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Conteúdo

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A NOTE ON THE CRITICAL EXPONENT OF A SEMILINEAR EVOLUTION EQUATIONS WITH EFFECTIVE SCALE-INVARIANT TIME-DEPENDENT DISSIPATION

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Abstract

The goal of this note is to find a critical exponent for a class of a semilinear evolution equations with effective scale-invariant time-dependent dissipation

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1+t} u_t = |u|^p, \;\; u(0,x) = 0, \;\; u_t(x,0) = u_1(x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{1}$$

where $\mu > 1$, $p > 1$ and $\sigma > 1$. A exponent p_c is critical when it is possible to prove global existence (in time) of small data energy solutions to $p > p_c$ and blow up in a finite time for $1 < p \leq p_c$. To achieve this goal we use a fixed point argument in a special operator defined on a suitable function space. So we shall derive optimal $L^p - L^q$ decay estimates, $1 \leq p \leq 2 \leq q \leq \infty$, for the solutions to (1).

The critical exponent p_c for the global (in time) existence of small data solutions to the Cauchy problem (1) is related to the long time behavior of solutions, which changes accordingly with μ. In this presentation, we consider effective dissipation, this means we suppose $\mu > 1$. Under the assumption of small initial data $u_1 \in L^1 \cap L^2$, we find the critical exponent $p_F(\sigma, n) = 1 + \frac{2\sigma}{n}$, which is known as a Fujita type exponent.

1 Introduction

Let us consider the following Cauchy problem for the semilinear σ-evolution equations with scale-invariant time-dependent effective dissipation

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1+t} u_t = |u|^p, \quad u(0,x) = 0, \quad u_t(0,x) = u_1(x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{2}$$

where $\mu > 1$, $\sigma > 1$ and $p > 1$. We discuss the global (in time) existence of small data energy solutions and blow up results to (2).

It is well known that the size of the parameter μ is relevant to describe the asymptotic behavior of solutions. When $\mu > 1$, this model is related to the semilinear corresponding heat type equation. This can be explain by the diffusion phenomenon.

For $\sigma = 1$ and constant coefficient case, in [6] the authors proved global existence of small data solutions for the semilinear damped wave equation $u_{tt} - \Delta u + u_t = |u|^p$, $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$, in the supercritical range $p > 1+2/n$, by assuming small initial data with compact support from the energy space. The compact support assumption on the initial data can be weakened. By only assuming initial data in Sobolev spaces, the existence result was proved in space dimensions $n = 1$ and $n = 2$ in [4], by using energy methods, and in space dimensions $n \leq 5$ in [5], by using $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$. Nonexistence of the global small data solution is proved in [6] for $1 < p < 1+2/n$ and in [8] for $p = 1+2/n$. The exponent $p_F(n) \doteq 1+2/n$ is well known as Fujita exponent and it is the critical index for the semilinear parabolic problem [3]: $v_t - \triangle v = v^p$ $v(0,x) = v_0(x) \geq 0$. The diffusion phenomenon between linear heat and linear classical damped wave models, see [5], explains the parabolic nature of classical damped wave models with power nonlinearities from the point of view of decay estimates of solutions. In [7] the author considered a more general model $u_{tt} - \Delta u + b(t)u_t = 0$, $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$, with a class of time dependent damping $b(t)u_t$ for which the critical exponent is still Fujita exponent $1 + 2/n$ for the associate semilinear Cauchy problem with power nonlinearity $|u|^p$ (see [2] and [1]).

The main goals in this presentation are to derive $L^p - L^q$ estimates and energy estimates for solutions to the linear Cauchy problem associated to ((2)) and to obtain the critical exponent for the global (in time) existence of small initial data energy solutions.

2 Main Results

The following results show us that for $\mu > 1$ the critical exponent for the Cauchy problem (2) is given by a Fujita type exponent $p_F(\sigma, n) = 1 + \frac{2\sigma}{n}$.

Theorem 2.1. *Let $\sigma > 1$, $n < 2\sigma$ and $\mu > \max\left\{\frac{n}{\sigma} + \frac{2n}{n+2\sigma}; 1\right\}$, $\mu \neq \frac{2n}{\sigma}$ and $\mu \neq \frac{n}{\sigma} + 2$. If $1 + \frac{2\sigma}{n} < p \leq \frac{n}{[2n-\sigma\mu]_+} \doteq \frac{q_0}{2}$, then there exists $\epsilon > 0$ such that for any initial data $u_1 \in \mathcal{A} = L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, $||u_1||_{\mathcal{A}} \leq \epsilon$, there exists a unique energy solution $u \in C([0,\infty), H^\sigma(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)) \cap C^1([0,\infty), L^2(\mathbf{R}^n))$ to (2). Moreover, for $2 \leq q \leq q_0$ the solution satisfies the following estimates*

$$||u(t,\cdot)||_{L^q} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(1-\frac{1}{q}\right)} ||u_1||_{\mathcal{A}}, \quad ||u(t,\cdot)||_{L^\infty} \lesssim (1+t)^{-\min\left\{\frac{n}{2\sigma}, \frac{\mu}{2}\right\}} ||u_1||_{\mathcal{A}},$$

$$\|u(t,\cdot)\|_{\dot{H}^\sigma} + \|\partial_t u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\min\left\{\frac{n}{2\sigma}+1, \frac{\mu}{2}\right\}} ||u_1||_{\mathcal{A}}, \quad \forall t \geq 0.$$

For the sake of simplicity, in the next result we restrict our analysis for integer σ.

Theorem 2.2. *Let $\sigma \in \mathbb{N}$, $\mu > 1$ and $1 < p \leq 1 + \frac{2\sigma}{n}$. If $u_1 \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} u_1(x)\, dx > 0$, then there exists no global (in time) weak solution $u \in L^p_{loc}([0,\infty) \times \mathbb{R}^n)$ to (2).*

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