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LEONARDO CABRAL

STABILITY ANALYSIS AND STABILIZATION OF DISCRETE-TIME PIECEWISE AFFINE SYSTEMS

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ABSTRACT

This work addresses the problems of global stabilization and local stability analysis of discrete-time piecewise affine (PWA) systems.

To tackle the global stabilization problem, this work considers a PWA state feedback control law, a recently proposed implicit PWA representation and piecewise quadratic (PWQ) Lyapunov candidate functions. Through Finsler's Lemma, congruence transformations and some structural assumptions, *quasi*-LMI sufficient conditions to ensure the global exponential stability of the origin of the closed-loop PWA system are derived from the stability conditions. An algorithm is proposed to solve the *quasi*-LMI conditions and compute the stabilizing gains.

Regarding the problem of local stability analysis, this work proposes a method to test the local nonnegativity of PWQ functions using the implicit representation. This method is used to assess the local stability of the origin of PWA systems by considering PWQ Lyapunov candidate functions. Estimates of the Region of Attraction of the Origin (RAO) are obtained as level sets of the Lyapunov function. Approaches to obtain maximized estimates of the RAO are therefore discussed.

Keywords: Piecewise affine systems, stability and stabilization, piecewise quadratic Lyapunov functions, semidefinite programming.

RESUMO

Este trabalho trata dos problemas de estabilização global e análise de estabilidade local de sistemas afim por partes (PWA, do inglês, Piecewise Affine) de tempo discreto.

Para tratar o problema de estabilização global, considera-se uma lei de controle do tipo realimentação de estados afim por partes, uma representação implícita de sistemas PWA e funções de Lyapunov quadraticas por partes (PWQ, do inglês, Piecewise Quadratic). Através do Lema de Finsler, transformações de congruência e algumas suposições de estrutura, condições suficientes na forma de *quasi*-LMIs para assegurar a estabilidade exponencial global da origem do sistema PWA em malha fechada são derivadas das condições de estabilidade. Um algoritmo para resolver as condições *quasi*-LMIs e computar os ganhos estabilizantes é proposto.

Quanto ao problema de análise local de estabilidade, um método para testar a não negatividade local de funções PWQ usando a representação implícita é proposto. Este método é então utilizado para verificar a estabilidade local da origem de sistemas PWA através de funções de Lyapunov PWQ. Estimativas da região de atração da origem (RAO, do inglês, Region of Attraction of the Origin) são obtidas como curvas de nível da função de Lyapunov. Abordagens para maximizar a estimativa da RAO são então discutidas.

Palavras-chave: Sistemas afim por partes, estabilidade e estabilização, funções de Lyapunov quadráticas por partes, programação semidefinida.

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LIST OF ABBREVIATIONS

CPWA	Continuous Piecewise Affine
ELC	Extended Linear Complementarity
GES	Globally Exponentially Stable
KKT	Karush-Kuhn-Tucker
LC	Linear Complementarity
LMI	Linear Matrix Inequality
MDL	Mixed Logical Dynamical
MMPS	Max-Min-Plus-Scaling
PWA	Piecewise Affine
PWL	Piecewise Linear
PWQ	Piecewise Quadratic
RAO	Region of Attraction of the Origin
ReLU	Rectifier Linear Unit
SDP	Semidefinite Programming

LIST OF SYMBOLS

Ω_i	i-th regions of the extended input-state space partition;
$\mathbb{R}$	Set of real numbers;
$\mathbb{R}^n$	n-dimensional vector space;
$\mathbb{R}^{n\times m}$	Set of matrices with dimensions $n \times m$;
Γ_i	i-th regions of the state space partition;
N_Γ	Number of regions in the partition;
$\mathcal{I}$	Set of indices of the regions in the partition;
$\mathcal{S}_{all}$	Set of all combinations of regions indices;
$\mathcal{S}_{fea}$	Set of combinations of region indices of feasible transitions, according to the system dynamics;
$\succeq$	Vector or matrix elementwise nonstrict inequality;
diag$(.)$	Block diagonal matrix composed by the arguments of diag;
$v_{(i)}$	i-th element of vector v;
x^+	Successor state;
$\mathbb{D}^n$	Set of diagonal matrices with dimensions $n \times n$;
$> (<)$	Used for symmetric matrices to indicate positive (negative) definiteness;
$A_{(i,j)}$	The scalar element of matrix A in the position (i, j);
$\mathcal{I}_0$	The set of indices of regions that contain the origin;
$\mathcal{I}_1$	The set of indices of regions that do not contain the origin;
Ker(A)	Set of all vectors v such that $vA = 0$;
$\bar{x}$	Extended state vector $[1\ \ x]^T$;
$\mathcal{D}_{[0,1]}$	Set of all diagonal matrices such that the entries are in $[0, 1]$;
$A_{i,j}$	The block of the matrix A in the position i, j;
$\lambda_{min}(A)$	The minimal eigenvalue of matrix A;
$\mathcal{L}_\rho$	The level set ρ of the Lyapunov function $V(x)$, i.e., $\{x \in \mathbb{R}^n \mid V(x) \leq \rho,\ \rho > 0\}$;
$\mathcal{D}$	set of the state space where $\Delta V(x) < 0$;

$\mathcal{L}_1$	The level set 1 of the Lyapunov function $V(x)$, i.e., $\{x \in \mathbb{R}^n \mid V(x) \leq 1\}$;
λ	Nonnegative scalars representing the magnitude of vectors in the optimization procedure of Algorithm 2;
Δ_λ	Step increase of vectors v_i in the optimization procedure of Algorithm 2;
A^k	Matrix A to the power of k;

CONTENTS

1 INTRODUCTION

A generic discrete-time piecewise affine (PWA) system is defined by partitioning the extended input-state space in different regions and associating each region with an affine state update equation, as expressed by

$$x^{+} = A_i x + B_i u + a_i \quad \forall \begin{bmatrix} x \\ u \end{bmatrix} \in \Omega_i \tag{1}$$

where x and $x^{+} \in \mathbb{R}^{n}$ are, respectively, the current and successor state, $u \in \mathbb{R}^{n_u}$ is the input and $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n_u}$ and $a_i \in \mathbb{R}^{n}$ define the dynamical behavior of the system within the region Ω_i.

As pointed by JOHANSSON (2003), commom nonlinear behaviors encountered in control systems are piecewise affine, such as switches, relays, deadzones and saturations. For example, a linear system with a single saturating actuator is a PWA system partitioned in three regions: linear region, positively saturated and negatively saturated. Hybrid systems that contains both analog (continuous) and logical (discrete) components can be equivalently described as PWA systems (HEEMELS; DE SCHUTTER; BEMPORAD, 2001) and smooth nonlinearities which are not piecewise affine can be conveniently approximated as such.

Thanks to its versatility to model several nonlinearities, SONTAG (1981) proposed the use of PWA systems as a systematic approach to numerical nonlinear control and this class of systems became an active topic of research throughout the last decades. The idea of using least squares identification to model continuous-time nonlinear systems as PWA systems was investigated by PAUL; PHILLIPS (1994). To obtain a discrete-time representation, the conventional method consists in discretizing the model in each region, but a more general method of discretization, able to deal with sliding modes, was studied by SCHWARZ *et al.* (2005). Stability analysis and stabilization for discrete-time piecewise linear systems (i.e without affine terms a_i in (1)) were studied by MIGNONE; FERRARI-TRECATE; MORARI (2000) using a Linear Matrix Inequality (LMI) approach and considering piecewise quadratic (PWQ) Lyapunov candidate functions. FENG (2002) extended previous results on global stability analysis to systems with affine terms. For continuous-time PWA systems, JOHANSSON (2003) studied the stability considering

PWQ and piecewise linear (PWL) Lyapunov candidate functions. Then, the stabilization of such systems and the estimation of the region of attraction were also tackled.

Recently, GROFF; VALMORBIDA; GOMES DA SILVA JR. (2019) formulated a new implicit representation based on ramp functions for discrete-time continuous PWA systems, where the partition depends only on the state and the vector field is continuous across the boundary of the polyhedral partition. When compared with the standard representation used in the works of MIGNONE; FERRARI-TRECATE; MORARI (2000) and FENG (2002), this implicit representation was proved advantageous in the global stability analysis problem (GROFF; VALMORBIDA; GOMES DA SILVA JR., 2020). Those advantages derive from the fact that the implicit representation does not require *a priori* knowledge of the possible transitions between regions. Also, the implicit representation is able to deal with uncertainties regarding the partition.

This dissertation investigates the use of the novel implicit representation in two problems: global stabilization and local stability analysis. For the stabilization problem, *quasi*-LMI sufficient conditions are formulated using Finsler's Lemma, congruence transformations and some structural assumptions to synthesize an affine state feedback law to stabilize the system. Those conditions are then solved using convex optimization tools in addition to an algorithm of grid search. Regarding the local stability analysis, a method is developed to ensure the local nonnegativity of PWQ functions and used to derive sufficient local stability conditions along with estimates of the region of attraction as level sets of PWQ functions.

The outline of this dissertation is as follows. Chapter 2 presents a literature review on PWA systems. First, PWA systems are categorized based on their characteristics and the class of discrete-time continuous piecewise affine (CPWA) systems is defined. Different representations are reviewed and previous results on the problems of global stability analysis and global stabilization are recalled. Chapter 3 addresses the problem of global stabilization. *Quasi*-LMI sufficient conditions are derived to synthesize an affine state feedback control law to stabilize the system. Those conditions can be solved using convex optimization tools and an additional grid search algorithm described. Chapter 4 considers the problem of local stability analysis. First, conditions to ensure the local nonnegativity of implicit PWQ functions are derived. Those conditions are applied to the analysis of the local stability of PWA systems, providing estimates of the Region of Attraction of the Origin (RAO) obtained as sub level sets of PWQ Lyapunov functions. Finally, chapter 5 concludes with final remarks and discusses future lines of research and perspectives.

2 LITERATURE REVIEW

This chapter presents an overview on piecewise affine (PWA) systems. Firstly, different representations and the relation among them, mainly focused in the explicit representation commonly used and a recently developed implicit representation, are introduced. Then, the main approaches to tackle the problem of global stability analysis and stabilization are recalled.

2.1 Considered Class of Piecewise Affine Systems

The class of discrete-time PWA systems is generically defined by partitioning the extended state-input space into polyhedral regions and associating with each region a different affine state update equation (CHRISTOPHERSEN, 2007), as in (1). However, this general definition is often narrowed into subclasses, such as PWA systems where the switching between different affine state update equations depends only on regions Γ_i of the state space. This subclass can be written as

$$x^+ = A_i x + B_i u + a_i \quad \forall x \in \Gamma_i \subset \mathbb{R}^n, \quad i \in \mathcal{I} = \{1, ..., N_\Gamma\}, \tag{2}$$

with $\mathcal{I}$ denoting the region index set and N_Γ the number of regions in the partition such that

$$\bigcup_{i \in \mathcal{I}} \Gamma_i = \mathbb{R}^n.$$

The subclass (2) where Γ_i is only state-dependent is frequently considered, such as in the works of MIGNONE; FERRARI-TRECATE; MORARI (2000), FENG (2002), JOHANSSON (2003) and GROFF; VALMORBIDA; GOMES DA SILVA JR. (2019).

Moreover, if the vector field of a PWA system is continuous over the boundary of the polyhedral partition, we refer to this subclass as Continuous Piecewise Affine (CPWA) system. The subclass of CPWA systems will be considered in the present dissertation and, for simplicity, it will be just referred as PWA.

2.2 Representations

There are different ways to represent a PWA system. This section reviews some representations and draws relations among them, with focus in the commonly used explicit represention (refered in this work as standard explicit representation) and the recently developed implicit one.

2.2.1 Standard explicit PWA representation

The standard explicit PWA representation traces back the work of SONTAG (1981) and was applied in the stability analysis and stabilization for instance by MIGNONE; FERRARI-TRECATE; MORARI (2000), FENG (2002) and JOHANSSON (2003). It is characterized by partitioning the state space into polyhedral regions represented by a finite set of inequalities and associating with each region an affine state update equation, that is

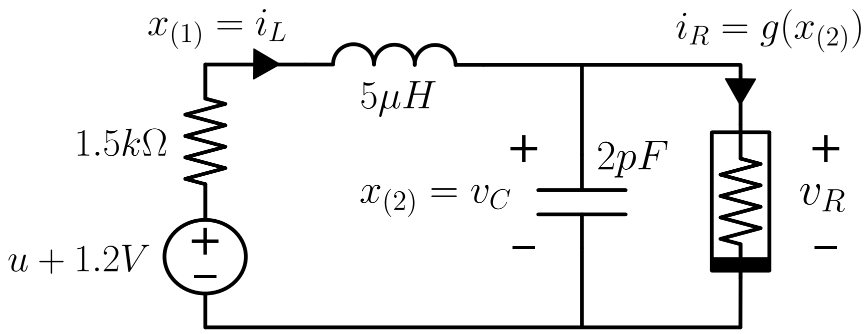
$$x^+ = A_i x + B_i u + a_i \quad \forall x \in \Gamma_i \subset \mathbb{R}^n, \quad i \in \mathcal{I} = \{1, ..., N_\Gamma\}, \tag{3a}$$

$$\Gamma_i = \{x \in \mathbb{R}^n \mid H_i x \succeq h_i\}, \tag{3b}$$

where matrix $H_i \in \mathbb{R}^{n_{ki} \times n}$ and vector $h_i \in \mathbb{R}^{n_{ki}}$ defines, for each polyhedral region Γ_i, the n_{ki} boundary hyperplanes. Alternatively, regions Γ_i can be represented by cone rays and vertices (FUKUDA; PICOZZI; AVIS, 2002) as done for stability analysis of continuous-time PWA systems by IERVOLINO; TANGREDI; VASCA (2017).

To illustrate the standard explicit representation (3) consider the following nonlinear circuit in Figure 1, proposed by RODRIGUES; BOYD (2005), whose continuous-time dynamical behavior is given by (4).

Figure 1 – Circuit with nonlinear resistor.



Source: RODRIGUES; BOYD (2005)

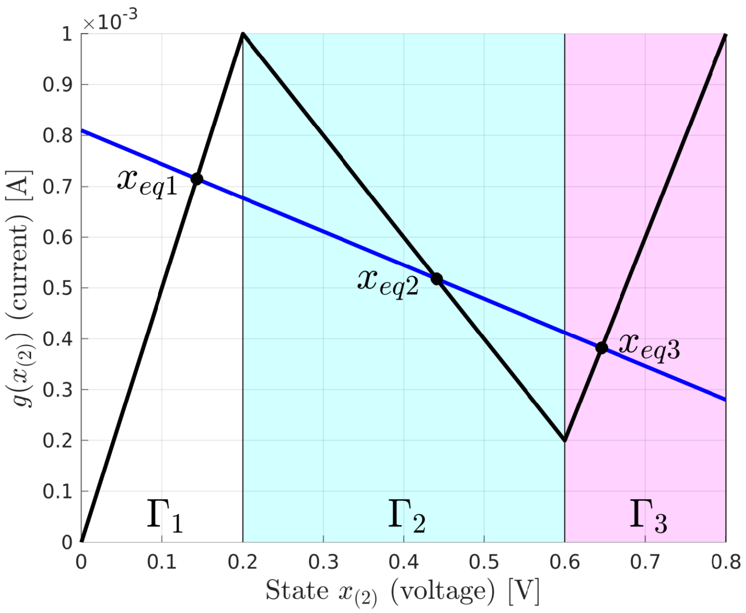
$$\dot{x} = \begin{bmatrix} -30 & -20 \\ 0.05 & 0 \end{bmatrix} x + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u + \begin{bmatrix} 24 \\ -50g(x_{(2)}) \end{bmatrix} \tag{4}$$

In (4) the time is expressed in 10^{-10} seconds, the state $x_{(1)}$ is the inductor current in milliamperes, $x_{(2)}$ is the voltage across the capacitor in Volts and input u is also given in

Volts. The nonlinear resistor voltage-current characteristic is given by $i_R = g(v_R)$ and modeled as the continuous PWA function depicted in Figure 2, from where we notice the existence of three regions, Γ_1, Γ_2 and Γ_3. The circuit load is given by the resistor connected in series with the input voltage source (1.5 kΩ for this example). In this case, the system presents three equilibrium points, x_{eq1}, x_{eq2} and x_{eq3}, one on each set of the partition. The partition is determined by the state variable $x_{(2)}$, which is equal to the voltage across the nonlinear resistor, as follows:

$$\Gamma_1 = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & -1 \end{bmatrix} x \geq -0.2 \right\} \quad \text{(i.e. } x_{(2)} \leq 0.2\text{)},$$
$$\Gamma_2 = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x \succeq \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix} \right\} \quad \text{(i.e. } 0.2 \leq x_{(2)} \leq 0.6\text{) and}$$
$$\Gamma_3 = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 1 \end{bmatrix} x \geq 0.6 \right\} \quad \text{(i.e. } x_{(2)} \geq 0.6\text{)}.$$

Figure 2 – Piecewise voltage-current characteristic of the nonlinear resistor (black) and the load given by the 1.5 kΩ resistor (blue) in Figure 1.



Source: RODRIGUES; BOYD (2005)

A discrete-time dynamic behavior can be derived from (4) using Euler discretization for each region Γ_i. A sampling period of $T \times 10^{-10}$ second leads to the following matrices and vectors

$$A_1 = \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1-0.25T \end{bmatrix},\ A_2 = \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1+0.1T \end{bmatrix},\ A_3 = \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1-0.2T \end{bmatrix},$$
$$a_1 = \begin{bmatrix} 24T \\ 0 \end{bmatrix},\ a_2 = \begin{bmatrix} 24T \\ -0.07T \end{bmatrix},\ a_3 = \begin{bmatrix} 24T \\ 0.11T \end{bmatrix} \text{ and } B_1 = B_2 = B_3 = \begin{bmatrix} 20T \\ 0 \end{bmatrix}.$$

Furthermore we have the regions Γ_1, Γ_2 and Γ_3 described, respectively, with

$$H_1 = \begin{bmatrix} 0 & -1 \end{bmatrix},\ H_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\ H_3 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$h_1 = -0.2,\ h_2 = \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix} \text{ and } h_3 = 0.6,$$

which completes the standard explicit PWA representation (3).

The standard explicit representation is often viewed as the most straightforward way of representing a PWA system. Its main advantage is its versatility, since it is possible to represent discontinuous systems (i.e. systems where the vector field is not continuous over the boundary of the polyhedral partition), systems with different matrices B_i for each region and also switched systems by including the input u in the set of inequalities. On the other hand, the main drawback is the complexity in problems such as stabilization and stability analysis, where is often necessary to know *a priori* what are the possible transitions among regions. Also, the standard explicit PWA representation is not suitable to handle uncertainties in the partition.

2.2.2 Implicit representation

The implicit representation considered in this work is the one recently proposed by GROFF; VALMORBIDA; GOMES DA SILVA JR. (2019). It takes the form

$$x^+ = F_1 x + F_2 \phi(y(x)) + Bu \tag{5a}$$
$$y(x) = F_3 x + F_4 \phi(y(x)) + f_5 \tag{5b}$$

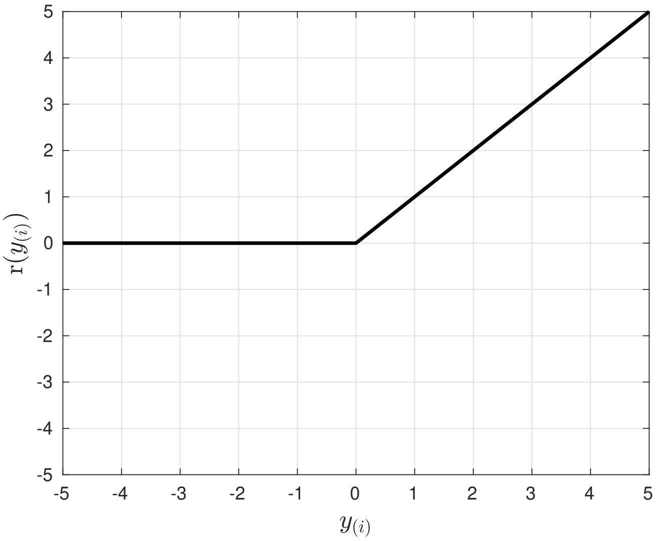
where x and $x^+ \in \mathbb{R}^n$ are, respectively, the current and successor state and $u \in \mathbb{R}^{n_u}$ is the input. The system is defined by matrices $F_1 \in \mathbb{R}^{n \times n}$, $F_2 \in \mathbb{R}^{n \times n_y}$, $B \in \mathbb{R}^{n \times n_u}$, $F_3 \in \mathbb{R}^{n_y \times n}$, $F_4 \in \mathbb{R}^{n_y \times n_y}$ and vector $f_5 \in \mathbb{R}^{n_y}$. Vector $y \in \mathbb{R}^{n_y}$ is the argument to the vector-valued ramp function $\phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$, which is defined elementwise in terms of the ramp function $r : \mathbb{R} \to \mathbb{R}$ as

$$\phi_{(i)}(y) = r(y_{(i)}) = \begin{cases} 0 & \text{if } \ y_{(i)} < 0 \\ y_{(i)} & \text{if } \ y_{(i)} \geq 0 \end{cases} \tag{6}$$

for each $i = 1, ..., n_y$, where $y_{(i)}$ and $\phi_{(i)}(y)$ are, respectively, the i-th element of vector y and vector function $\phi(y)$. The scalar ramp function $r(y_{(i)})$ is depicted in Figure 3.

Remark 1. *Since the vector field of the PWA system is assumed to be continuous across region boundaries, it is equivalent to interchange the strict and the nonstrict inequalities in* (6)*. In fact, it is more accurate to consider the boundary as belonging to both regions. See, for example, (RODRIGUES; BOYD, 2005).*

Figure 3 – Scalar ramp function $r(y_{(i)})$.



Source: The author

The partition is implicitly defined by (5b) and x belongs to the region Γ_i depending on the combination of elements of $\phi(y(x))$ that are activated (corresponding to $y_{(i)} \geq 0$) or inactivated (corresponding to $y_{(i)} < 0$). Hence, it is possible to represent $N_\Gamma = 2^{n_y}$ regions utmost, altought some of them may result in empty sets. To illustrate, consider the same nonlinear circuit in Figure 1 with continuous-time dynamic given by (4).

The three regions of Figure 2 can be constructed by inspection in the implicit representation with

$$F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.007 \\ 0 & 0.006 \end{bmatrix}, \; F_4 = 0 \text{ and } f_5 = \begin{bmatrix} 24 \\ -0.0014 \\ -0.0036 \end{bmatrix},$$

leading to the definition of the three regions as

$$\begin{aligned} \Gamma_1 &= \left\{x \in \mathbb{R}^2 \mid y_{(1)}(x) \geq 0,\; y_{(2)}(x) < 0,\; y_{(3)}(x) < 0\right\}, \\ \Gamma_2 &= \left\{x \in \mathbb{R}^2 \mid y_{(1)}(x) \geq 0,\; y_{(2)}(x) \geq 0,\; y_{(3)}(x) < 0\right\} \text{ and} \\ \Gamma_3 &= \left\{x \in \mathbb{R}^2 \mid y_{(1)}(x) \geq 0,\; y_{(2)}(x) \geq 0,\; y_{(3)}(x) \geq 0\right\}. \end{aligned}$$

Other combinations of active and inactive elements of $\phi(y(x))$ result in empty sets. To illustrate this situation, note that in this example the elements of $y(x)$ are given by

$$\begin{aligned} y_{(1)}(x) &= 24, \\ y_{(2)}(x) &= \begin{bmatrix} 0 & 0.007 \end{bmatrix} x - 0.0014 \quad \text{and} \\ y_{(3)}(x) &= \begin{bmatrix} 0 & 0.006 \end{bmatrix} x - 0.0036. \end{aligned}$$

Since element $y_{(1)}(x)$ is a positive constant, every possible region Γ_i defined by a combination containing an inactive $y_{(1)}$ (i.e. $y_{(1)} < 0$) is an empty set. On the other hand, a non-empty region such as Γ_2 is defined by

$$\Gamma_2 = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} y_{(1)} \geq 0 \\ y_{(2)}(x) \geq 0 \Leftrightarrow x_{(2)} \geq 0.2 \\ y_{(3)}(x) < 0 \Leftrightarrow x_{(2)} < 0.6 \end{array} \right\}.$$

Once F_3, F_4 and f_5 are settled, the state update equation can also be constructed by inspection, resulting in

$$F_1 = \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1-0.25T \end{bmatrix}, \; F_2 = \begin{bmatrix} T & 0 & 0 \\ 0 & 50T & -50T \end{bmatrix} \text{ and } B = \begin{bmatrix} 20T \\ 0 \end{bmatrix}.$$

To better illustrate the equivalence between this representation and the standard explicit one, take as an example region Γ_2, whose combination of active and inactive elements of $y_{(i)}$ was previously defined. Then, the dynamic in this region is written as

$$\begin{aligned} &F_1 x + F_2 \phi(y) \\ &= \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1-0.25T \end{bmatrix} x + \begin{bmatrix} T & 0 & 0 \\ 0 & 50T & -50T \end{bmatrix} \times \begin{bmatrix} 24 \\ 0.007x_{(2)} - 0.0014 \\ 0 \end{bmatrix} + \begin{bmatrix} 20T \\ 0 \end{bmatrix} u \\ &= \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1+0.1T \end{bmatrix} x + \begin{bmatrix} 20T \\ 0 \end{bmatrix} u + \begin{bmatrix} 24T \\ -0.07 \end{bmatrix} = A_2 x + B_2 u + a_2 \end{aligned}$$

In general, when F_4 does not have a particular structure (such as $F_4 = 0$ or an upper triangular matrix), (5b) is an implicit equation due to the algebraic loop. In such case it is necessary to ensure its well-posedness, i.e. to ensure the existence and uniqueness of solution for every $x \in \mathbb{R}^n$. This can be done by the following Proposition, whose proof can be found in (GROFF; VALMORBIDA; GOMES DA SILVA JR., 2019).

Proposition 1. *(GROFF; VALMORBIDA; GOMES DA SILVA JR., 2019) If there exists diagonal matrix $X \in \mathbb{D}^{n_y}$ such that*

$$-2X + XF_4 + F_4^T X < 0$$

then the implicit equation (5b) *is well-posed.*

Thanks to the relation between partition and dynamics that exists in the implicit representation (5), the partition enumeration and *a priori* knowledge of possible transitions among regions are not needed for stability analysis and stabilization. Moreover, uncertainties in the partition can be handled without further difficulties (GROFF; VALMORBIDA; GOMES DA SILVA JR., 2020). Those features constitute a major advantage when compared to the explicit representation, as will become clear in the next sections. However, a procedure to obtain a minimal implicit representation, that is, the implicit representation with the minimal amount of ramp functions, must be further investigated.

2.2.3 Relations between standard explicit and implicit representations

The implicit representation of the nonlinear circuit example presented in section 2.2.2 was obtained by inspection. In general, this can only be done when the order of the system and the number of regions in the partition are relatively small. This section will be focused in a brief discussion about the relation between the standard explicit representation and the implicit one.

As previously discussed, the partition in the implicit representation (5) is determined by (5b), namely $x \in \mathbb{R}^n$ will belong to a region Γ_i if $\phi(y(x))$ satisfies the combination of activated (i.e. $y_{(i)} \geq 0$) and inactivated (i.e $y_{(i)} < 0$) elements that defines Γ_i. Hence, we can write that

$$\phi(y(x)) = \Phi_i y(x) \quad \forall x \in \Gamma_i \subset \mathbb{R}^n,$$

where diagonal matrices $\Phi_i \in \mathbb{D}^{n_y}$ are defined by the combination of active and inactive elements of a given region as

$$\Phi_{i(j,j)} = \begin{cases} 1 \quad \text{if} \quad y_{(j)} \geq 0 \ \forall x \in \Gamma_i \\ 0 \quad \text{if} \quad y_{(j)} < 0 \ \forall x \in \Gamma_i \end{cases}$$

for $i \in \mathcal{I}$ and $j = 1, ..., n_y$. As $(I - F_4\Phi_i)$ is guaranteed to be non-singular if the system is well-posed (GROFF; VALMORBIDA; GOMES DA SILVA JR., 2020), it is therefore possible to rewrite (5b) for each region Γ_i as follows:

$$\begin{aligned} y(x) &= F_3 x + F_4 \Phi_i y(x) + f_5 \\ &= (I - F_4\Phi_i)^{-1} F_3 x + (I - F_4\Phi_i)^{-1} f_5. \end{aligned} \tag{7}$$

The combination of active and inactive elements of y associated with each region Γ_i defines inequalities on each element of y. Some of those inequalities are strict and some are nonstrict. However, as stated in Remark 1, it is possible to consider all of them as nonstrict. Consider now diagonal matrices $\Lambda_i \in \mathbb{D}^{n_y}$ having entries 1 or -1 depending on the combination of active and inactive elements of $\phi(y)$ defining the region Γ_i, as

$$\Lambda_{i(j,j)} = \begin{cases} 1 \quad \text{if} \quad y_{(j)} \geq 0 \ \forall x \in \Gamma_i \\ -1 \quad \text{if} \quad y_{(j)} < 0 \ \forall x \in \Gamma_i \end{cases}$$

for $i \in \mathcal{I}$ and $j = 1, ..., n_y$. This allows to write the regions Γ_i from an elementwise vector inequality, that is:

$$\Gamma_i = \{x \in \mathbb{R}^n \mid \Lambda_i y(x) \succeq 0\} \ \ \forall i \in \mathcal{I}, \tag{8}$$

where "lesser or equal to" inequalities had both sides multiplied by -1, resulting in all inequalities being of the "greater or equal to" type. Then, (7) and (8) allow to recover matrix H_i and vector h_i corresponding to the explicit description (3) of Γ_i, since

$$\Gamma_i = \{x \in \mathbb{R}^n \mid \Lambda_i (I - F_4\Phi_i)^{-1} F_3 x \succeq -\Lambda_i (I - F_4\Phi_i)^{-1} f_5)\} \ \forall i \in \mathcal{I},$$

from where the equivalence with the standard explicit representation (3) is

$$\begin{aligned} H_i &= \Lambda_i(I - F_4\Phi_i)^{-1}F_3 \quad \text{and} \\ h_i &= -\Lambda_i(I - F_4\Phi_i)^{-1}f_5. \end{aligned}$$

To complete the equivalente between (3) and (5) we rewrite (5a) as

$$\begin{aligned} x^+ &= F_1x + F_2\Phi_i((I - F_4\Phi_i)^{-1}F_3x + (I - F_4\Phi_i)^{-1}f_5) + Bu \\ &= (F_1 + F_2\Phi_i(I - F_4\Phi_i)^{-1}F_3)x + F_2\Phi_i(I - F_4\Phi_i)^{-1}f_5 + Bu \end{aligned}$$

for $i \in \mathcal{I}$, from where we obtain

$$\begin{aligned} A_i &= F_1 + F_2\Phi_i(I - F_4\Phi_i)^{-1}F_3, \\ a_i &= F_2\Phi_i(I - F_4\Phi_i)^{-1}f_5 \quad \text{and} \\ B_i &= B. \end{aligned}$$

The conversion from the standard explicit representation to the implicit one can be done by inspection in simple cases. In more general cases, it is possible to convert a CPWA system from the standard explicit representation to the canonical representation (CHUA; KANG, 1977) and, then, to the implicit representation (GROFF, 2020). The interested reader should consult those works. Relations between representations other than the standard explicit and the implicit are discussed next.

2.2.4 Other representations and their relations

Since PWA systems are common in different areas of interest, several representations for this class of systems were proposed, each one with their own advantages and drawbacks. For example, CHUA; KANG (1977) proposed the canonical PWA representation to deal with nonlinear circuit modeling related problems (CHUA, 1972). Also, hybrid systems were shown to be related to PWA systems by HEEMELS; DE SCHUTTER; BEMPORAD (2001) using five representations: mixed logical dynamical (MLD) systems, linear complementarity (LC) systems, extended linear complementarity (ELC) systems, standard piecewise affine (PWA) systems and max-min-plus-scaling (MMPS) systems. Note that the equivalence is sometimes guaranteed under additional assumptions and the interested reader shall consult those works for further information.

The parallel between the implicit representation and others was initially discussed by GROFF; VALMORBIDA; GOMES DA SILVA JR. (2020), where MMPS was related to the implicit representation, which, from (HEEMELS; DE SCHUTTER; BEMPORAD, 2001) can therefore be related to other representations. A more extense discussion can be found in (GROFF, 2020), where the relation among the implicit, MMPS and canonical representations was derived. The interested reader should consult those works for more details.

2.3 Global Exponential Stability Analysis

This section focuses on the global stability analysis of the origin of PWA systems. First, the global stability analysis problem is defined and necessary conditions for global stability (i.e. all trajectories in the state space converge asymptotically to the origin) are recalled. Then, using the standard explicit representation the stability analysis is reviewed for the special case of Piecewise Linear (PWL) systems and for the general case of Piecewise Affine (PWA) systems. Finally, the stability analysis is reviewed for PWA systems using the implicit representation. The stability analysis considered in this work is based on Lyapunov's second method, with a quadratic or piecewise quadratic (PWQ) structure for Lyapunov candidate functions. In this case, the existence of a positive definite candidate function whose difference along the trajectories is negative definite can be assessed through convex constraints formulated as Linear Matrix Inequalities (LMIs).

2.3.1 Necessary conditions for global stability

The global stability analysis problem can be stated as following: given a discrete-time PWA system, verify if all trajectories in the state space will converge asymptotically to the considered equilibrium point. Moreover, if the rate of convergence is bounded by an exponential decay, the system is named globally exponentially stable (GES). Without loss of generality, as a translation of the state coordinates system can always be done, we consider that the equilibrium point of interest is the origin.

Let the set of region indices $\mathcal{I}$ be defined by subsets $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, where $\mathcal{I}_0$ is the set of indices of regions containing the origin (including cases when the origin coincides with the region boundaries) and $\mathcal{I}_1$ is the set of indices of regions that do not contain the origin. Then, a necessary condition for global stability, expressed in the standard explicit representation (3), is $a_i = 0 \;\; \forall i \in \mathcal{I}_0$, since otherwise the origin would not be an equilibrium point (MIGNONE; FERRARI-TRECATE; MORARI, 2000).

Considering the implicit representation (5), a sufficient condition to ensure the origin is an equilibrium point is $f_5 \preceq 0$ (elementwise nonpositive), case where f_5 is the solution to the implicit equation

$$y(0) = F_3 \times 0 + F_4\phi(y(0)) + f_5 = f_5.$$

Hence, since $\phi(y(0)) = \phi(f_5) = 0$, we have that

$$x^+(0) = F_1 \times 0 + F_2\phi(y(0)) = 0$$

at the origin. Also, GROFF (2020) has proved that if the origin is an equilibrium point then there is always an implicit representation with $f_5 \preceq 0$ when F_4 is lower triangular and provided a method to obtain this representation. Note that if f_5 has positive elements, the origin will be an equilibrium point only if $F_2\phi(y(0)) = 0$ (i.e. $\phi(y(0)) \in \text{Ker}(F_2)$).

Finally, note that trying to deduce the stability or instability of a PWA system by the stability or instability of its subsystems (i.e. the dynamics on each region) is incorrect as stated by MIGNONE; FERRARI-TRECATE; MORARI (2000). For continuous-time, an example of unstable PWA system composed by stable subsystems is given by BRANICKY (1998), while an example of globally stable PWA system composed by unstable subsystem is given by UTKIN (1977).

2.3.2 Stability of PWL systems with the explicit representation

This subsection discusses the stability analysis for the special case of discrete-time Piecewise Linear (PWL) systems given by

$$x^+ = A_i x + B_i u \quad \forall x \in \Gamma_i \subset \mathbb{R}^n, \quad i \in \mathcal{I} = \{1, ..., N_\Gamma\}, \tag{9a}$$

$$\Gamma_i = \{x \in \mathbb{R}^n \mid H_i x \succeq 0\}. \tag{9b}$$

Note that PWL systems (9) are PWA systems (3) with all affine terms $a_i = 0$. Moreover, the partition in the PWL systems (9) is given by vector inequalities with $h_i = 0$ in (9b), rather than inequalities with possibly non-null terms h_i as in (3b). From this fact follows that every region of the partition contains the origin, that is, $\mathcal{I}_0 = \mathcal{I}$ while $\mathcal{I}_1 = \emptyset$.

One idea to assess the global stability of PWL systems (9) is to use a quadratic Lyapunov candidate function given by

$$V(x) = x^T P x \tag{10}$$

with $P = P^T \in \mathbb{R}^{n\times n}$ (JOHANSSON; RANTZER, 1998). In this case, the global exponential stability of the origin can be verified by the following Lemma.

Lemma 1. *(MIGNONE; FERRARI-TRECATE; MORARI, 2000) If there exists a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$ such that the LMIs*

$$P > 0 \tag{11a}$$

$$A_i^T P A_i - P < 0 \quad \forall i \in \mathcal{I} \tag{11b}$$

are satisfied, then the origin of the unforced ($u \equiv 0$) PWL system (9) *is globally exponentially stable.*

Proof. By pre and post multiplying (11a) by x^T and x, respectively, we obtain

$$V(x) > 0.$$

On the other hand, pre and post multiplying (11b) by x^T and x, respectively, leads to

$$V(x^+) - V(x) < 0,$$

from where the global exponential stability of the origin follows. ∎

A PWL system (9) whose stability is verified by Lemma 1 is called quadratically stable. However, the quadratic stability conditions (11) are very restrictive. One source of conservatism arises from the fact that a common matrix P must ensure the positivity of $V(x)$ and the negativity of $\Delta V(x) \triangleq V(x^+) - V(x)$ in every region of the partition. An example of PWL system with a globally stable origin whose stability cannot be assessed by a common quadratic Lyapunov function is given by FENG (2002).

Less restrictive conditions for the stability analysis can be obtained by considering a Piecewise Quadratic (PWQ) Lyapunov candidate function

$$V(x) = x^T P_i x \quad \forall x \in \Gamma_i,$$

with $P_i = P_i^T \in \mathbb{R}^{n \times n} \;\; \forall i \in \mathcal{I}$. In this case, the global exponential stability of the origin can be verified by the following Lemma.

Lemma 2. *(MIGNONE; FERRARI-TRECATE; MORARI, 2000) Let the set $\mathcal{S}_{all} = \mathcal{I} \times \mathcal{I}$ contain all combinations of region indices. If there exist symmetric matrices $P_i = P_i^T \in \mathbb{R}^{n \times n}$, for $i = 1, ..., N_\Gamma$, such that the LMIs*

$$P_i > 0 \quad \forall i \in \mathcal{I} \tag{12a}$$

$$A_i^T P_j A_i - P_i < 0 \quad \forall (i, j) \in \mathcal{S}_{all}. \tag{12b}$$

are satisfied, then the origin of the unforced ($u \equiv 0$) PWL system (9) *is globally exponentially stable.*

Proof. By pre and post multiplying (12a) by x^T and x, respectively, we obtain

$$x^T P_i x > 0 \quad \forall i \in \mathcal{I},$$

implying that $V(x) > 0$. On the other hand, pre and post multiplying (12b) by x^T and x, respectively, leads to

$$V(x^+) - V(x) < 0$$

for all combinations of regions indices. This accounts for transitions between regions (i.e. $i \neq j$ in (12b)) and steps where the trajectory does not leave the current region (i.e. $i = j$ in (12b)). Hence, $\Delta V(x)$ is negative definite for all trajectories, from where the global exponential stability of the origin follows. ■

A PWL system (9) whose stability is verified by Lemma 2 is called piecewise quadratically stable. The conditions (12) regarding PWQ stability are less conservative than (11) since different matrices P_i, for $i = 1, ..., N_\Gamma$, are responsible for the positive definiteness of $V(x)$ in different regions. However, there are still two main sources of conservatism in the PWQ stability conditions (12). The first one is the use of the set $\mathcal{S}_{all}$. This implies that (12b) ensures $\Delta V(x) < 0$ for all combinations of transitions between regions, even if some of them may be infeasible due to the system dynamics. To overcome this issue, a reachability analysis (BEMPORAD; FERRARI-TRECATE; MORARI, 2000) must be performed to compute, *a priori*, the set $\mathcal{S}_{fea}$ of feasible transitions. By replacing S_{all} by S_{fea} in (12) the stability analysis is less conservative, since impossible transitions among regions are removed from the negative definiteness test of $\Delta V(x)$.

The second main source of conservatism in (12) lays in the positivite definiteness condition (12a). Note that this condition requires each component $x^T P_i x$ of the PWQ Lyapunov function $V(x)$ to be positive definite for all state space and not only within its respective region Γ_i. Moreover, the inequality (12b) should be verified only if $x \in \Gamma_j$.

To obtain less conservative positive definiteness conditions and verify (12b) only if $x \in \Gamma_j$, it is possible to use the S-procedure (see Appendix A.1) to impose positivity of each component of $V(x)$ and negativity of each component of $\Delta V(x)$ only within its respective region. In order to do that, we take into account the term $H_i x$ from (9b). This term is used to define the regions of the partition and is elementwise nonnegative if and only if $x \in \Gamma_i$. From this fact the following Proposition is stated.

Proposition 2. *For any symmetric elementwise nonnegative matrix* $U = U^T \succeq 0 \in \mathbb{R}^{n_{ki} \times n_{ki}}$*, the following expression holds:*

$$x^T H_i^T U H_i x \geq 0 \quad \forall x \in \Gamma_i. \tag{13}$$

Proof. Note that from (9b) the term $H_i x$ results in a elementwise nonnegative vector (or a nonnegative scalar, depending on the dimensions of H_i) for any $x \in \Gamma_i$. Since the matrix U is assumed to be elementwise nonnegative, then (13) holds. ■

Proposition 2 allows to state the following Lemma regarding the exponential stability of PWL systems (9).

Lemma 3. *If there exist symmetric matrices* $P_i = P_i^T \in \mathbb{R}^{n \times n}$*, symmetric elementwise nonnegative matrices* $U_i = U_i^T \succeq 0 \in \mathbb{R}^{n_{ki} \times n_{ki}}$ *and* $Z_{ij} = Z_{ij}^T \succeq 0 \in \mathbb{R}^{n_{ki} \times n_{ki}}$*, for* $i, j = 1, ..., N_\Gamma$*, such that the LMIs*

$$P_i - H_i^T U_i H_i > 0 \quad \forall i \in \mathcal{I} \tag{14a}$$

$$A_i^T P_j A_i - P_i + H_i^T Z_{ij} H_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all} \tag{14b}$$

are satisfied, then the origin of the unforced ($u \equiv 0$) *PWL system* (9) *is globally exponentially stable.*

Proof. Pre and post multiplying (14a) by x^T and x, respectively, results in

$$x^T P_i x > x^T H_i^T U_i H_i x.$$

From Proposition 2 the term $x^T H_i^T U_i H_i x$ is a nonnegative scalar provided that $x \in \Gamma_i$ for any elementwise nonnegative matrix U_i, implying that

$$x^T P_i x > 0 \quad \forall x \in \Gamma_i \quad \text{and} \quad \forall i \in \mathcal{I},$$

that is, $V(x)$ is positive definite.

On the other hand, pre and post multiplying (14b) by x^T and x, respectively, leads to

$$x^T A_i^T P_j A_i x - x^T P_i x < -x^T H_i^T Z_{ij} H_i x.$$

Once again, from Proposition 2 the term $x^T H_i^T Z_{ij} H_i x$ is a nonnegative scalar for any elementwise nonnegative matrix Z_{ij} provided that $x \in \Gamma_i$. This implies that within each region Γ_i the function $\Delta V(x)$ is negative definite, from where the global exponential stability follows. ■

The terms $H_i^T U_i H_i$ and $H_i^T Z_{ij} H_i$ in conditions (14) relax the stability analysis problem, since the positive definiteness of the PWQ Lyapunov function $V(x)$ and the negative definiteness of the PWQ function $\Delta V(x)$ are only required within each respective region.

The following subsection addresses the stability analysis for the general case of PWA systems, where the results presented will be extended.

2.3.3 Stability of PWA systems with the explicit representation

To extended the stability analysis results presented in subsection 2.3.2 to the general case of PWA systems, consider the extended state vector $\bar{x} \triangleq [1 \quad x^T]^T$. Then, the PWA system (3) can be compactly written as

$$\bar{x}^+ = \bar{A}_i \bar{x} + \bar{B}_i u = \begin{bmatrix} 1 & 0 \\ a_i & A_i \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ B_i \end{bmatrix} u \quad \forall x \in \Gamma_i \subset \mathbb{R}^n \tag{15a}$$

$$\Gamma_i = \{x \in \mathbb{R}^n \mid \bar{H}_i \bar{x} = \begin{bmatrix} -h_i & H_i \end{bmatrix} \bar{x} \succeq 0\}. \tag{15b}$$

When we consider the extended state vector $\bar{x}$, the notation of a PWA system (3) becomes similar to a PWL system (9). However, it is important to note that the extended dynamic matrix $\bar{A}_i$ is not a Schur matrix (i.e. a matrix with absolute values of all eigenvalues strictly less than 1). Consequently, it is not possible to apply the quadratic or the piecewise quadratic stability conditions from Lemmas 1 and 2 for this extended PWL system representation, since this would lead to infeasible LMI constraints (CUZZOLA; MORARI, 2001).

To avoid infeasible conditions we must introduce terms to relax the problem, in a similar fashion to what was done in Lemma 3. In order to do that, consider the following Proposition, derived as an extended version of Proposition 2.

Proposition 3. *For any symmetric elementwise nonnegative matrix $U = U^T \succeq 0 \in \mathbb{R}^{n_{ki} \times n_{ki}}$, the following expression holds:*

$$\bar{x}^T \bar{H}_i^T U \bar{H}_i \bar{x} \geq 0 \quad \forall x \in \Gamma_i. \tag{16}$$

Proof. Note that from (15b) the term $\bar{H}_i \bar{x}$ results in an elementwise nonnegative vector (or scalar, depending on the dimensions of $\bar{H}_i$) for any $x \in \Gamma_i$. Since the matrix U is assumed to be elementwise nonnegative, then (16) holds. ■

From the Proposition 3, the term $\bar{H}_i\bar{x}$ is elementwise nonnegative if and only if $x \in \Gamma_i$. This term can then be used along with the S-procedure to propose conditions for the stability analysis of PWA systems. This is done by the following Lemma (FENG, 2002):

Lemma 4. *(FENG, 2002) Consider the PWA system* (3) *such that* $h_i = 0 \;\; \forall i \in \mathcal{I}_0$ *and a PWQ Lyapunov function*

$$V(x) = \begin{cases} x^T P_i x & \forall x \in \Gamma_i, \; i \in \mathcal{I}_0 \\ \bar{x}^T \bar{P}_i \bar{x} & \forall x \in \Gamma_i, \; i \in \mathcal{I}_1. \end{cases} \tag{17}$$

If there exist symmetric matrices $P_i \in \mathbb{R}^{n \times n}$ *for* $i \in \mathcal{I}_0$, $\bar{P}_i \in \mathbb{R}^{(1+n)\times(1+n)}$ *for* $i \in \mathcal{I}_1$, *symmetric elementwise nonnegative matrices* $U_i \in \mathbb{R}^{n_{ki} \times n_{ki}}$, $G_i \in \mathbb{R}^{n_{ki} \times n_{ki}}$ *and* $Z_{ij} \in \mathbb{R}^{n_{ki} \times n_{ki}}$ *such that the following LMIs are satisfied:*

$$P_i - H_i^T U_i H_i > 0 \quad \forall i \in \mathcal{I}_0 \tag{18a}$$
$$\bar{P}_i - \bar{H}_i^T U_i \bar{H}_i > 0 \quad \forall i \in \mathcal{I}_1 \tag{18b}$$
$$A_i^T P_i A_i - P_i + H_i^T G_i H_i < 0 \quad \forall i \in \mathcal{I}_0 \tag{18c}$$
$$\bar{A}_i^T \bar{P}_i \bar{A}_i - \bar{P}_i + \bar{H}_i^T G_i \bar{H}_i < 0 \quad \forall i \in \mathcal{I}_1 \tag{18d}$$
$$A_i^T P_j A_i - P_i + H_i Z_{ij} H_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all}, \; i,j \in \mathcal{I}_0 \tag{18e}$$
$$\bar{A}_i^T \bar{P}_j \bar{A}_i - \bar{P}_i + \bar{H}_i Z_{ij} \bar{H}_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all}, \; i,j \in \mathcal{I}_1 \tag{18f}$$
$$\bar{A}_i^T \bar{P}_j \bar{A}_i - \bar{P}_i + \bar{H}_i Z_{ij} \bar{H}_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all}, \; i \in \mathcal{I}_1, j \in \mathcal{I}_0 \tag{18g}$$
$$\bar{A}_i^T \bar{P}_j \bar{A}_i - \bar{P}_i + \bar{H}_i Z_{ij} \bar{H}_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all}, \; i \in \mathcal{I}_0, j \in \mathcal{I}_1 \tag{18h}$$

where we define

$$\bar{P}_j \triangleq \begin{bmatrix} 0 & 0 \\ 0 & P_j \end{bmatrix} \quad \textit{for} \quad j \in \mathcal{I}_0 \quad \textit{in (18g) and}$$
$$\bar{P}_i \triangleq \begin{bmatrix} 0 & 0 \\ 0 & P_i \end{bmatrix} \quad \textit{for} \quad i \in \mathcal{I}_0 \quad \textit{in (18h),}$$

then the origin of the PWA system (3) *is globally exponentially stable.*

The formal proof can be found in (FENG, 2002), but the outline is given next. Conditions (18a) and (18b) ensure the Lyapunov candidate function is positive for all x in the state space, partitioned in regions containing the origin and not containing the origin, respectively. The remaining conditions guarantee that $\Delta V(x)$ is decreasing along the trajectories for all possible transitions among regions. Equations (18c) and (18d) deal with the case when the state does not leave the current region in one step, (18e) tackles the case when a transition occurs between regions containing the origin while (18f) handles transitions between regions that do not contain the origin and, finally, equation (18g) addresses the case when the transition occurs between a region that does not contain the origin to a region that contains it and (18h) handles the other way around.

As mentioned before, to reduce conservatism, infeasible transitions due to the system dynamics can be removed from the stability conditions through a reachability analysis (BEMPORAD; FERRARI-TRECATE; MORARI, 2000).

2.3.4 Implicit representation

For the implicit representation (5) we can consider a piecewise quadratic (PWQ) Lyapunov function described as follows

$$V(x) = \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix} = \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}^T P \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}, \tag{19}$$

with $P_1 = P_1^T \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n_y}$ and $P_3 = P_3^T \in \mathbb{R}^{n_y\times n_y}$. Provided that $f_5 \preceq 0$, this Lyapunov candidate function has a quadratic upper bound given by the following Lemma.

Lemma 5. *(GROFF; VALMORBIDA; GOMES DA SILVA JR., 2020) If $f_5 \preceq 0$ and the implicit representation is well-posed, the function $V(x)$ in (19) has a quadratic upper bound given by $V(x) \leq \epsilon_{max} \|x\|^2$, with $\epsilon_{max} = \|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\|$ and $\sigma = max_{\Delta\in\mathcal{D}_{[0,1]}} \|\Delta(I - F_4\Delta)^{-1}F_3\|$, where $\mathcal{D}_{[0,1]}$ is the set of diagonal matrices with elements in $[0, 1]$.*

Proof. Let $\tilde{y}(x) \triangleq y(x) - f_5$. Then (5b) gives

$$\tilde{y}(x) = F_3x + F_4\phi(\tilde{y}(x) + f_5). \tag{20}$$

Since $f_5 \preceq 0$, then $0 \preceq \phi(\tilde{y}(x) + f_5) \preceq \phi(\tilde{y}(x))$ and it is possible to write $\phi(\tilde{y}(x) + f_5) = \Delta\tilde{y}(x)$, with $\Delta \in \mathcal{D}_{[0,1]}$. As the matrix $(I - F_4\Delta)$ is guaranteed to be non-singular for all $\Delta \in \mathcal{D}_{[0,1]}$ due to well-posedness assumption (GROFF; VALMORBIDA; GOMES DA SILVA JR., 2019), (20) implies that

$$\tilde{y}(x) = (I - F_4\Delta)^{-1}F_3x.$$

Thus, it follows that

$$\phi(y(x)) = \phi(\tilde{y}(x) + f_5) = \Delta\tilde{y}(x) = \Delta(I - F_4\Delta)^{-1}F_3x,$$

yielding an upper bound of $\|\phi(y(x))\|$ given by

$$\|\phi(y(x))\| \leq \sigma \|x\|$$

with $\sigma = \max_{\Delta\in\mathcal{D}_{[0,1]}} \|\Delta(I - F_4\Delta)^{-1}F_3\|$. Finally, from (19) it follows that

$$\begin{aligned} V(x) &\leq \|P_1\| \|x\|^2 + 2 \|P_2\| \|x\| \|\phi(y(x))\| + \|P_3\| \|\phi(y(x))\|^2 \\ &\leq (\|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\|) \|x\|^2 = \epsilon_{max} \|x\|^2 . \end{aligned}$$

■

To state Lyapunov-based conditions, it is necessary to ensure the positivity of such piecewise quadratic functions. One trivial way to do that, for the function presented in (19), is to require P to be positive definite. However, this would be very conservative since it would be valid for any function ϕ, and not only the considered vector ramp function ϕ.

To obtain less conservative conditions, specific properties of the considered vector function ϕ must be taken into account. These properties are inherited from properties of the scalar ramp function r, stated below and valid for any $y_{(i)} \in \mathbb{R}$.

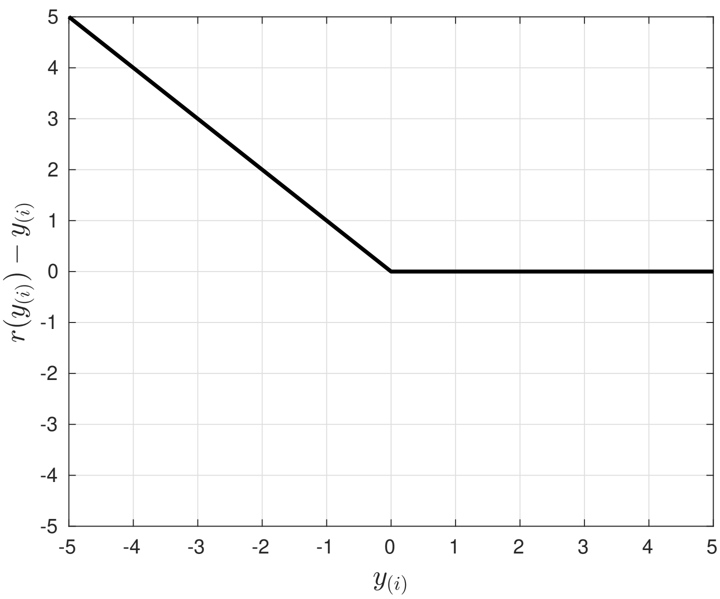
$$r(y_{(i)}) \geq 0 \tag{21a}$$

$$(r(y_{(i)}) - y_{(i)}) \geq 0 \tag{21b}$$

$$r(y_{(i)})(r(y_{(i)}) - y_{(i)}) = 0 \tag{21c}$$

Let $(r(y_{(i)}) - y_{(i)})$ be called the complement of $r(y_{(i)})$, with a graphic representation given in Figure 4. Properties (21a) and (21b) state the nonnegativity of both the ramp function and its complement, respectively. Property (21c) is a complementarity equality constraint, where the product of the ramp function by its complement is identically zero. As a brief remark, properties (21) can be derived from Karush-Kuhn-Tucker (KKT) optimality conditions that implicitly characterize the nonlinear function r. This approach was used for the saturation function by PRIMBS; GIANNELLI (2001) and specifically for the scalar ramp function r by GROFF; VALMORBIDA; GOMES DA SILVA JR. (2020).

Figure 4 – Complement of the ramp function, that is, $r(y_{(i)}) - y_{(i)}$.



Source: The author

From the properties (21) of the ramp function, the following Lemmas state several properties verified by the function ϕ. These properties will have a key role to derive conditions to guarantee that function $V(x)$ given by (19) is a Lyapunov function for the PWA system (5).

Lemma 6. *(GROFF, 2020) For any symmetric elementwise nonnegative matrix* $M \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$

$$s_1(M,y) = \begin{bmatrix} 1 \\ \phi(y) \\ \phi(y)-y \end{bmatrix}^T M \begin{bmatrix} 1 \\ \phi(y) \\ \phi(y)-y \end{bmatrix} \geq 0 \quad \forall y \in \mathbb{R}^{n_y} \tag{22}$$

Proof. From (6), function ϕ is defined elementwise in terms of scalar ramp function. Properties (21a) and (21b) ensures that all elements of ϕ and its complements are non-negative. Since, by definition, M contains only nonnegative elements, (22) holds for any $y \in \mathbb{R}^{n_y}$. ■

Lemma 7. *(GROFF, 2020) For any diagonal matrix* $T \in \mathbb{D}^{n_y}$

$$s_2(T,y) = \begin{bmatrix} 1 \\ \phi(y) \\ \phi(y)-y \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \phi(y) \\ \phi(y)-y \end{bmatrix} = 0 \quad \forall y \in \mathbb{R}^{n_y}. \tag{23}$$

Proof. Since T is a diagonal matrix, $s_2(T,y)$ can be written as

$$s_2(T,y) = 2\sum_{i=1}^{n_y} T_{(i,i)}\phi_{(i)}(y)(\phi_{(i)}(y) - y_{(i)}).$$

As function ϕ is defined elementwise in terms of scalar ramp function, from the complementarity equality constraint (21c), each term in the sum is identically zero. Thus, (23) holds for any $y \in \mathbb{R}^{n_y}$. ■

Remark 2. *Lemmas 6 and 7 were derived in (GROFF; VALMORBIDA; GOMES DA SILVA JR., 2019) using* $\phi(-y)$*. The results presented here are equivalent thanks to the following identity:* $\phi(-y) \equiv \phi(y) - y$*.*

Lemmas 6 and 7 refer to intrinsic properties of function ϕ, valid for any vector argument $y \in \mathbb{R}^{n_y}$. On the other hand, the following Lemma is based on the relation between $y(x)$ and x given by (5b).

Lemma 8. *(GROFF; VALMORBIDA; GOMES DA SILVA JR., 2020) Let* $n_\chi \triangleq 1+n+2n_y$ *and*

$$\chi(x) \triangleq \begin{bmatrix} 1 & x^T & \phi^T(y(x)) & (\phi(y(x)) - y(x))^T \end{bmatrix}^T \in \mathbb{R}^{n_\chi}.$$

For any vector $\zeta \in \mathbb{R}^{n_\zeta}$ *and matrix* $R \in \mathbb{R}^{n_\zeta \times n_y}$ *the following identity holds:*

$$s_3(R,\zeta,x) = 2\zeta^T R \begin{bmatrix} f_5 & F_3 & (F_4 - I) & I \end{bmatrix} \chi(x) = 0 \quad \forall x \in \mathbb{R}^n. \tag{24}$$

Proof. For each $x \in \mathbb{R}^n$, the state x belongs to some region Γ_i of the partition. Hence, from the discussion in section 2.2.2 and the well posedness assumption, it follows that x verifies (5b) and

$$\begin{aligned}\begin{bmatrix} f_5 & F_3 & (F_4 - I) & I \end{bmatrix} \chi(x) &= f_5 + F_3 x + (F_4 - I)\phi(y(x)) + (\phi(y(x)) - y(x)) \\ &= F_3 x + F_4 \phi(y(x)) + f_5 - y(x) = 0\end{aligned}$$

for any $x \in \mathbb{R}^n$. Thus, (24) holds. ■

The Lemmas previously presented in this section are instrumental to verify the global exponential stability of the origin of PWA systems, as stated by the following Theorem.

Theorem 1. *(GROFF; VALMORBIDA; GOMES DA SILVA JR., 2020) Consider the PWA system* (5) *with* $f_5 \preceq 0$ *and a PWQ Lyapunov candidate function* $V(x)$ *as in* (19)*. If there exist a symmetric matrix* $P \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $R_1 \in \mathbb{R}^{n_\chi \times n_y}$, $R_2 \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$, *elementwise nonnegative matrices* $M_1 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ *and* $M_2 \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$, *a positive scalar* ϵ_{min} *and* $\eta \in (0, 1)$ *such that*

$$(V(x) - \epsilon_{min} x^T x) - s_1(M_1, y(x)) + s_2(T_1, y(x)) + s_3(R_1, \chi(x), x) \geq 0 \tag{25}$$

and

$$-(V(x^+) - \eta V(x)) - s_1(M_2, \bar{y}(x)) + s_2(T_2, \bar{y}(x)) + s_3(R_2, \bar{\chi}(x), x) \geq 0 \tag{26}$$

with

$$\chi(x) = \begin{bmatrix} 1 \\ x \\ \phi(y(x)) \\ \phi(y(x)) - y(x) \end{bmatrix}, \quad \bar{y}(x) \triangleq \begin{bmatrix} y(x) \\ y(x^+) \end{bmatrix} \quad \textit{and} \quad \bar{\chi}(x) \triangleq \begin{bmatrix} 1 \\ x \\ \phi(\bar{y}(x)) \\ \phi(\bar{y}(x)) - \bar{y}(x) \end{bmatrix},$$

then the origin of system (5) *is globally exponentially stable.*

Proof. From Lemmas 5, 6, 7 and 8 and as $f_5 \preceq 0$, if (25) and (26) hold it respectively follows that

$$\epsilon_{min} \|x\|^2 \leq V(x) \leq \epsilon_{max} \|x\|^2 \tag{27a}$$

$$V(x^+) \leq \eta V(x) \tag{27b}$$

with $\epsilon_{max} = \|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\|$. Since $\eta \in (0, 1)$ and $V(x) > 0$, then $\Delta V(x) = V(x^+) - V(x) < 0$. Moreover, from (27b) we conclude that $V(x(k)) \leq \eta^k V(x(0))$, which, from (27a), implies that $\|x(k)\| \leq \sqrt{\epsilon_{max}/\epsilon_{min}} e^{\ln(\sqrt{\eta})k} \|x(0)\| \; \forall x \in \mathbb{R}^n$, from where the global exponential stability of the origin follows. ■

Remark 3. *The assumption of $f_5 \preceq 0$ in Theorem 1 ensures a quadratic upper bound to $V(x)$ without the need of additional constraints. However, as stated in subsection 2.3.1, it is possible for a PWA system in the implicit representation* (5) *to have the origin as an equilibrium point and positive entries in f_5. In this case, Theorem 1 can still be used to assess the exponential stability of the origin of PWA systems through the inclusion of one additional constraint to ensure a quadratic upper bound to $V(x)$: if there exist a diagonal matrix $T_\epsilon \in \mathbb{D}^{n_y}$, a symmetric elementwise nonnegative matrix $M_\epsilon \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, a matrix $R_\epsilon \in \mathbb{R}^{n_\chi \times n_y}$ and a positive scalar ϵ_{max} such that*

$$-(V(x) - \epsilon_{max}x^T x) - s_1(M_\epsilon, y(x)) + s_2(T_\epsilon, y(x)) + s_3(R_\epsilon, \chi(x), x) \geq 0, \tag{28}$$

then $V(x) \leq \epsilon_{max}x^T x \; \forall x \in \mathbb{R}^n$. In this case, from the Lemmas 6, 7 and 8 regarding the nonnegativity of PWQ functions, we conclude that

$$-(V(x) - \epsilon_{max}x^T x) \geq s_1(M_\epsilon, y(x)) \geq 0,$$

from where the fact that $V(x) \leq \epsilon_{max}x^T x \; \forall x \in \mathbb{R}^n$ follows.

Inequalities (25) and (26) can be represented as LMI constraints for a fixed value of η, as stated by the following Theorem.

Theorem 2. *Given $\eta \in (0,1)$, if there exist a symmetric matrix $P \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$ as in* (19)*, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $R_1 \in \mathbb{R}^{n_\chi \times n_y}$, $R_2 \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$,*

$$M_1 = \begin{bmatrix} M_{1_{1,1}} & M_{1_{1,2}} & M_{1_{1,3}} \\ \star & M_{1_{2,2}} & M_{1_{2,3}} \\ \star & \star & M_{1_{3,3}} \end{bmatrix} \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)},$$

$$M_2 = \begin{bmatrix} M_{2_{1,1}} & M_{2_{1,2}} & M_{2_{1,3}} \\ \star & M_{2_{2,2}} & M_{2_{2,3}} \\ \star & \star & M_{2_{3,3}} \end{bmatrix} \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$$

and a positive scalar ϵ_{min} such that the LMIs

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_1 - \epsilon_{min} I & P_2 & 0 \\ 0 & \star & P_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{1_{1,1}} & 0 & M_{1_{1,2}} & M_{1_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{1_{2,2}} & M_{1_{2,3}} - T_1 \\ \star & 0 & \star & M_{1_{3,3}} \end{bmatrix} + He\{R_1 Q_1\} \geq 0, \tag{29}$$

and

$$-\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & N_1 & N_2 & 0 \\ 0 & \star & N_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{2_{1,1}} & 0 & M_{2_{1,2}} & M_{2_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{2_{2,2}} & M_{2_{2,3}} - T_2 \\ \star & 0 & \star & M_{2_{3,3}} \end{bmatrix} + He\{R_2 Q_2\} \geq 0 \tag{30}$$

and the nonnegativity elementwise constraints

$$M_1 \succeq 0 \quad \textit{and} \quad M_2 \succeq 0 \tag{31}$$

are satisfied with

$$N_1 = F_1^T P_1 F_1 - \eta P_1,$$

$$N_2 = \begin{bmatrix} F_1^T P_1 F_2 - \eta P_2 & F_1^T P_2 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} F_2^T P_1 F_2 - \eta P_3 & F_2^T P_2 \\ \star & P_3 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} f_5 & F_3 & F_4 - I & I \end{bmatrix} \quad \textit{and}$$

$$Q_2 = \begin{bmatrix} f_5 & F_3 & F_4 - I & 0 & I & 0 \\ f_5 & F_3 F_1 & F_3 F_2 & F_4 - I & 0 & I \end{bmatrix},$$

then the origin of system (5) *with* $f_5 \preceq 0$ *is globally exponentially stable.*

Proof. Note that the terms $s_1(M_1, y(x))$, $s_2(T_1, y(x))$ and $s_3(R_1, \chi(x), x)$ in (25) can be written, respectively, as

$$s_1(M_1, y(x)) = \chi^T(x) \begin{bmatrix} M_{1_{1,1}} & 0 & M_{1_{1,2}} & M_{1_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{1_{2,2}} & M_{1_{2,3}} \\ \star & 0 & \star & M_{1_{3,3}} \end{bmatrix} \chi(x),$$

$$s_2(T_1, y(x)) = \chi^T(x) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_1 \\ 0 & 0 & T_1 & 0 \end{bmatrix} \chi(x) \quad \text{and}$$

$$s_3(R_1, \chi(x), x) = \text{He}\{\chi^T(x) R_1 Q_1 \chi(x)\}.$$

Thus, by pre and post multiplying (29) by $\chi^T(x)$ and $\chi(x)$, respectively, it follows that the inequality (25) is verified.

Moreover, since $\bar{y}(x)$ is given by

$$\bar{y}(x) \triangleq \begin{bmatrix} y(x) \\ y(x^+) \end{bmatrix} = \begin{bmatrix} F_3 \\ F_3 F_1 \end{bmatrix} x + \begin{bmatrix} F_4 & 0 \\ F_3 F_2 & F_4 \end{bmatrix} \phi(\bar{y}(x)) + \begin{bmatrix} f_5 \\ f_5 \end{bmatrix},$$

the term $s_3(R_2, \bar{\chi}(x), x)$ in (26) can be written as

$$s_3(R_2, \bar{\chi}(x), x) = \text{He}\{\bar{\chi}^T(x) R Q_2 \bar{\chi}(x)\}$$

with $\bar{\chi}(x) = [1 \;\; x^T \;\; \phi^T(\bar{y}(x)) \;\; (\phi^T(\bar{y}(x)) - \bar{y}(x))^T]^T$, as defined in Theorem 1. The remaining terms $s_1(M_2, \bar{y}(x))$ and $s_2(T_2, \bar{y}(x))$ in (26) can be equivalently written as

$$s_1(M_2, \bar{y}(x)) = \bar{\chi}^T(x) \begin{bmatrix} M_{2_{1,1}} & 0 & M_{2_{1,2}} & M_{2_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{2_{2,2}} & M_{2_{2,3}} \\ \star & 0 & \star & M_{2_{3,3}} \end{bmatrix} \bar{\chi}(x) \quad \text{and}$$

$$s_2(T_2, \bar{y}(x)) = \bar{\chi}^T(x) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_2 \\ 0 & 0 & T_2 & 0 \end{bmatrix} \bar{\chi}(x),$$

respectively. Lastly, the term $V(x^+) - \eta V(x)$ is equivalently represented by

$$V(x^+) - \eta V(x) = \bar{\chi}^T(x) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & N_1 & N_2 & 0 \\ 0 & \star & N_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{\chi}(x).$$

Thus, by pre and post multiplying (30) by $\bar{\chi}^T(x)$ and $\bar{\chi}(x)$, respectively, it follows that the inequality (26) is verified. Finally, constraints (31) ensure the elementwise nonnegativity of matrices M_1 and M_2 required by Theorem 1. ■

As stated in Remark 3, Theorem 2 can be applied if the origin of the PWA system (5) is an equilibrium point but the vector f_5 has positive entries. In order to do that, one additional convex constraint can be included to ensure a known quadratic upper bound to $V(x)$. This convex constraint is given by the following LMI: if there exist a symmetric matrix $P \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$ as in (19), diagonal matrix $T_\epsilon \in \mathbb{D}^{n_y}$, matrix $R_\epsilon \in \mathbb{R}^{n_\chi \times n_y}$, a symmetric elementwise nonnegative matrix $M_\epsilon \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ and a positive scalar ϵ_{max} such that

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -P_1 + \epsilon_{max} I & -P_2 & 0 \\ 0 & \star & -P_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{\epsilon_{1,1}} & 0 & M_{\epsilon_{1,2}} & M_{\epsilon_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{\epsilon_{2,2}} & M_{\epsilon_{2,3}} - T_\epsilon \\ \star & 0 & \star & M_{\epsilon_{3,3}} \end{bmatrix} + \text{He}\{R_\epsilon Q_1\} \geq 0 \quad (32)$$

is verified, then $V(x)$ is upper bounded by $\epsilon_{max} x^T x$. The proof follows directly by pre and post multiplying the expression in (32) by $\chi^T(x)$ and $\chi(x)$, respectively.

2.4 Stabilization of PWA Systems with the Explicit Representation

This section recalls the stabilization methods proposed in the literature regarding the special case of PWL systems and a PWL state feedback control law. Then, the difficulties of extending those methods to the general case of PWA systems are discussed.

2.4.1 Special case of PWL systems

In (MIGNONE; FERRARI-TRECATE; MORARI, 2000) stabilization methods considering a PWL state feedback control law are proposed. Those methods are based on the quadratic stability (11) and the piecewise quadratic stability (12) regarding the special case of PWL systems. Since the method based on quadratic stability is a particular case of the method based on the piecewise quadratic stability, let us recall only the latter.

Consider a PWL state feedback control law given by

$$u = K_i x \quad \forall x \in \Gamma_i. \tag{33}$$

Then, the PWL closed-loop system (9) and (33) reads

$$x^+ = (A_i + B_i K_i)x \quad \forall x \in \Gamma_i. \tag{34}$$

To assess the PWQ stability of the closed-loop system (34) the following Lemma is stated.

Lemma 9. *If there exist symmetric matrices $Q_i \in \mathbb{R}^{n\times n}$ and matrices $W_i \in \mathbb{R}^{n_u\times n}$ such that the following set of LMIs*

$$Q_i > 0 \quad \forall i \in \mathcal{I} \quad \textit{and} \tag{35}$$

$$\begin{bmatrix} Q_j & (A_i Q_i + B_i W_i) \\ \star & Q_i \end{bmatrix} > 0 \quad \forall (i,j) \in \mathcal{S}_{all} \tag{36}$$

is verified, then the origin of the closed-loop system (34) *is globally exponentially stable and the stabilizing gains are given by $K_i = W_i P_i$, where $P_i \triangleq Q_i^{-1}$.*

Proof. Since $P_i = Q_i^{-1}$, the set of LMIs (35) implies that $P_i > 0 \ \forall i \in \mathcal{I}$. Hence, the PWQ Lyapunov function $V(x) = x^T P_i x \ \ \forall x \in \Gamma_i$ is positive definite.

On the other hand, by pre and post multiplying (36) by the symmetric matrix

$$\begin{bmatrix} I & 0 \\ 0 & P_i \end{bmatrix},$$

and considering that $P_j = Q_j^{-1}$ and $K_i = W_i P_i$, we obtain

$$\begin{bmatrix} P_j^{-1} & A_i + B_i K_i \\ \star & P_i \end{bmatrix} > 0 \quad \forall (i,j) \in \mathcal{S}_{all}.$$

Using Schur complement, we note that this implies that

$$(A_i + B_i K_i)^T P_j (A_i + B_i K_i) - P_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all},$$

that is, $\Delta V(x)$ is negative definite, from where the global exponential stability of the origin follows for the stabilizing gains $K_i = W_i P_i$. ∎

This stabilization approach is restrictive. First, note that the set of all combinations of transitions $\mathcal{S}_{all}$ is considered in (36). However, in the stabilization case, it is not possible to perform the reachability analysis to know *a priori* the set of feasible transitions $\mathcal{S}_{fea}$. The impossibility arises from the fact that the control gains K_i may change the feasible transitions. Lacking of better knowledge, one needs to take into account the set $\mathcal{S}_{all}$ of all transitions, which may be conservative.

Another source of conservatism in the PWQ stabilization procedure of Lemma 9 is not to take into account the relaxation terms as in (14). The absence of those terms in (36) ensures that $\Delta V(x)$ is negative definite for any type of switching, and not only the switching that occurs due to the system dynamics.

If we take into account the relaxation terms, sufficient conditions to ensure the exponential stability of the origin of the closed-loop system is given by the following Lemma.

Lemma 10. *If there exist symmetric matrices $P_i = P_i^T \in \mathbb{R}^{n\times n}$, symmetric elementwise nonnegative matrices $U_i = U_i^T \succeq 0 \in \mathbb{R}^{n_{ki}\times n_{ki}}$ and $Z_{ij} = Z_{ij}^T \succeq 0 \in \mathbb{R}^{n_{ki}\times n_{ki}}$ and matrices $K_i \in \mathbb{R}^{n_u\times n}$ for $i, j = 1, ..., N_\Gamma$, such that the LMIs*

$$P_i - H_i^T U_i H_i > 0 \quad \forall i \in \mathcal{I} \tag{37a}$$

$$(A_i + B_iK_i)^T P_j (A_i + B_iK_i) - P_i + H_i^T Z_{ij} H_i < 0 \quad \forall (i,j) \in \mathcal{S}_{all} \tag{37b}$$

are satisfied, then the origin of the closed-loop PWL system (34) *is globally exponentially stable for the gains K_i.*

The proof follows directly from Lemma 3. However, in this case, applying the Schur complement leads to the following inequality:

$$\begin{bmatrix} P_j^{-1} & A_i + B_iK_i \\ \star & P_i - H_i^T Z_{ij} H_i \end{bmatrix} > 0.$$

Then, pre and post multiplying this inequality by the block diagonal matrix diag(I, Q_i), where $Q_i = P_i^{-1}$, does not lead to a convex constraint as it would in the case without relaxation terms.

Hence, it is not possible to derive convex or *quasi*-convex stabilization conditions from the PWQ stability with relaxation terms.

2.4.2 General case of PWA systems

In subsection 2.3.3 the quadratic and the PWQ stability analysis were extended to the general case of PWA systems. This was done by representing the PWA system as an extended PWL system (15). In this extended representation, the dynamic matrix $\bar{A}_i$ is not a Schur matrix. Consequently, it is not possible to apply the quadratic or the piecewise quadratic stability conditions from Lemmas 1 and 2 in this case, since this would lead to infeasible LMI constraints (CUZZOLA; MORARI, 2001). Hence, the use of relaxation terms are necessary to obtain feasible stability conditions.

However, as seen in subsection 2.4.1, if we take into account the relaxation terms, it is not possible to obtain convex nor *quasi*-convex stabilization conditions from the PWQ stability conditions. Also, to the author's knowledge, no stabilization methods based on the stability conditions (14) are proposed in the literature for PWL or PWA systems.

2.5 Final Remarks

This chapter reviewed the definition of PWA systems and different forms to represent them, with special emphasis on the traditionally used explicit representation (named, in this work, standard explicit representation) and a recently developed implicit representation. In the standard explicit representation the regions of the partition are defined by a finite set of explicit inequalities on the state variables. On the other hand, in the implicit representation the regions are defined by an implicit equation based on vector ramp functions.

Conditions to assess the global exponential stability based on convex feasibility problems were presented for both representations. One difficulty to address the stability analysis problem using the explicit representation is that not all transitions between regions are feasible due to the system dynamics. Hence, to relax this problem, a reachability analysis is necessary to obtain *a priori* knowledge about the set of possible transitions among regions.

The difficulties of deriving convex or *quasi*-convex stabilization conditions for PWA systems using the explicit representation based on the relaxed PWQ stability conditions were briefly discussed in Section 2.4. They derive mainly from the inclusion of relaxation terms, which prevents reaching convex stabilization conditions.

Based on what was presented so far, the next chapters will adress the problems of global stabilization and local stability analysis of PWA systems using the implicit representation.

3 GLOBAL STABILIZATION

In Section 2.3.4 sufficient conditions to verify if the origin of a PWA system in the implicit representation is globally exponentially stable were presented in Theorem 1. Then, Theorem 2 allows to assess the stability of the origin through a direct LMI feasibility test. However, the same cannot be done in the stabilization problem due to the product between variables. This chapter defines the stabilization problem and presents sufficient conditions that aims to eliminate or relax the nonlinearities coming from the analysis conditions when the control law gains are variables. Then, an algorithm based on the solution of Semidefinite Programming (SDP) problems (i.e. feasibility LMI problems) to compute a stabilizing controller is suggested and tested in numerical examples.

3.1 Problem Statement

Consider the open-loop PWA system given by the implicit representation

$$x^{+} = F_1 x + F_2\phi(y(x)) + Bu \tag{38a}$$

$$y(x) = F_3 x + F_4\phi(y(x)) + f_5 \tag{38b}$$

with a PWA feedback control law defined as

$$u = K_1 x + K_2\phi(y(x)) \tag{39}$$

with $K_1 \in \mathbb{R}^{n_u \times n}$ and $K_2 \in \mathbb{R}^{n_u \times n_y}$. The control law (39) is piecewise affine since the gain K_2 modifies the control action according to the active region Γ_i of the partition, that is, depending on the elements of the function $\phi(y(x))$ that are equal to zero or not. However, note that the control law does not alter the implicit equation (38b). As (39) does not change (38b), the well-posedness of the system is left unchanged, depending only on the matrix F_4. The proposed method also requires that the origin is an equilibrium point of the open-loop system (38). Due to those characteristics, the following Assumptions are made.

Assumption 1. *The algebraic loop in* (38b) *is well-posed, i.e. there is an unique solution to the implicit equation* (38b).

Assumption 2. *The implicit PWA system* (38b) *has* $f_5 \preceq 0$.

Assumption 1 ensures the closed-loop system is well-posed thanks to the open-loop well-posedness. Assumption 2 ensures that the origin of the open-loop system is an equilibrium point. In this case, the closed-loop system (38) and (39) reads

$$x^+ = (F_1 + BK_1)x + (F_2 + BK_2)\phi(y(x)) \tag{40a}$$

$$y(x) = F_3 x + F_4\phi(y(x)) + f_5. \tag{40b}$$

The goal is to provide conditions to compute gains K_1 and K_2 such that the origin of the closed-loop system (40) is globally exponentially stable.

3.2 Conditions for Stabilization

From (40), if gains K_1 and K_2 are fixed, Theorem 2 allows to assess the global exponential stability of the origin of the closed-loop system through a direct LMI feasibility test. However, considering the gains as variables, then Theorem 2 leads to non-convex conditions. The reason is twofold. First, in the closed-loop situation the terms N_1, N_2 and N_3 of Theorem 2 read

$$\begin{aligned} N_1 &= (F_1 + BK_1)^T P_1 (F_1 + BK_1) - \eta P_1, \\ N_2 &= \begin{bmatrix} (F_1 + BK_1)^T P_1 (F_2 + BK_2) - \eta P_2 & (F_1 + BK_1)^T P_2 \end{bmatrix}, \\ N_3 &= \begin{bmatrix} (F_2 + BK_2)^T P_1 (F_2 + BK_2) - \eta P_3 & (F_2 + BK_2)^T P_2 \\ \star & P_3 \end{bmatrix} \end{aligned}$$

from where we notice the product between the gains and the matrix defining the Lyapunov candidate function. Second, since $s_3(R_2, \bar{\chi}(x), x)$ is written in terms of $\bar{\chi}^T(x)$ and $\bar{\chi}(x)$ as

$$\text{He}\left\{ \bar{\chi}^T(x) R_2 \begin{bmatrix} f_5 & F_3 & F_4 - I & 0 & I & 0 \\ f_5 & F_3(F_1 + BK_1) & F_3(F_2 + BK_2) & F_4 - I & 0 & I \end{bmatrix} \bar{\chi}(x) \right\}$$

there is a product between variable R_2 and the matrix containing the gains K_1 and K_2.

To derive sufficient conditions to compute the stabilizing gains the idea is to write an extended version of Theorem 2 using Finsler's Lemma. This eliminates the non-convex condition introduced by the terms N_1, N_2 and N_3. Then, a fixed structure is imposed to variables R_1 and R_2, allowing to write *quasi*-convex conditions to stabilization after some congruence transformations.

3.2.1 Alternative stability conditions

This subsection states alternative stability conditions considering a vector $\xi(x) \in \mathbb{R}^{1+2n+2n_y}$ defined as

$$\xi(x) \triangleq \begin{bmatrix} 1 & x^T & (x^+)^T & \phi^T(y(x)) & (\phi(y(x)) - y(x))^T \end{bmatrix}^T.$$

Note that $\xi(x)$ is an extended version of $\chi(x)$ by the inclusion of the successor state x^+. The relation between x and x^+ expressed by (40a) allows to state an extended version of Lemma 8.

Lemma 11. *For any vector $\zeta \in \mathbb{R}^{n_\zeta}$ and matrix $R \in \mathbb{R}^{n_\zeta \times (n_y+n)}$*

$$\bar{s}_3(R,\zeta,x) = 2\zeta^T R \begin{bmatrix} f_5 & F_3 & 0 & (F_4 - I) & I \\ 0 & (F_1 + BK_1) & -I & (F_2 + BK_2) & 0 \end{bmatrix} \xi(x) \equiv 0 \tag{41}$$

is verified along the trajectories of the system (40).

Proof. From (40) it follows that

$$\begin{aligned} &\begin{bmatrix} f_5 & F_3 & 0 & (F_4 - I) & I \\ 0 & (F_1 + BK_1) & -I & (F_2 + BK_2) & 0 \end{bmatrix} \xi(x) \\ &\qquad = \begin{bmatrix} f_5 + F_3 x + (F_4 - I)\phi(y(x)) + (\phi(y(x)) - y(x)) \\ (F_1 + BK_1)x + (F_2 + BK_2)\phi(y) - x^+ \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Thus, (41) holds for any trajectory of the system (40). ■

Lemma 11 allows to represent the stability conditions in terms of an extended vector with an additional equality constraint. This constraint represents the relation between the inserted variable x^+ and the others. As a brief remark, Lemma 11 can also be derived using Finsler's Lemma (see Appendix A.2), as usually done in the control literature (OLIVEIRA; SKELTON, 2007). Then, Theorem 2 can be equivalently stated as follows.

Theorem 3. *Given $\eta \in (0,1)$, if there exist a symmetric matrix $P \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$ as in* (19)*, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $R_1 \in \mathbb{R}^{(1+2n+2n_y)\times(n_y+n)}$, $R_2 \in \mathbb{R}^{(1+2n+4n_y)\times(2n_y+n)}$, $M_1 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, $M_2 \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$ and a positive scalar ϵ_{min} such that the LMIs*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_1 - \epsilon_{min} I & P_2 & 0 \\ 0 & \star & P_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{1_{11}} & 0 & M_{1_{12}} & M_{1_{13}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{1_{22}} & M_{1_{23}} - T_1 \\ \star & 0 & \star & M_{1_{33}} \end{bmatrix} + He\{R_1 Q_1\} \geq 0, \tag{42}$$

and

$$\left[\begin{array}{ccc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta\tilde{P}_1 & 0 & \eta\tilde{P}_2 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{P}_1 & 0 & -\tilde{P}_2 & 0 & 0 \\ \hline 0 & \star & 0 & \eta\tilde{P}_3 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & -\tilde{P}_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] - \begin{bmatrix} M_{2_{1,1}} & 0 & 0 & M_{2_{1,2}} & M_{2_{1,3}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & M_{2_{2,2}} & M_{2_{2,3}} - T_2 \\ \star & 0 & 0 & \star & M_{2_{3,3}} \end{bmatrix} + He\{R_2 \bar{Q}_2\} \geq 0 \tag{43}$$

and the elementwise nonnegativity constraints

$$M_1 \succeq 0 \quad \textit{and} \quad M_2 \succeq 0 \tag{44}$$

are satisfied with

$$Q_1 = \begin{bmatrix} f_5 & F_3 & F_4 - I & I \end{bmatrix} \quad \textit{and}$$

$$\bar{Q}_2 = \begin{bmatrix} f_5 & F_3 & 0 & F_4 - I & 0 & I & 0 \\ f_5 & 0 & F_3 & 0 & F_4 - I & 0 & I \\ 0 & (F_1 + BK_1) & -I & (F_2 + BK_2) & 0 & 0 & 0 \end{bmatrix},$$

then the origin of the PWA system (5) *is globally exponentially stable.*

Proof. The procedure is similar to the proof of Theorem 2. Since (42) was not changed, it implies in (25). Moreover, pre and post multiplying (43) by $\bar{\xi}^T(x)$ and $\bar{\xi}(x)$, respectively, with $\bar{\xi}(x) \in \mathbb{R}^{1+2n+4n_y}$ defined as

$$\bar{\xi}(x) \triangleq \begin{bmatrix} 1 & x^T & (x^+)^T & \phi^T(\bar{y}(x)) & (\phi(\bar{y}(x)) - \bar{y}(x))^T \end{bmatrix}^T,$$

where $\bar{y}(x) = [y^T(x) \;\; y^T(x^+)]^T$, results in the following condition

$$-(V(x^+) - \eta V(x)) - s_1(M_2, \bar{y}(x)) + s_2(T_2, \bar{y}(x)) + \bar{s}_3(R_2, \bar{\xi}(x), x) \geq 0.$$

Since $\bar{s}_3(R_2, \bar{\xi}(x), x) \equiv 0$, then $\Delta V(x)$ is negative definite and the origin of the PWA system is globally exponentially stable. ■

3.2.2 Stabilization theorem

The advantage of Theorem 3 over Theorem 2 is that non-convex conditions introduced by N_1, N_2 and N_3 are eliminated. However there is still the product between the multiplier R_2 and matrix $\bar{Q}_2$, which contains the gains to be computed. To eliminate this non-convexity, some congruence transformations are made along with the choice of a particular structure for the multiplier R_2. The following Theorem formalizes that and presents sufficient conditions to ensure that the origin of the closed-loop system (40) is globally exponentially stable.

Theorem 4. *Given $\eta \in (0, 1)$, if there exist a symmetric matrix $\tilde{P} \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$, a positive definite symmetric matrix $\tilde{E} \in \mathbb{R}^{n\times n}$, $\tilde{M}_1 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, $\tilde{T}_1 \in \mathbb{D}^{n_y}$, $\tilde{M}_2 \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$, $\tilde{T}_2 \in \mathbb{D}^{2n_y}$, non-singular symmetric matrices $W_1 \in \mathbb{R}^{n\times n}$, $W_2 \in \mathbb{D}^{n_y}$, $W_3 \in \mathbb{D}^{n_y}$, $W_4 \in \mathbb{D}^{n_y}$ and $W_5 \in \mathbb{D}^{n_y}$, matrices $\tilde{K}_1 \in \mathbb{R}^{n_u\times n}$ and $\tilde{K}_2 \in \mathbb{R}^{n_u\times n_y}$ and scalars α, β and γ such that the matrix inequalities*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{P}_1 - \tilde{E} & \tilde{P}_2 & 0 \\ 0 & \star & \tilde{P}_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{M}_{1_{11}} & 0 & \tilde{M}_{1_{12}} & \tilde{M}_{1_{13}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \tilde{M}_{1_{22}} & \tilde{M}_{1_{23}} - \tilde{T}_1 \\ \star & 0 & \star & \tilde{M}_{1_{33}} \end{bmatrix} + He\{\tilde{R}_1 \tilde{Q}_1\} \geq 0 \tag{45}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta\tilde{P}_1 & 0 & \eta\tilde{P}_2 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{P}_1 & 0 & -\tilde{P}_2 & 0 & 0 \\ \hline 0 & \star & 0 & \eta\tilde{P}_3 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & -\tilde{P}_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{M}_{2_{11}} & 0 & 0 & \tilde{M}_{2_{12}} & \tilde{M}_{2_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \star & 0 & 0 & \tilde{M}_{2_{22}} & \tilde{M}_{2_{23}} - \tilde{T}_2 \\ \hline \star & 0 & 0 & \star & \tilde{M}_{2_{33}} \end{bmatrix} + He\{\tilde{R}_2\tilde{Q}_2\} \geq 0 \tag{46}$$

and the elementwise nonnegativity constraints

$$\Pi_1^{-1} \begin{bmatrix} \tilde{M}_{1_{11}} & 0 & \tilde{M}_{1_{12}} & \tilde{M}_{1_{13}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \tilde{M}_{1_{22}} & \tilde{M}_{1_{23}} \\ \star & 0 & \star & \tilde{M}_{1_{33}} \end{bmatrix} \Pi_1^{-1} \succeq 0 \tag{47}$$

$$\Pi_2^{-1} \begin{bmatrix} \tilde{M}_{2_{11}} & 0 & 0 & \tilde{M}_{2_{12}} & \tilde{M}_{2_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \star & 0 & 0 & \tilde{M}_{2_{22}} & \tilde{M}_{2_{23}} \\ \hline \star & 0 & 0 & \star & \tilde{M}_{2_{33}} \end{bmatrix} \Pi_2^{-1} \succeq 0 \tag{48}$$

are satisfied with

$$\Pi_1 = \Pi_1^T = diag(1, W_1, W_2, W_5),$$
$$\Pi_2 = \Pi_2^T = diag(1, W_1, W_1, W_2, W_2, W_3, W_4),$$

$$\tilde{R}_1 = \begin{bmatrix} 0 & 0 & \gamma I & I \end{bmatrix}^T,$$
$$\tilde{Q}_1 = \begin{bmatrix} f_5 & F_3W_1 & (F_4 - I)W_2 & W_5 \end{bmatrix},$$

$$\tilde{R}_2 = \begin{bmatrix} 0 & 0 & 0 & I & 0 & I & 0 \\ 0 & 0 & 0 & 0 & \beta I & 0 & I \\ 0 & I & \alpha I & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad and$$

$$\tilde{Q}_2 = \begin{bmatrix} f_5 & F_3W_1 & 0 & (F_4 - I)W_2 & 0 & W_3 & 0 \\ f_5 & 0 & F_3W_1 & 0 & (F_4 - I)W_2 & 0 & W_4 \\ 0 & (F_1W_1 + B\tilde{K}_1) & -W_1 & (F_2W_2 + B\tilde{K}_2) & 0 & 0 & 0 \end{bmatrix},$$

then the gains $K_1 = \tilde{K}_1W_1^{-1}$ *and* $K_2 = \tilde{K}_2W_2^{-1}$ *ensure that the origin of the closed-loop system* (40) *is globally exponentially stable.*

Proof. Consider (42) with the matrix $\epsilon_{min}I$ replaced by a symmetric positive definite matrix E. In this case, the following condition is ensured:

$$(V(x) - x^TEx) - s_1(M_1, y(x)) + s_2(T_1, y(x)) + s_3(R_1, \chi(x), x) \geq 0.$$

The positive definiteness of matrix E, along with Lemmas 6, 7 and 8 regarding the global nonnegativity of PWQ functions, ensures that

$$V(x) \geq x^T E x > 0$$

and the quadratic lower bound of $V(x)$ is given by $\lambda_{\text{min}}(E)\,\|x\|^2$, where $\lambda_{\text{min}}(E)$ is the minimal eigenvalue of E. The use of E instead of $\epsilon_{min} I$ allows for a subsequent change of variables.

Consider now the following structure for matrix

$$R_1 = \begin{bmatrix} 0 & 0 & \gamma W_2^{-1} & W_5^{-1} \end{bmatrix}^T.$$

Then, after pre and post multiplying (42) with $\epsilon_{min} I$ replaced by E, by the symmetric matrix Π_1, the term $\Pi_1 R_1 Q_1 \Pi_1$ becomes $\tilde{R}_1 \tilde{Q}_1$ and the following change of variables is considered to obtain the remaining terms of (45):

$$\begin{aligned}
&\tilde{E} \triangleq W_1 E W_1; && \tilde{M}_{1_{11}} \triangleq M_{1_{11}};\\
&\tilde{M}_{1_{12}} \triangleq M_{1_{12}} W_2; && \tilde{M}_{1_{13}} \triangleq M_{1_{13}} W_3;\\
&\tilde{M}_{1_{22}} \triangleq W_2 M_{1_{22}} W_2; && \tilde{M}_{1_{23}} \triangleq W_2 M_{1_{23}} W_3;\\
&\tilde{M}_{1_{33}} \triangleq W_3 M_{1_{33}} W_3; && \tilde{T}_1 \triangleq W_2 T_1 W_3;\\
&\tilde{P}_1 \triangleq W_1 P_1 W_1; && \tilde{P}_2 \triangleq W_1 P_2 W_2;\\
&\tilde{P}_3 \triangleq W_2 P_3 W_2.
\end{aligned} \tag{49}$$

Consider now (43) with the following particular structure for matrix R_2:

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & W_2^{-1} & 0 & W_3^{-1} & 0 \\ 0 & 0 & 0 & 0 & \beta W_2^{-1} & 0 & W_4^{-1} \\ 0 & W_1^{-1} & \alpha W_1^{-1} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

After pre and post multiplying (43) by the symmetric matrix Π_2 and considering the change of variables $\tilde{K}_1 \triangleq K_1 W_1$ and $\tilde{K}_2 \triangleq K_2 W_2$, the term $\Pi_2 R_2 Q_2 \Pi_2$ becomes $\tilde{R}_2 \tilde{Q}_2$ and the remaining terms in (46) are obtained from the following change of variables:

$$\begin{aligned}
&\tilde{M}_{2_{11}} \triangleq M_{2_{11}}; && \tilde{T}_2 \triangleq \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix} T_2 \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix};\\
&\tilde{M}_{2_{12}} \triangleq M_{2_{12}} \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix}; && \tilde{M}_{2_{13}} \triangleq M_{2_{13}} \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix};\\
&\tilde{M}_{2_{22}} \triangleq \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix} M_{2_{22}} \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix}; && \tilde{M}_{2_{23}} \triangleq \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix} M_{2_{23}} \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix};\\
&\tilde{M}_{2_{33}} \triangleq \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix} M_{2_{33}} \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix}.
\end{aligned} \tag{50}$$

Thus, (46) implies that (43) is satisfied, with $K_1 = \tilde{K}_1 W_1^{-1}$ and $K_2 = \tilde{K}_2 W_2^{-1}$. Finally, note that the elementwise constraints (47) and (48) ensure that the elementwise constraints (44) are satisfied, i.e., matrices M_1 and M_2 in Theorem 3 are elementwise nonnegative. Hence, the conditions from Theorem 4 imply in the conditions from Theorem 3, from where the global exponential stability of the origin follows. ■

The following section discusses an algorithm to compute the stabilizing gains K_1 and K_2 from Theorem 4.

3.3 Proposed Algorithm

Since constraints (45) to (48) are non-convex, it is important to discuss an algorithm to solve the feasibility problem defined by such constraints.

First, there is the product between variable matrices $\tilde{R}_1\tilde{Q}_1$ and $\tilde{R}_2\tilde{Q}_2$. Since matrices $\tilde{R}_1$ and $\tilde{R}_2$ have only a few scalar variables, then a gridding method can be used (RODRIGUES; BOYD, 2005), i.e., define a grid of values for α, β and γ and, for each point in the grid, (45) and (46) are LMIs. The grid is characterized by a minimal value (α_{min}, β_{min}, γ_{min}), a step value (α_s, β_s, γ_s) and a maximum value (α_{max}, β_{max}, γ_{max}) for each variable. In this work, these values were chosen based on preliminary tests.

We must also satisfy the elementwise constraints (47) and (48). Noting that W_2, W_3, W_4 and W_5 are diagonal matrices, the idea is to impose these matrices to be positive or negative definite and then add constraints on the corresponding elements of matrices $\tilde{M}_1$ and $\tilde{M}_2$. We have, therefore, 16 possible cases as described in Table 1. For instance, consider case 3 (i.e. $W_5 > 0$, $W_4 > 0$, $W_3 < 0$, $W_2 < 0$). In this case, from (49), we must impose the following elementwise constraints

$$\begin{aligned} &\tilde{M}_{1_{1,1}} \succeq 0, && \tilde{M}_{1_{1,2}} \preceq 0, \\ &\tilde{M}_{1_{1,3}} \preceq 0, && \tilde{M}_{1_{2,2}} \succeq 0, \\ &\tilde{M}_{1_{2,3}} \succeq 0 \quad \text{and} && \tilde{M}_{1_{3,3}} \succeq 0 \end{aligned}$$

to ensure that matrix M_1 is elementwise nonnegative. Taking into account (48), the same procedure must be applied to $\tilde{M}_2$ to ensure that matrix M_2 is elementwise nonnegative.

Hence, the idea is to check the feasibility of LMIs (45) and (46) on a grid on α, β and γ, considering the elementwise contraints associated to each one of the cases in Table 1. This is summarized in Algorithm 1.

Regarding the global exponential stability of the origin of (40), any pair K_1 and K_2 leading to a feasible solution to the LMIs associated with Algorithm 1 stabilizes the system. It is also possible to consider an optimization criterion, such as maximization of convergence rate of trajectories. This could be done by taking, among the set of feasible solutions, the one that leads to a minimal value for η.

Table 1 – Table of cases tested for matrices W_2 to W_5

Test Case	W_5	W_4	W_3	W_2		Test Case	W_5	W_4	W_3	W_2
0	>	>	>	>		8	<	>	>	>
1	>	>	>	<		9	<	>	>	<
2	>	>	<	>		10	<	>	<	>
3	>	>	<	<		11	<	>	<	<
4	>	<	>	>		12	<	<	>	>
5	>	<	>	<		13	<	<	>	<
6	>	<	<	>		14	<	<	<	>
7	>	<	<	<		15	<	<	<	<

Algorithm 1 Algorithm for stabilization of PWA Systems

for $\alpha = \alpha_{min} : \alpha_s : \alpha_{max}$ **do**
for $\beta = \beta_{min} : \beta_s : \beta_{max}$ **do**
for $\gamma = \gamma_{min} : \gamma_s : \gamma_{max}$ **do**
for testCase $= 0 : 15$ **do**
Solve the convex feasibility problem
composed by LMIs (45) and (46) and
the elementwise constraints (47) and (48)
associated with the case in Table 1.
if a feasible solution was found **then**
End algorithm. The stabilizing gains are
given by $K_1 = \tilde{K}_1 W_1^{-1}$ and $K_2 = \tilde{K}_2 W_2^{-1}$.
end if
end for
end for
end for
end for
if no feasible solution was found **then**
End algorithm. The PWA system cannot be stabilized by this method.
end if

Finally, note that Table 1 considers positive or negative definiteness constraints on matrices W_2, W_3, W_4 and W_5, casting 16 possible combinations of constraints irrespective of the size of the system or the number of regions in the partition. This same idea could be applied to the positivity or negativity of each diagonal element of matrices W_2, W_3, W_4 and W_5, leading to more degrees of freedom for matrices $\tilde{M}_1$ and $\tilde{M}_2$. However, in this case, the number of possible combinations of constraints is 2^{4n_y}, depending on the number of regions. This may be computationally prohibitive for some systems and testing only the 16 cases in Table 1 is often sufficient, as illustrated by the following numerical examples.

3.4 Numerical Examples

This section presents two numerical examples to illustrate the application of the method proposed in this chapter. All the examples in this section were solved using YALMIP (LÖFBERG, 2004), SeDuMi (STURM, 1999) and MATLAB.

3.4.1 Example 1

Consider the continuous-time nonlinear system presented in Section 8.1 of (MOREIRA *et al.*, 2020), whose dynamics is represented by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} f(x_{(2)})$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function given by

$$f(a) = \begin{cases} 0, & \text{if } \|a\| \leq 1 \\ \ln(a), & \text{if } a > 1 \\ -\ln(-a), & \text{if } a < -1 \end{cases}$$

for any $a \in \mathbb{R}$.

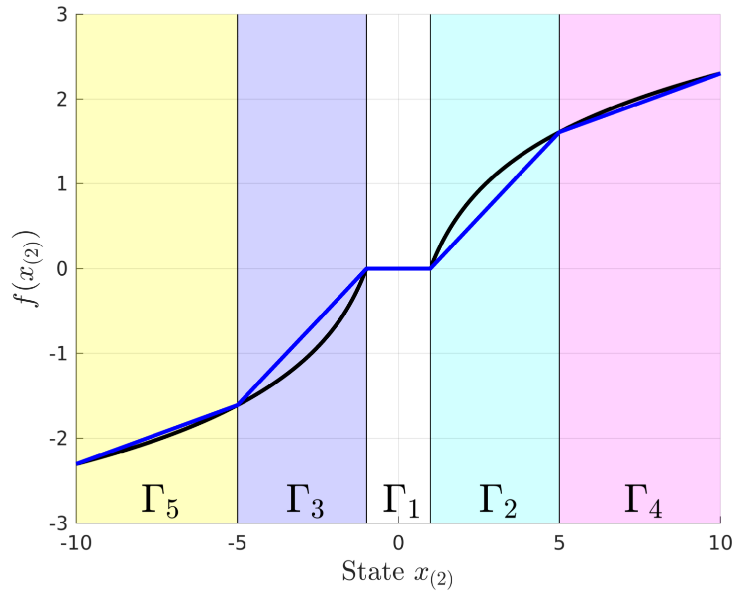
The continuous nonlinear function f can be approximated by a PWA function. For this example, it was considered a PWA function with five regions to model function f in the interval $-10 \leq x_{(2)} \leq +10$, as depicted in Figure 5. However, note that an arbitrarily good approximation can be obtained by considering more regions in this interval.

A discrete-time PWA system can be obtained by discretizing the system (using Euler discretization with $T = 0.5$) and considering the PWA approximation of the continuous nonlinearity in Figure 5. This procedure leads to the PWA system (38) with

$$F_1 = \begin{bmatrix} 1 & T \\ 4T & 1 - 0.25T \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ T & -T & -T & T \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0 & 0.4024 \\ 0 & 0.2638 \\ 0 & -0.4024 \\ 0 & -0.2638 \end{bmatrix}, \; F_4 = 0, \text{ and } f_5 = \begin{bmatrix} -0.4024 \\ -1.3190 \\ -0.4024 \\ -1.3190 \end{bmatrix}.$$

Figure 5 – Example 1: Nonlinear function $f(x_{(2)})$ (black) and a PWA approximation of $f(x_{(2)})$ with five regions (blue).



Source: The author

It should be noticed that the origin of the open-loop system is not globally exponentially stable. Thus applying Algorithm 1 with parameters $\eta = 0.9999$, $\alpha_{min} = -1.5$, $\alpha_s = 0.5$, $\alpha_{max} = 1.5$, $\beta_{min} = -1.5$, $\beta_s = 0.5$, $\beta_{max} = 1.5$, $\gamma_{min} = -1.5$, $\gamma_s = 0.5$ and $\gamma_{max} = 1.5$ results in the following global stabilizing gains

$$K_1 = \begin{bmatrix} -5.2350 & -3.9165 \end{bmatrix} \text{ and}$$
$$K_2 = \begin{bmatrix} -0.1614 & 0.1243 & 0.1625 & -0.1242 \end{bmatrix}$$

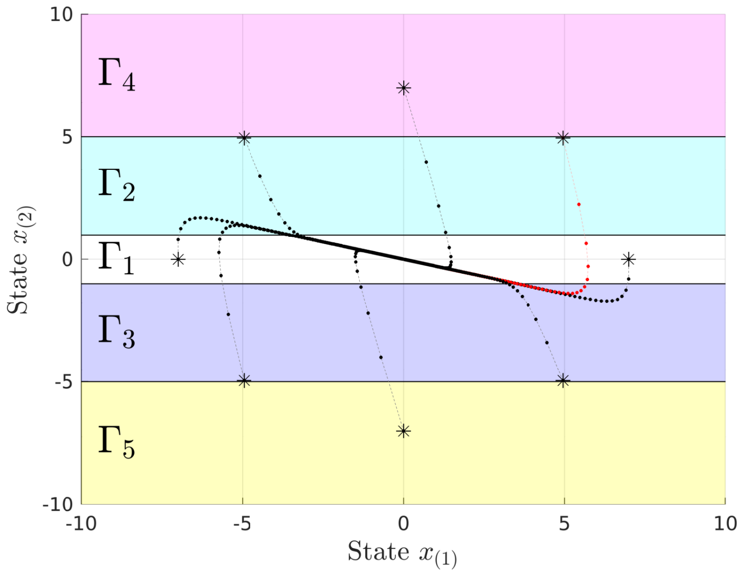
for $\alpha = 1.5$, $\beta = 1.0$, $\gamma = 1.5$ and test case 1. Some closed loop trajectories are shown in Figure 6 while Figure 7 describes the control input for one trajectory.

Note that, since the PWA approximation is only locally valid, the global exponential stabilization of the discrete-time system does not imply in the global stabilization of the continuous-time system. Nonetheless, this example shows the application of the proposed method.

3.4.2 Example 2

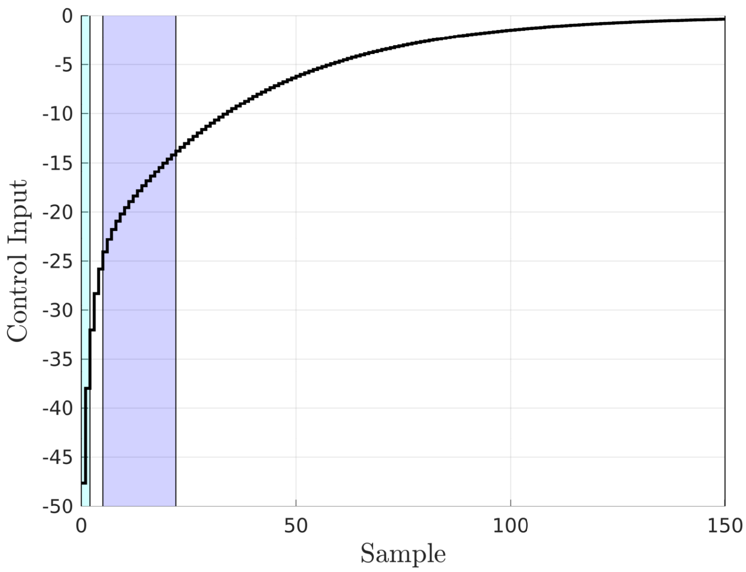
Recall the nonlinear circuit presented in Figure 1 and consider its discrete-time approximation as discussed in subsection 2.2.2 for $T = 0.03$. As shown in Figure 2, the system has three equilibrium points, one in each region. The goal is to make the equilibrium point in Γ_3 (i.e. $x_{eq3} = [0.3714 \;\; 0.6429]^T$) globally exponentially stable. To do that, we first apply a translation, from which the origin of the resulting system represented

Figure 6 – Example 1: Closed-loop trajectories (black dots) for a set of initial conditions (black stars). The trajectory highlighted in red has its control input depicted in Figure 7.



Source: The author

Figure 7 – Example 1: Control input for the trajectory highlighted in red in Figure 6.



Source: The author

with the implicit PWA representation (38) is given by

$$F_1 = \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1-0.2T \end{bmatrix},\ F_2 = \begin{bmatrix} 0 & 0 \\ -50T & 50T \end{bmatrix},\ B = \begin{bmatrix} 20T \\ 0 \end{bmatrix},$$
$$F_3 = \begin{bmatrix} 0 & -0.006 \\ 0 & -0.007 \end{bmatrix},\ F_4 = 0 \text{ and } f_5 = \begin{bmatrix} -0.0003 \\ -0.0031 \end{bmatrix}.$$

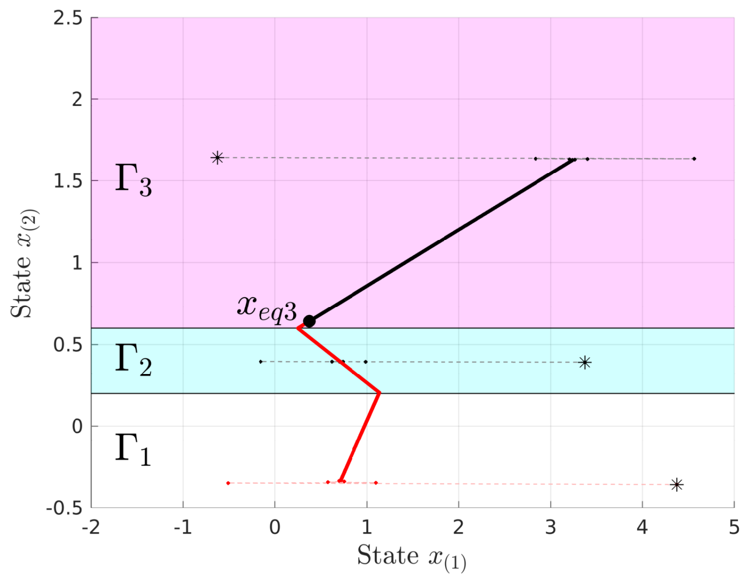
Note that the origin of the resulting system is the equilibrium point x_{eq3} of the original system and, in this case, $f_5 \preceq 0$ and Assumption 2 is met.

The algorithm was then applied with parameters $\eta = 0.9999$, $\alpha_{min} = -1.0$, $\alpha_s = 0.25$, $\alpha_{max} = 1.0$, $\beta_{min} = -1.0$, $\beta_s = 0.25$, $\beta_{max} = 1.0$, $\gamma_{min} = -1.0$, $\gamma_s = 0.25$ and $\gamma_{max} = 1.0$ resulting in the following global stabilizing gains

$$K_1 = \begin{bmatrix} -0.7123 & 7.4337 \end{bmatrix} \text{ and}$$
$$K_2 = 10^3 \begin{bmatrix} 1.9028 & -0.9602 \end{bmatrix}$$

for $\alpha = 1.0$, $\beta = 0.25$, $\gamma = 0.25$ and test case 1. Some closed loop trajectories are shown in Figure 8. The closed-loop trajectory highlighted in Figure 8 has its control input depicted in Figure 9.

Figure 8 – Example 2: Closed-loop trajectories (black dots) for a set of initial conditions (black stars). The trajectory highlighted in red has its control input depicted in Figure 9.

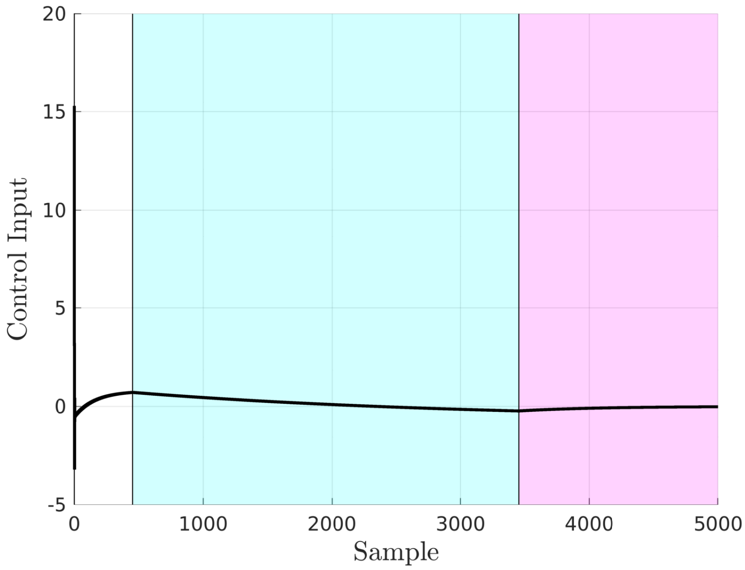


Source: The author

3.5 Final Remarks

This chapter addressed the stabilization of discrete-time PWA systems considering a PWA state feedback control law and a PWQ Lyapunov candidate function. The basic

Figure 9 – Example 2: Control input for the trajectory highlighted in red in Figure 8.



Source: The author

problem resides with the nonconvexity introduced by the stability conditions when the controller gains are variables, since this leads to product between variables. To overcome this issue, new stability conditions were derived for the extended vector $\xi(x)$ (which contains the successor state x^+) using Finsler's Lemma. Moreover, congruence transformations and some structural assumptions were performed in order to reach *quasi*-LMI sufficient conditions for the closed-loop global exponential stability of the origin. An algorithm, based on the solution of a set of LMI feasibility problems, was proposed to compute the stabilizing gains and was tested in numerical examples.

Differently from previous approaches in the literature, the stabilization method proposed in this work does not require *a priori* knowledge of possible transitions between regions. Moreover, the nonconvexity introduced by the relaxation terms in the explicit representation is avoided with our method and the presence of the affine term is taken into account without further difficulties.

4 LOCAL STABILITY ANALYSIS

Section 2.3.4 presented the global stability analysis for PWA systems using the implicit representation. This chapter proposes Lyapunov conditions to ensure local exponential stability of PWA systems and derives a method to estimate the Region of Attraction of the Origin (RAO) based on LMIs considering a PWQ Lyapunov candidate function.

4.1 Local Exponential Stability

The local asymptotic stability of a discrete-time nonlinear system is guaranteed if there is a Lyapunov candidate function $V(x)$ such that (ÅSTRÖM; WITTENMARK, 1997)

$$\begin{aligned} V(x) > 0 \quad & \forall x \in \mathcal{D} - \{0\} \\ \Delta V(x) < 0 \quad & \forall x \in \mathcal{D} - \{0\}. \end{aligned}$$

Moreover, if the Lyapunov candidate function also satisfies

$$\begin{aligned} \epsilon_{min} \|x\|^2 \leq V(x) \leq \epsilon_{max} \|x\|^2 \quad & \forall x \in \mathcal{D} - \{0\} \\ \Delta V(x) < -\epsilon_{\Delta} \|x\|^2 \quad & \forall x \in \mathcal{D} - \{0\} \end{aligned} \tag{51}$$

for positive scalars ϵ_{min}, ϵ_{max} and ϵ_{Δ} then, the origin is exponentially stable (KHALIL, 2002). In both cases, an estimate of the RAO is obtained as sub level sets of $V(x)$ given by $\mathcal{L}_{\rho} = \{x \in \mathbb{R}^n \mid V(x) \leq \rho,\ \rho > 0\} \subseteq \mathcal{D}$.

The following sections derive methods to estimate the RAO of discrete-time systems represented with the implicit representation.

4.2 Local Stability Analysis

Consider a PWQ Lyapunov candidate function $V(x)$ as in (19). Then, to ensure the local stability of a given PWA system origin it is necessary to test the local positivity of such functions and the local negativity of $\Delta V(x)$ in a set $\mathcal{D}$, as stated in (51). The following Lemma is instrumental to this task, being an extended version of Lemma 6 suitable for the local analysis.

Lemma 12. *Let the symmetric matrix $M(x) \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ be defined elementwise by locally nonnegative functions $m_{(i,j)} : \mathbb{R}^n \to \mathbb{R}$, such that $M_{(i,j)}(x) = m_{(i,j)}(x) \geq 0 \ \forall x \in \mathcal{D} \subseteq \mathbb{R}^n, \ i,j = 1,...,1+2n_y$. Then, it follows that*

$$s_1(M(x), y(x)) = \begin{bmatrix} 1 \\ \phi(y(x)) \\ \phi(y(x)) - y(x) \end{bmatrix}^T M(x) \begin{bmatrix} 1 \\ \phi(y(x)) \\ \phi(y(x)) - y(x) \end{bmatrix} \geq 0 \ \forall x \in \mathcal{D} \subseteq \mathbb{R}^n. \tag{52}$$

Proof. For any $x \in \mathcal{D} \subseteq \mathbb{R}^n$ we have $M(x) \succeq 0$, since each $m_{(i,j)}(x) \geq 0$, i, $j = 1,...,1+2n_y$. Since each element of $[1 \ \ \phi^T(y(x)) \ \ (\phi(y(x)) - y(x))^T]^T$ is nonnegative for any $y(x) \in \mathbb{R}^{n_y}$ thanks to properties (21a) and (21b) of the ramp function, (52) holds. ∎

Remark 4. *Note that Lemma 6 is a special case of Lemma 12 with $M(x)$ being a constant elementwise nonnegative matrix and, as a consequence, $\mathcal{D} = \mathbb{R}^n$.*

A matrix function $M(x)$ satisfying Lemma 12 will be called a locally elementwise nonnegative matrix in $\mathcal{D}$. Such matrix allows to state the following theorem regarding the local exponential stability of the origin of a PWA system.

Theorem 5. *Consider a PWQ Lyapunov candidate function $V(x)$ as in* (19)*. If there exist a symmetric matrix $P \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $R_1 \in \mathbb{R}^{n_\chi \times n_y}$, $R_2 \in \mathbb{R}^{n_{\bar\chi} \times 2n_y}$, locally elementwise nonnegative matrices $M(x) \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ and $\bar{M}(x) \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$ $\forall x \in \mathcal{D}$, a positive scalar ϵ_{min} and $\eta \in (0,1)$ such that*

$$(V(x) - \epsilon_{min} x^T x) - s_1(M(x), y(x)) + s_2(T_1, y(x)) + s_3(R_1, \chi(x), x) \geq 0 \tag{53}$$

and

$$-(V(x^+) - \eta V(x)) - s_1(\bar{M}(x), \bar{y}(x)) + s_2(T_2, \bar{y}(x)) + s_3(R_2, \bar{\chi}(x), x) \geq 0 \tag{54}$$

with

$$\chi(x) = \begin{bmatrix} 1 \\ x \\ \phi(y(x)) \\ \phi(y(x)) - y(x) \end{bmatrix}, \quad \bar{y}(x) \triangleq \begin{bmatrix} y(x) \\ y(x^+) \end{bmatrix} \quad \text{and} \quad \bar{\chi}(x) \triangleq \begin{bmatrix} 1 \\ x \\ \phi(\bar{y}(x)) \\ \phi(\bar{y}(x)) - \bar{y}(x) \end{bmatrix},$$

then the origin of the PWA system (5) *is locally exponentially stable and an estimate of the RAO is given by any sub level set of $V(x)$ contained in $\mathcal{D}$, that is $\mathcal{L}_\rho \subseteq \mathcal{D}$.*

Proof. From Lemma 5 it follow that exists a scalar ϵ_{max} such that

$$V(x) \leq \epsilon_{max} \|x\|^2 .$$

Moreover, if (53) holds, then, from Lemmas 7, 8 and 12, it follows that

$$(V(x) - \epsilon_{min}x^T x) \geq s_1(M(x), y(x)) \geq 0 \quad \forall x \in \mathcal{D} \subseteq \mathbb{R}^n$$
$$V(x) \geq \epsilon_{min} \|x\|^2 \quad \forall x \in \mathcal{D} \subseteq \mathbb{R}^n.$$

On the other hand, considering Lemmas 7, 8 and 12, (54) implies in

$$-\left(V(x^+) - \eta V(x)\right) \geq s_1(\bar{M}(x), \bar{y}(x)) \geq 0 \quad \forall x \in \mathcal{D} \subseteq \mathbb{R}^n$$
$$V(x^+) \leq \eta V(x) \quad \forall x \in \mathcal{D} \subseteq \mathbb{R}^n$$

ensuring the local exponential stability of the origin with any sub level set $\mathcal{L}_\rho \subseteq \mathcal{D}$ being an estimate of the RAO. ■

The goal now is to write conditions presented in Theorem 5 as LMI constraints. The basic problem resides with writing terms like $s_1(M(x), y(x))$ as quadratic terms in $\chi(x) = [1 \;\; x^T \;\; \phi^T(y(x)) \;\; (\phi(y(x)) - y(x))^T]^T$ and ensure the local elementwise nonnegativity of a matrix $M(x)$ with LMIs. Note that the other terms in conditions (53) and (54) can be written as quadratic terms similarly to what was done in Theorem 2. To deal with this problem, the idea is to consider the following structure for the locally elementwise nonnegative matrix $M(x)$ in (53) (or $\bar{M}(x)$ in (54)):

$$M(x) = \begin{bmatrix} M_{1,1}(x) & M_{1,2}(x) & M_{1,3}(x) \\ \star & M_{2,2} & M_{2,3} \\ \star & \star & M_{3,3} \end{bmatrix}, \tag{55}$$

where blocks $M_{2,2}$, $M_{2,3}$ and $M_{3,3}$ are constant elementwise nonnegative matrices of appropriate dimensions and

$$\begin{aligned} &M_{1,1} : \mathbb{R}^n \to \mathbb{R}, && M_{1,1}(x) = \chi^T(x) S_a \chi(x), \\ &M_{1,2}^T : \mathbb{R}^n \to \mathbb{R}^{n_y} && M_{1,2}(x) = \chi^T(x) S_\phi, \\ &M_{1,3}^T : \mathbb{R}^n \to \mathbb{R}^{n_y} && M_{1,3}(x) = \chi^T(x) S_{\bar{\phi}}. \end{aligned} \tag{56}$$

with $S_a = S_a^T \in \mathbb{R}^{n_\chi \times n_\chi}$, $S_\phi \in \mathbb{R}^{n_\chi \times n_y}$ and $S_{\bar{\phi}} \in \mathbb{R}^{n_\chi \times n_y}$. Then, the term $s_1(M(x), y(x))$ can be expressed as follows:

$$s_1(M(x), y(x)) = \chi^T(x) \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 \\ \star & \star & M_{2,2} & M_{2,3} \\ \star & \star & \star & M_{3,3} \end{bmatrix} + S_a + \text{He}\left\{ \begin{bmatrix} 0_{1\times n_\chi} \\ 0_{n\times n_\chi} \\ S_\phi^T \\ S_{\bar{\phi}}^T \end{bmatrix} \right\} \right) \chi(x),$$

which is a quadratic expression in $\chi(x)$.

Furthermore, to derive LMIs that ensure the local elementwise nonnegativity of $M(x)$, note that each element of vector functions $M_{1,2}(x)$ and $M_{1,3}(x)$ can be written as the following quadratic expressions in $\chi(x)$:

$$M_{1,2}(x)_{(i)} = \chi^T(x) S_{\phi(:,i)} = \chi^T(x) \begin{bmatrix} S_{\phi(:,i)} & 0 & 0 & 0 \end{bmatrix} \chi(x),$$
$$M_{1,3}(x)_{(i)} = \chi^T(x) S_{\bar{\phi}(:,i)} = \chi^T(x) \begin{bmatrix} S_{\bar{\phi}(:,i)} & 0 & 0 & 0 \end{bmatrix} \chi(x)$$

for $i = 1, ..., n_y$.

Now let the set $\mathcal{D}$ be defined by a quadratic expression in $\chi(x)$ as

$$\mathcal{D} = \{x \in \mathbb{R}^n \mid d(x) = \chi^T(x) D \chi(x) \geq 0\} \tag{57}$$

with $D \in \mathbb{R}^{n_\chi \times n_\chi}$.

Remark 5. *Regarding condition* (54)*, it is also useful to represent the set $\mathcal{D}$ as a quadratic expression in terms of $\bar{\chi}(x) = [1 \;\; x^T \;\; \phi^T(\bar{y}(x)) \;\; (\phi(\bar{y}(x) - \bar{y}(x))^T]^T$, with $\bar{y}(x) = [y^T(x) \;\; y^T(x^+)]^T$. In order to do that, define*

$$\bar{D} \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}^T D \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}. \tag{58}$$

Then, we can rewrite function $d(x)$ as

$$d(x) = \chi^T(x) D \chi(x) \equiv \bar{\chi}^T(x) \bar{D} \bar{\chi}(x).$$

Using the S-procedure (see Appendix A.1) the local nonnegativity of each element of $M(x)$ for $x \in \mathcal{D}$ can therefore be ensured from the following Lemma.

Lemma 13. *Consider a matrix function $M(x)$ as in* (55)-(56)*, $y(x)$ given by* (5b) *with $\phi(y(x))$ defined as in* (6) *and the set $\mathcal{D}$ as defined in* (57)*. If there exist a symmetric nonnegative matrix $M_a \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, a diagonal matrix $T_a \in \mathbb{D}^{n_y}$, $R_a \in \mathbb{R}^{n_\chi \times n_y}$ and a nonnegative scalar α_a such that*

$$(M_{1,1}(x) - \alpha_a d(x)) - s_1(M_a, y(x)) + s_2(T_a, y(x)) + s_3(R_a, \chi(x), x) \geq 0 \tag{59}$$

and if there exist symmetric elementwise nonnegative matrices $M_{\phi_i} \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ and $M_{\bar{\phi}_i} \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, diagonal matrices $T_{\phi_i} \in \mathbb{D}^{n_y}$ and $T_{\bar{\phi}_i} \in \mathbb{D}^{n_y}$, matrices $R_{\phi_i} \in \mathbb{R}^{n_\chi \times n_y}$ and $R_{\bar{\phi}_i} \in \mathbb{R}^{n_\chi \times n_y}$ and nonnegative scalar α_{ϕ_i} and $\alpha_{\bar{\phi}_i}$ for $i = 1, ..., n_y$, such that

$$(M_{1,2}(x)_{(i)} - \alpha_{\phi_i} d(x)) - s_1(M_{\phi_i}, y(x)) + s_2(T_{\phi_i}, y(x)) + s_3(R_{\phi_i}, \chi(x), x) \geq 0 \tag{60}$$

and

$$(M_{1,3}(x)_{(i)} - \alpha_{\bar{\phi}_i} d(x)) - s_1(M_{\bar{\phi}_i}, y(x)) + s_2(T_{\bar{\phi}_i}, y(x)) + s_3(R_{\bar{\phi}_i}, \chi(x), x) \geq 0 \tag{61}$$

are satisfied for $i = 1, ..., n_y$, then $M(x)$ is locally elementwise nonnegative in $\mathcal{D}$.

Proof. From Lemmas 6, 7 and 8 regarding the global nonnegativity of PWQ functions and the fact that $y(x)$ is given by (5b), (59) implies that

$$M_{1,1}(x) \geq \alpha_a d(x) \geq 0 \quad \forall x \in \mathbb{R}^n \subseteq \mathcal{D}$$

for any nonnegative scalar α_a. Similarly, (60) and (61) imply, respectively, that

$$M_{1,2}(x)_{(i)} \geq \alpha_{\phi_i} d(x) \geq 0 \quad \forall x \in \mathbb{R}^n \subseteq \mathcal{D}$$

and

$$M_{1,3}(x)_{(i)} \geq \alpha_{\bar{\phi}_i} d(x) \geq 0 \quad \forall x \in \mathbb{R}^n \subseteq \mathcal{D}$$

for each element $i = 1, ..., n_y$. Since the remaining blocks of $M(x)$ are elementwise nonnegative matrices, we conclude that $M(x)$ is locally elementwise nonnegative in $\mathcal{D}$. ■

Similarly, the ideas previously presented are applied to matrix $\bar{M}(x)$ in (54) with the same structure as given in (55)-(56). Then, using Lemma 13, we can now express the conditions of Theorem 5 as LMIs. The following Theorem formalizes that.

Theorem 6. *Consider $V(x)$ as in* (19). *If there exist symmetric matrices P, $S_a \in \mathbb{R}^{n_\chi \times n_\chi}$, $\bar{S}_a \in \mathbb{R}^{n_{\bar{\chi}} \times n_{\bar{\chi}}}$, $D \in \mathbb{R}^{n_\chi \times n_\chi}$ diagonal matrices $T_1 \in \mathbb{D}^{n_y}$, $T_a \in \mathbb{D}^{n_y}$, $T_{\phi_i} \in \mathbb{D}^{n_y}$, $T_{\bar{\phi}_i} \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $\bar{T}_a \in \mathbb{D}^{2n_y}$, $\bar{T}_{\phi_j} \in \mathbb{D}^{2n_y}$ and $\bar{T}_{\bar{\phi}_j} \in \mathbb{D}^{2n_y}$, matrices $S_\phi \in \mathbb{R}^{n_\chi \times n_y}$, $S_{\bar{\phi}} \in \mathbb{R}^{n_\chi \times n_y}$, $\bar{S}_\phi \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$, $\bar{S}_{\bar{\phi}} \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$, $R_1 \in \mathbb{R}^{n_\chi \times n_y}$, $R_a \in \mathbb{R}^{n_\chi \times n_y}$, $R_{\phi_i} \in \mathbb{R}^{n_\chi \times n_y}$, $R_{\bar{\phi}_i} \in \mathbb{R}^{n_\chi \times n_y}$, $R_2 \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$, $\bar{R}_a \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$, $\bar{R}_{\phi_j} \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$ and $\bar{R}_{\bar{\phi}_j} \in \mathbb{R}^{n_{\bar{\chi}} \times 2n_y}$, symmetric elementwise nonnegative matrices $M_{2,2} \in \mathbb{R}^{n_y \times n_y}$, $M_{3,3} \in \mathbb{R}^{n_y \times n_y}$, $\bar{M}_{2,2} \in \mathbb{R}^{2n_y \times 2n_y}$, $\bar{M}_{3,3} \in \mathbb{R}^{2n_y \times 2n_y}$ $M_a \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, $M_{\phi_i} \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, $M_{\bar{\phi}_i} \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, $\bar{M}_a \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$, $\bar{M}_{\phi_j} \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$ and $\bar{M}_{\bar{\phi}_j} \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$, elementwise nonnegative matrices $M_{2,3} \in \mathbb{R}^{n_y \times n_y}$ and $\bar{M}_{2,3} \in \mathbb{R}^{2n_y \times 2n_y}$, a positive scalar ϵ_{min}, a scalar $\eta \in (0,1)$ and nonnegative scalars α_a, α_{ϕ_i}, $\alpha_{\bar{\phi}_i}$, $\bar{\alpha}_a$, $\bar{\alpha}_{\phi_j}$ and $\bar{\alpha}_{\bar{\phi}_j}$, for $i = 1, ..., n_y$ and $j = 1, ..., 2n_y$, such that the following LMIs*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_1 - \epsilon_{min} I & P_2 & 0 \\ 0 & \star & P_3 - M_{2,2} & -M_{2,3} + T_1 \\ 0 & 0 & 0 & -M_{3,3} \end{bmatrix} - S_a - \mathbf{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ S_\phi^T \\ S_{\bar{\phi}}^T \end{bmatrix} \right\} + \mathbf{He}\{R_1 Q_1\} \geq 0, \tag{62}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -N_1 & -N_2 & 0 \\ 0 & \star & -N_3 - \bar{M}_{2,2} & -\bar{M}_{2,3} + T_2 \\ 0 & 0 & 0 & -\bar{M}_{3,3} \end{bmatrix} - \bar{S}_a - \mathbf{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ \bar{S}_\phi^T \\ \bar{S}_{\bar{\phi}}^T \end{bmatrix} \right\} + \mathbf{He}\{R_2 Q_2\} \geq 0 \tag{63}$$

$$S_a - \alpha_a D - \begin{bmatrix} M_{a_{1,1}} & 0 & M_{a_{1,2}} & M_{a_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{a_{2,2}} & M_{a_{2,3}} - T_a \\ \star & 0 & \star & M_{a_{3,3}} \end{bmatrix} + \mathbf{He}\{R_a Q_1\} \geq 0 \tag{64}$$

$$0.5He\left\{\begin{bmatrix} S^T_{\phi_{(:,i)}} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} - \alpha_{\phi_i} D - \begin{bmatrix} M_{\phi_{i1,1}} & 0 & M_{\phi_{i1,2}} & M_{\phi_{i1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{\phi_{i2,2}} & M_{\phi_{i2,3}} - T_{\phi_i} \\ \star & 0 & \star & M_{\phi_{i3,3}} \end{bmatrix} + He\{R_{\phi_i} Q_1\} \geq 0 \tag{65}$$

$$0.5He\left\{\begin{bmatrix} S^T_{\bar{\phi}_{(:,i)}} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} - \alpha_{\bar{\phi}_i} D - \begin{bmatrix} M_{\bar{\phi}_{i1,1}} & 0 & M_{\bar{\phi}_{i1,2}} & M_{\bar{\phi}_{i1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{\bar{\phi}_{i2,2}} & M_{\bar{\phi}_{i2,3}} - T_{\bar{\phi}_i} \\ \star & 0 & \star & M_{\bar{\phi}_{i3,3}} \end{bmatrix} + He\{R_{\bar{\phi}_i} Q_1\} \geq 0 \tag{66}$$

$$\bar{S}_a - \bar{\alpha}_a \bar{D} - \begin{bmatrix} \bar{M}_{a_{1,1}} & 0 & \bar{M}_{a_{1,2}} & \bar{M}_{a_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \bar{M}_{a_{2,2}} & \bar{M}_{a_{2,3}} - \bar{T}_a \\ \star & 0 & \star & \bar{M}_{a_{3,3}} \end{bmatrix} + He\{\bar{R}_a Q_2\} \geq 0 \tag{67}$$

$$0.5He\left\{\begin{bmatrix} \bar{S}^T_{\phi_{(:,j)}} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} - \bar{\alpha}_{\phi_j} \bar{D} - \begin{bmatrix} \bar{M}_{\phi_{j1,1}} & 0 & \bar{M}_{\phi_{j1,2}} & \bar{M}_{\phi_{j1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \bar{M}_{\phi_{j2,2}} & \bar{M}_{\phi_{j2,3}} - T_{\phi_i} \\ \star & 0 & \star & \bar{M}_{\phi_{j3,3}} \end{bmatrix} + He\{\bar{R}_{\phi_j} Q_2\} \geq 0 \tag{68}$$

$$0.5He\left\{\begin{bmatrix} \bar{S}^T_{\bar{\phi}_{(:,j)}} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} - \bar{\alpha}_{\bar{\phi}_j} \bar{D} - \begin{bmatrix} \bar{M}_{\bar{\phi}_{j1,1}} & 0 & \bar{M}_{\bar{\phi}_{j1,2}} & \bar{M}_{\bar{\phi}_{j1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \bar{M}_{\bar{\phi}_{j2,2}} & \bar{M}_{\bar{\phi}_{j2,3}} - \bar{T}_{\bar{\phi}_j} \\ \star & 0 & \star & \bar{M}_{\bar{\phi}_{j3,3}} \end{bmatrix} + He\{\bar{R}_{\bar{\phi}_j} Q_2\} \geq 0 \tag{69}$$

are satisfied with N_1, N_2, N_3, Q_1 and Q_2 as defined in Theorem 2 and $\bar{D}$ as defined in (58)*, then the origin of the PWA system* (5) *is locally exponentially stable and any sub level set $\mathcal{L}_\rho \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ is an estimate of the RAO.*

Proof. By pre and post multiplying (62) by $\chi^T(x)$ and $\chi(x)$, respectively, we obtain (53) with $M(x)$ as defined in (55)-(56). Moreover, by pre and post multiplying (63) by $\bar{\chi}^T(x)$ and $\bar{\chi}(x)$, respectively, we obtain (54) with $\bar{M}(x)$ having the same structure as (55)-(56).

Finally, it is necessary to ensure that both $M(x)$ and $\bar{M}(x)$ are locally elementwise nonnegative matrices in $\mathcal{D}$. By pre and post multiplying (64), (65) and (66) by $\chi^T(x)$ and $\chi(x)$, respectively, we obtain conditions (59), (60) and (61) of Lemma 13, ensuring that $M(x)$ is locally elementwise nonnegative in $\mathcal{D}$. On the other hand, by pre and post multiplying (67), (68) and (69) by $\bar{\chi}^T(x)$ and $\bar{\chi}(x)$, respectively, we obtain the conditions of Lemma 13 for $\bar{M}(x)$, ensuring its locally elementwise nonnegativity in $\mathcal{D}$.

Hence, any solution for the conditions presented in Theorem 6 is also a solution for the conditions of Theorem 5, from where the local stability of the origin follows. ∎

Note that any sub level set $\mathcal{L}_\rho \subseteq \mathcal{D}$ is contractive and, therefore, is an estimate of the RAO. However, the great interest lies on determining the best function $V(x)$, that is, the best matrix P leading to the largest estimate possible according to some criteria. Those criteria can be related, for instance, to the volume of the estimate or the inclusion of specific points of the state space in $\mathcal{L}_\rho$ through the use of additional constraints (TARBOURIECH *et al.*, 2011). Next subsections deal with this issue.

4.2.1 Choosing region $\mathcal{D}$

The estimate of the RAO is defined as any sub level set $\mathcal{L}_\rho$ of the Lyapunov function contained within the set $\mathcal{D}$, defined by a PWQ function in (57). This set plays, therefore, an important role in the estimation of the RAO. This subsection presents a brief discussion on how this set is choosen.

In this work we consider the set $\mathcal{D}$ defined by a level set of a quadratic function of the state x, given as follows.

$$\mathcal{D} = \left\{ x \in \mathbb{R}^n \;\middle|\; \chi^T(x) \begin{bmatrix} r_d^2 - x_c^T I x_c & x_c^T I & 0 & 0 \\ \star & -I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \chi(x) \geq 0 \right\}.$$

Note that $\mathcal{D}$ defines a hypersphere with radius r_d and center given by x_c. Furthermore, the values of r_d and x_c must ensure that the origin is included in $\mathcal{D}$.

In this work, the region $\mathcal{D}$ was chosen by the following procedure. We start with a small region $\mathcal{D}$ and test the feasibility of the LMIs proposed in Theorem 6. If the feasibility is verified, we proceed to consider larger regions $\mathcal{D}$. Otherwise, the last region $\mathcal{D}$ for which the feasibility of the conditions is verified is then used in the procedures described in the following subsections.

4.2.2 Optimizing the estimate of the RAO

Once the conditions in Theorem 6 are solved, i.e. a solution defined by matrices satisfying those conditions is found, we have a PWQ Lyapunov function $V(x)$ as in (19) which certifies the local stability of the PWA system origin (that is, it verifies conditions of Theorem 5). As mentioned before, any sub level set $\mathcal{L}_\rho$ of this Lyapunov function within region $\mathcal{D}$ is an estimate of the RAO. Then, for this Lyapunov function $V(x)$ the larger estimate of the RAO $\mathcal{L}_\rho^*$ is given by

$$\mathcal{L}_\rho^* = \max_\rho \mathcal{L}_\rho \quad \text{s.t.} \quad \mathcal{L}_\rho \subseteq \mathcal{D}. \tag{70}$$

Since $\mathcal{L}_\rho^*$ is computed for the obtained Lyapunov function $V(x)$ satisfying Theorem 5, the optimization procedure in (70) does not optimize the choice of $V(x)$ and, therefore, the shape of the level set $\mathcal{L}_\rho$.

However, from the set of possible solutions of Theorem 5, some of them may lead to larger estimates of the RAO. In order to compute these solutions, additional conditions can be included along with the local stability ones given in Theorem 5, allowing the computation of more suitable Lyapunov functions $V(x)$, that is, a more suitable matrix P among all feasible solutions.

One idea to optimize the shape of the estimate of the RAO is to maximize $\mathcal{L}_\rho$ in certain directions while ensuring that $\mathcal{L}_\rho$ is contained in $\mathcal{D}$, i.e. $\mathcal{L}_\rho \subseteq \mathcal{D}$. Without loss of generality, let us consider the specific sub level set $\mathcal{L}_1 = \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$ to estimate the RAO. Then, a sufficient condition to ensure that $\mathcal{L}_1$ is contained in $\mathcal{D}$ is given next.

Lemma 14. *If there exist a nonnegative scalar $\alpha_\mathcal{L}$, an elementwise nonnegative matrix $M_\mathcal{L} \succeq 0 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$, a diagonal matrix $T_\mathcal{L} \in \mathbb{D}^{n_y}$ and a matrix $R_\mathcal{L} \in \mathbb{R}^{n_\chi \times n_y}$ such that*

$$\alpha_\mathcal{L} D - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -P_1 & -P_2 & 0 \\ 0 & \star & -P_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{\mathcal{L}_{1,1}} & 0 & M_{\mathcal{L}_{1,2}} & M_{\mathcal{L}_{1,3}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{\mathcal{L}_{2,2}} & M_{\mathcal{L}_{2,3}} - T_\mathcal{L} \\ \star & 0 & \star & M_{\mathcal{L}_{3,3}} \end{bmatrix} + He\{R_\mathcal{L} Q_1\} \geq 0 \quad (71)$$

with Q_1 as in Theorem 2, then the sub level set $\mathcal{L}_1$ is contained in $\mathcal{D}$, that is, $\mathcal{L}_1 \subseteq \mathcal{D}$.

Proof. Pre and post multiply (71) by $\chi^T(x)$ and $\chi(x)$, respectively, and apply Lemmas 6, 7 and 8 regarding the global nonnegativity of PWQ functions to obtain

$$\alpha_\mathcal{L} d(x) - (1 - V(x)) \geq 0 \implies \alpha_\mathcal{L} d(x) \geq (1 - V(x)).$$

Note that for any $x \in \mathcal{L}_1$ we have $(1 - V(x)) \geq 0$, leading to

$$\alpha_\mathcal{L} d(x) \geq 0 \quad \forall x \in \mathcal{L}_1.$$

Since $\alpha_\mathcal{L}$ is a nonnegative scalar, this implies that $d(x) \geq 0 \;\; \forall x \in \mathcal{L}_1$, and thus it follows that $\mathcal{L}_1 \subseteq \mathcal{D}$. ■

On the other hand, we want to maximize $\mathcal{L}_1$ in certain directions. In order to do that, let those directions be encoded by a set vectors $\mathcal{V} = \{\lambda_1 v_1, ..., \lambda_k v_k\}$, where $v_i \in \mathbb{R}^n$ are unitary vectors and λ_i are nonnegative scalars, for $i = 1, ..., k$. Then, a necessary and sufficient condition to ensure that all vectors in the set $\mathcal{V}$ are contained in $\mathcal{L}_1$ is given by the following set of k constraints:

$$\begin{bmatrix} \lambda_i v_i \\ \phi(y(\lambda_i v_i)) \end{bmatrix}^T P \begin{bmatrix} \lambda_i v_i \\ \phi(y(\lambda_i v_i)) \end{bmatrix} \leq 1 \qquad \forall i = 1, ..., k. \quad (72)$$

The proof follows from the fact that (72) reads $V(\lambda_i v_i) \leq 1 \;\; \forall i = 1, ..., k$, implying that every vector $\lambda_i v_i \in \mathcal{V}$ is contained in $\mathcal{L}_1$.

Once the set of unitary vectors v_i, for $i = 1, ..., k$, is defined, that is, the directions in which we want to maximize the shape of $\mathcal{L}_1$ are chosen, it is possible to consider a criterion based on the magnitude of those vectors to maximize $\mathcal{L}_1$ in the specified directions. In this work the adopted criterion to maximize the shape of $\mathcal{L}_1$ is the sum of the nonnegative scalars λ_i. Hence, the optimization problem to maximize the estimate of the RAO is given by

$$\max_{\lambda_1,...,\lambda_k} \sum_{i=1}^{k} \lambda_i \quad \text{s.t. Constraints of Theorem 6, (71), (72) and } \lambda_i \geq 0. \tag{73}$$

Note that (72) is not a convex constraint since both λ_i and P are variables. Hence, the optimization problem (73) is *quasi*-convex. In this work, this *quasi*-convex optimization problem was solved using the following Algorithm.

Algorithm 2 Algorithm to maximize the shape of the estimate of the RAO

Define a value for the parameter Δ_λ (the step increment of the scalars λ_i).
Define an initial set of vectors $\mathcal{V}_0 = \{\lambda_1 v_1, ..., \lambda_k v_k\}$, with $\lambda_i \geq 0$ for $i = 1, ..., k$, such that the constraints of Theorem 6, (71) and (72) form a feasible SDP problem.
while the optimization criterion $\sum_{i=1}^{k} \lambda_i$ is increasing **do**
  for $i = 1, ..., k$ **do**
    $\lambda_i \leftarrow \lambda_i + \Delta_\lambda$ (that is, increase the magnitude of vector v_i).
    if the SDP problem becomes infeasible **then**
      $\lambda_i \leftarrow \lambda_i - \Delta_\lambda$ (that is, return v_i to its previous magnitude).
    end if
  end for
end while
End algorithm. The obtained estimate of the ROA is given by $\mathcal{L}_1$ and the optimal set of vectors $\mathcal{V}^*$ is given by $\mathcal{V}$ in the last iteration.

As a remark, note that a level set of a PWQ function is not necessarily a convex set. Hence, the procedure proposed in (73) and summarized in Algorithm 2 does not guarantee that the convex hull defined by the vectors in $\mathcal{V}$ (and similar sets such as $\mathcal{V}_0$ and $\mathcal{V}^*$) is contained in $\mathcal{L}_1$.

The following subsection deals with refining the original partition in order to obtain larger estimates of the RAO.

4.2.3 Analysis with modified partition

So far the local stability analysis was performed using the partition defined by the system dynamics, called original partition. However, a given PWA system can be represented with a new partition of the state space obtained by refining the original partition (IERVOLINO; TANGREDI; VASCA, 2017). First, it is necessary to define a method of refinement. Some algorithms for automated partition refinements are proposed in (IERVOLINO; VASCA; IANNELLI, 2015). In the present work we consider a partition refinement by considering successor instances of $y(x)$, i.e., the region in the original partition where future instances of $y(x)$ will be.

Let x^k denote the state k steps ahead ($k \geq 1$) and $y^k \triangleq y(x^k)$. Note that with this notation we have $x^+ = x^1$ and $y^0 = y(x)$. Then, for an unforced ($u \equiv 0$) PWA system the value of x^k can be computed by (5a) recursively, leading to

$$x^k = F_1^k x + \begin{bmatrix} F_1^{k-1}F_2 & F_1^{k-2}F_2 & \dots & F_1F_2 & F_2 \end{bmatrix} \phi(\hat{y}^{k-1})$$

where $\hat{y}^{k-1} \triangleq [(y^0)^T \;\; (y^1)^T \;\; \dots \;\; (y^{k-1})^T]^T$ contains previous values of the vector ramp function necessary to compute x^k. Each element of $\hat{y}^{k-1}$ is defined by an implicit equation, leading to

$$\hat{y}^{k-1} = \begin{bmatrix} F_3 \\ F_3F_1 \\ F_3F_1^2 \\ \vdots \\ F_3F_1^{k-1} \end{bmatrix} x + \begin{bmatrix} F_4 & 0 & 0 & \dots & 0 \\ F_3F_2 & F_4 & 0 & \dots & 0 \\ F_3F_1F_2 & F_3F_2 & F_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_3F_1^{k-2}F_2 & F_3F_1^{k-3}F_2 & F_3F_1^{k-4}F_2 & \dots & F_4 \end{bmatrix} \phi(\hat{y}^{k-1}) + \begin{bmatrix} f_5 \\ f_5 \\ f_5 \\ \vdots \\ f_5 \end{bmatrix}. \tag{74}$$

Remark 6. *Note that* (74) *can be written as* $\hat{y}^{k-1} = \hat{F}_1 x + \hat{F}_2\phi(\hat{y}^{k-1}) + \hat{f}_5$*, analogous to the implicit equation* (5b) *from the original PWA system. Since* $\hat{F}_4$ *has a lower block diagonal structure,* (74) *is well-posed if the original system is well-posed.*

The system dynamics (5a) can be expressed as a function of $\hat{y}^{k-1}$ as follows

$$x^+ = F_1 x + \begin{bmatrix} F_2 & 0 & 0 & \dots & 0 \end{bmatrix} \phi(\hat{y}^{k-1}) = \hat{F}_1 x + \hat{F}_2 \phi(\hat{y}^{k-1})$$

with $\hat{F}_1 \triangleq F_1$ and $\hat{F}_2 \triangleq [F_2 \;\; 0 \;\; 0 \;\; \dots \;\; 0]$. Considering $\hat{F}_3$, $\hat{F}_4$ and $\hat{f}_5$ as defined in Remark 6, the system with refined partition is described by

$$x^+ = \hat{F}_1 x + \hat{F}_2 \phi(\hat{y}^{k-1}) \tag{75a}$$

$$\hat{y}^{k-1} = \hat{F}_3 x + \hat{F}_4 \phi(\hat{y}^{k-1}) + \hat{f}_5. \tag{75b}$$

This refined representation of the original PWA system can be used to assess the local stability using Theorem 5, allowing PWQ Lyapunov functions $V(x)$ with an increased number of components. For example, a system with a single input subject to input saturation has an original partition with three regions: negative saturation, linear and positive saturation. Hence, the original partition of this example leads to PWQ Lyapunov functions defined with three quadratic components corresponding to each one of the regions. If we consider a refined partition composed by the successor instance of $y(x)$ (i.e. $y^1 = y(x^+)$) we obtain nine regions, originated from the combinations of original regions. Hence, the PWQ Lyapunov function $V(x)$ is now defined by 9 quadratic components, increasing the flexibility of possible solutions and possibly leading to larger estimates of the RAO. On the other hand, representing the system with a refined partition increases the numerical complexity of the SDP problem of Theorem 6, since we added new regions to the partition. Because of that, there is a trade-off between increasing the degrees of freedom and numerical complexity.

4.3 Numerical Example

This section provides a numerical example to illustrate the application of the proposed method. The results obtained are compared with other methods to estimate the RAO proposed in the literature.

4.3.1 Asymetric Saturation

The analysis of local stability are particularly important in feedback systems subject to input saturation when the open-loop dynamic is unstable. In such cases, the closed-loop stability is only local and some initial conditions will lead to trajectories that diverge to infinity or form a limit-cycle. Since those behaviors are unsought, an estimate of the RAO is necessary (TARBOURIECH *et al.*, 2011).

Consider a linear system with a stabilizing state feedback gain K subject to input saturation

$$x^+ = Ax + B\text{sat}(Kx, \mu_{min}, \mu_{max}) \tag{76}$$

where sat : $\mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$ is the saturation function defined by

$$\text{sat}(v, \mu_{min}, \mu_{max}) = \begin{cases} -\mu_{min(i)} & \text{if } v_{(i)} < -\mu_{min(i)} \\ v_{(i)} & \text{if } -\mu_{min(i)} \leq v_{(i)} \leq +\mu_{max(i)} \\ +\mu_{max(i)} & \text{if } v_{(i)} > +\mu_{max(i)} \end{cases}$$

and parameters $\mu_{min} \succeq 0$ and $\mu_{max} \succeq 0 \in \mathbb{R}^{n_u}$ are elementwise nonnegative vectors specifying the lower and upper saturation limits, respectively. The saturation is symmetric if $\mu_{min} = \mu_{max}$ and, in this case, several methods to assess the local stability and estimate the RAO can be found, for instance, in (TARBOURIECH *et al.*, 2011). In the more general case where $\mu_{min} \neq \mu_{max}$ the saturation is asymmetric and a method to assess the local stability for the continuous-time case is found in (LI; LIN, 2017) while a method based on the use of deadzone functions to assess the stability and estimate the RAO for discrete-time systems can be found in (GROFF; GOMES DA SILVA JR.; VALMORBIDA, 2019).

We show now how the results developed in this chapter can be applied to this problem. The feedback system subject to input saturation (76) can be written in the implicit PWA representation with

$$F_1 = A + BK, \quad F_2 = B\begin{bmatrix}-I & I\end{bmatrix},$$
$$F_3 = \begin{bmatrix}K \\ -K\end{bmatrix}, \quad F_4 = 0 \quad \text{and } f_5 = \begin{bmatrix}-\mu_{max} \\ -\mu_{min}\end{bmatrix},$$

for which (5) reads

$$\begin{aligned} x^+ &= (A + BK)x + B\begin{bmatrix}-I & I\end{bmatrix}\phi(y(x)) \\ y(x) &= \begin{bmatrix}K \\ -K\end{bmatrix}x + \begin{bmatrix}-\mu_{max} \\ -\mu_{min}\end{bmatrix}. \end{aligned} \tag{77}$$

Note that the implicit PWA representation for the saturation case (77) is always well-posed, since matrix $F_4 = 0$. Also note that (77) satisfies Assumption 2 and its origin is an equilibrium point.

Consider the numerical example of a second-order single-input discrete-time system subject to asymmetric input saturation from (GROFF; GOMES DA SILVA JR.; VALMORBIDA, 2019), given by (76) with

$$A = \begin{bmatrix}1.20 & 0.00 \\ -0.05 & 1.00\end{bmatrix}, \quad B = \begin{bmatrix}1 \\ 0\end{bmatrix}, \quad K = \begin{bmatrix}-1 & 1\end{bmatrix}, \quad \mu_{min} = 1 \quad \text{and} \quad \mu_{max} = 6.$$

The resulting closed-loop PWA system in the implicit representation is therefore given by

$$\begin{aligned} x^+ &= \begin{bmatrix}0.20 & 1.00 \\ -0.05 & 1.00\end{bmatrix}x + \begin{bmatrix}-1 & 1 \\ 0 & 0\end{bmatrix}\phi(y(x)) \\ y(x) &= \begin{bmatrix}-1 & 1 \\ 1 & -1\end{bmatrix}x + \begin{bmatrix}-6 \\ -1\end{bmatrix}. \end{aligned} \tag{78}$$

The partition of (78) is the original partition defined by the asymmetric input saturation and is depicted in Figure 10, where Γ_1 is the positive saturation region, Γ_2 is the linear region and Γ_3 is the negative saturation region.

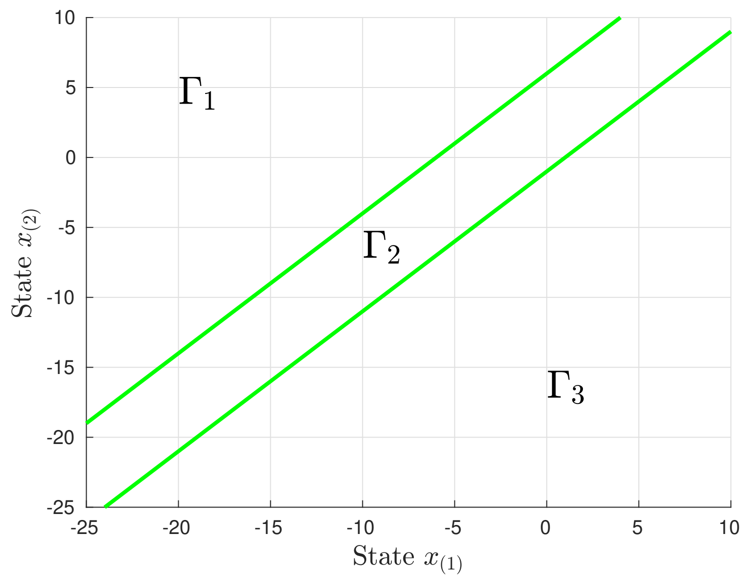
To assess the local stability of the system (78), SDP problems are built with the LMI constraints given by Theorem 6. In order to maximize the estimate of the RAO, the optimization problem (73) is considered and, therefore, the additional constraints (71) and (72) are taken into account. The optimization problem (73) is solved using Algorithm 2 with parameters

$$\mathcal{V}_0 = \left\{\begin{bmatrix}-1 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ -1\end{bmatrix}, \begin{bmatrix}1 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ 1\end{bmatrix}\right\} \quad \text{and} \quad \Delta_\lambda = 0.1. \tag{79}$$

An estimate of the RAO with area of 68.1 units squared was obtained from Algorithm 2 and is depicted in Figure 11. The optimal set of vectors $\mathcal{V}^*$ included in the estimate of the RAO found by Algorithm 2 is given by

$$\mathcal{V}^* = \left\{\begin{bmatrix}-13.0 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ -2.2\end{bmatrix}, \begin{bmatrix}2.5 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ 2.4\end{bmatrix}\right\}.$$

Figure 10 – Original partition of (78) defined by its asymmetric input saturation (green). Γ_1 represents the positive saturation, Γ_2 the linear region and Γ_3 the negative saturation.



Source: The author

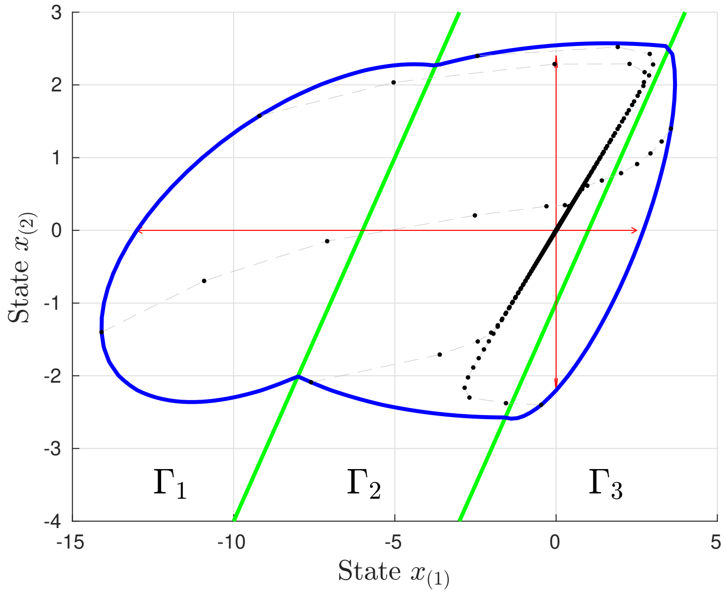
As stated in subsection 4.2.3, the local stability analysis can be performed considering a refined partition. To illustrate this idea, a new partition was generated for the system subject to asymmetric input saturation (78) by considering the one step ahead instance of $y(x)$. This refined partition is depicted in Figure 12 and the representation of system (78) with this partition is given by

$$
\begin{aligned}
x^+ &= \begin{bmatrix} 0.20 & 1.00 \\ -0.05 & 1.00 \end{bmatrix} x + \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \phi(\hat{y}(x)) \\
\hat{y}(x) &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -0.25 & 0 \\ 0.25 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \phi(\hat{y}(x)) + \begin{bmatrix} -6 \\ -1 \\ -6 \\ -1 \end{bmatrix}.
\end{aligned} \tag{80}
$$

The refined partition proposed in this example is composed by nine regions representing all two steps combinations for the three original regions. Table 2 associates each region of the refined partition depicted in Figure 12 with its respective combination of two regions of the original partition. For example, any trajectory starting within region Γ_5 of the refined partition in Figure 12 has its first sample in the linear region (input not saturated) and its second sample in the positive input saturation region of the original partition.

Note that the system (78) represented with the refined partition (i.e. (80)) has the same form of the implicit PWA representation (5). Hence, the same procedure used to estimate the RAO for the system represented with the original partition can be used with the extended partition. As a remark, note that (80) has a non-null matrix F_4 with a lower

Figure 11 – Estimate of the RAO (blue) considering the original partition given by the asymmetric saturation (green). The optimal set of vectors $\mathcal{V}^*$ included in the estimate of the RAO are shown in red. Examples of trajectories are shown as black dots.



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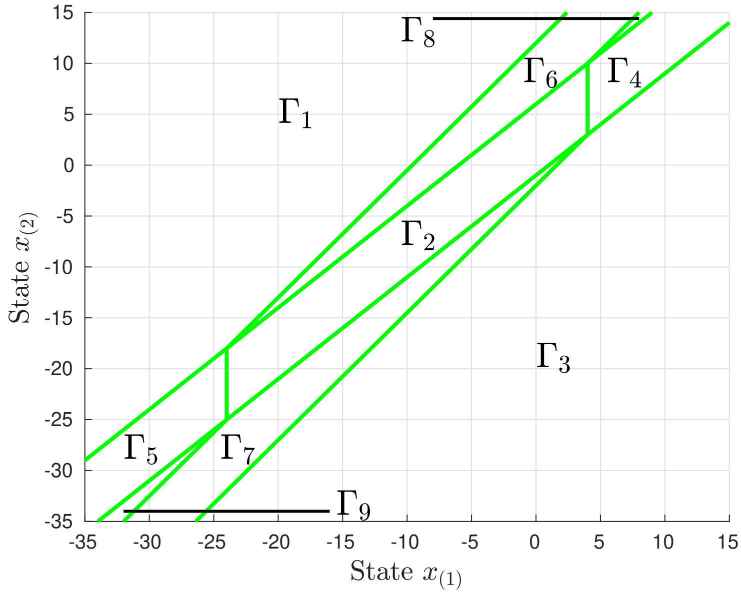
block diagonal structure and, as presented in subsection 4.2.3, in this case (80) inherits the well-posedness of the original system (78).

To estimate the RAO, the optimization problem (73), which includes the constraints of Theorem 6, (71) and (72), is solved again using Algorithm 2 and parameters (79). The estimate of the RAO obtained for the system represented with the refined partition is depicted in Figure 13 and has 71.2 units squared. Thus, the use of the refined partition in this numerical example lead to an increase of 4.6% in area when compared to the estimate of the RAO obtained with the original partition. The optimal set of vectors $\mathcal{V}^*$ included in the estimate of the RAO found by Algorithm 2 is given by

$$\mathcal{V}^* = \left\{ \begin{bmatrix} -13.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2.4 \end{bmatrix}, \begin{bmatrix} 3.0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} \right\}.$$

The method proposed in this chapter was compared with other techniques in the literature. This comparison is presented in Figure 14, where four estimates of the RAO were obtained using different methods: in black using a method for symmetric saturation from (TARBOURIECH *et al.*, 2011), where it was considered the worst case symmetric saturation (i.e. both input saturation limits μ_{min} and μ_{max} equal to 1), in red using a method suitable for asymmetric saturation from (GROFF; GOMES DA SILVA JR.; VALMORBIDA, 2019), in green using the formulation proposed for PWA systems with the implicit representation considering the original partition and, in blue, considering the refined partition with the one step ahead instance of $y(x)$ (partition depicted in Figure 12).

Figure 12 – Refined partition (green) considering one successor instance of $y(x)$.



Source: The author

The area of each estimate is, respectively, 37.3, 58.4, 68.1 and 71.2 units squared, meaning that the proposed method for PWA systems with the implicit representation and refined partition obtained an estimate of the RAO 21.9% larger, in area, than the method presented in (GROFF; GOMES DA SILVA JR.; VALMORBIDA, 2019). It is interesting to note that even considering the method proposed in this work with the original partition, the estimate of the RAO obtained is 16.6% larger than the method of (GROFF; GOMES DA SILVA JR.; VALMORBIDA, 2019).

4.4 Final Remarks

This chapter derived sufficient conditions to assess the local exponential stability of the origin of PWA systems with the implicit representation considering PWQ Lyapunov candidate functions. Those sufficient conditions were formulated as LMI constraints of a SDP problem, which allows to obtain estimates of the RAO given by sub level sets of the PWQ Lyapunov function computed as solution. Moreover, a *quasi*-convex optimization problem was formulated including additional constraints in order to compute larger estimates of the RAO.

A systematic procedure to refine the partition, that is, insert additional regions, was proposed based on future instances of vector $y(x)$. The idea is that the additional regions of the refined partition allow more flexibility for the solutions of the SDP problems. Each region of the refined partition is associated with a combination of regions in the original partition. For instance, this partition refinement in the case of feedback systems subject to input saturation is equivalent to create additional regions associated with the saturation

Table 2 – Regions of the refined partition in Figure 12 and the corresponding two steps combinations of the original partition.

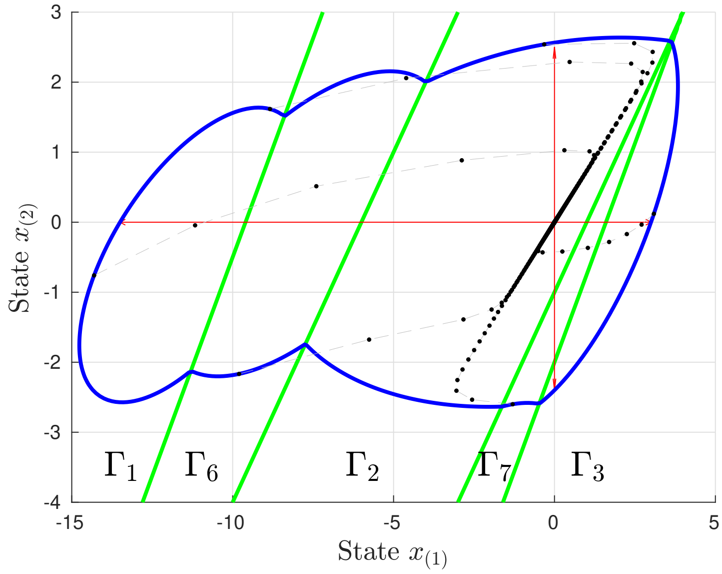
Region	$y(x)$	$y(x^+)$
Γ_1	Pos. Sat.	Pos. Sat.
Γ_2	Linear	Linear
Γ_3	Neg. Sat.	Neg. Sat.
Γ_4	Linear	Neg. Sat.
Γ_5	Linear	Pos. Sat.
Γ_6	Pos. Sat.	Linear
Γ_7	Neg. Sat.	Linear
Γ_8	Pos. Sat.	Neg. Sat.
Γ_9	Neg. Sat.	Pos. Sat.

of the current and future samples. The number of regions added in the refined partition depends on the number of future instances of $y(x)$ taken into account.

It is important to notice that the local stability analysis with the implicit PWA representation proposed in this work does not require enumeration of the regions or *a priori* knowledge of what regions are contained within the set $\mathcal{D}$.

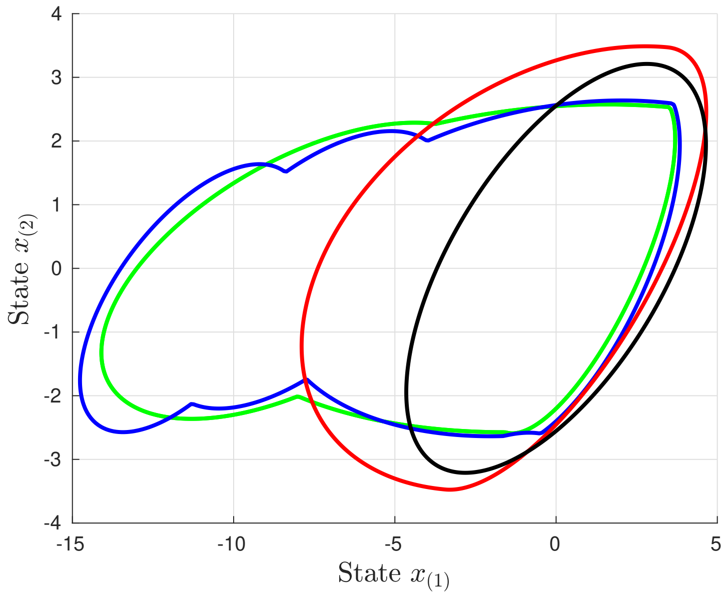
Finally, the method proposed was tested in a numerical example regarding the case of discrete-time systems subject to asymmetric input saturation. In this numerical example a larger estimate of the RAO was obtained when compared with other methods in the literature, demonstrating its usefulness. To the author's knowledge, there is no work in the literature regarding the local stability of discrete-time PWA systems with maximization of the shape of the RAO.

Figure 13 – Estimate of the RAO (blue) considering the extended partition with the successor instance of $y(x)$ (green). The optimal set of vectors $\mathcal{V}^*$ included in the estimate of the RAO are shown in red. Examples of trajectories are shown as black dots.



Source: The author

Figure 14 – Estimate of the RAO obtained by considering the worst case symmetric saturation (black), the asymmetric saturation (red), the PWA implicit representation with the original partition (green) and the PWA implicit representation with refined partition (blue).



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5 CONCLUSION

This work addressed the problems of global stabilization and local stability analysis of discrete-time continuous piecewise affine (CPWA) systems. To tackle those problems, a recently proposed implicit representation was used, since it was proved advantageous in the global stability analysis problem. Differently from the commonly used explicit representation, the novel implicit one does not require *a priori* knowledge of the possible transitions between regions in its stability analysis conditions.

Regarding the problem of global stabilization addressed in Chapter 3, a PWA state feedback control law and PWQ Lyapunov candidate functions were considered. The idea was to use the recently proposed implicit representation to avoid the nonconvexity introduced by the relaxation terms when the standard explicit representation is considered in the stabilization problem. With the implicit representation, *quasi*-LMI sufficient conditions to ensure the global exponential stability of the origin of the closed-loop PWA system were derived. Those conditions were obtained from the stability conditions presented in Chapter 2 through the use of Finsler's Lemma, congruence transformations and some structural assumptions. An algorithm based on convex optimization tools was proposed to solve the stabilization problem (i.e. compute the stabilizing gains). The method derived was tested in numerical examples and it provided a systematic approach to the stabilization problem of a class of PWA systems, which was not possible with the explicit representation. The results from this chapter were accepted for publication (CABRAL; GOMES DA SILVA JR.; VALMORBIDA, 2021).

Chapter 4 addressed the problem of local exponential stability analysis. To tackle this problem, the conditions for global stability analysis were generalized to ensure the nonnegativity of PWQ functions in a local context. This was achieved by extending the Lemma 6 regarding the nonnegativity of PWQ functions, allowing to build a locally nonnegative matrix function $\chi^T(x)M(x)\chi(x)$, that is, such that the PWQ term $\chi^T(x)M(x)\chi(x) \geq 0 \ \forall x \in \mathcal{D} \subseteq \mathbb{R}^n$. Then, this locally nonnegative PWQ term is used to assess the local exponential stability of the origin of PWA systems considering a PWQ Lyapunov candidate function through convex constraints written as LMIs. Estimates of the Region of Attraction of the Origin (RAO) are obtained as sub level sets of the Lyapunov function computed as a solution to a Semidefinite Programming (SDP) problem. Furthermore, procedures to maximize the estimate of the RAO based on additional constraints were also discussed. One of those procedures consists in a systematic method to refine the partition (i.e. include additional regions without altering the system's dynamic) based on the successor instances of vector $y(x)$. The idea is that the refined partition allows more flexible solution to the SDP problem, in the expense of numerical complexity. The proposed method was applied in the local stability analysis of a linear system subject to asymmetric saturation. The results obtained were compared with other techniques in the literature and it was observed that the method proposed in this work outperforms the other techniques when the area of the estimate of the RAO is used as a figure of merit. To the author's knowledge, there is no work in the literature regarding the local stability of discrete-time PWA systems with maximization of the shape of the RAO.

Since the implicit representation used in this work was recently proposed, PWA systems in this representation possesses several lines of research. Some of them are listed below:

- **Formulate other optimization problems for the local analysis**: it was pointed in subsection 4.2.2 that the optimization problem formulated to maximize the area of the estimate of the RAO is not a convex problem. Other optimization criteria should be investigated in order to formulate the optimization of the estimate of the RAO as a convex problem;

- **Refine the partition with backward steps**: the use of successor instances of $y(x)$ to systematically generate the refined partition is not the only procedure possible. For example, if the discrete-time system has a non-singular matrix F_1, then, it is possible to compute the system state in previous samples. Hence, the partition can be refined by also taking into account previous instances of $y(x)$, in addition to the successor instances as done in Chapter 4. The idea is that by considering the predecessor instances of $y(x)$, the refined partition has its additional regions inserted closer to the origin, which can be advantageous to estimate the RAO;

- **Create a MATLAB package to work with the implicit representation**: the creation of tools with good user interface to deal with the implicit representation of PWA systems is instrumental to make this representation available to the interested technical community, increasing the relevance of the methods proposed;

- **Continuous-time systems**: the application and extension of the results presented in this work for continuous-time PWA systems should be investigated.

- **Investigate other applications of PWA systems**: methods regarding PWA systems can be applied to problems in different areas of interest. For example, some neural networks use a model of neurons called Rectifier Linear Unit (ReLU). The ReLU activation function is the ramp function (6) depicted in Figure 3. Hence, such neural networks can be described as PWA systems and the methods proposed in this work can be applied.

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APPENDIX A LINEAR MATRIX INEQUALITIES

A linear matrix inequality (LMI) has the following form (BOYD *et al.*, 1994):

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} p_i F_i > 0$$

where $p \in \mathbb{R}^m$ is the variable and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{l \times l}$, $i = 0, ..., m$ are given. This implies that $F(x)$ is positive definite, i.e. for all $v \neq 0 \in \mathbb{R}^l$ we have $v^T F(x) v > 0$. If the inequality is nonstrict (i.e. is composed by $\geq$ instead of $>$), then this is a nonstrict LMI and the matrix $F(x)$ is called positive semidefinite, meaning that for any $v \in \mathbb{R}^l$ we have $v^T F(x) v \geq 0$.

A.1 S-Procedure

It is often necessary to constraint some quadratic function to be negative (or positive) whenever other quadratic functions are all negative (or positive) (BOYD *et al.*, 1994). In some cases, this can be expressed as an LMI in the data defining those quadratic functions, resulting in an LMI that is conservative but often an useful approximation of the original constraint. This is done by a method called S-procedure, presented below.

Let F_0, ..., F_p be quadratic functions of the variable $\zeta \in \mathbb{R}^n$:

$$F_i(\zeta) \triangleq \zeta^T T_i \zeta + 2u_i^T \zeta + v_i, \quad i = 1, ..., p$$

where $T_i = T_i^T$. Then, consider the following condition on F_0, ..., F_p:

$$F_0(\zeta) \geq 0 \quad \forall \zeta \mid F_i(\zeta) \geq 0, \quad i = 1, ..., p$$

If there exist nonnegative scalars τ_1, ..., τ_p such that

$$\forall \zeta \mid F_0(\zeta) - \sum_{i=1}^{p} \tau_i F_i(\zeta) \geq 0$$

then the original constraint holds. This derived constraint can be rewritten as

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix}.$$

A.2 Finsler's Lemma

Finsler's Lemma can be stated as

Lemma 15. *(FINSLER, 1936) Consider an euclidean space* $\mathbb{R}^k$ *(*$k \in \mathbb{N}$*) and let* $v \in \mathbb{R}^k$*,* $Q_a \in \mathbb{R}^{k \times k}$ *and* $Q_b \in \mathbb{R}^{k \times k}$*. There is* $\lambda \in \mathbb{R}$ *such that*

$$Q_a + \lambda Q_b > 0 \tag{81}$$

if and only if

$$v^T Q_a v > 0 \ \forall v \mid v^T Q_b v = 0, \ v \neq 0. \tag{82}$$

whose proof is found in (FINSLER, 1936).

Finsler's Lemma can also be stated in different forms, as stated by the next Lemma.

Lemma 16. *(OLIVEIRA; SKELTON, 2007) Consider an euclidean space* $\mathbb{R}^k$ *(*$k \in \mathbb{N}$*) and let* $v \in \mathbb{R}^k$*,* $Q_a \in \mathbb{R}^{k \times k}$*,* $Q_b \in \mathbb{R}^{m \times k}$ *such that rank*$(Q_b) < k$ *and* $\mathcal{N}\{Q_b\}$ *is a basis for the null-space of* Q_b*. The following statements are equivalent:*

$$v^T Q_a v > 0, \quad \forall Q_b v = 0, \quad v \neq 0 \tag{83a}$$

$$\mathcal{N}\{Q_b\}^T Q_a \mathcal{N}\{Q_b\} > 0 \tag{83b}$$

$$\exists \lambda \in \mathbb{R} \ \mid \ Q_a - \lambda Q_b^T Q_b > 0 \tag{83c}$$

$$\exists X \in \mathbb{R}^{k \times m} \ \mid \ Q + X Q_b + Q_b^T X^T > 0 \tag{83d}$$

The proof of Lemma 16 can be found in (OLIVEIRA; SKELTON, 2007). Those equivalent forms are useful in control problems since they allow to work with an extended state vector which includes the successor state x^+. This can be done since an additional algebraic constraint between the current state x and successor state x^+ is codified by the null space of matrix Q_b.