m})\urcorner$ is well-formed if and only if for each $j \leq n$ of the respective *r*-types $\ulcorner t_1, \ldots, t_n\urcorner$ of $\alpha_1, ..., \alpha_n$, either $v_j = t_j$ or the *r*-type is of t_j directly than that of v_j[128].

Thus, these definitions characterize a *cumulative* hierarchy of *r*-types: the domain of a variable α of *r*-type directly higher than that of a variable β will contain the domain of β. And non-predicative variables are allowed. This is already a clear departure from Whitehead and Russell's original formulation[129].

Church also introduces an explicit (predicative) axiom scheme of comprehension for propositional functions[130]:

$$\ulcorner(\exists\phi^{(t_1,...,t_m)/n})(\alpha_1^{t_1}), ..., (\alpha_m^{t_m}) : \phi^{(t_1,...,t_m)/n}(\alpha_1^{t_1}, ..., \alpha_m^{t_m}) \,.\equiv.\, A\urcorner,$$

where A is a well-formed formula such that: (i) the bound variables of A are all of order less than *n*; (ii) the constants occurring in A are of order not higher than *n*; and (iii) the free variables of A are of order not higher than *n*. And also an (explicit) axiom of comprehension for propositions[131]

$$\ulcorner(\exists p^{()/n}) : p^{()/n} \,.\equiv.\, A\urcorner,$$

where A is a well-formed formula such that: (i) the bound variables of A are all of order less than *n*; (ii) the constants occurring in A are of order not higher than *n.*[132]

[127] LANDINI, G., 2011a, p.113.
[128] CHURCH, A., 1976a, p.748.
[129] However, it must be observed at once that Church was completely aware that his formulation of type theory was *not* historically accurate. Church's paper follows the account he had given previously in his spectacular *Introduction to Mathematical Logic* (CHURCH, A., 1956), albeit with slight modifications (see CHURCH, 1976, p.747, footnote 2); there he claims explicitly that his exposition "[...] is not a reproduction of Russell's own account of semantical matters [...] But it is rather a first step of a contemplated attempt to fit the ramified functional calculi into our own semantical program and to provide for them "semantical rules" of a kind to which Russell would certainly not consent." (CHURCH, A., 1956, pp.347-8, footnote 577).
[130] CHURCH, A., 1976a, p.749.
[131] CHURCH, A., 1976a, p.749.
[132] Since this axiom introduces only propositions which are logically equivalent to what is expressed by some given formula **A**, it seems a stronger principle would be required in order to deal with intensional contexts (as in "Alexander doubts every proposition which Xerxes believes", etc.). Moreover, since Whitehead and Russell explicitly claim that only variables and constants can be flanked by identity, Church suggest in an earlier paper that a primitive connective for *propositional* identity should be introduced with accompanying axioms. For details see CHURCH, A., 1974 and also CHURCH, A., 1984. A critical discussion of this proposal and consideration of intensional logics inspired by Russell's views can be found in ANDERSON, A., 1986, 1989 and 1998.

As Church puts it, the core assumption of his reconstruction is that *Principia*'s system is "best represented by supposing sentences to denote propositions, taking propositions as values of the propositional variables, and properties and relations in intension as values of the functional variables"[133] - while allowing both predicative and non-predicative functional variables (in his sense that intends to capture that of *Principia*'s Introduction).

To complete his reconstruction, Church also introduces axioms of reducibility that mitigate the effects of the genuine *ramification* that characterizes his system, in terms of the following scheme[134]:

$$(\psi^{(t_1,\ldots,t_m)/n})(\exists\phi^{(t_1,\ldots,t_m)/1})(\alpha_1^{t_1},\ldots,\alpha_m^{t_m}) : \psi^{(t_1,\ldots,t_m)/n}(\alpha_1^{t_1},\ldots,\alpha_m^{t_m}) \,.\equiv.\, \phi^{(t_1,\ldots,t_m)/1}(\alpha_1^{t_1},\ldots,\alpha_m^{t_m})$$

This just asserts that given any *m*-ary function ψ of an arbitrary level *n*, there is a function ϕ co-extensive with ψ but of level 1; since the range of the functional variables of level 1 coincide with those of the simple theory of types, the introduction of the axiom scheme of Reducibility amounts, in practice, to abandoning the constraints of ramification altogether for *extensional* contexts.[135].

Sharing Church's assumption that *Principia* requires an ontology of propositions, Bernard Linsky has put forward an interpretation of the notion of propositonal function which seeks to vindicate Church's reconstruction in accordance with Russell's views on Logic and Metaphysics - in particular, Linsky seeks to reconcile the use of apparent propositonal variables with the multiple-relation theory of judgment. The core of Linsky's reading is that both propositions and propositional functions are logical constructions, and thus, must be understood as "structured entities"[136]. So, for instance, "a function $\hat{x}Rb$ will have the value aRb for argument" and "the structure of the function, including its constituents *R* and *b*, is carried along into the proposition which is its value"[137]. Thus, type distinctions are meant to reflect the structural complexity of a function or proposition: these entities have constituents, which are the building blocks from

133 CHURCH, A., 1956, pp.347-8, footnote 577.

134 CHURCH, A., 1976a, p.758.

135 This, however, should not be overstated; as Church observes, "the rejection of impredicative definition is annulled in extensional but not in intensional matters" (CHURCH, A., 1976a, p.758). In a footnote (CHURCH, A., 1976a, p.758, footnote 25) of his beautiful paper, Church makes a decisive point against a huge tradition of authors that claim that reducibility simply collapses ramified logic into simple type theory, thus fully restoring impredicativity. This, as it happens, is only true if we are *not* working within an intensional logic that intends to capture formally the behavior of locutions like "*Prior* believes that something he believes is false", "George IV believes something about Scott", etc.; the effects of Reducibility are simply irrelevant in such contexts where co-extensive functions and/or propopositions are *not* substitutable *salva veritate*. This is of some importance since, as Church also observed (CHURCH, A., 1974, pp.23-4) the ramified theory of types can be a very useful and sound philosophical tool for dealing with intensional paradoxes or so called *blindspots*, like Fitch's paradox of knowability and Moore's paradox and also paradoxes like the *Preface Paradox* (see, for instance, PRIOR., 1971, pp.84-89). On this topic, a small but growing literature has began to emerge since Bernard Linsky's *Factives, Blindspots and Some Paradoxes* (LINSKY, B., 1986); recently many have considered the topic both in favor or against the approach. See, for instance WILLIAMS, T., 2000, pp.280-3; PASEAU, A., 2008; LINSKY, B., 2009; CARRARA & FASSIO, 2011; RACLAVSKÝ, J., 2018.

136 LINSKY, B., 1999, p.77.

137 LINSKY, B., 1999, p.77.

which they are constructed and these constituents themselves may or may not be constructed entities, and so on, until we arrive at simple entities, namely, universals and particulars. And since any function or proposition is constructed from its constituents, their types "must be as high as any of its constituents"[138].

Accepting Church's claim that *Principia* requires a realist background, the three interpretative difficulties we found are considered by Linsky to be minor issues of exposition. Concerning the restriction to predicative variables, he writes that:

> While troubling, Russell's claim seems better interpreted as an oversight than as an expression of a startling logical doctrine. Russell clearly intends to show how the whole hierarchy of types is generated by an iterative process. For each type of argument, he thinks, we derive the ramification of the theory of types by quantifying over some variables. At each level a quantifier over all propositions or propositional functions raises the type of a proposition or function, even when its arguments are still of a lower type. Russell hypothesizes that this quantification which requires a change in type to observe the vicious circle principle is in fact minimal; it is the *only* way that higher orders functions can be derived.[139]

Linsky thinks the restriction should be seen as a mere "oversight" which results from Russell overlooking "the role of abstraction in producing new functions"[140], since he also takes it as unproblematic that the circumflex should be understood as a term-forming operator (this is understandable, since the comprehension axioms introduced by Church easily yield concretion principles required to treat the circumflex as an abstraction or term-forming operator). Indeed, Linsky also takes the divergence between the notions or predicativity from the Introduction and $*12$ as being "ultimately a minor issue that does not challenge the coherence of PM"[141], for two reasons. The first is that Linksy takes the issue as moot given the presence of axioms of reducibility; as he explains, though the strength of the axiom would vary according to which notion of predicativity is adopted, the end result would be the same, since every non-predicative function would end up having a predicative equivalent[142]; the second is not explicitly recognized by Linsky, but clearly follows from his account: if Russell was led to the predicativity constraint due to an oversight about "the role of abstraction in producing new functions"[143], assuming that all functions could be obtained from quantification over predicative functions, then the formulation of predicativity in $*12$ could indeed be just a result of this same oversight. There are serious reasons, however, to contest this account.

138 LINSKY, B., 1999, p.77.
139 LINSKY, B., 1999, pp.81-2.
140 LINSKY, B., 1999, p.82.
141 LINSKY, B., 1999, p.85.
142 LINSKY, B., 1999, p.85.
143 LINSKY, B., 1999, p.82.

4.3.2 Landini's Reconstruction of *Principia*'s Theory of Types

4.3.2.1 *Principia*'s Grammar as that of Simple Type Theory

Landini has provided substantive arguments to regard such reconstruction as historically inadequate and unfaithful to central aspects of *Principia*'s Logic. The claims that form the core of Landini's interpretation are the following:

1. There are no variables in *Principia* which are non-predicative: so called non-predicative functions are meant as schemata. That is: arguments to a predicate variable must agree in order\type; all predicate variables are predicative; and (Predicative) predicate variables must take predicative arguments.
2. Circumflexion was not a term forming operator, being "used only as a heuristic device in introducing and explaining the system"[144].
3. The Axiom (Scheme) of Reducibility is introduced as an impredicative comprehension axiom which assures that given any well-formed formula $A[x_1^{t_1}, ..., x_n^{t_n}]$, there is a predicative function $\phi!$ such that, for all $x_1^{t_1}, ..., x_n^{t_n}$, $\phi!(x_1^{t_1}, ..., x_n^{t_n})$ if, and only if $A[x_1^{t_1}, ..., x_n^{t_n}]$.
4. The intended official grammar - i.e., the grammar which is employed and presupposed throughout *Principia*'s numbered propositions - for the predicate variables or so-called 'predicative propositional functions' is that of the simple theory of types.
5. The discussion of types in the Introduction of *Principia* backed by the account of quantification in $*9$ presents Russell's attempt to provide an *informal* nominalistic substitutional semantics for the grammar of simple types which the body of the work employs. This nominalistic semantics, however, is *not* part of the formal system.

His reconstruction is evidently in stark constrast with those of Church and Linsky. To see how Landini argues in favor of these claims, we must discuss in some detail his reconstruction of *Principia*'s formal grammar and how it dissolves many of the issues which we discussed earlier.

In place of Church's *r-types*, Landini offers the following recursive characterization of *order\type symbols*[145]. According to Landini's reconstruction, the type aspect of an order\type symbol is any expression which satisfies the following recursive definition[146]: (i). "*o*" is an order\type symbol; If $\ulcorner t_1, \ldots, t_n \urcorner$ are order\type symbols, then $\ulcorner (t_1, \ldots, t_n) \urcorner$ is an order\type symbol; and finally, nothing else is an order\type symbol; also, the *order* of an order\type symbol is given by the following recursive definition[147]: the type symbol "*o*" has order 0; a type

144 LANDINI, G., 1998a, p.265.
145 Landini's rendition draws largely from William Hatcher's presentation of the theory of types in HATCHER, W., 1980.
146 LANDINI, G., 1998a, pp.255; also LANDINI, 2010, p.107.
147 LANDINI, G., 1998a, pp.255-6; also LANDINI, 2010, p.107.

symbol $\ulcorner(t_1, \ldots, t_m)\urcorner$ is of order $n+1$, where *n* is the highest order among the type symbols $\ulcorner t_1, \ldots, t_m\urcorner$.

A fundamental consequence of the above recursive characterization of type symbols is "that arguments to a predicate variable must match in order\type"[148]. And, in fact, what is characteristic of Landini's reconstruction of the syntax of *Principia* is that all variables are predicative with all atomic well-formed formulas having the following form:

$$\phi^{(t_1,\ldots,t_n)}(x^{t_1}, \ldots., x^{t_n})$$

Thus, according to his interpretation, *Principia*'s predicate variables are regimented by the structure of simple type-theory (with heterogeneous relations). In fact, to make this as clear is it can be made, the notation which Landini officially adopts for his reconstruction drops the ϕ's, ψ's, f's, g's, etc and simply puts adequate superscripts in lower-case italic letters, as in $x^{(o)}$, $x^{((o))}$, $x^{(o,o)}$ $x^{((o),o)}$, $x^{(o,(o))}$ $x^{((o),(o))}$. Thus, what Whitehead and Russell would express as "$\phi!x$" or "$\psi!(x,y)$" and "$f!(\phi!\hat{z})$" or "$g!(\phi!\hat{z},x)$" etc... Landini expresses as $x^{(o)}(x^o)$, $x^{(o)}(x^o)$, $x^{(o)}(x^o, z^o)$, $x^{((o))}(x^{(o)})$, $x^{((o),o)}(x^{(o)}, x^o)$ etc. To facilitate readability, Landini adopts the (often convenient) practice of separating the order aspect of an order\type symbol and of introducing Greek letters for (unofficial) predicate variables. Take the following, for instance:

"x^o" this is an individual variable of the lowest type; it can be re-written as ${}^o x^o$.

"$x^{(o)}$" this is a monadic predicate variable of the lowest type; it can be re-written as ${}^1\phi^{(o\backslash o)}$.

"$x^{(o,o)}$" this is a dyadic predicate variable of the lowest type; it can be re-written as ${}^1\phi^{(o\backslash o,o\backslash o)}$.

"$x^{((o))}$" this a monadic predicate variable which takes as arguments monadic predicate variable of the lowest type; it can be re-written as ${}^2\phi^{(1\backslash(o\backslash o))}$.

"$x^{((o),o)}$" this is a dyadic predicate variable which takes as arguments predicate variable of the lowest type and individual variables; it can be re-written as ${}^2\phi^{(1\backslash(o\backslash o),o\backslash o)}$.

and so on...

As we mentioned above, the axiom of Reducibility is thus understood as the (only) comprehension axiom of the work, and rendered in terms of the following scheme[149]:

$$(\exists\phi^{(t_1\ldots,t_n)})(\beta^{t_1}, \ldots, \beta^{t_n}) : \phi^{(t_1\ldots,t_n)}(\beta^{t_1}, \ldots, \beta^{t_n}) \,.\equiv.\, \mathrm{A},$$

where $\phi^{(t_1\ldots,t_n)}$ is not free in A. This, again is in accordance with Landini's claim that the grammar of *Principia* is that of simple theory of types with predicate variables that admit arguments

148 LANDINI, G., 1998a, p.256.
149 LANDINI, G., 1998a, p.257; also LANDINI, G., 2010, p.109.

of heterogeneous types: here the axiom (scheme) of Reducibility plays exactly the role of an impredicative comprehension axiom for its predicate variables.

Crucially, according this reconstruction *Principia*'s original so-called 'propositional variables' like p, q, r,etc., and so called 'real (free) functional variables', ϕ, ψ, f etc., (without the the shriek "!") are not variables at all. They are all schemata which stand for arbitrary well-formed formulas: 'propositional variables' are read as schematic letters for sentences while 'non-predicative functional variables' are schematic letters for open formulas. In other words, Landini's reconstruction draws a fundamental distinction which he claims to be overshadowed in *Principia*'s original presentation "by a particularly injurious misuse of typical ambiguity in *Principia*"[150]. As he explains:

> Typical ambiguity is only intelligible as a convention of suppressing order\type symbols; the formal language and deductive system must be statable without them. One cannot have indices on bound variables but drop them from free variables in theses, else the ramified type regimented rules of universal generalization and principles like universal instantiation become impossible to state. All free variables must come with indices. Whitehead and Russell's typically ambiguous variables must be understood to be schematic letters for formulae.
>
> Accordingly, a letter such as ϕ must not play a dual role - as predicate variable and as schematic letter. In our reformulation, all variables, bound or free, are alike fixed in order\type. It is schematic letters for *wffs*, letters such as "*A*", "*B*", "*C*", that realize Whitehead and Russell's use of the typically ambiguous "any".[151]

The first point made here is beyond any reasonable criticism: many of the imprecisions of *Principia*'s exposition of type theory arise because of Whitehead and Russell's reliance on typical ambiguity from the get-go instead of setting their notation for type distinctions from the outset with rules for suppressing them. The second point is the most controversial aspect of Landini's reading: according to him, there are two completely distinct uses of the term "propositional function" in *Principia*. In one sense, "propositional function" means a predicate variable that can be bound by quantifiers; in another sense, "propositional function" means a *schematic letter*.[152]. In other words, Landini fully accepts Quine's claim that the expression "propositional function" plays a dual role, but he does not accept that Russell was simply confused.

Now, what motivates such a reconstruction of *Principia*'s formal grammar? Put generally, Landini claims that the ambiguous use of the notion of 'propositional functionality' is more reasonably viewed as resulting from *carelessness* on Russell's part, not confusion. But Landini shows in meticulous detail that on the basis of the above interpretation he can dissolve many persistent interpretative problems that arise from the presentation of type theory in *Principia*'s Introduction, including those which we discussed in the previous section.

[150] LANDINI, G., 1998a, p.262.
[151] LANDINI, G., 1998a, p.263.
[152] Cf. LANDINI, G., 1998a, p.257; also LANDINI, G., 2010, pp.109-10.

Taking 'propositions' to be *sentences*, Landini can reasonably claim that when Whitehead and Russell assert that "*x* is "individual" when it is neither a proposition nor a function"[153] and that " "Matrix" or "predicative function" is a primitive idea"[154], they mean, respectively, that "individual variables are primitive symbols of the language"[155] and that "predicate variables are primitive signs of the formal language"[156].

Landini also reads the definition of predicativity of ∗12, i.e., that "a function is said to be predicative when it is a matrix" together with the claim that "the variables occurring in the present work [...] will all be either individuals or matrices"[157] as asserting that "the only terms of the language are to be individual variables and predicate variables; and the predicate variables have order\type indices in accordance with the definition of predicativity"[158]. This dissolves the problem of determining which definition of predicativity is the correct one: on this reading, the one presented in the introduction and in ∗12 coincide, since a function is a matrix when, and only when, it is a predicative variable, which, together with the matching requirement - i.e., the requirement that arguments to a predicate variable must agree both in order *and* type - means that all such functions are of the next order above that of their arguments. On his reading, the two definitions agree completely.

Concerning the role of the circumflex, if Landini's theses about the syntax of *Principia* are correct, he has a conclusive argument to deny that the circumflex notation was intended by Whitehead and Russell to be a term-forming operator: sharply distinguishing between schematic letters and predicate variables in the way that Landini does makes circumflexion superfluous. Furthermore, as Landini puts it, "since circumflexion always involves capping a variable, there can be no predicate terms which take a non-predicative term as an argument"[159]; this means that the problematic question we raised before: "how could the rules of universal generalization and instantiation be applied to non-predicative terms?" can be turned as an argument *against* circumflexion - such inference rules would simply *not* be applicable to non-predicative terms for the good reason that these are not terms at all![160].

4.3.2.2 The Semantic Justification for Types and Orders in *Principia*

The cogency of Landini's reconstruction of *Principia*'s syntax is not his only argument in favor of it. To support his interpretation, Landini also provides textual evidence which strongly indicates that *Principia*'s Introduction is an attempt to provide an informal *semantic* justification

[153] WHITEHEAD & RUSSELL, 1925, p.132 [1910, p.138]
[154] WHITEHEAD & RUSSELL, 1925, p.164 [1910, p.172]
[155] LANDINI, G., 1998a, p.263.
[156] LANDINI, G., 1998a, p.263.
[157] WHITEHEAD & RUSSELL, 1925, p.164 [1910, p.172]
[158] LANDINI, G., 1998a, p.263-4.
[159] LANDINI, G., 1998a, p.265.
[160] LANDINI, G., 1998a, pp.265-6.

for the grammar of simple types[161].

Following the lead of Kripke and Sainsbury[162], Landini argues that predicate variables are meant to be interpreted in terms of a nominalistic *substitutional* semantics. This claim is based on his reading of Whitehead and Russell's justification for the distinctions of *types* and *orders*, discussed, respectively in the sections *Why a Given Function Requires Arguments of a Certain Type* and *The Hiearchy of Functions and Propositions* of *Principia*'s Introduction. Following a terminology introduced by Charles Chihara, we may call the argument in favor of type-distinctions "the argument of direct inspection"[163], which is presented by Whitehead and Russell in the following terms:

> [...] direct consideration of the kinds of functions which have functions as arguments and the kinds of functions which have arguments other than functions will show, if we are not mistaken, that not only is it impossible for a function $\phi\hat{z}$ to have itself or anything derived from it as argument, but that if $\psi\hat{z}$ is another function such that there are arguments a with which both "ϕa" and "ψa" are significant, then $\psi\hat{z}$ and anything derived from it cannot significantly be arguments to $\phi\hat{z}$. This arises from the fact that a function is essentially an ambiguity, and that, if it is to occur in a definite proposition, it must occur in such a way that the ambiguity has disappeared, and a wholly unambiguous statement has resulted.[164]

The argument is "by direct inspection" because Whitehead and Russell think some crucial examples are enough to establish the above, general, point. They take the function "x is a man" and consider the possibility of stating, significantly, that "$\phi\hat{z}$ is a man"; according to them, in such an assertion:

> [...] there is nothing definite which is said to be a man. A function, in fact, is not a definite object which could be or not be a man; it is a mere ambiguity awaiting determination, and in order that it may occur significantly it must receive the necessary determination.[165]

Since the authors explicitly state that significance and thus, presumably, ambiguity, is a property of *signs* or expressions, a function in this context must be understood as an expression. As Landini points out, the best way to make sense of such a claim is by taking the letters ϕ, ψ, etc., to be interpreted in terms of a nominalistic semantics where every substitution must be made as if the expression "$\phi\hat{z}$" occurs as "falling *within*" the context in which it is substituted,

---

[161] Landini has put forward his intepretation in great detail in his books and a number of papers (LANDINI, G., 1998a; 2003; 2007; 2011; 2013;) Here I will follow what I take to be the best known exposition of his views, namely that of his *Russell's Hidden Substitutional Theory* (LANDINI, G., 1998a), but we will also draw some examples from his LANDINI, G., 2010 adapting the variations of notation to fit to his former work.

[162] KRIPKE, S, 1976, p.367-8; SAINSBURY, M, 1980, pp.31-3.

[163] CHIHARA, C., 1973, p.23-4.

[164] WHITEHEAD & RUSSELL, 1925, p.47 [1910, p.50]

[165] WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.50]

and *not* "falling under" as a term in subject position[166]. Thus, for instance, if "*x* is a man" is assigned to "$\phi\hat{x}$", the attempt to assert "$\phi(\phi\hat{x})$" would result in the following:

" *x* is a man is a man" or " ... is a man is a man"

both of which are simply nonsense. But as Landini observes, if "ψ" is assigned as "$(x)(...x...)$", which is of a higher-type than $\phi\hat{x}$, then to assert "$\psi(\phi\hat{x})$" is to assert "(x) . ϕx". As Landini puts it, it is in this way that "type indices on the predicate variables of the formal grammar are thereby philosophically justified"[167].

Landini's interpretation of how Whitehead and Russell justify the introduction of *orders* relies on how he interprets the hierarchy of different senses of "truth" and "falsehood" applied to elementary propositions, first-order propositions, and so on given in *Principia*'s Introduction. Landini takes the passage introducing the hierarchy to be, "unquestionably", "a recursive definition"[168], where the basis of the recursion is given in terms of the multiple-relation theory of judgment applied to atomic judgments. For Whitehead and Russell start by defining the truth of sentences like "*aRb*" which express elementary judgments in terms of the existence of a corresponding complex $a - R - b$ (the same applying for judgments involving any relation *R* and any combination of individuals $a_1, ..., a_n$). Thus, Landini takes Whitehead and Russell to be "offering a philosophical explanation and justification of the order part of the order\type indices on predicate variables by appeal to an informal semantics"[169] which is ultimately grounded in the multiple relation theory of judgment.

Put in terms of *Principia*'s semantic metalanguage, Landini claims that their idea is that if we go from 'top to bottom' eliminating the truth predicates of quantified formulas of order *n* in terms of some substitutional class of formulas of order $n-1$, no truth-predicate of order *n* or higher should remain. As he explains:

> [...] the hierarchy of senses of "truth" and "falsehood" are generated in such a way that they step downward, one level at a time, until the base notions of "truth" and "falsehood" for atomic *wffs* are reached. Since the truth-definition is offered as a semantic justification for the introduction of predicate variables with order\type indices, we get a philosophical explanation of the "matching requirement" - that is, the formal requirement that arguments to a predicate variable must match in their order indices as well as their type indices. The definition defines "truth$_n$" as applied to *wffs* containing at least one quantifier binding variables of order $n-1$ and no quantifiers binding variables of higher order; and "truth$_n$" means that in removing this quantifier (or all such if there are many) the resulting *wff* has "truth$_{n-1}$". If we could instantiate the variables

166 LANDINI, G., 1998a, p.279-80.
167 LANDINI, G., 1998a, p.280.
168 LANDINI, G., 1998a, p.283.
169 LANDINI, G., 1998a, p.286.

> of order $n-1$ to, say, $n-2$, the definition of "truth$_n$" would be lost. Higher senses of "truth" (and "falsehood") are recursively defined in such a way that they collapse step-by-step to the base of atomic "truth" (and "falsehood"). The matching requirement, therefore, is a product of the semantic justification - the recursive definition of "truth" set out in the Introduction of *Principia*.[170]

Landini takes distinctions of order to be grounded on a semantic interpretation for *Principia*'s variables that gives rise to a (recursively characterized) hierarchy of predicates of truth and falsehood. Landini is thus claiming that Whitehead and Russell had a *semantic* justification for their hierarchy of orders and the matching requirement: on their intended semantics for predicate variables, they are interpreted substitutionally and order distinctions determine which substitutions are legitimate for *formulas* within formulas[171]. At the basis of this recursive characterization is the multiple-relation theory of judgment applied to atomic judgments. Roughly, his recursive clause for quantified sentences of the form $(F)A$ is something along the following lines:

> $(F)A$ is true$_q$ if and only if $(F)(\ulcorner A \urcorner$ is true$_p)$, where *p* and *q* are appropriate indices for truth-predicates.

[170] LANDINI, G., 1998a, p.286.

[171] Indeed, on the basis of Whitehead and Russell's sketch of the hierarchy of the Introduction, Landini provides a full semantic interpreation for *Principia*'s language, on the basis of which he formulates a full recursive definition of truth and falsehood. Individual variables are interpreted *objectually* and functional variables interpreted *substitutionally.* In order to explain how this was meant to work in further detail, Landini introduces what he calls "very finely grained indices" that track what sort of formula can be substituted for functional variables of a determinate order\type. Though the full details need not detain us, looking at (and explaining) some examples will help. Landini puts, for instance "${}^{1.s;e.a}\phi^{(o\backslash o)}$" for "${}^{1}\phi^{(o\backslash o)}$"; this is a predicate variable that can only be substituted for a wff which contains exactly *m*-many bound individual variables and *a*-many quantifier-free propositional subordinate clauses (elementary propositions); ${}^{2.t;1.s;e.a}\phi^{(1\backslash(o\backslash o))}$ for ${}^{2}\phi^{(1\backslash(o\backslash o))}$; this functional variable can only be substituted for a wff which contains exactly exactly, *t*-many bound functional variables (of order\type $1\backslash(o\backslash o)$), *s*-many bound individual variables and *a*-many quantifier-free propositional subordinate clauses (elementary propositions);${}^{3.v;2.t;1.s;e.a}\phi^{(2\backslash(1\backslash(o\backslash o)))}$ for ${}^{3}\phi^{(2\backslash(1\backslash(o\backslash o)))}$; this functional variable can only be substituted for a wff which contains exactly exactly, *v*-many bound functional variables (of order\type $2\backslash(1\backslash(o\backslash o))$), *t*-many bound functional variables (of order\type $1\backslash(o\backslash o)$), *s*-many bound individual variables and *a*-many quantifier-free propositional subordinate clauses (elementary propositions); and so on. Thus, an atomic well-formed formula has the form ${}^{1.0;e.1}\phi^{(o\backslash o,\ldots,o\backslash o)}(x_1^o, \ldots, x_n^o)$. Landini is aware that his "very finely grained indices" were not adopted by Whitehead and Russell, and if they were, some complications would have to be introduced in *Principia*'s official grammar. His order indices keep track of the number *and* order of occurrences of quantifiers in a given formula; but as he observes, in practice, Whitehead and Russell meant for order indices on predicate variables to "[...] reflect *only* the order of truth in the recursion, first, second, third, etc., and *not* also the suborders within first, second, third, etc., which reflect the number of quantifiers of the given order" (LANDINI, G., 1998a, p.285. Our emphasis.). It is at this point that Landini's reconstruction also relies heavily on his interpretation of $*9$. In Landini's reconstruction of *Principia*'s informal semantics, the clauses where *A* is possibly a quantified formula, *A* must be in prenex normal form, since, as Landini puts it, "the recursion would not apply to wffs with quantified wffs occurring subordinate to a propositional connective". This fits perfectly with the core idea of *Principia*'s quantification theory of $*9$ which is meant to show how, by means of matrices, i.e., functions with no apparent variables, "propositions of any order can be reached" (WHITEHEAD & RUSSELL, 1910, p.133.).

With respect to predicate variables like "$\phi^{(o)}$", $\phi^{(o,o)}$, etc., which have either individual variables or constants as their range of arguments, the semantic interpretation founded on the multiple-relation assigns to these variables universals occurring as concepts or relating relations in complexes like *a-in-the-relation-R-to-b* or *a-having-the-propery-F*. But predicate variables of higher types are to be interpreted in substitutionally.

According to Landini's reconstruction, if we take, say, a typically ambiguous sentence of *Principia* like "$(\exists x)\phi!x$", this must be read as a scheme for formulas of *Principia*'s object language which can be obtained by restoring simple type-indices. Using Landini's preferred notation, we have, for instance:

$$(\exists x^{o})y^{(o)}(x^{o}) \tag{4.3.1}$$

$$(\exists x^{(o)})y^{((o))}(x^{(o)}) \tag{4.3.2}$$

$$(\exists x^{((o))})y^{(((o)))}(x^{((o))}) \tag{4.3.3}$$

Of course, as we just mentioned, instead of restoring type-indices, Whitehead and Russell employed the device of type-ambiguity, indicating differences of type in given contexts by employing different sorts of variables. So they would express 4.3.1, 4.3.2, 4.3.3, respectively, as something like the following:

$$(\exists x) \,.\, \phi!x, \tag{4.3.4}$$

$$(\exists \phi) \,.\, f!(\phi!\hat{x}) \tag{4.3.5}$$

$$(\exists g) \,.\, g!\{f!(\widehat{\phi!x})\} \tag{4.3.6}$$

According to the nominalist semantics which Landini's claims to be put forward in *Principia*'s Introduction, a sentence like 4.3.4 is true if there is some individual *a* to which some universal *U* assigned to "$\phi!$" applies to; a sentence like 4.3.5, in turn, is true if there is a universal *U* assigned to "$\phi!$" which applies to x^{o}. In sentences like 4.3.5 and 4.3.6 however, predicate variables are explained in terms of a substitutional semantics. Simplifying things, a formula like 4.3.6 is true if *some* possible substitution (for instance, putting "$(x) \,.\, \phi!x$" or "$(\exists x) \,.\, \phi!x$" in place of "$f!(\phi!\hat{x}$")) is true. And the same would apply to formulas of increasing complexity containing variables of yet higher types. For instance, the truth-conditions of a sentence which involves quantification over predicate letters of type (((o))), like:

$$(g)(\exists f) \,.\, g!\{(f!(\widehat{\phi!x}))\} \tag{4.3.7}$$

Will be given in terms of the possible substituends of formulas *within* this given formula. So 4.3.7 will be true if and only if *each* formula like:

$$(\exists f)[(\phi)(f!(\phi!\hat{x})]$$

$$(\exists f)[(\exists \phi)(f!(\phi!\hat{x}))]$$

etc., is true; while, say:

$$(\phi)(f!(\phi!\hat{x})$$

will be true iff *each* formula like:

$$(\phi)(x)(\phi!x)$$

$$(\phi)(\exists x)(\phi!x)$$

etc., is true. The point is that the senses of "truth" applicable to these sentences containing bound variables of higher types will, so to speak, collapse from top-to-bottom: the truth conditions of any sentence to which a given sense of truth is applicable will be given by some set of sentences to which a 'lower' sense of truth is applicable. This, according to Landini is the general idea behind the hierarchy of senses of truth and falsehood of *Principia*'s Introduction.

4.4 A Critical Discussion of the Approaches

4.4.1 The Multiple-Relation Analysis of judgment

We can summarize the points of agreement and disagreement between Cocchiarella, Linsky and Landini on the issues we have been considering so far with the following chart:

Ontology of *Principia*	Cocchiarella	Linsky	Landini
'Propositions' as entities:	No	Yes (Constructed)	No
Sentences as incomplete symbols:	Yes	Yes	Yes
'Propositional functions' as entities:	Yes	Yes (Constructed)	No
'Propositional functions' as universals:	Yes	No	No
Hierarchy of complexes	Yes	Yes	No
Syntax of *Principia*			
'Predicativity' constraint:	No	No	Yes
Two senses of 'predicativity':	Yes	Yes	No
'Predicativity' of the Introduction	Yes	Yes	Yes**
'Predicativity' of $*12$:	No	No	Yes**
Circumflex as a term-forming operator:	Yes	Yes	No

We saw that, in a sense, all of the above interpretative issues are entangled by the quesiton of how the phrase "propositional function" should be understood in *Principia* and furthermore, that this question can only be answered if we analyze with care Russell's multiple relation theory of judgment and the idea suggested in *Principia*'s Introduction that declarative sentences are a kind of incomplete symbols.

Cocchiarella, like Linsky, endorsed Church's rendition of *Principia*'s formal grammar apart from some reservations about the admission of propositional variables and a comprehension axiom for propositions[172]. Like Church and Linsky, Cocchiarella also thinks that there are two senses of 'predicativity' in *Principia* and he claims that the one of ∗12 is "less appropriate"[173] than that of the Introduction. And most importantly, Cocchiarella also takes type-distinctions to have ontological import.

The fundamental assumption involved in Cocchiarella's interpretation of Russell's multiple-analysis of judgment is that "without including propositional functions among the single entities contained in a judgment or belief complex, there would simply be no multiple relation theory of belief at all"[174]. Cocchiarella claims that this identification was already made in Russell's article *On the Regressive Method of Discovering the Premises of Mathematics*, where, Russell enumerates informally what he, at the time, viewed as the fundamental principles required to derive Pure Mathematics from Logic alone. Besides principles of pure deduction, i.e., laws of propositional and quantificational logic, we also find there the following:

> Any propositional function of x is equivalent to one assigning a property to x.[175]
>
> Any propositional function of x and y is equivalent to one asserting a relation between x and y.[176]

Cocchiarella takes these to be identifying monadic and dyadic propositional functions with attributes and relations-in-intension, respectively, with these being understood as universals, i.e., entities which "subsist separately in a timeless, Platonic realm"[177]. Cocchiarella claims that in *Mathematical Logic* "Russell attempted, albeit unsuccessfully, to do without them" and that "though they are not stated peremptorily in [PM], they are in any case clearly assumed there"[178].

All this is a misunderstanding, however. The 'principles' above are best read as informal statements of axioms of reducibility for monadic and dyadic functions. Indeed, just before introducing them, Russell comments:

> There remain two principles, less evident, but indispensable if we are both to avoid contradictions and to preserve ordinary mathematics. The first of these concerns the distinction between a general propositional function of x and a

172 COCCHIARELLA, N., 1980, p.106-109.
173 COCCHIARELLA, N., 1980, p.106.
174 COCCHIARELLA, N., 1980, p.108.
175 RUSSELL, B., 1907, p.281 / 579.
176 RUSSELL, B., 1907, p.281 / 579.
177 COCCHIARELLA, N., 1980, p.102.
178 COCCHIARELLA, N., 1980.

> function assigning a property to *x*. [...] The second principle similarly distinguishes between a general propositional function of *x* and *y* and one which asserts a relation between *x* and *y*.[179]

In fact, Russell alternatively referred to the Axiom of Reducibility in *Principia* as the "axiom of classes" since "[...] it retains as much of classes as we have any use for, and little enough to avoid the contradictions"[180]. Russell also frequently explained the assumption of classes as related to the assumption that one may legitimately speak about *all* the properties satisfied by a given object *x*, for the assumption of classes is that one may interchangeably speak of "*x* is an *F*" and "*x* belongs to the class of all *F*'s"; but Russell was also careful in making clear that this is an informal and imprecise way of stating the assumption - and, in fact, in *Principia* itself we find:

> If mathematics is to be possible, it is absolutely necessary (as explained in the Introduction, Chapter II) that we should have some method of making statements which will usually be equivalent to what we have in mind when we (*inaccurately*) speak of "all properties satisfied by *x*." [...] Hence, we must find, if possible, some method of reducing the order of a propositional function without affecting the truth or falsehood of its values. This seems to be what common-sense effects by the admission of *classes*.[181]

To be sure, Russell and Whitehead say, in that exact same passage above, that "a 'property' of *x*" *may* be defined as a propositional function satisfied by *x*"[182] but this is not the straightforward identification of a propositional function with a universal; the point is that in the presence of the Axiom of Reducibility the statements "*x* is ϕ" and "x belongs to the class of all *z* such that ϕz" are equivalent. Cocchiarella's identification of universals and propositional functions on the basis of such passages is unwarranted.

Furthermore, there is a serious difficulty which is faced by the identification of propositional functions with universals, which can be put very succinctly: universals are objects of acquaintance; but being such, they must be *ultimate* constituents of any analyzed judgment, and thus, they must be simple; but not all propositional functions are simple. In fact, as Hylton and Goldfarb correctly emphasize, *if* propositional functions are admitted as genuine intensional entities, they must be *structured* entities if we are to make sense of Russell's type-distinctions[183].

Bernard Linsky seems to recognize the difficulty we just considered, and, in a sense, Linsky's interpretation can be understood as a variant or modification of Cocchiarella's. Linsky, like Cocchiarella, understands type-distinctions as having ontological import, but the main departure from Cocchiarella's reading is his refusal to *identify* propositional functions with univer-

179 RUSSELL, B., 1907*b*, p.579
180 WHITEHEAD & RUSSELL, 1925, p.67 [1910, p.174].
181 WHITEHEAD & RUSSELL, 1925, p.166 [1910, p.173]. Our emphasis.
182 WHITEHEAD & RUSSELL, 1925, p.127 [1910, p.133]. Our emphasis.
183 Cf., for instance, HYLTON, 1980; 1991; GOLDFARB,, W., 1989.

sals[184]. Linsky does not take functions to be on a par with individuals (be them particulars or universals) and not even with facts.

Commenting on a claim made by Russell in the Introduction to second edition in connection with their contextual definitions of class symbols, that "[...] classes, as distinct from functions, lose even that shadowy being that they retain in ∗20"[185], Linsky writes:

> It appears that propositions and functions must also be granted a "shadowy" existence, enough to be quantified over and so [sic] the values of variables and constants, without there being any analysis or elimination of such talk by definition. The realist ontology I have found in Russell's logic suggests that this existence should be seen as some sort of dependence or supervenience on the basic ontology of particulars, universals and facts. Here "logical" construction has metaphysical import.[186]

Linsky elaborates this idea as follows:

> Propositions are not "single entities". They are constructed entities. Does the construction serve as a reduction? Russell does indeed avoid quantifying propositions in PM (with the notable exception of ∗14·3) usually allowing propositional variables to appear as "apparent" or free[187] variables. This practice invites the interpretation of the symbols as really schematic letters. But Russell does not maintain that propositions are merely linguistic entities such as sentences. The phrase that *expresses* a proposition, he says, is an incomplete symbol or expression, but the proposition is not an incomplete symbol or expression. Propositions are real, constructed entities, just as numbers are real, but constructed entities. The model here is the construction of numbers[188] from classes rather than definite descriptions. It looks as though, in addition to sense data and other particulars, universals and facts in Russell's ontology, one must also add functions and propositions with some sort of existence, albeit a derivative or secondary sort.[189]

The problem with Linsky's reading, however, is precisely the idea that propositions, functions and classes retain this "shadowy" sort of being or existence. The whole problem is that Linsky's interpretation appears to be re-introducing into Russell's metaphysics a view akin to the Meinongian thesis that *certain things have being but do not exist*, since as Linsky himself explains, classes, propositional functions and propositions are not real in the same sense that individuals are - they have this "shadowy", "derivative or secondary sort" of existence[190].

But this sort of ontological limbo, which Russell came to emphatically repudiate was meant to be obliterated by the theory of incomplete symbols. And indeed, even the conclusion

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[184] Cf. LINKSY, B., 1999, p.40.
[185] WHITEHEAD & RUSSELL, 1925, p.xxxix.
[186] LINSKY, B., 1999, p.126.
[187] This is a minor slip: "apparent" means *bound* variable, not free, Linsky must have meant "real" variable here.
[188] Linsky's views concerning cardinal numbers in *Principia* will be discussed in some detail in the final chapter.
[189] LINSKY, B., 1999, p.125.
[190] LINSKY, B., 1999, p.125.

that Russell was just confused about the status of the propositions and propositional functions, as Quine claims, seems more plausible than any interpretation that attributes such doctrine to *Principia*, since this view was so strongly disowned by Russell after *On Denoting*. For to say that *there are*, in a *genuine* sense, entities that are *not universals nor particulars nor facts* just seems to go against central aspects of Russell's views on Metaphysics in the period of 1910-1913. Thus, the plausibility of Linsky's reading depends on how the "metaphysical import" of constructions is to be understood. Are we to understand the theory of descriptions, the contextual definition of classes and the analysis of propositions in terms of the multiple relation theory of judgment as *eliminative* or *reductive* definitions? Linsky clearly takes them to be reductive. But what then, is the residue to be identified with propositional functions?

Linsky is entertaining the idea that the ontological import of type distinctions should be understood in terms of some notion of ontological dependency: an entity ϕ, say, a function of individuals, is of a higher type than an individual *a* because it is constructed from some totality of individuals; a function which takes functions as arguments is constructed from some given totality of functions, and so on; thus *Principia*'s hierarchy of types - understood from bottom up - should be interpreted as a hierarchy of entities of increasing complexity that is ultimately dependent - in a strong ontological sense - on the entities at the bottom. The problem with this reading is that there is no explicit textual evidence in *Principia* in favor of it[191]. On the contrary, many of the things that Whitehead and Russell assert seem to go strongly against it, for instance,

191 Interestingly, there is evidence indicating that Russell and Whitehead did entertain an approach for justifying type distinctions resembling that suggested by Linsky around the time that Russell was sending *Mathematical Logic* to the *American Journal of Mathematics*. The evidence in question appears in the letter Whitehead wrote to Russell on 16 June 1907, where he agrees that "the substitution theory is the proper explanatory starting point" of their system" (MOORE, G., 2014, p.lxxvii). There Whitehead provides an "outline" of Russell's "position" as he understood it (ibid., p.lxxvi), the main point being the idea that "the hierarchy of propositions appears to depend essentially on the *distinction between dependent and independent entities* - the dependent entities having in some sense an essential reference to totalities - and thence also on the *various modes of dependency*" (ibid.; our emphasis); Whitehead furthermore notes that "the independent entities (individuals) require no further logical discrimination" and that "Every entity (independent or dependent) must occur in a proposition containing it, in a manner specifically relevant to its peculiar type of *being*" (ibid., our emphasis). Now, it must be observed that there is room for dispute as to whether this corroborates Linsky's reading, since Whitehead also considers in the letter that "it is possible that all entities, not individuals, have no proper unity in any sense whatsoever; but that as they appear in propositions they are simply a grouping of ideas which conceal an alarming complexity of thought" (ibid.), which suggests that at this point he and Russell may have been considering that non-individuals (including propositional functions) could be considered as some sort of mental constructions. However, if we recall that in *Mathematical Logic* Russell was putting forward his last version of the Substitutional Theory embracing orders of propositions understood as *entities*, it makes sense to interpret the notion of dependency in ontological terms (with first-order propositions depending on individuals, second-order propositions depending on first-order propositions and so on, indefinitely). There is, in fact, another good reason to take Linsky's interpretation as a good fit for *Mathematical Logic*. Landini has observed that although *Principia*'s grammar cannot be adequately retrofitted into Church's r-types, that of *Mathematical Logic* can: Landini notes that all of Church's finely-grained distinctions among *r*-types of variables - including his non-predicative r-types that are not to be found in *Principia* - are captured by the Substitutional Theory retrofitted with orders of propositions as it appears in *Mathematical Logic* once the language of substitution is translated into the more "convenient" language of functional variables (cf. LANDINI, G., 2011a, p.160). All of this is quite striking and very interesting because it suggests that Linsky's interpretation may provide a compelling way to make sense of the metaphysics of a ramified theory of types of propositions such as that which Russell hinted at in *Mathematical Logic* and which Church systematically develops, even if no such hierarchy can be attributed to *Principia*.

when they claim that "[...] a function, in fact, is not a definite object" but a "*mere* ambiguity awaiting determination"[192]. In fact, it does not seem that Linsky's reading can account for the following passage, which is the core of the "argument by direct inspection" in favor of type distinctions:

> [...] when a function can occur significantly as argument, something which is not a function cannot occur significantly as an argument. But conversely when something which is not a function can occur significantly as an argument, a function cannot occur significantly. Take, e.g., "x is a man" and consider "$\phi\hat{y}$ is a man." There is nothing to eliminate the ambiguity which constitutes $\phi\hat{y}$; there is nothing definite which is said to be a man. A function, in fact, is not a definite object, which could be or not be a man; it is a mere ambiguity awaiting determination, and in order that it may occur significantly it must receive the necessary determination, which it obviously does not receive if it is merely substituted for something determinate in a proposition.*[193]

For in the footnote indicated by "*", we find:

> Note that statements concerning the significance of a phrase containing '$\phi\hat{z}$' concerns the symbol '$\phi\hat{z}$' and therefore do not fall under the rule that the elimination of the functional ambiguity is necessary to significance. Significance is a property of signs.[194]

Although nothing that is said here *directly* refutes Linsky's interpretation, the claim that a "function is an *ambiguity*", that it "requires determination", "that in order that *it may occur significantly* it must receive the necessary determination" and that it can be "*substituted for something determinate in a proposition*" together with the above footnote make a strong case for viewing the talk about propositional functions to be primarily viewed as talk about *Principia*'s formal *language*[195]. Of course, this same point also applies (perhaps more incisively) to Cocchiarella's reading: how could Russell think that propositional functions are entities that "subsist separately in a timeless, platonic realm"[196] and at the same type assert that "a function, in fact, is not a definite object" but "a mere ambiguity awaiting determination"[197]? Things just don't add up. As Kevin Klement has recently put it:

> It is pretty clear that if Russell had accepted an objectual understanding of higher-order quantification, he would be committed to many entities besides

192 WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.50].
193 WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.50].
194 WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.50], footnote.
195 This, obviously is not to say that Linsky's views don't have any interest in themselves - quite the contrary is evidently the case, in particular concerning his views on the similarities between the cumulative conception of sets and that of *Principia.* See chapter of this part.
196 COCCHIARELLA, N., 1980, p.102.
197 WHITEHEAD & RUSSELL, 1925, p.48 [1910, p.50].

> simple individuals and their properties and relations, entities entering into satisfaction relations unanalyzable into facts about simple individuals and their simple properties.[198]

But it is pretty clear - in *Principia*'s Introduction, *On Universals and Particulars* and *On Knowledge by Acquaintance and Knowledge by Description,* at least - that he didn't: Russell insists that universals and particulars exhaust the kinds of entities there are.

The above points have serious consequences for Cocchiarella's and Linsky's interpretations of Russell's multiple-relation theory of judgment. Cocchiarella explicitly viewed the hierarchy of propositional functions, i.e. attributes and relations in intension, as generating a hierarchy of judgment complexes. It seems that a hierarchy of complexes parallel to that of functions is also implied by Linsky's interpretation. But we have every reason to believe that there is no such hierarchy of complexes. To begin with, when Russell and Whitehead explain that the sense in which general judgments are true differs from the sense in which elementary judgments are true, they claim that:

> *If* ϕx is an *elementary* judgment, *it is true when it points to a corresponding complex*". *But* $(x)\phi x$ *does not point to a single corresponding complex*: the corresponding complexes are as numerous as the possible values of *x*.[199]

This makes it clear that Landini is right in claiming that "truth" explained in the fundamental sense of correspondence is *only* applicable to atomic/elementary judgments; other senses of "truth" applied to 'propositions' of any order are derived from this one elementary sense. If propositional functions could enter into complexes as (possibly complex) constituents, then there should be complexes of different order/type (with different senses of truth applicable to complexes of different type/orders). But we have good reasons to conclude that complexes are of an uniform 'type', or, better put, that there is no such thing as a type of complex. *Principia*'s Introduction indicates that there can be no such hierarchy of complexes, for when Whitehead and Russell introduce the metaphysical picture of the universe as consisting "of objects having various qualities and standing in various relations"[200] they also claim that "some of the objects which occur in the universe are complex"[201]. This strongly suggest that complexes are, as Landini puts it, "on a par with individuals"[202]. If they were not, how could we account for the claim that a "complex object "*a*-in-the-relation-*R*-to-*b*" may be capable of being *perceived*" and that "when perceived, it is perceived as *one* object"[203]?

[198] KLEMENT, K., 2018, p.163.
[199] WHITEHEAD & RUSSELL, 1925, p.46 [1910, p.48].
[200] WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.45].
[201] WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.45].
[202] LANDINI, G., 1998a, p.291.
[203] WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.45].

One could object that Russell never formulated in any detail an account of how his multiple relation theory would apply to general judgments (in particular those that are expressed by sentences containing bound predicate variables), and thus, that it is arguable that it does not follow from the above considerations that the multiple relation is not committed to a hierarchy of complexes. But there are at least two answers to this concern. First, as Landini observes, one may "hold that the multiple-relation theory is not part of the recursive truth (correspondence) definition"[204], but is meant only as *one possible alternative* to a realist view of propositions. As he puts it:

> On this view, the multiple-relation theory is only intended as a sketch which shows how it is possible to disarm a central argument for an ontology of propositions. It is intended to show, that is, how one might deny that belief (judgment) must be a relation between a mind and a proposition.[205]

Second, it is plausible to suppose that Russell thought that the *crucial* case for which the multiple relation analysis had to account for was the atomic/elementary case - and for this reason he never addressed the molecular and general cases in any systematic detail. As Kevin Klement observes:

> This lacuna in his theories of judgment is perhaps best explained by his assumption that it is only the words occurring in atomic or elementary judgments that "refer" or "mean" things in objective reality, and hence an account of the kind of truth involving the relationship between the mind and the world needs only tackle atomic or elementary judgments.[206]

And indeed, Klement correctly notes that the exposition of *Principia* itself suggests that this is the case[207]. Thus, the kind of dependency one finds in *Principia* between propositions of order $n + 1$ and those of order *n* is best understood as a *semantic* one[208]. The best way to make sense of the dependency one finds expressed in Whitehead and Russell's hierarchy of propositions is in terms of *truth conditions of sentences*: truth conditions for sentences of order $n + 1$ are given in terms of those of order *n.* Ultimately, of course, this process has to stop and truth is defined in terms of correspondence; but at this point we don't have any sort of hierarchy of entities and we are no longer talking about the truth or falsity of sentences - we are speaking of correspondences between judging complexes and facts.

So with respect to the role played by the multiple-relation theory in *Principia*'s Introduction, Landini's nominalistic interpretation has the upper hand against both Cocchiarella's

204 LANDINI, G., 1998a, p.291.
205 LANDINI, G., 1998a, p.291.
206 KLEMENT, K., 2018, p.161.
207 KLEMENT, K., 2018, p.162; cf. WHITEHEAD & RUSSELL, 1910, p.53.
208 KLEMENT, K., 2018, p.163.

and Linsky's interpretation: his interpretation completely dissolves the problem of explaining how propositional functions enter as constituents of judgments which does not have a satisfactory answer within the realist interpretations we considered.

4.4.2 Type Distinctions, the 'Universality of Logic' and the So-called 'Logocentric Predicament'

Modern formulations of propositional logic as a formal axiomatic system usually characterize their set of axioms using schemata that specify infinitely many axioms that share a certain form or by assuming a finite number of axioms and then introducing a rule of substitution. The Church-Linksy interpretation which we previously considered construes *Principia*'s propositional logic with propositional variables ranging over propositions conceived as intensional entities; Landini's interpretation, on the other hand, requires a schematic interpretation of *Principia*'s propositional variables and assumes no ontological counterparts for *Principia*'s so-called propositional variables. As Landini himself recognizes[209], however, despite Whitehead and Russell's insistence that there are no such things as propositions understood as singular logical subjects, from a purely syntactical point of view, it is somewhat unclear which route was intended by Whitehead and Russell in *Principia* (i.e., schemata or finite number of axioms) because they were simply too careless in distinguishing object-language from metalanguage.

In fact, the list of primitive notions of *Principia* also contains ideas which we would nowadays explicitly consider part of the metalanguage. Consider, in particular, the first four primitive notions introduced in section ∗1:

(Pi 1) *Elementary proposition*: "a proposition which does not involve any variables"[210].

(Pi 2) *Elementary propositional function*: "an expression containing an undetermined constituent, i.e., a variable, or several such constituents, and such that, when the undetermined constituent or constituents are determined, i.e, when values are assigned to the variable or variables, the resulting value of the expression in question is an elementary proposition"[211]

(Pi 3) *Assertion of a* proposition: "In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by "$\vdash . \ p$". The sign "$\vdash$ " is called the assertion-sign; it may be read as "it is true

209 LANDINI, G., 1998a, p.259.
210 WHITEHEAD & RUSSELL, 1925, p.91 [1910, p.95].
211 WHITEHEAD & RUSSELL, 1925, p.92 [1910, p.96].

that" (although philosophically this is not exactly what it means). The dots after the assertion-sign indicate its range;"[212]

(Pi 4) *Assertion of a propositional function*: "Let ϕx be a propositional function whose argument is *x*; then we may assert ϕx without assigning a value to *x*. [...] Thus when we assert ϕx, leaving *x* undetermined, we are asserting an ambiguous value of our function."[213]

If we draw a sharp distinction between "propositional function" as meaning a schematic letter versus a "propositional function" meaning a predicate variable[214], all of the 'primitive ideas' introduced above should be interpreted as metalinguistic[215]. A similar point applies to *Principia*'s formulation of propositional Logic. Whitehead and Russell assume the following principles under the guise of 'primitive propositions'[216]:

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[212] WHITEHEAD & RUSSELL, 1925, p.92 [1910, p.96].

[213] WHITEHEAD & RUSSELL, 1925, p.92 [1910, p.96].

[214] Apart from the evidence scattered in *Principia* and many of Russell's other published works and manuscripts which support this view which we previously discussed, another good reason for doing this is to also keep syntactic matters clearly distinguished from semantic ones. For, as we saw, Russell and Whitehead are not consistent in their use of these expressions, but many of the things they say are meant to apply to formulas, since they clearly intended to have a legitimate sense for the notion of 'proposition' (and 'propositional function') as some kind of expression - for instance, when they assert that a propositional function is "an expression" that may contain "an undetermined constituent, i.e., a variable" (WHITEHEAD & RUSSELL, 1927*a*, p.92 [1910, p.96]).

[215] That is, if we follow Landini, we conclude that the modern equivalent to *Principia*'s 'elementary proposition' would be an atomic or molecular well-formed sentence without any occurrences of bound variables. An equivalent to 'elementary propositional function' would be an open atomic or molecular well-formed formula, that is, expressions such as "$\phi!x$", "$\phi!(x,y)$", "$\sim \psi!x$", "$\phi!x \vee \psi!x$", "$\phi!x \supset \psi!x$" etc., while 'propositional functions' in general - as opposed to elementary and predicative ones - would be open formulas which are prefixed by quantifiers or which contain quantified clauses, such as "$(x) \,.\, \phi!(x,y)$" or "$\phi!x \vee (y) \,.\, \psi!(x,y)$". Notice, however, that Whitehead and Russell do not bother to introduce an explicit distinction between atomic formulas, e.g., formulas that do not contain propositional connectives and quantifiers and molecular formulas that contain propositional connectives as principal operators. Similar considerations apply to the notions of 'elementary proposition' and of 'proposition' generally: the former is just a sentence with no variables (free or otherwise), the latter is just a sentence which may be general or contain quantified clauses. Thus, the first two primitive notions are introducing syntatical categories of complex expressions. Concerning the notion of assertion, according to Whitehead and Russell, a special symbol for it is required for "distinguishing a complete proposition" which is *asserted* or affirmed (either as an axiom or as a consequence of their axioms) "from any subordinate propositions contained in it but not asserted" (WHITEHEAD & RUSSELL, 1927*a*, p.9 [1910, p.8-9].). The symbol $\vdash$ which was, of course, taken from Frege's *Begriffsschrift*, has a slightly different use in *Principia* than it had in Frege's system, where it was used to differentiate the mere consideration of the content of a possible judgment and the assertion of the content, with the symbol "$\vdash$" employed to mark the *assertoric force* in the latter. But, of course, in Frege's system the symbol "$\vdash$" was composed of two different symbols: the horizontal bar and the vertical bar. The former was used to indicate the mere consideration of the content of a judgment, while the latter indicated its assertion.

[216] WHITEHEAD & RUSSELL, 1925, pp.94-97 [pp.98-101]. Using disjunction and negation as primitives (WHITEHEAD & RUSSELL, 1927*a*, p.93 [1910, p.97]), they define material implication ($\supset$), conjunction (., :, :., ::, etc) and material equivalence ($\equiv$), in the usual ways, putting: "$p \supset q$" for "$\sim p \vee q$", "$p \,.\, q$" for $\sim (\sim p \vee \sim q)$ and "$p \equiv q$" for "$p \supset q \,.\, q \supset p$" (cf. WHITEHEAD & RUSSELL, 1925, pp.94; p.109; p.115 [1910, p.98; p.114; p.120]). This system was abandoned in *Principia*'s second edition, being substituted for the system of Jean Nicod which assumed the so-called 'Sheffer's stroke' as the sole primitive truth-functional operator with a single axiom; for details cf. SHEFFER, H., 1913; NICOD, J., 1917 and WHITEHEAD & RUSSELL, 1925, pp.xv-xix. As is also well known, the system of propositional logic of *Principia*'s first edition was used as the basis for the

$*1{\cdot}1$ Anything implied by a true elementary proposition is true. Pp

$*1{\cdot}11$ When ϕx can be asserted, where *x* is a real variable, and $\phi x \supset \psi x$ can be asserted, where *x* is a real variable, then ψx can be asserted, where *x* is a real variable. [and so on, for functions of several variables] Pp

$*1{\cdot}2$ $\vdash : p \vee p . \supset . p$ Pp

$*1{\cdot}3$ $\vdash : q . \supset . p \vee q$ Pp

$*1{\cdot}4$ $\vdash : p \vee q . \supset . q \vee p$ Pp

$*1{\cdot}5$ $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$ Pp[217]

$*1{\cdot}6$ $\vdash : q \supset r . \supset : p \vee q . \supset . p \vee r$ Pp

These propositions are first introduced as applying only to so-called '*elementary* propositions' that is, closed formulas containing no bound variables. The first of them, $*1{\cdot}1$, is Russell's sloppy formulation of the *rule* of *modus ponens*, while the second, $*1{\cdot}11$, acquires its significance in connection with the way Russell and Whitehead extend the validity of the axioms above to sentences containing apparent variables: it is meant as a rule of formal implication between 'propositional functions'. But given Whitehead and Russell's lack of care in handling and distinguishing the description of their formal system and the actual system, it is unclear whether the above are proper variables or (metalinguistic) schematic letters.

There are, however, some good reasons to think that the authors favored a schematic approach. Looking at a proof, say, of a familiar theorem like $p \supset \sim p . \supset . \sim p$, we find:

$*2{\cdot}01$ $\vdash : p \supset \sim p . \supset . \sim p$

Dem.

$$\left[\text{Taut}\frac{\sim p}{p}\right] \vdash : \sim p \vee p . \supset . \sim p \qquad (1)$$

$$[(1) . (*1{\cdot}01)] \vdash : p \supset \sim p . \supset . \sim p$$

As the authors explain, the notation "$\left[\text{Taut}\frac{\sim p}{p}\right]$" is used to mean "what "Taut" becomes when $\sim p$ is written in place of *p*"[218]. This notation is taken from Russell's paper *The Theory of Implication* published in 1906[219]. In that paper Russell formulated a rule of substitution that

first systematic investigation of the completeness of propositional logic in POST, E., 1921. For further historical details cf. also CHURCH, A., 1956, pp.156-9.

217 This proposition is redundant, as shown by Bernays. Cf. BERNAYS, P., 1926.

218 WHITEHEAD & RUSSELL, 1925, p.100 [1910, p.104]. "Taut" is their label for proposition $*1{\cdot}2$, that is $p \vee p . \supset . p$

219 Cf., for instance, RUSSELL, 1906*c*, p.34, proof of theorem $*3{\cdot}34$. The paper - published in the same journal in which Russell would later publish his seminal paper on the theory of types, the *American Journal of Mathematics* - is Russell's first published treatment in length of propositional logic. Some parts of the paper - those

applied both to propositional and individual variables, which he called the "principle of substitution of a dependent for an independent variable"[220] and he used the aforementioned notation to indicate substitutions. *Principia*, on the other hand, not only does *not* have a rule of substitution, but contains statements that deny the very legitimacy of formulating one, something that strongly suggest a schematic approach:

> The proofs of the earlier of the propositions of this number consist simply in noticing that they are instances of the general rules given in $*1$. In such cases these rules are not premises, since they assert any instance of themselves. [...] The recognition that a certain proposition is an instance of some general proposition previously proved or assumed is essential to the process of deduction from general rules, but cannot itself be erected into general rule, since the application required is particular, and no general rule can *explicitly* include a particular application.[221]

The terminology used, that of 'recognizing' a proposition as an instance of a previous one strongly indicates that they are taking their axioms as determined by schemata and that what they lacked - as in various other places of the work - is an adequate and explicit distinction between (meta)variables for arbitrary expressions of their formal language and the genuine variables of the language. Moreover - although this is also unclear - the last point of the paragraph also indicates a difference between recognizing an '*instance*' of a 'proposition' - in the sense that "$p \supset q \,.\supset.\, p \supset q$" is an instance of "$p \supset p$", or better yet, something like "$A \supset A$" - and of recognizing a case of universal instantiation, e.g., the passage from "$(x) \,.\, \phi x$" to "ϕa", since the latter can clearly be "erected into a general rule", namely $*10{\cdot}1$.

It must be observed, however, as Alonzo Church pointed out long ago[222], that Russell seems to have changed his mind yet again on the question, writing later in the *Introduction to Mathematical Philosophy*:

> The primitive propositions, whatever they may be, are to be regarded as asserted for all possible values of the variable propositions *p*, *q*, *r* which occur in them. We may therefore substitute for (say) *p* any expression whose value is always a proposition, e.g. not-*p*, "*s* implies *t*," and so on. By means of such substitutions we really obtain sets of special cases of our original proposition, but from a practical point of view we obtain what are virtually new propositions. The legitimacy of substitutions of this kind has to be insured by means of a non-formal principle of inference.

dealing with propositional logic without bound propositional variables - are quite similar to the first numbers of *Principia*, with minor notational variations (besides the mentioned rule of substitution). Unfortunately, the paper is, almost as a rule, neglected in secondary literature (an exception is LANDINI, 1998) and in historical notes in modern textbooks (CHURCH, 1956, p.157-8 is also an exception).

220 RUSSELL, B., 1906*c*, p.28.

221 WHITEHEAD & RUSSELL, 1925, p.98 [1910, p.102].

222 CHURCH, A., 1956, p.158.

In a footnote, he writes that "no such principle is enunciated in *Principia Mathematica*" and that "this would seem to be an omission"[223]. Church takes this as further evidence that *Principia* required a rule of substitution after all. But of course, these remarks were made long after *Principia* was written, and so are not conclusive.

Still, the best explanation to be found for Russell's hesitation to formulate a substitution rule is his reluctance to treat propositional variables as genuine variables: by the time of *Principia*'s publication he was done with propositions as entities and with the Substitutional Theory, so anything like the substitution rule of the 1906 article that treated propositional variables on a par with individual ones was out of the question[224]. Indeed, by 2 January 1911 he wrote to Jourdain:

> I no longer think it significant to deny $x \supset p$, where x is not a proposition. I think that, strictly, one ought not to use a single letter for a proposition, but always some such symbol as ϕx.[225]

This strongly suggests that at this point Russell was interpreting the letters p, q, r, etc., as mere place-holders for formulas of *Principia*'s language. So the schematic approach is surely the best interpretation.

This, as Landini points out, also corroborates his claim that the "the logical constants are statement connectives; they are flanked by *wffs* to form *wffs*"[226]. Interpreting their calculus in terms of a schematic approach as Landini suggests also helps to make sense of the following further primitive 'propositions' that are introduced in $*1$, and which are sometimes singled out by interpreters as having a problematic status[227]:

> $*1{\cdot}7$ If p is an elementary proposition, $\sim p$ is an elementary proposition. Pp
>
> $*1{\cdot}71$ If p and q are elementary propositions, $p \vee q$ are elementary propositions. Pp
>
> $*1{\cdot}72$ If ϕp and ψq are propositional functions which take elementary propositions as arguments, $\phi p \vee \psi q$ is an elementary propositional function. Pp

Once we understand that the letters p, q, r, must be read schematically, all the above 'primitive propositions' can be read as recursive clauses for determining the well-formed formulas of the formal system! That is, we may translate the above as a recursive definition of well-formed

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223 RUSSELL, B., 1919a, p.151. It is curious that Russell himself simply did not mention his 1906 article that *did* have the said rule, even if he thought it inadequate.

224 Landini claims that a schematic reading of propositional 'variables' is indispensable for an adequate reading of *Principia*'s quantification theory (LANDINI, 1998, p.258-9). Since his point is intimately connected to the difficulties involved with ' typical ambiguity' we will discuss it in the next section.

225 GRATTAN-GUINNESS, 1977, p. 136.

226 LANDINI, G., 1998a.

227 WHITEHEAD & RUSSELL, 1925, p.97 [1910, p.101].

formulas, something along the following lines: If A is a well-formed formula containing no variables, then so is $\sim A$; If A and B are well-formed formulas containing no variables, then so is $A \vee B$; If A and B are well-formed formulas containing no variables, then so are $A \supset B$, $A \equiv B$ and $A \,.\, B$[228]. An interpretation of 'primitive propositions' dealing with types along these lines also allows us to dissolve a common objection one finds in the literature according to which *Principia*'s exposition of type-theory violates its own grammar.

According to Landini, even if the formulation of propositional logic of *Principia* by itself does not provide enough evidence in favor of a schematic interpretation, "the quantificational part of ramified type theory militates against the approach of axioms and a rule of replacement", since "schematic order\type indices and the distinction between predicate variables and schematic letters for *wffs* are essential to the grammar and to the reducibility principles"[229]. In other words, if Landini's interpretation is correct, a schematic approach is required by what Whitehead and Russell intended typical ambiguity to accomplish[230].

Recall that one of the fundamental problems of *Principia*'s presentation of type-theory - both in the Introduction and in the body of the work - was the absence of any set of explicit rules for the restoration and suppression of type indices: instead of providing such rules, Whitehead and Russell simply stated that only predicative 'functional' variables could be bound by quantifiers; thus, type-distinctions were marked according to context as in "ϕx", "$\phi ! x$", "$f(\phi ! \hat{x})$", $f!(\phi ! \hat{x})$". Among the axioms of their calculus was the following, for instance:

$$*10{\cdot}1 \qquad \vdash : (x) \,.\, \phi x \,.\, \supset \,.\, \phi y$$

This stood for a whole class of axioms, to be determined according to the type of variables involved, for instance, "$(\phi) \,.\, f(\phi ! \hat{x}) \,.\, \supset \,.\, f(\psi ! \hat{x})$", where $\phi ! \hat{x}$ takes individual variables (i.e., variables of the lowest type) as arguments. This was their device of typical ambiguity. According to Landini, it is because of the the absence of explicit rules that Whitehead and Russell were forced to introduce further propositions like the following:

> $*9{\cdot}131$ Definition of "*being the same type*": We say that *u* and *v* "are of the same type" if (1) both are individuals, (2) both are elementary functions taking arguments of the same type, (3) *u* is a function and *v* its negation, (4) *u* is $\phi \hat{x}$ or $\psi \hat{x}$, and *v* is $\phi \hat{x} \vee \psi \hat{x}$, where $\phi \hat{x}$ and $\psi \hat{x}$ are elementary functions, (5) *u* is $(y) \,.\, \phi(\hat{x}, y)$ and *v* is $(z) \,.\, \phi(\hat{x}, z)$, where $\phi(\hat{x}, \hat{y})$ and $\psi(\hat{x}, \hat{y})$ are of the same type (6) both are elementary propositions, (7) *u* is a proposition and *v* is $\sim u$, or (8) *u* is $(x) \,.\, \phi x$ and *v* is $(y) \,.\, \psi y$ where $\phi \hat{x}$ and $\psi \hat{x}$ are of the same type.[231]

[228] Presumably, Whitehead and Russell intended that intensional operators could be added to the system. They would then also covered by clause $*1{\cdot}72$.

[229] LANDINI, G., 1998a, p.259.

[230] Cf. LANDINI, G., 1998a, p.259-63 and also our previous discussion in section 4.2.1.

[231] WHITEHEAD & RUSSELL, 1925, p.133 [1910, p.138].

Landini's view is that the above "is just a statement of grammatical well-formedness that governs the *wffs* of the system" and that such statements "are necessitated only by a quite deplorable misuse of typical ambiguity"[232]. His point is that propositions like the above - and many others, like $*9{\cdot}14{\cdot}15{\cdot}6{\cdot}61{\cdot}62{\cdot}63$ and $*10{\cdot}121{\cdot}122{\cdot}13$ - would be unnecessary if rules for restoring and suppressing type indices were given - much like propositions $*1{\cdot}7{\cdot}71{\cdot}72$ would be unnecessary had Whitehead and Russell grasped with more clarity the distinction between metalinguistic definitions of the grammatical rules of their symbolism and the proper axioms and definitions of their formal system.

Landini has compelling evidence in favor of his reading. In a letter Whitehead wrote to Russell in May of 1910 we find the former struggling with this exact issue:

> I don't feel sure that $*3\,.\,03$ is right as it stands - or at least as it is explained. It appears to me as if two ideas are muddled up together - namely, a true logical premise and a test which supplements the incompleteness of our symbolism.
>
> A true logical premiss must [be] such as would still be required, if our symbolism were complete and adequate. Now in such a case the type is always in evidence. E.g. let every letter representing an individual have *i* as subscript, then we have $\vdash \phi x_i$ and $\vdash .\ \psi x_i$, we do not need any axiom to assure that x_i and y_i are of the same type, and that any possible value of x_i is a possible value of y_i and vice versa.[233]

Proposition $*3{\cdot}03$ states the following[234]:

> Given two asserted elementary propositional functions "$\vdash .\ \phi p$" and "$\vdash .\ \psi p$" whose arguments are elementary propositions, we have "$\vdash .\ \phi\,.\,\psi p$".

As Landini observes, an analogue of the above is given by $*10{\cdot}13$, which states[235]:

> If $\phi\hat{x}$ and $\psi\hat{x}$ take arguments of the same type, and we have "$\vdash .\ \phi x$" and "$\vdash .\ \psi x$", we shall have "$\vdash .\ \phi x\,.\,\psi x$".

Thus, it is crystal clear that Whitehead is onto exactly the same point raised by Landini regarding the aforementioned problematic propositions dealing with types. This strongly suggests that wherever Whitehead and Russell assert that two so-called functions or propositions are of the same type or something along these lines, what they meant to say is something about the

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[232] LANDINI, G., 1998a, p.260. Of course, as Landini further observes, such statements "are entirely avoided if the formal grammar is set out" in terms of explicit rules for restoring and ommiting type indices.

[233] As quoted in LANDINI, G., 1998a, p.260.

[234] WHITEHEAD & RUSSELL, 1925, p.111 [1910, p.116].

[235] WHITEHEAD & RUSSELL, 1925, p.140 [1910, p.146]. Cf also LANDINI, G., 1998a, pp.259-61.

expressions of their formal grammar. While propositions like ∗1·7·71·72 give conditions for well-formedness of certain classes of closed sentences, those like ∗9·131 fill the gaps left by *Principia*'s problematic use of typical ambiguity. This in turn also strongly corroborates Landini's claim that they are not meant as substantial assertions about obscure entities, i.e., so called 'propositional functions'. For Whitehead himself recognizes that they constitute attempts to "supplement the incompleteness" of *Principia*'s notation.

Once we interpret *Principia*'s type-distinctions as put forward in propositions like ∗9·131 as purely grammatical statements along the lines suggested by Landini, a supposed fundamental problem or 'tension' that some authors detect in Russell's early writings on Mathematical Philosophy can be dissolved. The supposed general difficulty, as Peter Hylton puts it "arises from the attempt to state type theory within type theory"[236]. Since his views are quite representative of a widespread interpretation of Russell's views, it is worth quoting his arguments fully and considering them in some detail.

Hylton raises the difficulty in connection with several different aspects of *Principia*'s presentation of type-theory. One concerns notions that can be applied "across types"[237], as he put it. Concerning the notion of a 'predicative function', in particular, he argues that:

> [...] if the notion is to do any work at all, it must be truly applicable to some propositional functions and falsely applicable to others (i.e. there must be both predicative and non- predicative functions). But then a single propositional function ($\hat{\phi}$ is predicative) is applicable (truly or falsely) to entities of different type: this contravenes type restrictions.[238]

According to Hylton, another, even more serious version of the "violation of types" problem appears in *Principia*'s definition of 'being the same type as' given in $*9 \cdot 131$. He writes:

> Proposition ∗9·131 is a definition of 'being the same type as'. The primitive proposition (axiom) ∗9·14 makes essential use of this notion, asserting, 'If "ϕx" is significant, and if a is of the same type as *x*, then "ϕa" is significant, and vice versa'. This proposition, and others which depend upon it, are essential for Russell's extension of logic from (what we should call) the truth-functional to the quantificational. The relation 'is of the same type as' is one which itself violates type restrictions. The clause of ∗9·14 which states '*a* is of the same type as *x*' must be true in some cases and false in others (if it were always true, which it is not, it could be simply omitted; if it were always false the whole axiom would be unnecessary). If we take some fixed entity *a*, we can truly assert that *b*, say, is of the same type as *a*; we can also falsely, but with sense, assert that an entity of another type, *c*, say, is of the same type as *a*. So both *b* and *c* can be arguments to the same propositional function (that expressed by 'is of the same type as *a*'). By the 'vice versa' clause of $*9 \cdot 14$ itself, it

[236] HYLTON, P., 2005,p.76.
[237] HYLTON, P., 1990, p.317.
[238] HYLTON, P., 1990, p.317.

> ought to follow from this that *b* and *c* are themselves of the same type; but this is contrary to the initial assumption that *b* is of the same type as *a* while *c* is not. What this shows is that we cannot consistently treat 'is of the same type as', and other notions that Russell employs in setting up type theory, as being themselves subject to type theory. But Russell's conception of logic gives him no other way to treat them.[239]

Hylton also finds a fundamental tension related to the problem of 'stating type theory within type theory' in the following passage of *Principia*:

> We shall find that the unrestricted variable is still subject to limitations imposed by the manner of its occurrence, i.e. things which can be said significantly concerning a proposition cannot be said concerning a class or relation, and so on. But the limitations to which the unrestricted variable is subject do not need to be explicitly indicated, since they are the limits of significance of the statement in which the variable occurs, and are therefore intrinsically determined by this statement.[240]

The above statement, Hylton claims, attests that Russell was still commited in *Principia* with the view "that the fundamental notion of the variable is that of the unrestricted variable, ranging over everything"[241]; the problem, according to him, is that the above also implies that "restrictions imposed by limits of significance [...] go without saying: they do not need to be stated"[242]; but of course, they are stated - even in *numbered* propositions like ∗9·131 and many others we mentioned above.

The above points raised by Hylton are all very acute instances of what became known as the 'logocentric predicament'[243], i.e., the problem of not being able to state within Logic what is required for formulating the laws of Logic[244]. As formulated above, all of the problems raised against *Principia*'s theory of types crucially presuppose a radical interpretation of Russell's so-called 'Universalist' conception of Logic. In fact, according to Hylton propositions like ∗9·131 should be interpreted as substantial claims, not only because they are numbered, of course, but first and foremost because he assumes that the possibility of a metatheoretic perspective is "foreign to Russell's thought"[245]. Similarly, with respect to the crucial passage of *Principia* just quoted, Hylton finds it incoherent because he presumes that for Russell "logic applies to every statement, and thus also to statements which are intended to limit the scope of the variable used in other statements"[246]. So there are at least three issues entangled in Hylton's discussion: there is

239 HYLTON, P. 1990, p.317. This is same argument appears elsewhere in his writings, for instance, in HYLTON, P., 2005, pp.75-7 and pp.106-7.
240 WHITEHEAD & RUSSELL, 1925, p.4 [1910, p.4].
241 HYLTON, P. 1990, p.317.
242 HYLTON, P. 1990, p.318.
243 Cf. SHEFFER, H., 1926, p.228.
244 As Sheffer put it "In order to give an account of logic, we must presuppose and employ logic" (SHEFFER, H., 1926, p.228).
245 HYLTON, P., 2005, p.85.
246 HYLTON, P., 2005, p.75.

the issue of the possibility of a metatheoretical perspective in Logic; there is the issue of notions like 'is predicative' and 'is of the same type as' being applied to *entities* and *not to expressions* of *Principia*'s formal system; and then there is the issue of limitation (or absence thereof) in the range of the variables for the calculus of logic.

At this point it should be pretty clear why Landini's interpretation dissolves completely all of these issues: according to him, there is no ontology of propositional functions and ∗9·131 must be understood as statements about conditions of significance for expressions (well-formed formulas) of *Principia*'s language. Since the only variables which range over entities are individual variables, *there are no types of entities*. Therefore, there can be no such thing as an object language assertion about types and the variables of the calculus are indeed unrestricted.

Surely, those who accept Hylton's argument would refuse Landini's interpretation since the latter takes it that *Principia* embraces the distinction between language and metalanguage (at least in a somewhat tacit and/or unclear way). In the introduction of the present work and in our previous chapter on the Logic of the *Principles of Mathematics*, we had occasion to discuss somewhat exaggerated claims that are often made concerning the so-called 'Universalist' conception of Logic. We also had occasion to discuss and refer to Ian Proops's detailed critical discussion of these interpretations. As we saw, these radical claims do not hold water, for several different reasons[247]. This by itself seriously undermines Hylton's claim that there is a deep incoherence at the heart of *Principia* and Russell's conception of Logic. For once we drop the idea the idea that for Russell "Logic is all encompassing" in the radical sense advocated by Hylton (and many others) there is no reason to deny that of notions like 'is of the same type as', 'is a predicative function', etc., can interpreted - at least in some embryonic sense - as *metalinguistic*. And so the issues raised by Hylton disappear[248].

There is more that can be said in favor of Landini's claim that statements like ∗9·131 were meant by Whitehead and Russell to be purely grammatical statements. To appreciate this, we must look at what Whitehead and Russell say about types beyond *Principia*'s Introduction and its initial sections. The main sections that must be considered are those concerning *Principia*'s so-called 'relative types', discussed in part II, *Prolegomena to Cardinal Arithmetic* and in the brief - and often neglected - *Note on negative statements concerning types* of *Principia*'s second volume. What the discussion of those sections makes clear is that there are at least two ways of speaking of 'types' in *Principia*'s numbered propositions. There is one use of "types" which is, in fact, required for their development of Mathematics, the one introduced in ∗63, ∗64

[247] PROOPS, I., 2006. Most notably, Proops shows that the passages of Hylton's works display a tendency to conflate several different thesis under the label of "universality of logic"; the same applies to the writings of others who accept the same general interpretation, like Warren Goldfarb. Again, we add that if van Heijenoort, Hylton, Goldfarb were right, it would be quite hard to explain why Russell and Whitehead explicitly claim "the process of inference cannot be reduced to symbols".

[248] To be sure, Hylton's main point is correct: one "cannot consistently treat" expressions like "sameness of type", "is of the same type as", etc., on a par with the relation symbols and the predicates of *Principia*'s object-language on pain of incoherence. This is irreproachable. What is wrong is the assumption that Russell would or should admit such expressions in the object-language of *Principia.*

and $*65$[249]. It concerns what Whitehead and Russell called ‘relative types’. What is introduced there are - as the authors put it - notations that “serve to express the type of one variable in terms of the type of another”[250]. They introduce descriptive functions like “t_0‘α”, for instance, which means “the type of the elements of α”. What happens is that statements involving membership to a class like t_0‘α have a peculiar status: they are true whenever significant, for the cornerstone of *Principia*’s type theory is that any statement like “‘$x^t \epsilon \alpha^v$’ is false whenever $t \neq v$”. The very statement made in the last line here, however, is of a whole other sort than “$x \epsilon t_0$‘α”; an expression like “$x \epsilon t_0$‘α” is an expression of *Principia*’s object-language[251]. A statement like “ ‘$x^t \epsilon \alpha^v$’ is false whenever $v \neq (t)$” is a statement *about Principia*’s grammar. According to Landini’s interpretation, statements like $*9{\cdot}131$ are of the latter kind: they are statements about grammar.

Now, as we’ll discuss further, there are very important differences between the notion of ‘type’ as it appears in numbered propositions like $*9{\cdot}131$ and that of a ‘relative type’ which is treated in $*63$; still, the discussion of relative types in *Principia* unequivocally shows two things: not only that Whitehead and Russell did consider the possibility of treating statements about types as *grammatical* statements but also that they were onto the problem raised by Hylton; the problem, as they liked to put it, is that most statements involving types are “true whenever significant”[252] - which, by contraposition means that by the standards of *Principia*’s (object) language they can never be false but simply nonsensical. We find this in *Principia*’s second volume in the *Note on negative statements concerning types*:

> Statements such as “$x \sim \epsilon\, t$‘y” or “$x \sim \epsilon\, t_0$‘α” are always false when they are significant. Hence, when an object belongs to one type, there is no significant way of expressing what we mean when we say that it does not belong to some other type. The reason is that when, for example, t‘α and t_0‘α are said to be different, the statement is only significant if interpreted as applying to the symbols, *i.e.*, as meaning to deny that the two symbols denote the same class. We cannot assert that they denote different classes, since ‘t‘$\alpha = t_0$‘α’ is not significant, but we can deny that they denote the same class.”.[253]

Well, the moral to be drawn from such passages in connection with $*9{\cdot}131$ is quite straightforward: if Russell and Whitehead did consider the issue of statements of type having a very peculiar status in $*63$ and the *Note* of volume II and explicitly claimed that “propositions dealing with types acquire their importance largely from the fact that they can be interpreted as

249 This will be discussed in our chapter concerning the development of Arithmetic in *Principia*, but it is worth to extracting some points here.

250 WHITEHEAD & RUSSELL, 1925, p.400 [1910, p.419].

251 Strictly speaking, this is incorrect: there are good reasons to think that lower-case greek letters for classes are metalinguistic place holders for incomplete symbols; be that as it may, the same point put forward here holds since an expression like “$x \epsilon t_0$‘α” can be translated into *Principia*’s object-language by eliminating all incomplete symbols, while a phrase like “‘$x^t \epsilon \alpha^v$’ is false whenever $t \neq v$” cannot.

252 WHITEHEAD & RUSSELL, 1925, p.400 [1910, p.419].

253 WHITEHEAD & RUSSELL, 1927*a*, p.34 [1912, p.35].

dealing with the symbols"[254], why should the statements of $*9{\cdot}131$ be interpreted any differently? Indeed, the only reasonable explanation for the "largely" in the last quote is that Whitehead and Russell were aware that despite the fact that some statements which involve (*relative*!) types have some use in their development of Mathematics - those like "$x \in t_0 ‘\alpha$" - most of them are best interpreted as statements concerning conditions of significance, instead of assertions about entities of any sort. This, again, corroborates Landini's interpretation.

But at this point one may raise the following question: is it not possible also to deal satisfactorily with the issue raised by Hylton by accepting that *Principia* allows a 'metatheoretical perspective' while also accepting a realist view of 'propositional functions'? This seems to be the route adopted by Bernard Linsky. Attempting to answer Hylton's problem, while preserving a realist view of 'propositional functions', Linsky suggests that passages like $*9{\cdot}131$ are meant to "record some features of the logical form of a proposition, namely the chains of dependencies on various totalities that are uncovered in the analysis of the *constituents* of a propositional function"[255]. Thus, Linsky does not take numbered propositions like $*9{\cdot}131$ to be simply ruling out some expressions as *just* meaningless or nonsensical. According to him:

> [...] it is more accurate to take the talk of "significance" as describing simply what arguments a function can take, and thus as describing the dependencies of propositions on propositional functions and arguments.[256]
>
> Reading the talk of "meaninglessness" as only expressing violations of type also provide a response to those who find some deep incoherence in the very attempt to talk about the theory of types in general. [*Principia*] is not doomed to incoherence by somehow violating its own strictures on types in its very generalizations about types. Propositions about all types might be in violation of type theory and so meaningless in this technical sense, but not in the more general sense of being nonsense.[257]

So, at this point we may clarify the conflict between the two interpretations in the following terms. Statements like $*9{\cdot}131$, $*9{\cdot}14$, etc., *must* be read as metalinguistic statements that rule out expressions such as "$\phi!(\phi!\hat{x})$" and the like as ill-formed independently of the ontological status of so-called 'propositional functions'. This much seems to follow from Whitehead and Russell's claim that "propositions dealing with types acquire their importance largely from the fact that they can be interpreted as dealing with the symbols"[258]. According to Landini, proposi-

[254] WHITEHEAD & RUSSELL, 1927*a*, p.34 [1912, p.35].

[255] LINSKY, B., 1999, p.86. Our emphasis.

[256] LINSKY, B., 1999, p.86.

[257] LINSKY, B., 1999, p.86. Linsky, incidentally, also recognizes that there is a difficulty in reconciling $*9{\cdot}131$ with Church's reconstruction of *Principia* in terms of *r*-types. As he puts it: "It seems that while disjunctions will be of the type of their disjuncts, as will a formula and its negation, the argument structure and quantificational form of a function is relevant to its type. $*9{\cdot}131$ seems to require that if two two-place propositional functions are of different types, then the result of binding the same variable in each case will result in one-place propositional functions of different types. As a result, the division of propositional functions would be finer than the system of [Church's] *r*-types indicates." (LINSKY, B., 1999, p.86).

[258] WHITEHEAD & RUSSELL, 1927*a*, p.34 [1912, p.35].

tions like ∗9·131 are *just* meant as grammatical statements, not necessarily[259] having anything to do with types of entities. On Linsky's realist interpretation, on the other hand, numbered propositions dealing with types should be read as cutting both ways: they must assert grammatical statements embodying *metaphysical distinctions.*

But here, again, the evidence seems to stack in favor of Landini's interpretation. One main issue involved here is this: once type distinctions are interpreted as distinctions between ontological categories, there is a corresponding segmentation of the notion of logical generality, or 'variation', which must be accepted. And this is problematic in several ways. To begin with, if Linsky is correct, then there *is* something wrong in the passage quoted by Hylton above. For recall that there Russell claims that "the limitations to which the unrestricted variable is subject *do not need to be explicitly indicated*", because "they are the limits of significance of the statement in which the variable occurs, and are therefore *intrinsically* determined by this statement"[260]. On Landini's interpretation, there is no conflict between ∗9·131 and this passage because there are no types of entities and statements concerning types are not viewed as substantial claims of any sort. Thus, statements like "ϕ is of the same type as ψ" or "ϕ is not of same type as ψ" have truth-conditions in *Principia*'s metalanguage only so far as type-indices or symbolic conventions are concerned. If we accept Linksy's interpretation, on the other hand, it seems that statements like "ϕ is of the same type as ψ" or "ϕ is not of same type as ψ" are *substantial* claims which simply cannot be expressed in the object-language of *Principia* but *do need to be stated in the metalanguage.* Though this is not the severe issue raised by Hylton, it is a problem. For one thing, this goes against Whitehead and Russell's explicit claim that statements of types should be viewed as statements about symbols. For another, it is difficult to reconcile the metaphysical interpretation of type distinctions in *Principia* with the development of Russell's views from the *Principles* to *Principia* in a satisfactory way. As we discussed in the previous chapter, a fundamental driving force in the development of Russell's views in the period was his resolve to preserve what he called "the true formal variable" which has a completely unrestricted range. As we also discussed, it was this resolve to *preserve* the unrestricted variable that gradually led Russell to adopt an eliminativistic approach to classes, propositional functions via his Substitutional Theory[261]; but the available evidence also shows that it was this same wish to preserve the doctrine of the unrestricted variable that led him to abandon substitution and propositions as singular entities and to adopt the multiple-relation analysis of judgment[262]. These points again call the plausibility of Linsky's interpretation into question.

Furthermore, once we see that Hylton is wrong in claiming that Russell's views precluded him to accept any sort of metatheoretical perspective in Logic, we can follow Landini in distinguishing sharply the purely syntactical aspects of *Principia* - i.e., the formal system

[259] Cf. the point made below.
[260] WHITEHEAD & RUSSELL, 1925, p.4 [1910, p.4].
[261] Which, recall, was able to emulate the structure of the simple theory of types without any commitement to type distinctions between entities.
[262] Which, as we saw, is best interpreted as an attempt to eliminate 'propositions' in an ontological sense

which is build from the primitive numbered propositions - and the semantics which Russell (and Whitehead) intended for this formalism. Although it is certainly problematic or contentious to attribute to Russell and and Whitehead a clear or sharp grasp of this distinction as we have of it today, once we agree with Landini that *Principia*'s Introduction puts forward an informally sketched *interpretation* (i.e., a semantics) for *Principia*'s formal grammar, we must agree that they were able to separate this interpretation or semantics from the formal system: this much unquestionably follows from Whitehead and Russell's claim in the preface that their "logical system is *wholly* contained in the numbered propositions which are independent of the Introduction and the Summaries" and that "the Introduction and the Summaries are wholly explanatory, and form no part in the chain of deductions"[263]; this separation is also in accordance with Russell's later claim, concerning the Theory of Types as presented in *Principia*'s Introduction, that he "lay[s] no stress upon the particular form of that doctrine which is embodied in *Principia Mathematica*" , but that he remains "wholly convinced that without *some* form of the doctrine the paradoxes cannot be resolved"[264]. This, on its turn strongly corroborates the following point: whatever interpretation or semantics Russell (or Whitehead) intended for *Principia*'s formal system - be it a nominalistic substitutional semantics which Landini attributes to the Introduction to the first edition or a realistic semantics of ramified types of entities which Linsky attributes to that Introduction, against Landini - it seems that a clear distinction between these interpretations or semantics and the purely syntactical statements about types contained in the numbered propositions *ought* to be kept straight. Indeed, it seems that even if Hylton and Linsky are right in claiming, against Landini, that Russell intended a realistic semantics in *Principia*'s Introduction, the statements concerning types in the numbered propositions can - and should - be kept sharply distinguished from such a realistic interpretation, or in fact, any interpretation of the formal system. This seems the general outcome of the author's claim that their "logical system is *wholly* contained in the numbered propositions which are independent of the Introduction and the Summaries"[265].

Thus, to sum up. It seems that Landini's interpretation provides the best approach to make sense of *Principia*'s type distinctions as presented in the Introduction and section ∗9, not only because of the good amount of positive evidence supporting it that can be found scattered throughout *Principia*'s text and elsewhere, but also because it is able to make sense of several aspects of *Principia*'s exposition that are simply baffling or even incoherent in light of a realist interpretation. This interpretation is strongly corroborated by the fact that Russell was well aware of the difficulties involved in treating "sameness of type" on a par with genuine relations and it makes a very strong case for the view that *Principia* was not intended at all by Russell to have a type hierarchy of entities as far as the Introduction to the first edition is concerned. On

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263 WHITEHEAD & RUSSELL, 1925, p.vii [1910, p.viii]. Our emphasis.

264 RUSSELL, B., 1959, p.79.

265 WHITEHEAD & RUSSELL, 1925, p.vii [1910, p.viii]. Our emphasis. Indeed, Landini plausibly argues that Whitehead and Russell had *different* conflicting semantic interpretations of the formal system (cf. footnote 189 of the introduction of the present dissertation).

this account there is no incoherence in numbered propositions like $*9{\cdot}131$, deep or otherwise. The reason is that "sameness of type" and "difference of type" are meant to apply first and foremost to expressions, not to entities. And in any case, numbered propositions like $*9{\cdot}131$ are best interpreted as statements about grammar or formal syntax[266]: in Landini's own words, "they concern the formal grammar of *Principia* and the conditions for well-formedness of symbols"[267].

4.4.3 The Role and Status of the Axiom of Reducibility

One important issue which we considered in the introduction of the present work is the (universally recognized) problematic character of the Axiom of Reducibility. The Axiom has famously struck readers of *Principia* as an *ad hoc* way to recover important results from Classical Mathematics by circumventing the severe restrictions of the so-called 'ramified' theory of types. It seemed to most readers, as Quine put it, that "[...] the axiom of reducibility implies the superfluousness of the very distinctions that give it substance"[268]. As we noted in the Introduction, Quine yet again found that this situation was another problematic consequence of the unruly ambiguity of the notion of a 'propositional function' in *Principia*, claiming that by "failing to distinguish sharply between formula and object, he [Russell] did not think of the maneuver of letting a higher-order expression refer outright to a lower-order attribute or relation-in-intension"[269]. Once again echoing Gödel's criticism, Quine's point is that "[...] there is an excuse for orders only if a weak constructive theory is to be adhered to and the axiom of reducibility withheld"[270].

Both Landini's and Linksy's interpretations provide grounds for refusing Quine's negative assessment. Let us recapitulate how the Axiom of Reducibility is formulated according to the rival interpretations which we are considering[271]. According to Church's reconstruction which Linsky advocates, axioms of reducibility are given by the following scheme which employs his *r*-types:

$$(\psi^{(t_1,\ldots,t_m)/n})((\exists\phi^{(t_1,\ldots,t_m)/1}))(x_1^{t_1},\ldots,x_m^{t_m})[(\psi(x_1,\ldots,x_m)\equiv\phi(x_1,\ldots,x_m)]$$

In words, what this asserts is that given any so-called non-predicative ψ (i.e., of any arbitrary level), there is a predicative function ϕ (i.e., whose level is 1) which is equivalent to ψ for all arguments $x_1,\ldots,x_m$. On Landini' reconstruction, on the other hand, reducibility axioms are

[266] Much like $*1{\cdot}11$ is an attempt to state rules of inference that Whitehead and Russell explicitly recognize that cannot be expressed as laws *within* their symbolism.
[267] LANDINI, G., 1998a, p.259.
[268] QUINE, W., 1969, p.263. For other important references in this regard, cf. the previous footnote 47 of the Introduction of this dissertation.
[269] QUINE, W., 1969, p.254.
[270] QUINE, W., 1969, p.254.
[271] For a fuller discussion of how the Axiom of Reducibility fares according to Landini's interpretation and that of Linsky, cf., WAHL, R., 2011. Wahl addresses in detail many issues which we cannot discuss here, including Ramsey's criticisms and the relation between the Axiom of Reducibility and Leibniz's principle of the 'identity of indiscernibles'.

given by the following scheme, which employs Landini's simple type symbols[272]:

$$(\exists\phi^{(t_1,...,t_n)})(x_1^{t_1},...,x_1^{t_n})\{\phi(x_1,...,x_m) \equiv A[x_1,...,x_m]\},$$

where $A[x_1,...,x_m]$ is a formula containing free occurrences of $x_1,...,x_m$ and where ϕ does not occur free. What this asserts is that any formula *A* of *Principia*'s formal language which contains free occurrences of $x_1^{t_1},...,x_1^{t_n}$ is equivalent to an atomic formula $\phi^{(t_1,...,t_n)}(x_1^{t_1},...,x_1^{t_n})$, where each $t_1,...,t_n$ is a simple type-index. Understood in these terms, the above rendition of the Axiom of Reducibility gives us a fully *impredicative* axiom scheme of comprehension, in the sense that variables of the same (or higher) order\type of *F* can occur bound in *A* above.

Indeed, with respect to the Axiom of Reducbility, the main contrast between Church's reconstruction and Landini's is that, as we mentioned before, Church's reconstruction's attributes to *Principia* another implicit axiom scheme of *predicative* comprehension, namely:

$$(\exists\phi^{(t_1,...,t_m)/n})(x_1^{t_1},...,x_m^{t_m})\{\phi(x_1,...,x_m) \equiv A[x_1,...,x_m]\},$$

where the bound variables of A are all of order less than *n* and the free variables of A are of order not higher than *n*[273]. In practice, this contrast is not stark. Take as an example the definition of the concept Nc induct i.e., natural number. If we follow Church's rendition of *Principia*, the so-called 'function' Nc induct(α) would have to be defined first in terms of predicative comprehension, instantiated as follows:

$$(\exists\theta^{(t)/n+1})(\gamma^t)\{\theta^{(t)/n}(\gamma^t) \equiv (\phi^{(t)/n})[\phi^{(t)/n}(0^t)\,.\,\phi^{(t)/n}\alpha^t\,.\supset_\alpha.\,\phi^{(t)/n}(\alpha^t+1^t)^t:\supset.\,\phi^{(t)/n}\gamma^t]\}$$

This gives us a concept of natural number that is fixed to some given order *r*-type whose order is higher than the bound variable ϕ which occurs the instance of predicative comprehension as illustrated above. Reducibility is a further principle that gives us the fully impredicative notion of natural number. Things are more straightforward if we follow Landini's rendition[274]: comprehension for him is only given in terms of reducibility, which, of course, suffices alone for the full impredicative notion to be obtained. The stark contrast between their accounts of Reducibility comes up, of course, in the underlying syntax and semantics of each formulation. Church treats non-predicative functions as genuine variables, whereas Landini treats them as schemata. Church and, following him, Linsky, takes all functional variables to be interpreted in terms of a realistic semantics of intensional entities, whereas Landini thinks that the simple-type structure of *Principia*'s grammar was meant to be justified by an informal (substitutional) nominalistic semantics sketched by Russell in the Introduction.

The interpretation of the role and status of the Axiom of Reducibility within *Principia*'s formal system depends on how the notion of a propositional function is understood. Linsky ar-

[272] Russell and Whitehead explicitly refrained from assuming it for predicate variables of more than two argument places. This, of course is a very minor issue.

[273] Of course, if constants are included in a language, then the constants occurring in A should also be of an order not higher than *n*.

[274] For now we are ignoring the complications introduced by *Principia*'s No-Class Theory.

gues that once the intensional Logic of *Principia* is understood as a theory of propositional functions conceived as structured entities, then distinctions of orders are motivated independently of the purpose of recovering the basic principles of set theory. So according to him the appropriate response to Quine is to note that the ramified logic of *Principia* should *not* be viewed merely "as a contructivist hierarchy of classes but rather as a theory of propositional functions which includes as a part the theory of classes, but which does much more"[275]. Linsky points out that "[...] propositional functions of higher types are needed to capture intensional phenomena"[276] and argues that Quine's criticism holds water only if one forgets "the intensional nature of logic, and hence the use for all those additional, non-predicative propositional functions"[277]. Thus, Linsky's conclusion is that the Axiom of Reducibility must be understood as a "metaphysical principle" of "great generality"[278] introduced to reconcile the logicist development of classical mathematics with an intensional conception of Logic as the theory of propositions and propositional functions[279]. As we already discussed, such an interpretation of *Principia* is largely unwarranted for several reasons: it goes against the many metaphysical views which accompanied *Principia*'s multiple-relation analysis of judgment and it presupposes Church's reformulation of *Principia*'s syntax, whose many departures from *Principia* find no textual support.

The main point to be taken into consideration is that Whitehead and Russell did *not* intend any predicate variables apart from predicative ones to be bound by quantifiers. This point alone - which is almost never taken into serious consideration - seems to seriously undermine Linsky's interpretation. As we already discussed, Landini's reconstruction allows us to make sense of many aspects of *Principia*'s formal system that are otherwise bewildering. One such case was that of so-called 'non-predicative functions', which are best interpreted as schematic letters. Concerning the Axiom of Reducibility, instead of making it a metaphysical principle embedded within a theory with an already exuberant ontological import such as a realist ramified logic, Landini's rendition of the Axiom of Reducibility is more in line with Russell's claim that the Axiom of Reducibility is "[...] a smaller assumption than the assumption that there are classes"[280]. But this is not to say that the Axiom does not have a quite problematic status according to Landini's interpretation. It does. Roughly, the problem is this: the nominalistic substitutional semantics cannot validate the Axiom of Reducibility; as Landini himself recognizes, the semantics is too weak to account for the strength of the simple-types structure which is (by design) imposed by the Axiom of Reducibility[281]. So, if we accept Landini's view, how can the

[275] LINSKY, B., 1999, p.108.
[276] LINSKY, B., 1999, p.101.
[277] LINSKY, B., 1999, p.108.
[278] LINSKY, B., 1999, p.108.
[279] In holding this position Linsky is again following Church, who holds that: "If, following early Russell, we hold that the object of an assertion or belief is a proposition and then impose on propositions the strong conditions of identity which this requires, while at the same time undertaking to for- mulate a logic that will suffice for classical mathematics, we therefore find no alternative except for ramified type theory with axioms of reducibility" (CHURCH, A., 1984, p.521).
[280] WHITEHEAD & RUSSELL, 1925, p.58 [1910, pp.60-1].
[281] Cf. LANDINI, G., 1998, pp.293-4; 2011, pp. 134-5.

Axiom of Reducibility be reconciled with Russell's attempt to justify the distinctions of orders in *Principia*'s Introduction? The answer is that it can't.

To some this may seem like a good reason to refuse a substitutional interpretation of *Principia*'s predicate variables. Scott Soames[282], for instance, has recently denied that a "[...] substitutional reading of *Principia Mathematica* makes good philosophical or mathematical sense on its own, or coheres well with Russell's most important goals and doctrines"[283]. Soames's main concern can be put in the following terms. Suppose that we allow objectual quantification over individual variables at the bottom of the type hierarchy but adopt a substitutional account of higher-order variables. That is: rising to second-order quantification, we assume that given any predicate variable, its substitution class will be, as Soames puts it, "[...] all simple predicates used to construct atomic sentences, plus complex predicates"[284]; thus, a sentence like $(X_1)A[X_1]$ will be true if, and only if, "every sentence is true that results from substituting an occurrence of a predicate, simple or complex, associated with X_1", for every occurrence of "X_1" in *A*.[285]. In doing this we are restricting ourselves to predicates "X_1" which are *first-order definable* (in the modern sense). Similarly, we assume the range of substituends for third-order variables is framed in such a way that only second-order formulas can be obtained, and so on. As Soames explains:

> Looking at this from the outside (where we continue to allow ourselves to speak of sets), this means that our substitutional construal of second-order quantification parallels ordinary objectual second-order quantification over those sets that are extensions of first-level predicates of individuals (including complex predicates). This process is repeated for third-order quantification, except that here complex predicates are the only ones in the substitution class. This level mimics objectual quantification over those sets that are extensions of second-level predicates, members of which are sets of individuals that are extensions of first-level predicates. The hierarchy continues uniformly from there on.[286]

The problem with this procedure, as Soames puts it, is that "[...] whereas the objectual quantifiers range over all sets at a given level - both those that are extensions of predicates at that level (of the Russellian logical language) and those that are not - the substitutional quantifier mimics only quantification over the former"[287]. This mirrors very closely Landini's account of higher-order quantification as based on his interpretation of the hierarchy of senses of truth

[282] Professor João Vergílio Cuter has made similar considerations pertaining to a substitutional account of *Principia*'s higher-order quantifiers in my qualification exam for which I am grateful.

[283] SOAMES, S., 2014, p.519. Soames appreciates in his response that an interpretation that views Russell's quantifiers as substitutional "does not, of course, credit him with mastering all of what we now recognize to be the ins and outs of substitutional quantification and its difference from objectual quantification", but *only* claims to "to capture an important element in his thought, and to develop it more clearly, precisely, and consistently than he himself did" (SOAMES, S., 2014, p.519).

[284] SOAMES, S., 2014, p.520.

[285] SOAMES, S., 2014, p.520.

[286] SOAMES, S., 2014, p.520.

[287] SOAMES, S., 2014, p.520.

and falsehood given in *Principia*'s Introduction. Soames's concern is that the resulting logic would not be "[...] rich enough to provide a foundation for [...] higher mathematics"[288]. Soames exemplifies this with the case of Arithmetic: on the substitutional construal, the range of second-order quantifiers is restricted to those predicates that are first-order definable; and generally, the range of predicate variables of order $n + 1$ are restricted to expressions that are nth-order definable; since numbers are classes of classes, these should be defined by some third-order expression that is second-order definable. This just leads to an exact analogue to the problem we witnessed in connection with mathematical induction and Church's *r*-types:

> The crucial constraint on the substitutional quantifier leading to this result is one that excludes any expressions in the substitution class associated with a substitutional variable from containing that variable. (When this constraint is violated, truth cannot be defined for the original quantified sentence.) So when we have a substitutional interpretation of the quantification employed in the definition of 'N' and the statement of mathematical induction, the complex predicates that are substitutable for the third level predicate variable can't themselves contain such a variable. With this, we lose the simple Fregean way of understanding and proving mathematical induction. Thus, the strongest argument against the substitutional interpretation of quantification in *Principia Mathematica* is that it threatens the reduction there".[289]

Now, it must be observed that it is not clear whether Soames takes this be an objection against *interpreting Principia* in terms of a substitutional semantics or whether this is a criticism of *Principia* itself. But clearly there is a sense in which this issue cannot be raised as an objection to interpreting *Principia* in terms of a substitutional semantics, for the above problem just shows exactly why *Principia* requires the Axiom of Reducibility (and incidentally, just how powerful the axiom is). In other words, what Soames takes as a strong argument against the substitutional interpretation may just reflect a basic feature of the "ramified" structure which results from the hierarchy of senses of truth and falsehood that Russell proposes explicitly in *Principia*'s Introduction[290]. In other words, this was a difficulty which Russell and Whitehead explicitly recognized and accepted. It cannot be straightforwardly taken as an objection against *interpreting Principia* in terms of a nominalistic substitutional semantics.

But still, if Landini is right, then the difficulty which is so clearly explained by Soames *is* a real problem for Russell's intended semantics for *Principia*'s predicate variables: the problem

288 SOAMES, S., 2014, p.525.
289 SOAMES, S., 2014, p.522-3.
290 Of course, a similar point applies to Church's reconstruction, since his system also requires reducibility principles to recover classical mathematics, being much weaker than standard impredicative higher-order Logic. It must also be observed that one can adopt a substitutional interpretation of the quantifiers while also adopting Church's syntax. In doing this, the restrictions imposed by the substitutional account just reflect the predicativity constraint on the definability of functions. And in fact, as Allen Hazen pointed out, "the substitution interpretation of the variables works so nicely, indeed, that one is almost tempted to identify it as the intended interpretation of the ramified logics" (HAZEN, A., 1983, p.354.). For another approach for developing a predicative theory of types with a substitutional account of higher-order quantification cf. CHIHARA, C., 1973.

is that this semantics simply cannot justify the acceptance of the Axiom of Reducibility. Landini recognizes this, however. As he observes, what can be justified is the following predicative comprehension principle (which, incidentally, is akin to Church's, apart from the fact that he allows non-predicative variables)[291]:

$$(\exists\phi^{(t_1,...,t_m)})(x_1^{t_1},...,x_m^{t_m})\{\phi(x_1,...,x_m) \equiv A[x_1,...,x_m]\},$$

where: (i) $A[x_1,...,x_m]$ is a formula containing free occurrences of $x_1,...,x_m$; (ii) ϕ does not occur free; (iii) all bound variables are of a lower order than $(t_1,...,t_m)$; and (iv) the free variables are all of an order which is at most equal to that of $(t_1,...,t_m)$. According to Landini, the fully impredicative reducibility, on the other hand, is "an unwarranted [...] principle which assures that order vanishes altogether in extensional contexts in favor of the impredicative comprehension of simple type-theory"[292]. In other words, he recognizes that if we accept Russell's intended nominalistic semantics there is *nothing* which can assure us of the truth of such axiom in *Principia*, apart from the fact that desired consequences follow from it. But Whitehead and Russell themselves say as much, when they claim that the reasons in favor of it are "largely inductive" i.,e "many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it"[293].

So at this point we must carefully distinguish the following two questions:

1. What is the *informal justification provided* in *Principia*'s Introduction (which is *not part of the formal system*) for the grammar of types that the work embraces? and
2. What sort of interpretation for the grammar of simple types is required to sustain the mathematical edifice built up from $*12$ onward in *Principia*?

Given all the points we discussed so far, we may reasonably claim that although Russell intended something along the lines of the substitutional semantics formulated by Landini to justify the grammar of simple types which *Principia* adopts in practice, such semantics *cannot* account for the richness of *Principia*'s language. This is because what *Principia* requires is (at least) something as strong as an intensional simple type theory, that is, a theory where each open formula $\phi^{(o)}(y^o)$, $\phi^{((o))}(\psi^{(o)})$, $\phi^{(o,(o))}(x^o,\psi^{(o)})$, etc, determines an attribute or relation-in-intension. But this, of course, requires that predicate variables range over genuine entities, which amounts to embracing *actual types* of entities. So we would have a realist semantics for predicate variables

291 LANDINI, G., 2011a, p.135.
292 LANDINI, G., 2011a, p.135.
293 WHITEHEAD & RUSSELL, 1925, p.59 [1910, p.62]. As we shall discuss later, when viewed in the broader context of Russell's views on Logicism and Mathematical Philosophy in general, this is not as bad a justification as many suppose.

instead of a substitutional one. This, however, does not mean that Landini's interpretation is wrong, only that Russell's plan for justifying *Principia*'s simple type-grammar was a failure - which is, in fact, Landini's position[294].

4.4.4 The No-Class Theory

The discussion about the role of the Axiom of Reducibility leads us to *Principia*'s No-Class theory. Just like Russell's previous Substitutional Theory, the basic idea of *Principia*'s No-class Theory is to contextually eliminate class-expressions like "$\hat{z}\phi z$" and "$\hat{z}(\phi!z)$", treating them as incomplete symbols. The same is done for expressions "$\hat{x}\hat{y}\phi(xy)$", "$\hat{x}\hat{y}\phi!(xy)$" for relations-in-extension (which, following Landini we will refer to as "relations$_e$"). The treatment of class-expressions is given in $*20$ and that of relations$_e$ in $*21$. At the core of these sections lies a series of definitions on the basis of which Whitehead and Russell emulate a theory of simple types of classes and (homogeneous *and* heterogeneous) relations$_e$.

With respect to the No-class theory presented in $*20$, the most fundamental definitions are those which Russell had already given in *Mathematical Logic as Based in the Theory of Types*[295], namely[296]:

$*20{\cdot}01 \qquad f\{\hat{z}(\psi z)\} \,.\!=\!:\, (\exists\phi) : \phi!x \,.\equiv_x.\, \psi x : f\{\phi!\hat{z}\} \quad \text{Df}$

$*20{\cdot}02 \qquad x \,\epsilon (\phi!\hat{z}) \,.\!=\!.\, \phi!x \quad \text{Df}$

Definition $*20{\cdot}01$ gives the contextual elimination of class expressions like "$\hat{z}(\psi z)$" from any well-formed formula in which they occur, while $*21{\cdot}02$ is introduced to emulate abstraction or concretion principles. Observe that, in the above, theorems are framed using typically ambiguous *individual* variables - as we'll discuss below, this is a very important point.

Also following a convention of Russell's previous paper, in *Principia* Whitehead and Russell introduce lower-case Greek letters such as "α", "β", "γ", etc to serve, as they put it, "*merely* [...] as an abbreviation for an expression of the form $\hat{z}\phi z$"[297]. These lower case Greek

[294] Cf. LANDINI, G., 1998a, pp.291-294 and 2011, pp.135-6. It must be also be said, however, that there is a precise sense, in which a variant of Linsky's and Cocchiarella's interpretation, as based on Church's reconstruction of *Principia*, is vindicated. In a relatively recent paper, Edwin Mares formulated a formal semantics based on Linsky's interpretation which actually validates the Axiom of Reducibility. For details cf. MARES, E., 2007. It is important to notice however, that Mares departs drastically from Linky's interpretation with respect to two crucial points: he treats predicative propositional functions as universals which enter as constituents of complexes and his interpretation of non-predicative propositional functions is substititional, having no ontological counterparts for them in his intended models for the Axiom of Reducibility. Again, for details cf. MARES, E., 2007 and also WAHL, R., 2011, pp.54-55.

[295] RUSSELL, B., 1908, pp.613-616.

[296] As we shall briefly discuss below, these definitions have implicit scope markers.

[297] WHITEHEAD & RUSSELL, 1925, p.194 [1910, p.204]. Similarly, for relations they introduce capital Latin variables such as "*P*", " *Q*", "*R*", "*S*", etc., to serve as place-holders for expressions like "$\hat{x}\hat{y}\psi(x,y)$".

letters are introduced so that an arbitrary class can be represented by a single symbol, which is obviously very convenient. But *Principia*'s No-Class theory goes into further detail than Russell's previous paper regarding the use of these. Despite employing such lower-case Greek variables bound to quantifiers in *Mathematical Logic*, at no point in the paper Russell introduces explicit conventions or definitions which allow for their contextual elimination. In *Principia*, on the other hand, these 'variables' are to be eliminated according to the following definitions:

$*20{\cdot}07 \qquad (\alpha) \,.\, f\alpha \,.=.\, (\phi) : f\{\hat{z}(\phi!z)\} \quad \mathrm{Df}$

$*20{\cdot}071 \qquad (\exists\alpha) \,.\, f\alpha \,.=.\, (\exists\phi) : f\{\hat{z}(\phi!z)\} \quad \mathrm{Df}$

Just as important, Whitehead and Russell also have separate contextual definitions for classes of classes, that is, expressions for the extensions of 'functions' whose range of arguments include lower-case Greek variables, i.e., expressions like "$\hat{\alpha}(\psi\alpha)$":

$*20{\cdot}08 \qquad f\{\hat{\alpha}(\psi\alpha)\} \,.=:\, (\exists\phi) : \psi\alpha \,.\equiv_{\alpha}.\, \phi!\alpha : f\{\phi!\hat{\alpha}\} \quad \mathrm{Df}$

$*20{\cdot}081 \qquad \alpha \,\epsilon\, \psi!\hat{\alpha} = \psi!\alpha \quad \mathrm{Df}$

The above are just analogous to ∗20·01·02 applied to expressions for classes *of classes* instead of classes of individuals.

Now, as we already discussed in the introduction of the present work, the status of these definitions put forward in section ∗20 is a matter of serious debate in secondary literature, for many reasons. The main point of controversy, of course, is the status of expressions like "$\hat{z}(\psi z)$" and lower-case Greek letters; in particular, whether they should be read as *terms* of *Principia*'s language and whether they should be viewed as ranging over some kind of *abstracta.* And any answer to such issue, on its turn, would obviously have to fall back on an account of *Principia*'s problematic use of the phrase "propositional function" and of 'circumflected' expressions like "$\phi\hat{x}$" and "$\phi!\hat{x}$.

So, naturally, the two rival interpretations which we are discussing interpret the No-Class theory in radically different ways. Linsky follows Hylton and Goldfarb in characterizing so-called 'propositional functions'[298] as structured intensional entities. Thus, he does not take the No-Class theory to be an attempt to ontologically *eliminate* classes from Mathematics; on the contrary, according to him:

[298] And also so-called 'propositions'.

> The no-class theory provides a *replacement* of talk of classes by theses about some of the universe of propositional functions. The technique used is contextual definition, rather than explicit definition, but the effect is not any more of a wholesale ontological elimination than is the theory of definite descriptions.[299]

Again, in stark contrast, Landini holds that *Principia*'s No-Class theory is designed as a device of outright elimination. Since according to him there are no such things as 'propositional functions' understood as abstract entities, the definitions put forward in ∗20 are viewed by him as an attempt to eliminate discourse about classes completely with the help of the Axiom of Reducibility viewed as an impredicative comprehension axiom for *Principia*'s predicate variables.

How can this dispute be settled with respect to *Principia*'s no No-Class theory? As Landini has observed in a recent paper[300], simply analyzing the passages relevant to the main issues surrounding the problematic notion of 'propositional function' may not be enough, so what we must do is submit rival interpretations to an empirical tests: we must see which interpretation leads to a reconstruction of ∗20 which is best fit to achieve the goals Whitehead and Russell set out to achieve with their No-Class theory; we must see what happens when actual proofs in *Principia* are viewed in light of each interpretation; we must see which interpretation requires less modification in these proofs, and so on. Again, the evidence overwhelmingly favors Landini's view, and it is worth reviewing his arguments in detail.

Before proceeding, however, we need a little context of what is going on in section ∗20 - we must have a clear grasp of what Whitehead and Russell intended to prove in ∗20 where the contextual elimination of classes is given. There are three fundamental sets of theorems in section ∗20. The first is meant to show that the fundamental formal properties which are ordinarily expected of classes are preserved by their No-class theory. The most important theorems of this first group - and perhaps of sections ∗20 as a whole - are the following:

∗20·15 $\vdash :. \psi x \,.\equiv_x.\, \chi x : \equiv .\, \hat{z}(\psi z) = \hat{z}(\chi z)$

∗20·3 $\vdash : x \in \hat{z}(\psi z) \,.\equiv.\, \psi z$

∗20·31 $\vdash :. \hat{z}(\psi z) = \hat{z}(\chi z) : \equiv : x \in \hat{z}(\psi z) \,.\equiv_x.\, x \in \hat{z}(\chi z)$

These theorems show that the contextual elimination of expressions for classes and relations$_e$ can achieve what they were designed for: the recovery of extensional contexts from intensional ones. Theorem ∗20·15, in particular, as Whitehead and Russell explain embodies "[...] the essential properties of classes, and gives the justification of the definition ∗20·01"[301]; it is indeed *Principia*'s version of Frege's Basic Law V. Theorems ∗20·3 is an emulated abstraction

299 LINSKY, B., 1999, p.133.
300 LANDINI, G., 2013b, p.169-70.
301 WHITEHEAD & RUSSELL, 1925, p.191 [1910, pp.201].

principle and $*20{\cdot}31$ is an (also emulated) extensionality principle[302]. There is also a second group of theorems concerned with "dealing with both classes and descriptions", as Russell Whitehead put it. It is meant to prove results like the following, where class expressions interact with descriptions:

$$*20{\cdot}57 \qquad \vdash : \hat{z}(\phi z) = (\imath\alpha)(f\alpha) \,.\supset : g\{\hat{z}(\phi z)\} \,.\equiv.\, g\{(\imath\alpha)(f\alpha)\}$$

The third group of theorems is meant to to show that "variable classes satisfy all the primitive propositions assumed for variable individuals or functions" and thus that "classes of classes have all the formal properties of classes of individuals"[303]. The point is to show that the general laws of quantification and identity hold good when theorems are framed with lower-case Greek variables instead of individual variables, and thus that all previous results (of the first set of theorems) which hold good of classes of individuals can be assumed to hold also of classes of classes of individuals, classes of classes of classes and so on, indefinitely.

Now, to Landini's arguments. Throughout several different texts, Landini has put forward many strong arguments in favor of his interpretation of the No-Class theory. The three arguments we shall consider have at their core very simple points, namely:

1. *Principia*'s theory of incomplete symbols employs Russell's famous notion of 'scope' and carefully observing the role played by this notion shows that no incomplete symbol - class expressions included - should be considered as a genuine term of *Principia*'s language;
2. If *Principia*'s No-Class theory was simply designed to outright replace an ontology of classes for an ontology of propositional functions, then supplementary definitions like $*20{\cdot}08{\cdot}081$ would not have to be introduced to deal with classes of classes of individuals, classes of classes of classes of individuals, and so on - but they *are* needed;
3. Church's reconstruction of *Principia*'s syntax faces insurmountable difficulties dealing with scope in section $*20$.

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[302] It must be observed that both theorems $*20{\cdot}15$ and $*20{\cdot}3$ depend crucially on the Axiom of Reducibility (the demonstration of $*20{\cdot}15$ assumes $*20{\cdot}13$, which assumes $*13{\cdot}195$, which, in turn, assumes $*13{\cdot}101$; while the demonstration of $*20{\cdot}3$ introduces $*12{\cdot}1$ explicitly in the fourth line of the proof). The same applies exactly to the counterparts for relations$_e$. Of course, this couldn't be otherwise, since the bound variables which occur in the definiens of $*20{\cdot}01$ and $*21{\cdot}01$ must necessarily be restricted to *predicative* values in order to work as intended.

[303] WHITEHEAD & RUSSELL, 1925, p.196 [1910, p.206]. The first set of theorems is $*20 \cdot 1 - 43$, the second we mentioned is $*20 \cdot 5 - 59$; the third is $*20 \cdot 6 - 81$. Definitions and results that parallel those of $*20$ are given in $*21$ for relations$_e$. Since the definitions and theorems for the latter are almost completely analogous to those of classes, we shall only discuss the definitions and theorems which are concerned with the emulated theory of classes.

At the center stage of Landini's arguments is *Principia*'s notion of scope of a definite description and other incomplete symbols.

It is important to recall that the contextual elimination of classes and relations$_e$ is introduced in *Principia* as an extension of Russell's treatment of definite descriptions. The formal treatment of Russell's celebrated theory is given in ∗14[304]. The two main definitions are the following:

∗14·01 $[(\imath x)(\phi x)] \mathbin{.} \psi(\imath x)(\phi x) \mathbin{.}{=}\mathbin{:} (\exists b) : \phi x \mathbin{.} \equiv_x x = b : \psi b$ Df

∗14·02 $\text{E!}(\imath x)(\phi x) \mathbin{.}{=}\mathbin{:} (\exists b) : \phi x \mathbin{.} \equiv_x x = b : \psi b$ Df

The use of brackets "[...]" is very important in the above definitions: they mark what Russell calls the *scope* of descriptions. Taking the simplest of examples, let us look at the following:

$$[(\imath x)(\phi x)] \mathbin{.} \sim \psi(\imath x)(\phi x) \tag{4.4.1}$$

$$\sim [(\imath x)(\phi x)] \mathbin{.} \psi(\imath x)(\phi x) \tag{4.4.2}$$

In the first formula, the description has what Russell calls a primary occurrence, in the second, it has what he calls a secondary occurrence. Unpacking 4.4.1 and 4.4.2 according to $*14 \cdot 01$ would respectively yield the following:

$$(\exists y) : \phi x \mathbin{.} \equiv_x \mathbin{.} x = y : \sim\psi y$$

$$\sim (\exists y) : \phi x \mathbin{.} \equiv_x \mathbin{.} x = y : \psi y$$

If, for instance, we were dealing with an ambiguous sentence of ordinary language like "the ϕ is not equal to *z*", the difference of scope considered above would be given as follows:

[304] Besides its centrality in terms of how the basic ideas of the section are applied by the authors, section $*14$ stands out in the work for another important reason: it is the only number of the work where we find theorems in which bound propositional variables occur. The most important is this:

∗14·3 $\vdash \mathbin{.} p \equiv q \mathbin{.} \supset_{p,q} \mathbin{.} f(p) \equiv f(q) : \text{E!}(\imath x)(\phi x) : \supset :$

$$f\{[(\imath x)(\phi x)] \mathbin{.} (\imath x)(\phi x)\} \mathbin{.} \equiv \mathbin{.} [(\imath x)(\phi x)] \mathbin{.} f\{(\imath x)(\phi x)\}$$

This proposition establishes that the scope of an expression "$(\imath x)(\phi x)$", i.e., a definite description, is irrelevant in determining the truth-value of any proposition (or, better yet, *sentence*) in which it occurs when $\text{E!}(\imath x)(\phi x)$. The comments on it are are relevant in connection to the status of "propositions" and "propositional functions". The authors observe that "the use of propositions as apparent variables involves an apparatus not required elsewhere, and we have therefore not used this proposition in subsequent proofs". They explicitly claim that the reason for this is that they cannot legitimately introduce *p*, *q*, etc., as bound variables "without the explicit introduction of the hierarchy of propositions with a reducibility-axiom such as ∗12·1·11". In response to the "whole school of writers" who "state that *Principia* does not contain metatheorems" (KRIPKE, S., 2005, p.1013), Kripke aptly observes that the above theorem is "an explicit counterexample to their claim, though there are many others, and this is not even the most important counterexample" (KRIPKE, S., 2005, p.1013).

$$(\exists y) : \phi x . \equiv_x . x = y : y \neq z$$

$$\sim (\exists y) : \phi x . \equiv_x . x = y : y = z$$

In the first formula of each pair of formulas above, the scope of the description operator is the narrowest possible, in the second, it is the widest. Whitehead and Russell adopt the convention of generally omitting "explicit mention of scope"[305], for as they observe, "[...] the scope usually required is the smallest proposition enclosed in dots or brackets in which '$(\imath x)(\phi x)$' occurs"[306].

As Landini insists, the remarks made by Whitehead and Russell's when introducing the contextual elimination of expressions "$(\imath x)(\phi x)$" for definite descriptions strongly suggest that $*14$ is introducing a contextual elimination of expressions like $(\imath x)(\phi x)$ by employing the letters ψ, ϕ, etc., as *schematic letters*, not predicate variables. Take the following:

> For reasons explained in the Introduction, we do not define "the *x* which satisfies $\phi \hat{x}$" but we define any proposition in which this phrase occurs. Thus when, we say: " The term *x* which satisfies $\phi \hat{x}$ satisfies $\psi \hat{x}$" we shall mean: "There is a term *b* such that ϕx is true when, and only when, *x* is b, and ψx is true" That is, writing "$(\imath x)(\phi x)$" for "the term *x* which satisfies ϕx", $\psi(\imath x)(\phi x)$ is to mean:
>
> $(\exists b) : \phi x . \equiv_x x = b : \psi b$
>
> This, however, is not yet quite adequate as a definition, for when $(\imath x)(\phi x)$ occurs in a proposition which is part of a larger proposition, there is doubt whether the smaller or the larger proposition is to be taken as the "$\psi(\imath x)(\phi x)$".[307]

The only way to make sense of the above is by taking "$\psi(...)$" as a scheme standing for well-formed formulas. And, in fact, if ψ above were understood as a bindable object-language variable, there would be no need for scope markers[308]. As he also observes, definitions like $*20{\cdot}01$ *must* be read as containing implicit scope markers. And, in fact Whitehead and Russell explicitly say this:

> As in the case of $f(\imath x)(\phi x)$, so in that of $f\{\hat{z}(\phi z)\}$, there is an ambiguity as to the scope of $\hat{z}(\phi z)$ if it occurs in a proposition which itself is part of a larger proposition. But in the case of classes, since we always have the axiom of reducibility, namely
>
> $(\exists \psi) : \phi x . \equiv_x . \phi ! x$,
>
> which takes the place of E!$f(\imath x)(\phi x)$, it follows that the truth-value of any proposition in which $\hat{z}(\phi z)$ occurs is the same whatever scope we may give to $\hat{z}(\phi z)$, provided the proposition is an extensional function of whatever it may contain. Hence we may adopt the convention that the scope is to be always the

305 WHITEHEAD & RUSSELL, 1925, p.173 [1910, pp.181-2].
306 WHITEHEAD & RUSSELL, 1925, p.173 [1910, pp.181-2].
307 WHITEHEAD & RUSSELL, 1925, p.173 [1910, pp.181-2].
308 Cf. LANDINI, G., 2013b, p.191.

> smallest proposition enclosed in dots or brackets in which $\hat{z}(\phi z)$ occurs. If at any time a larger scope is required, we may indicate it by '$[\hat{z}(\phi z)]$' followed by dots, in the same way as we did for $(\imath x)(\phi x)$.[309]

Whitehead and Russell suppressed the scopes in accordance with the conventions of $*14$ - so the scope intended is always the narrowest possible[310]. Thus, $*20{\cdot}01$ should be read with scope markers that are simply omitted for readability.

More importantly, Landini points out that observing the need for (implicit or explicit) scope of descriptions - and incomplete symbols in general - shows that "$(\imath x)(\phi x)$" is not a term of *Principia*'s language and that the same applies to expressions for classes and relations$_e$[311]. Putting matters as directly as possible: Landini's point is that there is no such thing as a formula "$\psi[(\imath x)(\phi x)]$" in *Principia*'s object language![312] Landini exemplifies this with the following theorem:

$*14{\cdot}18$ $\vdash :. \mathrm{E!}(\imath x)(\phi x) \,.\supset :(x) \,.\, \psi x \,.\supset .\, \psi(\imath x)(\phi x)$

Restoration of scope markers and the elimination of "$(\imath x)(\phi x)$" in accordance with the above show this expression is not a term at all, and that "$\psi(\imath x)(\phi x)$" is a mere *scheme*. First, we restore the marker for a primary scope:

$$\mathrm{E!}(\imath x)(\phi x) \,.\supset :. (x) \,.\, \psi x : \supset : [(\imath x)(\phi x)] \,.\, \psi(\imath x)(\phi x)$$

Next, we eliminate the definitions, which yields:

$$(\exists b) : \phi x \,.\equiv_x .\, x = b :. \supset :: (x) \,.\, \psi x \supset :. (\exists b) : \phi x \,.\, x = b : \psi b$$

Similarly, let us take the following theorem:

$*14{\cdot}2$ $\vdash .\, (\imath x)(x = a) = a$

Restoring primary scope and eliminating "$(\imath x)(x = a)$" according to $*14{\cdot}01$ gives us:

$$(\exists b) : x = a \,.\equiv_x .\, x = b : b = a$$

Expressions of the form "$(\imath x)(\phi x)$" clearly do not behave as terms. The subjacent issue here is one raised by Gödel, who long ago complained that the order of elimination of incomplete symbols is *not* irrelevant. As Landini observes, since incomplete symbols like "$(\imath x)(\phi x)$" are

309 WHITEHEAD & RUSSELL, 1925, p.80 [1910, p.83].
310 Again, cf. WHITEHEAD & RUSSELL, 1925, p.173 [1910, pp.181-2].
311 LANDINI, G., 1998a, pp.165-171; see also LANDINI, G., 2011a, pp.117-124.
312 LANDINI, G., 2011a, pp.117-8.

not terms of *Principia*'s language, one *cannot* apply definitions framed with individual variables to them. Concerning theorem $*14{\cdot}2$ above, for instance, one *cannot* eliminate the identity sign directly according to *Principia*'s definition $*13{\cdot}01$, which is:

$$x = y \,.\!=\!:\, (\phi) \,.\, \phi!x \supset \phi!y \quad \text{Df}$$

This would yield:

$$(\phi) \,.\, \phi![(\imath x)(x = a)] \supset \phi!a$$

Which, of course, is an expression which Whitehead and Russell do allow; but Landini's point is that the above should *not* be taken as an expression of *Principia*'s object language and that it *cannot* be obtained directly from $*14{\cdot}2$ by applying $*13{\cdot}01$: that definition is framed with proper individual variables and thus cannot be applied to descriptions directly! And a similar point applies to expressions for classes: *none* of the expressions "$\hat{z}(\psi z)$", "$\hat{z}(\phi!z)$" are genuine terms. The pseudo-problems which arise if Landini's point is not followed show that he is right. Following Gödel, Carnap argued in *Meaning and Necessity*[313] that *Principia*'s definition of class expressions require modification in order to deal with the following issue. Assume, for the moment, that "Fx" asserts that *x* is featherless, "Bx" asserts that *x* is biped and "Hx" asserts that *x* is human. Assume further, then that the following is true:

$$(x)(Fx \,.\, Bx :\equiv .\, Hx) \tag{4.4.3}$$

$$(F\hat{x} \,.\, B\hat{x}) \neq H\hat{z} \tag{4.4.4}$$

Now, take the two sentences:

$$\{x : Hx\} = H\hat{z} \tag{4.4.5}$$

$$\{x : Hx\} \neq H\hat{z} \tag{4.4.6}$$

The conjunction of the above, Carnap argues, should be contradictory. It is not, however. If we apply $*20{\cdot}01$ to 4.4.5, we get:

$$\{x : Hx\} = H\hat{z} \,.\!\equiv:\, (\exists g)[(x)(gx \equiv Hx : g\hat{z} = H\hat{z})] \tag{4.4.7}$$

The right side of 4.4.7 is provable: just let $g\hat{z} = H\hat{z}$, from whence it follows that $gx \equiv Hx$ and generalize. Carnap recognizes that when applying $*20{\cdot}01$ "the smallest sentence or matrix in the actually given abbreviated notation is to be taken as corresponding to the left side in the definition"[314]. Thus, applying the definition to 4.4.6, he observes that we get:

$$\{x : Hx\} \neq H\hat{z} \,.\!\equiv:\, (\exists g)[(x)(gx \equiv Hx : g\hat{z} \neq H\hat{z})] \tag{4.4.8}$$

313 CARNAP, R., 1947.
314 CARNAP, R., 1947, p.149.

But the right side of 4.4.8 is provable from 4.4.3 and 4.4.4: just assume their conjunction $(x)(Fx \,.\, Bx : \equiv . \, Hx) : (F\hat{x} \,.\, B\hat{x}) \neq H\hat{z}$; then apply existential generalization on $F\hat{x}.B\hat{x}$. It turns out, then, that 4.4.5 and 4.4.6 are not only *not* contradictory - they are both true. Carnap finds this misleading, since the notation "suggests the interpretation of 4.4.6 as "$\hat{z}(Hz)$ is not identical with $H\hat{z}$"[315]. The problem, Carnap argues, is that Russell's definition was crafted so that "class expressions should be such that they can be manipulated as if they were names of entities"[316]. Carnap takes it that his "result makes this assumption doubtful", because it shows that "Russell's definition is not quite in agreement with the intended purpose"[317] What Carnap's example shows is that incomplete symbols should not be viewed as genuine terms and that the application of any definition (or theorem) framed in terms of individual variables like $*13{\cdot}01$ must wait the elimination of expressions like "$\hat{z}(\psi z)$".

The above leads us to Landini's second argument in favor of his interpretation of *Principia*'s emulation of classes of classes. As he observes, "if classes were themselves identified with propositional functions (i.e. entities of higher type) then these definitions [$*20{\cdot}01{\cdot}02$ and $*21{\cdot}01{\cdot}02$] would be sufficient"[318]. They are not, however. As we discussed, one of the main goals of section $*20$ is to recover basic principles governing extensional contexts. One important example that we mentioned is $*20{\cdot}3$, which, expressed with Landini's simple-type indices and with scope markers, is the following:

$$\vdash : [\hat{z}^t(\psi z^t)] \,.\, x^t \in \hat{z}^t(\psi z^t) \,. \equiv .\, \psi z^t$$

As Landini points out, *Principia* can and should (but doesn't) recover an analogue for classes of classes of the above, namely:

$$\vdash : [\hat{\beta}(f\beta)] \,.\, \alpha \in \hat{\beta}(f\beta) \,. \equiv_\alpha .\, f\alpha$$

This requires $*20{\cdot}08$. Let us follow Landini's reconstruction of the proof. First, we apply $*20 \cdot 07$ to eliminate lower-case Greek letters, getting[319]:

$$[\hat{\beta}(f\beta)] : [\hat{z}^t(\psi z^t)] \,.\, \hat{z}^t(\psi z^t) \in \hat{\beta}(f\beta) \,. \equiv .\, f\{\hat{z}^t(\psi z^t)\}$$

For convenience, let us drop the scope markers for class symbols like "$\hat{z}^t(\psi z^t)$" since here their occurrences are primary. Application of $*20 \cdot 08$ gets us:

$$(\exists g)(g!\alpha \,. \equiv_\alpha .\, f\alpha : \hat{z}^t(\psi z^t) \in g!\hat{\alpha}) \,. \equiv .\, f\{\hat{z}^t(\psi z^t)\}$$

Then, application of $*20{\cdot}01$ to the left side of the equivalence then gets us:

$$(\exists g)(g!\alpha \,. \equiv_\alpha .\, f\alpha : (\exists\phi)(\phi^{(t)}x^t \,. \equiv .\, \psi x^t : \phi^{(t)}\hat{x} \in g^{((t))}\hat{\alpha}) \,. \equiv .\, f\{\hat{z}^t(\psi z^t)\}$$

[315] CARNAP, R., 1947, p.149.
[316] CARNAP, R., 1947, p.149.
[317] CARNAP, R., 1947, p.150.
[318] LANDINI, G., 2011a, p.121.
[319] For readability, we have to employ more parenthesis for punctuation, reducing the number of dots, since the latter become very inconvenient in great number.

Then, applying ∗20·02, again, to the left side, we get:

$$(\exists g)(g!\alpha \,.\equiv_\alpha .\, f\alpha : (\exists\phi)(\phi^{(t)}x^t \,.\equiv.\, \psi x^t : g^{((t))}(\phi^{(t)}\hat{x})) \,.\equiv.\, f\{\hat{z}^t(\psi z^t)\}$$

We eliminate the remaining lower-case Greeks letters on the right side:

$$(\exists g)((\chi)(g!\{z^t(\chi^t z)\} \,.\equiv.\, f\{z^t(\chi^t z)\} : (\exists\phi)(\phi^{(t)}x^t \,.\equiv.\, \psi x^t : g^{((t))}(\phi^{(t)}\hat{x})) \,.\equiv.\, f\{\hat{z}^t(\psi z^t)\}$$

And then apply ∗20·01 again to the left side, which becomes:

$$(\chi)\left(\begin{array}{c} (\exists\phi)(\phi^{(t)}x^t \,.\equiv_x .\, \chi^t x \,.\, g^{((t))}(\phi^{(t)}\hat{x})) \\ \equiv \\ (\exists\phi)(\phi^{(t)}x^t \,.\equiv_x .\, \chi^t x \,.\, f(\phi^{(t)}\hat{x}))] \end{array}\right) : (\exists\phi)(\phi^{(t)}x^t \,.\equiv.\, \psi x^t : g^{((t))}(\phi^{(t)}\hat{x}))$$

embedded in "$(\exists g)(... \equiv f\{\hat{z}^t(\psi z^t)\})$". And if we apply ∗20·01 to the right side of the equivalence, we finally arrive at:

$$(\exists g)\left(\left(\left(\begin{array}{c} (\chi)((\exists\phi)(\phi^{(t)}x^t \,.\equiv_x .\, \chi^t x : g^{((t))}(\phi^{(t)}\hat{x})) \\ \equiv(\exists\phi)(\phi^{(t)}x^t \,.\equiv_x .\, \chi^t x : f(\phi^{(t)}\hat{x})) \end{array}\right)\right.\right.$$
$$\left.\left. : (\exists\phi)(\phi^{(t)}x^t \,.\equiv.\, \psi x^t : g^{((t))}(\phi^{(t)}\hat{x}))]\right) \equiv (\exists\phi)(\phi^{(t)}x^t \,.\equiv.\, \psi x^t : f(\phi^{(t)}x^t)\right)$$

The above gives the true form of abstraction for classes of classes of individuals in *Principia*, crucially relying on ∗20·08. And what does this very cumbersome formula tell us about the No-Class theory? Landini rightfully insists that it shows that the No-Class theory was a genuine attempt to get rid of classes as entities of any sort. The point, as Landini insists[320], is that ∗20·01 together with its auxiliary definitions enabled Whitehead and Russell to emulate a theory of *classes of entities* of any simple type and it is because classes of entities are *not* entities at all that ∗20·08 and auxiliary definitions are introduced, so that the theory can also emulate classes of classes of entities (of any simple type). There is simply no *term* of *Principia*'s object-language that outright takes the place of expressions for classes of individuals or classes of classes of individuals.

Now, Landini's third and most decisive argument also explores *Principia*'s notion of scope and the aims that ∗20 was supposed to achieve. The argument is very simple. Take ∗20·07, that is:

$$(\alpha) \,.\, f\alpha = (\phi) \,.\, f\{\hat{z}(\phi!x)\}$$

According to Church's rendition of *Principia*, "*f*" above is a genuine object language variable, not a schematic letter for well-formed formulas. This means that following Church's formulation a *primary scope* of "$\hat{z}(\phi!x)$" forces itself in the above formula. In other words, according to Church's reconstruction, restoring scope markers in ∗20·07 would have to give us:

$$(\alpha) \,.\, f\alpha = (\phi) : [\hat{z}(\phi!x)] \,.\, f\{\hat{z}(\phi!x)\}$$

[320] LANDINI, G., 2011a, p.122-3.

Landini's straightforward argument is that there are proofs that require a *secondary scope* in applications of $*20{\cdot}07$ in direct conflict with the above. One example he gives is the following theorem:

$$*20{\cdot}06 \vdash : (\exists\alpha) \,.\, f\alpha \,.\equiv.\sim \{(\alpha). \sim f\alpha\}$$

In *Principia*, this is proved as follows. By definition $*20{\cdot}071$, they have:

$$(\exists\alpha) \,.\, f\alpha \,.\equiv.\, (\exists\phi) \,.\, f\{\hat{z}(\phi!z)\}$$

By definition $*10{\cdot}01$ of the existential quantifier, they have:

$$(\exists\phi) \,.\, f\{\hat{z}(\phi!z)\} \,.\equiv.\sim [(\phi) \,.\sim f\{\hat{z}(\phi!z)\}]$$

From whence it follows:

$$(\exists\alpha) \,.\, f\alpha \,.\equiv.\sim [(\phi) \,.\sim f\{\hat{z}(\phi!z)\}]$$

Now, from definition $*20{\cdot}07$, they have:

$$(\alpha) \,.\sim f\alpha \,.\equiv.\, (\phi) \,.\sim f\{\hat{z}(\phi!z)\}$$

And thus:

$$\sim \{(\alpha) \,.\sim f\alpha\} \,.\equiv.\sim [(\phi) \,.\sim f\{\hat{z}(\phi!z)\}]$$

From this and above the desired proposition follows. What is relevant here is this: the application of $*10{\cdot}01$ shows that Whitehead and Russell allowed secondary occurrences of "$\hat{z}(\phi!z)$" in applications of $*20{\cdot}07{\cdot}071$, against Church's rendition[321].

All of this shows, as Landini emphasizes, that Russell meant to be working with an actual *No-Class* theory.

Indeed, this interpretation of the No-Class theory helps us to see a much more interesting picture of *Principia* in face of those who still accept Quine's concerns over Whitehead and Russell's use of "propositional function". As we mentioned above, Soames has claimed that the theory "remains dubious" as a device to contextually eliminate class expressions and that it raises "too many problems for too little result"[322]. Echoing Quine's views, Soames writes that:

> The entities to which classes are supposed to be reduced - propositional functions - are more taken for granted by Russell than seriously investigated. As already indicated, he speaks confusingly and inconsistently about them, and the view that seems to be uppermost in his mind - that they are expressions - is obviously inadequate. Although other choices - $extensional_1$ functions and gappy propositions - make more sense, he doesn't systematically explore them, and, as we have seen, they aren't promising candidates for achieving ontological economies anyway.[323]

321 Landini raises this same point in connection with the proof we considered above. Cf. LANDINI, G., 2013b, pp.200-1.

322 SOAMES, S., 2008, p.217.

323 SOAMES, S., 2008, p.217.

Similar remarks are easily found in recent literature[324], but Soames's interpretation - as presented in his response - gives a paradigmatic representation of how Russell's views are frequently misunderstood. We can now see that the root of all misunderstandings is the idea that what is defined by $*20 \cdot 01$ is "[...] a notation containing complex singular terms for classes"[325]. From this assumption, Soames is inevitably lead to the view that Russell was attempting a reduction of classes to so-called propositional functions.

Landini's interpretation provides us a much more interesting appraisal of the No-Class theory. For, as Soames correctly appreciates, "[...] whether or not an ontological reduction is achieved [by $*20{\cdot}01$ and its companion definitions] depends on how predicate quantifiers are understood"[326]. If we interpet *Principia*'s predicate variables in terms of a nominalistic semantics along the lines suggested by Landini we see clearly why Russell thought he was achieving a genuine ontological economy. Furthermore this interpretation also provides a compelling explanation as to why Russell abandoned the Substitutional Theory. The No-Class theory framed within the nominalistic substitutional semantics appears to have the same appeal of the Substitutional Theory of 1906 and more: the theory avoids any ontological commitments to classes, propositional functions and also to propositions[327]!

The above interpretation also explains why Russell's confidence in *Principia*'s contextual elimination of classes never wavered[328]. It is clear that even if we concede that Russell's intended plan for justifying the structure of simple types of *Principia*'s predicate variables in terms of a nominalistic substitutional semantics was a failure, we can still see that, on the above account, the No-Class Theory would still have a strong appeal to him. For even on a realistic semantics where each variable $\phi^{(o)}$, $\psi^{(o,o)}$, $\phi^{((o))}$, $\psi^{(o,(o))}$, etc., ranges over a totality of *universals*, the No-Class theory still avoids any commitments to classes and such interpretation would still preserve the idea that all variables are individual variables (although, of course, this would amount to abandoning Russell's doctrine that there is only *one* kind of individual variable with an unrestricted range). Surely, this shows that in a very important sense Quine was right all

[324] Following Peter Geach, who claims that the use of "propositional function" in *Principia* is "hopelessly confused and inconsistent" (GEACH, P., 1972, p.272), Richard Cartwright claims that "attempts to say what exactly a Russellian propositional function is, or is supposed to be, are bound to end in frustration" (CARTWRIGHT, R., 2005, p.915).

[325] SOAMES, S., 2008, p.214.

[326] SOAMES, S., 2008, p.214.

[327] This of course, dos not mean that the approach worked: the problem with it, as we just discussed, is that the need for the Axiom of Reducibility for the development of classical Mathematics undermines Russell's nominalistic semantics.

[328] Russell repeatedly referred to his contextual elimination of classes put forward in *Mathematical Logic* and *Principia*'s section $*20$ as giving the definite elimination of classes from the primitive vocabulary and assumptions of Mathematical Logic and Mathematics, even in posterior texts where he acknowledges that *Principia* does have serious problems (most notably the need for the Axiom of Reducibility). Cf., for instance, RUSSELL, B., 1919*a*, p.188; 1924, p.165; 1959, p.74-5. Indeed, even *before* Russell had settled for the approach put forward in *Principia* we have reason to believe that he and Whitehead took the approach of $*20$ to be established beyond doubt: on June of 1907 Whitehead wrote to Russell observing that "Your transition from intension to extension by means of [$*20 \cdot 01$] is beyond all praise. It must be right. That peculiar difficulty, which has worried us from the beginning is now settled forever" (MOORE, G., 2014, p.lxxxvi).

along when he claimed that a simple theory of types which purports to avoid any commitment to classes *and* to preserve most of classical Mathematics along the lines Russell intended must embrace simple type theory with an ontology of attributes and relations-in-intension. But then again, once we accept Landini's account of *Principia*'s Introduction we must conclude with him that Russell originally intended to completely avoid a hierarchy of types of entities of any sort and that Russell's plan to achieve that failed.

4.4.5 Russell's Later Remarks on 'Propositional Functions'

If we go beyond *Principia* for a discussion of the status of propositional functions and the ontological import of type distinctions, yet more evidence against Cocchiarella's and Linksy's interpretations can be found. In the article *On The Notion of Cause*, delivered as Russell's second *Presidential Address to the Aristotelian Society* on 4 November 1912[329] - thus right after the publication of *Principia*'s second volume - we find:

> [...] when it is worth saying that something "would be true under all circumstances", the something in question must be a propositional function, i.e. an *expression* containing a variable, and becoming a proposition when a value is assigned to the variable; the varying "circumstances" alluded to are then the different values of which the variable is capable.[330]

In Russell's abandoned *Theory of Knowledge* manuscript (the writing of which overlapped with the preparation of *Principia*'s third volume for publication) we find:

> By a "proposition", here, I mean a phrase which is grammatically capable of expressing a judgment; or one which, so far as form goes, might express a fact, though it may fail to do so owing to falsehood.[331]

If we turn to Russell's works published after *Principia*, the evidence that Russell took "propositional function" to be primarily understood as a kind of linguistic item is simply overwhelming. In *Introduction to Mathematical Philosophy*, we find:

> We mean by a "proposition" primarily *a form of words* which expresses what is either true or false. I say "primarily," because I do not wish to exclude other than verbal symbols, or even mere thoughts if they have a symbolic character. But I think the word "proposition" should be limited to what may, in some sense, be called "*symbols*," and further to such symbols as give expression to

329 SLATER, J., 1992, p.190.
330 RUSSELL, B., 1912, p.194. Our emphasis.
331 RUSSELL, B., 1913, p.80, footnote 1.

> truth and falsehood. [...] *A "propositional function," in fact, is an expression* containing one or more undetermined constituents, such that, when values are assigned to these constituents, the expression becomes a proposition. In other words, it is a function whose values are propositions.[332]

In *My Philosophical Development* we find Russell explicitly looking back at *Principia*'s notion of propositional function and accounting for it in purely linguistic terms:

> Whitehead and I thought of a propositional function as an *expression* containing an undetermined variable and becoming an ordinary *sentence* as soon as a value is assigned to the variable: '*x* is human', for example, becomes an ordinary sentence as soon as we substitute a proper name for '*x*'.[333]
>
> What logic requires is propositional functions, that is to say, expressions in which there are one or more variables and which are such that, when values are assigned to the variables, the result is a proposition.[334]

More importantly, in Russell's works after *Principia* we find him addressing the question of what the ontological counterpart of higher-order variables could be. There are two texts that provide *conclusive evidence* that at least *after Principia'*s publication, Russell would *not* endorse the idea that constructions are 'real' or 'exist' in any sense, nor the idea that type-distinctions have ontological import. The first is the 1918 set of lectures published as *The Philosophy of Logical Atomism*. There Russell states explicitly that "The theory of types is really a theory of symbols, *not* of things"[335]. When asked, by an unidentified member of the audience if he could "lump all those classes, and classes of classes, and so on, together"[336], Russell answered with the following:

> All are fictions, but they are different fictions in each case. When you say "There are classes of particulars", the statement "there are" wants expanding and explaining away, and when you have put down what you really do mean, or ought to mean, you will find that it is something quite different from what you thought. That process of expanding and writing down fully what you mean, will be different if you go on to "There are classes of classes of particulars." There are infinite numbers of meanings to "there are". The first *only* is fundamental, so far as the hierarchy of classes is concerned.[337]

Here he is explicitly acknowledging that *only* what we nowadays call first-order - i.e., individual - variables are *genuine variables* in the sense that their values are entities.

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332 RUSSELL, B., 1919a, p.155-6. Our emphasis.
333 RUSSELL, B., 1959, p.124. Our emphasis.
334 RUSSELL, B., 1959, p.167. Our emphasis.
335 RUSSELL, B., 1918, p.232. Our emphasis.
336 RUSSELL, B., 1918, p.233.
337 RUSSELL, B., 1918, p.233.

The exact same point is also discussed in more detail by Russell in an important - and often neglected article - from the fifties, *Logic and Ontology*, which is actually one of Russell's last technical works[338]. Besides introducing propositional functions again as *mere expressions*, the paper presents an account of the ontological commitments of a given theory which on the surface is much similar to that of Quine's[339]:

> I come now to the particular question of 'existence'. [...] I maintain that the only legitimate concept involved is that of $\exists$. This concept may be defined as follows: given an expression *fx* containing a variable, *x*, and becoming a proposition when a value is assigned to the variable, we say that the expression $(\exists x) \,.\, fx$ is to mean that there is at least one value of *x* for which *fx* is true. I should prefer, myself, to regard this as a definition of 'there is', but, if I did, I could not make myself understood.[340]

As Klement points out, however, "[...] it would be a mistake to read Russell as nothing more than a proto-Quinean"[341], for Russell did not take the ontological status of higher-order variables to be settled in the same way as that of first-order variables.

Russell introduces a distinction between the sense of the locution "there are ..." when applied to individual variables and when applied to higher-order terms, like numbers, explicitly warning that the latter use must not be understood as having any sort of 'Platonic' connotation:

> When we say 'there is' or 'there are', it does not follow from the truth of our statement that what we say there is or there are is part of the furniture of the world, to use a deliberately vague phrase. Mathematical logic admits the statement 'there are numbers' and metalogic admits the statement 'numbers are logical fictions or symbolic conveniences'. Numbers are classes of classes, and classes are symbolic conveniences. An attempt to translate $\exists$ into ordinary language is bound to land one in trouble, because the notion to be conveyed is one which has been unknown to those who have framed ordinary speech. The statement 'there are numbers' has to be interpreted by a rather elaborate process. We have first to start with some propositional function, say fx, then to define 'the number of things having the property *f*', then to define 'number' as 'whatever is the number of things having some property or other'. In this way we get a definition of the propositional function '*n* is a number', and we find that if we substitute for *n* what we have defined as '1', we have a true statement. This is the sort of thing that is meant by saying there is at least one number, but it is very difficult, in common language, to make clear that we are not making a Platonic assertion of the reality of numbers.[342]

338 The article was published in 1957 - Russell was already in his eighties. It illustrates very clearly how *ridiculous* is the claim that Russell never produced 'serious' or 'technical' philosophy after the 1920's or 1930's. A brief skim through Russell's *An Inquiry Into Meaning and Truth* from 1950 or the monumental *Human Knowledge: Its Scope and Limits* published in 1948 would also suffice.

339 There other passages that showcase this influence of Russell on Quine's views. One that is particularly striking is this: "[...] when you take any propositional function and assert of it that it is possible, that it is sometimes true, that gives you the fundamental meaning of 'existence'." (RUSSELL, B., 1918, p.204).

340 RUSSELL, B., 1957, p.235.

341 KLEMENT, K., 2018, p.156.

342 RUSSELL, B., 1957, p.234-5.

He observes that though "we cannot do without such words as 'precede' [...] such words do not seem to point at one of the bricks of the universe in the kind of way in which proper names can do"[343]. Russell acknowledges that predicate and relational expressions "serve a purpose in enabling us to assert facts which would otherwise be unstatable", but explicitly claims that it does not follow from this "that there is, *in any sense* whatever, a 'thing' called 'preceding' "[344].

Russell also addresses directly the question of how locutions like "there are ...", "there is a ... such that ..." should be understood when applied with predicate expressions when he acknowledges and addresses Quine's concerns over *Principia*'s bound predicate variables:

> Quine finds a special difficulty when predicates or relation-words appear as apparent variables. Take, for example, the statement "Napoleon had all the qualities of a great general". This will have to be interpreted as follows: 'whatever f may be, if "x was a great general implies fx, whatever x may be, then f(Napoleon)" '. This seems to imply giving a substantiality to f which we should like to avoid if we could. I think the difficulty real, and I do not know the answer. We certainly cannot do without variables that represent predicates or relation-words, but my feeling is that a technical device should be possible which would preserve the differences of ontological status between what is meant by names on the one hand, and predicates and relation-words on the other.[345]

Granted that this last passage makes it plain that Russell did not had a *definite* position on the ontological import of higher-order quantification, it is clear, as Kevin Klement points out, that the lectures on *Logical Atomism* and the article *Logic and Ontology* do present conclusive evidence that from 1918 onward "Russell thinks that first-order quantification is ontologically committing in a way that higher-order quantification is not"[346,347].

343 RUSSELL, B., 1957, p.236.

344 RUSSELL, B., 1957, p.237.

345 RUSSELL, B., 1957.

346 KLEMENT, K., 2018, p.164. Russell had already adressed the issue in passing in *An Inquiry Into Meaning and Truth*. There we find Russell explaining, in a rather careless way, that higher-order variables do not have substantial ontological counterparts besides expressions: "In a language of the second order, "$f(p)$ is true for every p" , '$f(\phi a)$ is true for every ϕ" , can be admitted as single sentences. This is familiar, and I need not dwell upon it. In the language of the second order, variables denote symbols, not what is symbolized."(RUSSELL, B., 1950, p.202). As Kevin Klement observes, this is somewhat misleading since it suggests an objectual interpretation of higher-order quantifiers over expressions which surely was not Russell's intent; this poor manner of expression can be overlooked, however, since, as Klement puts it, "Russell was writing for an audience that likely would not pick at this nit" (KLEMENT, K., 2018, p.164).

347 Klement departs from Landini, however, in two central aspects of his interprtetation. The first departure concerns the interpretation of *Principia*'s circumflex notation. Contrary to Landini's view, Klement thinks that "the circumflex notation does have a role to play in the object language of *PM*" (KLEMENT, K., 2013*a*, p.234), though this role is, admittedly, "*very narrow*": according to Klement there are circumflected terms in *Principia*'s language, but a term of this kind can only occur "as an argument to a higher-type propositional function *variable*" (KLEMENT, K., 2013*a*, p.234). Klement observes that "[...] the circumflex is still necessary"(KLEMENT, K., 2013*a*, p.234.) in some formulas of *Principia*'s original language to indicate that the predicate variable "ϕ" is occurring as an argument for the higher-order predicate variable "χ". And, in fact, formulas of this sort are easily found in *Principia*. The second issue of dispute between Landini and Klement concerns the substitutional interpretation of *Principia*'s predicate variables. Though Klement's is largely in agreement with Landini,

Now, though it must be also acknowledged at once that many of the texts quoted above were written and published long after *Principia* and one concern that may quite legitimately be raised is that all of them may be the result of Russell having changed his views, in particular due to the influence of the likes of Wittgenstein, Carnap, and most likely than anyone else, Quine. So the passages above must be handled with care: the views expressed in them cannot simply be projected into *Principia*. Still, Russell's later writings do provide strong evidence for interpretations which claim that the notion of "propositional function" must be understood primarily as being a purely linguistic notion - that is, either as meaning a bindable predicate variable (which receives simple-type indices) or a schematic letter.

4.5 Concluding Remarks

In this chapter we attempted to thoroughly consider distinct interpretative approaches that attempt to make sense of *Principia*'s problematic notion of a 'propositional function' in the face of the traditional criticisms put forward in secondary literature. There were two main approaches considered.

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he thinks that "Russell accepts a substitutional account for *all* types" (KLEMENT, K., 2018, p.164. Our emphasis.), including for *individual* variables. This claim is based on two sources of evidence. First, Klement observes that when Whitehead and Russell introduce the hierarchy of senses of truth and falsehood, their discussion "is not specifically targeted at first-order quantification". As in many other places in *Principia*, the point is rather that a sentence like "$(x).\phi x$" has 'second' truth or falsehood *relatively* to a sentence like "ϕa" which for the purposes of their exposition they are treating as being capable of first truth and falsehood. Second - and more importantly - Klement points to positive textual evidence that strongly indicates that Russell's also entertained the idea of interpreting first-order quantifiers substitutionally; cf., for instance, RUSSELL, B., 1950, p.196-7 and RUSSELL, B., 1918, in particular where Russell discussed his idea of a logically perfect language. Still, concerning the latter point there are good reasons for being skeptical about Klement's position. The main reason for skepticism is that Russell's conception of a "logically perfect language" as discussed in the lectures on Logical Atomism reverberates Wittgenstein's view in the *Tractatus* that what "the axiom of infinity is intended to say would express itself in language through the existence of infinitely many names with different meanings" (WITTGENSTEIN, L., 1922, §5.535.). So it is not at all implausible that Russell came to endorse this view due to his student's influence. And if that is the case, the view cannot be projected into *Principia*. A similar point can also be made with respect to Russell's remarks in *An Inquiry Into Meaning and Truth.* One important instance is the following passage of *An Inquiry into Meaning and Truth*, where Russell discusses how general propositions are obtained from particular ones by the operation of generalization: "Given any sentence containing either a name "*a*" or a word "*R*" denoting a relation or predicate, we can construct a new sentence in two ways. In the case of a name "*a*" , we may say that all sentences which result from the substitution of another name in place of "*a*" are true, or we may say that at least one such sentence is true. (I must repeat that I am not concerned with inferring true sentences, but only with constructing sentences syntactically, without regard to their truth or falsehood.) For example, from "Socrates is a man" we derive, by this operation, the two sentences "every thing is a man" and " something is a man" , or, as it may be phrased, " '*x* is a man' is always true" and " '*x* is a man' is some times true". The variable "*x*" here is to be allowed to take all values for which the sentence "*x* is a man" is significant, i.e., in this case, all values that are proper names" (RUSSELL, B., 1950, p.196). This clearly suggests a uniform subsitutional understanding of quantification. Russsell is explicitly defining the truth conditions of any sentence like "$((x))\ .\ \phi x$" or "$((\exists x)\ .\ \phi x$", where *x* is an individual variables, in terms of the truth or falsehood of all or some of the sentences which result from the substitution of *x* for a name. Thus, according to Klement, a first-order quantifier "carries existential import with it, because unlike other quantifiers, the substituends for its variable are proper names, and proper names must refer to something outside language in order to have meaning" (KLEMENT, K., 2018, p.165).

One was that of Bernard Linsky, who followed Church's reformulation of *Principia*'s grammar and interpreted it in terms of a realist conception of so-called propositions and propositional functions. But despite involving very interesting and fruitful ideas, his interpretation was found wanting for accounting for several important passages of *Principia* and other texts of Russell. The other was that of Gregory Landini, who claims that *Principia*'s official grammar for its predicate variables is that of the simple theory of types and that Russell attempted to justify the adoption of this grammar with an informal nominalistic semantics sketched in *Principia*'s Introduction.

We saw that Landini has provided us with an interpretation of Russell's views that accounts for both the (full) text of *Principia*'s Introduction and many remarks made in Russell's posterior works. His interpretation makes sense of Whitehead and Russell's 'direct inspection argument', of the distinctions between orders of so-called 'propositions' and furthermore, his interpretation affords us with the best approach to make sense of Russell's abandonment of 'propositions' (in the ontological sense) and the claim that sentences behave like incomplete symbols (i.e., like definite descriptions when flanked by truth-predicates). But most importantly, Landini's interpretation provides the best account available for the *radical change* that Russell's views went in the transition from the system of *Mathematical Logic* to that of *Principia.*

As Landini[348] observes, the editorial and scholarly work effected since the 70's by Grattan-Guinness[349] (and also by Douglas Lackey[350]) started a "revolution in Russell scholarship"[351] by presenting definitive evidence that there was much more to be said about what went on between *Insolubilia*, *Mathematical Logic* and *Principia*. Landini effected a second revolution by showing that the best way to make sense of what went on between the publication of these works is by reconstructing *Principia* in a way that: (i) does this by formulating *Principia*'s logical grammar on the terms envisaged by its authors and (ii) does not formulate *Principia*'s Introduction as defending any sort of type-hierarchy of *entities*.

Long live the revolution!

348 LANDINI, G., 2015.
349 Again, cf. GRATTAN-GUINNESS, I., 1974, 1977.
350 LACKEY, D. (ed.)., 1973.
351 LANDINI, G., 2015, p.177.

5 Logicism and the Development of Arithmetic in *Principia Mathematica*

> When a theory has been apprehended logically, there is often a long and serious labour still required in order to feel it: it is necessary to dwell upon it, to thrust out from the mind, one by one, the misleading suggestions of false but more familiar theories, to acquire the kind of intimacy which, in the case of a foreign language, would enable us to think and dream in it, not merely to construct laborious sentences by the help of grammar and dictionary.[1]

5.1 The Required Background Logic for the Development of Arithmetic in *Principia*

In our discussion of the logicism of *The Principles of Mathematics* in the first chapter, we saw that the two fundamental pillars of Russell's Logicism in that work were:

(I) The arithmetization[2] of the several branches of Pure Mathematics that Russell considers; and

(II) His proof that there is a set of terms that satisfy the Peano axioms.

In the *Principles*, it is these two accomplishments that serve as Russell's justification for the claim that the propositions of Pure Mathematics are propositions of Logic. In his early work, Russell relied essentially on Frege's construction to prove that there are infinitely many natural numbers. He intended to vindicate his definition of cardinal numbers by proving that: (1) "The existence of zero is derived from the fact that the null-class is a member of it"; (2) "the existence of 1 from the fact that zero is a unit-class (for the null-class is its only member)" and most importantly (3) "if n be a finite number, $n + 1$ is the number of numbers from 0 to n (both inclusive)"; thus establishing that "the existence-theorem follows for all finite numbers"[3]. He thought that "[...] throughout this process, no entities are employed but such as are definable in terms of the fundamental logical constants"[4].

The aim of the present chapter is to discuss in some detail why *Principia* is simply incapable of demonstrating (3) if the result is framed in such a way that it entails that the class of

1 RUSSELL, B., 1914*a*, p.105.
2 Again, the sense of "arihtmetization" presupposed is rather weak, given possible reservations Russell could have had with respect to what extent arithmetization is required for several branches of Mathematics (cf. GANDON, S., 2008; 2012; and also previous footnote 410 of chapter 2).
3 RUSSELL, B., 1903, p.497 §474.
4 RUSSELL, B., 1903, p.497 §474.

finite cardinal numbers is infinite (i.e., not inductive). We'll see that in *Principia*'s development of Arithmetic as a logical theory, no such proof that there is a set of terms that satisfy the Peano axioms is available anymore.

We shall also see, however, that this does not mean that Russell abandoned logicism. *Principia* embraces this view just as much as the *Principles* and any other of Russell's works on Mathematical Philosophy.

Before proceeding, however, we must briefly discuss what sort of background Logic can support *Principia*'s development of Mathematics, Arithmetic in particular. In the last two chapters we discussed interpretative issues concerning what sort of justification Russell meant for the formal grammar of *Principia*'s object language. On all essential points, we sided with Gregory Landini's interpretation[5], according to which Russell attempted, in *Principia*'s *Introduction*, to provide a somewhat complicated *informal* semantics (which is *not* part of *Principia*'s formal system) meant to justify a "*formal* syntax of simple type-regimented predicate variables"[6]. As we discussed, the core ideas involved in this interpretation consisted in the following points:

1. A schematic reading of so-called "propositional variables", i.e., expressions like "p", "q", "r", etc.
2. A schematic reading of so-called "non-predicative propositional functions", i.e., expressions like "ϕx", "ψx", "$f(x, y)$", etc.
3. A substitutional semantics for the bindable predicate variables of the object-language (the so called "predicative propositional functions"), i.e., expressions like "$\phi!$", "$\psi!$", "$f!$", etc.
4. The Axiom-Schemes of Reducibility $*12{\cdot}1\,.\,11$ understood as *impredicative comprehension axiom schemes*.

Such interpretation attributes to *Principia* the view according to which only individual variables range over genuine entities, keeping it in accordance with Russell's longstanding claim that "there is only one kind of being, namely being *simpliciter*"[7]. But, as discussed, Russell's attempt to justify the grammar of simple types in this way was a failure.

Somewhat shortening the story, this semantics *cannot* account for the richness that *Principia*'s language requires for the development of Mathematics - in particular where impredicative comprehension is needed. Russell's semantics failed to validate the axiom scheme of Reducibility[8]. Thus Russell failed to reconcile the development of classical Mathematics with his idea that a correct logical calculus should embrace only one sort of genuine variables which

5 Mainly as put forward in LANDINI, G., 1998a and 2011*a*.
6 LANDINI, G., 2016, p.1.
7 RUSSELL, B., 1903, p.449 §427.
8 Cf. LANDINI G., 1998, p.293-4; LANDINI, G., 2011*a*, pp.134-5; also our previous discussion in chapter 6.

range over a single domain of entities. So at this point we must carefully distinguish the two following questions:

(*a*) What is the *informal justification* (which, again, is *not* part of the formal system) actually provided by Russell in the Introduction for the grammar of simple types which *Principia* adopts?

and

(*b*) What is the *required* background Logic for sustaining the mathematical edifice built up from ∗12 onward in *Principia*?

Question (*a*), as we saw, is still very controversial (and will probably remain so): it is a matter of providing the best interpretation of *Principia*'s text given the available (and sometimes conflicting) background evidence left in Russell's published and unpublished works and correspondence. Our discussion of some of the available evidence in the last three chapters led us to side with Landini, although, of course, there is still room for dispute. Question (*b*), on the other hand, does not have such controversial status: what *Principia* requires is a background theory or foundation which is strong enough to (formally) recover the structure and theorems of the (impredicative) simple type theory of classes and relations-in-extension[9].

As is well known, several alternatives were formulated along the years after the publication of *Principia*'s first edition[10]. In his famous paper *The Foundations of Mathematics*[11], Frank Ramsey proposed his reformulation of *Principia*'s foundations by appealing to some of Wittgenstein's ideas and to his own new notion of an "extensional propositional function" and infinitely long conjunctions and disjunctions to get rid of Reducibility[12]. In his doctoral dissertation, *A System of Logistic*[13], Quine argued, following some ideas already developed in detail by Gödel[14], in favor of the adoption of a simple theory of types of classes as the primitive basis for *Principia* (along with several other modifications, like the adoption of the Wiener-Kuratowski definition of ordered pair/couple[15]). Similarly, some of Church's works[16] consist in attempts to overhaul the presentation of type theory in Russell's *Introduction*, putting in its place a rigorously formulated (realist) ramified theory of types of intensional entities, i.e., concepts and propositions (plus his version of the axiom-scheme of Reducibility).

9 Including both homogeneous *and* heterogeneous relations. See 7.3 below.
10 Cf. previous footnote 47 of the Introduction of the present work.
11 RAMSEY F., 1926.
12 For detailed discussions of Ramsey's views on the Theory of Types and the Axiom of Reducibility cf. POTTER, M., 2000, 206-22; LANDINI, G., 2011*a*, pp.360-8, WAHL, R., 2011. Russell himself wrote two reviews of Ramsey's works on Mathematical Philosophy (cf. RUSSELL, B., 1931 and 1932).
13 QUINE, W., 1934; cf. also QUINE, W., 1969, pp.239-265.
14 Cf. GÖDEL, K., 1933.
15 Cf., WIENER, N., 1914 and KURATOWSKI, K., 1921.
16 Cf. CHURCH, A., 1956, pp.346-356 and, especially CHURCH, A., 1976.

Each of the approaches mentioned above are attempts at overhauling *Principia*'s original foundations, clearing up the problematic or unclear aspects of Russell's *Introduction*[17] while keeping (most of) *Principia*'s development of Mathematics intact. *Formally*, all of the above reformulations are, arguably, up to the task. Philosophically, however, they are drastically departing from *Principia*, and this takes us to a further question, namely:

(*c*) Is there a (formally satisfactory) answer to question (*b*) that would also be philosophically satisfactory to Russell and/or Whitehead?

This is, of course, intimately related to question (*a*): in order to determine how *Principia*'s foundations should be fixed, one must determine what they actually are.

We saw in chapter 5 that Landini and Bernard Linsky had quite different views as to how *Principia's Introduction* should be read. Linsky argues that Church's rendition of *Principia*'s formal grammar in terms of his *r*-types can be viewed as faithful to what Russell meant, and Linsky further claims that a realist interpretation of so-called "propositional functions" is in accordance with what Russell (and, presumably, Whitehead) had in mind for justifying their system. According to what we called the "Church-Linsky interpretation", the simple-type surface grammar which is the working theory in *Principia* from $*12$ onward rests upon a realist *ramified* type theory of intensional entities. We also saw, however, that Landini, on the other hand, does not accept the Church-Linsky formal reconstruction as faithful to *Principia*'s text on very sound grounds[18] and holds what is, in a sense, the *opposite view*. According to Landini, the simple type grammar for predicate variables is meant to be *Principia*'s *official* formal grammar which Russell tried to informally (and unsuccessfully) justify in the *Introduction* with his nominalistic interpretation of its predicate variables. Thus, according to Landini, the best way to characterize *Principia*'s background logic is along the following lines[19]:

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17 Of course, in 1926, Russell himself sketched a reformulation of *Principia*'s foundations in the *Introduction to the Second Edition* (RUSSELL, B., 1927). The work was in the main inspired by Wittgenstein's ideas as presented in the *Tractatus*. He also discussed and incorporated some improvements made by Jean Nicod and Harry Sheffer. The modifications suggested in this second edition are very problematic, however, both by themselves and from a historical point of view, cf. POTTER, M., 2000, pp.195-205; LANDINI, G., 2011*a*, pp.341-368; LINSKY, B., 2011.

18 See our previous chapter 5.

19 Cf. LANDINI, G., 1998a and 2011*a*; also "Bertrand Russell: Logic," by Gregory Landini, *The Internet Encyclopedia of Philosophy*, ISSN 2161-0002, https://www.iep.utm.edu/, September 9 2020. Several points connected to his interpretation were kindly clarified to me by prof. Landini in personal correspondence, although, of course, any possible imprecision or error in the formulation above are entirely my responsibility. Landini has also pointed out to me that Whitehead probably had some very different ideas as to how *Principia*'s simple type grammar should be justified and he called my attention to several passages that indicate this, for which I'm also thankful. First, as he points out, there is evidence that Whitehead never fully bought Russell's nominalistic semantics for types sketched in the Introduction to the *first* edition just as he did not accept the revisions Russell proposed in the Introduction to the second edition (this latter point being well estabilished and hardly disputable cf. WHITEHEAD, A., 1926). In *Process and Reality*, for instance, we find Whitehead claiming that "Philosophy has been misled by the example of mathematics; and even in mathematics the statement of the ultimate logical principles is beset with difficulties, as yet insuperable, accompanied by the following footnote:

1. The formal grammar of *Principia*'s predicate variables is that of the simple theory of types (with both homogeneous and heterogeneous type indices, as in "$(o, (o))$", for instance)[20].

2. All variables range over individuals, which are stratified into *simple* types.

3. Each type of variable, starting from the lowest simple type *o* to any higher simple type, (o), (o, o), $((o))$, $(o, (o))$ and so on, *ad infinitum*, ranges over a totality of individuals of that corresponding simple type. Here, we may distinguish two important cases:

    a) Individual variables which have the lowest type index *o* range over the entities at the very bottom of the hierarchy, namely individuals of the lowest type.

    b) Predicate variables range over universals: either attributes or qualities of individuals (of some type indicated by the type-index of the variable) or relations-in-*intension* between individuals (also of some type indicated by the type-index of the variable).

4. Classes and relations-in-extension are not individuals of any simple type. Class expressions and class variables (lower-case Greek letters) and expressions for relations-in-extension and variables for relations-in-extension (latin capital letters in italic) are contextually eliminated by the definitions from $*20$ and $*21$. They are *not part of Principia*'s simple type grammar[21].

5. Class expressions and symbols for relations-in-extension *do not receive simple type-indices* as individual variables do, but only *relative type indices* (according to $*61 - *65$)[22] - and these notions of "type" are radically different: simple types are types of entities (i.e., types of *individuals*) while relative types concern types of classes and relations-in-extension *which are not assumed as entities of any sort at all.*

In what follows, we shall discuss *Principia*'s development of Arithmetic on the basis of the above interpretation of its background logic, but without departing from its original notation except where explicitly indicated.

"Cf. *Principia Mathematica*, by Bertrand Russell and A. N. Whitehead, Vol. I, Introduction and Introduction to the Second Edition. These introductory discussions are practically due to Russell, and in the second edition wholly so." (WHITEHEAD, A., 1929, p.8). Furthermore, Landini points out that several passages in *Principia* itself indicate that Whitehead did not entirely accept Russell's semantic interpretation. In the *Prefatory Statement* to Volume II, which we have historical reason to believe to be sole work of Whitehead (cf. RUSSELL, 1948, p.138) we find: "We often speak as though the type represented by small Latin letters were not composed of functions. It is, however, compatible with all we have to say that it should be composed of functions. It is to be observed, further that, given the number of individuals, there is nothing in our axioms to show how many predicative functions of individuals there are, i.e., their number is not a function of the number of individuals: we only know that their number is $\geq 2^{\mathrm{Nc}'\mathrm{Indiv}}$, where "Indiv" stands for the class of individuals" (WHITEHEAD & RUSSELL, 1912, p.vii); in another passage, Whitehead claims that "[...] it may be the case that the number of individuals is inductive, but the number of predicative functions of individuals is not inductive" (WHITEHEAD & RUSSELL, 1912, p.xxx).

20 Cf. LANDINI, G., 1998a, pp.255-257 for a full description of this grammar and also our discussion in the previous chapter.

21 See our discussion in the previous chapter 6.

22 See section 5.3 and 5.4 below.

Our reasons for accepting this as the correct characterization can be summarized as follows.

First, this is in accordance with the authors's explicit claim that the informal explanations given in the Introduction are not part of their formal system. Whitehead and Russell are crystal clear on this point. In the very preface we find:

> Our logical system is wholly contained in the numbered propositions, which are independent of the Introduction and the Summaries. The Introduction and the Summaries are wholly explanatory, and form no part of the chain of deductions.[23]

In fact, Russell was quite explicit on several occasions in claiming that he was never completely satisfied with the theory of types as it was presented in the first edition and that he was *only* convinced that *some* version of the theory of types should be adopted. In *My Philosophical Development*, for instance, we find:

> In the end, it became entirely clear to me that some form of the doctrine of types is essential. I lay no stress upon the particular form of that doctrine which is embodied in *Principia Mathematica* but I remain wholly convinced that without some form of the doctrine the paradoxes cannot be resolved.[24]

Second, the above realist interpretation of *Principia*'s type grammar can resonably be said to fit with Russell's view held at the time of writing and publishing *Principia* that individuals are the only genuine constituents of the universe, there being two kinds of individuals: particulars and universals (the latter category encompassing both qualities and relations).

Third, the above interpretation dissolves completely all the muddles involved with the vague and ambiguous notions of "propositional function" and "proposition" and it is also in accordance with the idea that there are no such thing as entities distinguished from individuals (either universals or particulars).

Fourth, the above interpretation is in complete agreement with Russell's repeated claim that *classes* (and relations-in-extension) *are not entities of any sort* and that talk of classes is only a *façon de parler*.

Before proceeding, we must also make a last observation on the notion of "propositional function". Landini aptly observes that "[...] lasting progress on understanding *Principia* can be made only if interpreters stop using the expression 'propositional function' as if it were a univocal referring term"[25]. In the present chapter we shall strictly follow this recommendation: when

23 WHITEHEAD & RUSSELL, 1925, p.vii [1910, p.viii].
24 RUSSELL, B., 1959, p.79.
25 LANDINI, G., 2013b, p.168.

discussing *Principia*'s text and numbered propositions we shall adopt the policy of uniformly translating the ambiguous notions of "propositional function" and "proposition" for *open* and *closed well-formed formula* (sentence), except when dealing with so-called "predicative functions", i.e. expression like "$\phi!\hat{x}$" which, as mentioned above, we will interpret as as *predicate* variables stratified in *simple types*.

Thus, following Landini, we may spell out an appropriate and historically accurate formal system for supporting the construction of Mathematics given in *Principia* as follows.

For now, let italic capital letters *A*, *B*, *C*, *D* both with and without numeric subscripts[26] be variables of our metalanguage ranging over well-formed formulas of *Principia*'s object language. Also for now, let ρ, σ and τ both with and without numeric subscripts[27] be metalinguistic variables standing for object language variables, that is, individual variables x, y, z ranging over individuals of the lowest simple type *o* or predicate variables ranging over individuals of some simple type (o), (o, o), $((o))$, $(o, (o))$, etc., other than those of the lowest simple type. The set of well-formed formulas (henceforth just *formulas*) is defined by means of the characterization of simple type symbols and the formation rules of molecular formulas by means of the propositional connectives and of general formulas by means of the universal quantifiers. That is, we can describe the basic vocabulary of the object-language in the following terms:

1. Dots ".", ":" ":.", "::", etc., for punctuation[28].
2. An infinite stock of individual variables x, y, z, x_1, y_1, z_1, ... accompanied by simple-type indices.
3. "$\sim$" for negation and "$\vee$" for disjunction.
4. "(σ^t)" for the universal quantifier, where σ is an individual variable of some simple-type index *t*.
5. Nothing else is part of the primitive vocabulary.

And where $t_1, ..., t_n$ are simple type indices as presented in our previous discussion, we have:

1. $\sigma^{(t_1,...,t_n)}(\rho_1^{t_1}, ..., \rho_n^{t_n})$ is a (atomic) formula.
2. If *A* is a formula, then $\sim A$ is a (molecular) formula.
3. If *A* and *B* are formulas, then $A \vee B$ is a (molecular) formula.

[26] To ensure an infinite stock of such variables.
[27] Again, to ensure an infinite stock of such variables.
[28] And also for conjunction, when it is defined by $*3 \cdot 01$.

4. If A is a formula containing free occurrences of a variable ρ, then $(\rho)A$ is a (general) formula.

5. Nothing else is a formula.

Then, the basic rules of inference, axioms and definitions of *Principia* are described in terms of the following *schemata*:

∗1·01	$A \supset B =_{\text{Df}} \sim A \vee B$
∗3·01	$A \,.\, B, A : B, A : B,$ etc... $=_{\text{Df}} \sim (\sim A \vee \ \sim B)$
∗4·01	$A \equiv B =_{\text{Df}} A \supset B \,.\, B \supset A$
∗1·1	From A and $A \supset B$, infer B.
∗1·2	$A \vee A \,.\supset .\, A$
∗1·3	$B \,.\supset .\, A \vee B$
∗1·4	$A \vee B \,.\supset .\, B \vee A$
∗1·6	$A \vee B : \supset : C \vee A \,.\supset .\, C \vee B$
∗10·01	$(\exists \sigma^t)A =_{\text{Df}} \sim (\sigma^t) \sim A$
∗10·1	$(\sigma^t)A(\sigma^t) \,.\supset .\, A[\sigma^t/\tau^t]$, where $A[\sigma^t/\tau^t]$ results from substituting every free occurrence of σ^t in $A(\sigma^t)$ for the variable τ^t.
∗10·11	From A infer $(\sigma^t)A$, where σ^t is variable that occurs free in A.
∗10·12	$(\sigma^t) \,.\, C \vee A(\sigma^t) : \supset \because \vee \,.\, (\sigma^t)CA(\sigma^t)$ where the variable σ^t does not occur free in C.
∗12·n	$(\exists \sigma^{(t_1,\ldots,t_n)})(\rho_1^{t_1},\ldots,\rho_n^{t_n}) : \sigma^{(t_1,\ldots,t_n)}(\rho_1^{t_1},\ldots,\rho_n^{t_n}) \,.\equiv .\, A$, where $\sigma^{(t_1,\ldots,t_n)}$ does not occur free in A.
∗13·01	$\tau^t = \rho^t =_{\text{Df}} (\sigma^{(t)}) : \sigma^{(t)}(\tau^t) \,.\equiv .\, \sigma^{(t)}(\rho^t)$

This completes the primitive basis of the formal system.

Below we re-frame - on the basis of the rigorously formulated formal system set above - *Principia*'s contextual elimination for class expressions "$\hat{\rho}^t B(\rho^t)$" and expressions for relations-in-extension "$\hat{\rho}^t \hat{\upsilon}^v B(\hat{\rho}^t, \hat{\upsilon}^v)$" set out in sections $*20$ and $*21$ (preserving scope markers)[29]:

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[29] Notice that none of the definitions introduce new terms into the language.

$*20{\cdot}01 \quad [\hat{\rho}^t B(\rho^t)][A(\hat{\rho}^t B(\rho^t))] =_{\text{Df}} (\exists\sigma^{(t)}) \mathbin{:.} (\rho^t) : \sigma^{(t)}(\rho^t) \,.\equiv.\, B(\rho^t) : A(\sigma^{(t)})$

$*20{\cdot}02 \quad \rho^t \,\epsilon\, \sigma^{(t)} =_{\text{Df}} \sigma^{(t)}(\rho^t)$

$*20{\cdot}03 \quad \text{Cls}(\alpha) =_{\text{Df}} (\exists\sigma^{(t)}) \,.\, \alpha = \hat{\rho}^t(\sigma^{(t)}(\rho^t))$

$*20{\cdot}07 \quad (\alpha)A(\alpha) =_{\text{Df}} (\sigma^{(t)}) : A(\hat{\rho}^t(\sigma^{(t)}(\rho^t))$

$*20{\cdot}071 \quad (\exists\alpha)A(\alpha) =_{\text{Df}} (\exists\sigma^{(t)}) : A(\hat{\rho}^t(\sigma^{(t)}(\rho^t))$

$*20{\cdot}08 \quad [\hat{\alpha}B(\alpha)][A(\hat{\alpha}B(\alpha))] =_{\text{Df}} (\exists\sigma^{(t)}) \mathbin{:.} (\alpha) : \sigma^{(t)}(\alpha) \,.\equiv.\, B(\alpha) : A(\sigma^{(t)})$

Notice that in the definitions above the lower-case Greek letters with simple type indices like $\sigma^{(t)}$ or ρ^t are not class-expressions but metalinguistic placeholders for individual variables. For relations-in-extension parallel definitions are easily provided:

$*21{\cdot}01 \quad [\hat{\rho}^t\hat{\upsilon}^v B(\hat{\rho}^t,\hat{\upsilon}^v)][A(\hat{\rho}^t\hat{\upsilon}^v B(\hat{\rho}^t,\hat{\upsilon}^v))] =_{\text{Df}} (\exists\sigma^{(t,v)}) \mathbin{:.} (\rho^t)(\hat{\upsilon}^v) : \sigma^{(t,v)}(\hat{\rho}^t,\hat{\upsilon}^v) \,.\equiv.\, B(\hat{\rho}^t,\hat{\upsilon}^v) : A(\sigma^{(t,v)})$

$*20{\cdot}02 \quad \tau_1^t\{\hat{\rho}^t\hat{\upsilon}^v\sigma^{(t,v)}(\hat{\rho}^t,\hat{\upsilon}^v)\}\tau_2^v =_{\text{Df}} \sigma^{(t)}(\tau_1^t,\tau_2^v)$

$*20{\cdot}03 \quad \text{Rel}(R) =_{\text{Df}} (\exists\sigma^{(t,v)}) \,.\, (\exists\sigma^{(t,v)}) \,.\, R = \sigma^{(t,v)}(\hat{\rho}^t,\hat{\upsilon}^v)$

$*20{\cdot}07 \quad (R)A(R) =_{\text{Df}} (\sigma^{(t,v)}) : A(\sigma^{(t,v)}(\hat{\rho}^t,\hat{\upsilon}^v))$

$*20{\cdot}071 \quad (\exists R)A(R) =_{\text{Df}} (\exists\sigma^{(t,v)}) \,.\, A(\sigma^{(t,v)}(\hat{\rho}^t,\hat{\upsilon}^v))$

$*20{\cdot}08 \quad [\hat{R}\hat{S}B(R,S)][A(\hat{R}\hat{S}B(R,S))] =_{\text{Df}}$

$$(\exists\sigma^{(t,v)}) \mathbin{:.} (R)(S) : \sigma^{(t,v)}(R,S) \,.\equiv.\, B(\hat{\rho}^t,\hat{\upsilon}^v) : A(\sigma^{(t,v)})$$

This concludes the contextual elimination of occurrences of class expressions and expressions for relations-in-extension.

The resulting formal system has no variables other than *individual* variables and no terms other than individual variables. The system is as strong as a system of standard (i.e., fully impredicative) higher-order logic with predicate variables of ascending simple types.

5.2 *Principia*'s Elementary ('Emulated') Theory of Classes and Relations$_{\text{e}}$

In *Principia*, what would generally be considered a treatment of elementary Arithmetic properly begins only in the second volume. The proofs of *Principia*'s analogues of the Dedekind-Peano postulates, in particular, are in $*120$, two hundred pages into volume 2, so as it is to be expected, lots of preliminary definitions and theorems of fundamental importance were laid down long before that.

In order to discuss the treatment of finite Arithmetic in *Principia*, focusing on the proofs of *Principia*'s analogues of the Dedekind-Peano postulates, we must draw on some definitions and theorems from the second half of the first volume which forms *Principia*'s part II, the 'Prolegomena' to cardinal arithmetic. In what follows we present the development of cardinal arithmetic in *Principia*'s second volume, drawing upon the definitions of volume I whenever relevant and/or necessary.

We may start by reviewing *Principia*'s treatment of the Algebra of sets or classes. Whitehead and Russell start by giving basic definitions for operations with classes at ∗22, with analogous versions for relations-in-extension (henceforth, relations$_e$) at ∗23:

∗22·01	$\alpha \subset \beta .=: x \epsilon \alpha . \supset_x . x \epsilon \beta$	Df	∗23·01	$R \mathbin{\dot\subset} S .=: xRy . \supset_{x,y} . xSy$	Df
∗22·02	$\alpha \cap \beta .=: \hat{x}(x \epsilon \alpha . x \epsilon \beta)$	Df	∗23·02	$R \mathbin{\dot\cap} S .=: \hat{x}\hat{y}(xRy . xSy)$	Df
∗22·03	$\alpha \cup \beta .=: \hat{x}(x \epsilon \alpha . \vee . x \epsilon \beta)$	Df	∗23·03	$R \mathbin{\dot\cup} S .=: \hat{x}\hat{y}(xRy . \vee . xSy)$	Df
∗22·04	$-\alpha = \hat{x}(x \sim \epsilon \alpha)$	Df	∗23·04	$\dot{-}R = \hat{x}\hat{y}\{\sim (xRy)\}$	Df
∗22·05	$\alpha - \beta = \alpha \cap -\beta$	Df	∗23·05	$R \dot{-} S = R \mathbin{\dot\cap} \dot{-}S$	Df

These give standard definitions of containment, intersection, union, complement and subtraction for classes with analogues for relations. Whitehead and Russell show that these definitions are sufficient within their logic to prove standard axioms for a Boolean algebra of classes. They define the universal and empty set/relation as:

∗24·01	$\mathrm{V} = \hat{x}(x = x)$	Df	∗24·02	$\Lambda = -\mathrm{V}$	Df
∗25·01	$\dot{\mathrm{V}} = \hat{x}\hat{y}(x = x . y = y)$	Df	∗25·02	$\dot{\Lambda} = \dot{-}\dot{\mathrm{V}}$	Df

And then demonstrate that Huntington's set of (independent) axioms are derivable in their system in both the theories of classes and relations[30]:

∗22·51	$\vdash . \alpha \cap \beta = \beta \cap \alpha$	∗22·51	$\vdash . R \mathbin{\dot\cap} S = S \mathbin{\dot\cup} R$
∗22·57	$\vdash . \alpha \cup \beta = \beta \cup \alpha$	∗23·57	$\vdash . R \mathbin{\dot\cap} S = S \mathbin{\dot\cup} R$
∗22·68	$\vdash . (\alpha \cap \beta) \cup (\alpha \cap \gamma) = \alpha \cap (\beta \cup \gamma)$	∗22·68	$\vdash . (R \mathbin{\dot\cap} S) \mathbin{\dot\cup} (R \mathbin{\dot\cap} T) = R \mathbin{\dot\cap} (S \mathbin{\dot\cup} T)$
∗22·69	$\vdash . (\alpha \cup \beta) \cap (\alpha \cup \gamma) = \alpha \cup (\beta \cap \gamma)$	∗22·69	$\vdash . (R \mathbin{\dot\cup} S) \mathbin{\dot\cap} (R \mathbin{\dot\cup} T) = R \mathbin{\dot\cup} (S \mathbin{\dot\cap} T)$
∗24·21	$\vdash . \alpha \cap -\alpha = \Lambda$	∗25·21	$\vdash . R \mathbin{\dot\cap} \dot{-}R = \dot{\Lambda}$
∗24·22	$\vdash . \alpha \cap -\alpha = \mathrm{V}$	∗22·22	$\vdash . R \mathbin{\dot\cup} \dot{-}R = \dot{\mathrm{V}}$
∗24·24	$\vdash . \alpha \cup \Lambda = \alpha$	∗22·24	$\vdash . R \mathbin{\dot\cup} \dot{\Lambda} = R$
∗24·26	$\vdash . \alpha \cap \mathrm{V} = \alpha$	∗22·26	$\vdash . R \mathbin{\dot\cap} \dot{\mathrm{V}} = R$

[30] HUNTINGTON, E. 1904. Huntington's work was explicitly written in continuation with Whitehead's *Universal Algebra.* He later also investigated an algebraic treatment of *Principia*'s propositional logic in HUNTINGTON, E. 1932.

Now they define the *existence* of classes and relations as non-emptiness:

$*24{\cdot}03 \quad \exists\,!\,\alpha \,.{=}.\, (\exists x) \,.\, x \,\epsilon\, \alpha \quad \text{Df} \qquad *25{\cdot}03 \quad \dot{\exists}\,!\,R \,.{=}.\, (\exists x, y) \,.\, xRy \quad \text{Df}$

Notice that in all of the above, when the *definiendum* is either an expression containing lower-case Greek letters or latin capitals in italic, the *definiens* is framed with individual variables, which allows for the restoration of simple type indices *only* to these individual variables, not to contextually defined expressions. To give a simple example, take $\alpha \cup \beta$, which by $*22{\cdot}03$, is just:

$\hat{x^t}(x^t \,\epsilon\, \alpha \,.\vee.\, x^t \,\epsilon\, \beta)$, where *x* is of some simple type *t*.

To fully restore simple type indices in the above expression, α and β must first be substituted for expressions of the form $\hat{x}(\phi!x)$, $\hat{x}(\psi!x)$, etc which also do not receive simple type indices. If, for instance, we let α and β be, respectively, $\hat{z}(\phi!z)$ and $\hat{z}(\psi!z)$, we get:

$\alpha \cup \beta$ for $\hat{x^t}\{x^t \,\epsilon\, \hat{z^t}(\phi^{(t)}!z^t) \,.\vee.\, x^t \,\epsilon\, \hat{z^t}(\psi^{(t)}!z^t)\}$

To get the fully restored expression we then apply $*20{\cdot}02$, getting[31]:

$$\hat{x^t}(\phi^{(t)}!x^t \vee \psi^{(t)}!x^t)$$

This may cause some confusion, since *Principia* allows for expressions representing a class of classes like "$\hat{\gamma}(\psi\gamma)$" to be substituted for lower-case Greek letters. Using our same example, but with α and β being $\hat{\gamma}(\phi\gamma)$ and $\hat{\gamma}(\psi\gamma)$, respectively, we would have $\alpha \cup \beta$ as:

$$\hat{\delta}\{\delta \,\epsilon\, \hat{\gamma}(\phi\gamma) \,.\vee.\, \delta \in \hat{\gamma}(\psi\gamma)\}$$

But of course, here simple type indices cannot be restored directly for δ, $\hat{\gamma}(\phi\gamma)$, etc. The same applies, *mutatis mutandis*, to expressions for relations$_e$ as they occur in "xRy", "$xR\alpha$", etc[32]. These can only receive *relative type indices*, which we shall discuss below, in connection with the typical ambiguity of the relation of similarity and cardinal numbers. For now, let us proceed with the elements of *Principia*'s treatment of classes and relations$_e$.

[31] Notice also that if we had put, say, "$\hat{z}(\psi z)$" in place of β, full restoration of simple type index would require that the *scheme* ψ be turned into a definite well-formed formula of *Principia*'s object language.

[32] Cf. LANDINI, G., 2016, pp.11-12.

Another important aspect of *Principia* is the fact the work does not employ (and of course, does not assume the existence of) 'functions' in the usual or ordinary mathematical sense. They have only what Russell calls 'descriptive functions' which are defined using the theory of descriptions and the emulated theory of relations-in-extension. The main definition is this:

$*30{\cdot}01 \qquad R\text{‘}y = (\imath x)(xRy) \quad \text{Df}$

This defines what can be read as '*the R of y*', as in 'the square of 2', the 'the father of Alexander', 'the greatest student of Plato', etc. Such expressions as "$R\text{‘}y$" purport to denote the unique *x* that bears the relation *R* to *y*; if either there is *no x* that has *R* to *y* or there is more than one*,* the expression simply does not denote anything. Yet, since such symbols are contextually defined via the device of incomplete symbols, any sentence in which they occur will still be meaningful if they fail to denote, as in "The rational root of 2". Whitehead and Russell also introduce a similar device for 'plural' descriptive functions, expressions which represents a whole *class* of things that bear a relation *R* with respect to a given class, as in "the sons of kings", "the prime factors of perfect numbers", etc. They have:

$*37{\cdot}01 \qquad R\text{“}\beta = \hat{x}\{(\exists x) \,.\, y \in \beta \,.\, xRy\} \quad \text{Df}$

where "$R\text{“}\beta$" is to be read as "the *R*'s of β's". They also introduce a notation for the referents and relata of given term with respect to a given relation (which is mostly used in connection with descriptive functions), namely[33]:

$*32{\cdot}01 \qquad \overrightarrow{R} = \hat{\alpha}\hat{y}\{\alpha = \hat{x}(xRy)\} \quad \text{Df}$

$*32{\cdot}02 \qquad \overleftarrow{R} = \hat{\beta}\hat{x}\{\beta = \hat{y}(xRy)\} \quad \text{Df}$

Here the first is the relation which holds between a class α of all *x* that bear *R* to *y* and the given term *y*; and the second is the relation which hold between the class β of all *y* to which *x* has the relation *R* and the given *y*. From the above they derive the followng which gives the form they actually employ with frequence:

33 Whitehead and Russell also introduce in ∗32·03·21 and ∗32·04·211 the notation, sg‘R and gs‘R, which are, respectively, equivalent to those of ∗32·01·02. The former are introduced merely for convenience of writing down formulas when dealing with more complicated relations-in-extension. For instance, they allow one to write sg‘$(R|S|Q \restriction \beta)$ instead of $\overrightarrow{R|S|Q \restriction \beta}$, since the former is more convenient.

$*32{\cdot}13 \quad \vdash . \overrightarrow{R}\text{‘}y = \hat{x}(xRy)$

$*32{\cdot}131 \quad \vdash . \overleftarrow{R}\text{‘}x = \hat{y}(xRy)$

This identifies $\overrightarrow{R}\text{‘}y$ with the class of all *x* such that *xRy*, e.g., the *referents of y*, and $\overleftarrow{R}\text{‘}x$ with the class of all *y* such that *xRy*, e.g., the *relata of x:* to give a simple example, if *R* is, for instance, the relation of *less than* among natural numbers, then $\overrightarrow{R}\text{‘}0$ is the empty set, whereas $\overleftarrow{R}\text{‘}0$ is the set of all natural numbers greater than 0. These are among the most used notations in the whole work.

Next, they introduce domains, converse domains and fields of relations in a similar fashion. First they introduce the proper definitions:

$*33{\cdot}01 \quad \mathrm{D} = \hat{\alpha}\hat{R}[\alpha = \hat{x}\{(\exists y) . xRy\}] \quad \mathrm{Df}$

$*33{\cdot}02 \quad \mathrm{Ɑ} = \hat{\beta}\hat{R}[\beta = \hat{y}\{(\exists x) . xRy\}] \quad \mathrm{Df}$

$*33{\cdot}03 \quad \mathrm{C} = \hat{\gamma}\hat{R}[\gamma = \hat{x}\{(\exists y) : xRy . \vee . yRx\} \quad \mathrm{Df}$

Then they obtain the more convenient notation using the device of plural descriptive functions:

$*33{\cdot}11 \quad \vdash . \mathrm{D}\text{‘}R = \hat{x}\{(\exists y) . xRy\}$

$*33{\cdot}111 \quad \vdash . \mathrm{Ɑ}\text{‘}R = \hat{y}\{(\exists x) . xRy\}$

$*33{\cdot}112 \quad \vdash . \mathrm{C}\text{‘}R = \hat{x}\{(\exists y) : xRy . \vee . yRx\}$

Number $*34$ introduces the notations for relative product, which is also much used (for instance, in section $*90$, which concerns the ancestral of a relation and in the whole treatment of ‘inductive’ relations). They have:

$*34{\cdot}01 \quad R|S = \hat{x}\hat{z}\{(\exists y) . xRy . yRz\} \quad \mathrm{Df}$

$*34{\cdot}02 \quad R^2 = R|R \quad \mathrm{Df}$

$*34{\cdot}03 \quad R^3 = R^2|R \quad \mathrm{Df}$

Then $*40$, which is the main number in section E of part I, introduces products (intersections) and sums (unions) of classes of classes (what we nowadays call ‘families’ of sets):

$*40{\cdot}01 \quad p\text{‘}\kappa = \hat{x}(\alpha \in \kappa . \supset . x \in \alpha) \quad \mathrm{Df}$

$*40{\cdot}01 \quad s\text{‘}\kappa = \hat{x}((\exists \alpha) . \alpha \in \kappa . x \in \alpha) \quad \mathrm{Df}$

Notice that in the above the lower-case κ is a class of classes which must be of a higher relative type than its members (which are themselves classes of lower relative type).

Now, starting part II of Volume I, the *Prolegomena to Cardinal Arithmetic*, the authors introduce singletons, or, as they call them, unit classes. They do this, however, in way that may be puzzling to the modern reader. They first introduce identity as a relation-in-*extension*:

$*50{\cdot}1 \qquad I = \hat{x}\hat{y}(x = y)$

The above is not to be confused with identity as a relation-in-intension, which is a genuine relation that holds *only between individuals* (of any given type). What is defined above is an incomplete symbol, another *façon de parler*. To define unit classes they then introduce the relation-in-extension ι which holds between a given term *x* and the class α of all *y* such that xIy, i.e.:

$*51{\cdot}01 \qquad \iota = \overrightarrow{I} \quad \text{Df}$

$*51{\cdot}11 \qquad \iota`x = \hat{y}(y = x) \quad \text{Df}$

As the authors explain, the only reason the latter two are introduced is notational convenience: the symbols "$\overrightarrow{I}$", "$\overrightarrow{I}`x$", etc., become cumbersome when nested within other notations like "$\breve{\overrightarrow{I}}$" or "$\breve{\overrightarrow{I}}`x$"[34].

Finally, before we proceed to *Principia's* development of Arithmetic, we must consider a last major definition, that of an ordered pair, or, as the authors call it, *ordinal couple*. Using one of their definitions from the section on the limitation of the fields of relations $*34$, namely, that of the Cartesian product of two classes α and β, they put:

$*35{\cdot}04 \qquad \alpha \uparrow \beta = \hat{x}\hat{y}(x \,\epsilon\, \alpha \,.\, y \,\epsilon\, \beta) \quad \text{Df}$

They claim a pair could have been defined as in $< x, y > =_{\text{Df}} \iota`x \uparrow \iota`y$. But since this notation "is cumbrous, and does not readily enable us to exhibit the couple as a descriptive function of *x* for the argument *y*, or vice-versa"[35], they introduce the following:

$*55{\cdot}01 \qquad x \downarrow y = \iota`x \uparrow \iota`y = \hat{x}\hat{y}(x \,\epsilon\, \iota`x \,.\, y \,\epsilon\, \iota`y) \quad \text{Df}$

[34] Cf. WHITEHEAD & RUSSELL, 1925, p.331 [1910, p.338]. Cf. also theorems $*51{\cdot}51{\cdot}511{\cdot}56$.
[35] WHITEHEAD & RUSSELL, 1925, p.366 [1910, p.383].

The above definition must be read carefully. Grattan-Guinness points out that since "the defining expression is a conjunction, and so commutative [...] the order itself is determined by that of the class abstraction operator", and so the definition brings "no real advance"[36]. The point can also be put in the following terms. Since "$\alpha \uparrow \beta$" defines a certain relation in extension, namely a relation which holds between x and y whenever $x \,\epsilon\, \alpha \,.\, y \,\epsilon\, \beta$, then $x \downarrow y$ defines a certain relation in extension which holds between *x* and *y* whenever $x \,\epsilon\, \iota\text{‘}x \,.\, y \,\epsilon\, \iota\text{‘}y$; the problem, of course, is that this is useless to introduce the notion of ordinal couple precisely because it defines it as some sort of relation - which has a 'direction' - thus it presupposes the very idea it was supposed to define.

But $*55\cdot01$ is not meant to *define* ordered pairs, only to introduce a convenient notation for them to be used in the (emulated) theory of relations-in-extension. Given a relation *xRy*, the order of the couple $x \downarrow y$ is ultimately taken as indefinable and given by the order of the arguments of the expression of the form "$\phi!(\hat{x}, \hat{y})$" which contextually defines the relation (in extension) *R* in question. The authors themselves unequivocally say as much when commenting on $*21\cdot01\cdot02$ and their conventions for substitution of variables:

> In $*21\cdot01$ the *alphabetical* order of *u* and *v* corresponds to the *typographical* order of $\hat{x}$ and $\hat{y}$ in $f\{\hat{x}\hat{y}\psi(x, y)\}$ [...] the mode of substitution depends upon the *alphabetical* order of the letters which have circumflexes and the *typographical* order of the letters.[37]
>
> If a capital Latin letter, say *R*, is used as an apparent variable, it is supposed that the *R* which occurs in the form "(*R*)" or "($\exists R$)" is to be replaced by "(ϕ)" or "$(\exists\phi)$", while the *R* which occurs later is to be replaced by $\hat{x}\hat{y}\phi!(x, y)$. [...] In virtue of the definitions $*21\cdot01\cdot02$ and the convention as to capital Latin letters, the notation "xRy" will mean "x has the relation *R* to *y*". This notation is practically convenient, and will, after the preliminaries, wholly replace the cumbruous notation $x\{\hat{x}\hat{y}\phi(x, y)\}y$.[38]

As a student of Russell in Cambridge, Norbert Wiener suggested the following, which was meant as an actual eliminative definition of the ordinal couple[39]:

$$\hat{x}\hat{y}\phi(x, y) = \hat{\alpha}\{(\exists x, y) \,.\, \phi(x, y) \,.\, \alpha = \iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}y\} \quad \text{Df}$$

But as some commentators observed, Russell was unimpressed[40]. Later, Kuratowski[41] formulated the now standard definition of an ordered pair $< x, y >$ as $\{\{x\}, \{x, y\}\}$, which in *Principia*'s notation is a little simpler than Wiener's:

36 GRATTAN-GUINNESS, I., 2000, p.394.
37 WHITEHEAD & RUSSELL, 1925, p.200 [1910, p.211].
38 WHITEHEAD & RUSSELL, 1925, p.201 [1910, p.212].
39 WIENER, N., 1914, p.225.
40 van HEIJENOORT, J., 1967, p.224; GRATTAN-GUINNESS, I., 2000, pp.420-1.
41 KURATOWSKI, 1921, p.171. Hausforff also put forward the definition $< x, y >= \{\{x, 1\}, \{y, 2\}\}$ where 1 and 2 are any objects such that $1 \neq 2$.

$$\hat{x}\hat{y}\phi(x,y) = \hat{\alpha}\{(\exists x,y) \,.\, \alpha = \iota`\iota`x \cup \iota`(\iota`x \cup \iota`y)\} \text{ Df}$$

Neither of these appealed to Russell however, for reasons we may never know with certainty, although they are easy to imagine. First and foremost, these only work to *homogeneous* relations, that is, relations which have domains and converse domains of the same relative type, but non-homogeneous relations are of fundamental importance to *Principia*'s development of Mathematics; to give a simple example, Cantor's power-class theorem asserts the non-existence of a non-homogeneous 1-1 correspondence between a class α and the class Cl'α of all sub-classes of α; in order for it work generally, Wiener had to introduce a family of auxiliary definitions accommodating non-homogeneous relations between individuals, classes, classes of classes, and so on, along the following lines[42]:

$$\hat{\alpha}\hat{y}\phi(\alpha,y) = \hat{\kappa}\{(\exists \alpha,y) \,.\, \phi(\alpha,y) \,.\, \kappa = \iota`(\iota`\iota`\alpha \cup \iota`\Lambda) \cup \iota`\iota`(\iota`\iota y \cup \iota`\Lambda)\} \quad \text{Df}$$

$$\hat{\kappa}\hat{y}\phi(\kappa,y) = \hat{\mu}\{(\exists \kappa,y) \,.\, \phi(\kappa,y) \,.\, \mu = \iota`(\iota`\iota`\alpha \cup \iota`\Lambda) \cup \iota`\iota`[\iota`\iota`(\iota`\iota y \cup \iota`\Lambda) \cup \iota`\Lambda]\} \quad \text{Df}$$

and so on...

$$\hat{x}\hat{\beta}\phi(x,\beta) = \hat{\kappa}\{(\exists x,\beta) \,.\, \phi(x,\beta) \,.\, \kappa = \iota`[(\iota`\iota`x \cup \iota`\Lambda) \cup \iota`\Lambda] \cup \iota`\iota`\iota`\beta\} \quad \text{Df}$$

$$\hat{x}\hat{\lambda}\phi(x,\lambda) = \hat{\mu}\{(\exists x,\lambda) \,.\, \phi(x,\lambda) \,.\, \mu = \iota`\{\iota`[(\iota`\iota`x \cup \iota`\Lambda) \cup \iota`\Lambda] \cup \iota`\Lambda\} \cup \iota`\iota`\iota`\lambda\} \quad \text{Df}$$

and so on...

But as a device meant to *simplify* their treatment, these do a poor job. Second, the above are inconvenient, since they unnecessarily raise the relative type of every relation-in-extension. Last, but not least, for Russell these definitions were completely irrelevant from a philosophical point of view, since to speak of relations *in extension* was a mere *façon de parler*. Ultimately, the intensional view of relations was fundamental:

> It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. [...] the facts seem to be that while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognizes and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to intension.[43]

42 WIENER, N., 1914, pp.225-6.

43 WHITEHEAD & RUSSELL, 1925, p.72 [1910, p.75]. Grattan-Guinness calls attention to this passage and notes that the point is further emphasized in a letter Russell wrote to Jourdain by 12 February 1907, where we find: "Rationals, and numbers which have sign, *are* relations, not classes (unless a relation is taken to be a class of couples). It is a mistake to say that all definitions define classes - all define propositional functions, is the right remark" (GRATTAN-GUINNESS, I., 1977, p.97).

This point is sometimes missed by interpreters of *Principia.* Rodríguez-Consuegra, for instance, claims that by the time of *Principia* Russell was "led to a reconsideration of the intensional view about relations and to a victory of the Peanian approach, including the interpretation of equivalence and propositional functions in terms of membership"[44]; that is, he assumes that *Principia* rejects an ontology of relations even from an intensional viewpoint.

He was probably led to this conclusion by a somewhat misleading comment that Russell makes in his correspondence with Jourdain. In April of 1910, Russell wrote:

> Until I got hold of Peano, it had never struck me that Symbolic Logic would be any use [sic] for the Principles of mathematics, because I knew the Boolian [sic] stuff and found it useless. It was Peano's ϵ together with the discovery that relations could be fitted into his system, that led me to adopt symbolic logic. [...] Oddly enough, I was largely guided by the belief that relations must be taken in *intension*, a view which I have since abandoned, though I have not the notations which it led me to adopt".[45]

This may suggest that in *Principia* the extensional viewpoint of relations (as classes of ordered couples) should be taken as fundamental. But the textual evidence in *Principia* simply does not admit such reading; in terms of ontological priority, the primacy of the intensional viewpoint is undeniable (despite the fact that there is some room for exegetical disputes concerning the status of propositional functions). The real point of Russell's letter is that in the *Principles* (and elsewhere), he thought that it was illegitimate to treat relations extensionally, a view which by the time of *Principia* he had abandoned since, through the use of incomplete symbols, one could do precisely that. As Landini observes, for Russell the whole problem before the discovery of the theory of incomplete symbols, was that he thought "the notion of an ordered "couple" logically odious"[46] because it could not be taken as a *single* entity of any sort. But the logic of *Principia*[47] can, through the device of incomplete symbols, recover a theory of relations in extension without assuming these as genuine entities. What Russell abandoned is the idea that it is illegitimate to talk about relations-*in-extension*, not the priority, much less the legitimacy of speaking of the intensional character of relations as *entities*.

5.3 The Definition(s) of Cardinal Number, Typicial Ambiguity and Relative Types

Obviously, the definition which is most fundamental to the development of Arithmetic is *Principia*'s version of Frege's celebrated definition of cardinal number. In their informal

44 RODRÍGUEZ-CONSUEGRA, F., 1991, p.195.
45 GRATTAN-GUINNESS, 1977, p. 132-3.
46 LANDINI, G., 1992, p.606-7.
47 As well as that of the substitutional theory.

summary, Whitehead and Russell initially present their version with the following formulas:

$$\text{Nc} = \overrightarrow{\text{sm}} \quad \text{Df}$$

$$\text{Nc}‘\alpha = \hat{\beta}(\beta \,\text{sm}\, \alpha) \quad \text{Df}$$

$$\text{NC} = \hat{\mu}\{(\exists\alpha) \,.\, \mu = \text{Nc}‘\alpha\} \quad \text{Df}$$

That is, Nc is defined as "[...] a relation, namely the relation of a cardinal number to any class of which it is the number"[48] and NC, the class of all cardinal numbers, is defined as "[...] the class of objects which are the cardinal numbers of something or other, i.e, objects which, for some α, are equal to Nc‘α"[49].

The typically ambiguous symbol "sm" stands for the relation$_e$ of similarity, which holds between two classes whenever there is a one-one correspondence between their members. Whitehead and Russell introduce *sm* as a relation that holds between *any* two classes α and β whenever there is a one-one relation *R* which has α as its domain and β as its converse domain. To properly understand how similarity and the relation of a class to its cardinal number are defined we must review some preliminary definitions. First, Whitehead and Russell have definitions of one-many, many-one and one-one relations:

$$*71{\cdot}01 \qquad \vdash .\, 1 \to \text{Cls} = \hat{R}(\overrightarrow{R}“\text{Ɑ}‘R \subset 1) \quad \text{Df}$$

$$*71{\cdot}02 \qquad \vdash .\, \text{Cls} \to 1 = \hat{R}(\overleftarrow{R}“\text{D}‘R \subset 1) \quad \text{Df}$$

$$*71{\cdot}03 \qquad \vdash .\, 1 \to 1 = \hat{R}(\overrightarrow{R}“\text{Ɑ}‘R \subset 1) \,.\, \overleftarrow{R}“D‘R \subset 1) \quad \text{Df}$$

With 1 defined as the set of all singletons[50] and $\alpha \to \beta$ defined as the class of relations *R* such that "$\overrightarrow{R}‘y \in \alpha$ whenever $y \in$Ɑ‘R, and $\overleftarrow{R}‘x \in \beta$ whenever $x \in$ D‘*R*"[51], that is:

$$*70{\cdot}01 \qquad \alpha \to \beta = \hat{R}(\overrightarrow{R}“\text{Ɑ}‘R \subset \alpha) \,.\, \overleftarrow{R}“\text{D}‘R \subset \beta \quad \text{Df}$$

This defines the class of relations$_e$ which have as referents and relata members of a given classes α and β respectively. Thus, for instance, $\alpha \to \beta \cap \text{Cls} \to 1$ is the class of all 'functions' from α into β, $\beta \to \alpha \cap \text{Cls} \to 1$ is the class of all 'functions' from β to α, etc. They then define the

48 WHITEHEAD & RUSSELL, 1927*a*, p.4 [1912, p.4].
49 WHITEHEAD & RUSSELL, 1927*a*, p.5 [1912, p.5].
50 Not yet identified with the cardinal number 1.
51 WHITEHEAD & RUSSELL, 1925, p.420 [1910, p.439]

class $\overline{\mathrm{sm}}$ of all one-one relations which have α as its domain and β as its converse domain and then define "sm" as the relation which holds between α and β whenever $\overline{\mathrm{sm}}$ is non-empty:

$$*73{\cdot}01 \qquad \alpha\,\overline{\mathrm{sm}}\,\beta = (1 \to 1) \cap \overleftarrow{\mathrm{D}}\text{‘}\alpha \,.\, \overleftarrow{\mathrm{Q}}\text{‘}\beta \quad \mathrm{Df}$$

$$*73{\cdot}02 \qquad \mathrm{sm} = \hat{\alpha}\hat{\beta}(\exists\,!\,\alpha\,\overline{\mathrm{sm}}\,\beta) \quad \mathrm{Df}$$

This yields the equivalence that would generally be used to define the relation of similarity, namely:

$$*73{\cdot}1 \qquad \vdash :\alpha\,\mathrm{sm}\,\beta \,.\equiv.\, (\exists R)\,.\, R\,\epsilon\, 1 \to 1 \,.\, \alpha = \mathrm{D}\text{‘}R \,.\, \beta = \mathrm{Q}\text{‘}R$$

In the above definitions, however, α and β are typically ambiguous and they *may or may not be the of the same (relative) type* and each case, in fact, generates a different relation$_{\mathrm{e}}$ of similarity. The same is true of the ocurrences of "ɪ" which receive their typical determination according to the relative types of α and β.

Let us take, for instance, the following example, using the modern notation $\{a_1, ..., a_n\}$ for the class whose members are $a_1, ..., a_n$: let $\alpha = \{a, b\}$, $\beta = \{c, d\}$ and $\gamma = \{\alpha, \beta\}$, where *a*, *b*, *c* and *d* are individuals. They are all similar. But the relation of similarity which holds between α and β is not the same as that which holds between, say, α and γ. For now[52], we may indicate the *relative* type indices for class expressions and relations$_{\mathrm{e}}$, with indices which resemble simple type indices like (i), (i, i), $((i))$, $((i), (i))$, etc., where *i* is a given simple type of individuals - but it must be kept in mind that none of these relative types of classes and $relations_e$ are simple types. Keeping this in mind, an sm which holds between, say α and β would have the index $((i), (i))$, being what Whitehead and Russell call a *homogeneous* relation of similarity. Similarity between α and γ, however, would have an index like $((i), ((i)))$ being what they call an *heterogeneous* relation of similarity. This point is very important because it allows them not only to state things like: "the class α of relative type (*i*) of individuals x^i is not similar to the class $\gamma^{((i))}$ of all classes of individuals x^i"[53], but because it actually gives rise to a taxonomy of different notions of cardinal numbers.

The actual definitions of the (completely ambiguous) cardinal number of a class and the class of cardinal numbers are, respectively:

$$*100{\cdot}01 \qquad \mathrm{Nc} = \overrightarrow{\mathrm{sm}} \quad \mathrm{Df}$$

[52] This is merely for the sake of exposition. Below we shall review the actual notation of relative types that is actually employed in *Principia*.

[53] Note, yet again, that this aspect of *Principia*'s theory of types would be either completely obliterated or made much more complicated by defining ordered pairs as classes, as we indicated in our discussion of the Wiener and Kuratowski's definitions of ordinal couples.

$*100{\cdot}02 \quad \mathrm{NC} = \mathrm{D}`\mathrm{Nc} \quad \mathrm{Df}$

Which, once unpacked by $*32{\cdot}01$ and $*33{\cdot}01$ give us:

$*100{\cdot}01' \quad \mathrm{Nc} = \hat{\kappa}\hat{\gamma}\{\kappa = \hat{\beta}(\beta\,\mathrm{sm}\,\gamma)\}$

$*100{\cdot}02' \quad \mathrm{NC} = \hat{\kappa}\{(\exists\beta)\,.\,\kappa\,\mathrm{Nc}\,\beta\}$

The first defines the relation which holds between a class γ and the class κ of all classes similar to it, e.g., $\hat{\beta}(\beta\, sm\, \gamma)$; the second defines the class NC of all such classes of classes κ which are the number of some given class. By employing the device of descriptive functions, we then have:

$*100{\cdot}1 \quad \vdash .\, \mathrm{Nc}`\alpha = \hat{\beta}(\beta\,\mathrm{sm}\,\alpha)$

The cardinal numbers 0 and 1 are defined respectively at sections *52 and *54 as the set that contains the empty set and the number one as the class of all unit classes:

$*54{\cdot}01 \quad 0 = \iota`\Lambda \quad \mathrm{Df}$

$*52{\cdot}01 \quad 1 = \hat{\alpha}\{(\exists x)\,.\,\alpha = \iota`x\} \quad \mathrm{Df}$

However, the actual proof that such numbers are, in fact, cardinals in the sense defined at $*100$ is only given in section $*101$, where we find the following theorems:

$*101{\cdot}1 \quad \vdash .\, 0 = \mathrm{Nc}`\Lambda$

$*101{\cdot}2 \quad \vdash .\, 1 = \mathrm{Nc}`\iota`x$

which presuppose, respectively, the following theorems from the section on similarity:

$*73{\cdot}48 \quad \vdash .\, 0 = \hat{\beta}(\beta\,\mathrm{sm}\,\Lambda)$

$*73{\cdot}45 \quad \vdash .\, 1 = \hat{\beta}(\beta\,\mathrm{sm}\,\iota`x)$

Thus: 0 is the number of any set that does not have any members, e.g., the set of all sets similar to the empty set, e.g., the set which contains only the empty set; the number 1 is the number of any set that has a single member, e.g., the set of all sets similar to some unit set, e.g., the set which contains only unit classes[54].

Now, all of the symbols defined above, however, are ambiguous as to their relative type. In particular, the occurrences of "sm" in the *definiendum* of some of them may be, once relative indices are restored, either homogeneous or heterogeneous. The same applies to the theorems on cardinals which are stated in a typically ambiguous manner. This point is of fundamental importance, because this possibility of restoring type indices in such different ways gives rise to different notions of cardinal numbers. As we explained, sm can occur either as a homogeneous or heterogeneous relation. This means that there are three possible cases which can occur for any classes α and β that are similar, namely:

1. α sm β and α and β are of the same type.
2. α sm β and the type of α is lower than the type of β.
3. α sm β and the type of α is higher than the type of β.

Each of these different kinds of similarity relations give rise to different relations Nc and, consequently, different notions of cardinal number. They generate, respectively, what Whitehead and Russell call 'homogeneous cardinals', 'ascending cardinals' and 'descending cardinals'.

Let us take the same example given before, letting $\alpha = \{a, b\}$, $\beta = \{c, d\}$ and $\gamma = \{\alpha, \beta\}$. Restoring type indices according to the informal conventions we employed considering this same example above and, again, letting *i* be some simple type of individuals, we would have an illustration of each case in 1, 2 and 3 below:

1. Take sm as it occurs in "α sm β" and use it to restore the typical determination in "Nc". Here Nc would be of relative type $((i), ((i)))$, with Nc'α being the class of all classes of relative type (*i*) similar to α, therefore, of relative type ((*i*)).
2. Take "sm" as it occurs in "β sm γ" and use it to restore the typical determination in "Nc". Here Nc would be of relative type $((i), (((i))))$, with Nc'β being the class of all classes of relative type ((*i*)) similar to β, therefore, of relative type (((*i*))).
3. Do the same with "sm" in "γ sm β", restoring relative typen indices for "Nc" accordingly. Here Nc would be of relative type $((((i))), (i))$, with Nc'γ being the class of all classes of relative type ((*i*)) similar to γ, therefore, of relative type ((*i*)).

[54] From their definition of succession, of course, it will follow that 2 will be the class of all couples, 3 the class of all trios, and so on, but this will be discussed on the next section.

In the above, Nc‘α is a *homogeneous* cardinal because the relative type of its members is the same as that of α; Nc‘β, on the other hand, is an *ascending* cardinal, since all of its members are of a relative type above that of β; Nc‘γ, on its turn, is a *descending* cardinal, since all of its members are of a relative type *below* that of γ. Notice, however, that, despite the terminology what is "homogeneous", "ascending" or "descending" are the similarity *relations*, not the cardinals which are defined in terms of them. Every class is homogeneous with respect to the relative types of their members: this is the cornerstone of *Principia*'s emulated simple type theory of classes. Each of these kind of cardinal numbers are dealt with separately in *Principia* in sections ∗103, ∗104 and ∗105[55], but Whitehead and Russell also speak of cardinals in ∗100 and ∗101 in a way that leaves it completely undetermined what kind they are. This, as we shall see, is done in a very problematic way.

Complete typical ambiguity is at odds with heterogeneous relations. To understand this, we must see in more detail that cardinal numbers generated by similarity relations of different relative types may have quite distinct characteristics. Whitehead and Russell were, of course, quite aware of that and explained the situation as follows:

> The only difficulties which arise in Cardinal Arithmetic in connection with the ambiguities of type of the symbols are those which enter through the use of the symbol sm, or of the symbol Nc, which is $\overrightarrow{sm}$. For it may happen that a class in one type has no class similar to it in some lower type (cf. ∗102·72·73). All fallacious reasoning in cardinal or ordinal arithmetic in connection with types, apart from that due to the mere absence of meaning in symbols, is due to this fact - in other words, to the fact that in some types ∃ ! Nc‘α is true, and in other types ∃ ! Nc‘α may not be true.[56]

This explanation occurs in the introductory note to the second volume entitled *Prefatory Statement Concerning Symbolic Conventions.* This is a famously problematic portion of *Principia*, often regarded as difficult and convoluted not only from the point of view of the historical circumstances which led to its writing[57], but also due to its content[58]. The *problem* presented, however, is not complicated, as a simple example will illustrate. Suppose that α is a class of relative type $((i))$, a class of classes of individuals of type *i*; one of its (*descending*) cardinal numbers will be, for instance, the class $\hat{\delta}(\delta \text{ sm}_{((i),((i)))}\ \alpha)$; call it, just for the sake of the present example, "N_ic‘α"; this class is the class of all classes of individuals (of simple type *i*) similar to α; suppose now that there is only one such individual of type *i* in the universe, call it *a*. Then

55 Section ∗106 also treats of cardinals of *relational types* which are cardinal numbers which contain classes of relations as elements.

56 WHITEHEAD & RUSSELL, 1927*a*, p.xiii [1912, p.xv].

57 GRATTAN-GUINNES, I., 1977, p.107; GRATTAN-GUINNES, I., 2000, p. 385; LANDINI, G., 2016.

58 This should cause some alarm being a statement about a work that has a quite (*only partially justified*) reputation for being dauntingly complex and convoluted. It *is* safe to say, however, that the *Prefatory Statement* is the most complicated part of *Principia*'s first thousand pages (we abstain from making a more general statement since that would require a discussion of *Principia*'s last thousand pages, which, as of yet, we are not prepared to offer).

the 'hierarchy' of relative types of classes which could be formed with the individuals of simple type *i* at the bottom would be this, where each 'type' T_v can be understood as a *class*, e.g., the class of things of type *v*[59]:

$T_i = \{a\}$

$T_{(i)} = \{\{a\}, \Lambda_{(i)}\}$

$T_{((i))} = \{\{\{a\}\}, \{\Lambda_{(i)}\}, \{\{a\}, \Lambda_{(i)}\}, \Lambda_{((i))}\}$

And so on, with every subsequent type $T_{(v)}$ having 2^n members, where *n* is the number of elements of the type t_v.

Suppose now that $\alpha = T_{((i))} - \Lambda_{((i))} = \{\{\{a\}\}, \{\Lambda_{(i)}\}, \{\{a\}, \Lambda_{(i)}\}\}$. The homogeneous cardinal $\mathrm{N_0c}$'α would then be the class of all classes of relative type $((i))$ similar to α, and, therefore, non-empty. Every ascending cardinal would also be non-empty. The descending cardinal, $\mathrm{N_ic}$'α however, would be empty: there are no classes of individuals similar to α.

The result does not depend in any way on the number of individuals at the bottom, however, but is a direct consequence of Whitehead and Russell's reconstruction of Cantor's theorem within *Principia*'s type theory[60]. Anticipating some notation that we have yet to explain, we find the following:

$*102{\cdot}72 \quad \vdash : \beta \subset \alpha \,.\supset.\, \sim (\beta \,\mathrm{sm}\, \mathrm{Cl}‘\alpha)$

$*102{\cdot}73 \quad \vdash .\, \mathrm{Nc}(\alpha)‘t‘\alpha = \Lambda$

The first is an equivalent statement of Cantor's power-class theorem: if β is a (not necessarily proper) subset of α, then there can be no one-one correspondence between β and the power class Cl'α of α; since two sets are said to have the same cardinal number (in some relative type) if they are similar, this means (in a sense not yet precisely defined in *Principia*[61]) that if β is a subset of α, the cardinality Nc'Cl'α of Cl'α is greater than Nc'α. The second theorem is an application of $*102{\cdot}72$ to the case where β is substituted for α and α is substituted for the class of all classes of the same type of α, that is, for *the relative type of* α, denoted by "$t‘\alpha$". $\mathrm{Nc}(\alpha)‘t‘\alpha$ is the class of all classes similar to $t‘\alpha$ of the same relative type as α. Since Cl'$\alpha \subset t‘\alpha$, we also have $\sim (\alpha \,\mathrm{sm}\, t‘\alpha)$ in virtue of $*102{\cdot}72$ and therefore, we have $\mathrm{Nc}(\alpha)‘t‘\alpha = \Lambda$. Descending cardinals like $\mathrm{Nc}(\alpha)‘t‘\alpha$ are always empty in virtue of Cantor's theorem.

59 We emphasize, again, that the notation *i*, (*i*), etc., is not used in *Principia* and that we use it only for the sake of presenting examples in the simplest possible way; again *class expressions do not have simple type indices*.

60 This point is extensively discussed in LANDINI, G., 2016.

61 Defintions of *greater*, *less*, etc. are introduced in $*117$. We'll discuss them below.

Thus, Nc‘α is, in a sense, empty in some types and non-empty in others. In virtue of this, if we take the completely ambiguous Nc‘α, several results which we would expect about it simply will not be provable due to its ambiguity and several apparently valid inferences will be blocked. Another (at first, surprising) example is Hume's Principle. The first half of the bi-conditional is proved for Nc‘α:

$*100{\cdot}321 \quad \vdash : \alpha \,\mathrm{sm}\, \beta \,.\supset.\, \mathrm{Nc}‘\alpha = \mathrm{Nc}‘\beta$

But as Whitehead and Russell explain, the second half of the bi-conditional cannot be so proved:

> Note that Nc‘α = Nc‘β . ⊃ . α sm β is not always true. We might be tempted to prove it as follows:
>
> | | | |
> |---|---|---|
> | ⊢ $*100{\cdot}1$ | ⊃⊢ :. Nc‘α = Nc‘β | . ≡ : γ sm α . $\equiv_\gamma$. γ sm β : |
> | [$*10{\cdot}1$] | | . ⊃ : α sm α . ≡ . α sm β |
> | [$*73{\cdot}3$] | | . ⊃ : α sm β |
>
> But the use of $*10{\cdot}1$ here is only legitimate when the ‘sm’ relation concerned is a homogeneous relation. If Nc‘α, Nc‘β are descending cardinals, we may have Nc‘$\alpha = \Lambda =$ Nc‘β without having α sm β.[62]

The first reason is quite simple: in order to correctly apply the rule of instantiation to a scheme $(x)\ .\ \phi x \supset \phi y$, "$\phi$" must be the same formula except for having occurrences of *x* substituted for *y*. If we restore type indices according to Whitehead and Russell's conventions[63], we see this is not the case for the first line of the above proof, which would be "$(\gamma) : \gamma\ \mathrm{sm}_{(\gamma,\alpha)}\ \alpha\ .\equiv.\ \gamma\ \mathrm{sm}_{(\gamma,\beta)}\ \beta$", while the second line is "$\alpha\ \mathrm{sm}_{(\alpha,\alpha)}\ \alpha\ .\equiv.\ \alpha\ \mathrm{sm}_{(\gamma,\beta)}\ \beta$"[64]. The second reason has to do with the peculiarities of descending cardinals which we just discussed: these may be empty in some (relative) types and not in others.

Before discussing in some detail the difficulties pointed out by Landini in *Principia*'s handling of typical ambiguity in cardinal Arithmetic, we must briefly review *Principia*'s official notation for relative types of classes and relations$_{\mathrm{e}}$. The notation we employed so far for making clear the importance of relative types is good enough for the purposes of exposition, but in order to get the record straight concerning the issues involved in *Principia*'s treatment of cardinal numbers, we need to be careful in reading *Principia*'s own notations. The main definitions for

62 WHITEHEAD & RUSSELL, 1927*a*, p.15 [1912, p.15].

63 WHITEHEAD & RUSSELL, 1927*a*, pp.5-6, 23-4 [1912, pp.5-6, 24-5]. They use $R_{(\alpha,\beta)}$ to indicate that the referents and relata of *R* have, respectively, the relative types of α and β.

64 Landini says that Whitehead does not provide a reason as to why line (2) cannot count as an instance of $*10{\cdot}1$ (LANDINI, G., 2016, p.18) but his comment that "the use of $*10{\cdot}1$ here is only legitimate when the ‘sm’ relation concerned is a homogeneous relation" indicates that Whitehead grasped clearly that it is a matter of keeping the same ϕ in the antecedent and consequent.

relative types of classes are the following[65]:

$*63{\cdot}01$	$t`x = \iota`x \cup -\iota`x$	Df	$*63{\cdot}02$	$t_0`\alpha = \alpha \cup -\alpha$	Df
$*63{\cdot}011$	$t^1`x = t`x$	Df	$*63{\cdot}03$	$t_1`\kappa = t_0`s`\kappa$	Df
$*63{\cdot}04$	$t^2`x = t`t`x$	Df	$*63{\cdot}05$	$t_2`\kappa = t_1`t_1`\kappa$	Df
	etc...			etc...	

Similar definitions are provided for relative types of relations$_e$ using ordinal couples:

$*64{\cdot}011$	$t^{11}`x = t`(t`x \uparrow t`x)$	Df	$*63{\cdot}01$	$t_{00}`\alpha = t`(t_0`\alpha \uparrow t_0`\alpha)$	Df
$*64{\cdot}012$	$t^{12}`x = t`(t`x \uparrow t^2`x)$	Df	$*63{\cdot}02$	$t_{01}`\alpha = t`(t_0`\alpha \uparrow t_1`\alpha)$	Df
$*63{\cdot}013$	$t^{21}`x = t`(t^2`x \uparrow t`x)$	Df	$*63{\cdot}021$	$t_{10}`\alpha = t`(t_1`\alpha \uparrow t_0`\alpha)$	Df
	etc...			etc...	

In their summary, Whitehead and Russell explain that the above notations "[...] serve to express the type of one variable in terms of the type of another"[66]. In the case of relative types of classes, they informally explain that $*63{\cdot}01$ defines "the type of which x is a member" and $*63{\cdot}02$ defines "the type in which α is contained"[67]. If we accept that *Principia*'s primitive grammar is that of simple types with *both* lowest type individual variables *and* predicate variables of every simple type (also standing for individuals) then the above must be read with a grain of salt: we must carefully distinguish genuine *simple* types (which differentiates the lowest type individuals from properties and relations which hold between them, properties of their properties, etc...) from *relative* types.

As Landini emphasizes[68], if *Principia* really embraces the grammar of simple types, then one cannot speak of these in its object-language: the key fact to be observed here is that the above definitions just introduce a useful notation to speak, for instance of the *relative type* of a term *x* to a class α; for instance, if we need to talk about the class of all individuals of the same type of *x*, the class of all classes of the same relative type as α, the class of all terms of the same type as the members of α, and so on ($*63{\cdot}01$, for instance, employs individual variables of a given simple type to define a class which is of a higher *relative* type than these individuals). The point is that each of the definitions in $*63$, $*64$ and $*65$ defines a class or relation$_e$ and none of them can be identified with an object of any simple type because none of them is in any simple type, since neither classes nor relation$_e$ are genuine entities which belong in *Principia*'s simple type structure[69].

[65] Recall that $s`\kappa$ is defined in $*40{\cdot}02$ as the sum of all members of a class of classes κ, that is, as the class $\hat{x}\{(\exists\alpha)\,.\,\alpha\,\epsilon\,\kappa\,.\,x\,\epsilon\,\alpha\}$.

[66] WHITEHEAD & RUSSELL, 1925, p.400 [1910, p.419].

[67] WHITEHEAD & RUSSELL, 1925, p.400 [1910, p.419].

[68] Cf. LANDINI, G., 2016, p.4, cf. also WHITEHEAD & RUSSELL, 1912, p.34.

[69] This is strongly corroborated by a passage from the *Note on negative statements concerning types* in volume 2: "Statements such as "$x \sim \epsilon\, t`y$" or "$x \sim \epsilon\, t_0`\alpha$" are always false when they are significant. Hence, when an

Now, using the definitions from $*63$ and $*64$, Whitehead and Russell introduce the following notations which indicate the various ways relative-type indexes are to be restored:

$*65{\cdot}01$ $\alpha_x = \alpha \cap t‘x$ Df

$*65{\cdot}02$ $\alpha(x) = \alpha \cap t‘t‘x$ Df

$*65{\cdot}03$ $R_x = (t‘x) \upharpoonleft R$ Df

$*65{\cdot}04$ $R(x) = (t^2‘x) \upharpoonleft R$ Df

$*65{\cdot}1$ $R_{(x,y)} = (t‘x) \upharpoonleft R \restriction t‘y$ Df

$*65{\cdot}11$ $R(x_y) = (t^2‘x) \upharpoonleft R \restriction t‘y$ Df

$*65{\cdot}12$ $R(x,y) = (t^2‘x) \upharpoonleft R \restriction (t^2‘y)$ Df

Several definitions and propositions which will be discussed in what follows rely heavily on the above, so they must be kept in mind.

5.4 Whitehead's vs Landini's Emendations

Another striking example of the problematic status of completely ambiguous occurrences of "Nc" (an example analyzed at lenght by Landini[70]) is given by the authors in the following passage:

> Formulae which are typically ambiguous, or only partially definite as to type, must not be admitted unless every significant interpretation is true. Thus, for example we may admit
>
> $\alpha \,\epsilon$Nc‘α
>
> because here ‘Nc’ must mean ‘Nc(α_α)’ so that the only ambiguity remaining is as to the type of α, and the formula holds of whatever type α may belong to, provided ‘Nc‘α’ is significant, i.e., provided α is a class. But we must not, from ‘$\alpha \,\epsilon$ Nc‘α ’ allow ourselves to infer ‘∃ !Nc‘α’. For here the conditions of significance no longer demand that ‘Nc’ should mean ‘Nc(α_α)’: it might just as well mean‘Nc(β_α)’. And as we saw, if β is a lower type than α, and α is sufficiently large of its type, we may have

object belongs to one type, there is no significant way of expressing what we mean when we say that it does not belong to some other type. The reason is that, when, for example, $t‘\alpha$ and $t_0‘\alpha$ are said to be different, the statement is only significant if interpreted as applying to the symbols, *i.e.*, as meaning to deny that the two symbols denote the same class. We cannot assert that they denote different classes, since ‘$t‘\alpha = t_0‘\alpha$’ is not significant, but we can deny that they denote the same class." (WHITEHEAD & RUSSEL, 1912, p.35).

70 Cf. LANDINI, G., 2016.

> $\mathrm{Nc}(\beta_\alpha)`\alpha = \Lambda$
>
> so that '$\exists\,!\mathrm{Nc}`\alpha$' is not admissible without qualification. Nevertheless, as we shall see in ∗100, there are a certain number of propositions to be made about a wholly ambiguous Nc or NC.[71]

At a first glance, it could seem that the authors are simply denying an instance of the following irreproachable deduction scheme:

[Hp]⊦	$\alpha \,\epsilon\, \mu$	(1)
[(1) . ∗10·24]⊦	$(\exists\beta) \,\epsilon\, \mu$	(2)
[24·03]⊦	$(\exists\beta)\beta \,\epsilon\, \mu \supset \mu \neq \Lambda$	(3)
[(2) . (3)]⊦	$\mu \neq \Lambda$	(4)

But as Landini showed in a meticulous study[72], matters are not so straightforward.

First, following Whitehead and Russell's convention for restoring relative type-indexes, we can pinpoint the reason why "[...] it is a fallacy to infer '$\exists!\mathrm{Nc}`\alpha$' from the (true) propositions '$\vdash .\ \alpha \,\epsilon\, \mathrm{Nc}`\alpha$' and '$\vdash .\ \alpha \,\epsilon\, \mathrm{Nc}`\alpha \supset \exists\,!\,\mathrm{Nc}`\alpha$'"[73]. The problem here lies in the non-uniform ambiguity of $\mathrm{Nc}`\alpha$. Take, for instance[74]:

[Hp]	$\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha$	(1)
[∗10·24]⊦	$\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha \supset (\exists\gamma) \,\epsilon\, \mathrm{Nc}(\beta)`\alpha$	(2)
[(1) . (2)]⊦	$\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha \supset \exists\,!\,\mathrm{Nc}(\beta)`\alpha$	(3)
[(1) . (3)]⊦	$\exists\,!\,\mathrm{Nc}(\beta)`\alpha$	(4)

This is as good as it gets: there is no problematic ambiguity and every step can be adequately justified. The following, however, is a fallacious deduction:

[Hp]	$\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha$	(1)
[∗10·24]⊦	$\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha \supset (\exists\gamma)\gamma \,\epsilon\, \mathrm{Nc}(\xi)`\alpha$	(2)
[(1) . (2)]⊦	$\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha \supset \exists\,!\,\mathrm{Nc}(\xi)`\alpha$	(3)
[(1) . (3)]⊦	$\exists\,!\,\mathrm{Nc}(\xi)`\alpha$	(4)

Line two introduces a pseudo-instance of ∗10·24, since the typical determination of $\mathrm{Nc}`\alpha$ is not uniform. Note that, apart from the confusing ambiguity which makes it appears as if the authors were denying an instance of *modus ponens*, this is as it should be: if, in the above deduction, $\mathrm{Nc}(\xi)`\alpha$ were a descending cardinal, it could be empty even if $\mathrm{Nc}(\beta)`\alpha$ is not - so $\exists\,!\,\mathrm{Nc}(\xi)`\alpha$ *ought not to follow from* $\alpha \,\epsilon\, \mathrm{Nc}(\beta)`\alpha$. The lesson to be learned here is that completely ambiguous "sm" and "$\mathrm{Nc}`\alpha$" must be handled with the highest caution or, better yet, avoided at all cost.

Unfortunately, as Landini showed in meticulous detail[75], Whitehead and Russell them-

71 WHITEHEAD & RUSSELL, 1912, p.12.
72 LANDINI, G., 2016.
73 WHITEHEAD & RUSSELL, 1912, p.34.
74 We use a simplified form of Landini's example (LANDINI, 2014, p.15).
75 Cf. LANDINI, G., 2016, pp.16-24.

selves were not careful enough. There are definitions and proofs in part III of volume 2 which are in need of serious revision - in fact, there are *invalid* 'proofs' in *Principia* that result from not observing the difficulties introduced by this sort of ambiguity. The most serious of all is perhaps the case of the following 'theorem' :

$*100{\cdot}3 \qquad \vdash . \alpha \in \mathrm{Nc}`\alpha$

This proposition is true whenever significant: for any κ, if it is meaningful to ask if either $\alpha \in \kappa$ or $\alpha \sim \in \kappa$, then the members of κ must be of the same relative type as α - this is the cornerstone of *Principia*'s emulated type theory of classes; and since α is similar to itself, it must belong the class of all classes similar to α.

As Whitehead and Russell themselves put it, in theorem $*100{\cdot}3$ "'Nc' must mean 'Nc(α_α)' so that the only ambiguity remaining is as to the type of α, and the formula holds of whatever type α may belong to, provided 'Nc'α' is significant, i.e., provided α is a class"[76]. As Landini observes, however, "[...] the characteristic of being true whenever significant is a poor guide to what is (and what ought to be) provable in *Principia*"[77]. In fact, *we may have true, significant, but not provable* propositions. The proof Whitehead and Russell had in mind is this:

$$[*100{\cdot}1] \vdash \quad \mathrm{Nc}`\alpha = \hat{\delta}(\delta\ \mathrm{sm}\ \alpha) \quad (1)$$

$$[*73{\cdot}3] \vdash \quad \alpha\ \mathrm{sm}\ \alpha \quad (2)$$

$$[(1)] \vdash \quad \alpha \in \mathrm{Nc}`\alpha \equiv \alpha \in \hat{\delta}(\delta\ \mathrm{sm}\ \alpha) \quad (3)$$

$$[(*20{\cdot}3) . (3)] \vdash \quad \alpha \in \mathrm{Nc}`\alpha \equiv \alpha\ \mathrm{sm}\ \alpha \quad (4)$$

$$[(2) . (4)] \vdash \quad \alpha \in \mathrm{Nc}`\alpha \quad (5)$$

The problem is that the occurrences of "sm" in line (2) must be different from those in the rest of the proof: the sm which defines the ambiguous, possibly heterogeneous, Nc'α must be indeterminate as to the relative type of its referents, that is, it must mean "$\mathrm{sm}_{(\delta,\alpha)}$" where the type of δ is left unassigned; conditions of significance, however require that the sm of line (2) be of type "$\mathrm{sm}_{(\alpha,\alpha)}$". This invalidates the proof: in order to get the desired result we should be able to prove $\alpha\ \mathrm{sm}_{(\delta,\alpha)} \alpha$, which - although also true whenever significant - is not provable[78].

The available historical record of the writing of volume 2 shows that Whitehead was aware[79] of some difficulties surrounding the completely ambiguous Nc'α and that fallacies could arise in connection with it. We shall address this briefly below[80]. For now, it is enough to observe

76 WHITEHEAD & RUSSELL, 1927*a*, p.12 [1912, p.12].
77 Cf. LANDINI, G., 2016, p.19.
78 Cf. LANDINI, G., 2016, p.17.
79 He reported difficulties in a letter to Russell dated 19 January 1911 (GRATTAN-GUINNESS, I., 2000, p.585). We shall discuss this letter below.
80 Again, for details the reader is referred to LANDINI, G., 2016. Also cf. GRATTAN-GUINNESS, I., 2000, pp.384-6 and 1977, p.107.

that the trouble is caused by their original sin of not presenting first their symbolism with determinate type-indices (both simple *and* relative) *and then* presenting rules for their suppression. Had they done this, the above problem (and so many other interpretative ones) would simply not arise.

The purpose of the *Prefatory Statement* is to amend issues related to the typical ambiguity of Nc and its variants, by introducing conventions for restoration and suppression of relative types indices of what Whitehead and Russell call 'formal numbers', e.g., any expression defined directly or indirectly by sm[81]. The goal was to secure that every theorem scheme - even those that contain non-uniformly ambiguous expressions like sm and Nc'α had what they called a "stable truth-value":

> A symbolic form has a *stable* truth-value if, after any assignment of types to the real variables, types can be assigned to the constant symbols so that the truth-value of the proposition thus obtained is the same as the truth-value of any proposition obtained by modifying it by the assignment of higher types to some or all of the constant symbols. This truth-value is the *stable* truth-value.[82]

In order to understand the conventions we need to consider in some detail their 'formal numbers' . They are classified in two sorts, namely *constant* and *functional*:

> A constant formal number is any constant symbol for which there is a constant α such that, in whatever type the constant symbol is determined, it is, in that type, identical with Nc'α. In other words, if σ be a constant symbol, then σ is a formal number provided that "truth" is the permanent truth value of $\sigma =$ Nc'α, for some constant α.
>
> The *functional* formal numbers are defined by enumeration; they are
>
> Nc'α, ΣNc'κ, ΠNc'κ, sm"μ, $\mu +_c \nu$, $\mu -_c \nu$, μ^ν
>
> where in each formal number the symbols α, κ, μ, ν occurring in it are called the arguments of the functional form even when they are complex symbols.[83]

The *occurrences* of formal numbers in formulas are also classified into five sorts, namely, *argumental*, arithmetical, equational, attributive and logical:

> The occurrence of σ in sm"σ is called an *argumental* occurrence
>
> The occurrence of σ as an argument of an *arithmetical* formal number (which may be a component of another formal number) or as one side of an arithmetic inequality is called an arithmetical occurrence.
>
> The occurrence of σ as one side of an equation is called an *equational* occurrence.

81 WHITEHEAD & RUSSELL, 1927*a*, p.xiii [1912, p.xv]; cf. also LANDINI, G., 2016, p.25.
82 WHITEHEAD & RUSSELL, 1927*a*, p.xiii [1912, p.xv].
83 WHITEHEAD & RUSSELL, 1927*a*, p.xiv [1912, p.xvi].

> The occurrence of σ in '$\xi \,\epsilon\, \sigma$' is called an *attributive* occurrence.
> Any other occurrence of σ is called a *logical* occurrence, so also is $\sigma = \Lambda$.[84]

It is worthwhile to take a look at some examples in order to get a clear grasp of the meaning of these distinctions. The idea is that any expression which contains an explicit or implicit occurrence of sm in its definition is a formal number; thus, Nc'α and its variants, and all class-expressions of *Principia* which contain occurrences of operations like *sum*, *subtraction* or *exponentiation* are formal numbers[85].

The difference between constant and functional formal numbers is simple enough: apart from type ambiguity, symbols like 1, 2, 3 etc., $1 +_c 1$, $1 +_c 2$, $\aleph_0$, $2^{\aleph_0}$ have definite meaning, while symbols like Nc'α, $\mu +_c 1$, $\mu +_c \nu$ do not, since they contain free class variables like α, μ, ν which must be determined in order for them to have a definite meaning. The first are constant, the latter functional. Now, the kinds of occurrences can be exemplified as follows:

1. Argumental: sm"Nc'α, sm"μ, sm"$\mu +_c 1$;
2. Arithmetical: $\mu^{\text{Nc`}\alpha}$, $\nu^{\mu +_c \nu}$, μ^2, Nc'$\alpha >$ Nc'β, Nc'$\alpha \geq$ Nc'β, $\mu +_c \nu$;
3. Equational: Nc'$\alpha =$ Nc'β, Nc'$\alpha = \mu$, Nc'$\alpha = \mu +_c \nu$, $\mu +_c \nu = \mu +_c \nu$; sm"$\mu =$ Nc'μ, sm"Nc'$\alpha = \mu$;
4. Attributive: $\beta \,\epsilon\,$ Nc'α, $\alpha \,\epsilon\, 1$, $\gamma \,\epsilon\, \mu +_c \nu$;
5. Logical: $\exists\,!$ Nc'α, $\exists\,!\, \mu +_c \nu$, Nc'$\alpha = \Lambda$,$\mu +_c \nu = \Lambda$, $\exists\,!$ sm"μ, sm"$\mu = \Lambda$;

Obviously, the same formal number can occur in different parts of a formula in different ways. Take for instance: Nc'$\alpha = \mu \,.\, \exists\,!$ Nc'α; the first occurrence of Nc'α is equational, the second logical; also, a formal number can occur within a formal number in complex ways; for instance, in $(\mu +_c \nu) +_c \nu = \xi$ the first occurrence of $\mu +_c \nu$ is arithmetical, while the occurrence of the whole formal number $(\mu +_c \nu) +_c \nu$ is equational; similarly in Nc'$\alpha +_c$ Nc'$\alpha =$ Nc'α, $\mu^{\text{Nc`}\alpha} =$ Nc'α, etc[86]. Among arithmetical formal numbers, Whitehead and Russell also distinguish *pure* arithmetical formal numbers, which have no component parts that are argumental occurrences of formal numbers.[87]

Now, for the conventions introduced in the *Prefatory Statement*: their purpose is to establish when two occurrences of a formal number "are said to be bound to each other"[88], that is,

[84] WHITEHEAD & RUSSELL, 1927*a*, p.xix [1912, p.xxi].

[85] They are absolutely explicit in the text that distinctions concerning formal numbers and their occurrences are about expressions: "[...] the distinction between formal numbers depends on the symbolism and not the entity denoted, and in considering them it is symbolic analogy and not denotation which is to be taken into account" (WHITEHEAD & RUSSELL, 1927*a*, pp.xiv-xv [1912, p.xvii].).

[86] WHITEHEAD & RUSSELL, 1927*a*, pp.xix-xx [1912, pp.xxii-xxiii].

[87] WHITEHEAD & RUSSELL, 1927*a*, p.xv [1912, p.xviii].

[88] WHITEHEAD & RUSSELL, 1927*a*, p.xxi [1912, p.xxiii].

when two occurrences of a formal number are to be identified as having the same relative type (index), but without "elaboration produced by the definition of types". For our purposes at the moment, the two conventions we must consider are the following:

> IT *All logical occurrences of the same formal number are in the same types; argumental occurrences are bound to logical and attributive occurrences; and if there are no argumental occurrences, equational occurrences are bound to logical occurrences.*[89]
>
> IIT *Whenever a formal number σ occurs, so that, if it were replaced by Nc'α, the actual type of Nc'α would by definition have to be adequate, then the actual type of σ is also to be adequate.*[90]

The most general and most important is IT and for now it is the only one we need to discuss; IIT will be considered when we discuss *Principia*'s definition of addition. If we go back to the question as to why "it is a fallacy to infer '$\exists!\mathrm{Nc}'\alpha$' from the (true) propositions '$\vdash . \alpha \epsilon \mathrm{Nc}'\alpha$' and '$\vdash . \alpha \epsilon \mathrm{Nc}'\alpha \supset \exists!\mathrm{Nc}'\alpha$'"[91] we can readily understand the point of IT.

Analyzing the occurrences of formal numbers in each of the sentences, we have in "$\alpha \epsilon \mathrm{Nc}'\alpha$" an *attributive* occurrence of Nc'α and in "$\alpha \epsilon \mathrm{Nc}'\alpha \supset \exists!\mathrm{Nc}'\alpha$" the first occurrence of Nc'α is *attributive* and the second *logical*; IT assures us that all logical occurrences of a formal number are bound but says nothing about attributive ones being bound to logical ones; thus, we know the inference to be fallacious without the need to fully restore type determinations. Another example, which is in fact discussed in some detail by the authors in connection with IT is a revised (and weaker) version of Hume's Principle:

∗103·35 $\vdash :. \exists!\mathrm{Nc}'\alpha . \vee . \exists!\mathrm{Nc}'\beta : \supset : \mathrm{Nc}'\alpha = \mathrm{Nc}'\beta . \equiv . \alpha \epsilon \mathrm{Nc}'\beta . \equiv \beta \epsilon \mathrm{Nc}'\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta$

The authors explain:

> In ∗103·35, IT directs the logical and equational occurrences of Nc'α to be in the same type, and similarly for Nc'β. Also "meaning" secures that the equational types of Nc'α and Nc'β are the same. Thus these four occurrences are all in one type, which has no necessary relation to the types of the attributive occurrences of Nc'α and Nc'β. Thus, using the notation of ∗65·04 to secure typical definiteness, ∗100·35 is to mean:

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89 WHITEHEAD & RUSSELL, 1927*a*, p.xxi [1912, p.xviii].

90 WHITEHEAD & RUSSELL, 1927*a*, p.xxiii [1912, p.xxv]. There are others, namely, conventions 'AT' and 'Infin T'. Like IT and IIT, each of these is intended to deal with the different difficulties that arise either from the ambiguity of formal numbers or from the possibility that a cardinal may exist in one type but not in another. For discussion of all of these conventions the reader is referred to Landini's study (LANDINI, G., 2016). Infin T will be briefly discussed in connection with the difficulties involved in proving the existence of finite cardinal numbers.

91 WHITEHEAD & RUSSELL, 1927*a*, p.12 [1912, p.12].

$\vdash :. \exists ! \mathrm{Nc}(\xi)`\alpha . \vee . \exists ! \mathrm{Nc}(\xi)`\beta : \supset : \mathrm{Nc}(\xi)`\alpha = \mathrm{Nc}(\xi)`\beta . \equiv . \alpha \epsilon \mathrm{Nc}(\alpha)`\beta .$
$\equiv \beta \epsilon \mathrm{Nc}(\beta)`\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta$

The types of these attributive occurrences are settled by the necessity of "meaning".[92]

Since for the purposes of developing the basics of the theory of natural numbers, e.g., finite cardinals, homogeneous and ascending cardinals mainly suffice, one could hope that the difficulties surrounding Nc'α would be avoided in their treatment. Unfortunately, this is not the case, as Landini showed[93].

The homogeneous cardinal number of a class α is defined as follows:

∗103·01 $\mathrm{N_0c}`\alpha = \mathrm{Nc}`\alpha \cap t`\alpha$ Df

The idea is to intersect Nc'α and $t`\alpha$, in order to fix the type of $\mathrm{N_0c}`\alpha$ as $t`t`\alpha$ or, more shortly, $t^2`\alpha$. Such a class of classes $\mathrm{N_0c}`\alpha$ ought to only be defined by a homogeneous similarity relation. Unpacking the definition they get:

∗103·11 $\vdash \beta \epsilon \mathrm{N_0c}`\alpha . \equiv . \beta \epsilon \mathrm{Nc}`\alpha . \beta \epsilon t`\alpha$

This is completely inadequate, as a quick look at some proofs of important results will show.

Homogeneous cardinals are important because several results are *provable* for them that have no analogues for Nc'α. The most important are, perhaps, the following:

∗103·12 $\vdash . \alpha \epsilon \mathrm{N_0c}`\alpha$

∗103·13 $\vdash . \exists ! \mathrm{N_0c}`\alpha$

∗103·42 $\vdash : \alpha \,\mathrm{sm}\, \beta . \equiv . \mathrm{Nc}(\beta)`\alpha = \mathrm{N_0c}`\beta$

The third theorem is the closest thing to Hume's Principle to be found in *Principia*. The second establishes that the homogeneous cardinal number of a given class α is never empty. The first is the most important and must obviously be provable: α *must* be a member of the class of all classes of the same relative type of α and similar to α. Whitehead and Russell's demonstration

92 WHITEHEAD & RUSSELL, 1927*a*, p.xxi [1912, p.xxiv]. Still, as Landini notes, IT would be dispensable if type indexes were not suppressed from the outset (LANDINI, G., 2016, p.27).
93 LANDINI, G., 2016.

of the first theorem would be something along these lines[94]:

$$[*103{\cdot}11]\vdash \quad \beta \epsilon \mathrm{N_0c}`\alpha \,.\, \equiv_\beta \,.\, \beta \epsilon \mathrm{Nc}`\alpha \,.\, \beta \epsilon t`\alpha \qquad (1)$$

$$[(1)\,.\,*10{\cdot}1]\vdash \quad \alpha \epsilon \mathrm{N_0c}`\alpha \,.\, \equiv \,.\, \alpha \epsilon \mathrm{Nc}`\alpha \,.\, \alpha \epsilon t`\alpha \qquad (2)$$

$$[*63{\cdot}103]\vdash \quad \alpha \epsilon t`\alpha \qquad (3)$$

$$[*73{\cdot}3]\vdash \quad \alpha\, \mathrm{sm}_{(\alpha,\alpha)}\alpha \qquad (4)$$

$$[*100{\cdot}1]\vdash \quad \alpha\, \mathrm{sm}\, \alpha \supset \alpha \epsilon \mathrm{Nc}`\alpha \qquad (5)$$

$$[(4)\,.\,(5)]\vdash \quad \alpha \epsilon \mathrm{Nc}`\alpha \qquad (6)$$

$$[(2)\,.\,(3)\,.\,(6)]\vdash \quad \alpha \epsilon \mathrm{N_0c}`\alpha \qquad (7)$$

The difficulty here seems to be none other than that encountered with theorem $*100{\cdot}3$: line 5 *must* be read as "$\alpha\, \mathrm{sm}_{(\xi,\alpha)}\alpha \supset \alpha \epsilon \mathrm{Nc}(\xi)`\alpha$". But "$\alpha\, \mathrm{sm}_{(\xi,\alpha)}\alpha$" is not provable, only "$\alpha\, \mathrm{sm}_{(\alpha,\alpha)}\alpha$" is. As Landini observes, the definition is useless because in order to show that $\alpha \epsilon \mathrm{N_0c}`\alpha$ cardinal is homogeneous, one must, in fact, make use of the illicit 'theorem' $\alpha \epsilon \mathrm{Nc}`\alpha$[95].

Although it is somewhat astonishing that such an error was made by the authors[96], the situation is easily remedied. As Landini showed, since what we want is the class of all classes similar to α and of the same type of α, we can simply parallel the (typically indefinite) definitions of $*100$ restoring relative type indexes according to the rules of $*63 - 65$[97]:

$$\mathrm{sm_o} = \hat{\beta}\hat{\gamma}\{\exists\,!(\beta \cap t_0`\gamma\, \overline{\mathrm{sm}}\, \gamma)\} \text{ Df}$$

$$\mathrm{N_oc} = \overrightarrow{sm_0} \text{ Df}$$

$$\mathrm{N_oc}`\alpha = \hat{\delta}(\delta\, \mathrm{sm_o}\, \alpha) \text{ Df}$$

These definitions solve the problem of typical (in)determination and allow for a corrected proof of $\alpha \in \mathrm{N_0c}`\alpha$ since they fix the relative type of sm to (α, α). Parallel definitions are also suitable (and necessary) for ascending and descending cardinals of any suffix:

$$\mathrm{sm}^i = \hat{\beta}\hat{\gamma}\{\exists\,!(\beta \cap t^i`\gamma\, \overline{\mathrm{sm}}\, \gamma) \quad \text{Df} \qquad \mathrm{sm}_i = \hat{\beta}\hat{\gamma}\{\exists\,!(\beta \cap t_i`\gamma\, \overline{\mathrm{sm}}\, \gamma) \quad \text{Df}$$

$$\mathrm{N}^i\mathrm{c} = \overrightarrow{\mathrm{sm}^i} \quad \text{Df} \qquad \mathrm{N}_i\mathrm{c} = \overrightarrow{\mathrm{sm}_i} \quad \text{Df}$$

$$\mathrm{N}^i\mathrm{c}`\alpha = \hat{\delta}(\delta\, \mathrm{sm}^i\, \alpha) \quad \text{Df} \qquad \mathrm{N}_i\mathrm{c}`\alpha = \hat{\delta}(\delta\, \mathrm{sm}_i\, \alpha) \quad \text{Df}$$

94 The proof is only is sketched, unfortunately. Had the authors gone on to fully state it we would have more information in order to realize what exactly had gone wrong. Landini again attributes the responsibility to Whitehead and claims that the problematic definitions are probably a result of his emendations (LANDINI, G., 2016, p.19). Although this is very plausible, it is still very puzzling how the mistakes got past both him and Russell. Perhaps the best explanation is their carelessness regarding syntactic matters. If our reading is correct, this is corroborated by the letter Whitehead wrote to Russell by January 19, 1911 quoted above.

95 LANDINI, G., 2016, p.19.

96 Landini finds it "difficult to believe" that such "[...] flawed definitions were in the original *Principia*" and claims that "they must have been part of Whitehead's emendations"(LANDINI, G., 2016, p.20). On this point, see the previous note 94.

97 LANDINI, G., 2016, p.19-20

Landini further introduces an additional definition $t^0‘\alpha =_{\mathrm{Df}} t_0‘\alpha$ in order for the ascending case to cover the homogeneous also[98]. We refrain from fully following him in this just to keep *Principia*'s use of "$\mathrm{N_0c}$" , "$\mathrm{N_0c}‘\alpha$" , etc., unaltered just to preserve and keep explicit, as Whitehead and Russell do, the difference between homogeneous and ascending cardinals (*Principia*'s notation could also be preserved while subsuming homogeneous under heterogeneous cardinals by putting $\mathrm{N_0c} =_{\mathrm{Df}} \mathrm{N^0c}$).

Be that as it may, the above provide us with corrected definitions of homogeneous, ascending and descending cardinal numbers of a given class α. In what follows we continue our discussion of *Principia*'s development of Arithmetic, calling attention to the reader at points where these modifications must be resorted to (they are of particular importance in the next section).

5.5 The Definitions of *Successor* and *Natural Number*

To fully understand how the Dedekind-Peano postulates are obtained we still need a definition of immediate successor and of finite cardinal. Let's start with the former. The relation of immediate successor is defined at $*110$ in three stages. As with their definition of cardinal number (and, in fact, most of their definitions) Whitehead and Russell want the most general definition possible. So in order to obtain the notion of immediate successor $\mu +_c 1$ of a cardinal μ as a particular case they define first the cardinal sum $\mu +_c \nu$ of any two cardinals μ and ν - either finite or infinite[99].

In order to do that, however, they define first what they call the 'arithmetical class-sum' of two classes α and β "[...] so as to give two mutually exclusive classes respectively similar to α and β, so that the number of terms in the logical sum of these two classes is the arithmetical sum of the numbers of terms in α and β respectively"[100]. They put:

$$*110{\cdot}01 \qquad \alpha + \beta = \downarrow (\Lambda \cap \beta)“\iota“\alpha \cup (\Lambda \cap \alpha) \downarrow “\iota“\beta \quad \mathrm{Df}$$

Now, despite its somewhat intimidating appearance, the idea behind this notation is simple and elegant[101]. If we break it down according to $*37{\cdot}01$ and $*51{\cdot}1$, we see that $\downarrow (\Lambda \cap \beta)“\iota“\alpha$ denotes the class of ordered pairs which we obtain from α if we pair the singleton of each member of α with the empty set of the same type of β; $(\Lambda \cap \alpha) \downarrow “\iota“\beta$ is the class of ordered pairs we obtain

98 LANDINI, G., 2016, p.21
99 In fact, "$\mu +_c \nu$" is significant even when μ or ν are not cardinals.
100 WHITEHEAD & RUSSELL, 1927*a*, p.72 [1912, p.75].
101 Russell himself gives a pretty straighforward explanation in RUSSELL, B., 1919, p.117-8. He gave an even better one employing *Principia's* notation to Jourdain in correspondence (GRATTAN-GUINNESS, 1977, p.119-20). Curiously, no explanation of the somewhat compact symbolism is given in *Principia.*

if we pair the empty set of the same type of α with the singleton of each member of β, where α and β are any classes of whatever (possibly distinct) relative types; the arithmetical sum of α and β is the union of theses classes of ordered pairs.

The motivation for this definition is twofold: to guarantee that the number of the resulting class is, in fact, the number which would be obtained by ‘summing up’ the members of the two classes even if they are not disjoint and to adjust the type of each ordered pair as to guarantee that the class of all of them forms an existent, legitimate class. Two simple examples will make these points clear. First, assuming for convenience that α and β are of the same relative type, let $\alpha = \{a, b, d\}$ and let $\beta = \{a, c, e\}$. Suppose we want to obtain directly the arithmetical-sum $\alpha + \beta$ by simply putting their members together, that is, by putting:

$$\alpha + \beta = \alpha \cup \beta = \{a, b, c, d, e\}$$

Obviously such definition would fail, because α and β are not disjoint and our class would be one member short of what we informally expected. If we follow Whitehead and Russell’s procedure, we see the problem is solved, since, using the standard notation for ordinal couples or ordered pairs, we would get:

$$\downarrow (\Lambda \cap \beta)``\iota``\alpha = \{< \{a\}, \Lambda >, < \{b\}, \Lambda >, < \{d\}, \Lambda >\}$$

$$(\Lambda \cap \alpha) \downarrow ``\iota``\beta = \{< \Lambda, \{a\} >, < \Lambda, \{c\} >, < \Lambda, \{e\} >\}$$

So that:

$$\downarrow (\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha) \downarrow ``\iota``\beta =$$

$$\{< \{a\}, \Lambda >, < \{b\}, \Lambda >, < \{d\}, \Lambda >, < \Lambda, \{a\} >, < \Lambda, \{c\} >, < \Lambda, \{e\} >\}$$

The resulting class $\alpha + \beta$ has the desired ‘number’ of members and is a legitimate class, since all of its members are of the same type. Suppose, however, that the classes α and β are of distinct types and that instead of pairing each singleton with the empty set of the type of the other class, we take the empty set in each case to be of the same type of its pair. Let, for instance, $\alpha = \{a, b\}$ and $\beta = \{\{a\}, \{b\}\}$, with *a* and *b* being individuals, so that:

$$\downarrow (\Lambda \cap \alpha)``\iota``\alpha = \{< \{a\}, \Lambda_\alpha >, < \{b\}, \Lambda_\alpha >\}$$

$$(\Lambda \cap \beta) \downarrow ``\iota``\beta = \{< \Lambda_{\alpha+1}, \{\{a\}\} >, < \Lambda_{\alpha+1}, \{\{b\}\} >\}$$

The attempt to unite the resulting classes would be blocked by the theory of types: if *a* is of type *n*, the classes $\{a\}$ and $\{b\}$ would be of type $n+1$; in this case the pairs $<\{a\},\Lambda>$ and $<\{b\},\Lambda>$ would be of type $(n+1)+1$; this is the reason why they had the following instead:

$$\downarrow(\Lambda \cap \beta)``\iota``\alpha = \{<\{a\},\Lambda_\beta>,<\{b\},\Lambda_\beta>\}$$

$$(\Lambda \cap \alpha)\downarrow``\iota``\beta = \{<\Lambda_\alpha,\{\{a\}\}>,<\Lambda_\alpha,\{\{b\}\}>\}$$

Every pair of the first class will be either of type α or β whichever is higher and every pair of the second class will be either of type α or β, whichever is higher, hence both of the same type. The definition is designed to deal with cases in which the summed classes are not disjoint and/or of different types.

We can see that the definition is adequate to capture (in precise terms) what we mean by *summing the members of two classes ignoring repetition*, given the two theorems:

$*110{\cdot}11 \quad \vdash . \downarrow(\Lambda \cap \beta)``\iota``\alpha \cap (\Lambda \cap \alpha)\downarrow``\iota``\beta = \Lambda$

$*110{\cdot}12 \quad \vdash . \downarrow(\Lambda \cap \beta)``\iota``\alpha \,\mathrm{sm}\, \alpha \,.\, (\Lambda \cap \alpha)\downarrow``\iota``\beta \,\mathrm{sm}\, \beta$

which establish that the classes $\downarrow(\Lambda \cap \beta)``\iota``\alpha$ and $(\Lambda \cap \alpha)\downarrow``\iota``\beta$ are not only disjoint, but are also similar to α and β respectively.

Now, with the definition of the arithmetical sum of two classes at hand they define the actual (cardinal) sum of two *homogeneous* cardinals as:

$*110{\cdot}02 \quad \mu +_c \nu = \hat{\xi}\{(\exists\alpha,\beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \xi \,\mathrm{sm}\, (\alpha+\beta)\} \quad \mathrm{Df}$

This defines the *cardinal* sum of the (homogeneous) cardinal number of a class α and the (homogeneous) cardinal number of a class β as the cardinal number of the arithmetical sum $\alpha+\beta$. The definition is framed so as to yield $\mu +_c \nu = \Lambda$ whenever μ or ν are not cardinals and it does not require that μ and ν be of the same relative type, only that they are classes of classes[102]. The requirement that μ and ν be homogeneous cardinals is due to the difficulties involved with descending cardinals which may be empty in some types. The authors explain:

> Suppose either μ or ν, say μ, is Λ. Then, by $*102{\cdot}73$, $\mu = \mathrm{Nc}(\zeta)`\alpha$, if ζ is of the appropriate type. Hence, if we had put

[102] WHITEHEAD & RUSSELL, 1927*a*, pp.63-4 [1912, pp.66-7].

> $\mu +_c \nu = \hat{\xi}\{(\exists\alpha, \beta) . \mu = \text{Nc}`\alpha . \nu = \text{Nc}`\beta . \xi \,\text{sm}\, (\alpha + \beta)\}$ Df,
>
> where the ambiguities of type involved in Nc'α and Nc'β may be determined as we please, we should have
>
> $\nu = \text{Nc}`\beta . \supset . t`\zeta + \beta \,\epsilon\, \mu +_c \nu$
>
> $\nu = \text{Nc}`\beta . \supset . t`\zeta + \beta \,\epsilon\, \Lambda +_c \nu$
>
> We should also have $t`t`\zeta + \beta \,\epsilon\, \Lambda +_c \nu$ and so on. Thus, $\Lambda +_c \nu$ would not have a definite value, i.e., it would not merely have typical ambiguity, which it ought to have, but it would not have a definite value even when its type was assigned. Thus such a definition would be unsuitable. For the above reasons, we put $\mu = \text{N}_0\text{c}`\alpha . \nu = \text{N}_0\text{c}`\beta$ in the definition, and obtain the typical ambiguity we desire by means of the typical ambiguity of the "sm" in "$\xi \,\text{sm}\, (\alpha + \beta)$".[103]

The point is again related to *Principia's* version of Cantor's theorem which assures that some cardinals are empty in some relative types and some comments are necessary in connection with the mentioned "typical ambiguity of the "sm" in "$\xi \,\text{sm}\, (\alpha + \beta)$". They require it in order to allow the sum of cardinals of any (possibly) distinct relative types. Unfortunately, this ambiguity also causes problems. As Whitehead and Russell explain "ξ is typically ambiguous, on account of the ambiguity of "sm"" and thus "$\mu +_c \nu$ is also typically ambiguous"[104]. This means that $\mu +_c \nu$ itself may be empty in some types when it is a descending cardinal, and also that $\mu +_c \nu$ is subject to the difficulties that surround completely ambiguous formal numbers like Nc'α. Again, we may follow a simple solution suggested by Landini, which is to always read $\mu +_c \nu$ in accordance with convention IT, making it typically definite; in other words we regard "$+_c$" as "$+_c(\sigma)$", or "$+_c^\delta$", for some definite relative type index σ, as in Nc(ξ)'α[105]. Thus, we fix $*110{\cdot}02$ as:

$$\mu +_c^\delta \nu = \hat{\xi}\{(\exists\alpha, \beta) . \mu = \text{N}_0\text{c}`\alpha . \nu = \text{N}_0\text{c}`\beta . \xi \,\text{sm}_\delta\, (\alpha + \beta)\} \quad \text{Df}$$

As Landini showed in his systematic study, this correction does no harm to the development of Arithmetic in *Principia* with the exception of theorems that also depend on ill-motivated supplementary definitions introduced by Whitehead and Russell in order to allow expressions of the form Nc'$\alpha +_c \mu$, Nc'$\alpha +_c$ Nc'β, etc.[106] Whitehead and Russell explain:

> The above definition of $\mu +_c \nu$ is designed for the case in which μ and ν are typically definite. But we must be able to speak of 'Nc'$\gamma +_c$ Nc'δ' and this must be a definite cardinal, namely Nc'$(\gamma + \delta)$. [...] It is obvious that we want our definition to yield
>
> $\text{Nc}`\gamma +_c \text{Nc}`\delta = \text{Nc}`(\gamma + \delta)$

[103] WHITEHEAD & RUSSELL, 1927*a*, pp.64-5 [1912, pp.67-8].
[104] WHITEHEAD & RUSSELL, 1927*a*, p.72 [1912, p.75].
[105] LANDINI, G., 2016, p.30.
[106] LANDINI, G., 2016, p.30-1.

> in all types, but in order to insure that this shall hold even when, for some values of ξ, $\mathrm{Nc}(\xi)`\gamma = \Lambda$, we must introduce two new definitions [...]
> whence $\vdash$: $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \mathrm{N_0c}`\alpha +_c \mathrm{N_0c}`\beta = \mathrm{Nc}`(\alpha + \beta)$.[107]

The authors apparently thought that $\mathrm{Nc}`\gamma +_c \mathrm{Nc}`\delta$ should not only be allowed as meaningful, but also that $\mathrm{Nc}`\gamma +_c \mathrm{Nc}`\delta = \mathrm{Nc}`(\gamma + \delta)$ ought to be provable; in order to ensure this, they introduced the following supplementary definitions mentioned in the quote above, namely:

$*110{\cdot}03 \quad \mathrm{Nc}`\alpha +_c \mu = \mathrm{N_0c}`\alpha +_c \mu \quad \mathrm{Df}$

$*110{\cdot}04 \quad \mu +_c \mathrm{Nc}`\alpha = \mu +_c \mathrm{N_0c}`\alpha \quad \mathrm{Df}$

Their purpose is to avoid the possibility of either $\mathrm{Nc}`\alpha = \Lambda$ or $\mathrm{Nc}`\beta = \Lambda$ when expressions like $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta$ are allowed. Again, if either $\mathrm{Nc}`\alpha$ or $\mathrm{Nc}`\beta$ are descending, this is always a possibility and in this case $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \mathrm{Nc}`(\alpha+\beta)$ may hold for some relative types and not for others[108]. A simple example can establish this. Let $\mathrm{Nc}`\alpha = \Lambda$ and $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \Lambda$. Since $+_c$ must be read as $+_c^{\xi}$ in accordance to convention IT, we should get $\mathrm{Nc}(\xi)`(\alpha+\beta) = \Lambda$. But if $\mathrm{Nc}(\xi)`(\alpha + \beta)$ is either a homogeneous or ascending cardinal, the theorem fails: in such cases $\mathrm{Nc}(\xi)`(\alpha+\beta)$ will *never* be empty. In order to secure $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta = \mathrm{Nc}`(\alpha+\beta)$, *descending* typical determinations for Nc must be excluded. Since homogeneous[109] cardinals are always non-empty, the supplementary definitions "exclude such determinations of the ambiguity"[110]. There are analogous supplementary definitions for cardinal *multiplication* ($*113{\cdot}04{\cdot}05$), *exponentiation* ($*116{\cdot}03{\cdot}04$) and for the definitions of *greater* and *less than* ($*117{\cdot}02{\cdot}03$). But, as Landini observes, these are all illicit for being in direct conflict with the definition of $\mathrm{Nc}`\alpha$. Simply put: one cannot introduce a definition that alters the *definiendum* of an *already defined* symbol! $\mathrm{Nc}`\alpha$ and $\mathrm{N_0c}`\alpha$ are *different* formal numbers. It must not be permissible to treat them as the same in some contexts and different in others.

The supplementary definitions, as Landini observes, are intimately connected with convention IIT, which must fall with them[111]. Convention IIT ensures that if a formal number σ "[...] were replaced by $\mathrm{Nc}`\alpha$, the actual type of $\mathrm{Nc}`\alpha$ would by definition have to be adequate, then the actual type of σ is also to be adequate"[112]. This is introduced in order to secure that $\exists\,!(\mu +_c \nu)$ when ν is substituted for symbols like $\mathrm{Nc}`\alpha$: the convention was introduced *to work*

[107] WHITEHEAD & RUSSELL, 1927*a*, p.65 [1912, p.68].
[108] LANDINI, G., 2016, p.31.
[109] Homogeneous cardinals are selected merely for convenience, since ascending cardinals would also do the trick. The point is just to ensure non-emptiness: "It does not make any difference to the value of $\mathrm{Nc}`\alpha +_c \mathrm{Nc}`\beta$ how the ambiguities of $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ are determined, so long as they are determined in a way that insures $\exists\,!\,\mathrm{Nc}`\alpha \,.\, \exists\,!\,\mathrm{Nc}`\beta$" (WHITEHEAD & RUSSELL, 1912, p.75-6).
[110] WHITEHEAD & RUSSELL, 1927*a*, p.72 [1912, pp.75-6].
[111] Cf. LANDINI, G., 2016, p.31.
[112] WHITEHEAD & RUSSELL, 1927*a*, p.xxiii [1912, p.xxv].

with the supplementary definitions $*110{\cdot}03{\cdot}04$, $*113{\cdot}04{\cdot}05$, $*113{\cdot}04{\cdot}05$, $*117{\cdot}02{\cdot}03$, which was intended to make the type of Nc'α adequate, e.g., would ensure that Nc'α is existent (as well as $\mu +_c$ Nc'α)[113]. But the supplementary definitions are spurious.

What makes this problematic - and relevant to understanding the development of finite Arithmetic in *Principia* - is the fact that important theorems rely on them. A very important example is the following[114]:

$$*110{\cdot}63 \quad \vdash : \mathrm{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma, y) \,.\, \gamma\, sm\, \alpha \,.\, y \sim \epsilon\, \gamma \,.\, \xi = \gamma \cup \iota`y\}$$

The authors attach much weight to this proposition, which is not only used very often in the treatment of finite Arithmetic but is also claimed to be the connection of "[...] mathematical induction for inductive cardinals with mathematical induction for inductive classes (cf. $*120$)"[115]. In fact, put in a simplified form in words, this proposition identifies the successor Nc'$\alpha +_c 1$ of Nc 'α as the class which contains every class ξ such that, for some γ that is similar to α and some *y* that is not a member of γ, if we add *y* to γ, the resulting class is ξ; in other words, the successor Nc'$\alpha +_c 1$ of the cardinal number Nc'α of a class α is such that, if we take any member γ of Nc'α and add to it (only) some *y* that does not belong to it, we get a member of Nc'$\alpha +_c 1$, or equivalently, if we take away (only) some *y* of a member of Nc'$\alpha +_c 1$, we get a member of Nc'α.

From this it is easy to see that, in the lowest possible type where cardinal numbers are definable, the number 1 will be emulated as the class of all classes containing a single individual of some simple type, the number 2 will be emulated by the class of all classes of pairs of individuals and so on, where, of course, classes are emulated in terms of the definitions of $*20$. Unfortunately, theorem $*110{\cdot}63$ is proved as follows:

113 Cf. WHITEHEAD & RUSSELL, 1927*a*, p.xxiii [1912, p.xxv].
114 Cf. LANDINI, G., 2016, p.34.
115 WHITEHEAD & RUSSELL, 1927*a*, p.82 [1912, p.85].

$$[*101\cdot2]\vdash \quad \text{Nc}`\alpha +_c 1 = \text{Nc}`\alpha +_c \text{Nc}`\iota`x \qquad (1)$$

$$[*110\cdot33]\vdash \quad \xi \,\epsilon\, \text{Nc}`\alpha +_c \text{Nc}`\beta \,.\equiv_\xi .\, (\exists\gamma,\delta) \,.\, \gamma\, sm\, \alpha \,.\, \delta\, sm\, \beta \,.\, \gamma \cap \delta = \Lambda \,.\, \xi = \gamma \cup \delta \qquad (2)$$

$$[(1).(2)]\vdash \quad \text{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma,\delta) \,.\, \gamma\, sm\, \alpha \,.\, \delta\, sm\, \iota`x \,.\, \gamma \cap \delta = \Lambda \,.\, \xi = \gamma \cup \delta\} \qquad (3)$$

$$[*73\cdot45]\vdash \quad 1 = \beta(\beta\, sm\, \iota`x) \qquad (4)$$

$$[(3).(4)]\vdash \quad \text{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma,\delta) \,.\, \gamma\, sm\, \alpha \,.\, \delta \,\epsilon\, 1 \,.\, \gamma \cap \delta = \Lambda \,.\, \xi = \gamma \cup \delta\} \qquad (5)$$

$$[*52\cdot1]\vdash \quad \alpha \,\epsilon\, 1 \,.\equiv_\alpha .\, (\exists y) \,.\, \alpha = \iota`y \qquad (6)$$

$$[(5).(6)]\vdash \quad \text{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma,\delta,y) \,.\, \gamma\, sm\, \alpha \,.\, \delta = \iota`y \,.\, \gamma \cap \delta = \Lambda \,.\, \xi = \gamma \cup \delta\} \qquad (7)$$

$$[*51\cdot211]\vdash \quad y \sim \epsilon\, \alpha \,.\equiv_y .\, \iota`y \cap \alpha = \Lambda \qquad (8)$$

$$[(7).(8)]\vdash \quad \text{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma,y) \,.\, \gamma\, sm\, \alpha \,.\, \gamma \cap \iota`y = \Lambda \,.\, \xi = \gamma \cup \delta\} \qquad (9)$$

$$[*13\cdot195]\vdash \quad y \sim \epsilon\, \alpha \,.\equiv_{\alpha,y} .\, \iota`y \cap \alpha = \Lambda \qquad (10)$$

$$[(9).(10)]\vdash \quad \text{Nc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma,y) \,.\, \gamma\, sm\, \alpha \,.\, y \sim \epsilon\, \gamma \,.\, \xi = \gamma \cup \iota`y\} \qquad (11)$$

The problem is that $*110\cdot33$, introduced in line two, depends on the following chain of theorems, all of which depend essentially on $*110\cdot03\cdot04$:

$*110\cdot3$ $\vdash .\, \text{Nc}`\alpha +_c \text{Nc}`\beta = \text{N}_0\text{c}`\alpha +_c \text{N}_0\text{c}`\beta = \text{Nc}`(\alpha + \beta)$

$*110\cdot31$ $\vdash : \gamma\, \text{sm}\, \alpha \,.\, \delta\, \text{sm}\, \beta \,.\supset .\, \text{Nc}`\gamma +_c \text{Nc}`\delta = \text{Nc}`\alpha +_c \text{Nc}`\beta$

$*110\cdot32$ $\vdash : \alpha \cap \beta = \Lambda \,.\supset .\, \text{Nc}`\alpha +_c \text{Nc}`\beta = \text{Nc}`(\alpha \cup \beta)$

As the authors themselves claim, it was in order to obtain $*110\cdot3$ that they introduced the supplementary definitions in the first place. And without $*110\cdot03\cdot04$ both $*110\cdot31$ and $*110\cdot32$ are illicit, since some descending cardinals are empty in virtue of Cantor's theorem. In fact, as Landini observes[116], both of the following theorems are illicit:

$*110\cdot33$ $\vdash \xi \,\epsilon\, \text{Nc}`\alpha +_c \text{Nc}`\beta \,.\equiv_\xi .\, (\exists\gamma,\delta) \,.\, \gamma\, sm\, \alpha \,.\, \delta\, sm\, \beta \,.\, \gamma \cap \delta = \Lambda \,.\, \xi = \gamma \cup \delta$

$*110\cdot331$ $\vdash \text{Nc}`\alpha +_c \text{Nc}`\beta = \hat{\xi}\{(\exists\gamma) \,.\, \gamma\, sm\, \alpha \,.\, \xi - \gamma\, sm\, \beta = \Lambda \,.\, \gamma \subset \xi\}$

At first this could seem very serious but, fortunately, their respective equivalents for homogeneous and ascending cardinals are recoverable using Landini's revised definitions[117]:

116 LANDINI, G., 2016, p.34.
117 Cf. LANDINI, G., 2016, p.34.

$*110{\cdot}33h \quad \vdash : \xi \,\epsilon\, \mathrm{N_oc}`\alpha +_c^\sigma \mathrm{N_oc}`\beta \,.\, \equiv_\xi \,.\, (\exists\gamma, \delta) \,.\, \gamma \,\mathrm{sm_o}\, \alpha \,.\, \delta \,\mathrm{sm_o}\, \beta \,.\, \gamma \cap \delta = \Lambda \,.\, \xi \,\mathrm{sm}_\sigma\, \gamma \cup \delta$

$*110{\cdot}331h \quad \vdash : \mathrm{N_oc}`\alpha +_c^\sigma \mathrm{N_oc}`\beta = \hat{\xi}\{(\exists\gamma) \,.\, \gamma \,\mathrm{sm_o}\, \alpha \,.\, \xi - \gamma \,\mathrm{sm}_\sigma\, \beta = \Lambda \,.\, \gamma \subset \xi\}$

$*110{\cdot}33a \quad \vdash : \xi \,\epsilon\, \mathrm{N}^i\mathrm{c}`\alpha +_c^\sigma \mathrm{N}^i\mathrm{c}`\beta \,.\, \equiv_\xi \,.\, (\exists\gamma, \delta) \,.\, \gamma \,\mathrm{sm}^i\, \alpha \,.\, \delta \,\mathrm{sm}^i\, \beta \,.\, \gamma \cap \delta = \Lambda \,.\, \xi \,\mathrm{sm}_\sigma\, \gamma \cup \delta$

$*110{\cdot}331a \quad \vdash : \mathrm{N}^i\mathrm{c}`\alpha +_c^\sigma \mathrm{N}^i\mathrm{c}`\beta = \hat{\xi}\{(\exists\gamma) \,.\, \gamma \,\mathrm{sm}^i\, \alpha \,.\, \xi - \gamma \,\mathrm{sm}_\sigma\, \beta = \Lambda \,.\, \gamma \subset \xi\}$

All that is required is that the relative type indices be kept right. The above, in turn also allow for the recovery of the following[118]:

$*110{\cdot}63h \quad \vdash : \mathrm{N_oc}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma, y) \,.\, \gamma \,\mathrm{sm_o}\, \alpha \,.\, y \sim \epsilon\, \gamma \,.\, \xi = \gamma \cup \iota`y\}$

$*110{\cdot}63a \quad \vdash : \mathrm{N}^i\mathrm{c}`\alpha +_c 1 = \hat{\xi}\{(\exists\gamma, y) \,.\, \gamma \,\mathrm{sm}^i\, \alpha \,.\, y \sim \epsilon\, \gamma \,.\, \xi = \gamma \cup \iota`y\}$

As long as we read $+_c 1$ here according to convention IT, there is no problem. Also, theorems concerning the successor of a cardinal number μ that rely on clauses which contain only argumental occurrences of μ as in sm"μ are not affected, as Landini points out, since their clauses ensure non-emptiness[119]. So we maintain the following theorems intact despite the need to drop $*116{\cdot}03{\cdot}04$ and modify $*110{\cdot}63$:

$*110{\cdot}631 \quad \vdash : \mu \,\epsilon\, \mathrm{NC} \,.\, \supset \,.\, \mu +_c 1 = \hat{\xi}\{((\exists\gamma, y) \,.\, \gamma \,\epsilon\, \mathrm{sm}``\mu \,.\, y \sim \epsilon\, \gamma \,.\, \xi = \gamma \cup \iota`y\}$

$*110{\cdot}632 \quad \vdash : \mu \,\epsilon\, \mathrm{NC} \,.\, \supset \,.\, \mu +_c 1 = \hat{\xi}\{(\exists\gamma, y) \,.\, y \,\epsilon\, \xi \,.\, \xi - \iota`y \,\epsilon\, \mathrm{sm}``\mu\}$

We can now pass to the definition of the set of finite cardinal numbers in order to complete the preliminaries for the proofs of the Peano postulates.

Similarly to Frege, in order to define the class of all finite cardinals (the class of natural numbers), Whitehead and Russell define the notion of "being an ancestor of a term with respect to a relation *R*". The main definition is, of course:

$*90{\cdot}01 \qquad R_* = \hat{x}\hat{y}\{x \,\epsilon\, C`R : \breve{R}``\mu \subset \mu \,.\, \supset_\mu \,.\, y \,\epsilon\, \mu\} \quad \mathrm{Df}$

In Russell and Whitehead's words this is to mean that "[...] xR_*y is to hold when *x* belongs to the field of *R*, and *y* belongs to every hereditary class to which *x* belongs; a hereditary class being a class such that $\breve{R}``\mu \subset \mu$"[120]. Of course, the 'intuitive' idea of the ancestral of a relation

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[118] Cf. LANDINI, G., 2016, p.34.
[119] LANDINI, G., 2016, 34.
[120] RUSSELL & WHITEHEAD, 1911, p.576.

is perhaps better explained in terms of relative products R^2, $R^2|R$, $R^3|R$ of a relation *R*. Let *R* be the relation *is a parent of* and let *a* and *b* be two terms in the field of *R*[121]. Take the following formulas:

$$aR^2b = (\exists x) : aRx \,.\, xRb$$

$$a(R^2|R)b = (\exists x, y) : aRx \,.\, xRy \,.\, yRb$$

$$a(R^3|R)b = (\exists x, y, z) : aRx \,.\, xRy \,.\, yRz \,.\, zRb$$

and so on...

These would state respectively that *a* is a grandparent of *b*, that *a* is a great grandparent of *b* and that a is a great-great grandparent of *b*; and, in general, to say that aR^nb holds is to say that *a* is an *ancestor* of *b n* steps removed: for $n = 1$, *a* is a parent of a parent of *b*, for $n = 2$ it is a parent of a parent of a parent, and so on. This is why Whitehead and Russell named R_* the "ancestral" of *R*.

The idea to be captured by a precise definition is that of being *n* steps away from a term in a series of terms in the field of a relation *R*. The notation $\breve{R}``\mu \subset \mu$ is somewhat compact. Recall that by the definition of plural descriptive functions from $*37$, we have:

$*37{\cdot}01{\cdot}1$ $\vdash : R``\mu = \hat{x}\{(\exists y) \,.\, y \,\epsilon\, \mu \,.\, xRy\}$

And by the definition of $\breve{R}$ as the converse of a relation *R* from $*31$ we have, then:

$*37{\cdot}105$ $\vdash : \breve{R}``\mu = \hat{x}\{(\exists y) \,.\, y \,\epsilon\, \mu \,.\, yRx\}$

From whence it follows:

$*37{\cdot}171$ $\vdash : \breve{R}``\mu \subset \mu \,.\, \equiv : x \,\epsilon\, \mu \,.\, xRy \,.\, \supset_{x,y} .\, y \,\epsilon\, \mu$

Since they also show:

$*90{\cdot}102$ $\vdash :. \breve{R}``\mu \subset \mu \,.\, x \,\epsilon\, \mu \,.\, \supset .\, y \,\epsilon\, \mu : \equiv : R``\mu \subset \mu \,.\, y \,\epsilon\, \mu \,.\, \supset_\mu .\, x \,\epsilon\, \mu$

[121] A brief account in these terms is given by Whitehead and Russell themselves (WHITEHED & RUSSELL, 1910, p.570-2). Recall that R^2 means the relative product of R and itself, that is $R|R$.

Then, the unpacked proposition that just embodies the definition and which is most frequently used is the following theorem:

$$*90{\cdot}11 \qquad \vdash :. \, xR_*y \,.\equiv: x \,\epsilon\, C\text{‘}R : z \,\epsilon\, \mu \,.\, zRw \,.\supset_{z,w} .\, w \,\epsilon\, \mu : x \,\epsilon\, \mu : \supset_\mu .\, y \,\epsilon\, \mu$$

Thus, xR_*y holds when *x* is in the field of the relation *R* and *y* belongs to every hereditary class to which *x* belongs, which is equivalent to the weak or non strict ancestral. Just like Frege's own version, R_* holds between two terms *x* and *y* whenever *y* belongs to every hereditary class μ to which *x* belongs or, equivalently, whenever *y* possesses every hereditary property possessed by *x*, as proved in the more familiar theorem scheme[122]:

$$*90{\cdot}112 \qquad \vdash :. \, xR_*y : \phi z \,.\, zRw \,.\supset_{z,w} .\, \phi w : \phi x : \supset .\, \phi y$$

Unlike Frege's version, however, the field of R_* is restricted to the field of *R*, that is, in order for *x* to have the relation $R_*$with some *y*, it must belong to the set: $\hat{x}\{(\exists y) \,.\, xRy \vee yRx\}$. The restriction is made for convenience and in order to avoid some undesirable consequences of not assuming it. On the one hand, assuming it entails reflexivity of *R*-ancestrality within the field of *R*, "but not elsewhere", as Whitehead and Russell put it[123]. On the other, this avoids an annoying fact that Frege's (weak) ancestral had to put up with: according to his version we had to accept that the posterity of, say, Julius Caesar with respect to the predecessor relation is the set containing Julius Caesar himself or, in his terminology, we had to accept that the *P*-series ending with Julius Caesar consists of him and no other object; restricting the possible *relata* of $R_*$to the field of *R* avoids such undesirable (albeit innocent) consequences.

Following closely the treatment of Frege, Whitehead and Russell want to define the set of finite cardinals as the *posterity of* 0 with respect to the relation $(+1_c)$ which holds between a cardinal number *n* and its immediate successor $n +_c 1$, where the posterity of a term *x* is the class $\overleftarrow{R}_*\text{‘}x$, that is, the class of all *y* of which *x* is an *R*-ancestor, *i.e.*, the class of all 'descendants' of *x* with respect to *R*. Informally, what they want then is to define the class of finite cardinals as the class of classes $\hat{\alpha}\{0 \,\epsilon\, \mu : \gamma \,\epsilon\, \mu \,.\supset_\gamma \gamma + 1 \,\epsilon\, \mu : \supset_\mu .\, \alpha \,\epsilon\, \mu\}$, *i.e.*, the class of all classes α that belongs to every hereditary class that belongs to zero[124]. Put in more precise terms, we first have, in virtue of $*90$:

$$x(+_c 1)_* y \,.\equiv:$$

[122] Most proofs - including that of mathematical induction at $*120$, where Arithmetic is actually developed - use this straightforward formulation of the ancestral.

[123] WHITEHEAD & RUSSELL, 1925, p.549 [1910, p.576].

[124] WHITEHEAD & RUSSELL, 1925, p.570 [1910, p.544].

$$x \in C`(+_c1) :. w \in \mu . w(+_c1) : \supset_{z,w} . z \in \mu : x \in \mu : \supset_\mu . y \in \mu$$

Then, using the definition of descriptive functions $R`y = (\iota x)(xRy)$ of $*30$ and of double descriptive functions $♀y = \hat{z}\hat{x}(z = x♀y)$ of $*38$[125], the relation of immediate *succession* relation can be put as[126]:

$$+_c1 = \hat{\alpha}\hat{\beta}(\alpha = \beta +_c 1) \quad \text{Df}$$

And from these then, we get the more straightforward:

$$x(+_c1)_* y . \equiv :$$

$$x \in C`(+_c1) :. w \in \mu \supset_w . w +_c 1 \in \mu : x \in \mu : \supset . y \in \mu$$

The class of natural numbers, denoted by "NC induct" is thus defined as the class of all classes (of classes) which bear the relation $(+_c1)_*$ to 0:

$*120{\cdot}01$ $\quad$ NC induct $= \hat{\alpha}\{\alpha(+_c1)_* 0\}$ $\quad$ Df

So we get:

$*120{\cdot}101$ $\quad \vdash :: \alpha \in \text{NC induct} . \equiv : \xi \in \mu . \supset_\xi . \xi +_c 1 \in \mu : 0 \in \mu : \supset_\mu . \alpha \in \mu$

That is, α is a finite cardinal number if, and only if, it belongs to all classes that are hereditary in the $+_c1$ series that begins with zero. Notice that despite the fact that these definitions are presented as apparently indefinite with respect to relative types, conditions of meaningfulness impose some restrictions. First and foremost, ξ and $\xi +_c 1$ must be of the same relative type otherwise it would be nonsense to speak of $\xi +_c 1$ as belonging to μ whenever ξ does. So in

[125] The symbol "♀" has a sort of unique and problematic status in *Principia*. The basic notation of $*38$ introduces what Whitehead and Russell refer to as a "double descriptive function" which stands for any "non-propositional function of two arguments, such as $\alpha \cap \beta$, $\alpha \cap \beta$, $R \mathbin{\dot\cap} S$, $R \mathbin{\dot\cup} S$ [...]" (WHITEHEAD & RUSSELL, 1910, p.311 [1925, p.296]). In short, "♀" functions basically as a variable for operators instead of a variable for functions. Since the above use in the informal explanation of the notion of the ancestral and induction is the only (non-problematic) instance of its use we need to consider of this notation, we shall not address it here, but we refer to an illuminating discussion in GRIFFIN, N., 2019 and also a forthcoming paper of Landini that will discuss $*38$ in length and detail; I leave him my thanks here for kindly letting me have a look at this unpublished paper, which was very helpful.

[126] Thus "$\alpha(+_c1)\beta$" reads "α precedes β immediately".

order to get a typically definite class NC induct, say, one whose members are of type η, type indexes can be restored only for 0 as in:

$$*120{\cdot}011 \quad \mathrm{N}_{\eta}\mathrm{C}\,\text{induct} = \hat{\alpha}\{\alpha(+_c 1)_* 0_{\eta}\}$$

$$*120{\cdot}102 \quad \vdash :. \, \alpha \,\epsilon\, \mathrm{N}_{\eta}\mathrm{C}\,\text{induct} \,.\equiv :\, \xi \,\epsilon\, \mu \,.\supset_{\xi} .\, \xi +_c 1 \,\epsilon\, \mu : 0_{\eta} \,\epsilon\, \mu :\supset_{\mu} .\, \alpha \,\epsilon\, \mu$$

Whitehead and Russell adopt the strange (and inadequate) procedure of first stating the ambiguous theorems and definitions and then the versions with explicit relative type indices. But as Landini emphasizes, the indefinite or ambiguous versions should be understood as the result of suppressing the indexes of ∗120·011·102[127].

5.6 The Peano Postulates and the Existence of Cardinal Numbers

5.6.1 Peano 1, 2, 4 and 5

At last, we may now review and discuss the proofs of the Peano postulates. In order of appearance in the text, *Principia*'s analogues of the Peano postulate are the following:

[∗120·11]⊦	$\alpha \,\epsilon\, \mathrm{NC}_{\eta}\,\text{induct} : \phi\xi \,.\supset_{\xi} .\, \phi(\xi +_c 1) : \phi 0_{\eta} :\supset .\, \phi\alpha$	Peano 5
[∗120·12]⊦	$0 \,\epsilon\, \mathrm{NC}\,\text{induct}$	Peano 1
[∗120·121]⊦	$\alpha \,\epsilon\, \mathrm{NC}_{\xi}\,\text{induct} \,.\supset .\, (\alpha +_c 1)_{\xi} \,\epsilon\, \mathrm{NC}_{\xi}\,\text{induct}$	Peano 2
[∗120·124]⊦	$\alpha +_c 1 \neq 0$	Peano 4

All of them are easily proved and are not affected at all by the difficulties surrounding the typical ambiguity of Nc‘α[128].

Peano 5[129] and 1 follow almost immediately from the definition of natural number. The

[127] Cf. LANDINI, G., 2016, p.41-51.

[128] See Appendix A for transcriptions from *Principia*'s proofs and (inevitably) loose translations into a modern notation and deductive system. Another (informal) reconstruction is considered in Appendix B. For transcriptions and translations of the main theorems from ∗90, see also Appendix A.

[129] We are following here Russell's own (not usual or standard) numbering of the Peano postulates put forward in *Introduction to Mathematical Philosophy*, namely (RUSSELL, B., 1919*a*, pp.5-6):
(1) Zero is a number;
(2) The successor of a number is a number;
(3) No two numbers have the same successor;
(4) Zero is not the successor of any number;
(5) Any property which belongs to zero and, also to the successor of every number which has the property, belongs to all numbers.

former is proved as the following theorem scheme:

$$[*120\cdot102]\vdash \quad \alpha\,\epsilon\,\mathrm{N}_\eta\mathrm{C\,induct}\,.\equiv.\,\alpha(+_c1)_*0_\eta \quad (1)$$
$$[*90\cdot112]\vdash \quad \alpha(+_c1)_*0:\phi\xi\,.\supset_\xi.\,\phi(\xi+_c1):\phi0:\supset.\,\phi\alpha \quad (2)$$
$$[(1).(2)]\vdash \quad \alpha\,\epsilon\,\mathrm{N}_\eta\mathrm{C\,induct}:\phi\xi\,.\supset_\xi.\,\phi(\xi+_c1):\phi0:\supset.\,\phi\alpha \quad (3)$$

It is worthwhile to note that the proof of $*90\cdot112$ used in line (2) relies on the contextual definition of classes from $*20$, being proved as follows:

$$[*90\cdot111]\vdash \quad xR_*y\,.\equiv:z\,\epsilon\,\mu.zRw\,.\supset_{z,w}.\,w\,\epsilon\,\mu\,.\,x\,\epsilon\,\mu:\supset_\mu.\,y\,\epsilon\,\mu \quad (1)$$
$$\left[(1)\frac{\hat{z}(\phi z)}{\mu}\right]\vdash \quad xR_*y\,.\equiv:z\,\epsilon\,\hat{z}(\phi z)\,.\,zRw\,.\supset_{z,w}.\,w\,\epsilon\,\hat{z}(\phi z)\,.\,x\,\epsilon\,\hat{z}(\phi z):\supset.\,y\,\epsilon\,\hat{z}(\phi z) \quad (2)$$
$$[*20\cdot02]\vdash \quad x\,\epsilon\,\hat{z}(\phi z)\,.\equiv_x.\,\phi x \quad (3)$$
$$[(2).(3)]\vdash \quad xR_*y\,.\equiv:\phi z\,.\,zRw\,.\supset_{z,w}.\,\phi w\,.\,\phi x:\supset.\,\phi y \quad (4)$$
$$[(4).(*3\cdot31)]\vdash \quad xR_*y:\phi z\,.\,zRw\,.\supset_{z,w}.\,\phi w\,.\,\phi x:\supset.\,\phi y \quad (5)$$

This means that for the theorem to have its full intended force, the Axiom of Reducibility is required - and so the same can be said, of course, about the principle of mathematical induction. Indeed, both theorems above are impredicative.

Peano 1 follows by a simple generalization on a tautology:

$$[*120\cdot101]\vdash \quad 0\,\epsilon\,\mathrm{NC\,induct}\,.\equiv:\xi\,\epsilon\,\mu\,.\supset_\xi.\,\xi+_c1\,\epsilon\,\mu:0\,\epsilon\,\mu:\supset_\mu.\,0\,\epsilon\,\mu \quad (1)$$
$$[(*3\cdot26).(*10\cdot11)]\vdash \quad \xi\,\epsilon\,\mu\,.\supset_\xi.\,\xi+_c1\,\epsilon\,\mu:0\,\epsilon\,\mu:\supset_\mu.\,0\,\epsilon\,\mu \quad (2)$$
$$[(1).(2)]\vdash \quad 0\,\epsilon\,\mathrm{NC\,induct} \quad (3)$$

Peano 2 is proved on the basis $*90\cdot172$, which asserts that $R|R_*\subset R_*$ [130]:

$$[*120\cdot102]\vdash \quad \alpha\,\epsilon\,\mathrm{N}_\xi\mathrm{C\,induct}\,.\equiv.\,\alpha(+_c1)_*0_\xi \quad (1)$$
$$[*90\cdot172]\vdash \quad \sigma R\gamma\,.\,\gamma R_*\beta:\supset_{R,\sigma,\beta,\gamma}.\,\sigma R_*\beta \quad (2)$$
$$\left[(2)\frac{+_c1,\alpha,0,\alpha+_c1}{R,\sigma,\beta,\gamma}\right]\vdash \quad \alpha+_c1(+_c1)\alpha\,.\,\alpha+_c1(+_c1)_*0_\xi:\supset.\,\alpha+_c1(+_c1)_*0_\xi \quad (3)$$
$$[(3).(.*38\cdot1)]\vdash \quad \alpha\,\epsilon\,\mathrm{N}_\xi\mathrm{C\,induct}:\supset.\,\alpha+_c1\,\epsilon\,\mathrm{N}_\xi\mathrm{C\,induct} \quad (4)$$

Using the induction schemes $*120\cdot11$ and $*120\cdot121$ they also prove the following impredicative

[130] For some explanations and translations of important theorems about the ancestral, see the appendices.

induction scheme $*120{\cdot}13$:

$$[*120{\cdot}121] \vdash \quad \xi \,\epsilon\, \mathrm{NC}_\eta \text{ induct} \,.\, \phi\xi \,.\, \supset_\xi \,.\, \phi(\xi +_c 1) : \supset : \tag{1}$$
$$\xi \,\epsilon\, \mathrm{NC}_\eta \text{ induct} \,.\, \supset_\xi \,.\, (\xi +_c 1)_\eta \,\epsilon\, \mathrm{NC}_\eta \text{ induct} \,.\, \phi(\xi +_c 1)_\eta$$

$$[*120{\cdot}12] \vdash \quad \phi 0_\eta \,.\, \supset \,.\, 0_\eta \,\epsilon\, \mathrm{NC}_\eta \text{ induct} \,.\, \phi 0_\eta \tag{2}$$

$$[*120{\cdot}11] \vdash \quad \alpha \,\epsilon\, \mathrm{NC}_\eta \text{ induct} : \phi\xi \,.\, \supset_\xi \,.\, \phi(\xi +_c 1) : \phi 0_\eta : \supset \,.\, \phi\alpha \tag{3}$$

$$\left[(3)\frac{\xi \in \mathrm{NC}_\eta \text{ induct} \,.\, \phi\xi}{\phi\xi}\right] \vdash \quad \alpha \,\epsilon\, \mathrm{NC}_\eta \text{ induct} : (1) \,.\, (2) : \supset \,.\, \alpha \,\epsilon\, \mathrm{NC}_\eta \text{ induct} \,.\, \phi\alpha \tag{4}$$

$$[(4)] \vdash \quad \alpha \,\epsilon\, \mathrm{NC}_\eta \text{ induct} : \xi \,\epsilon\, \mathrm{NC}_\eta \text{ induct} \,.\, \phi\xi \,.\, \supset_\xi \,.\, \phi(\xi +_c 1) : \phi 0_\eta : \supset \,.\, \phi\alpha \tag{5}$$

This is a form of induction that requires impredicative comprehension very frequently used by Whitehead and Russell. One important use is in the proof of theorem $*120{\cdot}57$ which we will discuss below.

Theorem $*120{\cdot}124$, *Principia*'s version of Peano 4, is proved by cases. There are two: (*a*) $\alpha = \Lambda$ and (*b*) $\exists\,!\,\alpha$. Whitehead and Russell must consider case (*a*) due to their definition of cardinal sum. Recall that they first define the arithmetical sum of two classes and cardinal sum at $*110$ and then derive the successor function as a particular case. Their definition of cardinal sum from $*110{\cdot}02$ does not require that the classes summed be cardinals and is designed to yield $\gamma +_c \delta = \Lambda$ whenever either γ or δ are not cardinals. So they must consider the case where $\alpha +_c 1$ is *not* a cardinal number, which results in $\alpha +_c 1 = \Lambda \neq 0$ as required[131]. The proof can be re-written as follows:

$[*110{\cdot}4] \vdash$	$\alpha +_c 1 \neq \Lambda \,.\, \supset \,.\, \alpha, 1 \,\epsilon\, \mathrm{NC} - \iota‘\Box$	(1)
$[(1) \,.\, \text{Transp}] \vdash$	$\alpha \sim \epsilon\, \mathrm{NC} \,.\, \supset \,.\, \alpha +_c 1 = \Lambda$	(2)
$[*101{\cdot}12] \vdash$	$0 \neq \Lambda$	(3)
$[(2) \,.\, (3)] \vdash$	$\alpha \sim \epsilon\, \mathrm{NC} \,.\, \supset \,.\, \alpha +_c 1 \neq 0$	(4)
$[*110{\cdot}632] \vdash$	$\alpha \,\epsilon\, \mathrm{NC} \,.\, \supset \,.\, \alpha +_c 1 = \hat{\xi}\{(\exists y) \,.\, y \,\epsilon\, \xi \,.\, -\iota‘y \,\epsilon\, \mathrm{sm}“\alpha\}$	(5)
$[(5)] \vdash$	$\alpha \,\epsilon\, \mathrm{NC} \,.\, \supset : \xi \,\epsilon\, \alpha +_c 1 \,.\, \supset \,.\, \xi \neq \Lambda$	(6)
$[*24{\cdot}63] \vdash$	$\Lambda \sim \epsilon\, \kappa \,.\, \equiv_\kappa : \beta \,\epsilon\, \kappa \,.\, \supset_\beta \,.\, \beta \neq \Lambda$	(7)
$[(7)] \vdash$	$\Lambda \sim \epsilon\, \alpha +_c 1 \,.\, \equiv : \xi \,\epsilon\, \alpha +_c 1 \,.\, \supset \,.\, \xi \neq \Lambda$	(8)
$[(6) \,.\, (8)] \vdash$	$\alpha \,\epsilon\, \mathrm{NC} \,.\, \supset \,.\, \Lambda \sim \epsilon\, \alpha +_c 1$	(9)
$[*54{\cdot}02] \vdash$	$\beta \,\epsilon\, 0 \,.\, \equiv_\beta \,.\, \beta = \Lambda$	(10)
$[(9) \,.\, (10)] \vdash$	$\alpha \,\epsilon\, \mathrm{NC} \,.\, \supset \,.\, \alpha +_c 1 \neq 0$	(11)
$[(4) \,.\, (11)] \vdash$	Prop.	(12)

[131] Thus the proof could be simplified by stating the theorem as "$\alpha \,\epsilon\, \mathrm{NC}$ induct $.\, \supset \,.\, (\alpha +_c 1) \neq 0$". Notice also that the proof only assumes that $\alpha \sim \epsilon\, \mathrm{NC}$ to show that $\alpha +_c 1 = \Lambda$. The assumption that $\Lambda \sim \epsilon\, \mathrm{NC}$ is false for every relative type due to descending cardinals and Cantor's power-class theorem $*102{\cdot}73$.

For the sake of uniformity and readability we dropped the notation "$\exists ! \alpha$" in this proof in favor of "$\alpha \neq \Lambda$". The law of transposition used in line two is $*2{\cdot}16$, which is $A \supset B . \supset . \sim B \supset \sim A$, the only elementary step explicitly indicated in the original proof. The first line of the proof, theorem $*110{\cdot}4$, when fully stated reads as $\exists ! \mu +_c \nu . \supset . \mu, \nu \in \mathrm{NC} - \{\Box\} . \mu, \nu \in \mathrm{N_0C}$, but the second conjunct in the consequent is irrelevant here, however. This completes the proof of the Peano postulates 1, 2, 4 and 5.

5.6.2 The Problem With Peano 3

Differently from the other Peano postulates, the third which states that different numbers have different successors cannot be proved in *Principia* without assuming some hypothesis as to the number of individuals of some simple type. To see why that is the case, let us consider the class NC induct at the lowest point in the 'type' hierarchy of classes in which it can be defined, that is, as a class of classes of individuals of the lowest type. Some basic definitions and theorems of *Principia*'s treatment of inductive and non-inductive (i.e., finite and infinite) classes. First they have[132]:

$*120{\cdot}02 \quad \mathrm{Cls\,induct} = s\text{‘}\mathrm{NC\,induct}$

$*120{\cdot}021 \quad \mathrm{Cls}_\xi\,\mathrm{induct} = s\text{‘}\mathrm{N}_\xi\mathrm{C\,induct}$

This defines the class of all inductive classes or classes that are finite in the ordinary sense, i.e. those classes which are members of a natural or inductive. Notice that Whitehead and Russell again introduce two definitions: one of the completely ambiguous "Cls induct" in terms of "s‘NC induct" and another which determinate as to relative type. From the above it is easily shown that:

$*120{\cdot}21 \quad \vdash \rho \in \mathrm{Cls\,induct} . \equiv . \mathrm{N_0c}\text{‘}\rho \in \mathrm{NC\,induct}$

That is, ρ is an inductive class if and only if its (homogeneous) cardinal number is a finite cardinal or natural number. As Whitehead and Russell observe, they cannot prove "$\rho \in \mathrm{Cls\,induct} . \equiv . \mathrm{Nc}\text{‘}\rho \in \mathrm{NC\,induct}$", again, because Nc‘$\rho$ may be empty in *some* types: to prove the completely ambiguous result they must prove beforehand that if the empty class is an inductive cardinal, then every class is inductive[133].

[132] Recall that the notation s‘ com from *Principia*'s volume 1 section E, $*40{\cdot}02$, which defines $s\text{‘}\kappa$ as $\hat{x}\{(\exists\alpha) : \alpha \in \kappa . x \in \alpha\}$, that is: if κ is a class of classes $s\text{‘}\kappa$ is the class of all classes that are members of κ.

[133] Cf. WHITEHEAD & RUSSELL, 1927*a*, p.206 [1912, p.213]. However, the empty class may be an inductive cardinal in one relative type and not in a higher relative. Thanks for Landini for calling this important detail to my attention.

Also, as they observe[134], they could have defined inductive classes in a way that more closely parallels their definition of finite cardinals, using the following as a definition instead of obtaining it as a theorem:

$$*120{\cdot}24 \quad \vdash :: \rho \in \text{Cls induct} \,.\, \equiv :. \, \eta \in \mu \,.\, \supset_{\eta,y} .\, \eta \cup \iota`y \in \mu : \Lambda \in \mu : \supset_{\mu} .\, \rho \in \mu$$

This defines the set of inductive classes as the set which contains Λ and every set obtainable from it by successively adding a element *y* to it. Both definitions are equivalent, however: in each case it follows that a class is inductive if, and only if, its (non-descending) cardinal number is inductive.

Now to the question of Peano 3. Let 'NC$\alpha_{(x^o)}$induct' be the class of all finite classes of individuals of the lowest type, with each of its members being the (homogeneous) cardinal number of some finite class of individuals of the lowest type. Suppose now that the number of individuals of the lowest type in the universe is finite. In more precise terms, suppose that, where "indiv" stands for the class of all individuals of the lowest type, we have:

$$N_0c`\text{indiv} \in \text{NC induct}$$

By the (corrected) definition of (homogeneous) successor we would then have:

$$N_0c`\text{indiv} +_c 1 = \hat{\xi}\{(\exists\gamma, y) \,.\, \gamma \,\text{sm}_0\, t`\text{indiv} \,.\, y \sim \in \gamma \,.\, \xi = \gamma \cup \iota``y\}$$

That is: the successor $N_0c`\text{indiv} +_c 1$ of the number of the class containing *all* individuals of the lowest type would be the class of all classes such that if we take any member $\gamma$ of $N_0c`\text{indiv}$ and add to it some *y* that does not belong to it, we get a member of $N_0c`\text{indiv} +_c 1$. But by our assumption that $N_0c`\text{indiv} \in \text{NC induct}$, there can be no such *y*, and so $N_0c`\text{indiv} +_c 1 = \Lambda$.

Put in simpler (albeit less precise) terms, if we let the number of individuals be a finite cardinal *m* and ask "how many classes containing $m+1$ individuals are there?", the answer must be *none*; for if there are exactly *m* individuals, then the class of all classes similar to a class with $m + 1$ individuals is the empty set, given that there is no class with $m + 1$ individuals.

The point is general and applies to every finite cardinal that follows $N_0c`indiv$ in the $+_c1$-series if $N_0c`\text{indiv} \in \text{NC induct}$. We would, in fact, have:

$$N_0c`\text{indiv} +_c 1 = \Lambda$$

[134] Cf. WHITEHEAD & RUSSELL, 1927*a*, p.207 [1912, p.214].

$$(\mathrm{N_0c`indiv} +_c 1) +_c 1 = \Lambda$$

$$\vdots$$

In this case, the third Peano postulate would be false: at least two different inductive or natural numbers, namely $\mathrm{N_0c`indiv}$ and $\mathrm{N_0c`indiv} +_c 1$ would have the same successor, namely Λ[135]. Here a comparison with the Fregean construction of the natural numbers is most relevant and it is worthwhile to review it. We shall do it adapting it to *Principia*'s notation[136].

What is characteristic of Frege's logicism is his thesis that numbers are *objects*, that is, entities which would be on a par, in terms of simple types, with Russell's individuals of the lowest type. Frege's first attempt to vindicate this conception of number is presented in §68 of the *Grundlagen*, where he defines[137]:

> The number which pertains to the concept F is the extension of the concept 'equinumerous with the concept F'.

We can use the notations "$F\hat{x}$", "$G\hat{x}$", etc., to represent the Fregeans concepts, and since we are primarily concerned here with Frege's definition of numbers as extensions of concepts, we can substitute them uniformly for their counterparts for *extensions* of concepts "$\hat{z}(Fz)$", "$\hat{z}(Gz)$", etc[138]. So, using *Principia's* notation we can put[139]:

$$\mathrm{Nc`}F\hat{x} = \hat{G}(G!\hat{x}\,\mathrm{sm}\,F\hat{x}) \quad \mathrm{Df}$$

which is the same as:

$$\mathrm{Nc`}\hat{z}(Fz) = \hat{\alpha}\{(\exists G) \,.\, \alpha = \hat{z}(G!z) \,.\, \alpha\,\mathrm{sm}\,\hat{z}(Fz)\} \quad \mathrm{Df}$$

135 Observe that if $\mathrm{N_0c`}t\mathrm{`indiv}$ is not inductive (infinite), then we would have $\mathrm{N_0c`}t\mathrm{`indiv} = (\mathrm{N_0c`}t\mathrm{`indiv}+_c 1)+_c 1$, since then $(\mathrm{N_0c`}t\mathrm{`indiv} +_c 1) = (\mathrm{N_0c`}t\mathrm{`indiv} +_c 1) +_c 1 = \aleph_n$ for some aleph *n*.

136 Our very brief exposition of Fregean Arithmetic will follow the Appendix of BOOLOS, G., 1998, pp.202-219, apart from the notation. We note that what follows is a just a brief presentation of what is generally taken to be Frege's definition of natural numbers and how this enables him to prove the infinity of numbers; we are *not* attempting to provide an exegesis of Frege's texts and so we do not compare and discuss different interpretations. For reasons of space we also do not discuss attempts at providing a "Neo-Fregean" foundation for Arithmetic at this point, though from a philosophical and a technical point of view, the issues are interesting (cf., for instance, KLEMENT, K., 2013*b*).

137 FREGE, G., 1950 [1884], §68.

138 There are, of course, many interpretative issues surrounding Frege's formal systems and his development of Mathematics within them and we obviously cannot consider such issues here here. For discussions of Frege's views, the reader is referred to DUMMETT, M., 1991, HECK, R., 2011, DUARTE, A., 2009 (unpublished doctoral dissertation) and LANDINI, G., 2012.

139 Frege's constructions can be more elegantly formulated, for instance, in terms of the calculus of λ-conversion, but for the sake of uniformity we'll stick with *Principia*'s notation.

Although, of course, the above are not the same as *Principia*'s "$\hat{z}(\phi!z)$", "$\hat{z}(\phi z)$", etc. Frege's extensions are individuals of the lowest type.

The number zero is defined as the number that pertains to a concept that is true of no objects; that is, zero is defined as the extension of the concept [*x* : *x* is different from itself][140] and the immediate successor relation is defined as the relation that holds between *n* and *m* when there exists a concept *F* and object *x* such that *x* is an *F* and *n* is the number of $\hat{z}(Fz)$ and m is the number of $\hat{z}(Fz.z \neq x)$[141]. These may expressed as:

$$0 = \text{Nc}`\hat{z}(z \neq z) \quad \text{Df}$$

$$nSm = (\exists F, x) : F!x \,.\, n = \text{Nc}`\hat{z}(F!z) \,.\, m = \text{Nc}`\hat{z}(F!z \,.\, z \neq x) \quad \text{Df}$$

Using Frege's own version of the ancestral of a relation *R,* this allows for a definition of the notion of *finite cardinal number* (i.e., natural number). To do so, Frege defines a concept *F* as hereditary with respect to a relation *R* as an *F* that satisfies the following condition:

$$xRy \,.\supset .\, F!x \supset_{x,y} F!y$$

And then the strict ancestral of a relation *R* as the relation that holds between *a* and *b* whenever they share all the hereditary properties of the *R* series, that is:

$$aR^{*}b = aRz \supset_z F!z : xRy \,.\supset_{x,y} .\, F!x \supset F!y :. \supset_F .\, F!b \quad \text{Df}$$

He also has the *weak* or *strict ancestral*, which we shall denote by "$R^{*=}$". This is the relation that holds between *x* and *y* whenever $xR^{*}y$ *or* $x = y$.

With these definitions at hand, Frege can identify the concept $\hat{x}$ *is a natural number* with the concept $\hat{x}$ *possesses every hereditary property possessed by zero and the successor of every number*, that is:

$$\mathbb{N}x = F!0 : zSy \,.\supset_{y,z} .\, F!y \supset F!z : \supset_F .\, F!x \quad \text{Df}$$

Or, more briefly: $xS^{*=}0$. Once induction is proved[142], these definitions allow for a proof of the proposition that the successor of the natural number *n* is the number which pertains to the

[140] FREGE, G., 1950 [1884], §74.
[141] FREGE, G., 1950 [1884], §76.
[142] In fact, induction follows immediately from the definition of natural number by means of the 'weak' or non-strict ancestral.

concept $\hat{x}S^{*=}n$, that is: the number succeeding *n* is the number which pertains to the concept *member of the S series which ends with n*.

Intuitively, this is somewhat clearly seen: the concept *member of the S series which ends with n* is true of (and only of) every number equal to, or less than, *n*. Since in Frege's arithmetic, as we have formulated it, numbers are treated as extensions of concepts and every concept has an extension correlated with it and understood as an *object*, we have the guarantee that the number zero exists: for we defined zero as the extension of the concept [X is equinumerous with the concept [x is different from itself]; but this number is an object - so the concept [less than or equal to 0] does not have an empty extension, for 0 is a member of it; but this guarantees the existence of the number 1: this number is the extension of the concept [X is equinumerous with the concept [x is equal to zero]], which, given the existence of zero, is also non-empty; but in that case, the extension of the concept [less than or equal to 1] will also be non-empty, for the concept is true of both 0 and 1 – and so the extension of this concept will contain both 0 and 1 and the number that pertains to the concept [less or equal to 1] is the number two, and so on, for *every natural number n*.

From this it follows that every number has a successor and that no two numbers have the same successor, and, therefore, that the natural number series has no last term. Roughly sketching a formal proof similar to what Frege seemed to have in mind[143], what we would have is the following induction. For the base step prove[144]:

$$[\text{Nc‘}\hat{x}(0S^{*=}x)]S0\text{, that is: }[\text{Nc‘}\hat{x}(x \leq 0)]S0.$$

For the induction step prove:

$$\{\text{Nc‘}\hat{x}[xS^{*=}\text{Nc‘}\hat{x}(xS^{*=}n)]\}S[\text{Nc‘}\hat{x}(xS^{*=}n)]\text{, that is:}$$

$$\{\text{Nc‘}\hat{x}[x \leq \text{Nc‘}\hat{x}(x \leq n)]\}S[\text{Nc‘}\hat{x}(x \leq n)].$$

This yields:

$$\mathbb{N}n \supset [\text{Nc‘}\hat{x}(nS^{*=}x)]Sn\text{, that is: }\mathbb{N}x \supset [\text{Nc‘}\hat{x}(x \leq n)]Sn.$$

143 There are difficulties, however, in determining what exactly Frege attempted to prove or could actually prove (cf. BOOLOS, G. 1998, pp.275-90).

144 Of course, this is easier said than done. For details see BOOLOS,G., 1998 [1994], pp.218-219 and also LANDINI, 2011, p.188-9 and, especially, HECK, R., 2011, p.54-62 and pp.69-89, where detailed proofs are given and compared and several interpretative issues concerning Frege's number theory are thoroughly analyzed.

The possibility of giving a proof along these lines presupposes a theory of numbers as extensions of equinumerous concepts understood as *objects*.

On the surface, *Principia*'s construction resemble Frege's. But, of course, Whitehead and Russell do not treat numbers as *objects* in the Fregean sense: they define the set of natural numbers as the set of all sets of equinumerous finite classes and no class is an individual (of any simple type). Indeed, in *Principia*'s type theory, the existence of a given cardinal number n depends on the existence of a certain number of individuals at the bottom of the type hierarchy (or in some simple type)[145]. Thus, Whitehead and Russell cannot generate numbers in the same way Frege does in his *Grundlagen* and *Grundgesetze*. In Frege's proof, it is essential that each number *n* can fall under the concept *less than or equal to n*, or, equivalently, that each number *n* be a member of the extension of the concept *less than or equal to n*.

This is impossible within the theory of types if we are to deal with numbers of uniform (non-ascending) relative types. Let $n \epsilon$ NC induct(n), that is, let *n* be a member of the class of all inductive numbers of the same relative type as *n*, so that each member of NC induct(n) will be a class of similar finite classes of some relative type below *n*. For convenience, let us take *n* as a class of finite classes of individuals (of the lowest possible type). In *Principia*, the 'Fregean' class:

$$\mathrm{N_0c}`\hat{x}(nS^{*=}x)$$

will be the cardinal number of the class of all finite cardinal numbers that are less than or equal to *n*, so this class must be *necessarily of a higher type* than *n*. Thus, there is simply no escaping the fact that the existence of cardinals in any relative type depends on the number of individuals of *some* simple type.

Frege's strategy allowed him to construct the numbers from scratch. In practice, his assumption that extensions are objects allowed him to let $0 = \iota`\Lambda$ and the successor of any Nc $`\alpha$ as Nc$`(\alpha \cup \{z\})$ for some $z \sim \epsilon\, \alpha$. This requires no existential assumptions beyond extensions. Take what we nowadays call the von Neumann ordinals, that is, the sequence:

$$\Lambda, \{\Lambda\}, \{\{\Lambda\}, \Lambda\}, \{\{\{\Lambda\}\, \Lambda\}, \{\Lambda\}, \Lambda\}, ...$$

where each term x is such that either $x = \Lambda$, or for some y in the sequence, $x = y \cup \{y\}$. This would be enough to instantiate every finite Fregean cardinal, since we have[146]:

$$0 = \mathrm{Nc}`\Lambda$$

[145] Cf. previous footnote 196 of the Introduction on Landini's and Elkind's forthcoming papers on alternatives to *Principia*'s Axiom of Infinity.

[146] Note that this construction is *not* von Neumann's: $\emptyset$ is *not* 0, $\{\emptyset\}$ is *not* 1, $\{\emptyset, \{\emptyset\}\}$ is *not* 2, etc, as they would be in Axiomatic Set Theory; but the series of von Neumann's ordinals can be used to show that every one of Frege's finite cardinal numbers is non-empty. Also, the Frege-Russell construction is impossible in any standard set theory like ZF, ZFC or NBG, etc, because each class Nc$`\{0\}$, Nc$`\{0, 1\}$, etc., is just too big to be a set, being either straightly banned or admitted just as a *proper* class. What the von Neuman sequence of ordinals illustrates

$$1 = \text{Nc}`\{\Lambda\}$$

$$2 = \text{Nc}`\{\Lambda, \{\Lambda\}\}$$

$$3 = \text{Nc}`\{\Lambda, \{\Lambda\}, \{\Lambda, \{\Lambda\}\}\}$$

$$\vdots$$

$$n + 1 = \text{Nc}`\{\Lambda, ..., \{\Lambda, ...\}\}$$

$$\vdots$$

Frege ingeniously showed that the set of numbers $\{0, 1, ..., n\}$ could be used to show that $n + 1$ is non-empty. So he had:

$$0 = \text{Nc}`\Lambda$$

$$1 = \text{Nc}`\{0\}$$

$$2 = \text{Nc}`\{0, 1\}$$

$$3 = \text{Nc}`\{0, 1, 2\}$$

$$\vdots$$

$$n + 1 = \text{Nc}`\{0, 1, 2, ..., n\}$$

$$\vdots$$

Of course, von Neumann's sets are ungrammatical in *Principia*'s simple (non-cumulative) theory of types, unless we restore relative type indices in a way that makes them entirely useless[147].

There are theorems in *Principia* that on the surface resemble some of the above, namely:

$*101{\cdot}1 \qquad \vdash 0 = \text{Nc}`\Lambda$

$*101{\cdot}2 \qquad \vdash 1 = \text{Nc}`\iota`x$

$*101{\cdot}31 \qquad \vdash 2 = \text{Nc}`\iota`(\iota`\iota`x \cup \iota`\Lambda)$

But as Landini notes, "[...] these hardly do justice to Frege"[148] and they are completely different from standard set theoretical constructions of finite numbers.

In the above, for instance, if *x* is an individual of lowest simple type, we can only have the guarantee that 2 is non-empty if it is of higher type than 1; and generally, without assumptions

clearly is that one can show that each finite Fregean cardinal is non-empty assuming naive set theory. Indeed, a non-historical and very elegant reconstruction of Frege's system which uses von Neumann's ordinals to prove that there are infinitely many natural numbers can be found in HATCHER, W., 1982, pp.85-6.

147 More on this point below.

148 LANDINI, G., 2016, p.48.

about the number of individuals (of some simple type), one can only show that $n + 1$ is non-empty in a higher type than that of *n*. This situation could be partly amended by accumulating types, allowing one to form *finite* sets of finite cardinal numbers using Frege's construction. But even apart from the philosophical issues which prevent *Principia* to let types accumulate, in order to really work[149], Frege's construction must, like that of Zermelo and von Neumann, "[...] shatter type ceilings"[150], as Quine once very aptly put it.

Whitehead and Russell themselves discuss the impossibility of giving a proof along the lines of Frege's based on their account of the *lesser than* relation. In section $*120$ on inductive cardinals, we find the following proposition[151]:

$$*120{\cdot}57 \qquad \vdash\colon \mu \in \mathrm{NC\,induct} - \iota`\Lambda \,.\supset .\, \mathrm{Nc}`\hat{\nu}(\nu \leq \mu) = \mu +_c 1$$

This is the closest thing *Principia* has to the main lemma of Frege's proof of the infinity of numbers: it asserts that the cardinal number of the class of all cardinals ν which are less than some given cardinal μ is equal to $\mu +_c 1$. But, of course, $*120{\cdot}57$ simply does not capture the strength of Frege's lemma. Concerning the result, the authors observe in the summary of its section that:

149 The cornerstone of simple, non-cumulative type theory is that a class is always of a (in *Principia*, a relative) type immediately above its members and that all (simple and relative) types are homogeneous. This blocks any (useful) attempt to construct natural numbers by taking, say, $0 = \Lambda$ and $S(x) = \{x\}$ (or, following von Neumann $S(x) = x \cup \{x\}$). This can be resolved by letting types be cumulative, i.e., eliminating the homogeneity requirement. This allows one to construct heterogeneous classes like $\{\Lambda, \{\Lambda\}\}$, $\{\Lambda, \{\Lambda\}, \{\{\Lambda\}\}\}$, etc. But still, they are of a type above any of their members. However, this is still not enough for the class of all natural numbers to be definable. The class would be such that $\Lambda \in N$ and $(x)(x \in N \supset S(x) \in N)$. But there can be no such class within any (finite) type *n*. Indeed, the legitimacy of such a set would require *transfinite* types, which Whitehead and Russell also do not allow. As is well known, the axiomatic set theory of Zermelo (and the so-called iterative conception of set in general) can be understood in terms of a generalization of the simple type-theory of types. The *locus classicus* (at least in terms of exposition) for this approach to Zermelo's Axiomatization as a generalization of Type Theory using cumulativity and transfinite types is Quine's *Set Theory and its Logic* (QUINE, W., 1969), but, of course, this possibility was well known to logicians well before that. Gödel had already fully developed this view in the 1930's (cf. GÖDEL, K., 1933). It seems also that in private correspondence the Mathematician Ralph Hawtrey suggested cumulative types to Russell around 1908. Hawytrey wrote: "[...] under the hierarchy which I stood out for when I was writing to you in the Spring, no Axiom of Reducibility is necessary. You simply state at the head of your page, "Let the type of the proposition of highest order hereafter occurring be high enough." Within the limit so laid down the propositions of different type can rub shoulders as they like, subject always to the condition that, when intelligibility requires it, they must be of different type. And any given proposition $(x)\phi x$ will be asserted for all *x*'s of types not higher than a maximum derived from the limit prescribed. The assumption of any limit of type becomes an unimportant formality and you can put it as far up the ω's as you like." (MOORE, G., (ed.)., 2014, p.lxxxvii.). The fundamental issue with this approach, however, is that for all intents and purposes the allowance of cumulative types ignores the distinction between individuals and sets (of *any type*), something that Russell would certainly be willing to do. I intend to explore this contrast between Russell's conception of a set and those of his contemporaries and successors (specially Zermelo and Gödel) in future research.

150 QUINE, W., 1969, p.283.

151 Where $>$ and $\geq$ are, respectively defined as follows:

$*117{\cdot}01\ \mu > \nu \,.=.\, (\exists \alpha, \beta) \,.\, \mu = \mathrm{N_0c}`\alpha \,.\, \nu = \mathrm{N_0c}`\beta \,.\, \exists!\, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`\beta \,.\, {\sim}\exists!\, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`\alpha \quad \mathrm{Df}$

$*117{\cdot}05\ \mu \geq \nu \,.=\colon \mu > \nu \,.\vee.\, \mu, \nu \in \mathrm{N_0C} \,.\, \mu = \mathrm{sm}``\nu \quad \mathrm{Df}$

Observe that "$\geq$" does not strictly means "less than or equal", for here μ and ν may perfectly well be of different types.

> A proof that all inductive cardinals exist has often been derived from $*120 \cdot 57$ (below). But according to the doctrine of types, this proof is invalid since "$\mu +_c 1$" in $*120 \cdot 57$ is necessarily of higher type than "μ".[152]

Very curiously, if we look at the rather long inductive demonstration of the theorem, we find the following as the first line of the proof:

$$\mathrm{Nc}`\hat{\nu}(\nu \leq 0) \,\epsilon\, 1$$

Which must be a typo. They must have meant $\mathrm{Nc}`\hat{\nu}(\nu \leq 0) = 1$. For the interested reader, we restore most of the steps of the proof:

[152] WHITEHEAD & RUSSELL, 1927*a*, p.186 [1912, p.192]. They also observe, in the body of the work, that "here "$\mu +_c 1$" is necessarily of a higher type than "μ", because it applies to a class of which μ is a member" (WHITEHEAD & RUSSELL, 1927*a*, p.221 [1912, p.228].). Landini finds these comments "odd". According to him, what is said above may suggest that the difference of type between μ and $\mu +_c 1$ is grounded in *Principia*'s disastrous acceptance of $*100 \cdot 3$, which would lead the authors to accept $\hat{\nu}(\nu \leq \mu) \,\epsilon\, \mathrm{Nc}`\hat{\nu}(\nu \leq \mu)$. The claim that $\mu +_c 1$ "applies to a class of which μ is a member" would then have to be explained as follows: assuming that $\mu \neq \Lambda$, we would have $\hat{\nu}(\nu \leq \mu) \,\epsilon\, \mu +_c 1$, and since $\mu \,\epsilon (\nu \leq \mu)$, it follows that $\mu +_c 1$ is of higher type than μ. But he finds this misleading given that, for any cardinal number μ, there is a class $\mathrm{Nc}`\hat{\nu}(\nu \leq \mu)$ which is of the same type as $\mu$ - it is the *possibly empty* descending cardinal $\mathrm{Nc}(t_0`\mu)`\hat{\nu}(\nu \leq \mu) = \mu +_c^{t_0`\mu} 1$. He claims that "we can salvage Whitehead's point by noting that it would be of no help in avoiding this situation to try to employ the homogeneous cardinal $\mathrm{N_0c}`\hat{\nu}(\nu \leq \mu)$ or the ascending cardinal $\mathrm{N^nc}`\hat{\nu}(\nu \leq \mu)$", since "neither are empty, but both are necessarily of higher relative type than μ" (LANDINI, G., 2016, p.48.). Given the way $*120 \cdot 57$ is actually proved, this seems to be most charitable interpretation of the passage in question.

[$*117\cdot511$] $\vdash$ $\mu \,\epsilon\, \mathrm{N_oC} - \iota\text{‘}0 \,.\equiv.\, \mu > 0$ (1)

[(1)] $\vdash$ $\mu \,\epsilon\, \mathrm{N_oC} - \iota\text{‘}0 \,.\supset_\mu .\sim (0 < \mu)$ (2)

[(2)] $\vdash$ $\hat{\nu}(\nu \leq 0) = \iota\text{‘}0$ (3)

(3) . ($*101\cdot2$) . ($*110\cdot641$) $\vdash$ $\mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq 0) = 1 = 0 +_c 1$ (4)

[$*110\cdot4$] $\vdash$ $\exists ! \,\mu +_c 1 \,.\supset.\, \mu, 1 \,\epsilon\, \mathrm{NC} - \{\Box\}$ (5)

[(5) . Transp] $\vdash$ $\mu = \Lambda \,.\supset.\, \mu +_c 1 = \Lambda$ (6)

[$*120\cdot429$] $\vdash$ $\nu \,\epsilon\, \mathrm{NC\ induct} \,.\supset:\, \mu > \nu \,.\equiv.\, \mu \geq \nu +_c 1$ (7)

[$*120\cdot442$] $\vdash$ $\alpha \,\epsilon\, \mathrm{NC\ induct} - \iota\text{‘}\Lambda \,.\, \beta \,\epsilon\, \mathrm{NC} - \iota\text{‘}\Lambda \,.\supset:$

$\alpha < \beta \,.\equiv.\, \sim(\alpha \geq \beta) : \alpha > \beta \,.\equiv.\, \sim(\alpha \leq \beta)$ (8)

[(7) . (8)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} \,.\, \exists ! \,\mu +_c 1 \,.\supset:\, \nu \leq \mu \,.\equiv.\, \nu < \mu +_c 1$ (9)

[(9)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} \,.\, \exists ! \,\mu +_c 1 \,.\supset.\, \hat{\nu}(\nu \leq \mu) = \hat{\nu}(\nu < \mu +_c 1)$ (10)

[$*117\cdot104$] $\vdash$ $\mu \geq \nu \,.\equiv:\, \mu > \nu \,.\vee.\, \mu, \nu \,\epsilon\, \mathrm{N_oC}. \mu = \mathrm{sm}\text{“}\nu$ (11)

[$*117\cdot105$] $\vdash$ $\mu \leq \nu \,.\equiv.\, \nu \geq \mu$ (12)

[(10) . (11) . (12)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} \,.\, \exists ! +_c 1 \,.\supset.\, \hat{\nu}(\nu \leq \mu) = \hat{\nu}(\nu \leq \mu) \cup \iota\text{‘}(\mu +_c 1)$ (13)

[$*120\cdot428$] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} \,.\, \exists ! +_c 1 \,.\, 1 \neq 0 \,.\supset.\, \mu +_c 1 > \mu$ (15)

[(15) . ($*101\cdot22$)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} \,.\, \exists ! +_c 1 \,.\supset.\, \mu +_c 1 \sim \epsilon\, \hat{\nu}(\nu \leq \mu)$ (16)

[$*110\cdot631$] $\vdash$ $\eta \,\epsilon\, \mathrm{NC} \,.\supset_\eta .\, \eta +_c 1 = \hat{\xi}\{(\exists \gamma, y) \,.\, \gamma \,\epsilon\, \mathrm{sm}\text{“}\eta \,.\, y \sim \epsilon\, \gamma \,.\, \xi = \gamma \cup \iota\text{“}y\}$ (17)

[(16) . (17)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} \,.\supset:$

$\mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu) = \mu +_c 1 \,.\supset.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu +_c 1) = \mu +_c 1 +_c 1$ (18)

I.p [(6) . (18)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} : \mu = \Lambda \,.\vee.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu) = \mu +_c 1 :\supset:$

$\mu +_c 1 = \Lambda \,.\vee.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu +_c 1) = \mu +_c 1 +_c 1$ (19)

[$*120\cdot13$] $\vdash$ $\mu \,\epsilon\, \mathrm{NC}_\eta \mathrm{\ induct} : \phi 0_\eta : \xi \,\epsilon\, \mathrm{NC}_\eta \mathrm{\ induct} \,.\, \phi\xi \,.\supset_\xi .\, \phi(\xi +_c 1) : \supset : \phi\mu$ (20)

I.s $\left[(20)\dfrac{\phi}{(19)}\right]$ $\vdash$ $0 = \Lambda \vee \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq 0) = 0 +_c 1 :$

$\mu \,\epsilon\, \mathrm{NC\ induct} : \mu = \Lambda \,.\vee.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu) = \mu +_c 1 : \supset :$

$\mu +_c 1 = \Lambda \,.\vee.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu +_c 1) = \mu +_c 1 +_c 1 :. \supset :$

$\mu \,\epsilon\, \mathrm{NC\ induct} : \supset .\, \mu = \Lambda \,.\vee.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu) = \mu +_c 1$ (21)

[(4) . (21)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} : \supset .\, \mu = \Lambda \,.\vee.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu) = \mu +_c 1$ (22)

[(22)] $\vdash$ $\mu \,\epsilon\, \mathrm{NC\ induct} : \supset .\, \mu \neq \Lambda \,.\supset.\, \mathrm{Nc}\text{‘}\hat{\nu}(\nu \leq \mu) = \mu +_c 1$ (23)

[(23)] $\vdash$ Prop. (24)

The above makes it clear why the attempt to replicate Frege's result within the theory of types is bound to fail. The inductive step is:

$\mu \epsilon \text{NC induct} : \exists! \mu \supset \text{Nc}`\hat{\nu}(\nu \leq \mu) = \mu +_c 1 : \supset : \exists! \mu +_c 1 . \supset . \text{Nc}`\hat{\nu}(\nu \leq \mu +_c 1) = \mu +_c 1 +_c 1$

this, on its turn is established by showing that the following lemma holds:

$$\mu \epsilon \text{NC induct} . \exists! \mu +_c 1 . \supset : \hat{\nu}(\nu \leq \mu +_c 1) = \hat{\nu}(\nu \leq \mu) \cup \iota`(\mu +_c 1)$$

from whence if follows:

$$\mu \epsilon \text{NC induct} . \exists! \mu +_c 1 : \supset . \text{Nc}`\hat{\nu}(\nu \leq \mu) = \mu +_c 1 . \supset . \text{Nc}`\hat{\nu}(\nu \leq \mu +_c 1) = \mu +_c 1 +_c 1$$

As with Frege's proof, the crucial step explores the fact that $\text{Nc}`\hat{\nu}(\nu \leq 0) = 1, \text{Nc}`\hat{\nu}(\nu \leq 1) = 2$, and so on. But in the above, the inductive step is framed in such a way that each cardinal $\mu +_c 1$ must be of a higher type than that of μ.

Thus, given the way they are constructed in the above proof, the formal numbers "$\text{Nc}`\hat{\nu}(\nu \leq \mu)$" and "$\mu +_c 1$" must be of different relative types: only in this way we can be sure that $\text{Nc}`\hat{\nu}(\nu \leq \mu)$ is not empty. This again, is because *Principia* simply cannot prove that there is a class containing infinitely many natural numbers without assuming something about the number of individuals (of some simple type).

5.6.3 Infin Ax and Finite Cardinal Arithmetic

The general point involved in what we just discussed is well known. Any attempt to replicate Frege's proof that NC induct is infinite violates the restrictions of the theory of types. In *Principia* there is simply no way to show that there is a set containing infinitely many natural numbers without further existential assumptions about the number of individuals (of some simple type). In order to deal with this, Whitehead and Russell introduce in section $*120$ their so-called 'Axiom of Infinity'. But, despite its misleading label, what we find at $*120 \cdot 03$ is the following *definition*:

$*120 \cdot 03$ Infin ax $= \alpha \epsilon \text{Nc induct} . \supset_\alpha . \exists ! \alpha$ Df

This introduces no proper axiom but, as the authors put it, "an arithmetical hypothesis"[153] that appears as a condition for existence theorems whenever necessary.

This hypothesis - which, put in informal terms - is equivalent to the assertion that the cardinality of the lowest type N_0c'indiv is non-inductive (or infinite) - is true if and only if no

[153] WHITEHEAD & RUSSELL, 1927*a*, p.203 [1912, p.210].

inductive cardinal number is identical to its successor[154], something which is established in the chain of theorems starting from $*120{\cdot}31$ to $*120{\cdot}33$. It is among these theorems that *Principia*'s close relatives of Peano 3 are to be found.

First, the authors show that if the successor of the cardinal number of a class α is (a) non-empty *and* (b) identical to the succesor of the cardinal number of a class β, then (c) the cardinal number of the class α is identical to the cardinal number of the class β and α and β are similar. That is:

$*120{\cdot}31 \quad \vdash : \exists ! \, \mathrm{Nc}`\alpha +_c 1 \,.\, \mathrm{Nc}`\alpha +_c 1 = \mathrm{Nc}\beta +_c 1 \,.\, \supset . \, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta \,.\, \alpha \mathrm{sm} \beta$

This is perhaps the closest thing to Peano 3 that *Principia* can prove. The demonstration can be reconstructed as follows:

$[*110{\cdot}63] \vdash$	$\mathrm{Nc}`\alpha +_c 1 = \mathrm{Nc}`\beta +_c 1 \,.\, \equiv \therefore$	
	$(\exists\gamma, y) \,.\, \gamma \,\mathrm{sm}\, \alpha \,.\, y \sim \epsilon \gamma \,.\, \xi = \gamma \cup \iota`y : \equiv_\xi :$	
	$(\exists\delta, z) \,.\, \delta \,\mathrm{sm}\, \beta \,.\, z \sim \epsilon \delta \,.\, \xi = \delta \cup \iota`z$	(1)
$[(1) \,.\, *10{\cdot}1] \vdash$	$\gamma \,\mathrm{sm}\, \alpha \,.\, y \sim \epsilon \gamma \,.\, \supset : (\exists\delta, z) \,.\, \delta \,\mathrm{sm}\, \beta \,.\, z \sim \epsilon \delta \,.\, \gamma \cup \iota`y = \delta \cup \iota`z$	(2)
$[*73{\cdot}72] \vdash$	$\gamma \cup \iota`y \,\mathrm{sm}\, \delta \cup \iota`z \,.\, y \sim \epsilon \gamma \,.\, z \sim \epsilon \delta \,.\, \supset . \, \gamma \,\mathrm{sm}\, \delta$	(3)
$[*73{\cdot}3] \vdash$	$\gamma \cup \iota`y = \delta \cup \iota`z \,.\, \supset . \, \gamma \cup \iota`y \,\mathrm{sm}\, \delta \cup \iota`z$	(4)
$[(2) \,.\, (3) \,.\, (4)] \vdash$	$\gamma \,\mathrm{sm}\, \alpha \,.\, y \sim \epsilon \gamma \,.\, \supset : (\exists\delta). \, \delta \,\mathrm{sm}\, \beta \,.\, \gamma \,\mathrm{sm}\, \delta$	(5)
$[*73{\cdot}32] \vdash$	$\gamma \,\mathrm{sm}\, \delta \,.\, \delta \,\mathrm{sm}\, \beta \,.\, \supset . \, \gamma \,\mathrm{sm}\, \beta$	(6)
$[*73{\cdot}32] \vdash$	$\alpha \,\mathrm{sm}\, \gamma \,.\, \gamma \,\mathrm{sm}\, \beta \,.\, \supset . \, \alpha \,\mathrm{sm}\, \beta$	(7)
$[(5) \,.\, (6)] \vdash$	$\gamma \,\mathrm{sm}\, \alpha . y \sim \epsilon \gamma \,.\, \supset . \, \gamma \,\mathrm{sm}\, \beta$	(8)
$[*73{\cdot}31] \vdash$	$\alpha \mathrm{sm}\, \gamma \equiv \gamma \,\mathrm{sm}\, \alpha$	(9)
$[(5) \,.\, (7) \,.\, (9)] \vdash$	$\gamma \,\mathrm{sm}\, \alpha \,.\, y \sim \epsilon \gamma \,.\, \supset . \, \alpha \,\mathrm{sm}\, \beta$	(10)
$[*100{\cdot}321] \vdash$	$\alpha \,\mathrm{sm}\, \beta \,.\, \supset . \, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$	(11)
$[(10) \,.\, (11)] \vdash$	Prop.	(12)

Observe that line (1) stands in need of correction. As we discussed, $*110{\cdot}63$ is not well put as it stands, being in need of typical determination in order to avoid the use of the illicit supplementary definitions $*110{\cdot}03{\cdot}04$. We may correct line one by employing homogeneous (or ascending[155]) analogues of $*110{\cdot}63$. Adopting the first alternative affords us with:

[154] In fact, could $\mathrm{N_0c}`t`\mathrm{indiv} \sim \epsilon\, \mathrm{NC\ induct}$ could be introduced as an axiom of infinity in, were it not for the fact "indiv" is not a proper expression of *Principia*'s object language. Also, Infin Ax ($*120{\cdot}03$) should be introduced first with relative indices and then have them omitted with rules of (relative) type ambiguity. Thanks for Gregory Landini for calling that to my attention.

[155] We are dealing with the homogeneous only for mere convenience. Ascending ones would serves as well.

(1') $\mathrm{N_0c}`\alpha +_c 1 = \mathrm{N_0c}`\beta +_c 1 \,.\equiv:$

$(\exists\gamma, y) \,.\, \gamma\,\mathrm{sm_0}\,\alpha \,.\, y \sim \epsilon\gamma \,.\, \xi = \gamma \cup \iota`y \,.\equiv_\xi .\, (\exists\delta, z) \,.\, \delta\,\mathrm{sm_0}\,\beta \,.\, z \sim \epsilon\delta \,.\, \xi = \delta \cup \iota`z$

which gets the proof going. Next, Whitehead and Russell show that if the non-empty sucessor of a cardinal number α is identical to the successor of the cardinal number β, then α identical to the class of all classes similar to the members of β and α is non-empty, that is:

$*120{\cdot}311 \quad \vdash: \exists!\alpha +_c 1 \,.\, (\alpha +_c 1) = (\beta +_c 1) \,.\supset.\, \alpha = sm``\beta \,.\, \exists!\alpha$

This is a slightly more convenient statement of $*120{\cdot}31$. Here, however, a warning is in order since the proof sketch which *Principia* has relies upon the following theorem:

$*103{\cdot}16 \quad \vdash: \mathrm{N_0c}`\alpha = \mathrm{Nc}`\beta \,.\equiv.\, \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$

Which is accompanied by a somewhat problematic comment:

> In this proposition, the equation '$\mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$' must be supposed to hold in any type for which it is significant. Otherwise, we might find a type for which $\mathrm{Nc}`\alpha = \Lambda = \mathrm{Nc}`\beta$, without having $\mathrm{N_0c}`\alpha = \mathrm{Nc}`\beta$.[156]

The authors correctly observe that if we let $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ be empty descending cardinals, then we could have $\mathrm{Nc}`\alpha = \Lambda = \mathrm{Nc}`\beta$ and $\mathrm{N_0c}`\alpha \neq \mathrm{Nc}`\beta$. But as Landini notes, this just shows the "right-to-left direction of this bi-conditional is, in fact, false"[157], for there is no guarantee that the right-side will 'hold in any type for which it is significant'[158]. Fortunately, $*120{\cdot}311$ goes without modification, since it requires only the left-to-right implication of $*103{\cdot}16$.

[156] WHITEHEAD & RUSSELL, 1927*a*, p.36 [1912, p.38].

[157] LANDINI, G., 2016, p.24.

[158] The proof of $*103{\cdot}16$ also assumes an illicit 'theorem', namely $*103{\cdot}31$ which, despite being true whenever significat, must be, like $*100{\cdot}3$, substituted either for its homogeneous or ascending equivalent. It also relies upon the illegitimate definition of homogeneous cardinals of $*103{\cdot}01$ which, on its turn, relies upon the illegitimate pseudo-theorem $*100{\cdot}3$(see LANDINI, G., 2016, p.21). To understand the *apparent* elementary mistake, we must make a brief digression about the other conventions which which are introduced in the *Prefatory Statement* (more specifically, about what motivated their introduction). Whitehead and Russell's claim that '$\mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$' must be supposed to "hold in any type for which it is significant" is to be read in accordance with convention AT (WHITEHEAD & RUSSELL, 1912, p.xxxi-xxxii), as they point out in the *Prefatory Statement*: "[...] in the few early propositions where AT is introduced, the fact is noted by stating that the equation hold "in *any* type". These propositions are $*103{\cdot}16$ and $*110{\cdot}71{\cdot}72$.". Ignoring the details which are not relevant to our present discussion, convention AT basically makes conditions of non-emptiness for cardinals eliminable in the satement of theorems. As the authors explain: "The result is that we may assume that the symbol representing inductive numbers are never null, and thereby obtain the stable truth-values of propositions about them." (WHITEHEAD & RUSSELL, 1912, p.295). A letter from Whitehead to Russell from 19 January 1911

Now, we can finally consider the theorems which establish the necessity and suffiency of Infin Ax for showing that no number identical to its predecessor. First, we have the theorem which establishes that if a finite cardinal number is non-empty, then it is different from its successor:

$*120\cdot32 \quad \vdash : \alpha \in \mathrm{NC\,induct} \,.\, \exists!\alpha : \supset . \, \alpha \neq (\alpha +_c 1)$

The (rather long) proof is by induction:

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strongly suggests that what motivated this was Whitehead's realization that there were fundamental difficulties involved in propositions with unstable truth-values like ∗103·16(GRATTAN-GUINNESS, I., 2000, p.584-5). And, in fact, Whitehead mentions about $*103 \cdot 16$ specifically: "Can you tell from your list of references where ∗103·16 is subsequently used? We may have a fine crop of fallacious proofs on our hands from it. But I hope not." (GRATTAN-GUINNESS, I., 2000, p.585). Grattan-Guinness reports that Russell replied the following: "∗117·107·108·24·31·54, ∗120·311 [In some, only the implication wh. holds anyhow is required.]" (GRATTAN-GUINNESS, I., 2000, p.585, footnote 4.).

$$[*101{\cdot}22] \vdash \quad 1 \neq 0 \qquad (1)$$

$$[*110{\cdot}641] \vdash \quad 1 +_c 0 = 0 +_c 1 = 1 \qquad (2)$$

$$\underline{\mathrm{B.s}}\,[(1) \,.\, (2)] \vdash \quad 0_\xi \neq 0_\xi +_c 1 \qquad (3)$$

$$[*120{\cdot}311] \vdash \quad \exists\,!\, \alpha +_c 1 \,.\, \alpha +_c 1 = \beta +_c 1 \,.\, \supset = \mathrm{Nc}`\alpha = \mathrm{Nc} \,.\, \alpha \mathrm{sm} \beta \,.\, \alpha = sm``\beta \qquad (4)$$

$$[(4) \,.\, \underline{*10{\cdot}1}] \vdash \quad \exists\,!\, \alpha +_c 1 \,.\, \alpha +_c 1 = \alpha +_c 1 +_c 1 \,.\, \supset \,.\, \alpha = sm``\alpha +_c 1 \qquad (5)$$

$$[*110{\cdot}44] \vdash \quad \mathrm{sm}``(\mu +_c \nu) = \mu +_c \nu \qquad (6)$$

$$[(6) \,.\, \underline{*13{\cdot}12}] \vdash \quad \exists\,!\alpha +_c 1 \,.\, \alpha +_c 1 = \alpha +_c 1 +_c 1 \,.\, \supset \,.\, \alpha = \alpha +_c 1 \qquad (7)$$

$$[(7)] \vdash \quad \alpha \,\epsilon\, \mathrm{NC} \,.\, \exists\,!\, \alpha +_c 1 \,.\, \alpha +_c 1 = \alpha +_c 1 +_c 1 \,.\, \supset \,.\, \alpha = \alpha +_c 1 \qquad (8)$$

$$[(8) \,.\, (\mathrm{Transp})] \vdash \quad \alpha \,\epsilon\, \mathrm{NC} \,.\, \exists\,!\, \alpha +_c 1 \,.\, \alpha \neq \alpha +_c 1 \,.\, \supset \,.\, \alpha +_c 1 \neq \alpha +_c 1 +_c 1 \qquad (9)$$

$$[*118{\cdot}2] \vdash \quad (\mu +_c \nu)_\xi = \hat{\eta}\{(\exists \gamma, \delta) \,.\, \mu = \mathrm{N_oc} \,.\, \nu = \mathrm{N_oc}`\delta \,.\, \eta\, \mathrm{sm}_\xi\, (\gamma + \delta)\} \qquad (10)$$

$$[(10)] \vdash \quad (\alpha +_c 1)_\xi = \hat{\eta}\{(\exists \gamma, \delta) \,.\, \alpha = \mathrm{N_oc} \,.\, 1 = \mathrm{N_oc}`\delta \,.\, \eta\, \mathrm{sm}_\xi\, (\gamma + \delta)\} \qquad (11)$$

$$[(10)] \quad \{(\alpha +_c 1)_\xi +_c 1\}_\xi = \hat{\eta}\{(\exists \gamma, \delta) \,.\, \alpha +_c 1 = \mathrm{N_oc} \,.\, 1 = \mathrm{N_oc}`\delta \,.\, \eta\, \mathrm{sm}_\xi\, (\gamma + \delta)\} \qquad (12)$$

$$[*118{\cdot}25] \vdash \quad (\mu +_c \nu +_c \varpi)_\xi = \{(\mu +_c \nu)_\xi +_c \varpi\}_\xi = \{\mu +_c (\nu +_c \varpi)_\xi\}_\xi \qquad (13)$$

$$[(13)] \vdash \quad (\alpha +_c 1 +_c 1)_\xi = \{(\alpha +_c 1)_\xi +_c 1\}_\xi = \{1 +_c (\alpha +_c 1)_\xi\}_\xi \qquad (14)$$

$$\left[\begin{matrix}(11) \,.\, (12) \,. \\ (14)\end{matrix}\right] \vdash \quad \alpha \,\epsilon\, \mathrm{NC}(\xi) \,.\, \exists\,!(\alpha +_c 1)_\xi \,.\, \alpha \neq (\alpha +_c 1)_\xi \,.\, \supset \,. \quad (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi \qquad (15)$$

$$[(9) \,.\, (15)] \vdash \quad \alpha \,\epsilon\, \mathrm{NC}_\xi \,.\, \exists\,!(\alpha +_c 1)_\xi \,.\, \alpha \neq (\alpha +_c 1)_\xi \,.\, \supset : \quad (\alpha +_c 1)_\xi = \Lambda \,.\, \vee \,.\, (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi \qquad (16)$$

$$[*110{\cdot}4] \vdash \quad \exists\,!\, \mu +_c \nu \,.\, \supset \,.\, \mu, \nu \,\epsilon\, \mathrm{NC} - \iota`\square \qquad (17)$$

$$[(17)] \vdash \quad (\alpha +_c 1)_\xi = \Lambda_\xi \,.\, \supset \,.\, \alpha, 1 \,\epsilon\, \mathrm{NC} - \iota`\square \qquad (18)$$

$$[(18).\mathrm{Transp}] \vdash \quad \alpha \sim \epsilon\, \mathrm{NC}(\xi) \,.\, \vee \,.\, \alpha = \Lambda_\xi : \supset \,.\, (\alpha +_c 1)_\xi = \Lambda_\xi \qquad (19)$$

$$\underline{\mathrm{I.p}}\,[(16) \,.\, (19)] \vdash \quad \alpha = \Lambda_\xi \,.\, \vee \,.\, \alpha \neq (\alpha +_c 1)_\xi : \supset : \quad (\alpha +_c 1)_\xi = \Lambda_\xi \,.\, \vee \,.\, (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi \qquad (20)$$

$$[*120{\cdot}11] \vdash \quad \alpha \,\epsilon\, \mathrm{N_\xi C}\ \mathrm{induct} : \phi 0_\eta : \phi\xi \,.\, \supset_\xi \,.\, \phi(\xi +_c 1) : \supset \,.\, \phi\alpha \qquad (21)$$

$$\underline{\mathrm{I.s}}\left[(21)\frac{\phi}{\underline{(20)}}\right] \vdash \quad \alpha \,\epsilon\, \mathrm{N_\xi C}\ \mathrm{induct} \therefore 0_\xi = \Lambda \vee 0_\xi \neq 0_\xi +_c 1 \therefore$$

$$\xi = \Lambda_\xi \vee \xi \neq (\xi +_c 1)_\xi : \supset_\xi \,.$$

$$(\xi +_c 1) = \Lambda_\xi \vee (\xi +_c 1)_\xi \neq \{(\xi +_c 1)_\xi +_c 1\}_\xi$$

$$:: \supset \,.\, \alpha = \Lambda_\xi \,.\, \vee \,.\, \alpha \neq (\alpha +_c 1)_\xi \qquad (22)$$

$$[(4) \,.\, (22)] \vdash \quad \alpha \,\epsilon\, \mathrm{N_\xi C induct} \,.\, \supset : \alpha = \Lambda_\xi \,.\, \vee \,.\, \alpha \neq (\alpha +_c 1)_\xi \qquad (23)$$

$$[(23)] \vdash \quad \mathrm{Prop.} \qquad (24)$$

Of course theorem $*120{\cdot}32$ does not assure us either that every finite cardinal *is* non-empty or that the succesor of a non-empty cardinal is non-empty.

Then $*120{\cdot}32$ is used as a lemma for a step in the proof of $*120{\cdot}322$, which, together with $*120{\cdot}321$, shows the necessity and sufficiency of Infin Ax for proving the basic existential theorems of finite cardinal Arithmetic:

$*120{\cdot}321$ $\vdash : \alpha \neq \alpha +_c 1 \,.\supset .\, \exists ! \alpha$

$*120{\cdot}322$ $\vdash :. \alpha \in \mathrm{NC\ induct} \,.\supset :\, \exists ! \alpha \,.\equiv .\, \alpha \neq (\alpha +_c 1)$

The first states that if a cardinal number is different from its successor then it is non-empty, while the second shows that if α is a finite cardinal, then it is empty if and only if it is different from its successor. The first is proved as follows:

$[*110{\cdot}4]\vdash$	$\exists ! \mu +_c \nu \,.\supset .\, \mu, \nu \in \mathrm{NC} - \iota$‘□	(1)
$[(1)]\vdash$	$\exists ! \alpha +_c 1 \,.\supset .\, \alpha, 1 \in \mathrm{NC} - \iota$‘□	(2)
$[(2)]\vdash$	$\exists ! \alpha +_c 1 \,.\supset .\, \exists ! \alpha$	(3)
$[(2)\,.\,\mathrm{Transp}]\vdash$	$\alpha = \Lambda \,.\supset .\, \alpha +_c 1 = \Lambda$	(4)
$[(3)\,.\,\mathrm{Transp}]\vdash$	Prop.	(5)

Observe that the above proof does not rely on any particular results for finite cardinal arithmetic, but only on their general theory of cardinal addition. They only sketch a demonstration of the second, since it is trivial. We can write it as the following:

$[*120{\cdot}32]\vdash$	$\alpha \in \mathrm{Nc\ induct} \,.\, \exists ! \alpha \,.\supset .\, \alpha \neq \alpha +_c 1$	(1)
$[(1)]\vdash$	$\alpha \in \mathrm{Nc\ induct} \,.\supset :\, \exists ! \alpha \,.\supset .\, \alpha \neq \alpha +_c 1$	(2)
$[*120{\cdot}321]\vdash$	$\alpha \neq \alpha +_c 1 \,.\supset .\, \exists ! \alpha$	(3)
$[(3)]\vdash$	$\alpha \in \mathrm{Nc\ induct} \,.\supset :\, \alpha \neq \alpha +_c 1 \,.\supset .\, \exists ! \alpha$	(4)
$[(2)\,.\,(4)]\vdash$	Prop.	(5)

And, finally, from the above theorems, they obtain:

$*120{\cdot}33$ $\vdash : \mathrm{Infin\ Ax} \,.\equiv :\, \alpha \in \mathrm{NC\ induct} \,.\supset_\alpha .\, \alpha \neq (\alpha +_c 1)$

which establishes that every inductive cardinal is different from its immediate sucessor if and only if no inductive cardinal is empty. Again, since the proof is trivial they offer only a sketch, but we can write as follows:

$[*120{\cdot}3]\vdash$	$\text{Infin ax} \mathbin{.} \equiv : \alpha \in \text{Nc induct} \mathbin{.} \supset \mathbin{.} \exists ! \alpha$	(1)
$[*120{\cdot}322]\vdash$	$\alpha \in \text{NC induct} \mathbin{.} \supset : \exists ! \alpha \mathbin{.} \equiv \mathbin{.} \alpha \neq \alpha +_c 1$	(2)
$[(1) \mathbin{.} (2)]\vdash$	$\text{Infin ax} \mathbin{.} \equiv : \alpha \in \text{Nc induct} \mathbin{.} \supset \mathbin{.} \alpha \neq \alpha +_c 1$	(3)
$[(3) \mathbin{.} (*10{\cdot}11)]\vdash$	Prop.	(4)

Curiously, as Landini observes[159], a more straightforward theorem like:

$$\vdash : \text{Infin Ax} \mathbin{.} \equiv : \alpha \in \text{NC induct} \mathbin{.} \supset_{\alpha,\beta} \mathbin{.} (\alpha +_c 1) = (\beta +_c 1) \supset \alpha = \beta$$

does not appear in *Principia*[160]. As we remarked, the closest analogues to this result are $*120{\cdot}31{\cdot}311$. Also, as Landini observes, the following which appears in a chain of theorems concerning subtraction is also very close:

$$*120{\cdot}41 \quad \vdash : \nu \in \text{NC induct} \mathbin{.} \exists ! \alpha +_c \nu \mathbin{.} \supset : \alpha +_c \nu = \beta +_c \nu \mathbin{.} \supset \mathbin{.} \alpha = sm``\beta$$

Be that as it may, *Principia*'s logical development of finite cardinal arithmetic must assume *some* axiom to the effect that the number of individuals of some simple type is not finite in order to establish the basic existential results of ordinary Arithmetic, e.g., that no two numbers have the same successor or that the successor relation is one-one among natural numbers, or yet that no natural number is identical to its immediate successor.

5.6.4 Infin Ax and Transfinite Numbers: $\aleph_0$ and ω

Given Russell's acceptance of arithmetization[161] of the several aforementioned branches of Pure Mathematics, aforementioned 'failure' also affects Russell's original claim made in the *Principles* that "from the class of the finite cardinal numbers themselves follows the existence of α_0 [*i.e.* $\aleph_0$], the smallest of the infinite cardinal numbers" and that "from the series of finite cardinals in order of magnitude follows the existence of ω, the smallest of infinite ordinals"[162]. Similar considerations apply to their theories of higher forms of number beyond the positive

159 Cf. LANDINI, G., 2016, p.44.

160 The above (provable) theorem should not be confused with $\alpha \in \text{NC induct} \mathbin{.} \supset_{\alpha,\beta} \mathbin{.} (\alpha +_c 1) = (\beta +_c 1) \supset \alpha = \beta$ that does not appear for the good reason that it is unprovable in *Principia*'s system!

161 Again the extent to which Russell's Logicism requires aritmetization is a somewhat problematic subject (cf. GANDON, S., 2008; 2012 and foonote 407 of chapter 2 on this topic in our previous chapter on the Logicism of the *Principles*).

162 RUSSELL, B., 1903, pp.496 §474.

integers, since ratios are (roughly) defined as certain classes of integers and reals as certain segments of ratios.

Following Cantor, in the *Principles* Russell defined $\aleph_0$ as the cardinal number of NC induct. As Russell and Whitehead explain, this procedure cannot be strictly followed in *Principia* because nothing assures us that there are infinitely many finite cardinals:

> Cantor defines $\aleph_0$ as the cardinal number of any class which can be put into one-one relation [sic] with the inductive cardinals. This definition assumes $\nu \neq \nu +_c 1$, when ν is an inductive cardinal; in other words, it assumes the axiom of infinity; for without this, the inductive cardinals would form a finite series with a last term, namely, Λ.[163]

Without Infin Ax, nothing would asssure us that the first aleph coincides with a finite cardinal. Their solution is to define $\aleph_0$ as the class of all classes that are domains of *progressions*; that is, $\aleph_0$ is defined the class of all classes which "can be arranged in progressions"[164]:

$$*123{\cdot}01 \qquad \aleph_0 = \mathrm{D}``\mathrm{Prog} = \hat{\alpha}\{(\exists R) : R \,\epsilon\, \mathrm{Prog} \,.\, \alpha = \mathrm{D}`R\} \quad \mathrm{Df}$$

Whereas a *progression* is defined as:

$$*122{\cdot}01 \qquad \mathrm{Prog} = (1 \rightarrow 1) \cap \hat{R}(\mathrm{D}`R = \overleftarrow{R}_*`B`R) \quad \mathrm{Df}$$

Where "B" denotes the relation *begins* and $B`R$ denotes the *beginner of R*. This is defined as:

$$*93{\cdot}01 \qquad B = \hat{x}\hat{P}\{x \,\epsilon\, \mathrm{D}`P - \mathrm{Q}`R\} \quad \mathrm{Df}$$

Thus: a relation *R* generates a progression if, and only if[165] (1) *R* is one-one; (2) *R* has a beginning, that is, a first term; and (3) the field of *R* is identical with the posterity of *R*'s first term $B`R$ with respect to *R*. This definition is embodied in the following proposition:

$$*122{\cdot}12 \qquad \vdash :: R \,\epsilon\, \mathrm{Prog} \,.\, \equiv :.$$

$$R \,\epsilon\, 1 \rightarrow 1 \,.\, \exists\,!\, B`R :. x \,\epsilon\, \mathrm{D}`R \,.\, \equiv_x : B`R \,\epsilon\, \alpha : z \,\epsilon\, \alpha \,.\, zRw \,.\, \supset_{z,w} \,.\, w \,\epsilon\, \alpha : \supset_\alpha \,.\, x \,\epsilon\, \alpha$$

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163 WHITEHEAD & RUSSELL, 1927*a*, p.260 [1912, p.268].
164 WHITEHEAD & RUSSELL, 1927*a*, p.260 [1912, p.268].
165 WHITEHEAD & RUSSELL, 1927*a*, p.246 [1912, p.253].

As Whitehead and Russell observe, to show that their definition of $\aleph_0$ is adequate, what must be proved is that (1) $\aleph_0 \,\epsilon\, \mathrm{NC}$, i.e., that $\aleph_0$ is a cardinal number; and (2) that if $\aleph_0 \neq \Lambda$, then $\aleph_0$ is the cardinal number of the class NC induct. For convenience, they introduce a definition of the relation which holds between a finite cardinal number μ and its successor $\mu +_c 1$ (of the same type as μ), symbolized by N:

$$*123{\cdot}02 \qquad N = \hat{\mu}\hat{\nu}\{\mu \,\epsilon\, \mathrm{NC\,induct} \,.\, \nu = (\mu +_c 1) \cap t_0{}^{\backprime}\mu\} \quad \mathrm{Df}$$

Their proof that this relation can generate a progression in the presence of Infin Ax is roughly divided in two parts. First, they show that $N \,\epsilon\, \mathrm{Cls} \to 1$, that $\exists\,!\, B{}^{\backprime}N$ and that $\overleftarrow{N}_*{}^{\backprime}B{}^{\backprime}N = \mathrm{D}{}^{\backprime}N$. This is done by the two following propositions:

$$*123{\cdot}21 \qquad \vdash .N \,\epsilon\, \mathrm{Cls} \to 1 \,.\, \mathrm{D}{}^{\backprime}N = \mathrm{NC\,induct} \mathrm{G}{}^{\backprime}N = \mathrm{NC\,induct} \,.\, -\iota{}^{\backprime}0 \,.\, B{}^{\backprime}N = 0$$

$$*123{\cdot}23 \qquad \vdash .\, \overleftarrow{N}_*{}^{\backprime}0 = \mathrm{NC\,induct} = \mathrm{D}{}^{\backprime}N$$

The second part of the proof consists in showing that $N \,\epsilon\, 1 \to \mathrm{Cls}$, and thus, that $N \,\epsilon\, 1 \to 1$. But as they observe, this amounts to proving Peano 3:

$$\vdash : \mu, \nu \,\epsilon\, \mathrm{NC\,induct} \,.\, \supset : \mu +_c 1 = \nu +_c 1 \,.\, \supset .\, \mu = \nu$$

Again Whitehead and Russell deal with this by introducing Infin Ax as a hypothesis.

Since section $*120$ establishes that Peano 3 is true if, and only if, the Infin Ax is true, the second half of their proof consists in showing that $N \,\epsilon\, \mathrm{Prog}$ if and only if $\mathrm{Nc}{}^{\backprime}(\mathrm{NC\,induct}) = \aleph_0$[166]. Thus, though they can prove that the usual properties expected of $\aleph_0$ hold, namely:

$$*123{\cdot}4 \qquad \vdash .\, \aleph_0 = \aleph_0 +_c 1$$

$$*123{\cdot}41 \qquad \vdash : \nu \,\epsilon\, \mathrm{NC\,induct} \,.\, \supset .\, \aleph_0 = \aleph_0 +_c \nu$$

$$*123{\cdot}411 \qquad \vdash : \nu \,\epsilon\, \mathrm{NC\,induct} \,.\, \supset .\, \aleph_0 = \aleph_0 -_c \nu$$

$$*123{\cdot}421 \qquad \vdash : \aleph_0 = \aleph_0 +_c \aleph_0 = 2 \times_c \aleph_0$$

[166] This is established by the following chain of theorems:
$*123{\cdot}25 \vdash : \mathrm{Infin\,ax}(x) \,.\, \supset .\, (t^3{}^{\backprime}x \uparrow N \upharpoonright t^3{}^{\backprime}x) \,\epsilon\, \mathrm{Prog}$
$*123{\cdot}26 \vdash : \mathrm{Infin\,ax}(x) \,.\, \supset .\, \mathrm{NC\,induct} \cap t^3{}^{\backprime}x \,\epsilon\, \aleph_0$
$*122{\cdot}36 \vdash : \exists\,!\, \mathrm{Prog} \cap t^{11}{}^{\backprime}x \,.\, \supset .\, \mathrm{Infin\,ax}$
$*123{\cdot}18 \vdash : \exists\,!\, \aleph_0(x) \,.\, \supset .\, \mathrm{Infin\,ax}$
$*123{\cdot}27 \vdash : \exists\,!\, \aleph_0(x) \,.\, \supset .\, \mathrm{NC\,induct} \cap t^3{}^{\backprime}x \,\epsilon\, \aleph_0$

$*123{\cdot}422 \quad \vdash : \nu \in \text{NC induct} - \iota`0 \,.\supset.\, \nu \times_c \aleph_0 = \aleph_0$

The *existence*, i.e., *non-emptiness* of such class depends crucially on the truth of the Axiom of Infinity: $\aleph_0$ is a non-empty cardinal if, and only if, there is a progression.

We have a similar situation with the smallest of Cantor's infinite *ordinal* number (or order-type of a series) ω. In *Principia*, ordinal numbers are defined as a particular case of Relation-Numbers treated in part IV of the second volume. Though this topic will be addressed in some detail below when we discuss the importance of the notion of *structure* in *Principia*, it is worth (and necessary) to review the basics of their treatment of ordinals in order to discuss the definition of ω. The fundamental definition is that of *ordinal similarity* between two relations *P* and *Q*:

$*151{\cdot}01 \quad P\,\overline{\text{smor}}\,Q = \hat{S}\{S \in 1 \to 1 \,.\, C`Q = \text{ᗡ}`S : xPy \,.\equiv_{x,y}.\, (\exists z, w) \,.\, xSz \,.\, ySw \,.\, zQw\}$ Df

Roughly, this defines two relations as ordinally similar if, and only if, there is an isomorphism function from *P* into *Q*[167]. Analogously to cardinal numbers, relation numbers are defined as classes of *ordinally similar relations*, starting from the relation Nr between a relation and the class of all relations ordinally similar to it[168]:

$*152{\cdot}01 \quad \text{Nr} = \overrightarrow{\text{smor}} \quad \text{Df}$

$*152{\cdot}02 \quad \text{NR} = \text{D}`\text{Nr} \quad \text{Df}$

$*152{\cdot}1 \quad \vdash . \,\text{Nr}`P = \hat{Q}(Q\text{smor}P)$

Ordinal numbers are then defined as the relation numbers of well-ordered series[169] (with a series defined as well-ordered if its generating relation *P* is a well-ordering relation, that is: (i) it generates a series[170] and (ii) every non-empty subset of its field has a minimal (first) element with respect to *P*[171]. In *Principia*'s notation this can be put as:

[167] This will be addressed in more detail below.

[168] Notice that similar considerations pertaining to typical ambiguity which we previously discussed must be kept in mind here.

[169] WHITEHEAD & RUSSELL, 1927*b*, p.1 [1913, p.1].

[170] *Principia*'s treatment of *series* is given in volume 2, part V. The basic definition is given in $*204$, where a relation *R* is defined *as serial* or as *generating a series* if it is (1) contained in diversity (or, as Russell sometimes says it, is aliorelative), that is $R \subset J$ or $xRy \,.\supset.\, x \neq y$; (2) is transitive, that is $R^2 \subset R$; and (3) is connected, that is $x, y \in C` \,.\supset:\, xRy \,.\vee.\, yRx \vee .\, x = y$. The authors put this succintly as:
$*204{\cdot}01$ Ser = Rl$`J \cap$trans $\cap$connexDf

[171] The notation $\text{min}_P`\alpha$ can be replaced for $B`(P \upharpoonright \alpha)$.

$$*250{\cdot}01{\cdot}121 \quad \begin{array}{ll} \vdash P \,\epsilon\, \Omega \; . \equiv : & P \,\epsilon\, Ser : \alpha \subset C`P \, . \, \exists ! \, \alpha \, . \supset_\alpha . \, \mathrm{E!} \, \mathrm{min}_P`\alpha \\ : \equiv : & P \,\epsilon\, Ser : \exists ! \, \alpha \cap C`P \, . \supset_\alpha . \, \mathrm{E!} \, \mathrm{min}_P`\alpha \end{array}$$

An ordinal number is then defined as the relation number of some well-ordered relation:

$$*251{\cdot}01 \quad \mathrm{NO} = \mathrm{Nr}``\Omega \quad \mathrm{Df}$$

The ordinal number ω, in its turn, is defined as the class of all relations *P* equal to R_{po} where R generates a progression[172]:

$$*263{\cdot}01 \quad \omega = \hat{P}\{(\exists R) \, . \, R \,\epsilon\, \mathrm{Prog} \, . \, P = R_{\mathrm{po}}\} \quad \mathrm{Df}$$

Roughly, this defines ω as the class of serial relations that generate progressions.[173] And now we can see precisely why the existence of infinite ordinals also depends on the Axiom of Infinity: without it, the relation *N* from $*123{\cdot}02$ fails to generate a progression. Whitehead and Russell explain this very succinctly in their summary of section $*263$:

> The axiom of infinity implies that "less to greater" with its field confined to inductive cardinals is a member of ω, or, what comes to the same thing but

[172] The notation R_{po} comes from section $*91$ which treats what Whitehead and Russell call "powers of relations"; the basic notion treated there is the already mentioned class of relations R, R^2, R^3, R^2, ... etc. The basic fact explored is that if the class is arranged in a series $R, R^2, R^3, R^2, ...$ each R^n bears to R^{n+1} the relation $|R$ (which comes from the notation of $*38$; see previous footnote 125); thus, given any relation *R* and any product R^n of it, *R* is an ancestral of R^n with respect to the relation $|R$; or yet, in other words: given any product R^n of a relation *R*, $R(|R)_* R^n$ holds (the reason for this is intuitively clear: to say that a is an ancestor of b with respect to *R* is just to say that some product R^n of *R* holds between *a* and *b* (thus, the ancestral itself could be defined in terms of the class of powers of a relation *R*). Now, the class of all such classes, that is, the class of all ancestors of *R* with respect to $|R$ is defined as:

$*91{\cdot}03 \; \mathrm{Pot}`R = \overrightarrow{(|R)_*}`R \; \mathrm{Df}$

This is the class of all powers of *R*. Whitehead and Russell also define the class of all powers including identity restricted to the field of *R*:

$*91{\cdot}04 \; \mathrm{Potid}`R = \overrightarrow{(|R)_*}`(I \upharpoonright C`R) \; \mathrm{Df}$

Thus, R_{po} is the relation-sum of all members of the class of all powers of *R*:

$*91{\cdot}05 \; R_{\mathrm{po}} = \dot{s}`\mathrm{Pot}`R = \hat{x}\hat{y}\{(\exists P) \, . \, P \,\epsilon\, \mathrm{Pot}`R \, . \, xPy\} \quad \mathrm{Df}$

And it is identical to R_* as they prove in $*91{\cdot}55$. Unpacking the above some more gives us the following which parallels the ancestral defined in $*90$:

$*91{\cdot}13 \vdash :: P \,\epsilon\, \mathrm{Pot}`R \, . \equiv :. \, S \,\epsilon\, \mu \, . \supset_S . \, S|R \,\epsilon\, \mu : R \,\epsilon\, \mu : \supset_\mu . \, P \,\epsilon\, \mu$

$*91{\cdot}15 \vdash :: P \,\epsilon\, \mathrm{Potid}`R \, . \equiv :. \, S \,\epsilon\, \mu \, . \supset_S . \, S|R \,\epsilon\, \mu \, .(I \upharpoonright C`R) \,\epsilon\, \mu : \supset_\mu . \, P \,\epsilon\, \mu$

$*91{\cdot}16 \vdash :: xR_{\mathrm{po}}y \, . \equiv :. \, (\exists P) :. \, S \,\epsilon\, \mu \, . \supset_S . \, S|R \,\epsilon\, \mu : R \,\epsilon\, \mu : \supset_\mu . \, P \,\epsilon\, \mu \, . \, .xPy$

Also, by applying the universal instantiation $\dfrac{\hat{S}(\phi S)}{\mu}$ and definition $*20{\cdot}02$ in $*91{\cdot}13{\cdot}15$, Whitehead and Russell get:

$*91{\cdot}171 \vdash :: P \,\epsilon\, \mathrm{Pot}`R \, . \, \phi S \, . \supset_S . \, \phi(S|R) : \phi R : \supset_\mu . \, \phi P$

$*91{\cdot}17 \; \vdash :: P \,\epsilon\, \mathrm{Potid}`R : \phi S \, . \supset_S : \phi(S|R) : \phi(I \upharpoonright C`R) : \supset_\mu . \, \phi P$

[173] Landini

> is easier to prove, that $\{(\text{NC induct}) \upharpoonright (+_c 1)\}_{\text{po}}$ is a member of ω ($*263{\cdot}12$). Thus the axiom of infinity for the type of x implies the existence of ω in the type $t^{33}{}^{\backprime}x$ ($*263{\cdot}132$); and generally, the existence of ω in any type of relations is equivalent to the existence of $\aleph_0$ in the type of their fields ($*263{\cdot}131$), because $\aleph_0 = \text{D}``\omega = C``\omega$ ($*263{\cdot}101$).[174]

The situation is the same as in the case of $\aleph_0$: if Infin Ax happens to be false, ω would be empty, so again, they cannot define ω outright as the order-type of the series of natural numbers. Instead, it is defined as the class of serial relations that generate a progression (of which there may be none that is actually instantiated).

5.7 A Digression: Infinity and the Multiplicative Axiom

Unbeknownst to Russell while writing the *Principles of Mathematics*, some results in the theory of infinite cardinal numbers also requires other assumptions with a problematic status besides the Axiom of Infinity. Finite classes are usually defined as they are in *Principia*: as classes which have *n* elements, for some $n \epsilon$ NC induct. But another common way one can define a class as infinite is this: α is infinite if, and only if α can be put into a one-one correspondence with a proper sub-class of itself. Russell himself adopted this as a definition of infinity in the *Principles*[175]. Classes that satisfy this latter notion of infinity are generally called *Dedekind-infinite*[176]. In the *Principles* and elsewhere, Russell thought that both definitions of infinite classes, that is, as non-inductive and reflexive, could be shown to be equivalent without any special assumptions[177]. This, unfortunately, is not the case, as Russell and Whitehead discovered around 1904: the proof of this result requires Zermelo's Axiom of Choice[178] or some of its many equivalents.

174 WHITEHEAD & RUSSELL, 1927*b*, p.143 [1912, p.143].

175 RUSSELL, B., 1937 [1903], p.121 §117.

176 Since Dedekind proposed it in his famous booklet (DEDEKIND, R., 1888, p.63). Cantor had also previously employed it (cf. MOORE, G., 1982, pp.22-6).

177 Whitehead and Russell thought so in 1902 when they wrote their first joint work (cf. WHITEHEAD & RUSSELL, B., 1902, pp.427-8, theorems $*2{\cdot}75{\cdot}76$) , but they implicitly employed an equivalent to what we now call the "Axiom of Countable Choice". Cf. footnote 178 below for more details.

178 As is well known, Zermelo introduced the axiom (as such) in his first proof that every set can be well-ordered (cf. ZERMELO, E., 1904, p.139-40). In his second proof that every set can be well-ordered Zermelo put the axiom forward in the sharp form that we nowadays usually refer to: "If *T* is a set whose elements all are sets that are different from 0 and mutually disjoint, its union $\mathcal{G}(T)$ includes at least one subset S_1 having one and only one element in common with each element of *T*" (ZERMELO, E., 1908, p.204); or, more closely to current terminology: "Given any family *T* of non-empty sets, there is a function *f* which assigns to each member *A* of *T* an element $f(A)$ of *A*" (MOORE, G., 1982, p.5). As Gregory Moore notes, Zermelo was the first to make explicit use of the axiom (MOORE, G., 1982, p.2), although Peano had considered it and rejected it as a legitimate principle, stating that "[...] one cannot apply infinitely many times an *arbitrary* rule by which one assigns to a class *A* an individual of this class, a *determinate* rule is stated here" (PEANO, G., 1957 [1890], p.210; MOORE, G., 1982, p.5). Russell and Whitehead's realization that a similar hypothesis was involved in several crucial results came around the same time. As Moore notes (MOORE, G. (ed.), 1993, p.16), in their first joint paper Whitehead and Russell implicitly assumed that every infinite class is the union of some family of denumerable classes, which is an equivalent of the Axiom of 'Countable' or 'Denumerable Choice'; they also implicitly employed another equivalent which they later coined as the 'Multiplicative Axiom': if *M* is a class of

In what follows we discuss why exactly that is the case by reviewing *Principia*'s treatment of Dedekind-infinite classes. We shall see that there is a parallel lesson that can be extracted from their use of the Axiom of Choice and their use of the Axiom of Infinity[179].

Whitehead and Russell call Dedekind-infinite classes *reflexive.* They define, respectively, reflexive classes and reflexive cardinal numbers as follows:

$$*124 \cdot 01 \qquad \text{Cls refl} = \hat{\rho}\{(\exists R) \,.\, R \,\epsilon\, 1 \to 1 \,.\, \text{Ɑ}`R \subset \text{D}` \,.\, \exists\,!\, \overrightarrow{B}`R \,.\, \rho = \text{D}`R\} \quad \text{Df}$$

$$*124 \cdot 02 \qquad \text{NC refl} = \text{N}_0\text{c}``\text{Cls refl} \quad \text{Df}$$

Principia can easily prove, without any special assumptions, that no reflexive cardinal is inductive and that no reflexive class is inductive, thus:

$$*124 \cdot 27 \qquad \vdash . \text{NC refl} \cap \text{NC induct} = \Box$$

$$*124 \cdot 271 \qquad \vdash . \text{Cls refl} \cap \text{Cls induct} = \Lambda$$

But the proof that every non-inductive class is reflexive requires the hypothesis which Whitehead and Russell called the 'Multiplicative Axiom', which asserts that given any class κ of mutually exclusive classes α, there is a class which contains exactly one element x of each element α of κ. This assumption is equivalent to the Axiom of Choice, to which Russell and Whitehead referred to as 'Zermelo's Axiom' since it was him that introduced it as an explicit principle for first time[180]. To have a minimally clear grasp of the role played by the Multiplicative Axiom in *Principia*'s theory of infinite cardinals, we must consider briefly its theory of cardinal multiplication[181].

non-empty mutually exclusive classes *A*, then there exists a class *C* having exactly one member in common with each *A* in *M*; the name 'Multiplicative' results from its implicit use in Whitehead's treatment of multiplication in the paper and elsewhere (*Principia* included). Russell seems to have explicitly recognized the Axiom as such in 1904. He told Jourdain the story as follows "As for the multiplicative axiom, I came on it so to speak by chance. Whitehead and I make alternate recensions of the various parts of our book, each correcting the last recension made by the other. In going over one of his recensions, which contained a proof of the multiplicative axiom, I found that the previous prop- osition used in the proof had surreptitiously assumed the axiom. This happened in the summer of 1904. At first I thought probably a proof could easily be found; but gradually I saw that, if there is a proof, it must be very recondite" (GRATTAN-GUINNESS, I., 1977, p.80). The fourth and fifth volumes of Russell's collected papers show him struggling over the Axiom, in particular whether there are sound philosophical *a priori* grounds from accepting its truth (cf., in particular, RUSSELL, B., 1906*g*). As we shall see, Russell never found any such *a priori* grounds for believing or disbelieving the Axiom, though he did not discard the possibility that such grounds could at some point be found (cf., for instance, RUSSELL, B., 1911*c*).

179 In a paper suggestively entitled "*On the Advantages of Honest Toil Over Theft*" (BOOLOS, G., 1998, pp.255-274), George Boolos discusses the use of these axioms and draws attention to some parallel considerations. As we shall see, however, Boolos fails to appreciate that Russell's conception of Logicism is quite different from that of Frege and this leads him to wrongly claim that *Principia* marks Russell's abandonment of the Logicist enterprise.

180 ZERMELO, E., 1904.

181 Which has its origins in the previous work of Whitehead

As with their theory of cardinal sum, Whitehead and Russell wanted the most general possible definition which preserved the usual formal properties of multiplication. But the case of products was more complicated than that of sums and they had to put forward *two* different definitions. First, they have:

$$*113{\cdot}02{\cdot}101 \quad \beta \times \alpha = \hat{R}\{(\exists x, y) \,.\, x \,\epsilon\, \alpha \,.\, y \,\epsilon\, \beta \,.\, R = x \downarrow y\} \text{ Df}$$ [182]

This, like the definition of the arithmetical sum $\alpha + \beta$ of two classes, defines a class of couples. Like the class $\alpha + \beta$ that is the arithmetical sum of α and β, the arithmetical product $\beta \times \alpha$ of these classes is intended to yield a class with the desired number of terms without assuming that α and β are disjoint or of the same type. As the authors explain, $*113{\cdot}02$ provides this because "for a given *y*, the class of couples we obtain is $\downarrow$ "y [...]"[183] that is, the class of all pairs $x \downarrow y$, where x is a member of α. The definition works because "[...] the number of such classes for varying *y*, is Nc‘β", which yields "[...] Nc‘β classes of Nc‘α couples, and $\beta \times \alpha$ is the logical sum of these classes of couples"[184]. On the basis of this definition of the arithmetical product of two classes, they define cardinal multiplication in a way that is parallel to the definition cardinal sum:

$$*113{\cdot}03 \qquad \mu \times_c \nu = \hat{\xi}\{(\exists \alpha, \beta) : \mu = \mathrm{N_0c}‘\alpha \,.\, \nu = \mathrm{N_0c}‘\beta \,.\, \xi \,\mathrm{sm}\, (\alpha \times \beta)\} \quad \text{Df}$$

This definition - like that of cardinal sum - applies to both finite and infinite numbers alike. However, it can only be extended to a finite number of factors $\mu_1 \times_c \mu_2 ... \times_c \mu_n$, and not to an infinite one[185]. In *Principia*, a definition which includes products of an infinite number of factors requires what Whitehead and Russell called *selections*[186].

A *selection from a class* κ *of classes* α is introduced by the authors as a class of *representatives x* from each member $\alpha \,\epsilon\, \kappa$, where a representative is a member of α picked by a *selector*, *i.e.* a relation *R* such that

$$\alpha \,\epsilon\, \kappa \,.\, \supset_\alpha .\, R‘\alpha \,\epsilon\, \alpha : \text{Ɑ}‘R = \kappa$$

[182] The official *definiens* is $s‘\alpha \downarrow$ "β but for our purposes this is condensed to the point of inconvenience.

[183] WHITEHEAD & RUSSELL, 1927*a*, p.101 [1912, p.104].

[184] WHITEHEAD & RUSSELL, 1927*a*, p.101 [1912, p.104].

[185] A clear informal survey of this point is given by Russell in the *Introduction to Mathematical Philosophy* (RUSSELL, B., 1919a, p.118-20).

[186] WHITEHEAD & RUSSELL, 1925, pp.478-9 [1910, pp.500-1]. A treatment of 'selections' given in terms of axiomatic set theory (employing only first-order logic) is given in the tenth chapter of Quine's classic book (QUINE, W., 1969).

Or, equivalently, an *R* such that:

$$R \in 1 \to \text{Cls} \,.\, R \subseteq \epsilon \,.\, \text{C}\text{'}R = \kappa$$

That is: given a class of classes κ, a selector *R* picks out an element *x* - which Whitehead and Russell call a *representative* of α - from each $\alpha \in \kappa$, thus forming what they call a *selection* or *selected class*[187]. To take a simple example, if $\kappa = \{\{a, b\}, \{c, d\}, \{e\}\}$, for instance, then $\{a, c, e\}$ is a selection from κ, where *a* is the representative of $\{a, b\}$, *c* is that of $\{c, d\}$ and *e* that of $\{e\}$; other possible selections would be $\{a, d, e\}$, $\{b, c, e\}$, etc.

The notion of a selection from a class of classes κ is obtained by Whitehead and Russell from the more general and fundamental notion of a selection from any given relation *P*, which they define as:

$$*80\cdot01 \qquad P_\Delta = \hat{\lambda}\hat{\kappa}\{\lambda = (1 \to \text{Cls})\text{Rl}\text{'}P \cap \overleftarrow{\text{C}}\text{'}\kappa\} \quad \text{Df}$$

from which it almost immediately follows that:

$$*80\cdot14 \qquad \vdash : R \in P_\Delta\text{'}\kappa \,.\, \equiv \,.\, R \in 1 \to \text{Cls} \,.\, R \subseteq P \,.\, \text{C}\text{'}R = \kappa$$

$$*80\cdot35 \qquad \vdash : R \in P_\Delta\text{'}\kappa \,.\, \supset \,.\, \text{D}\text{'}R = \hat{x}\{(\exists y) \,.\, y \in \kappa \,.\, x = R\text{'}y\}$$

The case which is relevant for determining the properties of selections from classes of classes, of course, is when P is the relation ϵ of class membership, for in that case, as Whitehead and Russell explain, "if $R \in \epsilon_\Delta\text{'}\kappa$,*R* picks out a *representative* $R\text{'}\alpha$ from each class α which is a member of κ"[188]. Also, in that case, we have:

$$\epsilon_\Delta\text{'}\kappa = \hat{R}\{(1 \to \text{Cls}) \cap \text{Rl}\text{'}\epsilon \cap \overleftarrow{\text{C}}\text{'}\kappa$$

This is the class of selective relations, or selectors, of the class of classes κ. Also, the class of all selected classes of κ will be D" $\in_\Delta$ 'κ which is just:

$$\hat{\alpha}\{(\exists R) \,.\, R \in \epsilon_\Delta\text{'}\kappa \,.\, \alpha = \text{D}\text{'}R\}$$

[187] WHITEHEAD & RUSSELL, 1925, pp.478-9 [1910, pp.500-1].
[188] WHITEHEAD & RUSSELL, 1925, p.479 [1910, p.501].

That is, the class of all domains of the members of ϵ_{Δ} ‘κ. This gives us enough of *Principia*’s theory of selections to appreciate the connection with cardinal multiplication. As the authors explain:

> It will be seen that, if $\alpha \in \kappa$, R‘α may be any member of α, and we get a different R for each different member of α. Thus if we keep the representatives of all the other members of κ unchanged, the number of selective relations to be obtained by varying the representative of α is the number of members of α. Hence the number of selective relations together may be fitly defined as the product of the numbers of terms possessed by the various members of κ. In case κ is finite or infinite, the product so defined obeys all the formal laws of multiplication.[189]

Thus, they define the product of *any* number of (possibly infinitely many) factors as:

$*114{\cdot}01 \qquad \Pi\text{Nc}‘\kappa = \text{Nc}‘\,\epsilon_{\Delta}\,‘\kappa \text{ Df}$

But not all classes of classes are what Whitehead and Russell call a *multiplicative* class (of classes). The classes of classes which *are* multiplicative are defined as those which have a non-empty class of selectors:

$*88{\cdot}02 \qquad \text{Cls}^2\text{Mult} = \hat{\kappa}\{\exists\,!\,\epsilon_{\Delta}\,‘\kappa\} \text{ Df}$

Thus, the following theorems, for instance, establish that classes of classes which have the empty set as a member are *not* multiplicative:

$*83{\cdot}1 \qquad \vdash : \exists\,!\,\epsilon_{\Delta}\,‘\kappa \,.\supset .\, \Lambda \sim\in \kappa$

$*83{\cdot}11 \qquad \vdash : \Lambda \in \kappa \,.\supset .\, \epsilon_{\Delta}\,‘\kappa = \Lambda$

Since Whitehead and Russell ingeniously realized that “[...] the number of selective relations together may be fitly defined as the product of the numbers of terms possessed by the various members of κ”[190], it follows that if some factor μ of any product $\Pi\text{Nc}‘\kappa$ is such that $\mu = \text{Nc}‘\Lambda = 0$, then the product $\Pi\text{Nc}‘\kappa = \text{Nc}‘\,\epsilon_{\Delta}\,‘\kappa = 0$ as well.

But Whitehead and Russell also eventually realized that they could *not* prove the following:

$\Lambda \sim\in \kappa \,.\supset .\, \exists\,!\,\epsilon_{\Delta}\,‘\kappa$

[189] WHITEHEAD & RUSSELL, 1925, p.479 [1910, p.501].
[190] WHITEHEAD & RUSSELL, 1925, p.479 [1910, p.501].

Which is equivalent to the assertion that any number of (possibly infinitely many) factors, none of which are equal to zero, is not equal to zero. This, as it turns out is, is also one of the many equivalents to the Axiom of Choice.

Indeed, they discovered that, generally, the question of whether an arbitrary (possibly infinite) class of classes κ is such that $\kappa \in \mathrm{Cls}^2\mathrm{Mult}$ or not is problematic - even in cases where it appears "evident" that they should be multiplicative. Perhaps the best example is that of classes of non-empty mutually exclusive classes[191]:

$$*84{\cdot}03 \qquad \mathrm{Cls\,ex}^2\mathrm{excl} = \mathrm{Cls}^2\mathrm{excl} - \overleftarrow{\epsilon}\text{`}\Lambda \ \mathrm{Df}$$

In such cases, it is very intuitive to assume that the following should be provable from logic alone:

$$\kappa \in \mathrm{Cls\,ex}^2\mathrm{excl} \,.\supset_\kappa .\, \exists\,!\, \epsilon_\Delta\text{`}\kappa$$

But as it happens, this is not so: this assumption is also equivalent to $\Lambda \sim \in \kappa \,.\supset .\, \exists\,!\, \epsilon_\Delta\text{`}\kappa$. Thus, Russell and Whitehead were led to introduce the following definition of a hypothesis that they called the *Multiplicative* Axiom:

$$*88{\cdot}03 \qquad \mathrm{Mult\,ax} = \kappa \in \mathrm{Cls\,ex}^2\mathrm{excl} :\supset_\kappa :.\, (\exists\mu) : \alpha \in \kappa \,.\supset_\alpha .\, \mu \cap \alpha \in 1 \quad \mathrm{Df}$$

This was used in their development of the theory of selections, multiplication and transfinite numbers much like the Axiom of Infinity was used in the theory of inductive cardinals: as an explicit undischarged hypothesis in the proof of an existence theorem wherever it was required.

This hypothesis is also needed for the theory of reflexive cardinals. It turns out that, as Whitehead and Russell put it, "[...] if a product of $\aleph_0$ factors, no one of which is zero, is never zero, then the two definitions of the finite and the infinite coincide"[192]. The proof of this result is rather long, but we can review the general idea. First, they define the class of multiplicative cardinals, that is, cardinals which are numbers of selections:

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[191] Classes of mutually exclusive and possibly empty classes are defined as follows:

$$*84{\cdot}01 \qquad \mathrm{Cls}^2\mathrm{excl} = \hat{\kappa}(\alpha, \beta \in \kappa \,.\, \alpha \neq \beta) \quad \mathrm{Df}$$

Observe that classes κ such that $\kappa \in \mathrm{Cls\,ex}^2\mathrm{excl}$ are important because otherwise it could happen that for some $\alpha, \beta \in \kappa$, that $\alpha \cap \beta \neq \kappa$; and in this case, α and β may have the same representative in a selection. Thus, as Russell puts it "there may be more selectors than selections" (RUSSELL, B., 1919a, p.120).

[192] WHITEHEAD & RUSSELL, 1927*a*, p.271 [1912, p.278].

$*124{\cdot}03 \quad \text{NC mult} = \text{NC} \cap \hat{\alpha}\{\kappa \,\epsilon\, \alpha \cap \text{Cls ex}^2\text{excl} \,.\, \supset_\kappa \,.\, \exists \,!\, \epsilon_\Delta \text{‘}\kappa\} \quad \text{Df}$

Now, an important property of inductive cardinals (which is easy to *see*, but tedious to prove) is that all of them are mutually exclusive, and if Infin Ax is assumed they are all non-empty. This is proved via the following proposition from the section on inductive cardinals and an immediate lemma concerning multiplicative cardinals:

$*120{\cdot}61 \quad \vdash : \kappa \,\epsilon\, \text{Cls induct} \,.\, \Lambda \sim\epsilon\, \kappa \,.\, \supset .\, \exists \,!\, \epsilon_\Delta \text{‘}\kappa$

$*124{\cdot}4 \quad :.\, \mu \,\epsilon\, \text{NC mult} \,.\, \equiv : \mu \,\epsilon\, \text{NC} : \kappa \,\epsilon\, \mu \cap \text{Cls ex}^2\text{excl} \,.\, \supset_\kappa \,.\, \exists \,!\, \epsilon_\Delta \text{‘}\kappa$

From this Whitehead and Russell obtain:

$*124{\cdot}41 \quad \vdash .\, \text{NC induct} \subset \text{NC mult}$

Also, two important facts follow from this. First: the (class) sum s‘NC induct of all finite cardinals will be such that its cardinal number is identical to $0 + 1 + 2 + ... + n + n + 1...$ Second, the cardinal number of the class of selected classes of NC induct will be the same as the cardinal number of the class of all selections of NC induct.

Now, the assertion that “[...] a product of $\aleph_0$ factors, no one of which is zero, is never zero” is just the claim that $\aleph_0 \,\epsilon\, \text{NC mult}$, and their proof is framed in these terms. That is: $\aleph_0 \,\epsilon\, \text{NC mult}$ implies the coincidence of the two definitions of infinity[193]. The demonstration is rather long and complicated, taking the bulk of $*124$, so we shall not reconstruct or review the full formal proof in detail. The final proposition is this:

$*124{\cdot}56 \quad \vdash : \aleph_0 \,\epsilon\, \text{NC mult} \,.\, \supset .\, -\text{Cls induct} = \text{Cls refl} \,.\, \text{N}_0\text{c} - \text{NC induct} = \text{NC refl}$

This just shows that the assumption that $\aleph_0$ is a multiplicative cardinal - which is equivalent to the assumption that “[...] a product of $\aleph_0$ factors, no one of which is zero, is never zero”[194] - implies that a class is infinite if, and only if it is Dedekind-infinite *i.e.*, reflexive.

To have a rough grasp of how this is proved in *Principia*, we may follow Russell’s sketch given in *Introduction to Mathematical Philosophy*[195]. Roughly, what we have is a (beautiful and instructive) proof in three stages which we can summarize as follows:

193 WHITEHEAD & RUSSELL, 1927*a*, p.271 [1912, p.278].
194 WHITEHEAD & RUSSELL, 1927*a*, p.271 [1912, p.278].
195 RUSSELL, B., 1919a, pp.127-9.

Stage 1: Let a class ρ be non-inductive. Given such class, show that for any $\nu \in$ NC induct, there is a proper non-empty sub-class δ of ρ such that Nc'$\delta = \nu$. That is: show that given any finite cardinal number, there is a proper sub-class of ρ that 'instantiates' that cardinal number. Show that the class of all classes of finite sub-classes of ρ form a progression; this is the crucial step of the first-stage of the proof: the basic idea is to show that the cardinal numbers 'formed' from ρ (by taking the class of all its finite sub classes) is contained in the power class of the power class Cl'Cl'ρ of ρ. This establishes the following: the class Cl'Cl'ρ has a subset which is a progression if ρ is not inductive. This is enough to show that: (i) if ρ is non-inductive then Cl'Cl'$\rho \in$ NC refl; (ii) if μ is a non-inductive cardinal then $2^{2^{\mu}}$ is a reflexive cardinal. [In *Principia*, this stage is given by propositions $*124{\cdot}51{\cdot}511$, $*124{\cdot}57$ and $*124{\cdot}6$]

Stage 2: Assume the Multiplicative Axiom restricted to classes of cardinality up to $\aleph_0$ i.e., $\aleph_0 \in$ NC mult. That is: that from any given class κ of non-empty mutually exclusive classes α, whose cardinal number is either equal to, or less than $\aleph_0$, we can form a class by taking a representative from each $\alpha \in \kappa$. Let κ be the the class of all finite cardinal numbers ρ_N formed from ρ in stage 1 above. Then: (i) each α will be a class of all finite sub-classes α_ν of cardinality ν of ρ; (ii) ρ_N is a class of mutually exclusive classes; (iii) ρ_N contains a progression; (iv) $\rho_N \subset$ Cl'Cl'ρ. But then, omitting the sub-class whose only member is Λ, and assuming the Mult Ax (up to classes with $\aleph_0$ terms), we can form, from ρ_N, a progression of classes $\alpha_1, \alpha_a ..., \alpha_n, ...$, where each α_j is an element of Cl'ρ. Thus, Cl'ρ contains a progression. This proves that if ρ is non-inductive then Cl'$\rho \in$ Cls refl and if μ is a non-inductive cardinal then 2^{μ} is a reflexive cardinal. [In *Principia*, this stage is given by propositions $*124{\cdot}512{\cdot}513{\cdot}514$]

The next stage is the most complicated: it consists in showing that another application of the Multiplicative axiom restricted to classes of cardinality up to $\aleph_0$ allows us to prove that any non-inductive class contains a progression and is thus reflexive.

Stage 3: Assume $\aleph_0 \in$ NC mult. First, show, by induction, that each class α_j, for $j > n$ in the progression $\alpha_1, \alpha_2 ..., \alpha_n, ...$ must introduce at least one new term with respect to some its predecessors. Base case: observe that if $\alpha_1 = \iota$'x, for some $x \in \alpha$, then α_2 must be either equal to ι'$x \cup \iota$'y or ι'$y \cup \iota$'z, for some $y, z \in \alpha$ and such that $y \neq z \neq x$; furthermore, observe that the next class α_3 can be such that no new term is introduced, e.g., $\alpha_3 = \iota$'$x \cup \iota$'$y \cup \iota$'z, but α_4 must necessarily introduce a new term w. Inductive step: observe that given the first n classes $\alpha_1, \alpha_2 ..., \alpha_n$, the maximum number of terms of the union of all such classes is given by the arithmetic progression $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$ (in the case where no term $x, y, z...$ is repeated); thus, the interval of classes $\alpha_n, ..., \alpha_{n(n+1)/2}$ could be formed without introducing any new terms from α into any of the classes α_i for $n < i \leq \dfrac{n(n+1)}{2}$; but the class α_{i+1}, on the other hand, must necessarily introduce a new term for it must have $\dfrac{n(n+1)}{2} + 1$ elements. The next step consists in showing that from the progression $\alpha_1, \alpha_2 ..., \alpha_n, ...$ we can form a subset of α which is also a progression: Russell does this by letting $\beta_1, \beta_2, ...\beta_3, ...$ be obtained from $\alpha_1, \alpha_2 ..., \alpha_n, ...$ by omitting "all those classes that are composed entirely of members that have occurred in previous classes"[196], and then, from this, another progression $\gamma_1, \gamma_2, ..., \gamma_n$ can be formed by letting each γ_i the part of β_i with 'new members', that is: each γ_i is such, for every *x*, if $x \,\epsilon\, \gamma_i$ then $x \sim \epsilon\, \beta_h$, where $h \leq i$. But from the above it follows that: (i) the class κ of all γ_i forms a class of mutually exclusive classes; and thus (ii) a selection ϵ_Δ 'κ can be formed from Γ which is also a progression; from whence it follows, since ϵ_Δ '$\kappa \,\epsilon\, \alpha$, that (iii) α contains a progression and so is reflexive. Therefore: assuming the axiom of countable choice, i.e, that $\aleph_0$ is a multiplicative cardinal, it follows that if α is a non-inductive class, it is reflexive; or equivalently: if $\nu \sim \epsilon$ NC induct, then $\nu \,\epsilon$ NC refl. [In *Principia*, this stage is given by propositions $*124{\cdot}52 - 55$]

In this proof, the first stage does not require in any way the Multiplicative Axiom, neither restricted to classes whose cardinal number is $\aleph_0$ nor otherwise.

The theorems which constitute the first stage of the proof are the following (where only the first conjuncts of $*124{\cdot}51$ and $*124{\cdot}511$ are relevant to what is proved):

$*124 \cdot 51$ $\vdash$:. $\rho \sim \epsilon\,$Cls induct . $Q = (\frown \text{Cl}`\rho)|N|\text{Cnv}`(\frown \text{Cl}`\rho)$. $\supset$.

$Q \,\epsilon\,$ Prog . D'$Q \subset$ Cl'Cl'ρ . D'$Q = (\frown \text{Cl}`\rho)$"NC induct

$*124 \cdot 511$ $\vdash$: $\rho \sim \epsilon\,$Cls induct . $\supset$. Cl'Cl'$\rho\,\epsilon\,$Cls refl . $(\frown \text{Cl}`\rho)$"NC induct $\epsilon \aleph_0 \frown$ Cls ex²excl

These lemmas are, if not the most important, the most interesting part of *Principia*'s treatment of reflexive cardinals, not only because they prove an original result discovered by the authors[197], but also because they exemplify perfectly Whitehead and Russell's handling of existence-theorems.

As they explain in their informal summary, assuming that ρ is non-inductive, they show that "it contains classes having ν terms, if ν is any inductive cardinal", so that for every inductive cardinal ν, $\upsilon \frown$ Cl'ρ is non empty. So each of these classes will be a class of sub-sets of ρ with ν elements; but this means that these classes $0 \frown$ Cl'ρ, $1 \frown$ Cl'ρ, ..., $\upsilon \frown$ Cl'ρ, ... "form a progression, which is contained in Cl'Cl'ρ"[198].

[196] RUSSELL, B., 1919a, p.129
[197] For proofs of their result given in the modern setting of Zermelo-Fraenkel set theory, see LEVY, I., 1979, p.80 and POTTER, M., 2004, p.
[198] WHITEHEAD & RUSSELL, 1927*a*, p.271 [1912, p.279].

This establishes the core of the beautiful result that some authors have come to call "the proper analogue of Frege's theorem"[199]: if ρ is non-inductive, then Cl'Cl'ρ is reflexive. When put in terms of cardinal numbers, the result is embodied in the following theorem:

$*124{\cdot}57 \quad \vdash : \mu \in N_0C - NC\,\text{induct} \,.\supset.\, 2^{2^\mu} \in NC\,\text{refl}$

The result is an almost direct consequence of $*124{\cdot}511$ and $*116{\cdot}72$ which is, in fact, a version of Cantor's power class theorem. Whitehead and Russell offer only a proof sketch, but one can easily reconstruct a short demonstration:

Hp =	$\mu \in N_0C - NC\,\text{induct}$	Df.
[Hp.$*120{\cdot}02$]$\vdash$	$\text{Hp} \,.\supset.\, (\exists\rho) : \rho \sim \in \text{Cls induct} \,.\, \mu = \text{Nc}`\rho$	(1)
[$*124{\cdot}511$]$\vdash$	$\rho \sim \in \text{Cls induct} \,.\supset.\, \text{Cl}`\text{Cl}`\rho \in \text{Cls refl}$	(2)
[$*116{\cdot}72$]$\vdash$	$\text{Nc}`\text{Cl}`\alpha = 2^{\text{Nc}`\alpha}$	(3)
[(3)]$\vdash$	$\text{Nc}`\text{Cl}`\text{Cl}`\rho = 2^{2^{\text{Nc}`\rho}}$	(4)
[(1) . (2) . (4)]$\vdash$	Prop.	(5)

From $*124{\cdot}511$, Whitehead and Russell also obtain the following:

$*124{\cdot}6 \quad \vdash : \rho \sim \in \text{Cls induct} \,.\equiv.\, \text{Cl}`\text{Cl}`\rho \in \text{Cls refl}$

Which has a straightforward proof:

[$*124{\cdot}511$]$\vdash$	$\rho \sim \in \text{Cls induct} \,.\supset.\, \text{Cl}`\text{Cl}`\rho \in \text{Cls refl}$	(1)
[$*120{\cdot}74$]$\vdash$	$\rho \in \text{Cls induct} \,.\supset.\, \text{Cl}`\rho \in \text{Cls induct}$	(2)
[(2)]$\vdash$	$\text{Cl}`\rho \in \text{Cls induct} \,.\supset.\, \text{Cl}`\text{Cl}`\rho \in \text{Cls induct}$	(3)
[$*124{\cdot}271$]$\vdash$	$\text{Cls refl} \cap \text{Cls induct} = \Lambda$	(4)
[(3) . (4)]$\vdash$	$\rho \in \text{Cls induct} \,.\supset.\, \text{Cl}`\text{Cl}`\rho \sim \in \text{Cls refl}$	(5)
[(1) . (5)]$\vdash$	Prop.	(6)

This provides a 'weak' or 'conditional' identification of non-inductive and reflexive classes without assuming any equivalent of the multiplicative axiom (countable or otherwise). And the parallel with the use of Infin Ax here is clear enough: Whitehead and Russell refrain from outright assuming the Multiplicative Axiom but employ it only as a hypothesis just like Infin Ax.

What we must consider next is how such approach can be philosophically justified in a way that is coherent with the assertion that Pure Mathematics is nothing but the development of

[199] DEMOPOULOS & CLARKE, 2005, pp.160-1.

Logic. In order to do this, we must see what is Russell's conception of Logicism and of Logic as a science.

5.8 Russell's Regressive Method

We may start by discussing what the Epistemology of Logicism (and Logic) is that underlies *Principia Mathematica*.

The first serious philosophical *and* mathematical attempt to establish that some branch of Mathematics is a branch of Logic was made by Frege, who aimed to show this with respect to Arithmetic and some portions of Analysis[200]. Frege's fundamental claim was that our knowledge of arithmetical propositions does not depend on any form of spatial or temporal intuition, but only upon recognition of the most basic laws of Logic. His project of developing a treatment of Arithmetic on a logical basis was conceived, first and foremost, as a continuation of the process of rigorization of Analysis started by mathematicians like Cauchy and Weierstrass and carried on by the likes of Cantor, Dedekind and Peano. Of course, what Frege realized - to a depth that was (and perhaps still is) unprecedented - was that the attempt to eliminate or reveal the appeal to spatial or temporal intuitions from mathematical reasoning required an analysis of the laws of reasoning themselves.

As he claimed in the first presentation of his *Begriffsschrift*, Frege's goal was to formulate a medium for the formulation of proofs that would "prevent anything intuitive from penetrating" them and "keep the chain of inferences free of gaps"[201]. This effort to formulate rigorous and gapless proofs was first directed at the laws of propositional logic and (first and higher-order) quantification theory together with a theory of sequences (based on Frege's notion of the ancestral of a relation). Later, in his *Grundgestze* Frege extended it to Arithmetic, attempting a "derivation of the simplest laws of Numbers by logical means alone"[202]. And, clearly by "logical means alone" Frege meant that every axiom of his system was a logical truth and every one of its inference rules was sound.

But despite being at its core a mathematical project - in the sense that ultimately its success depended on the possibility of obtaining a technical result - Frege's attempt to develop Arithmetic as Logic had a fundamental epistemic goal. Frege emphasizes that the logical analysis of the most fundamental concepts and laws of Arithmetic is meant to show that they are neither empirically grounded nor require some form of pure Kantian intuition in order to be known. This is clear, for instance, when, in in the Introduction to the *Grundgesetze*, he looks back to the *Grundlagen* and notes that:

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200 For a discussion of this point, see DUMMETT, M., 1991, pp.277-291.
201 FREGE, G., 1879, p.5.
202 FREGE, 2013 [1893], p.1

> In my *Grundlagen der Arithmetik*, I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition.[203]

Frege's main philosophical goal was to secure a privileged epistemic status to Arithmetic. This claim appears more or less explicitly in the opening of the *Grundgesteze*, when Frege writes:

> In the present book this is now to be established by deduction of the simplest laws of cardinal number by logical means alone. In order for this to be done convincingly, significantly higher demands have to be imposed on the conduct of proof than is usual in arithmetic. We have to mark out in advance a few modes of inference and consequence, and no step is allowed to occur which is not in accordance with one of them. Thus, in the transition to a new judgment, one is no longer to be satisfied, as mathematicians up until now nearly always have been, with its obvious correctness, rather it must be analyzed into the simple logical steps of which it consists, and these are often not particularly few. Herein, no presupposition can remain unnoticed; every required axiom must be uncovered. For it is precisely the presuppositions that are made tacitly or without clear awareness that bar insight into the *epistemological* nature of a law.[204]

Now, the question of whether Frege actually understood his project as having a fundamental epistemic component notwithstanding[205], the passage above may naturally suggest a particular picture of the essence of Logicism (to borrow a quote from one of Frege's greatest admirers)[206]. According to this picture, the ultimate goal of the logical analysis of Mathematics is to present proofs of fundamental theorems in such a way that each premise is to be justified either by being a logical self-evident axiom or by being obtainable from a logical axiom through the successive use of some (logically sound) rule of inference. And this is done to ensure that the proved proposition rests on ultimate, unshakable, foundations. Let us call this, following Andrew Irvine, 'Epistemic Logicism'[207].

Much of this cannot be adequately said of Russell's project. To be sure, there are some clear parallels between the objectives of Russell's project and the *modus operandi* for establishing them that were initially adopted by Russell with those of Frege, who, at some points suggests commitments to Epistemic Logicism. Again, first and foremost, Russell understood the attempt to ground Arithmetic in Logic as a continuation of the search for a rigorous foundation

[203] FREGE, 2013 [1893], p.1.
[204] FREGE, 2013 [1893], p.1. Our emphasis.
[205] It may be argued, for instance, that Frege *only* wanted to establish, against Kant, that no sort of temporal or spatial intuition ground knowledge of Arithmetic, which requires, so to say, only *logical intuition*. And in this case, the claim that Frege's project had a fundamental or substantial epistemological component would have to be weakened with respect to the picture that passages like the above suggest. Addressing this issue in any capacity here would require too large a detour, so we shall not commit to the idea that Frege actually meant something stronger. The passage above do seem to suggest, at least, a stronger epistemological emphasis, so we can refer to this 'picture of logicism' as at least inspired in or suggested by Frege's works.
[206] WITTGENSTEIN, L., 1953, p.1, §1.
[207] Cf. Again, cf. IRVINE, A., 1989.

for Mathematical Analysis effected in the Nineteenth Century - but he thought of it in a much more broad and ambitious way than Frege ever did. For Russell, the results obtained by Weierstrass and his successors showed that the theory of natural numbers could be used as the ultimate foundation upon which Real Analysis rests: that the whole of Pure Mathematics could be grounded on or reduced to Arithmetic - and at some points he even thought this could be extended to portions of Mechanics and Mathematical Physics[208]. Second, the crucial step towards establishing Logic as a foundation of Arithmetic was conceived - at least concerning its execution - precisely in the same way in which it was conceived by Frege: the goal Russell sets himself to achieve in the opening of the *Principles*, namely: to prove "[...] that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical concepts"[209] is the same goal envisioned by Frege. There was, in both the *Principles* and in *Principia*, a *technical* objective that was shared with Frege's work.

But there are stark differences in the Philosophical motivation for their projects which are, almost as a rule, neglected by interpreters of Russell. One important exception is Andrew Irvine who, over two decades ago, called attention to the crucial differences between Epistemic Logicism and Russell's Logicism[210]. What mainly differentiates Russell's conception of Logicism as realized in *Principia* and Frege's, as realized in his *Grundgesteze*, is that the former was written as a response to the difficulties that showed the latter to be a failure. It is because of Russell's paradox[211] that the Russellian project of developing Arithmetic as Logic acquired a completely distinct character[212] from Frege's (or at least as Frege is generally understood[213]). Epistemic Logicism, as Irvine puts it, aims at "[...] reducing "the problem of justifying mathematical belief" to "[...] the comparatively easier problem of justifying the self-evident principles of logic"[214]. This goal, which Russell shared with Frege before the discovery of the contradiction was shown to be unachievable precisely because of that discovery.

But differently from Frege, who took the discovery as a fatal blow to the attempt to develop Arithmetic as Logic, in the case of Russell his journey towards this goal was barely beginning. As we discussed in the previous chapter, after the publication of the *Principles*, Russell worked steadfastly in the search of a satisfactory logical theory which could serve as a founda-

208 It is worthile to observe, yet again, that in the *Principles* Russell's opposition to Kant was not regarding the thesis that Mathematics require some form of intuition - Russell's complaint was that Kant did not realize that the only form of intuition involved in the knowledge of (pure) Mathematics is *logical* intuition. Also, the points made in previous footnote 410 of chapter 2 should be kept in mind here.

209 RUSSELL, B., 1903, p.xiii.

210 Cf. IRVINE, A., 1989

211 And, of course, the many other antinomies and paradoxes that later caught Russell's attention in the course of development of his many logical theories.

212 This was already indicated in the *Principles*, during the writing of which Russell discovered his paradox, since there Russell acknowledges that he had "failed to perceive any concept fulfilling the conditions requisite for the notion of class"(RUSSELL, B., 1903, p.v-vi).

213 See previous footnote 205.

214 IRVINE, 1979, p.305. A similar view is described aptly by Susan Haack as the attempt to make mathematical propositions acquire "innocence by association" with logical ones (HAACK, 1978, p.10).

tion for Mathematics and solve the contradictions and, as we saw, this was no trivial task. We saw that the general issue that was brought up by the contradiction was the problem of salvaging what Russell viewed as the more fundamental achievements of Cantor, Dedekind, Peano and Frege once the general naive assumption that every propositional function defined a class (or an extension conceived as an object) was shown to be contradictory. Of course, Russell concluded that the fault was with the naive notion of class or extension, not with Mathematics; and indeed, as we saw, Russell shortly concluded that the very conception of an extension conceived as a kind of abstract object had no place in Mathematics and that discourse about classes should be paraphrased in terms of the more primitive vocabulary of Logic[215]; but this, in turn, raised another fundamental issue: how can complex logical theories which deal with this difficulty serve as a *foundation* for the propositions of ordinary Arithmetic which seem to have a much less problematic epistemic status? As Russell puts it:

> There is an apparent absurdity in proceeding, as one does in the logical theory of arithmetic, through many rather recondite propositions of symbolic logic, to the "proof" of such truisms as 2 + 2 = 4: for it is plain that the conclusion is more certain than the premises, and the supposed proof therefore seems futile.[216]

Ignoring, for the moment, the obvious observation that there is much more going on in a treatise like *Principia* than the proof of such truisms, it is plain that those who endorse a logicist analysis of Mathematics must have an answer to this question. Or, at the very least, an answer is needed if the analysis is to give any insight into the nature of mathematical knowledge.

An idea which was more or less implicit and only superficially developed in the works of Frege and Russell *before* the discovery of Russell's Paradox is the idea that *self-evidence* could answer this question: that despite being laws which are discovered much later in the axiomatic development of a science, the laws of Mathematical Logic enjoy the greatest degree of self-evidence; but for Russell, the contradiction in Frege's system showed this idea to be as untenable as the naive conception of set. Indeed, by the time *Principia* was done, self-evidence became *almost irrelevant* as a criterion for an axiom to be accepted as primitive or fundamental:

> But in fact self-evidence is never more than a part of the reason for accepting an axiom, and is never indispensable. The reason for accepting an axiom, as for accepting any other proposition, is always largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it. If the axiom is apparently self-evident, that only means, practically, that it is nearly indubitable; for things have been thought to be self- evident and have yet turned out to be false. And if the axiom itself is nearly indubitable that merely adds to the inductive evidence derived from the fact that its consequences

215 Like that of Russell's Substitutional Theory or *Principia*'s No-Class Theory.
216 RUSSELL, 1907, p.272.

> are nearly indubitable: it does not provide new evidence of a radically different kind. Infallibility is never attainable, and therefore some element of doubt should always attach to every axiom and to all of its consequences. In formal logic, the element of doubt is less than in most sciences, but it is not absent, as appears from the fact that the paradoxes followed from premises which were not previously known to require limitations.[217]

So, in order to answer the above question, namely "what is the point of proving that 2 + 2 = 4 through recondite laws of logic", Russell came to rely on a methodology that became known as the *regressive method* of discovering premises in Mathematics[218].

This method consists in a two-stage analysis of mathematical theories whose concepts are vague, imprecise, confused or even contradictory and whose propositions (as well as the deductive relations that hold among them) could, consequently, be better articulated. The first stage consists in the identification of a minimal cluster of concepts and axioms for the theory; the second consists in the reconstruction of the theory starting from this minimal group of primitive concepts and axioms, in order to obtain the greatest number of results of the original theory, while ridding it of its conceptual and logical defects[219].

Thus, Russell took the logicist enterprise as standing in a field of investigation that has a very peculiar nature: it presupposes data to analyze, namely, Mathematical Theories - but its main essential purpose is neither to discover new truths deducible from the theory nor to provide reasons to believe the theory to be true (though, of course, nothing excludes the possibility that one of these things can eventually happen as a result of logical investigations). Rather, it proceeds in the opposite direction of discovering the premises of Mathematics - and, in fact, *this* field of inquiry was the discipline which, in fact, Russell called *Mathematical Philosophy*:

> Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analyzing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterizes mathematical philosophy as opposed to ordinary mathematics.[220]

[217] WHITEHEAD & RUSSELL, 1925, p.59 [1912, p.62].

[218] Russell generally spoke of the regressive method of discovering the premisses *of* Mathematics (for instance, RUSSELL, B, 1907), but this seems to convey a narrower idea of what his methodology consists in; the point is not merely that the premisses *of* Mathematics can be discovered via the logistic investigation, but also that this investigation itself is a (more or less) typical mathematical activity.

[219] RUSSELL, 1907, p.272.

[220] RUSSELL, B., 1919a, p.1.

From a technical point of view, the goal of this enterprise was still akin to that originally set out by Frege: to define mathematical notions in terms of a logical vocabulary and prove the basic principles of mathematics assuming only logical axioms. But, as Irvine has emphasized long ago[221], Frege's 'Epistemic' conception of Logicism differs radically from that the epistemological outlook behind the regressive conception or methodology which underlies Russell's Logicism.

Russell's methodology greatly approximates the natural and the mathematical sciences, since the regressive investigation can only provide *inductive* support to the most basic principles of Mathematics[222]:

> We tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true. But the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science.[223]

And it was this epistemological outlook that was behind the development of Mathematics of *Principia Mathematica*. Russell and Whitehead were quite explicit in affirming that most of the conclusions that followed from their axioms have a greater "degree of self-evidence" than the axioms themselves, and that the possibility of deriving such conclusions was the main reason to adopt the axioms of their system, as indicated in the quote above[224].

Thus, *Principia* was never meant as a defense of "Epistemic Logicism" a thesis that may also fairly be called "Classical Logicism" and that has strong foundational undertones, as attested in the quotes from Frege's works above. But despite that, many authors take for granted that Russell's attempt to derive Mathematics from *Principia*'s axioms aims at making ordinary Mathematics more certain, secure or well-founded than it was before. This *may* be correct regarding Frege's goal with his logical investigations, but it is quite inadequate regarding Russell.

Indeed, in the context of Russell's general philosophical outlook, the inverse is actually the case: a logical system is to be regarded as more certain or well-founded according to how much ordinary mathematics can be developed within the system. Of course, were it not for the

221 IRVINE, A., 1989.

222 RUSSELL, 1907, p.274.

223 RUSSELL, 1907, p.273.

224 Similar remarks are scatterd throughout *Principia*'s informal observations. In the preface, for instance, we find: "The chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e. it must lie in the fact that the theory in question enables us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premises because true consequence follow from them, than for believing the consequences because they follow from the premises."(WHITEHEAD & RUSSELL, 1925, p.v [1910, p. v]).

contradictions, Frege's system would have been a paragon for logicism: Frege's construction allowed him to prove something Russell could not prove, that there are infinitely many finite cardinal numbers. But in virtue of the discovery of the contradictions at the foundations of his theory of extensions, such as Russell's paradox, Frege's system - and naive set theory, in general - had to be abandoned.

One aspect of Russell's regressive conception of Logicism that cannot be understated is that for him it is not the business of the Mathematical Logician as such to determine whether Mathematics as a whole or some of its portions are true or should be taken as true. The business of Mathematical Logic and the goal of logicism is to analyze mathematical theories, by (a) clarifying which propositions are necessary (and sufficient) for their development; and (b) purging them from possible contradictions. But the amount of Mathematics that can be derived from the theory is not an absolute standard of evaluation: it must be weighted against the threat of contradiction. In *Principia*, this is stated clearly in the very first page of the work, where Whitehead and Russell put among their main objectives:

(a) "[...] effecting the greatest possible analysis of the ideas with which it deals and of the processes by which it conducts its demonstrations, and at diminishing to the utmost the number of the undefined ideas and *primitive* propositions"[225]; and

(b) "[...] solve the paradoxes which, in recent years, have troubled students of symbolic logic and the theory of aggregates"[226].

There is no foundational claim in *Principia* because the work was not meant to be part of any foundationalist program in the Foundations of Mathematics. *Principia* is mainly a work in *regressive* Mathematics which aims to establish that Pure Mathematics leads back to purely logical assumptions.

Russell also emphasizes this aspect of his thought even more in a paper of the mid-twenties, *Logical Atomis*m, which is often neglected, despite being of first-rate importance: it puts *Principia* and many of Russell's technical achievements within the context of his broader views on Mathematical Philosophy (and Philosophy of Science in general: we find in that paper the idea that Science in general deals with structure, to be later developed in the monumental *Human Knowledge: Its Scope and Limits*[227]). There Russell emphasizes yet again the regressive method and is adamant about the importance of distinguishing the *epistemic* and *logical* dimension of the work in Mathematical Philosophy:

225 WHITEHEAD & RUSSELL, 1925, p.1 [1910, p.1].
226 WHITEHEAD & RUSSELL, 1925, p.1 [1910, p.1].
227 RUSSELL, B., 1948.

> When pure mathematics is organized as a deductive system - i.e. as the set of all those propositions that can be deduced from an assigned set of premises - it becomes obvious that, if we are to believe in the truth of pure mathematics, it cannot be solely because we believe in the truth of the set of premises. Some of the premises are much less obvious than some of their consequences, and are believed chiefly because of their consequences. This will be found to be always the case when a science is arranged as a deductive system. It is not the logically simplest propositions of the system that are the most obvious, or that provide the chief part of our reasons for believing in the system. [...] the same thing happens in the pure realm of logic; the logically first principles of logic—at least some of them—are to be believed, not on their own account, but on account of their consequences. The epistemological question: "Why should I believe this set of propositions?" is quite different from the logical question: "What is the smallest and logically simplest group of propositions from which this set of propositions can be deduced?" Our reasons for believing logic and pure mathematics are, in part, only inductive and probable, in spite of the fact that, in their logical order, the propositions of logic and pure mathematics follow from the premises of logic by pure deduction. I think this point important, since errors are liable to arise from assimilating the logical to the epistemological order, and also, conversely, from assimilating the epistemological to the logical order. The only way in which work on mathematical logic throws light on the truth or falsehood of mathematics is by disproving the supposed antinomies. This shows that mathematics may be true. But to show that mathematics is true would require other methods and other considerations.[228]

What are these "other" considerations involved in showing that the propositions of Mathematics are actually true? Well, what is required is a criterion for determining whether there are logical grounds for believing the truth or falsity of propositions like the Axiom of Infinity and the Multiplicative Axiom or whether such propositions should be considered as purely empirical and, hence, outside of the scope of Pure Mathematics and part of *Applied* Mathematics.

5.9 Infin Ax and Russell's Conception of Pure Mathematics

In preface to the second edition of the *Principles of Mathematics*, written three decades after its original publication, Russell states:

> The fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never since seen any reason to modify.[229]

Almost than a decade *earlier*, in the closing chapter of his *Introduction to Mathematical Philosophy*, we find the following passage:

> If there are still those who do not admit the identity of logic and mathematics, we may challenge them to indicate at what point, in the successive definitions and deductions of *Principia Mathematica*, they consider that logic ends

[228] RUSSELL, B., 1924, p.129-130. Irvine, who long ago called attention to this important and often neglected passage notes that it also "shows how closely Russell's mathematical epistemology was integrated within his general theory of knowledge" (IRVINE, A., 1989, p.316).

[229] RUSSELL, B., 1937, p.xxxi.

> and mathematics begins. It will then be obvious that any answer must be quite arbitrary.[230]

There are authors, however, who answered Russell's challenge by pointing at *Principia*'s section ∗120, specifically, ∗120·03 which defines the axiom of infinity. William and Martha Kneale, for instance, claim that since "the axiom cannot be described as a truth of Logic in any reasonable use of that phrase", its use in *Principia* "amounts in effect to the abandonment of Frege's project of exhibiting arithmetic as a development of logic"[231]. There are others who endorse the idea that Russell simply abandoned Logicism (without realizing, apparently) in favor of a view that became known in the literature as "If-Thenism". According to this view, every mathematical (or arithmetical) truth is derivable from logical axioms plus non-logical (existence) axioms such as the Axiom of Infinity. Alan Musgrave, for instance, claims that Russell retreated to such a position:

> By using the If-thenist manoeuvre, Russell arrives at a position which is far-removed from his original logicism. The claim that all so-called mathematical axioms can be deduced from logical axioms (thesis (B) of old-style logicism) is weakened to read: (B*) either an apparently primitive proposition of mathematics can be deduced from logical axioms or it is not to be regarded as a primitive proposition at all but only as the antecedent of various conditional statements [...] It turned out, in fact, that only a fragment of arithmetic (finite arithmetic) could be 'reduced to logic' in the way that old-style logicism demands. The rest of mathematics could be 'reduced to logic' only if the If-thenist manoeuvre was applied to it first[232].

But this is a complete trivialization of Logicism – for it can be applied to any branch of knowledge whatsoever formulated as a deductive system[233]. Even authors that are a little more sympathetic to Russell's work and to the logicist project in general, like George Boolos, tried to understand Russell in a more charitable way, realizing that he did not treat Infin Ax as a genuine axiom, but only took it as an antecedent to ordinary theorems (such as Peano 3) where necessary[234].

But since this in itself does not account for how Russell and Whitehead understood the need for the axiom and how it affects the claim that Arithmetic is a branch of Logic, even more charitable readers like Boolos remained puzzled: seeing an unresolvable tension between passages like the one from *Introduction to Mathematical Philosophy* quoted above[235] and the use of Infin Ax, Boolos concluded that *Principia* was *not*, after all, a work meant to "vindicate full-

230 RUSSELL, B., 1919a, p.194-5.
231 KNEALE & KNEALE, 1984, p.669.
232 MUSGRAVE, A., 1977, p.112. See also PUTNAM, H., 1975, p.20.
233 LANDINI, G., 2011*a*, p.97.
234 This much is certainly correct: it follows from the fact that the so-called *Axiom* of Infinity is simply not introduced as an axiom in *Principia*, as we already observed.
235 RUSSELL, B., 1919a, p.194-195.

fledged Logicism"[236]. His explanation for Russell's more bold remarks is that they should be read as "careless, if not propagandistic"[237]. But if Boolos's assessment is correct, Russell expressed himself quite inconsistently for quite some time, and what is even worst, apparently without even noticing it. This, although better than the conclusion that Russell adopted *if-thenism*, is still highly unsatisfactory. The root of the problem is that Boolos, like Musgrave, Kneale & Kneale and many others, does not consider the possibility that Russell's conception of Logicism is not the one they take for granted.

Russell's endorsement of Logicism long after *Principia*'s publication is not "careless" or "propagandistic".[238]. Russell never abandoned Logicism, nor did he change his conception of it: it was always the thesis according to which all the laws of pure mathematics are laws of logic. What happened is that Russell was led to revise his conception of pure mathematics. If we go back to where we started the present work, to the Logicism of the *Principles*, we find the laws of Pure Mathematics defined as the class of all *true* propositions *r* of the form "*p* implies *q*", where *p* and *q* are *propositional functions* containing any number of variables $x, ..., z$ and *r* asserts that *p* implies *q* is true for all the values of $x, ..., z$[239]. What Russell came to realize is that this definition is, in a sense, too broad. In the preface to the second edition to the *Principles* - the same place where Russell claims *he never abandoned* the logicist thesis - we find:

> [...] the absence of non-logical constants, though a necessary condition for the mathematical character of a proposition, is not a sufficient condition. Of this, perhaps, the best examples are statements concerning the number of things in the world. Take, say: "There are at least three things in the world". [...] This statement can be enunciated in purely logical terms, and it can be logically proved to be true of classes of classes of classes: of these there must, in fact, be at least 4, even if the universe did not exist. For in that case there would be one class, the null-class; two classes of classes, namely, the class of no classes and the class whose only member is the null class; and four classes of classes of classes, namely the one which is null, the one whose only member is the null class of classes, the one whose only member is the class whose only member is the null class, and the one which is the sum of the two last. But in the lower types, that of individuals, that of classes, and that of classes of classes, we cannot logically prove that there are at least three members. From the very nature of logic, something of this sort is to be expected; for logic aims at independence of [sic] empirical fact, and the existence of the universe is an empirical fact. It is true that if the world did not exist, logic-books would not exist; but the existence of logic-books is not one of the premises of logic, nor can it be inferred from any proposition that has a right to be in a logic-book.[240]

The reason for this change should be evident by now: because of the contradictions,

[236] BOOLOS, G., 1998, pp.271.
[237] BOOLOS, G., 1998, p.272.
[238] BOOLOS, G., 1998, p.272. This somewhat unfortunate remark is found in an otherwise insightful and instructive paper on *Principia*'s development of Arithmetic and the theory of infinite cardinal numbers. We shall have occasion to discuss it in some detail below.
[239] RUSSELL, B., 1937 [1903], p.1 §1. Here we are using the paraphrasis provided in our first chapter, section 1.3.4.1.
[240] RUSSELL, B., 1937, p.vii-viii.

Russell was driven to eliminate more and more the ontological commitments to so-called logical objects. As we saw, the abandonment of the Substitutional Theory marked also the abandonment of a conception of Logic fully committed to *proving* that there are infinitely many *entities*. This is why *Principia* has a no-class theory *and* a no-proposition theory. What Russell concluded, then, is not that logicism as he understood it is false, but that results whose truth which depends on the existence of particulars - either abstract or concrete - do not belong to pure mathematics.

One might argue that this is circular - that Russell would seem to be simply *defining* Pure Mathematics as what can be derived from Logic and then declaring logicism to be true. But this would be a gross oversimplification of his view. As Gregory Landini, and following him, Kevin Klement have argued[241], the way out of the puzzle is not merely to realize that Russell did not assume the axiom of infinity as a proper axiom in *Principia*'s theory of Arithmetic, but also that Russell did not take it as axiomatic to Logicism to show that propositions involving existential claims about numbers, like the third Peano Postulate, are logical truths. In fact - perhaps in a rather hasty way[242] - Russell explicitly recognizes that existential assumptions involved in theorems like Peano 3[243] are *empirical*:

> [...] such existence-theorems, with certain exceptions, are, I should now say, examples of propositions which can be enunciated in logical terms, but can only be proved or disproved by empirical evidence[244].

The reason for this is that Russell gave up the idea that it is the business of Logic to guarantee the existence of an infinity of objects like classes or propositions[245].

For Russell, one of greatest achievements made with the use of the regressive method was Peano's and Dedekind's 'reduction'[246] of the indefinable notions of Arithmetic to the three notions of zero, number and successor and of the primitive axioms to the five propositions we

241 LANDINI, G.,2011, p.193; KLEMENT, K., 2013*b*, p.156.

242 Cf. previous footnote 196 of the Introduction on Landini's and Elkind's forthcoming papers on alternatives to *Principia*'s Axiom of Infinity.

243 Recall that we following the numbering of the Peano postulates which Russell employs in *Introduction to Mathematical Philosophy* (Cf. RUSSELL, B., 1919*a*, pp.5-6.

244 RUSSELL, B., 1937, p.viii.

245 Here it must be noted that the issue may lay in our epistemology of Logic which may be too weak and not necessarily with the contingent status of the Axiom of Infinity, i.e., it may be a logical necessity that there are infinitely many entities of the lowest simple type but even this were a logically necessary truth, it may be unknowable to us. Thanks for Landini for calling my attention to this point.

246 Cf. RUSSELL, B., 1903, p.111-12 §107. Now, one must not be misled by Russell's praise of the achievement of Dedekind and Peano, however, into thinking that Russell accepted the widely held view the 'reduction' leads to a theory of natural numbers understood as objects, or as Landini puts it, abstract particulars - Russell did not; on the contrary, he ended up rejecting the view shared by Peano, Dedekind and Frege that numbers are objects; the fundamental merit of the reduction, for Russell, was one of conceptual economy and of a better articulation of Arithmetic and other branches of Pure Mathematics as axiomatic theories. Also concerning this latter point, one must not be misled by the weak idea of reduction we are employing here into thinking that Whitehead and Russell accepted that every branch of Pure Mathematics - including Geometry - is simply 'reduced' to the theory of natural numbers; again, on this point cf. GANDON, S., 2008 and 2012.

nowadays call Peano Postulates. Of course, the attempt by Frege and (and later, Russell) to define Peano's primitive notions and prove his primitive propositions on a purely logical basis was a further, even more important, step. And until Russell's discovery of his contradiction, naive set theory seemed to provide the basic framework upon which Peano's Arithmetic could be built upon. But the contradiction showed that the principles of that theory were in need of thorough revision. This revision led - through and a long and arduous road - to the theory of types which made him reconsider the claim made in the *Principles*[247] that the *existence*-theorems of mathematics - those that involve the axiom of Infinity (and Choice which was not yet discovered) - could be proved on a logical basis.

Perhaps the best source for appreciating this is a lecture Russell delivered to the *Société Mathématique de France*, in March 1911[248]. In the paper Russell asserts that "[...] pure mathematics can be expressed and proved entirely in terms of ideas and axioms of logic"[249], but tempers this claim with the following qualification:

> [...] a reduction to logic can only be obtained if we cease to demand with too much insistence that there exist objects which verify the hypotheses whose consequences we are considering. It will happen occasionally that such objects can be constructed *a priori*; for example, we can construct *a priori* a class having a finite number of arbitrary terms. [...] But most existence theorems (which, by the way, are not necessary for the truth of other theorems, but only for their importance) require data which are not purely logical. In pure mathematics, two existence axioms provide just about all the existence theorems one might want. These two axioms are: I. The axiom of infinity; 2. The multiplicative axiom, otherwise known as Zermelo's axiom.[250]

He also goes on to say that "these two axioms cannot be proved using logic alone" and that they "are not intuitively obvious".[251]. Yet, this does not mean that Russell abandoned logicism - either unknowingly or otherwise.

Russell came to the conclusion that there are theorems of ordinary Mathematics- like those that assert the existence of infinitely many inductive cardinal numbers - which are *not* propositions of Pure Mathematics, because they have *empirical* presuppositions[252]. Thus, what Russell is attesting here is that axiom of infinity is "purely empirical"[253]:

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[247] RUSSELL, B., 1903, pp.497-8 §474.

[248] This is a very important and instructive piece that was not made widely available until Grattan-Guinness published a translation of it in his splendid commentary of the Russell-Jourdain correspondence (RUSSELL, B., 1911*c*, p.41-53; GRATTAN-GUINNESS, I., 1977, pp.161-174.). Grattan-Guinness's translation is more precise than that given in the *Collected Papers* so we provide references to it jointly with that of the *Papers*.

[249] RUSSELL, B., 1911*c*, p.43.

[250] RUSSELL, B., 1911*c*, p.43; GRATTAN-GUINNESS, 1977, p.162-3

[251] RUSSELL, B., 1911*c*, p.43; GRATTAN-GUINNESS, 1977, p.162-3

[252] Again, for possible alternative approaches for addressing this, see previous footnote 196 of the Introduction on Landini's and Elkind's forthcoming papers on alternatives to *Principia*'s Axiom of Infinity.

[253] RUSSELL, B., 1911*c*, p.52; GRATTAN-GUINNESS, 1977, p.173.

> For any ν, whether is a finite or an infinite cardinal, it is *a priori* possible that it is the number of individuals in the universe. But, according to empirical evidence, given the divisibility of physical objects, it appears artificial to suppose that there is a finite number of individuals in the universe. It does not seem to me that the empirical data are sufficient to prove that the number of individuals is not finite, but they suffice to show that the finitist hypothesis is much more difficult and less simple than the other. And the logic of the infinite shows that the finitist hypothesis is not at all preferable *a priori*. I conclude from this that, for the reasons which usually decide scientific hypotheses, it is better to assume that the number of individuals is infinite, while keeping in mind, however, that this hypothesis could be false, even though the sensible evidence is such that one could never know that it is false.[254]

In fact, when Russell observes above that finite classes can be constructed *a priori*, he cautiously qualifies this by noting that this can be done "*only* if we admit as *a priori* the axiom that there exists at least one object, or some equivalent axiom"[255]. Making even this assumption, Russell would later observe, is a "defect in logical purity"[256]. The reason for this is that such axioms have an ontological import of *individuals*, *i.e*, "beings of the actual world" which by the time of *Principia*'s publication Russell had recognized as the mark of the non-logical.

Now, it is worth pointing out that Russell was "perceptive"[257] , as Grattan-Guinness puts it, of important differences between the Multiplicative Axiom and the Axiom of Infinity. To begin with, Russell did not view the Axiom of Choice and its equivalent, the Axiom of Choice as *empirical* hypotheses:

> There is a difference between the axiom of infinity and the multiplicative axiom which is important from the vantage point of the theory of knowledge. The multiplicative axiom has the form and the nature of the axioms of logic, for there can be no empirical demonstration of its truth. The considerations which bear on the truth or falsehood of the multiplicative axiom are considerations of logic, *a priori* considerations.[258]

Russell also thought that "there is no reason to believe that the multiplicative axiom is true, whereas, for empirical reasons, it appears probable that the axiom of infinity is actualized in the real world"[259]. Russell also thought that the need for the Axiom of Infinity was a result of *Principia*'s type distinctions and he observed that it was perfectly possible that "by modifying the theory of types, the axiom of infinity would become unnecessary"[260].

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[254] RUSSELL, B., 1911*c*, p.52; GRATTAN-GUINNESS, 1977, p.173.
[255] RUSSELL, B., 1911*c*, p.43; GRATTAN-GUINNESS, 1977, p.162-3.
[256] RUSSELL, B., 1919a, p.205, footnote 1.
[257] GRATTAN-GUINNESS, I., 1991, p.109.
[258] RUSSELL, B., 1911*c*, p.52; GRATTAN-GUINNESS, 1977, p.173.
[259] RUSSELL, B., 1911*c*, p.53; GRATTAN-GUINNESS, 1977, p.174.
[260] RUSSELL, B., 1911*c*, p.52-3; GRATTAN-GUINNESS, 1977, p.173-4. Whether Russell was *strictly* right about this, however, is certainly debatable. As already noted (cf. footnote 196 of the Introduction): Landini argues in a forthcoming paper that what *Principia* requires is a new - philosophically motivated - axiom which allows the proof of Peano 3 in a sufficiently high type, not a modification in the simple type structure; Elkind, on the

Still, Russell thought both ‘axioms’[261] should be used *only* as explicit hypotheses[262], albeit for different reasons. The Axiom of Infinity is used only as a hypothesis because at that point Russell viewed it as an empirical assumption, and thus, as part of *applied* Mathematics.[263] The Multiplicative Axiom is considered as a hypothesis because it has no intrinsic plausibility and is employed and assumed only on the basis of its desirable consequences.

5.10 Numbers as Logical Constructions and Logic as the Science of Structure

The appreciation of the difference in epistemological outlook between Classical logicism and its Russellian counterpart opens the way for a different interpretation of the role of Infin ax in *Principia*. And, in fact, it allows for a better understanding of Russell’s Mathematical Philosophy in general. The important difference we just discussed can be summarized in the following: for Russell, the regressive approach of discovering premises in mathematics does not start from a privileged set of primitive notions and axioms that have a privileged epistemic status to secure a firmer foundation for Mathematics (or Arithmetic, in particular) but proceeds in the opposite direction, by lending credibility to the primitive propositions through the derivation of ordinary Mathematics.

Thus, in somewhat rough terms, we may say that the classical logicist wants to provide a logical foundation for (almost) all of ordinary Arithmetic, while Russell aimed at *reconstructing* Arithmetic within a logical framework. The classical logicist assumes from the outset that some propositions (like Peano postulates) must be provable from logic alone. In a way, the aim of Russell’s logicist project, as presented in *Principia Mathematica*, was a more modest one, namely, to determine how far one can go reconstructing Mathematics in terms of logical axioms and definitions without being guided by the assumption that numbers must be logical objects - or, as Landini puts it, without being guided the traditional “metaphysical” view that numbers are objects or abstract particulars[264]. And this objective is not incompatible with the discovery that additional hypothesis such as Infin ax. are required to fully develop mathematics. As Kevin Klement has aptly put it:

> [...] Russell’s project was a reconstructive one; his aim was not to preserve all aspects of the original ways of conceiving and talking about mathematical so-

other hand, puts forward a modified version of simple-type theory which allows both ascending and descending simple types which, in turn, allows a proof of infinity when a suitable axiom - with quite a different import than *Principia*’s Infin Ax - is added to the system.

[261] Again, recall that ‘Infin Ax’ is a defined statement, not a proper axiom and the same applies to ‘Mult Ax’.

[262] RUSSELL, B., 1911, p.53; GRATTAN-GUINNESS, 1977, p.174

[263] Again, whether Russell was right about this is certainly debatable, cf. previous footnote 196.

[264] Cf. LANDINI, G., 2011*b*.

> called entities. If a different logical form better represents the form of the actual fact hinted at by the original doctrine, it is to be preferred[265].

So, again, this "reconstruction" is *not* guided by any assumption that Logic must provide an ontology of numbers as objects and a proof that there must infinitely many of these objects. To see how coherent this position is, we may compare Russell's attempt at reconstructing Mathematics in terms of Logic to the attempt (which Russell also indulged at some point[266]) to reconstruct the discourse about ordinary physical objects in terms of perceptual vocabulary, that is, as someone who maintains that physical objects are conjectures or constructions based on our acquaintance with sense-data, that is, as someone who maintains that the common sense theory of physical objects can be reconstructed from a theory of sense-data. Is it enough to point out that the existence of ordinary physical objects does not follow from the existence of sense-data in order to refute this position? Of course, the answer must be no, unless one assumes that the existence of physical objects *must* be taken as a fundamental theoretical assumption or a consequence of one.

The attempt to construct a theory of physical objects as a theory about sense data does not (and should not) presuppose that every single proposition we assume true about physical objects can be deduced from true propositions about sense data, but just establishes how far a theory about physical objects can be carried out assuming only truths about sense-data as the "material" for constructing physical objects. We have an analogous situation in the case of the derivation of the Peano postulates in *Principia*'s theory of types. Russell's logicism cannot be considered a failure for not being able to prove some particular arithmetical proposition (albeit an important one from the perspective of ordinary mathematics), because its aim is to reconstruct arithmetic in another framework that does not presuppose the truth of any arithmetical proposition and, more importantly, one that does not assume an ontology of mathematical *objects* on a par with individuals (of any simple type).

We saw that this approach is motivated by a concern with consistency which is inseparable from concerns with ontological commitments to some sort of abstract objects - or as Landini prefers to call them, abstract particulars: invariably, the threat of paradox with which Russell struggled sprang from assuming some sort of ontology - be it of classes, so-called 'propositional functions' (that is properties and relations-in-intension that are type free) or propositions - that was too broad. For Russell, the last straw in this regard was the discovery of the paradoxes of the substitutional theory.

When Russell was committed to an ontology of propositions (as non-linguistic entities) he was able to prove something like the axiom of infinity, by "manufacturing $\aleph_0$ entities"[267]. This "manufacturing" was done as follows. First, Russell assumed that for any entity *a*, there

265 KLEMENT, K., 2013*b*, p.156.
266 RUSSELL, 1914 is, of course, the *locus classicus* of such an enterprise.
267 LACKEY, D., 1973, p.203

exists the true proposition that asserts $a = a$; but given the fact that no proposition can occur within itself (which was guaranteed by the restrictions of the theory of propositions he held at the time[268]), *a* must not be the proposition that asserts $a = a$. He concluded, therefore, that there exists, besides the entity *a*, another entity *p* that asserts that the equality holds. Once this reasoning is allowed (and it must be allowed once an ontology of propositions is allowed) it can be proved that, given any finite number *n*, the number of entities that exists is not *n*, taking $p_0 = (a = u)$ and $p_{n+1} = (p_n = u)$ and showing by induction that each p_n is a distinct entity from p_{n+1}. From this it follows that there must be a Dedekind-infinite totality of entities distinct from each other, and thus, that there are $\aleph_0$ entities from logic alone. However, as we already discussed, by the time of *Principia Mathematica* Russell had already abandoned any sort of untyped ontology of propositions, propositional functions or classes taken as *objects*. Thus, this proof was simply not available anymore. Russell wrote to Jourdain in June of 1907 explaining the situation:

> Consideration of the paradox of the liar and its analogues led me to be chary of treating propositions as entities. I therefore no longer regard as valid the proof of the existence of $\aleph_0$ entities by putting
>
> $p_0 = (u = u)$ [Df], $p_{n+1} = (p_n = u)$ [Df]
>
> [...] I now think that existence-theorems beyond the finite require a definite assumption that the number of entities is not finite. I put:
>
> Infin ax $.{=}.\ \alpha \epsilon \mathrm{NC} \,.\, \supset_\alpha \exists\,!\, \alpha(\mathrm{Indiv}))$ Df where
>
> NC Induct = $\hat{\alpha}\{0 \epsilon \beta \,.\, \gamma \epsilon \beta \,.\, \supset_\gamma . \gamma + 1 \epsilon \beta : \supset_\beta . \alpha \epsilon \beta\}$ Df
>
> and α(Indiv) means "an α composed of individuals" (as opposed to classes, relations, etc.). All existence-theorems that can reasonably be expected to follow from *Infin ax*; i.e. ω, η, $\aleph_0$, $2^{\aleph_0}$, $\aleph_v$ (for finite *v*), etc. I do not assume *Infin ax.* to be true but state it explicitly as hypothetical wherever I use it. It goes with *Mult ax*, which together make up the two essentially arithmetical axioms.[269]

The logicism of the *Principles* was an attempt to derive Mathematics from a logical theory that was committed to classes and propositions. The logicism of the substitutional era was an attempt to reconstruct Mathematics within a logical theory that was committed to a realist theory of propositions which was capable to emulate a theory of classes and relations-in-extension. But by the time of *Principia*, both *classes* and *propositions* were abandoned as genuine entities. How, then is the reconstruction of Mathematics in the work to be understood?

Following Landini we claim that "[…] *Principia* is following out the implications of the new Cantorion notion of number, short of any commitment to an ontology of classes"[270]. According to this reading, Russell's logicism aims at reconstructing arithmetic as Cantorian set theory short of a commitment to sets as entities on par with individuals (of whatever type).

[268] LACKEY, D., 1973, pp.203-4.
[269] GRATTAN-GUINNESS, I., 1977, p.105-6.
[270] LANDINI, G., 2016, p.43.

Numbers are constructed in terms of relations of one-one correspondence. This reconstruction shows that an infinity of individuals must be taken as a hypothesis in order to secure that the third Peano postulate holds.

Of course one can criticize, refute, or question the attempt to reconstruct some theory in terms of another. One can question the legitimacy of basing a theory of physical objects on a theory of sense-data by questioning the obscure character of these entities. That is a legitimate criticism which, in fact, could be made regarding the status of classes as logical constructions in Russell's theory of types. For instance, one may easily imagine a Platonist who is convinced of the intrinsic plausibility of, say, the Iterative Conception of Set[271] arguing that the notion of a set *as an entity* must be taken as fundamental, for one reason or another. However, simply pointing out that something which we generally take as a truth or that we think should be taken as a truth cannot be derived from the theory without further assumptions is not a legitimate criticism in this context. Pointing out that Peano 3 is not a theorem of *Principia* does not refute Russell's Logicism.

According to Russell's new conception of Pure Mathematics, theorems like Peano 3 belong - if true - to *applied* Mathematics and, as Landini has argued, Russell's position is that those who think otherwise are (perhaps, incorrectly[272]) committed to a (metaphysical) view of numbers as *objects*[273]. What led Russell to this conclusion, of course, was his gradual realization that Logic must work with the method of *construction* rather than *postulation*. This idea is encapsulated in Russell's methodological guideline of "substituting, wherever possible, logical constructions for inferred entities"[274], to which he sometimes referred as "*the* supreme maxim in scientific philosophizing"[275]. Again, in the very important and often neglected article *Logical Atomism*, Russell reviewed methodological aspects that guided, in general, his views in Mathematical Philosophy. In that text Russell describes this idea as "a heuristic maxim" that guided this gradual reduction of the ontological commitments of his Logic, and therefore, his conception of Pure Mathematics:

> One very important heuristic maxim which Dr. Whitehead and I found, by experience, to be applicable in mathematical logic, and have since applied to various other fields, is a form of Occam's Razor. When some set of supposed entities has neat logical properties, it turns out, in great many instances, that

[271] As presented, for instance, in BOOLOS, G., 1971 or POTTER, M., 2006.

[272] Cf. LANDINI, G., 2011*b*. Landini argues that the core of Logicism - which can, in a sense, be seen as a framework shared by Frege and Russell - is the idea that a correct analysis of the concept of number must be given in terms of the existence of one-one correlations between concepts or attributes conceived extensionally within the structure of the simple theory of types and he argues that, viewed in this light, the requirement that numbers must be viewed as objects is a "metaphysical prejudice" that can be dispensed with.

[273] Again, cf. LANDINI, G., 2011*b*, pp.199-204.

[274] The first explicitly formulated appearance of this "maxim" was in Russell's paper *On the Relation of Sense-Data to Physics* (RUSSELL, B., 1914*b*), originally written for a presentation at Johns Hopkins University on 20 April 1914 (cf. SLATER, J. (ed.), 1986, p.3).

[275] RUSSELL, B., 1914*b*, p.11. Our emphasis.

> the supposed entities can be replaced by purely logical structures composed of entities which have not such neat properties. In that case, in interpreting a body of propositions hitherto believed to be about the supposed entities, we can substitute the logical structures without altering any of the detail of the body of propositions in questions. This is an economy, because entities with neat logical properties are always inferred, and if the propositions in which they occur can be interpreted without making this inference, the ground for the inference fails, and our body of propositions is secured against the need of a doubtful step. The principle may be stated in the form: "Wherever possible, substitute constructions out of known entities for inference to unknown entities".[276]

Indeed, there are several applications of this "heuristic maxim" within Mathematical Philosophy throughout his works. As Russell recounts, the first time he applied the maxim was in connection with he called "the principle of abstraction", or "the principle which dispenses with abstraction", as he also observes in *Our Knowledge of the External World*[277]. Calling attention to a point he had already made in the *Principles*, Russell observes that "we are apt to infer that such [equivalence] relations arise from possession of some common quality" - but that such supposition is unnecessary since:

> [...] all the formal purposes of a common quality can be served by membership of the group of terms having the said relation to a given term. Take magnitude, for example. Let us suppose that we have a group of rods, all equally long. It is easy to suppose that there is a certain quality, called their length, which they all share. But all propositions in which this supposed quality occurs will retain their truth-value unchanged if, instead of "length of the rod *x*" we take "membership of the group of all those rods which are as long as *x*".[278]

This point, in fact, brings us back to Russell's Principle of Abstraction[279] that we discussed in the first chapter on the Logicism of the *Principles* and it is worth making a brief digression in order to discuss its role in *Principia*. In Part I, section C on one-may, many-one, one-one relations, specifically at number ∗72, entitled "Micellaneous Propositions", we find the following:

$$*72{\cdot}66 \qquad \vdash : S^2 \subset S \,.\, S = \breve{S} \,.\equiv.\, (\exists R) \,.\, R \,\epsilon\, \mathrm{Cls} \to 1 \,.\, S = R|\breve{R}$$

Which, for the sake of readibility, can be put less compactly as follows:

$$*72{\cdot}66' \qquad \vdash . \, xSy \,.\, ySz \,.\supset_{x,y,z} xSz : wSv \,.\equiv_{w,v}.\, vSw :\equiv ::$$

$$(\exists R) \,.\, R \,\epsilon\, \mathrm{Cls} \to 1 : xSy \,.\equiv_{x,y,z}.\, (\exists z) \,.\, xRz \,.\, yRz$$

276 RUSSELL, B., 1924, p.164.

277 RUSSELL, B., 1914, p.42.

278 RUSSELL, B., 1924, p.326-7.

279 I.e., the principle which asserts that any equivalence relation *R* can be analyzed into (or implies the existence of) a many-one relation *S* that is equivalent to the relative product of *R* and its converse $\breve{R}$

This is *Principia*'s version of the Principle of Abstraction and according to Whitehead and Russell, it "[...] embodies a great part of the reasons for our definitions of the various kinds of numbers"[280]. Let's see why.

Recall that in the *Principles*, the definition of the cardinal number of a class β as the class of all classes similar to β is given in terms of Russell's Principle of Abstraction: the principle which asserts that every transitive symmetrical relation[281] can be analyzed in terms of the relative product of a many-one relation *R* and its converse $\breve{R}$. Similarly, here, the principle states that "every relation which has the formal properties of equality, i.e. which is transitive and symmetrical is equal to the relative product of a many-one relation and its converse"[282]; again, the idea behind the proof is to show that if a symmetric and transitive relation *S* holds between *x* and *y*, there is a many-one relation (function) *R* and a term α such that *x* bears *R* to α and *y* has $\breve{R}$ to *y*, or, as the authors put it, that "whenever the relation *S* holds between *x* and *y*, there is a term α such that $xR\alpha \,.\, yR\alpha$, where *R* is a many-one relation"[283]. Before we discuss what role this principle plays in the justification of *Principia*'s definition, let's see very briefly how the theorem is proved. Since the proof is somewhat complex, mainly because the use of the condensed notation Cnv'$(\overleftarrow{S} \restriction \mathrm{D}`S)$ which makes it hard to read (and even harder to reproduce/typeset), and since parts of it are only sketched, it is a worthwhile exercise to reconstruct it. The principle, proposition $*72{\cdot}66$ is proved via the following chain of theorems:

$*72{\cdot}62$ $\quad \vdash : R \,\epsilon\, 1 \to \mathrm{Cls} \,.\, S = R|\breve{R} : \supset . \, S^2 = S \,.\, S = \breve{S}$

$*72{\cdot}621$ $\quad \vdash :. \, R \,\epsilon\, 1 \to \mathrm{Cls} : \supset : y(\breve{R}|R)z \,. \equiv . \, R`y = R`z$

$*72{\cdot}622$ $\quad \vdash :. \, R \,\epsilon\, \mathrm{Cls} \to 1 : \supset : y(R|\breve{R})z \,. \equiv . \, \breve{R}`y = \breve{R}`z$

$*72{\cdot}63$ $\quad \vdash : R \,\epsilon\, \mathrm{Cls} \to 1 \,.\, S = R|\breve{R}. : \supset . \, S^2 = S \,.\, S = \breve{S}$

$*72{\cdot}64$ $\quad \vdash : S^2 = S \,.\, S = \breve{S} \,.\, R = \mathrm{Cnv}`(\overleftarrow{S} \restriction \mathrm{D}`S) : \supset . \, R \,\epsilon\, \mathrm{Cls} \to 1 \,.\, S = R|\breve{R}$

$*72{\cdot}65$ $\quad \vdash : S^2 = S \,.\, S = \breve{S} \,. \equiv . \, (\exists R) \,.\, R \,\epsilon\, \mathrm{Cls} \to 1 \,.\, S = R|\breve{R}$

The proof of $*72{\cdot}66$, the Principle of Abstraction itself, is only sketched with a reference to theorem $*72{\cdot}65$. Theorem $*72{\cdot}64$ is the main lemma of the chain of theorems: it states that if *S* is equal to its relative product and its converse and *R* is equal to the converse of $\overleftarrow{S}$ restricted to the domain of *S*, then *R* is a many-one relation (a function) such that *S* is equal to the relative product of *R* and its converse (where $\overleftarrow{S}$ is the relation of which holds between the class of *relata* of *x* to *x*, defined at $*32{\cdot}02$ as $\hat{\beta}\hat{x}\{\beta = \hat{y}(xSy)\}$). Since the notation Cnv'$(\overleftarrow{S} \restriction \mathrm{D}`S)$ is

[280] WHITEHEAD & RUSSELL, 1925, p.442 [1910, p.463].
[281] Thus in *Principia* Russell shows that *S* does not need to be reflexive to prove the principle of abstraction. Thanks to Landini for calling my attention to this detail.
[282] WHITEHEAD & RUSSELL, 1925, p.442 [1910, p.463].
[283] WHITEHEAD & RUSSELL, 1925, p.442 [1910, p.463].

somewhat condensed, it is worth to unpack it to see clearly what is going on in the proof. By the definition of $R \upharpoonright \beta$, we have:

$$\overleftarrow{S} \upharpoonright \text{D}‘S = \hat{\beta}\hat{x}\{\beta \overleftarrow{S} x \,.\, x \,\epsilon\, \text{D}‘S\}, \text{ and thus: } \beta(\overleftarrow{S} \upharpoonright \text{D}‘S)x \,.\equiv:\, \beta \overleftarrow{S} x \,.\, x \,\epsilon\, \text{D}‘S$$

Then, theorem $*76{\cdot}64$ can be expressed more conveniently:

$$\vdash : S^2 = S \,.\, S = \breve{S} \,.\, R = \text{Cnv}‘(\hat{\beta}\hat{x}\{\beta \overleftarrow{S} x \,.\, x \,\epsilon\, \text{D}‘S\}) \,.\supset.\, R \,\epsilon\, \text{Cls} \to 1 \,.\, S = R|\breve{R}$$

Recall that what the Principle of abstraction asserts is this: if *S* is a symmetric and transitive relation, then there is many-one relation *R* such that *xSy* if, and only if there is a term α such that $xR\alpha$ and $yR\alpha$; but as the authors observe, "$*72{\cdot}64$ shows that this term α may be taken to be $\overleftarrow{S}‘x$, which is equal to $\overleftarrow{S}‘y$"[284]; To get the Principle of Abstraction, then, we then need the right to left implication of the following theorem:

$*34{\cdot}81$ $\quad \vdash R = \breve{R} \,.\, R^2 \subset R \,.\equiv.\, R = \breve{R} \,.\, R^2 = R$

From this and $*72{\cdot}65$, the Principle of Abstraction - $*72{\cdot}66$ - immediately follows. And we can now see why it serves as a justification for their definition of cardinal number. Concerning their definitions of number *in general*, they explain:

> This principle embodies a great part of the reasons for our definitions of the various kinds of numbers; in seeking these definitions, we always have, to begin with, some transitive symmetrical relation which we regard as sameness of number; thus by $*72{\cdot}64$ the desired properties of the numbers of the kind in question are secured by taking the number of an object to be the class of objects to which the said object has the transitive symmetrical relation in question. It is in this way that we are led to define cardinal numbers as classes of classes, and ordinal numbers as classes of relations.[285]

The point being made here is this: let *S* be a symmetric and transitive relation and let *xSy*; what the proof of $*72{\cdot}64$ shows is that whenever this happens we have the guarantee that there is a function that maps *x* and *y* to a unique α, letting $\alpha = \overleftarrow{S}‘x = \overleftarrow{S}‘y$, which (by $*32{\cdot}02$) is simply: $\alpha = \hat{z}(xSz) = \hat{z}(ySz)$. Thus, given some symmetric and transitive relation *S* that is used as the standard of numerical equality that holds between any two classes β and γ (in the case of cardinals), or relations P and *Q* (in the case of ordinals), there is always a unique class which is

[284] WHITEHEAD & RUSSELL, 1925, p.442 [1910, p.463].
[285] WHITEHEAD & RUSSELL, 1925, p.442 [1910, p.463].

the number of such classes and relations, namely $\alpha = \hat{\delta}(\beta S \delta) = \hat{\delta}(\gamma S \delta)$ in the case of cardinals and $\Omega = \hat{R}(PSR) = \hat{\delta}(QSR)$ in the case of ordinals. To put the point even more generally: in *Principia* the Principle of Abstraction allows one to dispense the *postulation* of an entity which is the property common to all members of an equivalence class by putting the equivalence class itself in its place. Of course, if such a property were to be postulated, and thus assumed as a primitive notion, axioms for it would have to be introduced too, thus increasing the number of non-analyzed notions and undemonstrated propositions. If one can define *nominally* the notion of cardinal number with the assurance that the definition yields uniqueness, such assumptions can be avoided. This is what is assured by the Principle of Abstraction. The authors themselves say this unequivocally:

> The chief merits of this definition are (1) that the formal properties which we expect cardinal numbers to have result from it; (2) that unless we adopt this definition or some more complicated and practically equivalent definition, it is necessary to regard the cardinal number of a class as indefinable. Hence the above proposition avoids a useless indefinable with its attendant primitive propositions.[286]

Thus, the general idea here - according to our reconstruction of the proof of $*72{\cdot}66$ and our interpretation of Whitehead and Russell's remarks - is precisely the same employed in Russell's 1901 paper *On the Logic of Relations and* in the *Principles*, where we find:

> Mathematically, a number is nothing but a class of similar classes: this definition allows the deduction of all the usual properties of numbers, whether finite or infinite, and is the only one (so far as I know) which is possible in terms of the fundamental concepts of general logic. [...] Wherever Mathematics derives a common property from a reflexive, symmetrical and transitive relation, all mathematical purposes of the supposed common property are completely served when it is replaced by the class of terms having the given relation to a given term; and this is precisely the case presented by cardinal numbers.[287]

Again, in *Principia*, the point involves *dispensing* with definitions by abstraction or by postulates. From a mathematical point of view the Russell's nominal definition is preferable because it allows the derivation of the expected formal properties of cardinal numbers without further assumptions, in particular that of a specific ontology of *numbers*. But, of course, the definition is also justified philosophically for it tells *what* numbers are: the definition identifies a *unique* thing which is *the* number of a class α, namely the class of all classes similar to α. With respect to Russell's use of the Principle of Abstraction, these are the two fundamental points at

[286] WHITEHEAD & RUSSELL, 1927*a*, p.4 [1912, p.4].
[287] RUSSELL, B., 1937 [1903], p.116 §111.

stake in the 1901 article, in the *Principles* and in *Principia*[288]. The same point re-appears in the text *Logical Atomism* which led us to this digression, where Russell explicitly claims that his definition of cardinal number - anticipated by Frege - is intended to preserve the formal properties expected of cardinals without *postulating* any specific entities which satisfy these properties:

[288] Rodriguez-Consuegra disagrees. As we had occasion to briefly discuss in the first chapter, he thinks that there are three distinct phases which the Principle of Abstraction goes through in the development of Russell's Mathematical Philosophy. The first was that of the 1901 article, where, supposedly, the role of the principle was meant to *strengthen* the method of abstraction and not to eliminate it" (RODRIGUEZ-CONSUEGRA, F., 1987, p.149); the second is that of the *Principles* where the Principle gives a "genuine foundation" (RODRIGUEZ-CONSUEGRA, F., 1987, p.194). for the logicist treatment of cardinals leading Russell and was actually used to "[...] criticize definitions by abstraction" (RODRIGUEZ-CONSUEGRA, F., 1987, p.192). He enumerates three roles the Principle of Abstraction in that work: (i) "the older and more philosophical one according to which properties are really other terms of certain relations;" (ii) "the technical one through which equivalence relations may be analyzed into asymmetrical relations;" (iii) "the Ockhamian version allowing the elimination of supposed existing entities by replacing them with the corresponding classes"(RODRIGUEZ-CONSUEGRA, F., 1987, p.192). The latter point concerns what Rodriguez-Consuegra calls the " definitive constructive method from 1914 onwards" (RODRIGUEZ-CONSUEGRA, F., 1987, p.192.), namely that of substituting constructions for postulated entities, as paradigmatically exposed in *Our Knowledge of the External World* in explicit connection with the Principle of Abstraction (RUSSELL, B., 1914, pp.33-4) Finally, the third and final stage of the principle is given in *Principia.* According to Rodriguez-Consuegra, *Principia* "contains a definitive statement of this last version" - the one which emphasizes conceptual and ontological economy. But despite Russell's allegation that the principle "embodies a great part of the reasons" for their definition of cardinal number, Rodriguez-Consuegra thinks "the supposed philosophical content of the principle is referred to only by mere respect to a venerable (but now useless) tradition" (RODRIGUEZ-CONSUEGRA, F., 1991, p.192). This is because: "[...] the desired properties of the numbers of the kind in question are secured by taking the number of an object to be the class of objects to which the said object has the transitive symmetrical relation in question. Nevertheless, the supposed philosophical ground is completely avoided in fact because the property is replaced by the class of objects, proposed only on the pragmatic basis that the class in question has the same mathematical properties. Thus the principle is reduced to a mere description of an actually used method without being applied as a real foundation in any formal deduction. That is to say, all definitions and demonstrations in PM would be exactly the same without the principle of abstraction" (RODRIGUEZ-CONSUEGRA, F., 1991, p.193.) Indeed, he points out, quite correctly, that the "the principle is not actually used and not even mentioned in the rest of the book" (RODRIGUEZ-CONSUEGRA, F., 1991, p.192). This, however, is something that Whitehead and Russell themselves are prone to recognize; right before starting the sequence of lemmas for ∗72·66 they observe: "The following propositions lead up to the "principle of abstraction" (∗72·66), which, though not explicitly referred to in the sequel, has a certain intrinsic interest, and generalizes a type of reasoning frequently employed by us."(WHITEHEAD & RUSSELL, 1910, p.473). But this does not warrant his radical conclusion that "the final destiny of the principle of abstraction possibly was the (secret) admission by Russell to have been pursuing an illusion, and therefore the admission that to replace a property by another term and a certain relation was hopeless" (RODRIGUEZ-CONSUEGRA, F., 1991, p.203.). This conclusion is grounded on the mistaken belief that the Principle of Abstraction was retained in *Principia* "*only* as [sic] vague foundation in terms of conceptual economy" (RODRIGUEZ-CONSUEGRA, F., 1991, p.203). But in the passage of *Principia*'s second volume that indicates the "chief merits" (WHITEHEAD & RUSSELL, 1912, p.4) of Russell's definition of cardinal number, we find the following observation: "The grounds in favor of this definition will be found at lenght in *Principles of Mathematics*, part II" (WHITEHEAD & RUSSELL, 1927*a*, p.4 [1912, p.4], footnote †). This makes it clear that Russell still thought, when writing *Principia*, that his arguments in favor of the definition, as presented in the *Principles*, were sound. So why would Russell refer to part II of the *Principles* as offering the philosophical foundation of that definition if he changed his mind so radically about the role of the Principle of Abstraction? Rodriguez-Consuegra seems to think that there is no continuity in the use of the principle except as a means of "ontological or conceptual economics", but this seems unwarranted given the explicit reference Russell makes to his previous work. This reading also anachronistically projects a use of the Principle of Abstraction in the *Principles* that was neither emphasized nor explicitly made in there. What Rodriguez-Consuegra

> A very important example of the principle is Frege's definition of the cardinal number of a given set of terms as the class of all sets that are "similar" to the given set—where two sets are "similar" when there is a one-one relation whose domain is the one set and whose converse domain is the other. Thus a cardinal number is the class of all those classes which are similar to a given class. This definition leaves unchanged the truth-values of all propositions in which cardinal numbers occur, and avoids the inference to a set of entities called "cardinal numbers", which were never needed except for the purpose of making arithmetic intelligible, and are now no longer needed for that purpose.[289]

In the context of *Principia*, however, the Principle of Abstraction also allows Russell to dispense numbers as *objects* and, in fact, as entities of any sort, for, as Russell himself observes right after the passage quoted above, "[...] classes themselves can be dispensed with by similar methods"[290], by which he means the contextual definitions of *Principia*'s section $*20$ - which, on their turn, afford what may very well be the most important application of the principle of avoiding postulated entities in favor of constructions, as Russell also notes[291].

Another very important case in which Russell applied this method or procedure was that of the real numbers, and we'll also do well to review briefly how they are dealt with in *Principia.* According to the classical theory of irrational numbers put forward by Cantor and Dedekind, an irrational number should be defined as a *limit* of either a convergent sequence of ratios of natural numbers (Cantor) or as a "cut" that partitions the set of ratios of natural numbers (Dedekind).

What is common to both methods is that the series of real numbers is characterized as a *Dedekindian series*. In *Principia*'s volume 2, part V, a series is defined as Dedekindian "when it is such that every class has either a maximum or a sequent with respect to it"[292]:

$$*214{\cdot}01 \qquad \text{Ded} = \hat{P}\{(\alpha) \,.\, \alpha \,\epsilon\, \text{Cl}\text{`}\text{max}_P \cup \text{Cl}\text{`}\text{seq}_P\} \quad \text{Df}$$

As Whitehead and Russell explain, when *P* is a transitive relation - which is always the case with a *serial* relation - then the assumption that *P* is Dedekindian is equivalent to the hypothesis that

calls the "Ockhamian [...] elimination of supposed existing entities" (RODRIGUEZ-CONSUEGRA, F., 1987, p.192.) became a serious concern for Russell *after* the discovery of the contradiction; what is explicitly taken as problematic and emphasized in the *Principles* about the postulation of a property defined by abstraction is not that this approach is ontologically onerous but that it does not ensure uniqueness. Indeed, this aspect of the use of the principle is not emphasized or explicitly made in *Principia* itself! What Whitehead and Russell *do* say is that defining cardinal numbers nominally dispenses the need of assuming further primitive notions and propositions (WHITEHEAD & RUSSELL, 1927*a*, p.4 [1912, p.4]); but numbers are still identified with classes, just like in the *Principles*. The Ockhamian economy achieved by the Principle of Abstraction in *Principia* is at best a secondary (not emphasized) gain in comparison to the mathematical issue of securing a definition which yields the expected properties of cardinals numbers and the philosophical issue of defining a unique entity which is *the* cardinal number of a class. In fact, it is only in *Our Knowledge of the External World* that Russell says explicitly that the use of the Principle of Abstraction "clears away incredible accumulations of metaphysical lumber"(RUSSELL, B., 1914, p.33) and it is only in *Logical Atomism* that Russell speaks of the use of the Principle as an application of his "heuristic maxim" (RUSSELL, B., 1924, p.164.

289 RUSSELL, B., 1924, p.165.

290 RUSSELL, B., 1924, p.165.

291 RUSSELL, B., 1924, p.165-6.

292 WHITEHEAD & RUSSELL, 1927*a*, p.659 [1912, p.684].

"that every segment of *P* which has no maximum has a limit"[293]. Of course, there are segments of rational numbers defined in terms of convergent ratios or in terms of partitions of the set of ratios that are *not* rational numbers: to take an ancient example, we have $\sqrt{2}$. The sequence of all ratios less than $x^2 = 2$ has no rational limit. But what right do we have to say that there is an *irrational* one, a *cut* in the ratios that fill this gap?

Russell was convinced that the identification of irrationals with limits was problematic, claiming that there is "no reason whatever to suppose that there are any irrational numbers in the above sense"[294]; he also criticized the supposition that there are limits to segments of ratios with no maximum (i.e., the assumption that the series of reals is Dedekindian) in the *Introduction to Mathematical Philosphy*, where he puts the point very succinctly:

> From the habit of being influenced by spatial imagination, people have supposed that series must have limits in cases where it seems odd if they do not. Thus, perceiving that there was no rational limit to the ratios whose square is less than 2, they allowed themselves to "postulate" an irrational limit, which was to fill the Dedekind gap. Dedekind, in the above-mentioned work, set up the axiom that the gap must always be filled, i.e. that every section must have a boundary. It is for this reason that series where his axiom is verified are called "Dedekindian." But there are an infinite number of series for which it is not verified.[295]

Famously, following this criticism, he observed that "the method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil"[296].

The now standard solution achieved with "honest toil" which Russell produced in the *Principles* and which was later perfected in *Principia* was to construct real numbers as classes of rationals that "have all the mathematical properties commonly assigned to real numbers"[297]: this is done by defining a real number as the lower section of a Dedekind "cut", that is: a real number is any segment of ratios with no lower limit; rational real numbers are those segments

[293] WHITEHEAD & RUSSELL, 1927*a*, p.659 [1912, p.684]. The notation for maximum and minimum of a class with respect to a relation are defined in ∗91

$*93{\cdot}02\ \mathrm{min}_P = \mathrm{min}(P) = \hat{x}\hat{\alpha}(x \in \alpha \cap C`P - \breve{P}``\alpha)$ Df

$*93{\cdot}0021\ \mathrm{max}_P = \mathrm{max}(P) = \mathrm{min}(\breve{P})$ Df

In volume 2, the definitions of upper and lower limits are given in ∗207 together with definitions for "limit or maximum" and "limit or minimum" :

$*207{\cdot}001\ \mathrm{lt}(P)`\alpha = \mathrm{prec}_P \upharpoonright (-\mathrm{Cl}`max_P)$Df

$*207{\cdot}002\ \mathrm{tl}(P)`\alpha = \mathrm{prec}_P \upharpoonright (-\mathrm{Cl}`min_P)$ Df

$*207{\cdot}003\ \mathrm{limax}_P = \mathrm{max}_P \dot{\cap} \mathrm{lt}(P)$ Df

$*207{\cdot}004\ \mathrm{limin}_P = \mathrm{min}_P \dot{\cup} \mathrm{tl}(P)$ Df

In words, the first defines *x* as an upper limit of α in *P* if "α has no maximum and *x* is in the sequent of α" and the second defines *x* as a lower limit of α in *P* if "α has no minimum and *x* is in the sequent of α" (WHITEHEAD & RUSSELL, 1927*a*, p.575 [1912, p.594]).

[294] RUSSELL, B., 1903, pp.270 §258.

[295] RUSSELL, B., 1919*a*, p.53.

[296] RUSSELL, B., 1919*a*, p.53.

[297] RUSSELL, B., 1903, pp.270 §258.

which have an upper limit while irrational real numbers are segments of ratios which have neither a lower nor an upper limit.

In *Principia* we find the following semi-formal explanation:

> [...] the properties which we desire real numbers to have will result if we identify them with segments of H [the relation *less than* for rationals], and give the name "rational real numbers" to segments of the form $\overrightarrow{H}$‘X, i.e., to segments which have ratios as limits. Thus $\overrightarrow{H}$‘X is the rational real number corresponding to the ratio X, and a real number in general is of the form H“λ, where λ is a class of ratios. H“λ will be *irrational* when λ has no limit or maximum in H.[298]

Commenting on this definition in *Introduction to Mathematical Philosophy*, Russell writes:

> The above definition of real numbers is an example of "construction" as against "postulation," of which we had another example in the definition of cardinal numbers. The great advantage of this method is that it requires no new assumptions, but enables us to proceed deductively from the original apparatus of logic.[299]

Now, at this point it should come as no surprise for the reader that Russell's construction of the real numbers relies on Infin Ax to deliver the desired formal properties of them, most notably, that they form a Dedekindian series. The relation H is *Principia*'s formal rendition of the *less than* relation for rational numbers of a given type (excluding zero), and is defined in $*304$:

$*304{\cdot}02 \qquad H = \hat{X}\hat{Y}\{X, Y \,\epsilon\, \text{Rat def} \,.\, X <_r Y\}$ Df

As Whitehead and Russell observe:

> If the axiom of infinity does not hold, H and H' will be finite series"[300], for if, for some ν, "$\nu +_c 1$ is the greatest integer in a given type ($\mu > 1$), the first term of H is $1/\nu$ and the last is $\nu/1$ ($*304{\cdot}281$). In a higher type, we shall get a larger series for H, but at no stage shall we get an infinite series. If, on the other

[298] WHITEHEAD & RUSSELL, 1927*b*, p.317 [1913, p.317].

[299] RUSSELL, B., 1919a, p.73

[300] H' is the relation of *less than* for rationals of a given type including zero. The definition of $<_r$ in general is given as follows:

$*304{\cdot}01 \qquad X <_r Y = (\exists \mu, \nu, \rho, \sigma) \,.\, \mu, \nu, \rho, \sigma \,\epsilon\, \text{NC ind} \,.\, \sigma \neq 0 \,.\, \mu \times_c \sigma = \nu \times_c \rho . X = \mu/\nu \,.\, Y = \rho/\sigma$ Df

This just explores the fact that two ratios μ/ν and ρ/σ are identical whenever $\mu \times_c \sigma = \nu \times_c \rho$.

> hand, the axiom if infinity does hold, H is a compact series $(*304\cdot3)$ without beginning or end $(*304\cdot31)$ and having $\aleph_0$ terms in its field.[301]

If Infin Ax is false, not only the series of ratios does not have its most basic property - that between any two ratios X and Y such that $X <_r Y$, there is a ratio Z such that $Z <_r X$ and $Y <_r Y$ - but it becomes a *finite* series; but then, since reals are defined as segments of ratios, there will also be only finitely many real numbers. This is the price of choosing honest toil over theft.

As we saw in the present chapter, this deal was settled from the moment classes and relations-in-extension were constructed from individuals stratified within the superstructure of the simple theory of types. The question, then, is why adopt this framework to develop Mathematics?

Even apart from his failed hopes of grounding *Principia*'s development of Mathematics upon a nominalistic interpretation of its grammar, Russell's choice of treating classes as logical constructions comes from the realization that the alternatives to this were - or so he thought - either logically or philosophically unsound. On the one hand, the admission of classes as entities on a par with individuals led to paradox if adequate primitive propositions were not assumed. Again, in *Logical Atomism* Russell makes it clear that the construction of classes from individuals was a fundamental step - perhaps *the* most important one - in resolving the contradictions:

> Perhaps even more important is the fact that classes themselves can be dispensed with by similar methods. Mathematics is full of propositions which seem to require that a class or an aggregate should be in some sense a single entity—e.g. the proposition "the number of combinations of *n* things any number at a time is 2^n". Since 2^n is always greater than n, this proposition leads to difficulties if classes are admitted because the number of classes of entities in the universe is greater than the number of entities in the universe, which would be odd if classes were among entities. Fortunately, all the propositions in which classes appear to be mentioned can be interpreted without supposing that there are classes. This is perhaps the most important of all the applications of our principle. (See *Principia Mathematica*, $*20$.)[302]

[301] WHITEHEAD & RUSSELL, 1927*b*, p.278 [1913, p.278]

[302] RUSSELL, B., 1924, p.326. The exact same point is repeated in *My Philosophical Development*: "Without going into difficult technical details, it is possible to explain the broad principles of the theory of types. Perhaps the best way of approaching the theory is by examination of what is meant by a 'class'. [...] Suppose you have *n* objects before you and you wish to know how many ways there are of choosing none or some or all of the *n*. You will find that the number of ways is 2^n. To put it in logical language: a class of n terms has 2^n sub-classes. This proposition is still true when *n* is infinite. What Cantor proved was that, even in this case, 2^n is greater than *n*. Applying this, as I did, to all the things in the universe, one arrives at the conclusion that there are more classes of things than there are things. It follows that classes are not 'things'. But, as no one quite knows what the word 'thing' means in this statement, it is not very easy to state at all exactly what it is that has been proved. The conclusion to which I was led was that classes are merely a convenience in discourse. I was already somewhat bewildered on the subject of classes at the time when I wrote *The Principles of Mathematics*. I expressed myself, however, in those days, in language which was more realistic (in the scholastic sense) than I should now think suitable." (RUSSELL, B., 1959, p.80-1)

The alternative to the method of construction was that of postulation, exemplified in its standard forms in the axiomatic set theories of Zermelo and von Neumann, later improved by Fraenkel, Bernays, Gödel and many others. Russell's reluctance to adopt such route was, of course, not merely technical, but philosophical (I think no one would dispute how simple the development of Arithmetic within an axiomatic theory like ZF is in comparison with *Principia* or, to a *much* higher degree of disparity, any one of Russell's many versions of the Substitutional Theory of classes and relations).

As we already had occasion to discuss, in his recounting of the *desiderata* for an adequate solution of the contradictions in *My Philosophical Development*, Russell enumerates three fundamental ones: the first, "which was absolutely imperative, was "that the contradictions should disappear"[303]; the second "which was highly desirable, though not logically compulsive, was that the solution should leave intact as much of mathematics as possible"[304]; and the third - which Russell thought "difficult to state precisely", was that "the solution should, on reflection, appeal to what may be called 'logical common sense', i.e., that it should seem, in the end, just what one ought to have expected all along"[305]. As far as we know, Russell did not correspond with Zermelo[306] or any one of the first pioneers involved in the creation of the now standard formulations of axiomatic set theory, nor did he criticized explicitly their approach in published works[307]. But he did correspond and commented on the works of one of the greatest expositors of axiomatic Set Theory, namely Quine[308].

Quine - who was profoundly influenced by *Principia* - wrote his doctoral dissertation *A System of Logistic*[309], as an attempt to remedy what he regarded as the defects of *Principia*. Quine's many emendations and innovations in that work reflect the needs of the Mathematician: the adoption of "an extensional version of propositions, construed as sequences", "the abandonment of propositional functions in favor of classes" , and, of course, "the elimination of the axiom of Reducibility"[310]. Russell read it with attention and replied with "the highest admiration" for the work, observing that it had "reformed many matters as to which I [Russell] had always been uncomfortable"[311]. But in his reply Russell also made the following observation:

303 RUSSELL, B., 1959, p.79.
304 RUSSELL, B., 1959, p.79.
305 RUSSELL, B., 1959, p.79-80.
306 It seems that the only comments Russell ever made about Zermelo's Axiomatization of Set Theory - apart, of course, from Zermelo's Axiom of Choice and his well-ordering theorem - occur in the correspondence with Jourdain (cf. GRATTAN-GUINNESS, I., 1977, pp.108-10).
307 Russell did make some remarks about Zermelo's views in his correspondece with Jourdain. Unfortunately, apart from discussions of Zermelo's Axiom of Choice and his Well-Ordering Theorem, all we have are two letters where Russell briefly considers Zermelo's Axiom of Separation, the consistency of his axioms and his construction of natural numbers (GRATTAN-GUINNESS, 1977, p.108-9). Although the details are interesting, Russell's comments in those letters clearly display his conviction that the theory of types was *the* solution for the contradictions.
308 Who also formulated some not-so-standard systems of axiomatic set theory which also received much attention in the literature.
309 QUINE, W., 1934.
310 QUINE, W., 1988, p.226-7.
311 QUINE, W., 1988, p.226.

> In reading you I was struck by the fact that,in my work, I was always being influenced by extraneous philosophical considerations. Take e.g. descriptions. I was interested in "Scott is the author of Waverley", and not only in the descriptive functions of PM. If you look up Meinong's work, you will see the sort of fallacies I wanted to avoid; the same applies to the ontological argument.[312]

Presumably commenting on Quine's systems ML and NF[313], Russell writes *In My Philosophical Development*:

> The third condition is not regarded as essential by those who are content with logical dexterity. Professor Quine, for example, has produced systems which I admire greatly on account of their skill, but which I cannot feel to be satisfactory because they seem to be created *ad hoc* and not to be such as even the cleverest logician would have thought of if he had not known of the contradictions. On this subject, however, an immense and very abstruse literature has grown up, and I will say no more about its finer points.[314]

Though it is completely adequate to say that systems of Set Theory like ZF and NBG *do* possess some "intuitive" appeal (for lack of a better word) which a system like Quine's NF and ML do not, it is also fair to say that all axiomatic systems of Set Theory share a common feature which Russell certainly thought to be *ad hoc*: they *postulate* entities - genuine entities, *individuals* - that satisfy some conception of set which provide Mathematics with the necessary (and generally more than sufficient) amount of existential theorems[315]. By the time *Principia* was done, Russell could not accept any theory which postulated the existence of classes, treating them on par with individuals, as a basis for Pure Mathematics.

The need for the axiom of infinity springs directly from the very nature of Russell's definitions: they are eliminative definitions. They are meant to secure the structural (formal) properties of the ordinary non-analyzed notions they substitute without presupposing that *any entities whatsoever satisfy them*. To show that there are objects that satisfy them, one must construct them. Russell's logicism is the thesis that all the 'entities' with which Pure Mathematics is concerned, namely cardinal numbers, integers, ratios, series, progressions, real numbers, ordinal numbers, or what have you, are all *logical* constructions.

But as Russell notes "[...] logical constructions, like all other constructions, require materials"[316] and these materials, must be the "ultimate constituents of the world"[317]. But the existence

312 QUINE, W., 1988, p.226. Russell's reply is dated by Quine as 6 June 1935.

313 QUINE, W., 1940; 1937; HATCHER, 1982, pp.213-36.

314 RUSSELL, B., 1959, p.80.

315 Obviously, we are not endorsing the claim that there can't be a philosophical case in favor of axiomatic systems of Set Theory, in particular ZF and the so-called iterative conception of set: the *locus classicus* for such discussion are BOOLOS, G., 1971 and PARSONS, C., 1975, both reprinted in BERNACERAFF & PUTNAM, 1984.

316 RUSSELL, B., 1924, p.330.

317 RUSSELL, B., 1924, p.326. As Russell vividly explains in the *Introduction to Mathematical Philosophy*: "Pure logic, and pure mathematics (which is the same thing), aims at being true, in Leibnizian phraseology, in all

or nonexistence of the world is not the business of Logic. Logic does not study the World as it *is*, neither does Mathematics (although, Logic and Mathematics are *applicable* to the world). In the 1911 article in which Russell discusses the Axioms of Infinity and Choice and how they fit within his new conception of Pure Mathematics, he observes that an "[...] *individual* signifies *a being of the actual world, as opposed to the beings of logic*"[318]. Grattan-Guinness finds it somewhat puzzling that Russell holds at the same time that an infinity of individuals is necessary for the development of Mathematics as part of Logic, while accounting for them as "beings in the actual world, as *opposed* to beings of logic"[319]. According to him, it "[...] seems strange for a logicist, especially for one who wrote elsewhere that 'in pure mathematics, as such we do not consider actual objects existing in the actual world'"[320]. But despite the puzzlement, Grattan-Guinness's assessment is on spot : "he [Russell] seems to have regarded such assumptions as antecedent to, but not part of, pure mathematics"[321].

But what then is the subject-matter of Logic and Pure Mathematics in *Principia*? In *Logical Atomism*, Russell asserts that "in science, structure is the main study"[322]. Logic and pure mathematics, being the most general sort of sciences, are the study of the most general sort of structure. Surprisingly, Russell's claim that "[...] in science, structure is the main study"[323] is almost never considered with his accompanying observation that "[...] the mathematical definition and study of structure (under the name of "relation- numbers") form Part IV of *Principia Mathematica*"[324]. Indeed, Part IV of *Principia* is widely neglected[325] - even by Russellian scholars - despite the fact that Russell himself considered it one his most original and relevant contributions to Mathematical Logic, precisely because it gives a logicist account of the notion of *structure*:

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possible worlds, not only in this higgledy-piggledy job-lot of a world in which chance has imprisoned us. There is a certain lordliness which the logician should preserve: he must not condescend to derive arguments from the things he sees about him." (RUSSELL, B., 1919a, p.192)

318 GRATTAN-GUINNESS, I., 1977, p.106.

319 GRATTAN-GUINNESS, I., 1977, p.106-7.

320 GRATTAN-GUINNESS, I., 1977, p.107. Also, according to Grattan-Guinness, Russell's description of *Infin ax* as an "arithmetical axiom" rather than a logical one "seems an unnecessary phrase for a logicist for whom anything essentially arithmetical would certainly be logical also"(GRATTAN-GUINNESS, I., 1977, p.106), and so he is puzzled by Russell's apparently unclear position regarding the "axiom". Our discussion in the previous section makes it clear why there is no need for puzzlement here. Following Landini, we argued that Russell stopped regarding existential theorems that require the axiom of Infinity and Choice as parts of *pure* mathematics; so results like Peano 3 are part of *applied* Arithmetic.

321 GRATTAN-GUINNESS, I., 1977, p.107.

322 RUSSELL, B., 1924, p.340.

323 RUSSELL, B., 1924, p.

324 RUSSELL, B., 1924, p.340. This is a topic which is also very important for a correct appreciation of Russell's views on the Philosophy of Science in general; to give a well-known example, Russell claims in *The Analysis of Matter* that "[...] wherever we infer from perceptions, it is only structure that we can validly infer; and structure is what can be expressed by mathematical logic, which includes mathematics" (RUSSELL, 1927, p.254) and that "The only legitimate attitude about the physical world seems to be one of complete agnosticism as regards all but its mathematical properties" (RUSSELL, 1927, p.270-1). A well-known objection to such a view was famously put forward by Max Newman (NEWMAN, M., 1928). For a careful discussion of the Russell-Newman controversy, cf. LANDINI, G., 2018.

325 So much so that in *My Philosophical Development* Russell starts the chapter on the mathematical aspects of *Principia* noting how disapponting it was for the authors to realize that *Principia* was not seriously studied as a mathematical work (see RUSSELL, B., 1959, p.86).

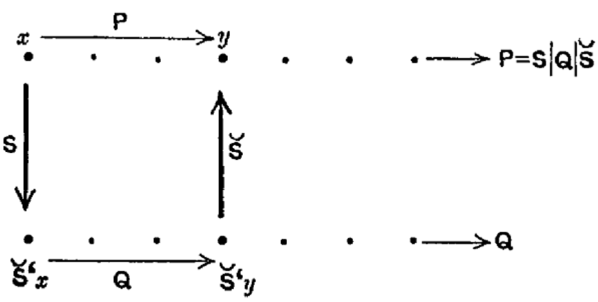
> From the mathematical point of view this was my most important contribution to the work. What I called 'relation-numbers' were numbers of an entirely new sort of which ordinal numbers were a very specialized example. I found that all the formal laws which are true of ordinal numbers are true of this far more general kind. I found, also, that relation-numbers are essential to the understanding of structure. 'Structure' is one of those phrases, like 'and so on' or 'series', which are familiarly employed in spite of the fact that no precise significance is attached to them. By means of relation-arithmetic the concept 'structure' can be precisely defined.[326]

We discussed relation-numbers very briefly just now. To understand Russell's claim we must go into the topic in a little more detail. The fundamental definition of Relation Arithmetic is that of a relation between relations which Whitehead and Russell call *ordinal similarity* or *likeness*[327]:

$$*151\cdot01 \qquad P \,\overline{\text{smor}}\, Q = \hat{S}\{S \,\epsilon\, 1 \rightarrow 1 \,.\, \mathrm{C}`Q = \text{ᗡ}`S : P = S|Q|\breve{S}\} \quad \text{Df}$$

This defines a relation which holds between two relations *P* and *Q* whenever there is a one-one relation *S* which correlates the fields of *P* and *Q* such that *xPy*, if and only if there is a *z* and a *w*, such that xSz, zQw and ySw, that is: if *P* holds between *x* and *y*, then for some *z* and *w* for which *Q* holds, there is *S* maps *x* to *z* and *y* to *w* and vice-versa, where $z = \breve{S}`x$, $w = \breve{S}`y$, $x = S`z$ and $y = S`z$. The authors explain it in some detail as follows (with one of *Principia*'s few graphs):

> Two series generated by the relations *P* and *Q* respectively are said to be ordinally similar when their terms can be correlated as they stand, without change of order.



> In the accompanying figure, the relation *S* correlates the members of C'P and C'Q in such a way that if xPy, then $(\breve{S}`x)Q(\breve{S}`y)$, and if zQw, then $(S`z)Q(S`w)$. It is evident that the journey from *x* to *y* (where *xPy*) may, in such a case, be taken by going first to $\breve{S}`x$, thence to $\breve{S}`y$, thence back to *y*, so that $xPy \,.\equiv .\, x(S|Q|\breve{S})y$, i.e. $P = S|Q|\breve{S}$. Hence to say that *P* and *Q* are

[326] RUSSELL, B., 1959, p.95.

[327] The official definition given is: $P \,\overline{\text{smor}}\, Q = S\{S \,\epsilon\, 1 \rightarrow 1 \,.\, \mathrm{C}`Q = \text{ᗡ}`S \,.\, P = S;Q\}$ Df; where $S;Q$ is defined in ∗150·01 as: $S;Q = S|Q|\breve{S}$ Df. The following theorem just embodies the definition:
∗151·11 $x(S|Q|\breve{S})y \,.\equiv_{x,y} .\, (\exists z, w) \,.\, xSz \,.\, ySw \,.\, zQw\}$.

> ordinally similar is equivalent to saying that there is a one-one relation *S* which has C‘Q for its converse domain and gives $P = S|Q|\breve{S}$. In this case we call *S* a *correlator* of *Q* and *P*.[328]

We saw briefly that relation numbers are defined as classes of *ordinally similar relations*, starting from the relation Nr between a relation and the class of all relations ordinally similar to it, in a manner which is exactly analogous to the definition of cardinal numbers:

$*152{\cdot}01 \quad \mathrm{Nr} = \overrightarrow{\mathrm{smor}} \quad \mathrm{Df}$

$*152{\cdot}02 \quad \mathrm{NR} = \mathrm{D}‘\mathrm{Nr} \quad \mathrm{Df}$

$*152{\cdot}01 \quad \mathrm{Nr}‘P = \hat{Q}(Q\mathrm{smor}P)$

And most results proved in the elementary portions of Relation Arithmetic also have exact analogues of the theory of cardinal numbers[329].

Now, given *Principia*’s definitions and logic of relations, to say that two relations have (or, as Russell sometimes put, *generate*) the same structure is to say that they are ordinally similar, which is to say that they have the same relation number. The structure of a relation *is* its relation-number. Clearly, this definition captures exactly what is meant informally by the “structure” of a relation. For what it defines is an equivalence relation[330] which holds between two relations *P* and *Q* if and only if they share all the properties which do not depend on the specific terms between which the relation holds: if two P and Q are ordinally similar, and thus have the same relation number, we can easily show that if *P* is either transitive or not, symmetric or not, serial or not, many-one, one-many, one-one, implied by diversity or not, etc., then so is *Q* and vice-versa. Russell also makes this point quite concisely in *My Philosophical Development*[331]:

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328 WHITEHEAD & RUSSELL, 1927*a*, p.295 [1912, p.301]

329 Cf. WHITEHEAD & RUSSELL, 1912 [1927*a*], part IV; the summaries of sections A, B, and C make this quite clear.

330 The relation smor is reflexive, symmetrical and transitive:
$*151{\cdot}13\ Q\,\mathrm{smor}\,Q$
$*151{\cdot}14\ P\,\mathrm{smor}\,Q \equiv Q\,\mathrm{smor}\,P$
$*151{\cdot}15\ P\,\mathrm{smor}\,Q.Q\,\mathrm{smor}\,R\,.\supset.\,P\,\mathrm{smor}\,R$

331 This also noted in *Introduction to Mathematical Philosophy* (RUSSELL, B., 1919a, p.59-61). Observe also that, although in *Principia* only relation-numbers of dyadic relations were discussed, Russell was obviously aware that *n*-adic relations also have structures and are meant to be encompassed by the theory of relation-numbers. He briefly points this out in *My Philosophical Development*, where he sketches how *Principia*’s treatment could be so extended: “When two relation-numbers are ordinally similar, we can say that they generate the same ’structure’, but structure is a somewhat more general conception than this [...] It is, however, fairly easy to see how the conception of structure can be generalized. Suppose that *P* and *Q* are no longer dyadic but triadic relations. There are many familiar examples of such relations, for example, *between* and *jealousy*. We shall say of P and Q that they have the same structure if their fields can be correlated so that whenever x, y, z, in that order, have the relation *P*, their correlates, in the same order, have the relation *Q*, and vice versa.” (RUSSELL, B., 1959, p.99.).

> Structure is important for empirical reasons, but there are also purely logical reasons for its importance. When two relations have the same structure, their logical properties are identical, except such as depend upon the membership of their fields. I mean by 'logical properties', properties such as can be expressed in logical terms, not only such as can be proved by logic. Take, for example, the three characteristics by which serial relations are defined viz. that they are asymmetrical, transitive and connected. These characteristics can be expressed in logical terms; and if a relation has any one of them, so has every relation which is ordinally similar to it. Each relation-number, whether finite or infinite, is a logical property of any relation which has this number. Broadly speaking, anything that you can say about a relation, without mentioning the terms between which it holds and without bringing in any property that cannot be expressed in logical terms, will be equally true of any relation similar to the one with which you start.[332]

The distinction emphasized here is between characteristics or properties which can be *proved* by logic and those that can only be *expressed* by logic is of fundamental importance and illustrates clearly what the subject-matter of logic is as understood in *Principia Mathematica*. *Principia* tells us that the "universe consists of objects having various qualities and standing in various relations"[333] and it attempts to capture the framework for the study of "relational structure"[334], as Landini puts it, in terms the grammar of the simple theory of types. *Principia*'s numbered propositions, however, do *not* tell us what exactly are individuals of the lowest simple type because that is a matter of *interpreting Principia*'s grammar, hence a subject that does not belong in the numbered propositions which describe the formal system[335].

This is an important point which must be bear in mind in considering the status of the so-called 'Axiom' of Infinity in *Principia*. From the point of view of *Principia*'s numbered propositions, "Infin Ax" is just a statement; if the variables of lowest simple type are understood as ranging *solely* over concrete particulars, then its truth depends on the existence of such entities; if, however, one considers a semantics which includes concrete *and* abstract particulars among the possible values of the variables of lowest simple type, then Infin Ax can be made true in virtue of the existence of such entities. This, however, is a matter of *interpreting Principia*'s formalism and the numbered propositions do not address how individuals of the lowest type should be conceived[336]. Be that as it may, the subject of Logic and Mathematics, according to *Principia*, is relational structure and the work does not make any assumption about the existence of individuals of lowest simple type besides those required by classical logic (i.e., that there is at least one such individual).

We saw that by the time of the publication of *Principia*'s first edition, Russell wanted no commitments to abstract particulars and he hinted at the idea that individuals of lowest type should be viewed as "beings of the actual world" whose existence he considered an *empirical*

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332 RUSSELL, B., 1959, p.99-100.
333 WHITEHEAD & RUSSELL, 1925, p.43 [1910, p.45]
334 LANDINI, G., 2011*b*, p.197.
335 Thanks to Landini for calling my attention to this important point.
336 Thanks again to Landini for calling my attention to this important point.

matter[337]. So at that time Russell's answer to the question concerning what are individuals of the lowest simple type seems to be that they are concrete particulars. In this case it is not the business of Logic to decide which (and more importantly, *how many*) such particulars exist and which qualities and relations they instantiate (i.e., to determine which our how many particulars have this or that property or bear some relation or other); if, for instance, the world consisted of only two concrete particulars, then three-place relations contained in diversity that hold among would not be instantiated, the same being the case for dyadic relations if there was only one particular; to give some another simple example, if monism were true (in the lowest simple type) no asymmetric relation between entities of that simple type would be instantiated. Logic does not tell us anything as to which (or how many) particulars exist, nor what relations they have to one another. What logic can tell us is what follows from the assumption that there are a certain number of objects and relations (with certain determinate properties, i.e., transitivity, symmetry, etc.) which hold between them: for these turn out to be truths about *structure*.

It is important to observe, however, that according to the conception of Logic which is being ascribed to Russell here, which abstains from asserting the existence of (more than one) individual of the lowest type, Logic *does have ontological commitments*, for it follows that Logic is committed to the existence of logical - i.e., *structural* - properties of relations even if these properties and relations are not instantiated[338]; for instance, whether there is any symmetric or transitive relation holding between concrete particulars it is not the business of logic to decide, but it is the business of Logic to tell us that there is such a thing as transitiveness as a property which applies to relations. Similar considerations apply to relations (in higher simple types) which hold between relations (in lower simple types) in virtue of their structural properties. This, however, seems in complete accordance with Russell's realism about universals and its bearing on Russell's views on Logic and Mathematics. As Russell once put it, what is

337 RUSSELL, B., 1911*c*, p.52; GRATTAN-GUINNESS, 1977, p.173. Again, it must be bear in mind, as Landini notes (cf. previous footnote 196 of the introduction), that Whitehead and Russell had different semantic interpretations of *Principia*; also, it seems that at this point my views depart slightly from Landini's. In the last chapter we argued, following Landini, that according to Russell's original semantic interpretation put forward in *Principia*'s Introduction, predicate variables are to interpreted substitutionally. But as we discussed in the last chapter and the initial section of the present chapter, *Principia* requires a realist interpretation in order to validate the Axiom of Reducibility. Landini has recently pointed out that Whitehead gives hints of a realistic interpretation according to which predicate variables - i.e., variables of simple type other than the lowest - range over universals and according to which both particulars *and* universals are among individuals of the lowest simple type. A possible point of contention here is whether Russell would accept this latter idea around the time *Principia*'s first edition was published; Landini argues that Russell came to reject this possibility in virtue of the influence of Wittgenstein, which lead him to accept the view that variables of lowest simple type range *only* over concrete particulars (cf., for instance, LANDINI, G., 2011*a*, p.343). However, the passage from the 1911 paper *On the Axioms of the Infinite and the Transfinite*, where Russell suggests that individuals of lowest type should be viewed as "beings of the actual world" and that the Axiom of Infinity should be regarded as "empirical" seems to go against this view and suggest that Russell had already rejected the idea that, in a realist interpretation of *Principia*'s predicate variables, universals are among entities of the lowest simple type shortly after *Principia*'s first volume was published. Be that as it may, the more substantial remaining point made below that Russell viewed Logic as the general science of structure which does commit to the existence of any kind of abstract or concrete particulars stands irrespective of this minor point.

338 Here I must thank Landon Elkind who called my attention to this point.

relevant for Logic and Mathematics is "[...] a world of universals and of truths which do not bear directly on such and such a particular existence"[339]. Similarly, in the *Problems of Philosophy*, for instance, Russell wrote:

> It must be taken as a fact, discovered by reflecting upon our knowledge, that we have the power of sometimes perceiving such relations between universals, and therefore of sometimes knowing general *a priori* propositions such as those of arithmetic and logic.[340]

Thus, this interpretation shows that it is possible to reconcile what some interpreters[341] have taken as irreconcilable aspects of Russell's views, namely: his eliminativistic or nominalistic approach to numbers, classes, relations-in-extension and his general Platonistic conception of universals. The point to be noted is that the former are not what Russell called 'ultimate' residue of analysis: they can be explained away in such a way that what remains are the notions of Logic conceived as the science of structure. *Principia* attempts to 'capture' the laws of Logic as the science of structure in terms of the theory of types with impredicative comprehension embodied in the Axiom (scheme) of Reducibility: this commits Russell to a conception of Logic that does have ontological commitments. These commitments, however, are not to concrete or abstract *particulars* - i.e., to what Frege called *objects*. In a realist a realist interpretation of *Principia*'s simple type grammar, variables with simple type indices other than the lowest range over *universals* - i.e., properties and relations-in-intension. Logic is committed to the existence of universals (which are individuals stratified in simple types).

The interpretation proposed here also helps to disarm an allegedly fatal objection to developing Arithmetic within the Theory of Types that appears recurrently on secondary literature: the fact that in *Principia* natural numbers are "replicated" or "repeated" in each type[342]. This fact, which some authors find "intuitively repugnant"[343] would indeed be a problem for anyone who assumes that the subject matter of Arithmetic is an ontology of numbers understood as *objects*. Fortunately, the subject-matter of Arithmetic, being a branch of Logic, is *structure*. The relevant properties of cardinals and relation-numbers which are the concern of Logic and Mathematics are the same in each type. They "repeat" because within the theory of types the same structure can be replicated by constructing numbers (either cardinal numbers or relation-numbers) of ascending types. This is not to say that there are many different entities called cardinal or ordinal numbers in different types: there are no such entities in any of *Principia*'s simple types. By itself the fact that numbers "repeat" in each type in *Principia* is of no more ontological significance than the fact that in each type we have 'different' quantifiers. According to the interpretation

339 RUSSELL, B. 1911*a*, p.39.
340 RUSSELL, B., 1912, p.105.
341 Cf., for instance, COFFA, A., 1991, p.128.
342 For recent example, cf. SOAMES, S., 2014, p.492.
343 HATCHER, W., 1982, p.123.

which we have followed, endorsed and attempted to strengthen here, mostly due to Landini, the whole point of *Principia*'s enterprise is to recover the structure of the theory of cardinal and ordinal numbers within a theory whose vocabulary and axioms avoid any sort of existence assumptions concerning abstract or concrete particulars. Given the conception of Logic which we are, following Landini, attributing to Russell around 1910, *Principia*'s Logicism is the thesis that Pure Mathematics requires no assumption about abstract or concrete particulars, because they are concerned only with truths about structures generated by relations. *Principia*'s Logicism is the thesis that the subject matter of both Logic and Pure Mathematics is relational structure[344].

[344] Which, in a realist interpretation of *Principia*'s simple type structure is given by relations-in-intension conceived as universals (which are individuals of some simple type) whose existence is asserted by the Axiom of Reducibility understood as an impredicative axiom of comprehension for attributes and relations-intension.

6 Concluding Remarks and Assessments

This dissertation was concerned with the development of Russell's views on Mathematical Philosophy starting with Russell's early conception of Logicism as presented in *The Principles of Mathematics* and culminating in what we may call Russell's mature conception of Logicism as presented in the first edition of *Principia Mathematica*. The main objective of the dissertation as sketched in the prefatory note and explained in some detail in the Introduction was to defend Gregory Landini's revolutionary interpretation[1] of the development of Russell's views - and, in particular, Landini's view that Russell intended a nominalistic substitutional interpretation (in the modern sense) for *Principia*'s predicate variables in the Introduction to its first edition - against the orthodox interpretations of authors like Alonzo Church[2], Peter Hylton[3] and Warren Goldfarb[4] and also, more generally, interpreters who view that Introduction as putting forward a ramified hierarchy of types of so-called 'propositional functions' understood as abstract entities, like Nino Cocchiarella[5] and Bernard Linsky[6]. This main task of exposing, defending and, to some extent, expanding Landini's interpretation was carried out by following three intertwined threads that run through the development of Russell's views between the publication of the *Principles* and that of *Principia*, namely: Russell's conception of Logic as a genuine science, his views on the ontological commitments of Logic and the idea that Pure Mathematics - Arithmetic in particular - is nothing but a development of Logic. Hence the title of the dissertation: *Logic*, *Ontology and Arithmetic*.

The division of the dissertation into roughly two parts corresponds, roughly, to two stages of the general argument I tried to present in favor of Landini's revolutionary interpretation against the orthodox views. Apart from context setting for the second part, the first part was mainly intended to refute or reject, on the basis of careful consideration of Russell's manuscripts now available in the fifth volume of his *Collected Papers*, what is - unfortunately, still - the received view of Russell's 'ontological development' with respect to the period between the discovery of 'The Contradiction' and the abandonment of the Substitutional Theory of Classes and Relations. The bulk of our discussion focused on Russell's attempt, after the discovery of the contradiction, to reconcile what turned out to be three irreconcilable goals, namely:

1. Formulate a calculus for Logic which preserved the central doctrine of the *Principles of Mathematics*, according to which the grammar of Logic should embrace only one style of

[1] As put forward in LANDINI, G., 1998 and 2011*a*.
[2] CHURCH, A., 1976*a*.
[3] HYLTON, P., 1990.
[4] GOLDFARB, W., 1989.
[5] COCCHIARELLA, N., 1980.
[6] LINSKY, B., 1999.

entity variable, thus reflecting the thesis that there is only one kind being or entity.

2. Formulate a calculus for Logic a consistent formal system which avoided the contradictions like Russell's paradox of predication and self-membership to a class.

3. Formulate a calculus for Logic strong enough to work as a foundation for classical Mathematics - in particular, such that it yielded without extra-logical hypothesis a system strong enough to prove that there exists a set of terms satisfying the Dedekind-Peano postulates.

As we discussed, Russell's efforts throughout 1903-1906 to reconcile these goals culminated in in the formulation of his Substitutional Theory of Classes and Relations which, in turn, was grounded on his newly discovered theory of incomplete symbols. The Substitutional Theory in its prime form embraced only one style of entity variables ranging over individuals (including particulars and universals) and propositions; it also embraced a radically nominalistic approach to class-expressions and predicate variables in order to avoid the contradictions; and it also afforded a proof of a theorem of infinity. The theory came very close to satisfying Russell's desiderata short of consistency: as we now know, it was plagued by variants of the paradox which Landini unearthed from Russell's manuscripts and dubbed the "p_o/a_o paradox". This paradox and its variants, we argued, following Cocchiarella's and Landini's lead against the orthodox interpretation, showed that there was a conflict between Cantor's power class theorem and the Substitutional Theory. Indeed, it was this paradox that led Russell to embrace a version of the Substitutional Theory with orders of propositions in the paper *Mathematical Logic as Based to Theory of Types* and which led to theory's demise, not variants of what we nowadays refer to as semantic paradoxes like the 'liar' paradox. With respect to this latter point, we also considered Graham Steven's and Jolen Galaugher's arguments in favor of a somewhat intermediate view that aims at finding a place for what Russell referred to as the "purely logical" version of the Liar within the development of Russel's views. I concluded, however, that although their points are solid, Steven's and Galaugher's arguments do not impugn Landini's main contentions, and that, in fact, they indicate that Russell most likely took this "pure form" of the liar seriously because of its structural similarity to the Cantorian constructions involved in the formulations of the p_o/a_o. We also concluded, again endorsing a main point of Landini's interpretation, that although there may have been philosophical issues surrounding Russell's abandonment of the ontology of propositions around 1909-1910, the main cause of this shift in his views was the p_o/a_o paradox.

The second part of the dissertation was concerned with the aftermath of the demise of Russell's Substitutional Theory and its ontology of propositions and it was itself divided in two main stages (whose arguments are, to some extent, independent). The first stage concerned the Theory of Types as Russell and Whitehead present (and defend) it in the Introduction to the first edition of *Principia*. Against Church's orthodox reconstruction[7] (and its variants[8]) of the theory in terms of his *r*-types, we argued in favor of Landini's interpretation, according to which Russell attempted, in *Principia*'s Introduction, to provide a nominalistic substitutional interpretation of *Principia*'s predicate variables[9]. We also endorsed Landini's claim that *Principia* formal grammar - that explained and put forward in its numbered propositions - should be viewed as that of simple type theory. This interpretation, we found, has ample textual support and affords the best approach to make sense of many aspects of *Principia*'s exposition of type theory and its reconstruction of set theory from higher-order logic on the basis of the contextual definitions from $*20$ and $*21$. We also concluded with Landini, however, that Russell's nominalistic semantics cannot validate the Axiom of Reducibility, which in light of the distinctions of order that are introduced *informally* in *Principia*'s Introduciton on the basis of the hierarchy of senses of truth and falsehood is nothing but an *ad hoc* principle that can only be justified on the basis of its consequences - i.e., the development of the theory of classes, mathematical induction and the theory of real numbers. Thus, the somewhat underwhelming - but nonetheless correct from the historical development of Russell's views - conclusion which we reached was that despite Russell's efforts to construct a hierarchy of types that could preserve his central doctrine of the unrestricted variable and the univocity of being, there is an important sense in which Quine was, after all, correct in claiming that *Principia* ends up committed to a theory of types of attributes and relations-in-intension, his claim that Russel was simply confused about use and mention of so-called 'propositional functions' notwithstanding: *Principia* does require, in order to reconcile the development of classical Mathematics with Russell's metaphysics[10], an interpretation of the theory of types according to which each open formula of the language determines a universal (either a property or a relation-in-intension).

The second stage of the second part concerned the development of Arithmetic and the defense of Logicism that Whitehead and Russell put forward in *Principia*. The focus of the discussion, of course, was on the proof of *Principia*'s analogues of the Dedekind-Peano postulates and the difficulties surrounding the proof of a theorem of infinity which, in turn, led them to introduce the inappropriately named 'Axiom' of Infinity. Against the negative assessments according to which Russell abandoned or trivialized the logicist thesis, we argued that Russell was attempting what we called, following Kevin Klement[11], a *reconstruction* of Arithmetic on a logical basis, not a derivation of all ordinary Arithmetic from logical foundations. We concluded thus, that Russell was not a logicist in the same sense that Frege was[12]. Following Landini, we

7 CHURCH, A., 1976*a*.

8 I.e., those of Bernard Linsky (LINSKY, B., 1999) and Nino Cocchiarella (COCCHIARELLA, N., 1980).

9 And according to which so called 'non-predicative' functional variables and propositional variables are schematic letters for well-formed formulas, i.e., part of metalanguage.

10 I.e., his view that the ultimate furniture of the universe consists of individuals among which there are universals and particulars.

11 KLEMENT, K., 2013*b*.

12 In particular because Frege's Logicism was the view that numbers are *logical objects*. Russell, on the other hand rejects the idea that numbers are objects or, as Landini puts it, abstract particulars.

argued that Logicism, according to Russell's conception of it, is the thesis according to which Arithmetic (and Pure Mathematics in general) can be reconstructed as part of logic, conceived as the general science of *structure*[13]. Thus our assessment regarding Russell's use of the Axiom of Infinity in *Principia* is that it is a coherent aspect of his general conception of the relations between Logic, Pure Mathematics and Mathematical Philosophy, not something that reveals his logicism as a failure. In sum, our conclusion was that Russell may have failed to show that Logicism as he envisaged it could demonstrate everything he thought desirable, but he never failed to live up to his own standard of logical correctness. Against the suggestion that Russell may have given in to the convenience of theft over honest toil, our conclusion is that what he thought was *desirable* but could not prove he left as a hypothesis, but this was no thievery: it was the culmination of his honest toil, in accordance with Russell's methodological principles of avoiding postulated entities in favor of constructions, numbers themselves being treated as such.

Now, if the arguments presented in both parts of this dissertation are sound, then what we established is that the interpretation of Russell's writings[14] on Mathematical Philosophy advocated here is the most plausible and compelling in light of the available evidence. This result, of course, is of historical and exegetical nature and it does not allow us to decide whether Russell's project did actually succeed as a viable stance in the Philosophy of Mathematics and whether, in case of a negative answer, there are prospects for fixing its most serious flaws - if there are any. In what remains of this concluding section, I want to make a few remarks about this. Evidently, I will not even attempt to come close to exhaust all the fundamental relevant points for assessing Russellian Logicism as I characterized it here - for this would require at least a whole new dissertation; but I want to briefly address what I take to be the hard core of Russellian Logicism that may afford a tenable Philosophy of Arithmetic and what - from an 'external' perspective, so to say - may be considered the most serious difficulties that Russell's views, as they were presented here, face.

As we characterized it here, the most fundamental trait of Russell's logicism - which, in fact, pending on perspective, consists in its main virtue or flaw - is that it completely rejects Frege's framing of the task of establishing number theory as the province of Logic. Frege famously wrote, in his correspondence with Russell, that the "[...] the possibility of placing arithmetic on a logical foundation" hanged in balance with the question of "how do we apprehend logical objects"[15]: the fundamental issue for Frege's conception of logicism was that of justifying the introduction of numerical terms as *singular* terms standing for *logical objects* into his system[16]. Russell's version of the Logicist reconstruction of Arithmetic, in its turn, challenges the

[13] LANDINI, G., 2011, pp.173-174.

[14] Writings, that is, roughly from the period of 1903-1911.

[15] FREGE, G., 1980, p.140.

[16] Frege failed to fulfill this task because his system was inconsistent when adduced by his Basic Law V and because he could not come to grips with a solution for the famous Caesar problem when such singular terms were introduced via contextual definition by appealing to what Michael Dummett has called "the original equi-

widespread assumption that Logicism died with Frege precisely because of this problem. For instance - in what is surely one of the best and most important contemporary works on Logicism and, in fact, on the Philosophy of Mathematics in general - Michael Dummett wrote that:

> The logicist thesis failed because of its inability to justify the existence of mathematical *objects*, more particularly of systems of objects satisfying the axioms of the theories of natural numbers and of real numbers.[17]

This, as we saw, is not a fair diagnosis of Russell's version of the logicist enterprise: numbers for Russell are *not* objects; cardinal numbers are defined as classes of similar classes and classes are treated as logical fictions or constructions whose symbolic expressions are to be eliminated in terms of the contextual definitions of *Principia*'s section ∗20. This point however, is not missed by Dummett, who, at one of the many points in his book where he draws parallel considerations about Frege's views and those of Russell and Whitehead, writes:

> The problem how to introduce abstract objects would have been avoided if Frege could have dispensed with mathematical objects altogether by construing numbers of various kinds as concepts of second or higher order, beginning by building on the rejected definitions of §55 (with the third of them suitably amended). This was in effect Russell and Whitehead's solution, or would have been if *Principia* had been developed within the simple theory of types, rather than the ramified theory required by the vicious circle principle. Arithmetical theorems would then have been interpreted as yet more unproblematically in character, and as admitting of yet more direct application.[18]

Dummett's mention of paragraph §55 of Frege's *Grundlagen* refers, of course, to the attempt of introducing numbers as higher-order concepts that are completely analogous to what we now nowadays refer to as 'numerically definite quantifiers'. There are, in fact, different ways[19] to translate *Principia*'s construction to this approach which dispenses with Whitehead and Russell's use of "surrogate" classes, as Dummett puts it. One such translation or inspired reconstruction of numbers as higher-order quantifier-like concepts has been put forward by Landini who on its basis develops a robust argument in favor of the view that the rejection of Logicism on the grounds that Logic cannot supply an ontology of infinitely many numbers as objects is based on a misguided or biased "intuition that it is necessary that there are infinitely many numbers - a

valence" and George Boolos called "Hume's Principle".

17 DUMMETT, M., 1991, p.307. Elsewhere, Dummett made a similar point as follows: "Logicism, represented first by Frege and then by Russell and Whitehead, failed because it combined three incompatible aims: to keep mathematics uncontaminated by empirical notions; to represent it as a science, that is, as a body of *truths*, and not a mere auxiliary of other sciences; and to justify classical mathematics in its entirety" (DUMMETT, M., 1991, p.312).

18 DUMMETT, M., 1991, p.225-6; Dummett also puts the same point forward in p.302.

19 Compare, for instance, LANDINI, G., 2011*b* and KLEMENT, K., 2015. I'll be following closely Landini's definitions, but my notation of choice is closer to that of Klement.

de re intuition of a non-logical necessity governing special metaphysical objects (numbers)"[20]. Altering Landini's notation[21], but keeping the letter of his definitions intact, the foundation of such approach can be given along the following lines[22]:

(Func) $\text{Func}(R) =_{\text{Df}} (x, y, z)(((R(x, y) \wedge R(x, z)) \supset y = z)$

$(1 \to 1)$ $1 \to 1(R) =_{\text{Df}} \text{Func}(R) \wedge (x, y, z)(((R(x, y) \wedge R(x, z)) \supset x = z)$

$(\approx)$ $\lambda x : \phi x \approx \lambda x : \psi x =_{\text{Df}}$

$$(\exists R)[(1 \to 1(R) \wedge (x)((\exists y)R(x, y) \equiv \phi x) \wedge (y)((\exists x)R(x, y) \equiv \psi y)]$$

(Card) $\text{Card}(\lambda x : \phi x, \lambda x : \psi x) =_{\text{Df}} \lambda x : \phi x \approx \lambda x : \psi x$

(0) $0(\lambda x : \phi x) =_{\text{Df}} \text{Card}(\lambda x : \phi x, \lambda x : x \neq x)$

(P) $\lambda\phi : m(\lambda x : \phi x)P\lambda\phi : n(\lambda x : \phi x) =_{\text{Df}}$

$$(\exists\psi)(\exists z)\left(\begin{array}{c} \psi z \\ (\phi)[n(\lambda x : \phi x) \equiv \text{Card}(\lambda x : \phi x, \lambda x : \psi x)]\wedge \\ (\phi)[m(\lambda x : \phi x) \equiv \text{Card}(\lambda x : \phi x, \lambda x : \psi x \wedge x \neq z)] \end{array}\right)$$

$(<)$ $\lambda\phi : m(\lambda x : \phi x) < \lambda\phi : n(\lambda x : \phi x) =_{\text{Df}}$

$$(F)\left(\begin{array}{c} [(m(\lambda x : \phi x)P\lambda\phi : p(\lambda x : \phi x) \wedge F(\lambda\phi : m(\lambda x : \phi x)) \supset F(\lambda\phi : p(\lambda x : \phi x))) \wedge \\ q(\lambda x : \phi x)P\lambda\phi : r(\lambda x : \phi x) \wedge F(\lambda\phi : m(\lambda x : \phi x)) \supset F(\lambda\phi : r(\lambda x : \phi x))] \supset \\ F(\lambda\phi : n(\lambda x : \phi x)) \end{array}\right)$$

$(\leq)$ $\lambda\phi : m(\lambda x : \phi x) \leq \lambda\phi : n(\lambda x : \phi x) =_{\text{Df}}$

$$\lambda\phi : m(\lambda x : \phi x) < \lambda\phi : n(\lambda x : \phi x) \vee (\psi)(n(\lambda x : \phi x) \equiv m(\lambda x : \phi x))$$

$(\mathbb{N})$ $\mathbb{N}(\lambda\phi : n(\lambda x : \phi x)) =_{\text{Df}} \lambda\phi : 0(\lambda x : \phi x) \leq \lambda\phi : n(\lambda x : \phi x)$

Details apart, the relevant point, of course, is that such a translation can be made and that definitions which employ structured variables like the above bring to the surface a point that can easily be missed when reading *Principia* from $*20$ onward[23], namely: that at its heart *Principia*'s approach to number-expressions amounts to treating them as higher-order concepts analogous to

[20] LANDINI, G., 2011, p.172..

[21] I find Landini's use of subscripts in his structured variables somewhat inconvenient. I'll be employing a variant of the lambda notation for structured variables, although the lambda-abstracts are not singular terms here.

[22] Cf. LANDINI, G., 2011*b*, pp.176-8. In the above each lambda-abstract like "$\lambda x : \phi x$" stands for a nominalized concept or attribute in extension - they are *not* singular terms then - to which higher-order concepts like 0, 1, 2, ... $\mathbb{N}$, etc apply.

[23] That is, when functional expression practically vanish from the work giving place to the convenient notation of classes.

quantifiers. The question then, is: is such a position tenable as a defense of Logicism, in particular when compared with other variants of Logicism which assume that numbers must be objects after all[24]?

Despite being apparently sympathetic the approach, Dummett, like many others, thinks not and gives a paradigmatic diagnosis as to why he thought that Whitehead and Russell failed to establish Logicism:

> Their attempt ran against the difficulty that would have supplied the only valid ground for Frege's insistence that numbers *are* genuine objects, the impotence of logic (at least as they understood it) to guarantee that there are sufficiently many surrogate objects for the purposes of Mathematics, forcing them to make assumptions far from being logically true, and probably not true at all: to secure the infinity of the natural-number sequence, they had to assume their axiom of infinity, and to secure the completeness of the system of real numbers, they had to assume the axiom of reducibility.[25]

The criticism presented here is the standard one: *Principia*'s reconstruction of Arithmetic in terms of higher-order concepts cannot provide a proof of the infinity of the natural numbers and *Principia*'s formulation of the theory of types requires the Axiom of Reducibility for the development of Classical Mathematics[26]. The prospects of rejecting Russell's Logicism are quite different, however, on the basis of each of these 'faults'.

As we discussed - somewhat extensively - the fact that *Principia* does not provide a theorem of infinity but only a conditional result, by itself, does nothing to impugn Russell's Logicism[27]. If we are to reject Russsell's view that numbers are logical constructions or a *Principia*-inspired construction like that of Landini, what is required are positive reasons to regard such a reconstruction as inadequate, incoherent or insufficient; merely requiring that a proof of infinity should be forthcoming begs the question, precisely because this involves assuming that numbers have to be objects after all.

The central issue involved in determining whether the need for an Axiom of Infinity is a vice or a virtue, then, concerns whether there are good reasons for rejecting Russell's view that numerical terms are *not* singular terms standing for objects. What is at stake, I think, is the tenability of the position now referred to in the literature as "Syntactic Reductionism". We may borrow Kevin Klement's sharp statement of it:

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24 Even more specifically, I will be concerned with the views put forward by Crispin Wright in the modern classic *Frege*'s *Conception of Numbers as objects* (WRIGHT, C., 1983) which has gained immense notoriety.

25 DUMMETT, M., 1991, p.302.

26 It must be noted that there is a slip in what Dummett says here. It is quite misleading to say that Whitehead and Russell required the axiom of Reducibility for the development of the theory of real numbers for, although the statement is correct, as is well known, without the axiom of Reducibility *Principia* cannot recover mathematical induction and so the arithmetic of the natural numbers collapses without it.

27 Furthermore, there is also the aforementioned possibility (cf. footnotes 196 of the Introduction and footnote 260 of chapter 5) of adding new axioms to *Principia* that yield the desired results and which can be philosophically motivated.

> **Syntactic reductionism** is the view that what appear to be terms for abstract objects are not genuine syntactic terms at all; in a full or correct representation of the syntax of sentences in which they appear they can be shown not to be genuine individual terms at all.[28]

Russell, of course, fully endorsed this view with respect to numerical terms[29]. The opposite stance - which amounts to holding what is referred to in the literature as the "syntactic priority thesis" - consists in the view that everything that is required for a class of expressions to function as singular terms referring to objects is an appropriate criterion for determining the truth-conditions of (certain classes of) sentences containing them. One famous statement of this view is that of Crisipin Wright who puts the view forward both on behalf of Frege and of his own version of Logicism in the modern classic *Frege's Conception of Numbers as Objects*[30]:

> Frege requires that there is no possibility that we might discard the preconceptions inbuilt into the syntax of our arithmetical language, and, the scales having dropped from our eyes, as it were, find in reality there are no natural numbers, that in our old way of speaking we had not succeeded in referring to anything. Rather, it has to be the case that when it has been established, by the sort of syntactic criteria sketched, that a given class of terms are functioning as singular terms, and when it has been verified that certain appropriate sentences containing them are, by ordinary criteria, true, then it follows that those terms genuinely refer. And, being singular terms, their reference will be to objects. There is no further, intelligible question, whether such terms really have a reference whether there are really such objects.[31]

As is well known, Wright advocates what became known in the literature as Neo-Fregean Logicism, which consists in holding that the following 'abstraction' principle for number terms provides, in the context of impredicative second-order logic, an adequate justification for the introduction of numerical expressions as singular terms:

$$(F)(G)(\#(\lambda x : Fx) = \#(\lambda x : Gx) \equiv \lambda x : Fx \approx \lambda x : Gx)$$

If one accepts this as an analytic truth about numbers and concedes Wright's point that settling the truth-conditions of sentences containing singular terms settles the question of whether or not they have a reference, then the above does seem to provide exactly the sort of criterion required to claim that numerical terms - e.g., "$\#(\lambda x : Fx)$" - are singular terms referring to objects and, as is well known, the above not only settles this but also allows for a proof the Dedekind-Peano

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28 KLEMENT, K., 2015, p.94; for a general critical discussion of the position cf. HECK, R., 2011, pp.180-199.

29 And, of course, many other classes of expressions which he took as incomplete symbols instead of singular terms.

30 WRIGHT, C., 1983.

31 WRIGHT, 1983, p.14

postulates, including the result that there are infinitely many numbers conceived as objects. But why should we accept the Priority Thesis[32]?

According to Neo-Fregeans[33], Frege's adequate response comes from one of his central methodological maxims put forward in the *Grundlagen*, his famous 'context principle':

> Never ask for the meaning [*Bedeutung*] of a word in isolation, but only in the context of a sentence.[34]

The disagreement between Russellians who hold the Syntactic Reductionism and the Fregeans who hold the syntactic priority thesis would, putting things very roughly, amount to this: Fregeans hold that there is a priority of syntactic categories (Name, predicate, etc...) with respect to ontological categories (object, property, etc...) because there is a priority of truth over reference and this view is grounded or licensed by the Context Principle, i.e., the idea that the reference of a category of expressions can only be settled after settling the truth-conditions of sentences in which they occur by appeal to some appropriate criterion. Granting this characterization of Frege's Context Principle, the content of the syntactic priority thesis is that questions pertaining to the reference of a certain group of terms (such as numerals) are questions "internal to language", as Dummett puts it[35]. Russellians who accept the eliminative treatment of numerical expressions, on the other hand, must refuse the priority thesis and thus, the view that the context principle - interpreted in such a robust fashion - has the unrestricted validity or strength that Wright claims. How could this be done?

Well: it's complicated. As Kevin Klement points out, settling this dispute would amount to settling "[...] many of the fundamental questions in philosophical semantics and metaphysics"[36]. I'll risk, however, pointing to two very rough sketches of ways of answering.

32 Another important question concerns whether we should concede that the so-called Hume's principle is an analytic truth about numbers - or at least that it has such a privileged status as a principle of number-theory (cf., for instance, BOOLOS, G., 1998, pp.301-14 and DUMMETT, M., 1998, pp.367-387 for the main lines of criticism and WRIGHT & HALE, 2001, for their extensive replies to the standard criticisms). Frege himself rejected the idea that Hume's principle provides a strong enough principle for settling the truth conditions of statements of number-expressions as singular terms - this was the outcome of his famous discussion in the *Grundlagen* of the notorious Caesar problem, which is also a pressing problem for Neo-Fregeans (cf., for instance, WRIGHT & HALE, 2001, pp.335-396 for an attempt at a dissolution of the issue).

33 Cf., WRIGHT, C., 1983, p.50-2.

34 FREGE, G., 1884, p.x.

35 Dummett puts this point forward in the following terms: "The context principle, as enunciated in *Grundlagen*, can be interpreted as saying that questions about the meaning (*Bedeutung*) of a term or class of terms are, when legitimate, internal to the language. We know the meaning of a term, say, "the Equator", when we know the conditions for the truth of any sentence containing it; that is all we need to know, and all we can know. Hence, to determine the meaning of a term, what we have to do, and all we do, is to fix the senses of sentences in which it occurs. Reference therefore does not consist in a mental association between the term and the object, considered as apprehended by the mind independently of language; nor can it consist, we may add, in the existence of a causal chain leading from the object to an utterance of the term. It follows that any legitimate question about the meaning of a term, that is, about what we should call its reference, must be reducible to a question about the truth or otherwise of some sentence of the language." (DUMMETT, 1991, p.192).

36 KLEMENT, K., 2013, p.150.

The first, and most promising, I think, is simply to reply that a *logical* analysis of the concept of number is *possible*[37] without recourse to a conception of numbers as objects and that this analysis avoids many of the severe issues which arise when one attempts to justify the assumptions of numbers as objects on a logical basis. A case for Russellian Logicism on this basis has, in fact, been made by Klement[38], who discusses in some detail how the main problems that affect the so-called Neo-Fregean form of Logicism are either solved or dissolved by the Russellian approach. Thus, in a nutshell: conceptual and ontological economy, if anything, favors the Russellian position, although, obviously, this by itself is does not settle the issue.

Settling the issue, would, I think, involve a second - much harder - path for the Russellian, which is to accept that she owes us an explanation of some linguistic or extra-linguistic criterion that allows her, at least locally, to refuse the syntactic priority thesis. Needless to say that coming up with such a (satisfactory) explanation or solution is nothing short of a Herculean task - for, again, this would involve tackling or settling some of the most intractable philosophical issues out there. The pressing question for the Russellian is to establish a philosophically sound motivation for, so to say, shave off mathematical theories with Occam's razor: the problem, however, is that simply pointing to conceptual or ontological economy begs the question against the Fregean, who does have a philosophically motivated criterion for taking number-expressions as singular terms. Furthermore, whether a eliminativistic approach to number-expressions like that of *Principia* or *Principia*-inspired forms are really more simple or economical is itself a controversial issue. Klement nicely sums this up when he claims that:

> An abstract science that treats its special expressions in an eliminative, nominalist, way is metaphysically simpler than one that postulates *abstracta* as genuine references of its terms, or as values of ontologically committing variables. But this is perhaps to take too narrow a view of what is involved in the intellectual virtue of 'simplicity'. It is not simply a matter of how many, or how many kinds, of things are presupposed. It may have to do with the simplicity and unity of the basic principles assumed in the theory, how well integrated such a theory is with other theories, and so on.[39]

Klement also points, however, to reasons to be optimistic about the merits of the Russellian approach, for as he observes "[...] the advantages, even the advantages regarding simplicity, are not limited to a reduction in what is quantified *over*"[40]. As he notes, a case can be made for adopting an eliminative approach to apparent singular terms of a theory like that of the

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37 It is important to emphasize, as Klement aptly does, that what is at stake for the Russellian is a syntactic reductionist account of number-expressions within an artificial and/or ideal formal language; as Klement puts it the relevant question "[...] is not whether or not ordinary language embodies syntactic reductionism, but whether or not anything important is lost (and indeed, perhaps, whether anything is gained) in number theory or other abstract science if we choose to employ an artificial language that decidedly does." (KLEMENT, K., 2015, p.110).

38 Cf. KLEMENT, K., 2013*b* and 2015.

39 KLEMENT, K., 2015, p.111.

40 KLEMENT, K., 2015, p.111.

Arithmetic of natural numbers based on the following prospects of theoretical economy and/or simplicity:

1. **Epistemological Economy:** It "[...] simplifies our epistemology in so far as we do not need an account of special *de re* knowledge of abstract objects"[41].
2. **Philosophico-Semantical Economy:** "It simplifies our semantic theories, in so far as we do not need a special theory of reference for such terms"[42].
3. **Prospects of Consistency:** "Depending on the abstract entities eliminated, it may simplify our logic, as certain abstract objects (extensions, ordinal numbers) are prone to lead to contradictions if taken realistically"[43].
4. **Prospects of Applicability:** "It may simplify our account of the applicability of abstract studies to the concrete world, as our account of abstract objects resolves such discourse into quantification over concrete entities (and concepts applicable to them)"[44]

But again, it must be emphasized that the issue is not settled just by pointing that the Russellian approach has these virtues for, as already noted, the issues involved here are deeply problematic and complex. Nonetheless, I think it is established beyond reasonable doubt that the Russellian approach which advocates Syntactic Reductionism with respect to Arithmetic along the lines we considered both in the body of the present dissertation and in the present concluding section does have strong virtues and a strong internal coherence.

We are not done, however, for there remains the more broad question regarding the necessity of the Axiom of Reducibility. Given the interpretation of *Principia* we endorsed, the problem of justifying the presence of this axiom in the system amounts to this: can Russell's conception of Logic as a science and the metaphysics which accompanies this conception of Logic justify the introduction of impredicative axioms of comprehension? This is obviously a very complex question that raises just as many if not more fundamental and complex issues in almost every branch of Philosophy. For our purposes here, this question may be broken down into a more tractable (but still complex) separate sub-question for which I can sketch a discussion and prospects of future investigation, namely: Is there a tension between Russell's conception of Logic, the metaphysics which underlies it and the formal system which he adopts to embody it - i.e., the simple theory of types of attributes and relations-in-intension with impredicative axioms of comprehension?

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41 KLEMENT, K., 2015, p.111.
42 KLEMENT, K., 2015, p.111.
43 KLEMENT, K., 2015, p.111.
44 KLEMENT, K., 2015, p.111.

In order to address this, I want to take as my starting point the two following strikingly different takes on higher-order Logic, which, I think, strike unequivocally at the very heart of the issue:

> The doctrine of logicism [...] is not a doctrine about the reducibility of mathematics to a theory of membership (in a class or set) but about the reducibility of mathematics to a theory of predication, and in particular to a theory about the concepts which predicates stand for in their role as predicates. For this reason both Frege and Russell maintain that the logistic framework within which classical mathematics is to be represented must consist at least of a second order predicate logic where quantification is not only with respect to the role of singular terms but to that of predicates as well. The distinction between predicates and singulars terms, it must be emphasized, is fundamental to both Frege's and Russell's forms of logicism; and to fail to attend to this distinction is to fail to understand the nature of predication in either framework.[45]

> [...] tendency to see set theory as logic has depended early and late on overestimating the kinship between membership and predication. An intermediate notion, attribution of attributes, insinuates itself and heightens the illusion of continuity In the innocent '*Fx*' of the logic of quantification, the schematic letter '*F*' stands in place of a predicate. Or, more explicitly, the combination '*Fx*' stands in the place of an open sentence in '*x*' ; whether the sentence has '*x*' on one side and an isolated predicate on the other is of no moment. What is important is that in writing '*F*' and '*Fx*' we are just schematically simulating sentences and their parts; we are not referring to predicates or other strings of signs, nor are we referring to attributes or sets. Some logicians, however, have taken a contrary line, reading '*F*' as an attribute variable and '*Fx*' as '*x* has *F*' . Some, fond of attributes, have done this with their eyes open; others have been seduced into it by a confusion. The confusion begins as a confusion of sign and object; a confusion between mentioning a sign and using it. Instead of seeing '*F*' steadfastly as standing in place of an unspecified predicate, our confused logician sees it half the time as naming an unspecified predicate. Thus '*F*' gains noun status, enabling him to read '*Fx*' as '*x* has *F*' without offending his grammatical ear. Having got this far, he can round out his confusion by calling *F* an attribute. This attunes his usage to that of the unconfused but prodigal logician who embraces attributes with his eyes open. The prodigal logician is identifiable with Frege. The confused logician could be Russell, despite his great contributions.[46]

The merits of what is being with respect to Frege and Russell's actual views notwithstanding, I see an important and deep disagreement between Cocchiarella and Quine which, I think points to why there is indeed an unsolvable tension in Russell's Logicism. The core of the issue, I think does not concern the legitimacy of quantifying over *abstracta* such as relations and properties or whether logicians should indulge in commitments with such entities (or entities of any sort for that matter). Quine would certainly claim that one should avoid at all costs surrendering to the temptations of such "creatures of darkness", but that is not what I have in mind. The fundamental point underlying Quine's criticism is deeply Russellian. In a nutshell: for Quine if we are to talk about properties or relations *as entities* these should be referred to by the only

45 COCCHIARELLA, N., 1986, p.198-9.

46 QUINE, W., 1986, p.66.

means which we have at our disposal to talk about anything whatsoever, namely by means of object variables (i.e., what we now call 'first-order' variables) or singular terms/proper names. Indeed, let me be even more clear as to why Quine's point as deeply Russellian. If we really take Quine seriously, then the very phrase "first-order variable" is just a pleonasm, for a correct notation should - as Russell argued in the *Principles of Mathematics* - adopt one and only one style of variable, namely individual variables ranging over whatever "can be counted as one". Were we to Russellianize Quine's *dictum*, we could say: to be is to be a value of the unrestricted variable.

Cocchiarella, on the other hand, better appreciates Frege's prodigal path and recognizes that it is precisely a fundamental distinction between the role fulfilled by predicates and singular terms that is required by Logicism. The use of bound letters *F* which Logicism requires is one that accepts that "predicates are not singular terms or what Frege called 'proper names' in his extended sense" and that "the role of concepts in predication is not that of objects or individuals to which other objects or individuals stand in a relation of exemplification"[47]. This is *radically* anti-Russellian. Perhaps nowhere this is more clearly displayed than in Frege's explanation of the incomplete character of predicates - and their non-linguistic and also incomplete counterpart, concepts - in terms of the metaphor of 'saturation'. As we saw, Russell found the very statement of this idea contradictory.

The most fundamental underlying disagreement that can be extracted from the passages of Quine's and Cocchiarella's quoted above is whether Logic should embrace an ultimate cleavage between proper names and individual variables - which stand for objects, i .e., saturated entities - on the one hand, and predicates standing for incomplete entities, i.e., concepts, on the other. Apart from his doctrine of concept-correlation and the introduction of courses-of-values or extensions into his system, Frege's logic embraced this cleavage in the most strictest way possible. Frege did *not* view higher-level functions in complete analogy to first-level ones whose arguments were objects. As he explains in On *Concept and Object*:

> Second-level concepts, which concepts fall under, are essentially different from first-level concepts, which objects fall under. The relation of an object to a first-level concept that it falls under is different from the (admittedly similar) relation of a first-level to a second level concept. (To do justice at once to the distinction and to the similarity, we might perhaps say: An object falls under a first-level concept; a concept fall within a second-level concept.)[48]

That is why Frege employed structured variables for second-level functions like "$M_{\alpha,\beta}(f(\alpha,\beta))$". A function as something incomplete is a radically different sort of thing than an object and it cannot occur as an argument of another function in the same way as on object does: a function is *essentially predicative* and thus one cannot - to smuggle Russell's most central logical notion

47 COCCHIARELLA, N., 1986, p.199.
48 FREGE, G., 1892, p.201.

- make a function a logical subject in the same way one makes an object a logical subject. As we had already said following Hylton and borrowing a Wittgensteinian saying: It is here that we hit bedrock[49].

Like Frege, Russell also wanted to ground Mathematics on Logic conceived as a theory of predication - or, more appropriately in the case of Russell, a theory of property and relation exemplification - (instead of a theory of classes) as embodied in the hierarchy of levels, i.e., the simple theory of types, but Russell vehemently resisted accepting Frege's interpretation of this hierarchy in terms of the distinction between complete/saturated and incomplete/unsaturated entities. Russell resisted this view in the *Principles* given his conviction that whatever can be mentioned must be a genuine logical subject and thus, a term; and it was, in fact, this difficulty which precluded him from accepting the theory of types as put forward in the *Appendix B* of the *Principles* and to reject propositional functions as entities in the same work. Moreover it was precisely this difficulty that led Russell to develop his Substitutional Theory of Classes and Relations, which afforded a way of embracing a hierarchy of levels (of both classes and so-called propositional functions) without assuming these as any sort of entities (either complete or incomplete) and thus, without betraying his most central logical doctrine of the unlimited or unrestricted variable. The approach did not work, as we saw, and Russell scrapped the Substitutional Theory in its entirety, never mentioning it again after *Mathematical Logic*. Thus came *Principia*'s Introduction, which was Russell's ultimate attempt to reconcile a hierarchy of levels with the doctrine of the unrestricted variable and the thesis that whatever *is* is an individual. According to the interpretation which Landini's offers and which we endorsed - according to which Russell was offering a nominalistic semantics for typed predicate variables in *Principia*'s Introduction - Russell's attempted to follow Frege in characterizing the occurrence of a 'predicative propositional function' - i.e., a predicate variable - as an argument of another in terms of *falling within* instead of falling under without accepting an ontology of incomplete entities. Russell's attempt was a failure[50], however, and one that struck at the very heart of the Logicist enterprise: it requires comprehension axioms like the Axiom of Reducibility, without which classical Mathematics cannot be recovered.

Indeed, it is this aspect of Russell's logicism more than any other that elicited the most sanguine and strident criticisms, in particular by those who favored constructivistic approaches to the foundations of Mathematics. Weyl, for instance, wrote the following:

> With his axiom of reducibility Russell [...] abandoned the road of logical analysis and turned from the constructive to the existential-axiomatic standpoint, - a complete *volte-face*.* After thus abolishing the several levels of properties, he still has the hierarchy of types: primary objects, their properties, properties of their properties, etc. And he finds that this alone will stop the known antinomies. But in the resulting system mathematics is no longer founded on logic,

[49] HYLTON, P., 1990, p.220; WITTGENSTEIN, L., 1953, p.91 §217.

[50] Because his intended nominalistic substitutional semantics cannot validate Reducibility.

> but on a sort of logician's paradise, a universe endowed with an "ultimate furniture" of rather complex structure and governed by quite a number of sweeping axioms of closure. The motives are clear, but belief in this transcendental world taxes the strength of our faith hardly less than the doctrines of the early Fathers of the Church or of the scholastic philosophers of the Middle Ages.[51]

Even those who were prone to resist Weyl's "Bolshevik"[52] impetus of tearing down classical Mathematics shared this skeptical reaction: Carnap, for instance, deemed Ramsey's attempt to salvage the structure of Simple Type theory by outright rejecting distinction of orders and introducing the so-called 'extensional propositional functions' in order to validate the Axiom of Reducibility a "theological"[53] form of Logicism. It is also with respect to this point that Quine's usual and well-known arguments against the theory of types and higher-order logic show their real strength, for it is at this point - i.e., when impredicative principles of comprehension are introduced or assumed - that one may reasonably question whether one is not dealing, after all, with "Set Theory in sheep's clothing"[54].

Even in light of the charitable interpretation of *Principia* we have endorsed, a variant of such concerns turn out to be well founded, even if we concede that *Principia* does manage to eliminate commitments to an ontology of classes. As we discussed, also following Landini's lead, Russell's desiderata for what a logical theory should accomplish forced the acceptance of a version of the theory of types in which: (i) predicate variables are structured in terms of simple types; (ii) predicate variables may occur both in predicate *and* subject positions; (iii) impredicative comprehension is assured by assuming that each open well-formed formula of the system determines an attribute or relation represented by a predicate variable. This is because Russell needed type distinctions - in some form or another - in order to deal with the contradictions, most crucially his own paradox, which may arise in terms of the vocabulary of classes but also in terms of predication; the goal of preserving classical Mathematics, on its turn, required impredicative comprehension; and since Russell's required the preservation of "logical common sense", which amounted to preserving the univocal conception of being according to which whatever *is* must be an individual, which, in turn, precluded him from accepting anything like Frege's notion of an incomplete entity, predicate variables had to be conceived as ranging over universals which themselves had to be counted as individuals of some simple type. So although *Principia*'s predicate variables are not wolves disguised as sheep exactly in the sense Quine claims, they are certainly not sheep in wolve's clothing[55] either.

The question, then is: can this interpretation of the theory of types which we are attributing to Russell here be motivated as a theory that deserves the label of Logic? My take is that

51 WEYL, H., 1946, p.6.
52 RAMSEY, F., 1926, p.56.
53 CARNAP, R., 1931, p.50.
54 QUINE, W., 1986, p.66.
55 Cf. QUINE, W., 1986, p.68-9. This is an expression Quine uses to refer to the opposite tendency of dressing "Set Theory in sheep's clothing", i.e., that of "over-acknowledgment", of speaking "[...] ostensibly of sets or attributes where logic in a narrower sense would have sufficed" (QUINE, W., 1986, p.68).

it cannot be so motivated. The root of the problem, I think, is one that has been hinted by Landini somewhat recently but which is also underlying the passage from Quine's *Philosophy of Logic* quoted above and which also appears in Dummett's penetrating commentaries on Frege[56]. Landini writes:

> The theory of simple types of attributes in extension is well-motivated by the conception that attributes have a predicable nature only. The simple-type indices on predicate variables track the predicable nature of attributes. Indeed, we have already had occasion to note that, quite independently of any paradox of classes/sets, Frege's conception of functions as 'unsaturated' led him to adopt a formal language of structured variables for this hierarchy of levels (simple types). Simple-type theory rests on solid philosophical foundations. The solid philosophical foundations of simple-type theory are obliterated by those who reject Frege's distinction between concept and object. Imposing a metaphysics in which attributes are themselves objects undermines *Principia*'s logicism, making simple-type theory *ad hoc*. Both Frege and Russell agreed that a simple-type theory of objects is little more than an *ad hoc* dodge of the paradoxes plaguing logicism. We now see that it also undermines *Principia*'s logicism by obliterating its thesis that mathematics (and logic) are abstract sciences of the structures given by relations, not by objects.[57]

The problem which Landini hints at here is that interpreting the theory of simple types in terms of a hierarchy which has at the bottom variables ranging over particulars that can only occur as subjects of predicate variables and then goes up, so to say, into an infinitely ascending stair of universals with both a predicable and predicative nature dismantles the very motivation for the simple type structure that both Frege and Russell employ for their development of Arithmetic. This is the point underlying the abyss that lies between Cocchiarella's and Quine's perspectives on higher-order logic: the former accepts Frege's doctrine of incompleteness while the latter does not. Quine is on firm ground when he refuses to accept letters *F*, *G*, *R*, *S*, etc... as ranging over universals or similar *abstracta*; for, as we already observed, if there are such and we accept the Russellian doctrine that whatever *is* is an individual, then they must be values of the same variables as other sorts of individuals - the only reason for not doing so being the need of dodging contradictions - which seems exactly the predicament that Russell ends up finding himself when *Principia*'s predicate variables with simple type indices are interpreted ranging over attributes and relations-in-intension[58]. This, perhaps, is the main reason why Quine cannot but see Russell's acceptance of bound predicate variables as a product of confusion.

The fundamental issue here is that the very distinction between universals and particulars as Russell conceived it is at odds with the theory of types that the contradictions and the demands of classical Mathematics forced upon him. This, as Michael Dummett has pointed long ago, is a

56 DUMMETT, M., 1967; 1981.
57 LANDINI, G., 2011, p.200.
58 That is, in stark opposition to the nominalistic semantics that Russell originally had in mind in the Introduction to the first edition.

difficulty (like many others surrounding the distinction between universals and particulars[59]) that Frege's logic obliterated completely, for, as Dummet puts it, "Frege's distinction between objects and concepts cuts clean across the traditional method of posing the problem of universals"[60]. Dummet writes:

> Traditionally a particular is that which can only be named (in order to have something predicated of it), whereas a universal can either be predicated of a particular or have some higher-level universal predicated of it; the dispute concerns whether, and in what sense, universals are 'real'. From this standpoint, therefore, the universal, redness, is denoted by the word 'red' equally when it is used as an adjective, as in 'The carpet is red', and as a noun, as in 'Red is a primary colour'; [...] For Frege such an approach was erroneous from the outset. The word 'red' used as a noun is a proper name and must stand for an object; a predicate like '... is red', on the other hand, is an expression of such a totally different kind that we cannot suppose it to be correlated with an entity of the same sort at all.[61]

The point here is that if we are willing to accept something like the hierarchy of levels whose structure is given along the lines of the simple theory of types, it seems that the the dichotomy of universals and particulars must be supplanted by that of concept and object - not merely because of the simple reason that the latter seems philosophically more economical and cogent - but more crucially because the very rationale for the hierarchy of levels which is embodied in the theory of simple types requires the distinction. It is no wonder that Frege, in his philosophical masterpiece, the *Grundlagen*, also included the following among his fundamental methodological guidelines:

> Never lose sight of the distinction between concept and object.[62]

Russell not only does not accept the distinction which this principle puts forward, but his most cherished and fundamental logical doctrine precludes him from even acknowledging the consistency - and perhaps the very ineligibility of the distinction.

Does this point, however, completely undermines the logical reconstruction of Arithmetic that is carried out in *Principia* and the other methodological principles that guide it, e.g.,

[59] One such problem, as Dummett also puts it, is that of explaining "[...] how sentences succeed in actually *saying* anything, true or false, for the sentence has appeared to degenerate into a list: Russell expresses this by saying that the proposition disintegrates under analysis" (DUMMETT, M., 1981, p.174); the problem here is indeed one Russell knew all too well. When he held a realistic view of propositions, the problem amounted to the question about what, after all, is the 'glue' that holds a proposition together. When he abandoned propositions and adopted his multiple-relation analysis of judgment, the problem became that of explaining how a relation 'relates' the terms of a judging complex, an issue which was famously involved in the abandoning of Russell's *Theory of Knowledge* manuscript.

[60] DUMMETT, M., 1978, p.99.

[61] DUMMETT, M., 1967, p.99-100.

[62] FREGE, G., 1884, p.x.

Russell's "Heuristic Maxim" which yields his syntatic reductionism with respect to the terms of number theory, and the view that numbers are "logical constructions"? Absolutely not. In fact, I would argue that a Fregean*[63] interpretation of the formal language of *Principia* lends a stronger plausibility to *Principia*'s eliminative approach.

For one thing, once the Axiom of Reducibility is re-interpreted in terms of a Fregean* conception of concepts and relations, the very statement of the axiom of amounts to an assumption that is shared those who employ some version of second and higher-order Logic like that presupposed by those who hold a Neo-Fregean or Abstractionist form of Logicism, namely, an impredicative principle of comprehension for variables of arbitrary simple type whose range comprises concepts and relations conceived as essentially predicative, i.e., not objects or individuals any sort. This, of course does not settle the complex issues surrounding second and higher-order logics, in particular those surrounding impredicativity, but it makes the starting point of the Russellian approach the same as that of many who advocate other forms of Logicism and those who assume that impredicative second and higher-order logics are worthy of the name[64].

Moreover, if one follows Frege in assigning to predicates and relation-symbols a reference that is itself incomplete or 'unsaturated', a strategy for further justifying a Russellian form of syntactic reductionism with respect to number terms becomes available. As we discussed, the fundamental issue with syntactic reductionism centers around motivating the elimination paraphrase on the face of the intrinsic plausibility of treating such terms as singular terms. The paradigmatic approach for refusing reductionism is that put forth by Wright, who refuses "the possibility that" once we "discard the preconceptions inbuilt into the syntax of our arithmetical language" we may "[...] find in reality there are no natural numbers, that our old way of speaking we had not succeeded in referring to anything"[65]; Wright finds this possibility unacceptable because he assumes that if we have an appropriate criterion for determining the truth conditions of sentences containing singular number-terms this settles the question of whether or not numbers are objects. Well, once we ascribe reference to predicates and we show that number terms may after all be conceived as higher-order concepts whose conditions of assertibility are exactly analogous to that of numerically definite quantifiers, we may distinguish a criterion for the existence of numbers as *objects* from a criterion of existence for numbers as concepts. This suggestion appears in the following passage of Dummett's commentary of Frege's Philosophy of Language:

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[63] Recall (cf. previous footnote 251 of chapter 2) that given the peculiarities of Frege's actual formulation of his Logic I refrained from calling a *predicate* calculus with structured variables a proper Fregean language, but found it more appropriate to call it a Fregean* language.

[64] And although argument by consensus is certainly not decisive, it must be noted that both of these groups are quite well-represented in the literature.

[65] WRIGHT, C., 1983, p.14.

> The criterion for the existence of a number considered as an object thus differs from the criterion for the existence of the corresponding second-level concept in just the same way as the criterion for the existence of a color considered as an object differs from the criterion for the existence of the corresponding first-level concept. Suppose there were only 8 objects in the universe. Then there would still be such a thing as there being nine things of a certain kind (nine objects falling under a given concept). The second-level predicate "There are just nine a's such that $\phi(a)$" would still have a reference: it would be irrelevant that there would be no concept to which it applied. But the symbol "9", when used as a singular term, as the name of an object, would have no reference: for if "9" is defined, say, as the "number greater than 8", there would be no concept such that the number of objects falling under it was one greater than 8.[66]

If we follow Russell in contextually eliminating number-terms by employing higher-order logic - as do Landini and Klement in their *Principia*-inspired reconstructions of Arithmetic - and follow Frege in assigning a reference to predicates that is itself incomplete, i.e., *not an object*, then we have precisely a way of avoiding the situation that Wright finds unacceptable - i.e., that we end up discovering that number-term did not after all have a reference: as he put it, that "[...] the scales having dropped from our eyes, as it were, and in reality there are no natural numbers, that our old way of speaking we had not succeeded in referring to anything"[67]. On the contrary, number-terms *do* have a reference, only not to objects. Embedded within a Fregean* conception of predicates and relations, Russellian number-terms refer to quantifier-like concepts, so they have a reference, only not to objects.

This, of course, does not settle the question of whether Logicism *must* dispense with a conception of numbers as objects nor whether it requires it and, again, it is needless to say that formidable problems remain and that new ones arise for the Russellian approach when such a move is made. Also, it is clear by this point that we have drifted away considerably from Russell's own views on Logic, Semantics and Metaphysics. We have, in particular, left behind the doctrine that above all others shaped the development of Russell's views on Mathematical Philosophy, namely the doctrine of the unrestricted variable and the univocity of being. Still, the approach does justice to a central aspect of Russell's views on Mathematical Philosophy which was ironically developed precisely to preserve this doctrine that is left behind at this point, namely: the idea that number-terms are incomplete symbols, not singular terms. The approach suggested here provides a way to keep Logicism alive and without the need to resort to any *postulated* specific ontology of numbers. Numbers are construed in terms of quantificational structures, as Landini puts it[68], and the vocabulary of Arithmetic remains that of higher-order logic. Despite possible objections and problems that can be raised, I think we may thus claim that, after all, that there may be still a quite substantial sense in which Russell's Logicism is preserved as a viable research program and also that the works of many contemporary Russellian scholars, in particular those of Landini and Klement, have established quite clearly that engaging Rus-

66 DUMMETT, M., 1981, p.263.
67 WRIGHT, C., 1983, p.14.
68 LANDINI, G., 2011*b*, p.183.

sell's works on Mathematical Philosophy and Russellian-inspired approaches to the Philosophy of Mathematics are worthwhile serious consideration and further research from contemporary philosophers of Mathematics, in particular to those who recognize or are willing to defend that Logicism may not be dead after all.

As is well known, Quine dedicated his finest work on Set Theory - *Set Theory and its Logic* - to Russell. As is characteristic of Quine, one specific trait of that work - which he, in fact inherited from Russell - is the effort put in minimizing existential commitments. When notified in personal correspondence that the book was written in his honor, Russell - who was ninety years old then - was flattered. As Quine reports, Russell responded at the end of October of 1962, with a letter that shows a mixture of appreciation, concern and enthusiasm wrapped in his characteristic wit and humor:

> I am highly honored and much pleased that you wish to dedicate your forthcoming book to me. If Kennedy and Khrushchev permit, I shall read it as soon as I get it, but, at present, it looks very doubtful whether any book not yet published ever will be.
>
> I shall be interested to see how you manage with weakened existence axioms. I always particularly admire your symbolic virtuosity and I am sure you will manage the job better than anybody else would[69].

The above display of wit also shows that more than five decades after Russell set himself the task of establishing Logicism, he still kept his concerns about the ontological commitments of Logic, Set Theory and Mathematics - concerns that I tried to emphasize as among the most fundamental for the development of his thought - close to his chest, even under the threat of nuclear fallout. Also, a year after after receiving the book, Russell wrote the following to him:

> Thank you warmly for sending me your book on Set Theory, and still more warmly for the dedication and inscription, both of which gave me the greatest pleasure. [...] I found your dedication particularly gratifying because so many logicians now-a-days seem to consider *Principia Malhemalica* worthless. I do not at all mind any number of emendations, but I like to think that, considering its date, it was not without merit. It is comforting to find that you think so.[70]

I hope that this dissertation may have also contributed, at least a little, in showing that *Principia Mathematica*[71] can be said to be many things, but worthless is not one of them.

[69] QUINE, W., 1988, p.229.
[70] QUINE, W., 1988, p.229-30.
[71] And Russell works on Mathematical Philosophy in general, for that matter.

Note on the Organization and Annotation of the Bibliography

The bibliography below is divided in three parts. The first contains all referenced texts of Russell, including published papers, books and unpublished manuscripts, plus the works co-authored by Russell and Whitehead. The second consists of collections of Russell's papers. The third consists of works by other authors. The bibliography includes works quoted and referenced in footnotes; there are a few sources that were consulted but not explicitly referenced (e.g., textbooks in Mathematical Logic and Set Theory). All references to published papers and unpublished manuscripts of Russell are to the volumes of Russell's *Collected Papers*. In each published paper the original reference is enclosed within brackets, as in:

> RUSSELL, B. *On Denoting*. In: URQUHART, A. (ed.). The Collected Papers of Bertrand Russell vol 3: Foundations of Logic 1903-05. London: Routledge, 1994, pp. [First published in: *Mind*, vol. 14, n. 56, 1905, pp. 479-493] [**1905*a***]

In some primary sources of historical importance other than Russell's works this is also indicated. If a paper by Russell was first published in an idiom other than English, this is enclosed in brackets as follows:

> RUSSELL, B. *General Theory of Well-Ordered Series.* In: MOORE, G. The Collected Papers of Bertrand Russell Volume 3: Towards *The Principles of Mathematics*. London: Routledge, 1993, pp.384-421. [First published as: Theorie generale des series bien-ordonnées. In: *Revue de mathematiques*, vol.8, pp.12-43.] [**1902*a***]

Again, this is also indicated for other important primary sources. If a paper of Russell which was written in an idiom other than english was first published before the *Collected Papers* this is also indicated, as follows:

> RUSSELL, B. On the Axioms of the Infinite and the Transfinite. In: SLATER, J. (ed.). The Collected Papers of Bertrand Russell vol.6: 1909-13. London: Routledge, 1992, pp.41-53. [Orignally published as: Sur les axiomes de l'infini et du transfini. In: *Séances Soc. Math. France*, 1911, pp.22-35; first published in english in: GRATTAN-GUINNESS, I. Dear Russell, Dear Jourdain. London: Duckworth, 1977, pp.162-175.] [**1911*c***]

If a cited work of Russell is an unpublished manuscript this is also disclosed in brackets. If the manuscript was previously published in a volume other than the *Collected Papers*, this is indicated as follows:

RUSSELL, B. *On The Substitutional Theory of Classes and Relations*. In: MOORE, G. The Collected Papers of Bertrand Russell Volume 5: Towards *Principia Mathematica*. London: Routledge, 2014, pp.243-261. [Manuscript. first published in: LACKEY, D. (ed.). *Essays in Analysis*. New York: Routledge, 1973, pp.165-189.] [**1906*a***]

This is also indicated for some primary sources other than works of Russell (mainly works of Frege and Peano). For works of Russell and original sources of historical importance, the original year of publication is *always* used in citations throughout the present work. As indicated above, the year used for citations is enclosed within brackets in bold accompanied by italic lowercase letters whenever more than one work by the same author with the same publication year is referenced. No conventions for abbreviations are adopted.

Below follows a quick reference list of Russell's cited works[72]:

1901*a*	*Recent Work on the Principles of Mathematics* (or *Mathematics and the Metaphysicians*).
1901*b**	*Recent Italian Work on the Foundations of Mathematic.*
1902	*The Logic of Relations with Some Applications to the Theory of Series.*
1903	*The Principles of Mathematics.*
1904*a*	*Meinong's Theory of Complexes and Assumptions.*
1904*b*	*The Existential Import of Propositions.*
1905*a*	*On Denoting.*
1905*b*	*On Some Difficulties in the Theory of Transfinite Numbers and Order Types.*
1905*c**	*On Fundamentals.*
1905*d**	*Necessity and Possibility.*
1905*e**	*Letter to Hardy on Substitution.*
1905*f**	*On Substitution* (22 December 1906).
1905*g*	*On the Relation of Mathematics to Symbolic Logic.*
1906*a**	*On The Substitutional Theory of Classes and Relations.*

72 Those references marked with * are unpublished papers or manuscripts. As observed in the annotation of the bibliography, all citatiotions of Russell's published and unpublioshed papers (and manuscripts) refer to the *Collected Papers*. Of course, for complete bibliographical information the reader should consult the bibliography at the end of the dissertation.

1906*b* *The Paradoxes of Logic* (or *Insolubilia and Their Solution Through Symbolic Logic*).

1906*c* *The Theory of Implication.*

1906*d** *On Substitution* (April/May 1906).

1906*e** *Logic in Which Propositions are Not Entities.*

1906*f** *The Paradox of the Liar.*

1906*g** *Multiplicative Axiom.*

1906*h** *List of Propositions.*

1906*i** *On the Functional Theory of Propositions, Classes and Relations.*

1907*a* *On the Nature of Truth.*

1907*b* *The Regressive Method of Discovering the Premises of Mathematics.*

1907*c** *A Paradox of the Substitutional Theory* (Letter to Hawtrey from 22 January).

1907*d** *Types.*

1907*e** *On Types.*

1907*f** *Notes on Types.*

1907*g** *Fourth Theory.*

1907*h** *Fundamentals*.

1907*i** *Individuals.*

1908 *Mathematical Logic as Based on the Theory of Types.*

1910*a* *The Theory of Logical Types.*

1910*b* *On the Nature of Truth and Falsehood.*

1911*a* *The Philosophical Importance of Mathematical Logic*.

1911*b* *Knowledge by Acquaintance and Knowledge by Description.*

1911*c* *On the Axioms of the Infinite and the Transfinite.*

1911*d* *Analytic Realism*.

1912*a* *On the Relations of Universals and Particulars.*

1912*b* *The Problems of Philosophy*.

1912*c** *What is Logic?*

1913* *Theory of Knowledge: The 1913 Manuscript*.

1914*a* *Our Knowledge of the External World as a Field for Scientific Method in Philosophy.*

1914*b* *The Relation of Sense-Data to Physics.*

1918 *The Philosophy of Logical Atomism.*

1919*a* *Introduction to Mathematical Philosophy.*

1919*b* *On Propositions: What They Are and How They Mean.*

1924 *Logical Atomism.*

1937 *Introduction to the second edition of The Principles of Mathematics.*

1944 *My Mental Development.*

1950 *An Inquiry Into Meaning and Truth.*

1957 *Logic and Ontology.*

1959 *My Philosophical Development.*

1967 *The Autobiography of Bertrand Russell.*

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Appendix

APPENDIX A – The Ancestral and the Peano Postulates in *Principia*

A.1 Guidelines for Reading This Appendix

We do not include the steps or theorems that would *fully* justifiy the introduction of the previous theorems in a line of proof for that would require *a lot* of space. We do include, however, some supplementary steps to make clear the general outline of the proofs. For a proof of the Peano postulates in a modern (simple) theory of types of sets that is faithful to *Principia*'s main traits, see the next Appendix.

We follow the general style of proofs employed by William Hatcher in his *The Logical Foundations of Mathematics*[1]. Our goal was to preserve the general structure of Russell and Whitehead's proofs while balancing readabilty and concision. The guidelines for reading the translated proofs are the following.

***l,l'*,MP.** This indicates the use of the rule of *Modus Ponens* from two previous lines.

Taut. This indicates that the justification for introducing a formula on the line *l* is the fact that it is a tautology.

***l*,Taut,MP.** This Indicates a step that uses some tautological substitution and/or some tautology (or tautologies) and the rule of Modus Ponens.

This will be used just like Whitehead and Russell use their Transp. justification for introducing lines in proofs, as a means to shorten proofs when the the steps abbreviated are mere use of

1 HATCHER, W., 1980.

propositional logic [we include elimination of hypothesis in conditional proofs under this case].

For example :

$\{...\}l.$	$(p \wedge q) \longrightarrow (Fx \longrightarrow Gx)$	[...]
$\{...\}l'.$	$(p \wedge q) \longrightarrow (\neg Fx \vee Gx)$	l, l' Taut, MP.

Abbreviates :

$\{...\}l.$	$(p \wedge q) \longrightarrow (Fx \longrightarrow Gx)$	[...]
$\{...\}m$	$(Fx \longrightarrow Gx) \longrightarrow (\neg Fx \vee Gx)$	Taut.
$\{...\}n$	$([l] \longrightarrow ([m] \longrightarrow ((p \wedge q) \longrightarrow (\neg Fx \vee Gx))$	Taut.
$\{...\}p$	$[m] \longrightarrow ((p \wedge q) \longrightarrow (\neg Fx \vee Gx)$	$l, n,$ MP.
$\{...\}l'.$	$(p \wedge q) \longrightarrow (\neg Fx \vee Gx)$	$m, p,$ MP.

***l*,Pl.** This indicates a step that uses laws of predicate logic (or predicate logic and propositional logic).

For example :

$\{...\}l.$	$(\exists y)Hy \longrightarrow (x)(Fx \longrightarrow Gx)$	[...]
$\{...\}l'.$	$(x)((\exists y)Hy \longrightarrow (Fx \longrightarrow Gx))$	l, Pl.

Abbreviates :

$\{...\}l.$	$(\exists y)Hy \longrightarrow (x)(Fx \longrightarrow Gx)$	[...]
$\{...\}m.$	$[l] \longrightarrow (x)((\exists y)Hy \longrightarrow (Fx \longrightarrow Gx))$	Pl Theorem.
$\{...\}l'$	$(x)((\exists y)Hy \longrightarrow (Fx \longrightarrow Gx))$	$l, m,$ MP.

***l*,*l'*,Pl.** This indicates a step that uses laws of predicate logic with identity.

For example :

$\{...\}l.$	Fa	[...]
$\{...\}l'$	$a = b$	[...]
$\{...\}l''.$	Fb	l, l', Pl $=$.

Abbreviates :

$\{...\}l.$	Fa	[...]
$\{...\}m.$	$a = b$	[...]
$\{...\}n.$	$a = b \longrightarrow (Fa \longleftrightarrow Fb)$	Pl $=$ Theorem.
$\{...\}l'$	Fb	$l, m, n,$ Taut, MP.

$*m{\cdot}n$ This indicates that the justification for introducing the formula is the fact that it is previously proved theorem of *Principia.*

Example :

$\{...\}l.$	$0 \in \mathbb{N}$	$*120 \cdot 12.$

[...]Df. This indicates that the justification for introducing the formula is a definition of *Principia* (and the fact that the *definiendum* can always be substitued by the *definiens*).

Example :

$\{...\}l.$	$0 \in \mathbb{N}$	[...]
$\{...\}l$	$\{\alpha : \alpha = \emptyset\} \in \mathbb{N}$	$l,$ 0Df.

***l*,UG.** This indicates the use of the rule of Universal Generalization (i.e., *from* ϕ,*infer* $(x)\phi$).

***l*,UI.** This indicates the use of the rule of Universal Instantiation (i.e., *from* $(x)\phi[x]$,*infer* $\phi[x/y]$).

***l,m,*Ext.** This indicates the use of laws equivalent to or derived from the fact that two classes α and β are identical iff they have the same members.

Example :

$\{...\}l.$	$(x)(Fx \longleftrightarrow Gx)$	[...]
$\{...\}m$	$\alpha = \{x : Fx\}$	[...]
$\{...\}l$	$\alpha = \{x : Gx\}$	$l, m,$ Ext.

***l*,Comp.** This indicates that some principle of comprehension for concepts or relations is needed for this step.

Example :

$\{...\}l.$	$(F)(Fx \longrightarrow (Fx \vee Gx)$	[...]
$\{...\}m.$	$(\exists y)(Hx \wedge Rxy) \longrightarrow (\exists y)((Hx \wedge Rxy) \vee Gx)$	$Comp, l\left[\dfrac{(\exists y)Hz \wedge Rzy}{Fz}\right]$, UI.

A.2 The Ancestral

This appendix transcribes and translates the proofs of the most useful theorems about Whitehead and Russell's version of Frege's ancestral. The main definition is:

$*90.01 \qquad R_* =_{df} \{< x, y >: x \in C(R) \wedge (\mu)[(z, w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu)]\}$

Which is embodied in the following theorem:

$*90 \cdot 111 \quad xR_*y \longleftrightarrow [(x \in C(R) \wedge (\mu)((z, w)(z \in \mu \wedge zRw \longrightarrow w \in \mu)) \longrightarrow y \in \mu)]$

This is the most used proposition in $*90$ and is proved just by doing what we did, unpacking the successive definitions ∗90·01, ∗31·02 and ∗37·01.

Now, passing to the actual theorems, first we have:

∗90·112 $\vdash :. xR_*y : \phi z \,.\, zRw \,.\supset_{z,w} .\, \phi w : \phi x : \supset .\, \phi y$

Dem.

$\vdash .\, *90 \cdot 111 \dfrac{\hat{z}(\phi z)}{\mu} .\supset$

$\vdash :: xR_*y \,.\quad \supset :.\quad z \,\epsilon\, \hat{z}(\phi z) \,.\, zRw \,.\supset_{z,w} .\, w \,\epsilon\, \hat{z}(\phi z) : x \,\epsilon\, \hat{z}(\phi z) : \supset : y \,\epsilon\, \hat{z}(\phi z) :.$

[∗20·03] $\quad \supset :.\quad \phi z \,.\, zRw \,.\supset_{z,w} .\, \phi w : \phi x : \supset .\, \phi y \qquad$ (1)

$\vdash .\, (1) \,.\, \mathrm{Imp} \,.\quad \supset .\quad \vdash .\,$ Prop

We may translate it as follows:

$*90 \cdot 112 \qquad (xR_*y \wedge (z)(w)((\phi z \wedge zRw) \longrightarrow \phi w) \wedge \phi x) \longrightarrow \phi y$

Λ1. $xR_*y \longleftrightarrow$
$x \in C(R) \wedge (\mu)[(z, w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu)]$ $\qquad *90 \cdot 111, R_*$Df.

Λ2. $xR_*y \longrightarrow$
$(\mu)[(z, w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu)]$ $\qquad$ 1, Taut, MP.

Λ3. $xR_*y \longrightarrow$
$(z, w)((z \in \{x : \phi x\} \wedge zRw) \longrightarrow w \in \{x : \phi x\})$
$\longrightarrow y \in \{x : \phi x\})$ $\qquad 2[\mu/\{x : \phi x\}]$, UG.

Λ4. $x \in \{x : \phi x\} \longleftrightarrow \phi x$ $\qquad \in$ Df, $*20 \cdot 03$.

Λ5. $xR_*y \longrightarrow [(z, w)((\phi z \wedge zRw) \longrightarrow \phi w) \longrightarrow \phi y)]$ $\qquad$ 3, 4, Taut, MP.

Λ6. $[xR_*y \wedge (z, w)((\phi z \wedge zRw) \longrightarrow \phi w)] \longrightarrow \phi y$ $\qquad$ 5, Taut, MP.

Next, we have the theorem which asserts the reflexivity of R_* within the field of *R*:

$*90{\cdot}12 \;\vdash : x \,\epsilon\, C\text{‘}R \,.\equiv.\, xR_*x$

Dem.

$\vdash . *90{\cdot}1 . \quad \supset\vdash : xR_*x . \supset . x \,\epsilon\, C\text{‘}R$ (1)

$\vdash . *3{\cdot}27 . *10{\cdot}11 . \quad \supset\vdash : \breve{R}\text{“}\mu \subset \mu . x \,\epsilon\, \mu . \supset_\mu . x \,\epsilon\, \mu :$

$[*3{\cdot}21] \quad \supset\vdash :. x \,\epsilon\, C\text{‘}R . \quad \supset : \quad x \,\epsilon\, C\text{‘}R . \breve{R}\text{“}\mu \subset \mu . x \,\epsilon\, \mu . \supset_\mu . x \,\epsilon\, \mu :$

$[*90{\cdot}1] \quad \supset : \quad xR_*x$ (2)

$\vdash . (1) . (2) . \supset\vdash .$ Prop

We can translate it as:

$*90 \cdot 12 \qquad x \in C(R) \longleftrightarrow xR_*x$

Λ1. $xR_*y \longleftrightarrow$
$x \in C(R) \wedge (\mu)[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu)]$ — $*90 \cdot 1, R_*$Df.

Λ2. $xR_*x \longrightarrow x \in C(R)$ — 1, Taut, MP.

Λ3. $[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu) \wedge x \in \mu] \longrightarrow x \in \mu$ — Taut.

Λ4. $(\mu)[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu) \wedge x \in \mu] \longrightarrow x \in \mu$ — 3, UG.

Λ5. $x \in C(R) \longrightarrow (x \in C(R) \wedge [4])$ — Taut.

Λ6. $(x \in C(R) \wedge [4)] \longleftrightarrow xR_*x$ — $*90{\cdot}1, R_*$Df.

Λ7. $x \in C(R) \longrightarrow xR_*x$ — 5, 6, Taut, MP.

Λ8. Prop. — 2, 7, Taut, MP.

Next we have the theorem which asserts that any relation *R* is contained in its ancestral:

$*90{\cdot}151 \;\vdash . R \in R_*$

Dem.

$*11{\cdot}1 . \supset\vdash :: z \,\epsilon\, \mu . zRw . \supset_{z,w} . w \,\epsilon\, \mu : \quad \supset :. x \,\epsilon\, \mu . xRy . y \,\epsilon\, \mu :.$

$[\text{Exp.Comm}] \quad \supset :. xRy . \supset : x \,\epsilon\, \mu . \supset . y \,\epsilon\, \mu$ (1)

$\vdash . (1) . \supset \quad \vdash :: xRy . \quad \supset :. z \,\epsilon\, \mu . zRw . \supset_{z,w} . w \,\epsilon\, \mu : x \,\epsilon\, \mu : \supset_\mu . y \,\epsilon\, \mu$ (2)

$\vdash . (2) . *10{\cdot}11 . 21 . \supset \quad \vdash :: xRy . \quad \supset :. z \,\epsilon\, \mu . zRw . \supset_{z,w} . w \,\epsilon\, \mu . x . \mu : \supset_\mu . y \,\epsilon\, \mu :.$

$[*90{\cdot}111 . *33{\cdot}17] \quad \supset :. xR_*y :: \supset\vdash .$ Prop

Its translation is straightforward:

$*90 \cdot 151 \quad (x)(y)(xRy \longrightarrow xR_*y)$

$\Lambda 1.\quad (z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow ((x \in \mu \wedge xRy) \longrightarrow y \in \mu)$ PL, $*$II $\cdot$ I.

$\Lambda 2.\quad [(x \in \mu \wedge xRy) \wedge (z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu)] \longrightarrow y \in \mu$ I, Taut, MP.

$\Lambda 3.\quad xRy \longrightarrow$
$[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \wedge x \in \mu \longrightarrow y \in \mu]$ 2, Taut, MP.

$\Lambda 4.\quad (\mu)[3]$ 3, UG.

$\Lambda 5.\quad xRy \longrightarrow$
$(\mu)[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \wedge x \in \mu \longrightarrow y \in \mu]$ 4, PL.

$\Lambda 6.\quad xR_*y \longleftrightarrow$
$x \in C(R) \wedge (\mu)[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu)]$ $*90 \cdot 111$.

$\Lambda 7.\quad xRy \longrightarrow (x \in C(R) \wedge y \in C(R))$ $*33 \cdot 17$.

$\Lambda 8.\quad xRy \longrightarrow$
$[x \in C(R) \wedge (\mu)[(z,w)((z \in \mu \wedge zRw) \longrightarrow w \in \mu) \longrightarrow y \in \mu)]]$ 5, 7, Taut, MP.

$\Lambda 9.$ Prop. $8, *90 \cdot 01, R_*$Df.

Next we have a group of propositions which are expressed using *Principia*'s notation of relative products. Recall that the relative product of two relations *R* and *S* is defined as:

$*34{\cdot}01 \qquad R|S = \hat{x}\hat{y}\{(\exists y)xRy \,.\, ySz\} \quad \text{Df}$

Which, translated, is the relation *R*|*S* such that:

$*34 \cdot 1' \qquad xR|Sz \longleftrightarrow (\exists y)(xRy \wedge ySz)$

Almost all the proofs are straightforward. The only (somewhat laborious) proof is that of the proposition which asserts that the relative product R/R_* of a relation *R* and its ancestor is contained in the ancestor of *R* ($*90{\cdot}172$), because it is proved via the stronger proposition which asserts that the relative product of *R-ancestry* and itself is contained in itself, i.e., R_* is transitive. First we have:

$*90\cdot16 \vdash . R|R_* \subset R_*$

Dem.

$*11\cdot1 . \supset \vdash :. z \epsilon \mu . zRw \supset_{z,w} . w \epsilon \mu : \supset : y \epsilon \mu . yRv . \supset . v \epsilon \mu$ (1)

$\vdash . *90\cdot11 . *10\cdot1 .$ Fact $. \supset$

$\vdash :: xR_*y . yRv . \supset :. z \epsilon \mu . zRw \supset_{z,w} . w \epsilon \mu : x \epsilon \mu : \supset . y \epsilon \mu . yRv$ (2)

$\vdash . (1) . (2) . \supset$

$\vdash :: xR_*y . yRv . \supset :. z \epsilon \mu . zRw \supset_{z,w} . w \epsilon \mu : x \epsilon \mu : \supset . v \epsilon \mu.$ (3)

$\vdash . (3) . *10\cdot11\cdot21 . *90\cdot111 . \supset$

$\vdash : xR_*y . yRv . \supset . xR_*v$ (4)

$\vdash . (4) . *10\cdot11\cdot21 . *34\cdot1 . \supset .$ Prop

This asserts that the relative product of *R* and its ancestral is contained in the ancestral of *R*, that is: for every *x* and *z* such that there is a *y* such that *x* has the relation *R* to *y* and *y* is an ancestral of *z* with respect to *R*, *x* is an ancestral of *z* with respect to *R*. Once again, we may translate it following *Principia*'s proof very closely:

$*90\cdot16 \quad (x)(v)[(\exists y)(xR_*y \wedge yRv) \longrightarrow xR_*v]$

1. $(z)(w)[z \in \mu \wedge zRw \longrightarrow w \in \mu] \longrightarrow (y \in \mu \wedge yRv \longrightarrow v \in \mu)$ PL.
2. $xR_*y \longrightarrow$
   $(z,w)(z \in \mu \wedge zRw \longrightarrow w \in \mu) \longrightarrow y \in \mu)$ $*90\cdot111$
3. $xR_*y \wedge yRv \longrightarrow$
   $[(z,w)(z \in \mu \wedge zRw \longrightarrow w \in \mu) \wedge x \in \mu \longrightarrow y \in \mu \wedge yRv]$ 2, Taut, MP.
4. $xR_*y \wedge yRv \longrightarrow$
   $[(z)(w)(z \in \mu \wedge zRw \longrightarrow w \in \mu) \wedge x \in \mu \longrightarrow v \in \mu]$ 1, 3, Taut, MP.
5. $(\mu)[4]$ 4, UG.
6. $xR_*y \wedge yRv \longrightarrow$
   $(\mu)[(z)(w)(z \in \mu \wedge zRw \longrightarrow w \in \mu) \wedge x \in \mu \longrightarrow v \in \mu]$ 5, PL.
7. $xR_*y \longrightarrow x \in C(R)$ $*33\cdot17$.
8. $xR_*v \longleftrightarrow$
   $[(x \in C(R) \wedge (\mu)((z,w)(z \in \mu \wedge zRw \longrightarrow w \in \mu)) \longrightarrow y \in \mu)]$ $*90\cdot111$
9. $xR_*y \wedge yRv \longrightarrow xR_*v$ 7, R_*Df.
10. $(y)(xR_*y \wedge yRv \longrightarrow xR_*v)$ 9, UG.
11. $(\exists y)(xR_*y \wedge yRv) \longrightarrow xR_*v$ 10, PL.
12. $(x)(v)[(\exists y)(xR_*y \wedge yRv) \longrightarrow xR_*v]$ 11, UG.

Next, we have:

$*90{\cdot}161 \vdash : S \Subset R_* . \supset . S|R \Subset R_*$

Dem.

$\vdash . *34{\cdot}34 . \supset \vdash : \text{Hp} . \supset . S|R \Subset R_*|R$ (1)

$\vdash . *90{\cdot}16 . \supset \vdash . \text{Prop}$

This states that if a relation *S* is contained in the ancestor of *R*, then the relative product of *S* and *R* is contained in the ancestor of *R*. This is quite easily proved from a $*90 \cdot 16$ and a lemma from $*34$ which assserts that if *P*, *Q*, *T*, *U* are relations such that *P* is contained in *Q* and *T* is contained in *U*, then the relative product of *P* and *T* is contained in the relative product of *Q* and *U*:

$*90 \cdot 161$	$(x)(y)(xSy \longrightarrow xR_*y) \longrightarrow (x)(z)((\exists y)(xSy \wedge yRz) \longrightarrow xR_*z)$	
$\{1\}1.$	$(x)(y)(xSy \longrightarrow xR_*y)$	Hp.
$\Lambda 2.$	$(P,Q,U,T)\{[(w,v)(wPv \longrightarrow wQv) \wedge (w,v)(wTv \longrightarrow wUv)]$	
Λ	$\longrightarrow (w,v)[(\exists u)(wPu \wedge uTv) \longrightarrow (\exists u)(wQu \wedge uUv)]\}$	$*34 \cdot 34.$
$\Lambda 3.$	$(Q,U,T)\{[(w,v)(wSv \longrightarrow wQv) \wedge (w,v)(wTv \longrightarrow wUv)]$	
Λ	$\longrightarrow (w,v)[(\exists u)(wSu \wedge uTv) \longrightarrow (\exists u)(wQu \wedge uUv)]\}$	$[P/S]$UI.
$\Lambda 4.$	$(U,T)\{[(w,v)(wSv \longrightarrow wR_*v) \wedge (w,v)(wTv \longrightarrow wUv)]$	
Λ	$\longrightarrow (w,v)[(\exists u)(wSu \wedge uTv) \longrightarrow (\exists u)(wR_*u \wedge uUv)]\}$	$[Q/R_*]$UI.
$\Lambda 5.$	$(T)\{[(w,v)(wSv \longrightarrow wR_*v) \wedge (w,v)(wTv \longrightarrow wRv)]$	
Λ	$\longrightarrow (w,v)[(\exists u)(wSu \wedge uTv) \longrightarrow (\exists u)(wR_*u \wedge uRv)]\}$	$[U/R]$UI.
$\Lambda 6.$	$(w,v)(wSv \longrightarrow wR_*v) \wedge (w,v)(wRv \longrightarrow wRv)$	
Λ	$\longrightarrow (w,v)[(\exists u)(wSu \wedge uRv) \longrightarrow (\exists u)(wR_*u \wedge uRv)]\}$	$[T/R]$UI.
$\Lambda 7.$	$(w,v)(wRv \longrightarrow wRv)$	PL.
$\{1\}8.$	$(w,v)[(\exists u)(wSu \wedge uRv) \longrightarrow (\exists u)(wR_*u \wedge uRv)]$	1, 7, MP.
$\Lambda 9.$	$(w,v)[(\exists u)(wR_*u \wedge uRv) \longrightarrow wR_*v]$	$*90 \cdot 16$
$\{1\}10.$	$(w,v)[(\exists u)(wSu \wedge uRv) \longrightarrow wR_*v]$	9, PL.
$\Lambda 11.$	Prop.	1, 10, Taut, MP.

Observe that the only thing that extends the proof are the successive separate applications of UI in steps 3-6, which we made explicit in order to make clear what is going on. This proposition is proved as a lemma for the next, which has more direct applications, for instance in the theory of series:

$*90{\cdot}162 \quad \vdash . R|R \Subset R_*$ [$*90{\cdot}151{\cdot}161$]

This asserts that for every x and z, if there is a y such that x has R to y and y has R to z, then x is an ancestor of z with respect to R. The case of finite Arithmetic makes its meaning clear: let R be the relation of immediate predecessor; for every x and z, if there is a y such that x precedes y and y precedes z, then x is an ancestor of z with respect to the predecessor relation. TransLating the proof we have the quite straightforward:

$*90 \cdot 162 \qquad (x)(z)((\exists y)(xRy \wedge yRz) \longrightarrow xR_*z)$

$\Lambda 1. \quad (x)(y)(xRy \longrightarrow xR_*y) \qquad *90 \cdot 151.$

$\Lambda 2. \quad (S)[(x)(y)(xSy \longrightarrow xR_*y) \longrightarrow (x)(z)((\exists y)(xSy \wedge yRz) \longrightarrow xR_*z)] \quad *90 \cdot 161, \text{UG}.$

$\Lambda 3. \quad (x)(y)(xRy \longrightarrow xR_*y) \longrightarrow (x)(z)((\exists y)(xSy \wedge yRz) \longrightarrow xR_*z) \qquad [S/R]\text{UI}.$

$\Lambda 4. \quad \text{Prop.} \qquad 1, 3, \text{MP}.$

The next two proofs are, by far, the trickiest to translate. First, we have:

$*90{\cdot}163 \qquad \vdash . \breve{R}``\overleftarrow{R}`x \subset \overleftarrow{R_*}`x \; [*37{\cdot}301 \, . \, {\cdot}32{\cdot}19 \, . \, *90{\cdot}16]$

This proposition, together with ∗37·301 that is referenced in its proof is a good example of how several of *Principia*'s propositions just indicate how the same facts can be written in different ways. Theorem ∗37·301 is the counterpart for ∗37·3, both conerning respectively the notations $\overrightarrow{R}$ and $\overleftarrow{R}$, respectively. They only provide proof for ∗37·3, but it is trivial to retrace step by step how they would prove ∗37·301: "

$*37{\cdot}3 \vdash . \{\text{sg}`(P|Q)\}`z = P``\overrightarrow{Q}`z$

Dem.

$\vdash *32{\cdot}23{\cdot}13 \, . \supset$

$\vdash . \{\text{sg}`(P|Q)\}`z \quad = \hat{x}\{(P|Q)\}$

$[*34{\cdot}1] \quad = \hat{x}\{(\exists y) \, . \, xPy \, . \, yQz\}$

$[*32{\cdot}18] \quad = \hat{x}\{((\exists y) \, . \, xPy \, . \, y \in \overrightarrow{Q}`z\}$

$[*37{\cdot}01] \quad = P``\overrightarrow{Q}`z \, . \supset \vdash \text{Prop}$

$*37{\cdot}301 \vdash . \{\text{gs}`(P|Q)\}`x = \breve{Q}``\overleftarrow{P}`x$

Dem.

$\vdash *32{\cdot}231{\cdot}131 \, . \supset$

$\vdash . \{\text{gs}`(P|Q)\}`x \quad = \hat{z}\{(P|Q)\}$

$[*34{\cdot}1] \quad = \hat{z}\{(\exists y) \, . \, xPy \, . \, yQz\}$

$[*32{\cdot}181] \quad = \hat{z}\{(\exists y) \, . \, y \in \overleftarrow{P}`z \, . \, yQz\}$

$[*31{\cdot}11] \quad = \hat{z}\{(\exists y) \, . \, y \in \overleftarrow{P}`z \, . \, z\breve{Q}y\}$

$[*37{\cdot}01] \quad = \breve{Q}``\overleftarrow{P}`x$

Now, with the compact notation $\breve{Q}``\overleftarrow{P}`x$ translated as the simpler $\hat{z}\{(\exists y) \, . \, xPy \, . \, yQz\}$, the meaning of theorem ∗90·163 is made clear: it is just the class counterpart of proposition ∗90·16; it asserts that for every z, if z belongs to the class of all v such that $(\exists y)(xR_*y \wedge yRv)$, then z belongs to the class of all v such that xR_*v. That is:

$*90 \cdot 163 \quad (z)[z \in \{v : (\exists y)(xR_*y \wedge yRv)\} \longrightarrow z \in \{v : xR_*v\}]$

Which is equivalent to $(w, v)[xR_*w \wedge wRv \longrightarrow xR_*v]$. As Whitehead and Russell put it, this amounts to showing that $\{z : xR_*z\}$ is an hereditary class, something that is used in the proof of the next, very important proposition, in which ∗90·163 is referenced. Whitehead and Russell have:

∗90·17 ⊦ . $R_*|R_* = R_*$

Dem.

⊦ . ∗90·13 .	⊃⊦ : xR_*y.	⊃ . $xR_*y \,.\, yR_*y$	
[∗35·5 . ∗10·24]		⊃ . $xR_*\|R_*y$	(1)
⊦ . ∗90·163·1 $\frac{\overleftarrow{R}‘x}{\mu}$.	⊃⊦ :. yR_*z.	⊃ : $y \in \overleftarrow{R}‘x \,.\supset .\, z \in \overleftarrow{R}‘x$:	
[∗32·181]		⊃ : $xR_*y \,.\supset .\, xR_*z$	(2)
⊦ . (2) . Imp.	⊃⊦ : $xR_*y \,.\, yR_*z \,.\supset .\, xR_*z$:		
[∗11·11 . ∗34·55]	⊃⊦ : $R_*\|R_* \subset R_*$		(3)
⊦ . (1) . (3) . ⊃⊦ . Prop			

The theorem states that the relative product of $R_*$and itself is identical with R_*. The interesting half of the theorem are the steps (2) and (3) which actually show that R_* is transitive and then derive the desired result using theorem ∗34·55, which is:

∗34·55 $\quad R|R \subset S \,.\equiv :\, xRy \,.\, yRz \,.\supset_{x,y,z} .\, xSz$ [∗34·5 . 10·21]

The theorem is proved using *Principia's* theorem ∗10·23 which is the familiar scheme from quantification theory:

$*10 \cdot 23 \quad (x)(\mathbf{A}[x] \longrightarrow \mathbf{B}) \equiv [(\exists x)\mathbf{A}[x] \longrightarrow \mathbf{B}]$ where *x* does not occur free in B.

We can translate the interesting half of ∗90·17 as a conditional proof and make explicit several ommited steps of the original (second half) of the proof to make it more readable:

$*90 \cdot 17$	$(x)(z)[(\exists y)(xR_*y \wedge yR_*z) \longleftrightarrow xR_*z]$	
$\{1\}1.$	xR_*y	Hp.
$\{2\}2.$	yR_*z	Hp.
$\Lambda 3.$	$yR_*z \longrightarrow$	
	$(\mu)((w)(v)(w \in \mu \wedge wRv \longrightarrow v \in \mu) \wedge y \in \mu \longrightarrow z \in \mu$	$*90 \cdot 111.$
$\{2\}4.$	$[(w)(v)(w \in \{u : xR_*u\} \wedge wRv \longrightarrow v \in \{u : xR_*u\}) \wedge$	2, Taut, MP,
	$y \in \{u : xR_*u\}] \longrightarrow z \in \{u : xR_*\}$	$3[\mu/\{u : xR_*u\}]$UI.
$\{2\}5.$	$[(w)(v)(xR_*w \wedge wRv \longrightarrow xR_*v) \wedge xR_*y] \longrightarrow xR_*z$	$4, \in$ Df.
$\{2\}6.$	$((xR_*y \wedge yRz \longrightarrow xR_*z) \wedge xR_*y) \longrightarrow xR_*z$	$5, [w/y; v/z]$UI.
$\{2\}7.$	$(xR_*y \wedge yRz \longrightarrow xR_*z) \longrightarrow (xR_*y \longrightarrow xR_*z)$	6, Taut, MP.
$\Lambda 8.$	$xR_*y \wedge yRz \longrightarrow xR_*z$	$*90 \cdot 163.$
$\{2\}9.$	$xR_*y \longrightarrow xR_*z$	8, 7, MP.
$\{1,2\}10.$	xR_*z	2, 9, MP.
$\Lambda 11.$	$(xR_*y \wedge xR_*y) \longrightarrow xR_*z$	$1 - 9$, Taut, MP.
$\Lambda 12.$	$(x)(y)(z)[(xR_*y \wedge xR_*y) \longrightarrow xR_*z]$	11, UG.
$\Lambda 13.$	$(R,S)\{(x)(z)[(\exists y)(xRy \wedge yRz) \longrightarrow xSz] \longleftrightarrow$	
	$(x)(y)(z)[(xRy \wedge yRz) \longrightarrow xSz]$	$*34 \cdot 55.$
$\Lambda 14.$	Prop.	$13[R/R_*; S/R_*]$UI,
		12, 13, Taut, MP.

Finally, we have the theorem which asserts that the relative product of *R* and its ancestral R_* is contained in R_*:

$*90{\cdot}172 \vdash . R|R_* \subset R_*$

Dem.

$\vdash . *90{\cdot}151 . \supset \vdash . R|R_* \subset R_*|R_*$ (1)

$\vdash . *90{\cdot}17 . \supset \vdash .$ Prop

Which has a straightforward translation:

$*90 \cdot 172$	$(x)(z)((\exists y)(xRy \wedge yR_*z) \longrightarrow xR_*z)$	
$\{1\}1.$	$(\exists y)(xRy \wedge yR_*z)$	Hp.
$\Lambda 2.$	$(w)(y)(wRy \longrightarrow wR_*y)$	$*90 \cdot 151.$
$\Lambda 3.$	$(\exists y)(xRy \wedge yR_*z) \longrightarrow (\exists y)(xR_*y \wedge yR_*z)$	1, 2, PL.
$\Lambda 4.$	$(x)(z)[(\exists y)(xRy \wedge yR_*z) \longrightarrow (\exists y)(xR_*y \wedge yR_*z)]$	3, UG.
$\Lambda 5.$	$(x)(z)[(\exists y)(xR_*y \wedge yR_*z) \longleftrightarrow xR_*z]$	$*90 \cdot 17.$
$\Lambda 6$	Prop	

Note, that using the scheme ∗10·23 (or theorem ∗34·55) it follows easily from theorems ∗90·16, ∗90·162, ∗90·17 and ∗90·172 their counterparts with universal closure, as in:

$*90 \cdot 16$	$(x)(y)(z)[(xR_*y \wedge yRz) \longrightarrow xR_*z]$	
1.	$(x)(z)[(\exists y)(xR_*y \wedge yRz) \longrightarrow xR_*z]$	$*90 \cdot 16$
2.	$(\exists y)(wR_*y \wedge yRv) \longrightarrow xR_*v$	1, UI.
3.	$[(\exists y)(wR_*y \wedge yRv) \longrightarrow xR_*v] \longrightarrow (y)[(wR_*y \wedge yRv) \longrightarrow xR_*v]$	$*10 \cdot 23.$
4.	$(w)(y)(v)[(wR_*y \wedge yRv) \longrightarrow xR_*v]$	2, 3, MP.

The exact same application of ∗10·23 yields the same for the remaining theorems.

A.3 Peano Postulates in *Principia*

The present appendix contains a transcription of *Principia*'s version of the Peano postulates 1, 2, 4 and 5 with their respective proofs, and also the chain of theorems that 'use' the axiom of infinity in order to 'prove' the third postulate.

We also offer more 'unpacked' translations of the proofs so that the previous theorems being used in them be more apparent, ignoring the subtleties involved with relative types (indicated by Greek subscripts as in "NC_ξ induct"). We do not restore type-indexes because the types in question are clear from context.

Let us see the proofs of the Peano Postulates in the order they appear in *Principia*. First, we have :

∗120·11 $\vdash :. \alpha \epsilon \mathrm{NC}_\eta \text{ induct} : \phi\xi \,.\supset_\xi .\, \phi(\xi +_c 1) : \phi 0_\eta : \supset .\, \phi\alpha$

[∗120·102 . ∗90·112]

This is Peano 5, the postulate of induction stated in its most usual form, which follows almost directly from the definition of $\mathbb{N}$ (the proof is pratically the same as that of theorem, only applied to the succesor relation $*90 \cdot 112$). We translate it as:

Peano 5 $(F)[(F0 \wedge (\xi)(F(\xi) \longrightarrow F(s(\xi)) \longrightarrow (\alpha \in \mathbb{N} \longrightarrow F\alpha)]$

Proof :

$\Lambda 1.$ $\alpha \in \mathbb{N} \longrightarrow$
$[(\mu)((0 \in \mu \wedge (\xi)(\xi \in \mu \longrightarrow s(\xi) \in \mu)) \longrightarrow \alpha \in \xi)]$ — $\mathbb{N}$Df.

$\Lambda 2.$ $\alpha \in \mathbb{N} \longrightarrow$
$[0 \in \{\beta : F\beta\} \wedge$
$(\xi)(\xi \in \{\beta : F\beta\} \longrightarrow s(\xi) \in \{\beta : F\beta\}))$
$\longrightarrow \alpha \in \{\beta : F\beta\}]$ — $3[\mu/\{\beta : F\beta\}]$, UI.

$\Lambda 3.$ $(\gamma)(\gamma \in \{\delta : F\delta\} \longleftrightarrow F\gamma)$ — $\in$ Df.

$\Lambda 5.$ $\alpha \in \mathbb{N} \longrightarrow [(F0 \wedge (\xi)(F\xi \longrightarrow Fs(\xi))) \longrightarrow F\alpha]$ — 2, 3, Taut, MP.

$\Lambda 6.$ $[(F0 \wedge (\xi)(F\xi \longrightarrow Fs(\xi))) \longrightarrow (\alpha \in \mathbb{N} \longrightarrow F\alpha)]$ — 5, Taut, MP.

$\Lambda 7.$ Prop. — 7, GU.

Next, we have:

$*120{\cdot}12 \quad \vdash\!:\!.\ 0 \,\epsilon\, \mathrm{NC\ induct} \left[*120{\cdot}101 \frac{0}{\alpha}\right]$

This is the first postulate and it also follows almost directly from the definition of $\mathbb{N}$. We translate it as:

Peano 1 $0 \in \mathbb{N}$

Proof :

$\Lambda 1.$ $\alpha \in \mathbb{N} \longleftrightarrow (\mu)((0 \in \mu \wedge (\xi)(\xi \in \mu \longrightarrow s(\xi) \in \mu)) \longrightarrow \alpha \in \xi)$ — $\mathbb{N}$Df.

$\Lambda 2.$ $0 \in \mathbb{N} \longleftrightarrow (\mu)((0 \in \mu \wedge (\xi)(\xi \in \mu \longrightarrow s(\xi) \in \mu)) \longrightarrow 0 \in \xi)$ — $1[\gamma/0]$, UI.

$\Lambda 3.$ $0 \in \mu \wedge (\xi)(\xi \in \mu \longrightarrow s(\xi) \in \mu)) \longrightarrow 0 \in \xi$ — Tautology

$\Lambda 4.$ $(\mu)((0 \in \mu \wedge (\xi)(\xi \in \mu \longrightarrow s(\xi) \in \mu)) \longrightarrow 0 \in \xi)$ — 3, GU.

$\Lambda 5.$ Prop. — 4, $\mathbb{N}$Df, Taut, MP.

Next, we have:

$*120{\cdot}121 \quad \vdash\!:\ \alpha \,\epsilon\, \mathrm{NC}_\xi\ \mathrm{induct} \,.\supset.\, (\alpha +_c 1)_\xi \,\epsilon\, \mathrm{NC}_\xi\ \mathrm{induct}\ [*90{\cdot}172 \,.\, *120{\cdot}102]$

Which we translate as a less compact, but also straightforward proof:

Peano 2 $(\alpha)(\alpha \in \mathbb{N} \longrightarrow s(\alpha) \in \mathbb{N})$

Proof :

$\{1\}1.$	$\alpha \in \mathbb{N}$	Assumption.
$\Lambda 2.$	$\alpha \in \mathbb{N} \longleftrightarrow \alpha S_* 0$	$120 \cdot 102/\mathbb{N}$Df.
$\{1\}3.$	$\alpha S_* 0$	MP, 1, 2.
$\Lambda 4.$	$(z, y, x)(xSy \wedge yS_* z) \longrightarrow xS_* z$	$90 \cdot 172.$
$\Lambda 5.$	$(y, x)(xSy \wedge yS_* 0) \longrightarrow xS_* 0$	$4[z/0]$, UI.
$\Lambda 6.$	$(x)(xS\alpha \wedge \alpha S_* 0) \longrightarrow xS_* 0$	$5[y/\alpha]$UI.
$\Lambda 7.$	$(s(\alpha)S\alpha \wedge \alpha S_* 0) \longrightarrow s(\alpha)S_* 0$	$6[x/s(\alpha)]$UI.
$\Lambda 8.$	$(\gamma)(s(\gamma)S\gamma)$	sDf, PL.
$\Lambda 9$	$s(\alpha)S\alpha$	8, UI.
$\{1\}10.$	$s(\alpha)S_* 0$	3, 7, Taut, MP.
$\{1\}11.$	$s(\alpha) \in \mathbb{N}$	10, $\mathbb{N}$Df.
$\Lambda 12.$	Prop.	(1) − (10), Taut, MP, GU.

Next, we have:

$*120 \cdot 124 \quad \vdash . \alpha +_c 1 \neq 0$

Dem.

$\vdash . * \cdot 110 \cdot 4 .$ Transp $. \supset \vdash : \alpha \sim \epsilon \mathrm{NC} . \quad \supset . \quad \alpha +_c 1 = \Lambda$

$[*101 \cdot 12] \qquad \supset . \quad \alpha +_c 1 \neq 0 \qquad (1)$

$\vdash \cdot * 110 \cdot 632 . \supset \vdash :. \alpha \epsilon \mathrm{NC} : \xi \epsilon \alpha +_c 1 . \supset . \exists ! \xi :$

$[*24 \cdot 63] \qquad \supset : \quad \xi \sim \epsilon \alpha +_c 1$

$[*54 \cdot 102] \qquad \supset : \quad \alpha +_c 1 \neq 0 \qquad (2)$

$\vdash . (1) . (2) . \supset$Prop

This is te fourth postulate, which asserts that zero succeds no natural number. We translate it as:

Peano 4 $(\alpha)(s(\alpha) \neq 0)$

Proof : (Proof by cases)

$\{1\}1.$	$\alpha \notin \text{NC}$	$[Case(a) : \alpha \notin \text{NC}]$
$\Lambda 2.$	$s(\alpha) \neq \emptyset \longrightarrow (\alpha \in \text{NC} \wedge \alpha \neq \{\emptyset\})$	$* \cdot 110 \cdot 4.$
$\Lambda 3.$	$\alpha \notin \text{NC} \longrightarrow s(\beta) = \emptyset$	Taut, 2, MP.
$\{1\}4.$	$s(\beta) = \emptyset$	1, 3, MP.
$\{1\}5.$	$0 \neq \emptyset$	$*101 \cdot 12$
$\{1\}6.$	$s(\alpha) \neq 0$	4, 5, PL $=$.
$\{7\}7.$	$\alpha \in \text{NC}$	$[Case(b) : \alpha \in \text{NC}]$
$\Lambda 8.$	$(\mu)(\mu \in \text{NC} \longrightarrow (s(\mu) = \{\xi : (\exists y)(y \in \xi \wedge \xi - \{y\} \in \alpha\}))$	$*110 \cdot 632$
$\Lambda 9.$	$\alpha \in \text{NC} \longrightarrow (s(\alpha) = \{\xi : (\exists y)(y \in \xi \wedge \xi - \{y\} \in \alpha\})$	8, UI.
$\{7\}10.$	$s(\alpha) = \{\xi : (\exists y)(y \in \xi \wedge \xi - \{y\} \in \alpha\}$	7, 9, MP.
$\{7\}11.$	$(\xi)(\xi \in s(\alpha) \longrightarrow \xi \neq \emptyset)$	10, $\emptyset$Df, PL.
$\Lambda 12.$	$(\kappa)(\emptyset \notin \kappa \longleftrightarrow (\xi)(\xi \in \kappa \longrightarrow \xi = \emptyset))$	$*24 \cdot 63$
$\Lambda 13.$	$\emptyset \notin s(\alpha) \longleftrightarrow (\xi)(\xi \in s(\alpha) \longrightarrow \xi = \emptyset)$	$12[\kappa/s(\alpha)]$UI.
$\{7\}14.$	$\emptyset \notin s(\alpha)$	11, 13, Taut, MP.
$\Lambda 15.$	$(\xi)(\xi \in 0 \longleftrightarrow \xi = \emptyset)$	$*54 \cdot 102$
$\Lambda 16.$	Prop.	15, Ext.

The following proposition embodies the definition of the Axiom of Infinity. We need it in oder to prove the remaining lemmas for *Principia's* version of Peano 3:

$*120{\cdot}3$ $\vdash$:. Infin ax . $\equiv$: $\alpha \,\epsilon\,$ NC induct . $\supset_\alpha$. $\exists\,!\,\alpha$ [($*120{\cdot}03$)]

Passing now to the lemmas for *Principia*'s version of Peano 3, we have:

$*120{\cdot}31$ $\vdash$: $\exists\,!\,$Nc‘$\alpha +_c 1$. Nc‘$\alpha +_c 1 =$ Nc$\beta +_c 1$. $\supset$. Nc‘$\alpha =$ Nc‘β . αsmβ

Dem.

$\vdash . *110{\cdot}63 . \supset \vdash :. \mathrm{Nc}`\alpha +_c 1 = \mathrm{Nc}`\alpha +_c 1 . \equiv :$

$(\exists \gamma, y) . \gamma \,\mathrm{sm}\, \alpha . y \sim \epsilon \gamma . \xi = \gamma \cup \iota`y . \equiv_\xi . (\exists \delta, z) . \delta \,\mathrm{sm}\, \beta . z \sim \epsilon \delta . \xi = \delta \cup \iota`z :$

$[*10{\cdot}1] \supset : \gamma \,\mathrm{sm}\, \alpha . y \sim \epsilon \gamma . \quad \supset . \quad (\exists \delta, z) . \delta \,\mathrm{sm}\, \beta . z \sim \epsilon \delta . \gamma \cup \iota`y = \delta \cup \iota`z$

$[*73{\cdot}72{\cdot}3] \quad \supset . \quad (\exists \delta, z) . \delta \,\mathrm{sm}\, \beta . \gamma \,\mathrm{sm}\, \delta .$

$[*73{\cdot}32] \quad \supset . \quad \gamma \,\mathrm{sm}\, \beta$

$[*73{\cdot}32] \quad \supset . \quad \alpha \,\mathrm{sm}\, \beta$ (1)

$\vdash . *110{\cdot}63 \supset \vdash :. \mathrm{Hp} . \supset . (\exists \gamma, y) . \gamma \,\mathrm{sm}\, \alpha . y \sim \epsilon \gamma$ (2)

$\vdash . (1) . (2) . *100{\cdot}321 . \supset . \mathrm{Prop}$

This theorem states that if the successor of the cardinal number of a class α is (a) non-empty *and* (b) identical to the succesor of the cardinal number of a class β, then (c) the cardinal number of the class α is identical to the cardinal number of the class β and α and β are similar.

Peano 3.1 $\quad s(\mathrm{c}(\alpha)) \neq \emptyset \wedge s(\mathrm{c}(\alpha)) = s(\mathrm{c}(\beta)) \longrightarrow$
$$(\mathrm{c}(\alpha) = \mathrm{c}(\beta) \wedge \alpha \approx \beta)$$

Proof :

$\{1\}1.$	$s(\mathrm{c}(\alpha)) \neq \emptyset$	Assumption.
$\{2\}2.$	$s(\mathrm{c}(\alpha)) = s(\mathrm{c}(\beta)$	Assumption.
$\Lambda 3.$	$(\mu)(s(\mathrm{c}(\mu)) = \{\xi : (\exists y)(y \in \xi \wedge \xi - \{y\} \approx \mu\})$	$*110 \cdot 63.$
$\Lambda 4.$	$s(\mathrm{c}(\alpha)) = s(\mathrm{c}(\beta)) \longleftrightarrow$ $(\xi)[(\exists \gamma, y)(\gamma \approx \alpha \wedge y \notin \gamma \wedge \xi = \gamma \cup \{y\}) \longleftrightarrow$ $(\exists \delta, z)(\delta \approx \beta \wedge z \notin \delta \wedge \xi = \delta \cup \{z\})]$	2, 3, Several Steps.
$\{2\}5.$	$(\xi)[(\exists \gamma, y)(\gamma \approx \alpha \wedge y \notin \gamma \wedge \xi = \gamma \cup \{y\}) \longleftrightarrow$ $(\exists \delta, z)(\delta \approx \beta \wedge z \notin \delta \wedge \xi = \delta \cup \{z\})]$	4, Taut, Mp.
$\{1\}6.$	$s(\mathrm{c}(\alpha)) \neq \emptyset \longrightarrow (\exists \gamma, y)(\gamma \approx \alpha \wedge y \notin \gamma)$	$*110 \cdot 63.$
$\{2\}7.$	$(\gamma \approx \alpha \wedge y \notin \gamma) \longrightarrow$ $(\exists \delta, z)(\delta \approx \beta \wedge z \notin \delta \wedge (\gamma \cup \{y\}) = (\delta \cup \{z\}))$	$5[\xi/\gamma \cup \{y\}]$, UI.
$\{1,2\}8.$	$(\exists \delta, z)(\delta \approx \beta \wedge z \notin \delta \wedge (\gamma \cup \{y\}) = (\delta \cup \{z\}))$	1, 6, 7, Taut, MP.
$\Lambda 9.$	$(\gamma \cup \{y\}) = (\delta \cup \{z\}) \longrightarrow (\gamma \cup \{y\}) \approx (\delta \cup \{z\})$	$*73 \cdot 3.$
$\Lambda 10.$	$(\gamma \cup \{y\}) \approx (\delta \cup \{z\}) \wedge y \notin \gamma \wedge z \notin \delta \longrightarrow \gamma \approx \delta$	$*73 \cdot 72.$
$\{1,2\}11.$	$\gamma \approx \delta$	8 – 10, PL, Taut, MP.
$\Lambda 12.$	$\gamma \approx \delta \longrightarrow (\delta \approx \beta \longrightarrow \gamma \approx \beta)$	$*73 \cdot 32.$
$\Lambda 13.$	$\alpha \approx \gamma \longrightarrow (\gamma \approx \beta \longrightarrow \alpha \approx \beta)$	$*73 \cdot 32.$
$\{1,2\}15.$	$\alpha \approx \beta$	11 – 13, Taut, MP.
$\{1,2\}16.$	$\alpha \approx \beta \longrightarrow (\mathrm{c}(\alpha) = \mathrm{c}(\beta))$	$*100 \cdot 321.$
$\Lambda 17.$	Prop.	1 – 16, Taut, MP.

This proposition is a lemma for the next, **Peano 3.11**, which is:

$*120{\cdot}311 \quad \vdash : \mathrm{Nc}'(\alpha +_c 1) \neq \Lambda \,.\, (\alpha +_c 1) = (\beta +_c 1) \,.\, \supset .\, \alpha = sm``\beta \,.\, \exists ! \alpha$

[$*120{\cdot}31 \,.\, *110{\cdot}4 \,.\, *103{\cdot}16{\cdot}4{\cdot}2$]

This theorem states that if the non-empty sucessor of cardinal number α is identical to the successor of a cardinal number β, then α and β are similar and α is non-empty. We translate it as a more explicit proof:

Peano 3.11 $(s(\alpha) \neq \emptyset \wedge s(\alpha) = s(\beta)) \longrightarrow (\alpha = \beta \wedge \alpha \neq \emptyset)$

Proof :

$\{1\}1.$ $s(\alpha) \neq \emptyset$ Assumption.

$\{2\}2.$ $s(\alpha) = s(\beta)$ Assumption.

$\Lambda 3.$ $(\gamma, \delta)[s(\mathrm{c}(\gamma)) \neq \emptyset \wedge s(\mathrm{c}(\gamma)) = s(\mathrm{c}(\delta))$

Λ $\longrightarrow (\mathrm{c}(\gamma) = \mathrm{c}(\delta) \wedge \gamma \approx \delta)]$ $*120 \cdot 31$

$\Lambda 4.$ $(\mu, \nu)((\mu +_c v) \neq \emptyset \longrightarrow ((\mu, v \in \mathrm{NC} - \{\emptyset\} \wedge \mu, v \in \mathrm{NC}))$ $*110 \cdot 4$

$\Lambda 5.$ $s(\alpha) = \alpha +_c 1$ sDF

$\Lambda 6.$ $s(\beta) = \beta +_c 1$ sDF

$\{1\}7.$ $\alpha +_c 1 \neq \emptyset$ $1, 5, \mathrm{PL} =.$

$\{1,2\}8.$ $\beta +_c 1 \neq \emptyset$ $1, 2, 6, \mathrm{PL} =.$

$\{1,2\}9.$ $\alpha, \beta \in \mathrm{NC} - \{\emptyset\} \wedge \alpha, \beta \in \mathrm{NC}$ $7, 8, 4\mathrm{PL} =$

$\{1,2\}10.$ $\alpha, \beta \in \mathrm{NC} - \{\emptyset\} \longrightarrow \alpha, \beta \neq \emptyset$ Extensionality.

$\{1,2\}11.$ $\alpha, \beta \neq \emptyset$ 9, 10, Taut, MP.

$\Lambda 12.$ $(\mu)(\mu \in \mathrm{NC} \longleftrightarrow (\exists \xi)(\mu = (\xi))$ $*103 \cdot 2$

$\{1,2\}13.$ $(\exists \gamma)(\alpha = \mathrm{c}(\gamma))$ $9, 12, \mathrm{PL} =.$

$\{1,2\}14.$ $(\exists \delta)(\beta = \mathrm{c}(\delta))$ $9, 12, \mathrm{PL} =.$

$\{1,2\}15.$ $s(\alpha) = s(\mathrm{c}(\gamma)) = s(\mathrm{c}(\delta))$ $1, 13, 14, \mathrm{PL} =.$

$\{1,2\}16.$ $\mathrm{c}(\gamma) = \mathrm{c}(\delta) \wedge \gamma \approx \delta$ $3, 15, \mathrm{PL} =.$

$\{1,2\}17.$ $\mathrm{c}(\gamma) = \mathrm{c}(\delta) = \alpha = \beta$ $13, 14, 16, \mathrm{PL} = .$

Λ Prop. $1 - 17$, Taut, MP.

The proposition is proved as a lemma for **Peano 3.2**, which is:

$*120 \cdot 32 \quad \vdash : \alpha \in \mathrm{NC\ induct} \,.\, \exists ! \alpha \,.\, \supset \,.\, \alpha \neq \alpha +_c 1$

Dem.

$\vdash . *101{\cdot}22 . *110{\cdot}641 . \supset \vdash . \quad 0_\xi \neq 0_\xi +_c 1$ (1)

$\vdash . *120{\cdot}311 . *110{\cdot}44 \cdot \supset \vdash : \quad \alpha \epsilon \mathrm{NC} . \exists ! \alpha +_c 1 . \alpha +_c 1 = \alpha +_c 1 +_c 1 . \supset . \alpha = \alpha +_c 1 :$

[Transp] $\supset \vdash : \alpha \epsilon \mathrm{NC} . \exists ! \alpha +_c 1 . \alpha \neq \alpha +_c 1 . \supset . \alpha +_c 1 \neq \alpha +_c 1 +_c 1 .$

[*118·2·25] $\supset \vdash : \alpha \epsilon \mathrm{NC}_\xi . \exists ! (\alpha +_c 1)_\xi . \alpha \neq (\alpha +_c 1)_\xi . \supset . (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi$ (2)

$\vdash . (2) . \supset \vdash :. \alpha \epsilon \mathrm{NC}_\xi . \exists ! (\alpha +_c 1)_\xi . \alpha \neq (\alpha +_c 1)_\xi . \supset :$

$(\alpha +_c 1)_\xi = \Lambda . \vee . (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi$ (3)

$\vdash . *110{\cdot}4 . \mathrm{Transp} \supset \vdash :. \alpha \sim \epsilon \mathrm{NC}_\xi . \vee . \alpha = \Lambda_\xi : \supset . (\alpha +_c 1)_\xi = \Lambda_\xi$ (4)

$\vdash . (3) . (4) . \supset \vdash :. \alpha = \Lambda_\xi . \vee . \alpha \neq (\alpha +_c 1)_\xi : \supset :$

$(\alpha +_c 1)_\xi = \Lambda_\xi . \vee . (\alpha +_c 1)_\xi \neq \{(\alpha +_c 1)_\xi +_c 1\}_\xi$ (5)

$\vdash . (1) . (5) . *120{\cdot}11 . \supset :. \alpha \epsilon \mathrm{NC}_\xi \mathrm{induct} . \supset : \alpha = \Lambda_\xi . \vee . \alpha \neq (\alpha +_c 1)_\xi :. \supset \vdash \mathrm{Prop}$

This is a slightly stronger theorem than **Peano 3.1** and **3.11**. It states that if a finite cardinal number is non-empty, then it is different from its successor. Still, it does not assure us either (a) that every finite cardinal *is* non-empty or (b) that the succesor of a non empty cardinal is non-empty. Our translation turns out to be simpler than the original since our notational modifications dispense with the need of assuming the commutativity of homogeneous cardinal addition ($*110 \cdot 2 \cdot 25$):

Peano 3.2	$(\alpha)((\alpha \in \mathbb{N} \wedge \alpha \neq \emptyset) \longrightarrow \alpha \neq s(\alpha))$	
Proof :	(Induction)	
$\Box$1.	$(\beta)[(s(\alpha) \neq \emptyset \wedge s(\alpha) = s(\beta)) \longrightarrow (\alpha = \beta \wedge \alpha \neq \emptyset)]$	Peano 3.11
Λ2.	$(s(\alpha) \neq \emptyset \wedge s(\alpha) = s(s(\alpha))) \longrightarrow (\alpha = s(\alpha) \wedge \alpha \neq \emptyset)$	$2[\beta/s(\alpha)]$, UI.
Λ3.	$\neg(\alpha = s(\alpha) \wedge \alpha \neq \emptyset) \longrightarrow$	
Λ	$\neg(s(\alpha) = \emptyset \vee s(\alpha) \neq s(s(\alpha)))$	2, Taut, MP, PL = .
Λ4.	$(\alpha \neq s(\alpha) \vee \alpha = \emptyset) \longrightarrow (s(\alpha) = \emptyset \vee s(\alpha) \neq s(s(\alpha)))$	3, Taut, MP, PL = .
Λ5.	$(\alpha = \emptyset \vee \alpha \neq s(\alpha)) \longrightarrow (s(\alpha) = \emptyset \vee s(\alpha) \neq s(s(\alpha)))$	4, Taut, MP, PL = .
[I.p]6.	$(\alpha)[(\alpha = \emptyset \vee \alpha \neq s(\alpha)) \longrightarrow$	
Λ	$(s(\alpha) = \emptyset \vee s(\alpha) \neq s(s(\alpha))]$	5, UG.
[B.s]7.	$0 \neq s(0)$	$*101 \cdot 22$
Λ8.	$0 = \emptyset \vee 0 \neq s(0)$	7, Taut, MP..
[I.s]9.	$(F)[(F0 \wedge (\beta)(F\beta \longrightarrow Fs(\beta)) \longrightarrow (\alpha \in \mathbb{N} \longrightarrow F\alpha)]$	Peano 5.
[Ind]10.	$0 = \emptyset \vee 0 \neq s(0) \ \wedge$ [6]	9, Comp, UI,
Λ	$\longrightarrow [\alpha \in \mathbb{N} \longrightarrow (\alpha = \emptyset \vee \alpha \neq s(\alpha))]$	$\left[\dfrac{\gamma = \emptyset \vee \beta \neq s(\gamma)}{F\gamma}\right]$
Λ11.	$\alpha \in \mathbb{N} \longrightarrow (\alpha = \emptyset \vee \alpha \neq s(\alpha))]$	8, 6, 10, Taut, MP.
Λ12.	$\alpha \in \mathbb{N} \longrightarrow (\alpha \neq \emptyset \longrightarrow \alpha \neq s(\alpha))$	11, Taut, MP.
Λ13.	$(\alpha \in \mathbb{N} \wedge \alpha \neq \emptyset) \longrightarrow \alpha \neq s(\alpha))$	12, Taut, MP.
Λ14.	Prop.	13, UG.

The proposition is used in the proof of **Peano 3.22**, which is the final lemma for ∗120·33. Below we have the proofs of ∗120·321 and ∗120·322:

∗120·321 ⊢ : $\alpha \neq \alpha +_c 1$. ⊃ . ∃ ! α

Dem.

⊢ . ∗110·4 . Transp . ⊃⊢ : $\alpha = \Lambda$. ⊃ . $\alpha +_c 1 = \Lambda$ (1)

⊢ . (1) . Transp . ⊃⊢ . Prop

∗120·322 ⊢ :. Infin ax . ⊃ :. ∃ ! α . ≡ . $\alpha \neq \alpha +_c 1$

[∗120·3·322]

We translate them as follows:

Peano 3.21 $\alpha \neq s(\alpha) \longrightarrow \alpha \neq \emptyset$

Proof : (Proof by cases)

$\{1\}1.$	$\alpha \neq s(\alpha)$	Assumption.
$\{2\}2.$	$s(\alpha) = \emptyset$	$[Case(a)]$
$\{2\}3.$	$\alpha \neq \emptyset$	1, 2, PL $=$.
$\{3\}4.$	$s(\alpha) \neq \emptyset$	$[Case(b)]$
$\Lambda 5.$	$s(\alpha) = \alpha +_c 1$	sDf.
$\Lambda 6.$	$(\mu, \nu)((\mu +_c v) \neq \emptyset \longrightarrow ((\mu, v \in \text{NC} - \{\emptyset\} \wedge \mu, v \in \text{NC}))$	$*110 \cdot 4$
$\{3\}7.$	$\alpha \in \text{NC} - \{\emptyset\}$	4, 5, 6, PL $=$.
$\{3\}8.$	$\alpha \neq \emptyset$	7, Ext.
$\Lambda 9.$	Prop.	1 $-$ 8, Taut, MP.

Peano 3.22 $\alpha \in \mathbb{N} \longrightarrow (\alpha \neq \emptyset \longleftrightarrow \alpha \neq s(\alpha))$

Proof :

$\{1\}$ 1.	$\alpha \in \mathbb{N}$	Assumption.
$\{2\}$ 2.	$\Rightarrow \alpha \neq \emptyset$	Assumption.
$\Lambda 3.$	$(\alpha)((\alpha \in \mathbb{N} \wedge \alpha \neq \emptyset) \longrightarrow \alpha \neq s(\alpha))$	Peano 3.2
$\{1, 2\}$ 4.	$\alpha \neq s(\alpha)$	1, 2, 3, PL.
$\{5\}$ 5.	$\Leftarrow \alpha \neq s(\alpha)$	Assumption.
$\Lambda 6$	$(\alpha)(\alpha \neq s(\alpha) \longrightarrow \alpha \neq \emptyset)$	Peano 3.21
$\{1, 5\}7.$	$\alpha \neq \emptyset$	5, 6, PL.
$\Lambda 8.$	Prop.	1 $-$ 7, Taut, MP.

Peano 3.21 states that if a cardinal number is different from its successor it is non-empty (the proof has two cases α as empty and α as non-empty). **Peano 3.22** states that if α is a finite cardinal, then it is empty if and only if it is different from its successor. Both **Peano 3.21** and **Peano 3.22** are lemmas for 3*, that is ∗120·33:

∗120·33 $\vdash$:. Infin ax . $\equiv$. $\alpha \,\epsilon\,$ NC induct . $\supset_\alpha$. $\alpha \neq \alpha +_c 1$

[∗12·3·322]

This theorem establishes that different numbers have different successors iff Infin.Ax is true. It is the closest to Peano 3 that *Principia* can prove. We translate the proof as follows:

Peano 3∗ Infin.Ax $\longleftrightarrow (\alpha)(\alpha \in \mathbb{N} \longrightarrow \alpha \neq s(\alpha))$

Proof :

$\{1\}1.$	$\Rightarrow$Infin.Ax	Assumption.
$\{1\}2.$	$(\alpha)(\alpha \in \mathbb{N} \longrightarrow \alpha \neq \emptyset)$	Infin.Ax.Df.
$\Lambda 3.$	$(\alpha)(\alpha \in \mathbb{N} \longrightarrow (\alpha \neq \emptyset \longleftrightarrow \alpha \neq s(\alpha))$	Peano 3.22.
$\{1\}4.$	$(\alpha)(\alpha \in \mathbb{N} \longrightarrow \alpha \neq s(\alpha))$	$1-3$, PL.
$\{5\}5.$	$\Leftarrow(\alpha)(\alpha \in \mathbb{N} \longrightarrow \alpha \neq s(\alpha))$	Assumption.
$\{5\}6.$	$(\alpha)(\alpha \in \mathbb{N} \longrightarrow \alpha \neq \emptyset)$	3, 5, PL.
$\Lambda 7.$	Prop.	$1-3$, PL.

This completes the proof (and our translation) of *Principia*'s analogues of the Peano postulates.

APPENDIX B – *Principia*-Style Arithmetic

B.1 The Language and Background Logic

Our language is that of the simple theory of types, that is of higher-order logic[1]. Aside from standard comprehension axioms for concepts and relations, we assume, for convenience purposes of avoiding contextual or eliminative definitions of class expressions, axioms of comprehension for sets, which are also stratified in simple types - although an eliminative approach to these notations could be adopted. Our goal here is to capture closely the way *Principia* develops Arithmetic in the most concise and simplest way possible and for this the notation of classes indispensable. Apart from the assumptions of Standard higher-order Logic, then, our main axioms are the following:

Axiom 1. ***Comprehension for sets.*** *Let x, α be variable-terms of type t and $t+1$ respectively. Then, for every concept F* there is an α such that *for every x , $x \in \alpha$ iff only $F(x)$ is true. In symbols:* $(F^{t+1})(\exists \alpha^{t+1})(x^t)(x^t \in \alpha^{t+1} \equiv Fx^t)$

We take first-order variables x, y, z,etc., to range over *individuals* that is, objects of the lowest type, namely *i*. So, informally, sets of individuals are of type $i+1$, sets of sets of individuals of type $i+2$ and so on. We also add an axiom-scheme of extensionality for sets:

Axiom 2. ***Extensionality for sets***: *Let α and β be sets. α and β are identical iff they have the same members. In symbols:* $(\alpha^{t+1} = \beta^{t+1}) \equiv (x^t)(x^t \in \alpha^{t+1} \equiv x^t \in \beta^{t+1})$.

To make the proofs more readable we omit type-indexes since it is clear from context what the relative types of the variables must be.

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1 We employ the term "order" here as in "first-order Logic", "second-order Logic" and so on, not in the sense it is sometimes employed in the study of ramified systems of second and higher-order Logics. This coincides with the most usual use Russell makes of the term "type" when applied to so-called propositional functions. For an instructive discussion of the many uses of the notions of *type*, *order* and *level* see HAZEN, A., 1983, p.396, footnote 1. The basic principle that guides the formulations os such theories is that every variable has a *type* which determines its domain or range: a variable of type *t* ranges over things of type *t* and nothing else. We start with first order variables which range over *individuals* in the sense of Russell or *objects* in the sense of Frege and ascend to concepts of arbitrarily large order. The hierarchy is simple in the sense that each type is disjoint or not cumulative: a given formula *Fx* is only meaningful if *x* is of a type immediately lower than *x*, that is, the range of a variable of type *t* does not include the range of any variable of lower type. We assume as our background logic classical standard (impredicative) higher-order logic of any arbitrarily large order.

B.2 Definitions for Finite Arithmetic

Definition 1. [Domains, Converse Domains and Fields of Relations - Russell 1901; Similarity] Let *R* be any relation and α, β sets *of the same type*. Then:

The *domain* of *R* is the set of all terms *x* such that for some *y*, *xRy*. In symbols: $\text{Dom}(R) =_{\text{df}} \{x : (\exists y)xRy\}$;

The *converse domain* of *R* is the set of all terms *y* such that for some *x*, *xRy*. In symbols[2]: $\text{Cont}(R) =_{\text{df}} \{y : (\exists x)xRy\}$;

The *field* of a relation *R* is the union of its domain and converse domain. In symbols: $\text{Campus}(R) =_{\text{df}} \{z : z \in \text{Dom}(R) \vee z \in \text{Cont}(R)\}$

R is a one-one relation iff for every *x* such that *xRy*, there is exactly one *y* and for every *y* such that *xRy* there is exactly one *x*. In symbols: $1 \to 1(R) =_{\text{df}} (x)(y)(z)(xRy \wedge xRz \supset y = z) \wedge (x)(y)(z)(xRy \wedge zRy \supset z = x)$

α is similar (or equinumerous) to β iff there is a one-one relation *R* such that α is the domain of *R* and β is the converse domain of *R*. In symbols: $\alpha \approx \beta =_{\text{df}} (\exists R)(\text{Dom}(R) = \alpha \wedge \text{Cont}(R) = \beta \wedge 1 \to 1(R))$

Remark: We adopt the restriction that α and β must be of the same type in order to simplify our treatment. In the terminology of *Principia* this means that every cardinal number with which we shall be concerned $\text{Nc}(\alpha)$ is *homogeneous*: all its members are of the same type as the set α. This makes several results provable about $\text{Nc}(\alpha)$ without the need to make type symbols explicit (such as $\alpha \in \text{Nc}(\alpha)$). It also facilitates the treatment of finite arithmetic, since it collapses the class of all classes similar with α with its homogeneous cardinal number: if *non*-homogeneous relations of similarity were allowed this would not be the case, since then the cardinal number of α could be of a higher or lower type than α. Also, if non-homogeneous similarity relations were allowed, the cardinal number of α could be empty in some types and not in others: for suppose there was only one individual, *a*; there are types in which the cardinal number of a given class such as $\text{Nc}(\{\{a\}, \Lambda\})$ would be empty, namely in the type of the sets of individuals; but in every other type it would be non-empty.

Next we have the Frege-Russell definition of cardinal number.

Definition 2. [Cardinal Number of a Set - Frege 1884; Russell 1901] *The cardinal number of a set α or* $\text{Nc}(\alpha)$ *is the set of all sets similar to α.*

In symbols: $\text{Nc}(\alpha) =_{\text{df}} \{\beta : \alpha \approx \beta\}$.

[2] We take "Cont" from the Latin "*contrarium*", in order to avoid Whitehead and Russell's *very* inconvenient inverted D.

Definition 3. [Zero - Frege 1884; Russell 1901; Whitehead & Russell 1912] 1: Let Λ be the empty set, defined as the set of all things that are not self identical. (ii) *The cardinal number 0 is the cardinal number of the empty set.*

In symbosl we have $\Lambda =_{df} \{x : x \neq x\}$ and $0 =_{\text{df}} \text{Nc}(\Box)$.

Definition 4. [Successor - Frege 1884; Whitehead & Russell 1912] 1: The *immediate successor* $\text{Nc}(\alpha) +_c 1$ of a cardinal number $\text{Nc}(\alpha)$ is the number of elements of the set which consists of the elements of α together with an element *x* which does not belong to α; in other words, the successor of $\text{Nc}(\alpha)$ is the class of all classes γ such that, for some *x*, if we take away some unique member of γ, we get a member of $\text{Nc}(\alpha)$. 2: A cardinal number *m* precedes a cardinal number *n* whenever there is a set δ and an individual *x* such that $m = \text{Nc}(\delta)$ and $n = \text{Nc}(\delta \cup \{x\})$, where $x \notin \delta$ (this definition is introduced merely for notation convenience; see the corollary below).

In symbols we have[3] $\text{Nc}(\alpha) +_c 1 =_{\text{df}} \{\gamma : (\exists x)(x \in \gamma \wedge \gamma - \{x\} \in \text{Nc}(\alpha))\}$ and $\alpha P \beta =_{\text{df}} (\exists \delta)(\exists x)(x \notin \delta \wedge \alpha = \text{Nc}(\delta) \wedge \beta = \text{Nc}(\delta \cup \{x\}))$.

Corollary. *α precedes β iff β is the immediate successor of α. That is: $\alpha P \beta \equiv \alpha +_c 1 = \beta$.*

Remark. We are following Russell simplified definition given in *Introduction to Mathematical Philosophy*, where he puts: "The successor of the number of terms in the class α is the number of terms in the class consisting of α together with *x*, where *x* is any term not belonging to the class". This coincides with the definition we shall use. PM, however, does not use this simple definition, but:

$*110{\cdot}01 \qquad \alpha + \beta = \downarrow (\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha) \downarrow ``\iota``\beta \quad \text{Df}$

$*110{\cdot}02 \qquad \mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) \,.\, \mu = \text{N}_0\text{c}`\alpha \,.\, \nu = \text{N}_0\text{c}`\beta \,.\, \xi \,\text{sm}\, (\alpha + \beta)\} \quad \text{Df}$

The second defines the actual sum of two cardinal numbers, while the first defines what Whitehead and Russell call the arithmetical sum of two classes. These somewhat complicated definitions are designed to deal with some difficulties introduced by the theory of types and to ensure that the arithmetical sum of two non-disjoint classes yields a class with the intended number[4]. We avoided such complications by giving up generality. We did not not define the operation of cardinal sum of any two cardinals in general, only the succesor function/relation. In other words, we did *not* obtain $+_c 1$ (or immediate successor) as a particular case. Whitehead and Russell develop a whole theory of addition which applies both to finite and infinite numbers and

3 Here "$\gamma - \{x\}$" abbreviates $\gamma \cap \overline{\{x\}}$.

4 Russell explains these definitions in *Introduction to Mathematical Philosophy* (pp.117-119). Another explanation can be found in the correspondence with Philip Jourdain (GRATTAN-GUINNESS, 1977, pp.119-120).

which accomodates $+_c 1$ as a particular case. Their theory is based on an account of "equinumerosity" or "similarity" as a possibly heterogeneous relation, that is, a relation which may hold between classes of any types (possibly distinct). As we noted in the begining (first remark) we are avoiding these subtleties to simplify our treatment.

Next, we have the celebrated definition of the Ancestral of a relation *R*:

Definition 5. [Ancestral of a Relation - Frege 1879, 1884; Whitehead & Russell 1910] Let *R* be a relation, F be a concept and α a set. Then:

5.1: *F is hereditary in the R-series* or *R-hereditary* iff it belongs to *x* whenever if *xRy* and *Fx*, then *Fy*. In symbols: $\text{Her}(F, R) =_{\text{df}} (x)(y)(Fx \wedge xRy \supset Fy)$.

5.2: α is *hereditary in the R-series* or *R-hereditary* iff its defining property is *R-hereditary*. In symbols: $\text{Her}(\alpha, R) =_{\text{df}} (\exists F)(\text{Her}(F, R) \wedge \alpha = \{x : Fx\})$.

5.3: *a* is an *R-ancestor* of *b* iff *a* belongs to the field of *R* and falls under every *R-hereditary* concept unde which *a* falls. In symbols: $aR_*b =_{\text{df}} (F)(Fa \wedge a \in \text{Campus}(R) \wedge \text{Her}(F, R) \supset Fb)$.

5.4: The *R-posterity of a* is the set of all *y* such that *a* is an *R-ancestor* of *y*. In symbols: $\overleftarrow{R}_*(a) =_{\text{df}} \{w : aP_*w\}$.

Corollary 1. *a is an R-ancestor of b iff b belongs to every hereditary class to which a belongs. That is:* $aR_*b \equiv (\alpha)(a \in \alpha \wedge a \in \text{Campus}(R) \wedge \text{Her}(\alpha, R) \supset b \in \alpha)$

Definition 6. [Natural Number - Frege 1884; Whitehead & Russell 1912] 1. The set of natural numbers, that is, of finite cardinals or inductive numbers is the posterity of zero with respect to the relation of immediate predecessor. In Symbols:

$$\text{NC induct} =_{\text{df}} \overleftarrow{P}_*(0) =_{\text{df}} \{\alpha : 0P_*\alpha\} =_{\text{df}} \{\alpha : (F)(F0 \wedge \text{Her}(F, P). \supset: F\alpha)\}$$

B.3 Some Facts About the Ancestral

Fact 1. $(v)(w)(vRw \supset vR_*v)$

Dem. Assume vRw. By the definition of R_*, xR_*y iff $Fv \wedge v \in \text{Campus}(R) \wedge (x)(y)(Fx \wedge xRy \supset Fy) \supset Fw$. If for some *w* we have vRw, it follows that $v \in \text{Campus}(R)$. Assume Fv and $(x)(y)((Fx \wedge xRy) \supset Fy)$. By UI we get $Fv \wedge vRw \supset Fw$. Hence Fw. ■

Fact 2. $(v)(w)(z)(vR_*w \wedge wR_*z \supset vR_*z)$.

Dem. Assume vR_*w and wR_*z. By vR_*w, $Fv \wedge Her(F, R)$ entails Fw. By wR_*z, $Fw \wedge Her(F, R)$ entails Fz. Hence, vR_*w and wR_*z imply $Fv \wedge Her(F, R) \supset Fz$. ■

Fact 3. $(v)(w)(z)(vR_*w \wedge wRz \supset vR_*z)$.

Dem. Assume vR_*w and wRz. By fact 1 above $wRz \supset wR_*z$, hence we have $vR_*w \wedge wR_*z$; and again, by the previously proved fact 2 we get vR_*z. ■

B.4 Peano Postulates 1, 2, 4 and 5

In what follows we play very loose with theorems of the predicate calculus and the algebra of classes (especially concerning results such as $\gamma - \{x\} \approx \{y\} \supset (\exists z)(\gamma = \{z\})$, $x \notin \gamma \wedge y \notin \delta \wedge \gamma \cup \{x\} \approx \delta \cup \{y\} \supset \gamma \approx \delta$, etc.).

Theorem. [Whitehead & Russell 1912 $*120 \cdot 12$ **- Peano 1]** $0 \in \text{NC induct}$

Dem. By definition, $0 \in \text{NC induct}$ iff $(F)(F0 \wedge Her(F, P) \supset F0)$. Since $F0 \wedge Her(F, P) \supset F0$ what we want follows by UG. ■

Theorem. Peano 2. $(\alpha)(\alpha \in \text{NC induct} \wedge \alpha P \beta . \supset . \beta \in \text{NC induct})$

Dem. Assume that $\alpha \in \text{NC induct}$ and $\alpha P \beta$. By definition, we have $0P_*\alpha$. Thus, by fact 1 about the ancestral, we have $\alpha P \beta \supset \alpha P_* \beta$; then again, by fact 3 about the ancestral we have $0P_*\alpha \wedge \alpha P \beta \supset 0P_*\beta$ and hence $0P_*\beta$. But for any γ, $\gamma \in \text{NC induct}$ iff $0P_*\gamma$ and that's it. ■

Theorem. [Whitehead & Russell 1912 $*120 \cdot 121$ **- Alternative Peano 2]** $(\alpha)(\alpha \in \text{NC induct} \supset \alpha +_c 1 \in \text{NC induct})$

Dem. Immediate from Peano 2 and the fact that $\alpha P \beta \equiv \alpha +_c 1 = \beta$ (corollary from the definiton of successor). ■

Theorem. [Whitehead & Russell 1912 $*120 \cdot 124$ **- Peano 4]** $(\alpha) \sim \alpha P 0$

Dem. Assume that for some α, $\alpha P 0$. That is: for some δ, $\alpha = \text{Nc}(\delta) = 0$ and $\alpha P 0$. By the definition of zero and extensionality, $\delta \in 0$ iff $\delta = \Lambda$. But by the definition of successor $\alpha P 0$ iff for some δ and some $x \notin \delta$, $\alpha = \text{Nc}(\delta)$ and $0 = \text{Nc}(\delta \cup \{x\})$. Contradiction. ■

Theorem. Peano 5. $(F)(F0 \wedge (\alpha)(F\alpha \wedge \alpha P \beta \supset F\beta) \supset (\gamma)(\gamma \in \text{NC induct} \supset F\gamma))$

Dem. Assume $F0$ and that F is hereditary, that is $F0 \wedge (\alpha)(F\alpha \wedge \alpha P \beta \supset F\beta)$. By definition, $\gamma \in \text{NC induct}$ iff $F0 \wedge (\alpha)(F\alpha \wedge \alpha P \beta) \supset F\gamma$. Therefore $\gamma \in \text{NC induct} \wedge F0 \wedge (\alpha)(F\alpha \wedge \alpha P \beta \supset F\beta) \supset F\gamma$. By propositional logic and UG we are done. ■

Corollary. [Whitehead & Russell 1912 $*120 \cdot 13$ **- Alternative Peano 5].** $(F)(F0 \wedge (\alpha)(F\alpha \supset F(\alpha +_c 1)) \supset (\gamma)(\gamma \in \text{NC induct} \supset F\gamma))$.

Dem. Immediate from Peano 5 and the fact that $\alpha P \beta \equiv \alpha +_c 1 = \beta$ (corollary from the definiton of successor). ■

B.5 The Axiom of Infinity and the Existence of Cardinal Numbers

Conjecture. [Whitehead & Russell 1912 $*120{\cdot}03$ Axiom of Infinity] *The emtpy set is not a natural number. That is:* $(\alpha)(\alpha \in \text{NC induct} \supset \alpha \neq \Lambda)$. *For short we shall call it* Infin Ax.

In order to show that Peano 3 is equivalent to it Infin Ax we need to prove some lemmas. We follow the structure of *Principia*'s proofs as close as possible.

Lemma. [Whitehead & Russell 1912 $*100{\cdot}321$ - First Half of Hume's Principle] *If* $\alpha \approx \beta$*, then* $\text{Nc}(\alpha) = \text{Nc}(\beta)$.

Dem. Assume $\alpha \approx \beta$, since $\approx$ is an equivalence relation, we have $\alpha \approx \beta \supset (\gamma \approx \alpha \equiv \gamma \approx \beta)$. Since, by the definition of cardinal number, we have $(\delta)(\gamma \in \text{Nc}(\delta) \equiv \gamma \approx \delta)$, we also have: $\alpha \approx \beta \supset (\gamma \in \text{Nc}(\alpha) \equiv \gamma \in \text{Nc}(\beta))$; which, by predicate logic is equivalent yields $\alpha \approx \beta \supset (\gamma)(\gamma \in \text{Nc}(\alpha) \equiv \gamma \in \text{Nc}(\beta))$; which by extensionality gives us $\alpha \approx \beta \supset \text{Nc}(\alpha) = \text{Nc}(\beta)$. ■

Lemma. [Whitehead & Russell 1912 $*120{\cdot}31$ - Peano 3.1] *If the sucessor* $\text{Nc}(\alpha) +_c 1$ *of the cardinal number* $\text{Nc}(\alpha)$ *is non-empty and identical to* $\text{Nc}(\beta) +_c 1$*, then* $\text{Nc}(\alpha) = \text{Nc}(\beta)$. *That is:* $\text{Nc}(\alpha) +_c 1 \neq \Lambda \wedge \text{Nc}(\alpha) +_c 1 = \text{Nc}(\beta) +_c 1 \supset \alpha \approx \beta \wedge \text{Nc}(\alpha) = \text{Nc}(\beta)$.

Dem. Assume, (1). $\text{Nc}(\alpha) +_c 1 \neq \Lambda$ and (2). $\text{Nc}(\alpha) +_c 1 = \text{Nc}(\beta) +_c 1$. By the definition of successor, we have:

$$(\gamma)(\gamma \in \text{Nc}(\alpha) +_c 1 \equiv (\exists x)(x \in \gamma \wedge \gamma - \{x\} \in \text{Nc}(\alpha))\})$$

$$(\gamma)(\gamma \in \text{Nc}(\alpha) +_c 1 \equiv (\exists x)(x \in \gamma \wedge \gamma - \{x\} \in \text{Nc}(\beta))\})$$

By our first assumption, we have some δ such that $\delta \in \text{Nc}(\alpha) +_c 1$, and by extensionality and our second assumption we have $\delta \in \text{Nc}(\beta) +_c 1$. Now, by the definition of successor, for some $x \in \delta$ and some we have $\delta - \{x\} \in \text{Nc}(\alpha)$ and $\delta - \{x\} \in \text{Nc}(\beta)$, therefore $\delta - \{x\} \approx \alpha$ and $\delta - \{x\} \approx \beta$. But since $\delta \approx \delta$ and thus $(\delta - \{x\}) \approx (\delta - \{x\})$, we get $\alpha \approx \beta$. But if $\alpha \approx \beta$ we have, by the previous lemma, $\text{Nc}(\alpha) = \text{Nc}(\beta)$. ■

Lemma. [Whitehead & Russell 1912 $*120{\cdot}311$ - Peano 3.11] *If* $\alpha +_c 1$ *is a non empy cardinal and* $\alpha +_c 1 = \beta +_c 1$, then $\alpha = \beta$ and $\alpha \neq \Lambda$. *That is:* $(\alpha +_c 1 \neq \Lambda \wedge \alpha +_c 1 = \beta +_c 1 \supset \alpha = \beta \wedge \alpha \neq \Lambda$.

Dem. Assume (1). $\alpha +_c 1 \neq \Lambda$ and (2). $\alpha +_c 1 = \beta +_c 1$. Again, by the definition of successor we have, for some sets γ and δ:

$$\alpha +_c 1 = \beta +_c 1 = \text{Nc}(\gamma) +_c 1 = \text{Nc}(\delta) +_c 1$$

From the previously proved lemma Peano 3.11 and the first assumption we get $\alpha = \beta$. By the definition of successor, for some γ and some *x*, we have $\alpha = \mathrm{Nc}(\gamma - \{x\})$ and since $\gamma - \{x\} \approx \gamma - \{x\}$, also $\gamma - \{x\} \in \alpha$. Now, either $\gamma - \{x\} = \Lambda$ or $\gamma - \{x\} \neq \Lambda$. If, $\gamma - \{x\} = \Lambda$, then $\Lambda \in \alpha$ and $\alpha \neq \Lambda$. If $\gamma - \{x\} \neq \Lambda$, then α is also non-empty. ■

Lemma. [Whitehead & Russell 1912 $*120{\cdot}32$ **- Peano 3.2]** *If* α *is a non empty finite cardinal then* α *is different from* $\alpha +_c 1$. *That is:* $\alpha \in \mathrm{NC\,induct} \wedge \alpha \neq \Lambda \supset \alpha \neq \alpha +_c 1$.

Dem. Induction. Base step: 0 is a non-empty finite cardinal number. By Peano 4, $0 \neq 0 +_c 1$. Inductive step: from Peano 3.11, UI and propositional logic we get[5] $(\alpha = \Lambda \vee \alpha \neq \alpha +_c 1) \supset (\alpha +_c 1 = \Lambda \vee \alpha +_c 1 \neq \alpha +_c 1 +_c 1)$. Instantiating "$F\alpha$" in the induction axiom as $\alpha = \Lambda \vee \alpha \neq \alpha +_c 1$, we get:

$$(0 = \Lambda \vee 0 \neq 0 +_c 1) \wedge (\alpha = \Lambda \vee \alpha \neq \alpha +_c 1) \supset$$

$$(\alpha +_c 1 = \Lambda \vee \alpha +_c 1 \neq \alpha +_c 1 +_c 1) \supset (\gamma)(\gamma \in \mathrm{NC\,induct} \supset \gamma = \Lambda \vee \gamma \neq \gamma +_c 1)$$

Since, by propositional logic, $\gamma \in \mathrm{NC\,induct} \supset \gamma = \Lambda \vee \gamma \neq \gamma +_c 1$ is equivalent to $\gamma \in \mathrm{NC\,induct} \wedge \gamma \neq \Lambda \supset \gamma \neq \gamma +_c 1$ we are done. ■

Lemma. [Whitehead & Russell 1912 $*120{\cdot}321$ **- Peano 3.21]** *If* α *is finite a cardinal different from* $\alpha +_c 1$, *then* α *is non-empty. That is:* $\alpha \in \mathrm{NC\,induct} \wedge \alpha \neq \alpha +_c 1 \supset \alpha \neq \Lambda$.

Dem. Assume $\alpha \in \mathrm{NC\,induct}$ and $\alpha \neq \alpha +_c 1$. There are two cases to be considered. (a) $\alpha +_c 1 = \Lambda$ and (b) $\alpha +_c 1 \neq \Lambda$. Assume $\alpha +_c 1 = \Lambda$. By hypothesis, $\alpha \neq \alpha +_c 1$, therefore $\alpha \neq \Lambda$. Assume $\alpha \in \mathrm{NC\,induct}$ and $\alpha +_c 1 \neq \Lambda$. For some δ, $\alpha = \mathrm{Nc}(\delta)$ and $\alpha +_c 1 = \mathrm{Nc}(\delta) +_c 1 = \{\gamma : (\exists x)(x \in \gamma \wedge \gamma - \{x\} \in \mathrm{Nc}(\delta))\}$; this means that there is some δ' such that $(\gamma - \{x\}) \approx \delta$ and $\delta' \in \mathrm{Nc}(\delta)$. ■

Remark. Notice that in the above proof, even if $\delta' = \Lambda$ the result holds, since in that case $\mathrm{Nc}(\delta) = \{\Lambda\}$ which is non-empty. Notice also that in such sentences like $\{\Lambda\} \neq \Lambda$, the two occurrences of "Λ" are, strictly speaking, different, since if we were to restore type-indexes they would have to have different types.

Lemma. [Whitehead & Russell 1912 $*120{\cdot}322$ **- Peano 3.22]** *If* α *is a finite cardinal, then* α *is non-empty iff* $\alpha +_c 1$ *is different from* α. *That is:* $\alpha \in \mathrm{NC\,induct} \supset (\alpha \neq \Lambda \equiv \alpha \neq \alpha +_c 1)$.

Dem. Assume $\alpha \in \alpha \in \mathrm{NC\,induct}$ and $\alpha \neq \Lambda$. By Peano 3.2 we have $\alpha \neq \alpha +_c 1$. Conversely, assume $\alpha \in \alpha \in \mathrm{NC\,induct}$ and $\alpha \neq \alpha +_c 1$, then by Peano 3.21 $\alpha \neq \Lambda$. ■

[5] First substitute β for $\alpha +_c 1$, thus getting: $(\alpha +_c 1 \neq \Lambda \wedge \alpha +_c 1 = \alpha +_c 1 +_c 1) \supset (\alpha = \alpha +_c 1 \wedge \alpha \neq \Lambda)$. Then, by contraposition, the laws of De Morgan, and the equivalence $A \vee B \equiv B \vee A$, we get: $(\alpha = \Lambda \vee \alpha \neq \alpha +_c 1) \supset \sim (\alpha +_c 1 \neq \Lambda \wedge \alpha +_c 1 = \alpha +_c 1 +_c 1)$. And applying De Morgan again gets us: $(\alpha = \Lambda \vee \alpha \neq \alpha +_c 1) \supset (\alpha +_c 1 = \Lambda \vee \alpha +_c 1 \neq \alpha +_c 1 +_c 1)$.

Corollary. [Whitehead & Russell 1912 $*120{\cdot}33$ - Peano 3.3]. Infin Ax *is true iff every number is different from its successor. That is:* $(\alpha)(\alpha \in \text{NC induct} \supset \alpha \neq \Lambda) \equiv (\alpha)(\alpha \in \text{NC induct} \supset \alpha \neq \alpha +_c 1)$.

Dem. Assume Infin Ax and $\alpha \in$ NC induct. Then α is non-empty. By Peano 3.22, if α is non-empty it differs from its successor. Conversely, assume $\alpha \in$ NC induct and $\alpha \neq \alpha +_c 1$. Then, by Peano 3.22 α is non-empty. ■

Remark. This establishes that no finite cardinal number is identical to its immediate successor iff the Axiom of Infinity is true, and thus, that different numbers have different successors iff the Axiom of Infinity is true.

B.6 Principia's Most Famous Theorem: 1+1=2

Here we prove the famous and "occasionally useful"[6] proposition which asserts that one plus one equals two, or, equivalently, that the number two is the immediate successor of one. First we define 1 and 2:

Definition 7. [Whitehead & Russell 1910 $*52{\cdot}01$, $*54{\cdot}02$] $1 =_{\text{df}} \{\alpha : (\exists x)(\alpha = \{x\})\}$; $2 =_{\text{df}} \{\alpha : (\exists x, y)(x \neq y \wedge \alpha = \{x\} \cup \{y\})\}$.

In volume one of *Principia* Whitehead and Russell prove the following proposition which is sometimes mastaken[7] for *Principia*'s equivalent of $1 + 1 = 2$:

Lemma. [Whitehead & Russell 1910 $*54{\cdot}43$] $\alpha, \beta \in 1 \supset (\alpha \cap \beta = \Lambda \equiv \alpha \cup \beta \in 2)$

Dem. Assume $\alpha, \beta \in 1$. Then for some x, y $\alpha = \{x\}$ and $\beta = \{y\}$. Assume $\alpha \cap \beta = \Lambda$. Then, extensionality and predicate logic yield $x \neq y$. But then, by the definition of 2 as $\{\gamma : (\exists x, y)(x \neq y \wedge \gamma = \{x\} \cup \{y\})\}$, we have $\{x\} \cup \{y\} \in 2$ and thus $\alpha, \beta \in 1 \supset (\alpha \cap \beta = \Lambda \supset \alpha \cup \beta \in 2)$. Conversely, assume $\alpha, \beta \in 1$ and $\alpha \cup \beta \in 2$. By the definition of 1, $\alpha = \{x\}$ and $\beta = \{y\}$ and by the definition of 2 we have $x \neq y$. But then, by predicate logic and extensionality, we have $\alpha \cap \beta = \Lambda$ and, therefore, $\alpha, \beta \in 1 \supset (\alpha \cup \beta \in 2 \supset \alpha \cap \beta = \Lambda)$. ■

Remark. As the authors explain, "from this proposition it will follow, when arithmetical addition has been defined, that 1 + 1 = 2" [8]. The actual proof of $1 +_c 1 = 2$ occurs in the second volume. Curiously it does not reference this lemma, but another theorem, namely:

[6] WHITEHEAD & RUSSELL, 1912, p.82.
[7] MARION, M., 2012, p.143
[8] WHITEHEAD & RUSSELL, 1910, p.379.

$$54 \cdot 3 \qquad 2 = \{\alpha : (\exists x)(x \in \alpha \wedge \alpha - \{x\} \in 1\}$$

which actually just embodies the definition of successor which we are using here. The reason as to why they proceed in this way is clear: the proof becomes trivial. For the sake of their observation in the first volume, we offer the following proof via the lemma $*54{\cdot}43$.

Proposition. [Whitehead & Russell 1912 $*110{\cdot}643$] $1 +_c 1 = 2$

Dem. Assume $\gamma \in 2$, that is $(\exists x)(\exists y)(x \neq y \wedge \gamma = \{x\} \cup \{y\})$. If $x \neq y$ and $\gamma = \{x\} \cup \{y\}$,then $\gamma - \{x\} = \{y\}$. By definition, $\{y\} \in 1$, and since $\gamma - \{x\} = \{y\}, \gamma - \{x\} \in 1$. Thus, we have $(\exists x)(x \in \gamma \wedge \gamma - \{x\} \in 1)$. If $\gamma \in 2$, then $\gamma \in 1 +_c 1$. Conversely assume $\gamma \in 1 +_c 1$, that is, $(\exists x)(x \in \gamma \wedge \gamma - \{x\} \in 1)$. If $(\exists x)(x \in \gamma \wedge \gamma - \{x\} \in 1)$, then $(\exists x)(\exists y)(x \in \gamma \wedge \gamma - \{x\} = \{y\})$. By lemma $*54 \cdot 43$, we have $\{x\}, \{y\} \in 1 \supset (\{x\} \cap \{y\} = \Lambda \equiv \{x\} \cup \{y\} \in 2)$. By definition, for any *x* and *y*, $\{x\}, \{y\} \in 1$ is true. Thus, $\{x\} \cap \{y\} = \Lambda \equiv \{x\} \cup \{y\} \in 2$. If $x \in \gamma \wedge \gamma - \{x\} = \{y\}$, then $x \neq y$. But if $x \neq y$, then $\{x\} \cap \{y\} = \Lambda$ and, therefore, $\{x\} \cup \{y\} \in 2$ Since $\{x\} \cup \{y\} = \gamma$, it follows $\gamma \in 2$ and we are done. ■

B.7 Finite, Infinite and Russell’s Theorem

Definition 8. [Whitehead & Russell 1912] 1. A set is finite in the ordinary sense or finite or yet inductive iff it belongs to every class λ which contains the empty set, and, for every set β and every individual *a*, if β belongs to λ, so does $\beta \cup \{a\}$; 2. A set is infinite in the ordinary sense, or non-inductive or yet infinite iff it is not finite. In Symbols:

$$Fin(\alpha) =_{\text{df}} (\lambda)(\Lambda \in \lambda \wedge (\beta)(\beta \in \lambda \supset \beta \cup \{a\} \in \lambda). \supset .\alpha \in \lambda)$$

$$Infin(\alpha) =_{\text{df}} \sim Fin(\alpha);$$

Corollary 2. *1. A set α is finite iff* $\text{Nc}(\alpha) \in$ NC induct. *2. Let* indiv *be the set of individuals. If* $Infin$(indiv)*, then* Infin Ax *is true.*

Definition 9. [Dedekind 1888; Cantor 1895] 1. A set α is infinite in the sense of *Dedekind or reflexive* iff it can be put into a one-one correspondence with a proper, subset of itself; 2. A set is finite in the sense of Dedekind if it is not infinite in the sense of Dedekind. In Symbols:

$$\text{Refl}(\alpha) =_{\text{df}} (\exists R)(\exists \beta)(1 \rightarrow 1(R) \wedge \text{Dom}(R) = \alpha \wedge \text{Cont}(R) = \beta \wedge \beta \subset \alpha);$$

$$\text{Irrefl}(\alpha) =_{\text{df}} \sim \text{Refl}(\alpha);$$

Lemma 1. [Whitehead & Russell 1912] *If α is a non-inductive set, then there is an injective correlation between* NC induct *and the power set of the power set of α. That is:* $\sim Fin(\alpha) \supset (\exists \xi)(\xi \subset \wp(\wp(\alpha)) \wedge \xi \approx$ NC induct$)$.[9]

[9] We follow the proof in BOOLOS, G., 1998, pp.262-3 and DEMOPOULOS & CLARKE, 2005, pp.160-1.

Dem. $\Lambda \subset \alpha$. Therefore there is a $\beta \subset \alpha$ such that Nc$(\beta) = 0$. Suppose now there is some set $\gamma \subseteq \alpha$ such that Nc$(\gamma) = n$, for some $n \in$ NC induct. Since, by hypothesis, β is non-inductive, there is an element *a* of α such that Nc$(\gamma \cup \{a\}) = n +_c 1$. By induction, we know then that, for every *n*, there is a γ, such that Nc$(\gamma) = n$. Thus we can identify the Russellian number *n* as $S_n =_{\text{df}} \{\gamma : \gamma \subseteq \alpha \wedge \text{Nc}(\gamma) = n\}$ and the class of all S_n as NC induct. By definition, it follows that, for every *n*: (a) $S_n \neq \Lambda$;(b) $S_n \subseteq \wp(\alpha)$; (c) for all $m = n$, $S_m \neq S_n$; Therefore, since $\wp(\alpha) \subseteq \wp(\wp(\alpha))$, we have an injection of NC induct into $\wp(\wp(\alpha))$. ■

Corollary 3. *If* indiv *is non-inductive, then* NC induct *is reflexive.*

Dem. If indiv is non-inductive, then the successor function which maps each $S_n = \{\gamma : \gamma \subseteq \text{indiv} \wedge \text{Nc}(\gamma) = n\}$ to S_{n+1}, given that $Infin(\text{indiv}) \supset$ Infin Ax $\supset$ Peano 3. Thus, the successor function is one-one. Let NC induct $- 0$ be the set of all S_n except 0; since this set can be correlated one-one with NC induct, NC induct is reflexive. ■

Theorem. [Whitehead & Russell 1912] *Russell's Theorem.* *If λ is non-indutive, then $\wp(\wp(\lambda))$ is reflexive.*

Dem. By corollary above, if λ is non-inductive, the set *r* of all $S_n = \{\gamma : \gamma \subseteq \lambda \wedge \text{Nc}(\gamma) = n\}$ is reflexive. Since $r \subset \wp(\wp(\lambda))$, by lemma 2 follows $\wp(\wp(\lambda))$. ■