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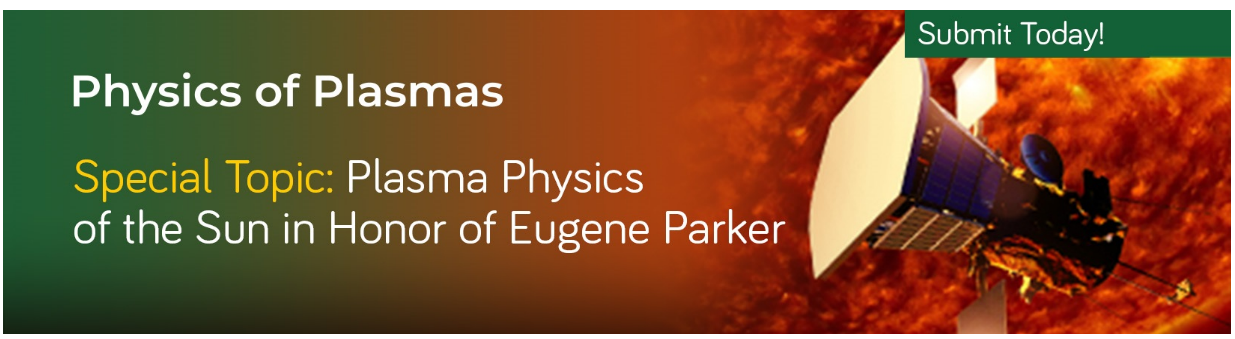

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ABSTRACT

Electrostatic waves with frequencies that are integer multiples of the electron plasma frequency have been observed since the early days of laboratory experiments on beam–plasma interactions, and also in experiments made in the space environment. These waves have also appeared in numerical experiments, and can be explained in the context of weak turbulence theory. This paper presents results obtained by numerical solution of the equations of weak turbulence theory, which show the coupled time evolution of the amplitudes of harmonic waves and of the amplitudes of Langmuir and ion acoustic waves, and the time evolution of the electron distribution function. The results are obtained considering a two-dimensional geometry, considering harmonics up to $n = 5$, and are consistent with earlier results obtained by one-dimensional analyses.

I. INTRODUCTION

It is common knowledge that two normal modes of electrostatic oscillations occur in unmagnetized plasmas. At high frequencies, there is a mode in which the electron population oscillates, while the ion population, composed of much more massive particles, can be considered at rest. These high frequency oscillations are known as electron plasma waves, or Langmuir waves. These waves can be excited by an electron beam passing through the plasma, with frequency close to the electron plasma frequency, $\omega_{pe} = (4\pi n_{e0} e^2/m_e)^{1/2}$, where n_{e0} is the electron number density at equilibrium, and e and m_e are the electron charge and mass, respectively. At low frequencies, there are oscillations in which the ion population oscillates, carrying along the electron population. Due to their much smaller inertia, the electron population is almost at equilibrium with the ion population. These oscillations bear resemblance with ion sounds in neutral gases, which is why they are known as ion acoustic waves, or ion sound waves. Langmuir waves and ion acoustic waves will be denominated as L waves and S waves, respectively, in the present paper.

However, it has been observed that plasmas can exhibit other electrostatic oscillations, which appear as high frequency waves with frequencies that are integral multiples of the electron plasma frequency. Such harmonic waves are not predicted by plasma analysis based on the fluid approach, and are also not predicted by analysis based on linear kinetic theory.

The occurrence of electrostatic harmonic waves has been observed in laboratory experiments based on beam–plasma interactions, since the 1960s.[1–5] These waves have also been observed in experiments in which electron beams have been inserted into ionospheric environments by equipment on board a rocket.[6] Moreover, computer simulations have also obtained such harmonic waves, in the recent decades.[7–15]

Electrostatic harmonics has been addressed in the context of a relatively recent formulation of weak turbulence theory (WT). The basic formulation of WT theory has been mostly developed between the decades of 1950 and 1970, with significant developments made by researchers of the former USSR. Details about the development of the early versions of WT theory have been collected in well-known textbooks.[16–25] However, the WT theory has also been rediscussed in more recent years, starting with the work of Yoon in the year 2000.[26] The formulation has been developed starting from first principles, and a variety of mechanisms have been discussed and incorporated into the set of equations of the theory. The initial step in this recent development took into account the presence of L and S modes, and also the presence of the first harmonic of L waves, with frequency close to $2\omega_{pe}$. This harmonic of the L wave, not predicted by the dispersion relation from linear theory, was obtained as an additional root by the inclusion of nonlinear terms into the dispersion relation.[26] Electrostatic harmonics of higher order, with frequencies close to integer multiples of the electric plasma frequency, $n\omega_{pe}$, have also been described, by more accurate approaches to the formulation.[15,27–29] Other developments of the theory can be mentioned, such as the incorporation of effects of single-particle fluctuations[29] and the incorporation of effects due to binary particle interactions, both in the equations for particle distributions and in the equations for the time evolution of wave amplitudes.[30,31] These developments have been hitherto restricted to an electrostatic formulation, but electromagnetic oscillations also have been included into the formalism, up until now in analysis that did not take into account the occurrence of electrostatic harmonics.[32,33] A complete formulation, which simultaneously takes into account electrostatic waves, both normal modes and nonlinear harmonics, electromagnetic waves, effects due to single-particle fluctuations, and effects due to binary collisions, has not yet been discussed and utilized in the literature.

The approach utilized in Ref. 26 was purely theoretical. As far as we are aware, the first numerical solution of the equations that appeared in the paper by Yoon (2000) was that of Ref. 34, which contained a one-dimensional (1D) solution for the time evolution of the spectra of L and S waves, and also of the harmonic $n = 2$ of the L waves. The formulation utilized in Ref. 34 did not include the effect of spontaneous fluctuations. These effects due to spontaneous fluctuations were for the first time taken into account in an analysis that included electrostatic harmonics in Ref. 35, still in the context of a 1D formulation. Another instance of numerical solution of the equations of WT theory in 1D can be found in Ref. 28, which discussed the evolution of multiple Langmuir harmonics, still without taking into account the effects due to spontaneous fluctuations.

The search for numerical solutions of the WT equations developed according to the formulation presented in Ref. 26, using a 2D geometry, can be traced back to the work by Ziebell *et al.*[36] Reference 36 presented an analysis of the evolution of electrostatic L and S modes and of the electron velocity distribution, with results obtained by taking into account quasilinear effects and the effect of nonlinear scattering, as well as the effect of spontaneous fluctuations. The possibility of occurrence of Langmuir harmonics was not taken into account. As far as we are aware, the only instance of 2D analysis of WT equations accounting for the electrostatic harmonics was that presented in Ref. 37, which discussed the beam–plasma instability taking into account the occurrence of the harmonic $n = 2$ of the Langmuir mode.

In the present paper, we further investigate the beam–plasma instability using WT theory, considering a 2D geometry and taking into account the occurrence of several electrostatic harmonics. As in Ref. 37, we do not include in the formulation the occurrence of electromagnetic waves. The justification for this simplified approach is that the nonlinear interaction between electrostatic waves and electromagnetic waves is expected to become meaningful only after significant growth of the wave amplitudes. Therefore, this interaction is not expected to affect the early evolution of electrostatic harmonic modes, which is the focus of the present work. The structure of this paper is as follows: In Sec. II, we briefly present the theoretical formulation to be utilized for the analysis. Section III presents the results of the numerical analysis. The results obtained are summarized in Sec. IV.

II. THEORETICAL FORMULATION AND NUMERICAL SETUP

Details about the development of the equations of WT theory, including those describing the evolution of electrostatic harmonics, can be found in published papers in the literature. Here, we will only reproduce the basic set of equations that will be utilized in the analysis. This set of equations has already appeared in Ref. 37, but is reproduced here for the sake of completeness. We present these equations using nondimensional variables, which are more suitable for numerical analysis, as follows:

$$w \equiv \frac{\omega}{\omega_{pe}}, \quad \tau \equiv t\omega_{pe}, \quad \mathbf{q} \equiv \frac{\mathbf{k}v_{te}}{\omega_{pe}}, \quad \mathbf{u} \equiv \frac{\mathbf{v}}{v_{te}},$$

where $v_{te} = (2T_e/m_e)^{1/2}$ is the electron thermal speed, T_e being the temperature defined in energy unit. We also utilize normalized distribution functions and wave spectra as follows:

$$\Phi_a(\mathbf{u}) = v_{te}^3 F_a(\mathbf{v}), \quad \mathcal{E}_{\mathbf{q}}^{\sigma\alpha} = \frac{(2\pi)^2 g}{m_e v_{te}^2}\frac{I_{\mathbf{k}}^{\sigma\alpha}}{\mu_{\mathbf{k}}^{\alpha}}.$$

The equations of WT theory that will be utilized are a set of coupled equations for the amplitudes of electrostatic waves (L and S waves and harmonic modes) and for the velocity distribution functions for plasma particles. Using the dimensionless variables, the equation for the time evolution of the fundamental L wave can be written as follows:

$$\begin{aligned}
\frac{\partial \mathcal{E}_{\mathbf{q}}^{\sigma L}}{\partial \tau} = & \left\{ \mu_{\mathbf{q}}^L \frac{\pi}{q^2} \int d\mathbf{u}\, \delta(\sigma w_{\mathbf{q}}^L - \mathbf{q}\cdot\mathbf{u}) \right. \\
& \left. \times \left(g\,\Phi_e(\mathbf{u}) + (\sigma w_{\mathbf{q}}^L)\, \mathbf{q}\cdot \frac{\partial \Phi_e(\mathbf{u})}{\partial \mathbf{u}}\, \mathcal{E}_{\mathbf{q}}^{\sigma L} \right) \right\}_{Lql} \\
& + \left\{ 2\sigma \mu_{\mathbf{q}}^L w_{\mathbf{q}}^L \sum_{\sigma',\sigma''=\pm1} \int d\mathbf{q}' \frac{\mu_{\mathbf{q}'}^L \mu_{\mathbf{q}-\mathbf{q}'}^S (\mathbf{q}\cdot\mathbf{q}')^2}{q^2 q'^2 |\mathbf{q}-\mathbf{q}'|^2} \right. \\
& \times \left[\sigma w_{\mathbf{q}}^L \mathcal{E}_{\mathbf{q}'}^{\sigma' L} \mathcal{E}_{\mathbf{q}-\mathbf{q}'}^{\sigma'' S} - \left(\sigma' w_{\mathbf{q}'}^L \mathcal{E}_{\mathbf{q}-\mathbf{q}'}^{\sigma'' S} + \sigma'' w_{\mathbf{q}-\mathbf{q}'}^L \mathcal{E}_{\mathbf{q}'}^{\sigma' L} \right) \mathcal{E}_{\mathbf{q}}^{\sigma L} \right] \\
& \left. \times\, \delta(\sigma w_{\mathbf{q}}^L - \sigma' w_{\mathbf{q}'}^L - \sigma'' w_{\mathbf{q}-\mathbf{q}'}^S) \right\}_{LdLS} \\
& + \left\{ \sigma w_{\mathbf{q}}^L \sum_{\sigma'} \int d\mathbf{q}' \int d\mathbf{u} \frac{\mu_{\mathbf{q}}^L \mu_{\mathbf{q}'}^L (\mathbf{q}\cdot\mathbf{q}')^2}{q^2 q'^2} \right. \\
& \times \delta\left[\sigma w_{\mathbf{q}}^L - \sigma' w_{\mathbf{q}'}^L - (\mathbf{q}-\mathbf{q}')\cdot\mathbf{u} \right] \\
& \times \left[g\left(\sigma w_{\mathbf{q}}^L \mathcal{E}_{\mathbf{q}'}^{\sigma' L} - \sigma' w_{\mathbf{q}'}^L \mathcal{E}_{\mathbf{q}}^{\sigma L} \right) [\Phi_e(\mathbf{u}) + \Phi_i(\mathbf{u})] \right. \\
& \left.\left. + \frac{m_e}{m_i} \mathcal{E}_{\mathbf{q}'}^{\sigma' L} \mathcal{E}_{\mathbf{q}}^{\sigma L} (\mathbf{q}-\mathbf{q}')\cdot \frac{\partial \Phi_i(\mathbf{u})}{\partial \mathbf{u}} \right] \right\}_{LsLL},
\end{aligned} \tag{1}$$

where

$$\mu_{\mathbf{q}}^{L}=1\,,\quad \mu_{\mathbf{q}}^{S}=\frac{q^{3}}{2^{3/2}}\sqrt{\frac{m_e}{m_i}}\left(1+\frac{3T_i}{T_e}\right)^{1/2},$$
$$g=\frac{1}{2^{3/2}\,(4\pi)^{2}\,\hat{n}\,\lambda_{De}^{3}},\quad \lambda_{De}^{2}=\frac{T_e}{4\pi\hat{n}e^{2}}=\frac{v_{te}^{2}}{2\omega_{pe}^{2}}.$$

In the above, λ_{De} is the Debye length and $1/(\hat{n}\lambda_{De}^{3})$ represents the plasma parameter.

The first term on the right-hand side of Eq. (1), denoted by the subscript Lql, describes the spontaneous and induced emission effects for the L mode (the induced emission is also known as the quasilinear effect). The second term, denoted as $LdLS$, describes the effect of three-wave decay involving L and S mode waves. The third term, denoted by $LsLL$, describes the scattering process involving L waves.

The equation describing the time evolution of the S mode is the following:

$$\begin{aligned}\frac{\partial\mathcal{E}_{\mathbf{q}}^{\sigma S}}{\partial\tau}=&\Bigg\{\mu_{\mathbf{q}}^{S}\frac{\pi}{q^{2}}\int d\mathbf{u}\,\delta(\sigma w_{\mathbf{q}}^{S}-\mathbf{q}\cdot\mathbf{u})\\&\times\left[g[\Phi_e(\mathbf{u})+\Phi_i(\mathbf{u})]+(\sigma w_{\mathbf{q}}^{L})\left(\mathbf{q}\cdot\frac{\partial\Phi_e(\mathbf{u})}{\partial\mathbf{u}}\right.\right.\\&\left.\left.+\frac{m_e}{m_i}\mathbf{q}\cdot\frac{\partial\Phi_i(\mathbf{u})}{\partial\mathbf{u}}\right)\mathcal{E}_{\mathbf{q}}^{\sigma S}\right]\Bigg\}_{Sql}\\&+\Bigg\{\sigma w_{\mathbf{q}}^{L}\sum_{\sigma',\sigma''}\int d\mathbf{q}'\,\frac{\mu_{\mathbf{q}}^{S}\mu_{\mathbf{q}'}^{L}\mu_{\mathbf{q}-\mathbf{q}'}^{L}\left[\mathbf{q}'\cdot(\mathbf{q}-\mathbf{q}')\right]^{2}}{q^{2}\,q'^{2}\,|\mathbf{q}-\mathbf{q}'|^{2}}\\&\times\left[\sigma w_{\mathbf{q}}^{L}\mathcal{E}_{\mathbf{q}'}^{\sigma'L}\mathcal{E}_{\mathbf{q}-\mathbf{q}'}^{\sigma''L}-\left(\sigma' w_{\mathbf{q}'}^{L}\mathcal{E}_{\mathbf{q}-\mathbf{q}'}^{\sigma''L}+\sigma'' w_{\mathbf{q}-\mathbf{q}'}^{L}\mathcal{E}_{\mathbf{q}'}^{\sigma'L}\right)\mathcal{E}_{\mathbf{q}}^{\sigma S}\right]\\&\times\delta(\sigma w_{\mathbf{q}}^{S}-\sigma' w_{\mathbf{q}'}^{L}-\sigma'' w_{\mathbf{q}-\mathbf{q}'}^{L})\Bigg\}_{SdLL}.\end{aligned}\tag{2}$$

The first term on the right-hand side of Eq. (2), denoted as Sql, describes the spontaneous emission and quasilinear effects. The second term, designated as $SdLL$, describes the three-wave decay process.

The equation for the time evolution of the amplitudes of the harmonics of L waves can be written as follows:[29,35]

$$\frac{\partial\mathcal{E}_{\mathbf{q}}^{Ln}}{\partial\tau}=\frac{\gamma_{\mathbf{q}}^{Ln}+\nu_{\mathbf{q}}^{Ln}}{1+\eta_{\mathbf{q}}^{Ln}}\mathcal{E}_{\mathbf{q}}^{Ln},\tag{3}$$

where

$$\gamma_{\mathbf{q}}^{Ln}=n^{2}\frac{\pi}{q^{2}}\int d\mathbf{u}\,w_{\mathbf{q}}^{Ln}\,\mathbf{q}\cdot\frac{\partial\Phi(\mathbf{u})}{\partial\mathbf{u}}\,\delta(\sigma w_{\mathbf{q}}^{Ln}-\mathbf{q}\cdot\mathbf{u}),\tag{4}$$

$$\begin{aligned}\nu_{\mathbf{q}}^{Ln}=&\,n^{3}\int d\mathbf{q}'\,\frac{a_{\mathbf{q},\mathbf{q}'}^{n}\,\mu_{\mathbf{q}'}^{L(n-1)}\left(w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}\right)}{|\epsilon(\mathbf{q}-\mathbf{q}',w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)})|^{2}}\\&\times\mathcal{E}_{\mathbf{q}'}^{L(n-1)}\int d\mathbf{u}\,\frac{\mathbf{q}-\mathbf{q}'}{|\mathbf{q}-\mathbf{q}'|^{2}}\cdot\frac{\partial\Phi_e}{\partial\mathbf{u}}\\&\times\delta\left[w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}-(\mathbf{q}-\mathbf{q}')\cdot\mathbf{u}\right],\end{aligned}\tag{5}$$

$$\begin{aligned}\eta_{\mathbf{q}}^{Ln}=&\,\frac{n^{3}}{\pi}\int d\mathbf{q}'\,a_{\mathbf{q},\mathbf{q}'}^{n}\,\mu_{\mathbf{q}}^{\prime L(n-1)}\,\mathcal{E}_{\mathbf{q}'}^{L(n-1)}\\&\times\mathrm{Re}\left[\epsilon^{-2}(\mathbf{q}-\mathbf{q}',w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)})\right],\end{aligned}\tag{6}$$

with the following auxiliary expressions:

$$\begin{aligned}a_{\mathbf{q},\mathbf{q}'}^{n}=&\left\{(n-1)q^{2}\left[\mathbf{q}'\cdot(\mathbf{q}-\mathbf{q}')\right]+nq'^{2}\left[\mathbf{q}\cdot(\mathbf{q}-\mathbf{q}')\right]\right.\\&\left.+n(n-1)|\mathbf{q}-\mathbf{q}'|^{2}(\mathbf{q}\cdot\mathbf{q}')\right\}^{2}\times\left[n^{2}(n-1)^{2}qq'|\mathbf{q}-\mathbf{q}'|\right]^{-2},\end{aligned}\tag{7}$$

$$\begin{aligned}&\left|\epsilon\left(\mathbf{q}-\mathbf{q}',w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}\right)\right|^{2}\\&=4\left[\left(w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}-w_{\mathbf{q}-\mathbf{q}'}^{L1}\right)^{2}+\pi\xi_{\mathbf{q}-\mathbf{q}'}^{6}e^{-2\xi_{\mathbf{q}-\mathbf{q}'}^{2}}\right],\quad \xi_{\mathbf{q}-\mathbf{q}'}=\frac{w_{\mathbf{q}-\mathbf{q}'}^{L1}}{|\mathbf{q}-\mathbf{q}'|},\end{aligned}\tag{8}$$

$$\mathrm{Re}\,\epsilon^{-2}\left(\mathbf{q}-\mathbf{q}',w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}\right)=\frac{\left(w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}-w_{\mathbf{q}-\mathbf{q}'}^{L1}\right)^{2}}{\left|\epsilon\left(\mathbf{q}-\mathbf{q}',w_{\mathbf{q}}^{Ln}-w_{\mathbf{q}'}^{L(n-1)}\right)\right|^{2}}.\tag{9}$$

In Eq. (3), the term with $\gamma_{\mathbf{q}}^{Ln}$ represents the quasilinear effect and the term with $\nu_{\mathbf{q}}^{Ln}$ represents the effect of wave–particle scattering.[29,35]

The equations for the time evolution of velocity distribution functions for the particles ($a=e$ for electrons and $a=i$ for the ions) can be written as follows:

$$\begin{aligned}\frac{\partial\Phi_a(\mathbf{u})}{\partial\tau}=&\,\frac{e_a^{2}}{e^{2}}\frac{m_e^{2}}{m_a^{2}}\sum_{\sigma}\sum_{\alpha=L,S}\int d\mathbf{q}\left(\frac{\mathbf{q}}{q}\cdot\frac{\partial}{\partial\mathbf{u}}\right)\\&\times\mu_{\mathbf{q}}^{\alpha}\,\delta(\sigma w_{\mathbf{q}}^{\alpha}-\mathbf{q}\cdot\mathbf{u})\left(g\frac{m_a}{m_e}\frac{\sigma w_{\mathbf{q}}^{L}}{q}\Phi_a(\mathbf{u})+\mathcal{E}_{\mathbf{q}}^{\sigma\alpha}\frac{\mathbf{q}}{q}\cdot\frac{\partial\Phi_a(\mathbf{u})}{\partial\mathbf{u}}\right).\end{aligned}\tag{10}$$

The first term on the right-hand side describes the effects of spontaneous fluctuations, and the term with the velocity derivative describes the process of quasilinear diffusion.

The dispersion relations for plasma normal modes L and S in terms of nondimensional variables are given by the following expressions:

$$w_{\mathbf{q}}^{L}=\left(1+\frac{3}{2}q^{2}\right)^{1/2},\quad w_{\mathbf{q}}^{S}=\frac{qA}{(1+q^{2}/2)^{1/2}},\tag{11}$$

where

$$A=\frac{1}{\sqrt{2}}\left(\frac{m_e}{m_i}\right)^{1/2}\left(1+\frac{3T_i}{T_e}\right)^{1/2}.$$

For the harmonic waves, we utilize an approximate form of the dispersion relation as follows:

$$\begin{aligned}w_{\mathbf{q}}^{L_n}=&\,n+\frac{3}{4}\left\{q_\perp^{2}+[q_z-(n-1)q_0]^{2}\right\}\\&+\frac{3}{4}(n-1)q_0^{2}+\varepsilon_{\mathbf{k}}^{(n)},\end{aligned}\tag{12}$$

where $q_0=v_{te}/v_f=1/u_f$, with $v_{te}=(2T_e/m_e)^{1/2}$ being the thermal speed of the background electrons and v_f being the drift velocity of a beam of electrons moving through the plasma.

For more details on the derivation of the above equations of WT theory, the reader is referred to Refs. 29, 32, and 33. For details on the derivation of the approximate form of the dispersion relation, Eq. (12), the reader is referred to Appendix.

III. NUMERICAL ANALYSIS

The objective of this paper is to investigate situations in which the ions are considered stationary, and electrons as well as the waves evolve in time. The ion distribution in 2D velocity space in dimensionless form is assumed to be

$$\Phi_i(\mathbf{u}) = \frac{1}{\pi}\frac{T_e}{T_i}\frac{m_i}{m_e}\exp\left(-\frac{m_i}{m_e}\frac{T_e}{T_i}u^2\right). \tag{13}$$

The initial electron distribution function is assumed to be made of a Maxwellian background population and a forward-propagating beam component, with number density assigned by n_f. In 2D and using dimensionless variables, the electron distribution is given as follows:

$$\Phi_e(\mathbf{u},0) = \frac{1}{\pi}\left(1-\frac{n_f}{n_0}\right)\exp\left[-u_\perp^2 - \left(u_\parallel - \frac{v_M}{v_{te}}\right)^2\right] + \frac{1}{\pi}\frac{n_f}{n_0}\frac{T_e}{T_f}\exp\left\{-\frac{T_e}{T_f}\left[u_\perp^2 + \left(u_\parallel - \frac{v_f}{v_{tf}}\right)^2\right]\right\}, \tag{14}$$

where $u_\parallel$ and $u_\perp$ are the components of the normalized velocity along the direction of the beam and perpendicular to the direction of the beam, respectively. $v_{tf} = (2T_f/m_e)^{1/2}$ is the forward-beam thermal speed and v_M is the drift velocity associated with the background electrons. The drift velocity for the background v_M is chosen in such a way that it guarantees zero net drift velocity for the total electron distribution, i.e., $v_M = -(n_f v_f)/(n_0 - n_f)$.

The initial spectra of the normal modes are given by the following expressions, obtained from the balance between quasilinear and spontaneous effects, assuming the initial plasmas at thermodynamic equilibrium:

$$\mathcal{E}^{\sigma L}_{\mathbf{q}}(0) = \frac{g}{2\,(w^L_{\mathbf{q}})^2},$$

$$\mathcal{E}^{\sigma S}_{\mathbf{q}}(0) = \frac{g}{2w^L_{\mathbf{q}}\,w^S_{\mathbf{q}}}\frac{\exp\left(-\zeta^2_{\mathbf{q}}\right) + \rho^{1/2}\exp\left(-\rho\,\zeta^2_{\mathbf{q}}\right)}{\exp\left(-\zeta^2_{\mathbf{q}}\right) + (T_e/T_i)\,\rho^{1/2}\exp\left(-\rho\,\zeta^2_{\mathbf{q}}\right)}, \tag{15}$$

$$\zeta^2_{\mathbf{q}} = \frac{(w^S_{\mathbf{q}})^2}{q^2}, \quad \rho = \frac{m_i}{m_e}\frac{T_e}{T_i}.$$

For the initial spectra of the harmonics of Langmuir waves, we assume arbitrary levels, following the procedure adopted in Ref. 28. It is assumed that the initial level of each harmonic n is described by a Gaussian expression, centered at $q \simeq nq^{L1}$, with the width in normalized wavenumber space given by Δ. The quantity q^{L1} is the normalized wavenumber where the fundamental L mode is located,

$$I^{Ln}(q) = \frac{I^n}{\sqrt{\pi}\Delta}\exp\left(-\frac{(q-nq^{L1})^2}{\Delta^2}\right), \tag{16}$$

where I^n is given by $I^n = I^1 e^{-\beta(b-1)} n^{-\alpha}$, and I^1, α, and β are constants that can be chosen arbitrarily. The combination of an exponential function and a power law expression allows for a variety of relative amplitudes between the initial spectra of the harmonic modes. Of course, in the normalized equations, we utilize the normalized spectra, given by

$$\mathcal{E}^{\sigma Ln}_{\mathbf{q}} = \frac{(2\pi)^2 g}{m_e v_{te}^2}\frac{I^{Ln}_{\mathbf{k}}}{\mu^\alpha_{\mathbf{k}}}, \tag{17}$$

where $\mu^{Ln}_{\mathbf{k}} = 1$.

Equations (1)–(3) for the waves (L, S, and Ln), and Eq. (10) for the electrons, are solved in the 2D wavenumber space and 2D velocity space, by employing a splitting method with fixed time step for the evolution of the distribution and a Runge–Kutta method with the same fixed time step for the wave equations. The ion distribution is assumed to be fixed during the time evolution of the system.

For all the numerical examples to be discussed subsequently, we use the normalized time interval $\Delta\tau = 0.1$. We employ 41×41 grids for the components of the normalized wave vector that are perpendicular and parallel to the direction of the beam, $q_\perp$ and $q_\parallel$, respectively, with $0 < q_\perp = k_\perp v_{te}/\omega_{pe} < 2.0$, and $0 < q_\parallel = k_\parallel v_{te}/\omega_{pe} < 2.0$. For the velocities, we use a 53×107 grid for the $(u_\perp, u_\parallel) = (v_\perp/v_{te}, v_\parallel/v_{te})$ space, covering the velocity range $0 < u_\perp = v_\perp/v_{te} < 12$ and $-12 < u_\parallel = v_\parallel/v_{te} < 12$. For subsequent numerical solutions, we assume the plasma parameter given by $(\hat{n}\lambda^3_{De})^{-1} = 5.0\times 10^{-3}$, and assume that the beam velocity is $v_f/v_{te} = 5.0$, with beam temperature given by $T_f/T_e = 1.0$ and ratio of electron and ion temperature $T_e/T_i = 7.0$. The relative density of the beam is assumed to be $n_f/n_e = 2.0\times 10^{-4}$.

For the definition of the arbitrary initial level of the harmonic waves, given by Eq. (16), we utilize the following parameters:

$$\alpha = 4, \quad \beta = 5, \quad \Delta = 0.2, \quad I^1 = 2.0\times 10^{-5}. \tag{18}$$

These values are not the same as those used in Ref. 28, where the initial levels are also given by Eq. (16). The values used in the present paper were chosen in order that the time evolution of the harmonics in 2D is similar to the evolution seen in numerical simulations and in solutions obtained with the 1D formulation.[15,28,35]

In the numerical analysis leading to the results that are shown in the ensuing figures, we have used Eq. (10) for the electron distribution, Eq. (2) including only the term *Sql*, Eq. (1) including the terms *Lql* and *LsLS*, and Eq. (3) neglecting the influence of the term $\nu^{Ln}_{\mathbf{q}}$, which represents the effect of wave–particle scattering.

Figure 1 shows the electron velocity distribution as a function of the normalized parallel and perpendicular velocities ($u_\parallel$ and $u_\perp$, respectively). Figure 1(a) shows the distribution at $\tau = 0$, with the presence of the beam added to the background plasma. Figure 1(b) shows the distribution at $\tau = 2000$. It is seen that there is a plateau already clearly formed in the region between the beam and the background population. That means that by the time $\tau = 2000$, the positive derivatives in the velocity distribution have already vanished, and the quasilinear growth of Langmuir modes, fundamental and harmonics, no longer occurs.

Figures 2 and 3 depict the normalized intensity of the spectra of Ln waves, in logarithmic scale, as a function of the components of the normalized momentum, $q_\parallel$ and $q_\perp$. Figure 2(a) shows the spectra at normalized time $\tau = 500$. In that figure, it is possible to see the growth of the fundamental L wave, at $q_\parallel \simeq 0.2$. In Fig. 2(b), the wave spectra are shown at $\tau = 1000$, and it is possible to observe the growth of the L mode, in comparison with Fig. 2(a), and also observe the appearance of the first harmonic, at $q_\parallel \simeq 0.4$.

In Fig. 3(a), one sees the spectra of waves for $\tau = 1500$. The figure shows the presence of the fundamental mode and of all harmonics that

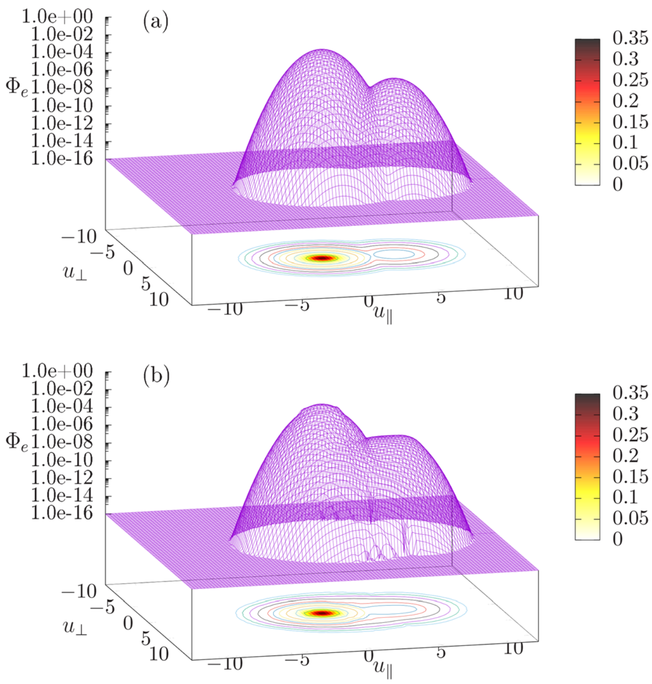

FIG. 1. Normalized electron distribution function vs $u_\perp = v_\perp/v_e$ and $u_\parallel = u_\parallel/v_e$, obtained taking into account the evolution of the fundamental mode and the following four harmonics of Langmuir waves. (a) Initial distribution ($\tau = 0$); (b) distribution at $\tau = 2000$.

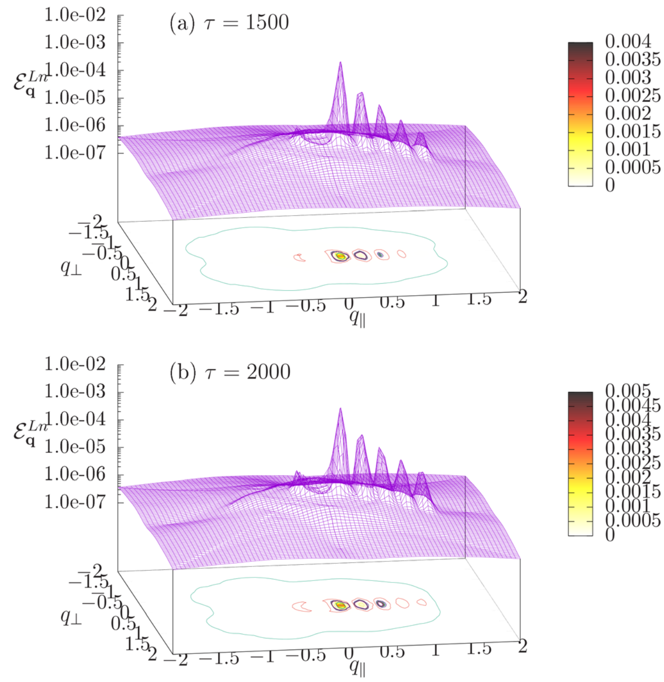

FIG. 3. Normalized wave intensity $\mathcal{E}^{Ln}$ vs $q_\perp = k_\perp v_{te}/\omega_{pe}$ and $q_\parallel = k_\parallel v_{te}/\omega_{pe}$, at normalized times (a) $\tau = 1500$ and (b) $\tau = 2000$. Results obtained taking into account scattering effects and spontaneous and induced emissions, for $n = 1$, and induced emission for harmonics $n \geq 2$.

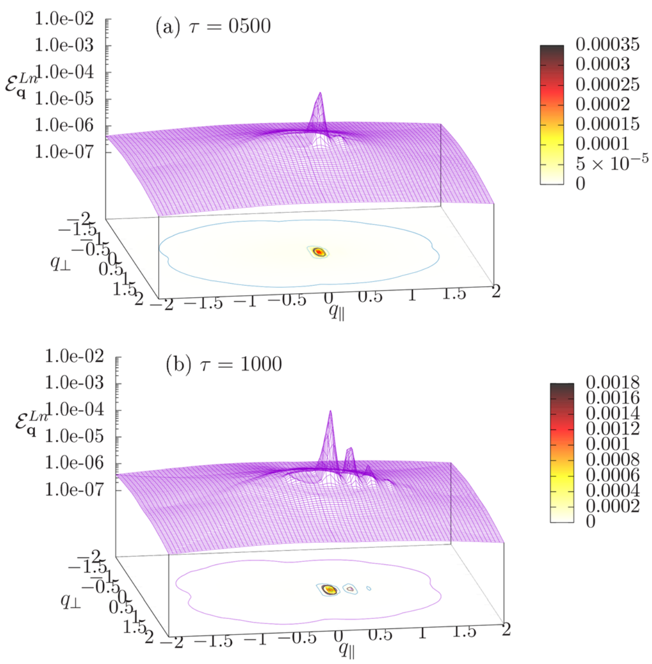

FIG. 2. Normalized wave intensity $\mathcal{E}^{Ln}$ vs $q_\perp = k_\perp v_{te}/\omega_{pe}$ and $q_\parallel = k_\parallel v_{te}/\omega_{pe}$, at normalized times (a) $\tau = 500$ and (b) $\tau = 1000$. Results obtained taking into account scattering effects and spontaneous and induced emissions, for $n = 1$, and induced emission for harmonics $n \geq 2$.

have been considered in the analysis, that is the peaks corresponding to $n = 2$, 3, 4, and 5. One can also see the appearance of the peak corresponding to backward-propagating Langmuir waves, which we denote as L'. These backward waves occur due to scattering of waves in the fundamental mode. In Fig. 3(b), one sees the wave spectra at $\tau = 2000$. By comparison with Fig. 3(a), it is noticed that the fundamental peak and the harmonic peaks are already at the saturation level. The L' peak representing the backward waves is more prominent in Fig. 3(b) than in Fig. 3(a), and a ring-like structure connecting the forward and the backward peaks is also more noticeable in Fig. 3(b). Both the backward peak and the ring-like structure are due to the effect of the scattering of the fundamental mode. It is seen that the harmonic peaks are located at positions such $q_\parallel \simeq n q_\parallel^{L1}$, as expected.

Another view of the wave spectra is seen in Fig. 4, which shows contour plots of the wave spectra at $\tau = 2000$, in the plane formed by coordinates $q_\parallel$ and $q_\perp$. Figure 4 represents the projection of Fig. 3(b), and shows the fundamental and the harmonic modes, up to $n = 5$, as well as the intensity of the peak corresponding to backward waves. As pointed out in connection with Fig. 3(b), the peaks appearing in the wave spectra are already at the amplitude of saturation. The level curves indicate that the intensities $\mathcal{E}^{Ln}_{\mathbf{q}}$ decrease with the value of n, displaying a behavior similar to that observed in previous 1D analyses.[28,35] It is also seen that the width of each harmonic peak along the $q_\perp$ coordinate decreases with increase in the harmonic number n.

In Fig. 5, we show the maximum value of intensity of the peaks corresponding to each harmonic, in logarithmic scale, as a function of normalized time, from $\tau = 0$ up to $\tau = 2000$. Figure 5 shows the rapid initial growth of the harmonic modes, which occurs due to resonance

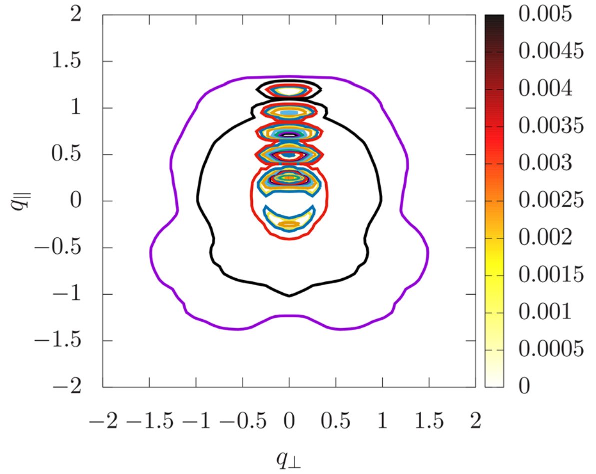

FIG. 4. Surface projection of the normalized wave intensity $\mathcal{E}^{Ln}$, in the plane formed by coordinates $q_\perp = k_\perp v_{te}/\omega_{pe}$ and $q_\parallel = k_\parallel v_{te}/\omega_{pe}$, at $\tau = 2000$. Results obtained taking into account scattering effects and spontaneous and induced emissions, for $n = 1$, and induced emission for harmonics $n \geq 2$.

in the region of velocity space where there is a positive derivative of the distribution function. For all modes, the growth becomes slower along the time evolution, due to the formation of a plateau in the electron distribution function. Figure 5 shows that near $\tau = 2000$, the harmonic modes are already near saturation, as already indicated by other figures. It is observed in Fig. 5 that the rate of initial growth increases with n, in agreement with the dependence on n that is seen in the expression for the growth rate of the harmonic modes, Eq. (4). If the results depicted in Fig. 5 are compared with those shown in Fig. 1 of Ref. 28, it is seen that the behavior obtained is similar, although in our 2D analysis we had to use parameters α, β, and I^1 of Eq. (16) with values different from those used in the 1D analysis made in Ref. 28.

It may be mentioned that, if the parameters α, β, and I^1 are used with the same values as those used in the 1D approach of Ref. 28, we

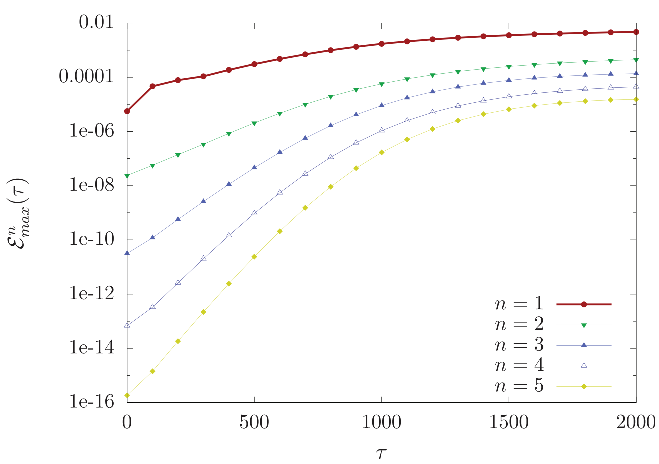

FIG. 5. Maximum value of the normalized wave intensity $\mathcal{E}^{Ln}_{max}(\tau)$, for the modes characterized by $n = \{1, 2, 3, 4, 5\}$, as a function of normalized time τ, $0 \leq \tau \leq 2000$. Initial values are given by Eq. (15) for $n = 1$ and Eqs. (16) and (17) for $n > 1$.

obtain for the maximum value of the amplitudes curves that cross each other with the evolution of time. That is, with those values the results obtained display maximum intensities of the harmonics that are crescent with n, instead of decreasing with n, as in Fig. 5. These results indicate that the question relative to the initial level of the harmonic modes is still an unresolved issue, which deserves further investigation.

IV. SUMMARY

In the present paper, we have utilized a set of coupled equations, derived in the context of the weak turbulence theory, in order to discuss the time evolution of the spectra of waves that are harmonics of Langmuir waves, coupled to the evolution of Langmuir and ion acoustic waves and to the evolution of the electron velocity distribution. The analysis has been made considering two-dimensional geometry. The harmonic waves are not normal modes of the plasma, described by the dispersion relation derived by linear theory, but can be described by a dispersion relation that incorporates nonlinear effects. In the analysis presented in this paper, we have considered harmonics up to $n = 5$, assuming arbitrary initial levels, and obtained results that are consistent with results obtained in 1D analysis that can be found in the literature. The results obtained also can be qualitatively compared with simulation studies performed by Rhee *et al.*[13] In that work, the authors performed a full electromagnetic simulation of the dynamical evolution of a beam–plasma system. In spite of the fact that in our work we have considered only the evolution of the electrostatic modes, thereby not including the nonlinear interaction with electromagnetic waves, our findings can still be compared with the full electromagnetic simulation. For example, Fig. 2(a) of Ref. 13 shows electrostatic emissions up to the second harmonic that closely follow the theoretical dispersion relations we are employing in our work. One of the reasons why Rhee *et al.* were not able to detect emissions at higher harmonics of ω_{pe} can be related to the finite time resolution invariably associated with numerical simulations. Such limitations can artificially restrict the generation of higher frequency modes. On the other hand, our numerical solutions do not suffer from such constraints and we are thus able to reproduce an arbitrary number of harmonic eigenmodes. Our results can also be qualitatively compared with Fig. 5(a) of Rhee *et al.*[13] In this figure, the simulation results show that the harmonic modes underwent an exponential growth during the linear phase of the evolution, with a subsequent saturation at later times. These findings are in qualitative agreement with our Fig. 5. Notwithstanding the differences between both approaches, we conclude that our results are in general agreement with the numerical experiments performed by Rhee *et al.*

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: APPROXIMATED DISPERSION RELATION FOR ELECTROSTATIC HARMONICS

The dispersion relation for the Langmuir harmonic modes can be written as follows:[27,29]

$$\omega_{\mathbf{k}}^{Ln} = \omega_{pe}\left(n + \varepsilon_{\mathbf{k}}^{(n)} + \frac{3}{2}k^2\lambda_{De}^2 + 3\frac{\theta_{\mathbf{k}}^{(n)}}{\varepsilon_{\mathbf{k}}^{(n)}}\lambda_{De}^2\right), \tag{A1}$$

where

$$\begin{aligned}\varepsilon_{\mathbf{k}}^{(n)} &= \frac{n^2}{2(n^2-1)}\frac{e^2}{m_e^2\omega_{pe}^4}\int d\mathbf{k}'a_{\mathbf{k},\mathbf{k}'}^{(n)}I^{L(n-1)}(\mathbf{k}'),\\ \theta_{\mathbf{k}}^{(n)} &= \frac{n^2}{2(n^2-1)}\frac{e^2}{m_e^2\omega_{pe}^4}\int d\mathbf{k}'a_{\mathbf{k},\mathbf{k}'}^{n}I^{L(n-1)}(\mathbf{k}')\\ &\quad\times\left(k\prime^2 - \mathbf{k}\cdot\mathbf{k}' + \frac{\theta_{\mathbf{k}'}^{(n-1)}}{\varepsilon_{\mathbf{k}'}^{(n-1)}}\right),\end{aligned} \tag{A2}$$

with

$$\begin{aligned}a_{\mathbf{k},\mathbf{k}'}^{n} &= \{(n-1)k^2[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')] + nk'^2[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]\\ &\quad + n(n-1)|\mathbf{k}-\mathbf{k}'|^2(\mathbf{k}\cdot\mathbf{k}')\}^2\\ &\quad\times\left[n^2(n-1)^2kk'|\mathbf{k}-\mathbf{k}'|\right]^{-2}.\end{aligned} \tag{A3}$$

For evaluation of the integrals that appear in Eqs. (A2), we write the quantity $a_{\mathbf{k},\mathbf{k}'}$ in a different form. We start by writing $\mathbf{k}'$ as a function of the components parallel and perpendicular to $\mathbf{k}$ as follows:

$$\mathbf{k}' = k'_{\parallel}\hat{\mathbf{k}} + \mathbf{k}'_{\perp k} = \frac{\mathbf{k}\cdot\mathbf{k}'}{k}\hat{\mathbf{k}} + \frac{\mathbf{k}\times(\mathbf{k}'\times\mathbf{k})}{k^2}. \tag{A4}$$

Using Eq. (A4), $a_{\mathbf{k},\mathbf{k}'}^{(n)}$ can be written as follows:

$$a_{\mathbf{k},\mathbf{k}'}^{(n)} = \frac{\left\{kk'^2 + \left[(n^2-1)k^2 + n(n-2)k'^2 - 2n(n-1)kk'_{\parallel}\right]k'_{\parallel}\right\}^2}{n^4(n-1)^4\left(k^2 - k'^2 - 2kk'_{\parallel}\right)k'^2}.$$

Regarding the intensity of the harmonic waves, which appear in the integrand of Eqs. (A2), we follow Ref. 28 and utilize the following argument, which applies to the case of beam–plasma instability: the growth of Langmuir waves occurs for $k_0 \approx \omega_{pe}/v_f$, where k_0 is the wavelength of the wave resonating with the beam particles, with velocity near v_f. In this case, the wave intensities of the fundamental and harmonic Langmuir waves can be written as follows:

$$I_{\mathbf{k}}^{L} = \frac{I^L}{\pi^{3/2}\delta^3}e^{-(k_z-k_0)^2/\delta^2}e^{-k_\perp^2/\delta^2}, \tag{A5}$$

where I^L is the intensity of the waves for unit volume; δ is the width of the wave spectrum; and k_z and $k_\perp$ are, respectively, the component of the wavevector along the z axis and the modulus of the component perpendicular to the z axis. This expression can be further approximated, considering very narrow spectrum, as follows:

$$\begin{aligned}\frac{e^{-(k_z-k_0)^2/\delta^2}}{\pi^{1/2}\delta} &\simeq \delta(k_z-k_0),\\ k_\perp\frac{e^{-k_\perp^2/\delta}}{\pi\delta^2} &\simeq \delta(k_\perp).\end{aligned}$$

Using these approximations in the equation for $\varepsilon_{\mathbf{k}}^{(n)}$, which is the first of Eqs. (A2), we obtain, for the case $n = 2$,

$$\begin{aligned}\varepsilon_{\mathbf{k}}^{(2)} &\approx \frac{\pi}{12}\frac{I^Le^2}{m_e^2\omega_{pe}^4}\int_{-\infty}^{\infty}dk'_z\int_0^{\infty}dk'_\perp\\ &\quad\times\frac{\left[kk\prime^2 + (3k^2 - 4kk'_{\parallel})k'_{\parallel}\right]^2}{(k^2 + k'^2 - 2kk'_{\parallel})k'^2}\delta(k'_z-k_0)\delta(k'_\perp).\end{aligned} \tag{A6}$$

The wave vectors can be written in terms of components parallel and perpendicular to the z axis as follows:

$$\begin{aligned}\boldsymbol{k} &= k_\perp(\cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}) + k_z\hat{\boldsymbol{z}},\\ \boldsymbol{k}' &= k_\perp{}'(\cos\phi'\hat{\boldsymbol{x}} + \sin\phi'\hat{\boldsymbol{y}}) + k_z{}'\hat{\boldsymbol{z}},\\ k_\parallel{}' &= \frac{k_\perp k_\perp{}'}{k}\cos(\phi-\phi') + \frac{k_zk_z{}'}{k}.\end{aligned}$$

Moreover, the component of $\mathbf{k}'$ perpendicular to $\mathbf{k}$ satisfies the following expression:

$$\begin{aligned}k^2\mathbf{k}'_{\perp k} &= \left[k_z(k_zk'_\perp\cos\phi' - k_\perp k'_z\cos\phi) + k_\perp^2k'_\perp\sin(\phi-\phi')\sin\phi\right]\hat{\mathbf{x}}\\ &\quad + \left[k_z(k_zk'_\perp\sin\phi' - k_\perp k'_z\sin\phi) - k_\perp^2k'_\perp\sin(\phi-\phi')\cos\phi\right]\hat{\mathbf{y}}\\ &\quad + k_\perp\left[k_\perp k'_z - k_zk'_\perp\cos(\phi-\phi')\right]\hat{\mathbf{z}}.\end{aligned}$$

Taking into account the conditions $\delta(k'_z-k_0)\delta(k'_\perp)$, we obtain

$$\begin{aligned}\mathbf{k}'_{\perp k} &\to -\frac{k_zk_\perp k_0}{k_2}(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) + \frac{k_\perp^2k_0}{k^2}\hat{\mathbf{z}},\\ k\prime^2 &\to k_0^2, \quad k'_\parallel \to \frac{k_z}{k}k_0.\end{aligned}$$

Using these results in the integrals appearing in the first of Eqs. (A6), and performing the integrations, we obtain

$$\varepsilon_{\mathbf{k}}^{(2)} \approx \frac{\pi}{12}\frac{e^2I^L}{m_e^2\omega_{pe}^4}\frac{\left[kk_0 + (3k^2 - 4k_zk_0)\frac{k_z}{k}\right]^2}{k^2 + k_0^2 - 2k_zk_0}. \tag{A7}$$

Similar approximations can be made in the evaluation of $\theta_{\mathbf{k}}^{(2)}$. Using the second equation in Eqs. (A2), with $\theta^{(1)} = 0$, one obtains

$$\begin{aligned}\theta_{\mathbf{k}}^{(2)} &\approx \frac{\pi}{12}\frac{e^2I^L}{m_e^2\omega_{pe}^4}\int_{-\infty}^{\infty}dk'_z\int_0^{\infty}dk'_\perp\frac{\left[kk'^2 + (3k^2 - 4kk'_\parallel)k'_\parallel\right]^2}{(k^2 + k'^2 - 2kk'_\parallel)k'^2}\\ &\quad\times\left(k'^2 - kk'_\parallel\right)\delta(k'_z-k_0)\delta(k'_\perp),\end{aligned}$$

which after integration leads to

$$\theta_{\mathbf{k}}^{(2)} = \frac{\pi}{12}\frac{e^2I^L}{m_e^2\omega_{pe}^4}\frac{\left[k^2k_0 + (3k^2 - 4k_zk_0)k\right]^2}{k^2\left(k^2 + k_0^2 - 2k_zk_0\right)}\left(k_0^2 - k_zk_0\right). \tag{A8}$$

Using Eqs. (A7) and (A8), the following relationship is obtained:

$$\theta_{\mathbf{k}}^2 = \varepsilon_{\mathbf{k}}^{(2)}\left(k_0^2 - k_z k_0\right).$$

Using these results into Eq. (A1), the dispersion relation for the case $n = 2$ can be written as follows:

$$\frac{\omega^{L2}}{\omega_{pe}} = 2 + \frac{3}{2}\lambda_{De}^2\left[k_\perp^2 + (k_z - k_0)^2 + k_0^2\right] + \varepsilon_{\mathbf{k}}^{(2)}.$$

We can write Eq. (A7) as a function of the plasma parameter g, the volume of the Debye sphere $V_{De} = \frac{4\pi}{3}\lambda_{De}^3$, and the electron temperature T_e as follows:

$$\varepsilon_{\mathbf{k}}^{(2)} = \frac{\pi}{12}\frac{e^2 I^L}{m_e^2\omega_{pe}^4}\frac{\left[k^2 k_0 + (3k^2 - 4k_z k_0)k\right]^2}{k^2\left(k^2 + k_0^2 - 2k_z k_0\right)} = \frac{\pi}{\sqrt{2}}g\frac{V_{De}I^L}{T_e}\lambda_{De}^2\frac{\left[k^2 k_0 + (3k^2 - 4k_z k_0)k\right]^2}{k^2\left(k^2 + k_0^2 - 2k_z k_0\right)}.$$

It is seen that, if $g \ll 1$ and $I^L V_{De} \ll T_e$, the quantity $\varepsilon_{\mathbf{k}}^{(2)}$ is very small and can be neglected in the dispersion relation.

The analysis that has been made for the case $n = 2$, can be applied to Langmuir harmonic modes of higher order, with the wave intensities written as follows:

$$I_{\mathbf{k}}^{Ln} = \frac{I^{Ln}}{\pi^{3/2}\delta^3}e^{-(k_z - nk_0)^2/\delta^2}e^{-k_\perp^2/\delta^2}.$$

Therefore, considering very narrow spectra,

$$\varepsilon_{\mathbf{k}}^{(n)} \approx \frac{(n^2-1)^{-1}}{\pi n^2(n^2-1)}\frac{e^2 I^{Ln}}{m_e^2\omega_{pe}^4}\int_{-\infty}^{\infty}dk'_z\int_0^{\infty}dk'_\perp \times\left[\delta\left(k'_z - (n-1)k_0\right)\delta(k'_\perp) \times\frac{\left\{kk'^2 + \left[(n^2-1)k^2 + n(n-2)k'^2 - 2n(n-1)kk'_\parallel\right]k'_\parallel\right\}^2}{n^4(n-1)^4\left(k^2 - k'^2 - 2kk'_\parallel\right)k'^2}\right].$$

Using the delta functions, the quantities k'^2 and $k'_\parallel$ can be evaluated as $k'^2 = (n-1)^2 k_0^2$, $k'_\parallel = (n-1)\frac{k_z}{k}k_0$, and $\varepsilon_{\mathbf{k}}^{(n)}$ becomes the following:

$$\varepsilon_{\mathbf{k}}^{(n)} = \frac{(n^2-1)^{-1}}{\pi n^2(n-1)^2}\frac{I^{L(n-1)}e^2}{m_e^2\omega_{pe}^4}\frac{\left\{k^2k_0 + \left[(n+1)k^2 + n(n-2)(n-1)k_0^2 - 2n(n-1)k_zk_0\right]k_z\right\}^2}{\left[k^2 + (n-1)^2k_0^2 - 2(n-1)k_zk_0\right]k^2} = \frac{6\sqrt{2}}{\pi}\frac{(n^2-1)^{-1}}{n^2(n-1)^2}g\frac{I^{L(n-1)}V_{De}}{T_e}\lambda_{De}^2\frac{\left\{k^2k_0 + \left[(n+1)k^2 + n(n-2)(n-1)k_0^2 - 2n(n-1)k_zk_0\right]k_z\right\}^2}{\left[k^2 + (n-1)^2k_0^2 - 2(n-1)k_zk_0\right]k^2}. \tag{A9}$$

Taking into account that $gI^{L(n-1)}V_{De}/T_e \ll 1$, it is seen that $\varepsilon_{\mathbf{k}}^{(n)}$ is a quantity much smaller than unity. Applying similar procedures to the quantity $\theta_{\mathbf{k}}^{(n)}$ gives

$$\theta_{\mathbf{k}}^{(n)} = \frac{(n^2-1)^{-1}}{\pi n^2(n^2-1)^4}\frac{e^2}{m_e^2\omega_{pe}^4}\int_{-\infty}^{\infty}dk'_z\int_0^{\infty}dk'_\perp k'_\perp \times\left[\frac{I^{L(n-1)}e^{-(k'_z-k_0)^2/\delta^2}e^{-k_\perp'^2/\delta^2}}{\pi^{3/2}\delta^3}\left(k'^2 - kk'_\parallel + \frac{\theta_{\mathbf{k}'}^{(n-1)}}{\varepsilon_{\mathbf{k}'}^{(n-1)}}\right)\frac{\left\{kk'^2 + \left[(n^2-1)k^2 + n(n-2)k'^2 - 2n(n-1)kk'_\parallel\right]k'_\parallel\right\}^2}{n^4(n-1)^4\left(k^2 - k'^2 - 2kk'_\parallel\right)k'^2}\right],$$

which leads to

$$\theta_{\mathbf{k}}^{(n)} \approx \frac{(n^2-1)^{-1}}{\pi n^2(n^2-1)}\frac{e^2I^{Ln}}{m_e^2\omega_{pe}^4}\frac{\left\{k^2k_0 + \left[(n+1)k^2 + n(n-2)(n-1)k_0^2 - 2n(n-1)k_zk_0\right]k_z\right\}^2}{\left[k^2 + (n-1)^2k_0^2 - 2(n-1)k_zk_0\right]k^2} \times\left((n-1)k_0^2 - (n-1)k_zk_0 + \left.\frac{\theta_{\mathbf{k}'}^{(n-1)}}{\varepsilon_{\mathbf{k}'}^{(n-1)}}\right|_{k'_z=(n-1)k_0}\right). \tag{A10}$$

Using Eqs. (A9) and (A10), one obtains

$$\frac{\theta_{\mathbf{k}}^{(n)}}{\varepsilon_{\mathbf{k}}^{(n)}} = (n-1)\left[(n-1)k_0^2 - k_zk_0\right] + \left.\frac{\theta_{\mathbf{k}'}^{(n-1)}}{\varepsilon_{\mathbf{k}'}^{(n-1)}}\right|_{k'_z=(n-1)k_0}. \tag{A11}$$

Taking into account that $\theta_{\mathbf{k}'}^{(1)} = 0$, one obtains

$$\frac{\theta_{\mathbf{k}}^{(2)}}{\varepsilon_{\mathbf{k}}^{(2)}} = -k_0^2,$$

$$\frac{\theta_{\mathbf{k}}^{(3)}}{\varepsilon_{\mathbf{k}}^{(3)}} = 2\left[2k_0^2 - k_z k_0\right] + \frac{\theta_{\mathbf{k}'}^{(2)}}{\varepsilon_{\mathbf{k}'}^{(2)}}\bigg|_{k_z'=2k_0} = 3k_0^2 - 2k_z k_0,$$

$$\frac{\theta_{\mathbf{k}}^{(4)}}{\varepsilon_{\mathbf{k}}^{(4)}} = 3\left[3k_0^2 - k_z k_0\right] + \frac{\theta_{\mathbf{k}'}^{(3)}}{\varepsilon_{\mathbf{k}'}^{(3)}}\bigg|_{k_z'=3k_0} = 6k_0^2 - 3k_z k_0,$$

$$\frac{\theta_{\mathbf{k}}^{(5)}}{\varepsilon_{\mathbf{k}}^{(5)}} = 4\left[4k_0^2 - k_z k_0\right] + \frac{\theta_{\mathbf{k}'}^{(4)}}{\varepsilon_{\mathbf{k}'}^{(4)}}\bigg|_{k_z'=4k_0} = 10k_0^2 - 4k_z k_0,$$

$$\frac{\theta_{\mathbf{k}}^{(6)}}{\varepsilon_{\mathbf{k}}^{(6)}} = 5\left[5k_0^2 - k_z k_0\right] + \frac{\theta_{\mathbf{k}'}^{(5)}}{\varepsilon_{\mathbf{k}'}^{(5)}}\bigg|_{k_z'=5k_0} = 15k_0^2 - 5k_z k_0, \ldots.$$

By induction, we obtain the following:

$$\frac{\theta_{\mathbf{k}'}^{(n)}}{\varepsilon_{\mathbf{k}'}^{(n)}}\Big|_{k_z'=nk_0} = -\frac{n(n-1)}{2}k_0^2, \tag{A12}$$

so that Eq. (A11) can be re-written as follows:

$$\frac{\theta_{\mathbf{k}}^{(n)}}{\varepsilon_{\mathbf{k}}^{(n)}} = \frac{1}{2}n(n-1)k_0^2 - (n-1)k_z k_0. \tag{A13}$$

Therefore, the dispersion relation can be written as follows:

$$\frac{\omega^{Ln}}{\omega_{pe}} = n + \frac{3}{2}\lambda_{De}^2\left\{k_\perp^2 + [k_z - (n-1)k_0]^2\right\} + \frac{3}{2}(n-1)\lambda_{De}^2 k_0^2 + \varepsilon_{\mathbf{k}}^{(n)}, \tag{A14}$$

where $\varepsilon_{\mathbf{k}}^{(n)}$ is given by Eq. (A9). Since $\varepsilon_{\mathbf{k}}^{(n)}$ is a small quantity, it can be neglected in numerical analyses that utilize the dispersion relation, and therefore one obtains the dispersion relation given by Eq. (12).

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