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# EVENT-TRIGGERED SYNCHRONIZATION OF SATURATED LUR'E TYPE SYSTEMS

Porto Alegre
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Dissertation presented to Programa de Pós-Graduação em Engenharia Elétrica of Universidade Federal do Rio Grande do Sul in partial fulfillment of the requirements for the degree of Master in Electrical Engineering.
Area: Control and Automation

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This dissertation was considered adequate for obtaining the degree of Master in Electrical Engineering and approved in its final form by the Advisor and the Examination Committee.

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# ABSTRACT

This dissertation addresses the problem of master-slave synchronization of nonlinear discrete-time Lur'e systems subject to input saturation via event-triggered control (ETC) techniques. Synchronization, which is considered a remarkable property in the physics literature specially when chaotic systems are under investigation, is achieved through the stabilization of the error between the states of the master and the slave system. Regarding the Lur'e type nonlinearity, two different cases are studied along this work: generic slope-restricted state-dependent nonlinearity and piecewise-affine function. In the ETC paradigm, the control signal is updated aperiodically only after the occurrence of an event, which is generated according to a triggering criterion that depends on the evaluation of a triggering function. In the emulation-based design, a synchronization error feedback controller is given *a priori* and the task is to compute the event generator parameters ensuring performance and closed-loop stability. On the other hand, in the co-design approach the event generator and the control law are simultaneously designed. Theoretical results are obtained for three types of event-triggered mechanism (ETM), namely: static, dynamic and relaxed. In the last case, practical synchronization conditions are derived as form of ultimately boundedness stability. In order to tune the parameters of the event-based strategy, optimization problems are formulated aiming to reduce the number of control signal updates (number of events) with respect to a time-triggered implementation. Numerical simulations are presented to illustrate the application of the proposed methods.

**Keywords: Master-slave synchronization, Event-triggered control, Nonlinear Lur'e systems, Piecewise-affine nonlinearities, Saturation.**

# RESUMO

Esta dissertação aborda o problema de sincronização mestre-escravo de sistemas Lur'e não lineares de tempo discreto sujeitos à saturação de entrada via técnicas de controle baseado em eventos. A sincronização, que é considerada uma propriedade importante na literatura de Física especialmente quando sistemas caóticos são investigados, é alcançada através da estabilização do erro entre os estados do mestre e do sistema escravo. Em relação à não linearidade do tipo Lur'e, dois casos diferentes são estudados ao longo do trabalho: não linearidade genérica dependente do estado e restrita em inclinação e função afim por partes. No paradigma de controle baseado em eventos (ETC), o sinal de controle é atualizado aperiodicamente apenas após a ocorrência de um evento, que é gerado de acordo com um critério de disparo que depende da avaliação de uma função de disparo. No projeto baseado em emulação, um controlador por realimentação do erro de sincronização é dado *a priori* e a tarefa é calcular os parâmetros do gerador de eventos garantindo desempenho e estabilidade em malha fechada. Na abordagem de *co-design*, o gerador de eventos e a lei de controle são projetados simultaneamente. Resultados teóricos são obtidos para três tipos de mecanismo de geração de eventos (ETM), nomeadamente: estático, dinâmico e relaxado. Neste último caso, condições de sincronização prática são derivadas como uma forma de estabilidade ultimamente limitada. Para sintonizar os parâmetros da estratégia baseada em eventos, problemas de otimização são formulados visando reduzir o número de atualizações do sinal de controle (número de eventos) em relação a uma implementação *time-triggered*. Simulações numéricas são apresentadas para ilustrar a aplicação dos métodos propostos.

**Palavras-chave: Sincronização mestre-escravo, Controle baseado em eventos, Sistemas Lur'e não lineares, Função afim por partes, Saturação.**

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# LIST OF ABBREVIATIONS

| | |
|---|---|
| ETC | Event-Triggered Control |
| ETM | Event-Triggered Mechanism |
| LF | Lyapunov Function |
| LMI | Linear Matrix Inequality |
| MSS | Master-Slave Synchronization |
| PWA | Piecewise-Affine |
| RA | Region of Attraction |
| RAS | Region of Asymptotic Stability |
| TTC | Time-Triggered Control |
| UUB | Uniformly Ultimately Bounded |

# LIST OF SYMBOLS

| | |
|---|---|
| $\mathbb{N}$ | The set of natural numbers including zero. |
| $\mathbb{R}$ | The set of real numbers. |
| $\mathbb{R}^+$ | The set of non-negative real numbers. |
| $\mathbb{N}_k$ | The set of natural numbers from $1$ to $k$. |
| $\mathbb{R}^n$ | The set of real-valued vectors with $n$ elements. |
| $\mathbb{R}^{n\times m}$ | The set of real-valued matrices with $n$ rows and $m$ columns. |
| $\mathbb{S}^n$ | The set of all real-valued symmetric matrices of dimension $n \times n$. |
| $I$ | A identity matrix of appropriate dimensions. |
| $0$ | A zero matrix of appropriate dimensions. |
| $diag\{A, B\}$ | The block-diagonal matrix composed from matrices $A$ and $B$. |
| $A^\top, v^\top$ | The transpose of matrix $A$ or the vector $v$, respectively. |
| $A_{(i)}, v_{(i)}$ | The $i$-th row of matrix $A$ or the vector $v$, respectively. |
| $A > (<)\, 0$ | Matrix $A$ is positive (negative-)definite. |
| $A^{-1}$ | The inverse of matrix $A$. |
| $tr(A)$ | The trace of matrix $A$. |
| $rank(A)$ | The rank of matrix $A$. |
| $\star$ | The symmetric blocks within a matrix. |
| $c \in \mathcal{C}$ | $c$ is an element of the set $\mathcal{C}$. |
| $\mathcal{C} \subset \mathcal{D}$ | $\mathcal{C}$ is a subset of $\mathcal{D}$. |
| $\mathcal{E}_\gamma(P)$ | The ellipsoidal set $\mathcal{E}_\gamma(P) = \{\zeta \in \mathbb{R}^n : \zeta^\top P\zeta \leq \gamma^{-1}\}$ defined from matrix $P = P^\top > 0 \in \mathbb{R}^{n\times n}$ and scalar $\gamma > 0$. |
| $\|x\|$ | The Euclidean norm $\sqrt{x^\top x}$. |

# CONTENTS

# 1 INTRODUCTION

Motivated by the wide interest of industry and academy, the master-slave synchronization (MSS) of dynamical systems has become an active research topic over the last years, specially in the context of nonlinear systems. One of the main reasons is that synchronization is associated with several practical engineering applications such as robotic manipulators (FATHALLAH; ABDELHEDI; DERBEL, 2017), satellites (FARZAMI; ESMAELZADEH; TAHERI, 2015), telesurgery (NIJMEIJER; RODRIGUEZ-ANGELES, 2003), data encryption in secure communication (YANG; CHUA, 1997), pattern recognition (HOPPENSTEADT; IZHIKEVICH, 2000) and neural networks (CHEN; CHEN; ZENG, 2021), just to mention a few. In summary, the main objective of the MSS is to drive the trajectories of the slave system asymptotically to those of the master and to ensure that they will remain in synchrony with each other. When the task is the computation of a control law to guarantee such behavior, then the synchronization can be interpreted as a problem analogous to the synthesis of state observers (NIJMEIJER; MAREELS, 1997). Following the ideas presented by PECORA AND CARROLL (1990) in their seminal work on synchronization, where the MSS can be addressed using the Control Systems framework, a variety of solutions has been proposed to deal with this problem: robust control (SUYKENS; CURRAN; CHUA, 1999), adaptive control (HAO *et al.*, 2020), switched control (ZHU; ZHENG; ZHOU, 2020), time-triggered control (TTC) (ZANI, 2018), among others.

The development of generic approaches to the analysis of stability or stabilization of nonlinear dynamical systems is still a hard task to be faced by the control community. This is justified because a diversity of nonlinear phenomena may arise in these systems, for instance, nonlinearities can degrade the performance of a control loop or even lead to divergent trajectories of the controlled system. In this sense, a typical solution is to investigate classes of nonlinear systems rather than generic nonlinear ones. The class of Lur'e systems, for example, has been widely studied along the last decades mainly because it can represent a significant amount of nonlinear systems of practical interest. Some classical examples of Lur'e systems are Chua's circuit, $n$-scroll and hyperchaotic attractors (ZHANG; LU; ZHENG, 2011), Swarm models (GAZI; PASSINO, 2003), just to mention a few. This class of nonlinear systems can take advantage of the Lur'e format and the sector condition satisfied by the system nonlinearity to derive conditions in the form of linear matrix inequalities (LMIs) that guarantee the MSS (SUYKENS; VANDEWALLE, 1997). The application and effectiveness of some design methods based on these conditions can be seen in (JIN; LU, 2021), (ZHANG *et al.*, 2021) and references cited therein. In the particular case where the system nonlinearity is modeled by a piecewise-affine (PWA) function, specialized stability conditions can be derived as in the works by HU; HUANG; LIN (2004), HU AND LIN (2005) and ZANI (2018). In this context, it is expected to obtain less conservative results because more information about the nonlinearity is taken into account when compared to the generic Lur'e type nonlinearity. Some examples that fit in this formulation are saturation function, hysteresis, relay and deadzone.

In many real applications, however, the asymptotic MSS may not be achievable or it may be too restrictive giving rise to the so-called practical synchronization problem, i.e., MSS with a residual error in steady-state. In this context, one of the first works to investigate the MSS with bounded error in steady-state was (SUYKENS; CURRAN; CHUA, 1999). In this work, a robust synthesis criterion was proposed to achieve MSS of continuous-time Lur'e systems with parameters mismatch. Nonetheless, as the master and the slave are nonidentical systems, the asymptotic synchronization is not possible. Thus, the slave trajectories initiated in a given neighborhood of the master initial states will converge in finite-time to a closer vicinity of the master trajectories. This is equivalent to say that the synchronization error trajectories, i.e., the difference between the master's and the slave's trajectories, are uniformly ultimately bounded (UUB) to a set near the origin (KHALIL, 1992). Based on similar ideas, a fuzzy adaptive observer-based controller was used by BOULKROUNE AND M'SAAD (2011) to ensure the practical MSS of chaotic systems with deadzone nonlinearities affecting the input. On the other hand, SU *et al.* (2022) proposed a practical fixed-time synchronization approach of chaotic continuous-time neural networks subject to the effect of disturbances and time-delay.

In recent years, the implementation of modern control systems over shared digital communication networks has given rise to what is known as networked control systems, which offers several advantages when compared to the traditional point-to-point implementation, such as: easy of installation, maintenance and expansion, more flexibility in terms of architecture and reduced cost, just to mention a few (HESPANHA; NAGHSHTABRIZI; XU, 2007). In order to deal with limitations that typically arise in networked control systems, for instance, bandwidth limitation of shared networks (ZHANG *et al.*, 2020), power consumption in wireless networks (AKYILDIZ *et al.*, 2002) and packet dropouts (MASTANI; RAHMANI, 2021), several strategies that update the control signal aperiodically have been developed. Among them, the event-triggered approach can reduce data transmissions between the distributed control elements (plant, controller and sensors) while ensuring performance and closed-loop stability (TABUADA,

2007; HEEMELS; JOHANSSON; TABUADA, 2012). Differently from the traditional time-triggered strategy, where the control signal is updated at predefined periodic instants, the main idea of the event-triggered control (ETC) consists in updating it only when a triggering condition is fulfilled, which generally requires monitoring variables of the system in a periodic or continuous fashion (HEEMELS; DONKERS; TEEL, 2011). With respect to the design methods, two general approaches can be found in the ETC literature: emulation and co-design. In the emulation approach, a stabilizing controller (determined, for example, through eigenvalue assignment or optimal control techniques) under a time-triggered policy is assumed to be given *a priori*. Then, the task is to synthesize only the event-triggered mechanism (ETM) while maintaining closed-loop stability and performance requirements. In the co-design method, however, the objective is to design the controller and the ETM simultaneously (HEEMELS; JOHANSSON; TABUADA, 2012), which is a more challenging problem and covered in less references (specially for nonlinear systems) (ABDELRAHIM *et al.*, 2018).

In this context, one of the major challenges lies in choosing suitable triggering criteria allowing to achieve closed-loop stability, satisfactory performance and, at the same time, reducing the number of events. A large amount of results in the ETC literature has been derived based on the seminal work by TABUADA (2007), which employs a static ETM to determine the instants when control tasks should be executed. The idea is to update the control signal whenever the static triggering function, which depends on the current and last transmitted state, verifies a triggering rule. On the other hand, GIRARD (2015) proposed a new class of ETMs, which adds an internal dynamic variable into the triggering criterion, aiming to obtain larger reductions in the number of events. In this case, the event generator is not only based on the current state and last transmitted state value, but also dependent on the additional dynamic variable. Recently, SBARBARO; GOMES DA SILVA JR.; MOREIRA (2020) proposed a relaxed triggering criterion by adding an extra parameter into the triggering rule. This idea was derived in the context of reference tracking and disturbance rejection to avoid that the ETC policy degenerates to a TTC one. By virtue of that, a practical approach using positive invariance arguments was proposed, leading to a trade-off between the reduction of control updates and the tracking error in steady-state.

The physical limitation to provide signals with unlimited amplitude makes saturation an ubiquitous problem when dealing with engineering systems. In this sense, a relevant aspect is the design of controllers capable of ensuring closed-loop stability and performance in the presence of actuator saturation. In addition to being a widely encountered nonlinearity in real applications, its effects in the closed-loop system are associated with multiple equilibria points, limit cycles, performance degradation or even instability (TARBOURIECH *et al.*, 2011). Thus, it is of fundamental importance to consider the saturation effects when dealing with real implementations of MSS. On this topic, XU *et al.* (2021), for example, propose an adaptive ETM to ensure the asymptotic MSS in the particular context of continuous-time neural networks with time-varying delay and input constraints based on a Lyapunov-Krasovskii functional. On the other hand, REHAN (2013) addressed the design of a state feedback controller to achieve the MSS of continuous-time chaotic oscillators under input saturation and disturbance. WANG; DING; LI (2021) investigate an event-triggered dynamic control strategy with anti-windup action to achieve the MSS of continuous-time neural networks with actuator saturation. Additionally, ZHENG *et al.* (2022) studied the event-based master-slave consensus problem of first-order multi-agent systems using the graph theory, where a two-step stabilization procedure is proposed to deal with the saturated closed-loop system.

Based on the aforementioned considerations, the general objective of this work is to derive systematic design techniques for asymptotic and practical MSS of nonlinear saturated Lur'e type systems via ETC strategies. In view of this general objective, the following specific objectives can be highlighted:

- Develop LMI-based stability conditions for generic slope-restricted Lur'e systems and for the particular case of PWA systems.

- Propose systematic design methodologies to compute the control systems parameters using an emulation-based approach and also a co-design based approach.

- Investigate two classes of ETM: static and dynamic.

- Derive stability conditions to achieve the practical MSS via relaxed ETMs.

The technical challenges to be faced throughout this dissertation are related to the nonlinear nature of the closed-loop system and the goal of proposing a systematic design methodology. One source of difficulty is caused by the system nonlinearity that affects both the master and the slave, and another by the input constraint to which the slave system is subjected. In the case of practical synchronization, an additional challenge is the proposition of conditions to ensure that the synchronization error trajectories converge to a set in the neighborhood of the origin.

The text of this dissertation is organized as follows. After this introduction, Chapter 2 presents some background concepts and details the primary aspects related to the subject of study, namely: the synchronization problem, the saturation handling and the ETC paradigm. Then the text is divided into two parts: while Part I addresses the synchronization of generic slope-restricted Lur'e systems, Part II deals with PWA systems. Part I starts with Chapter 3, where the asymptotic MSS problem is formally defined and LMI-based conditions are derived to guarantee the regional stabilization of the synchronization error. In addition, optimization problems are formulated to compute the parameters of the event-based strategy aiming to reduce the control updates. In Chapter 4, the asymptotic approach is extended to practical synchronization, where the trajectories of the closed-loop system are now ultimately confined in a set near the origin. Part II, which starts in Chapter 5, follows a similar organization. Furthermore, each of chapters from 3 to 6 presents constructive synchronization conditions both in co-design and emulation contexts, and numerical examples are presented to illustrate the theoretical results and the proposed design methods. Finally, Chapter 7 states the concluding remarks and shows perspectives of future investigation.

# 2 PRELIMINARIES

This chapter presents some preliminary concepts used throughout the text. Section 2.1, for instance, focuses on saturated systems. Afterwards, Section 2.2 introduces the MSS problem and also details the classes of systems under consideration. Finally, Section 2.3 shows the differences between the types of ETC strategies investigated in this work.

## 2.1 Saturated Systems

The physical limitation to provide signals with unlimited amplitude makes saturation an ubiquitous problem when dealing with practical systems. By virtue of that, over the past few decades, the control community has been devoting a significant effort to develop systematic design methods using rigorous proofs to handle the stability analysis and stabilization of saturated systems. In this sense, a series of results dealing with this topic can be seen in (TARBOURIECH *et al.*, 2011) and references therein.

First, it is important to mention that there are several models to describe the effects of saturation in the closed-system stability, each with its own degree of conservatism. The typical ways are: sector nonlinearity model (traditional and generalized), regions of saturation, polytopic model, and norm-bounded parametric uncertainty. When using the classical sector nonlinearity model the stability is normally ensured for a larger class of systems, which may be potentially conservative. Therefore, in the context of this work the generalized sector approach is applied because it allows to obtain conditions valid only for deadzone functions, which should result in less conservative stability conditions. Thus, consider a linear discrete-time dynamical system as follows:

$$x[k+1] = Ax[k] + Bu[k], \quad x[0] = x_0 \tag{1}$$

where $x[k] \in \mathbb{R}^n$ is the state vector; $u[k] \in \mathbb{R}^m$ is the saturated control input; $A$ and $B$ are constant real-valued matrices of appropriate dimensions that determine the closed-loop system dynamics. The saturation function is defined as

$$u_{(r)}[k] = sat(v_{(r)}[k], u_{0_{(r)}}) = \begin{cases} u_{0_{(r)}} & \text{if} \quad v_{(r)}[k] > u_{0_{(r)}} \\ v_{(r)}[k] & \text{if} \quad -u_{0_{(r)}} \leq v_{(r)}[k] \leq u_{0_{(r)}}, \quad r \in \mathbb{N}_m \\ -u_{0_{(r)}} & \text{if} \quad v_{(r)}[k] < -u_{0_{(r)}} \end{cases} \tag{2}$$

with $u_0 \in \mathbb{R}^m$ being the saturation levels of the control input and $v[k]$ is the controller output. Defining a deadzone nonlinearity

$$\psi(v[k], u_0) = v[k] - sat(v[k], u_0) \tag{3}$$

and considering a state feedback controller as follows

$$v[k] = Kx[k], \tag{4}$$

then the resulting closed-loop dynamics can be written as

$$x[k+1] = (A + BK)x[k] - B\psi(Kx[k], u_0), \tag{5}$$

by substituting (4) in (3), and then in the open-loop system (1).

From this point, the stability can be analyzed considering $\psi(\cdot\,,\cdot)$ a sector-bounded nonlinearity and using the concepts of absolute stability for Lur'e type systems. However, by employing a generalized sector condition, which is valid only for deadzone functions, less conservative results can be obtained. This result is captured in the following lemma:

**Lemma 1** (Generalized sector condition - TARBOURIECH *et al.*, 2011)**.** *If $v$ and $w$ are elements of the polyhedral set $\mathcal{S}_0(v - w, u_0) = \{v, w \in \mathbb{R}^m : |v_{(r)} - w_{(r)}| \leq u_{0_{(r)}}, r \in \mathbb{N}_m\}$, then the nonlinearity $\psi(v, u_0)$ satisfies the following inequality:*

$$\Upsilon_\psi = \psi^\top(v, u_0)J[\psi(v, u_0) - w] \leq 0 \tag{6}$$

*for any diagonal positive definite matrix $J \in \mathbb{R}^{m\times m}$.*

Considering the control law (4) and assuming that $w = Gx[k]$, then the relation (6) becomes

$$\Upsilon_\psi = \psi^\top(Kx[k], u_0)J[\psi(Kx[k], u_0) - Gx[k]] \leq 0. \tag{7}$$

Now, (7) is verified provided that the state $x$ belongs to the following polyhedral set

$$\mathcal{S}_0 = \{x \in \mathbb{R}^n : |(K - G)_{(r)}x| \leq u_{0_{(r)}}, r \in \mathbb{N}_m\}. \tag{8}$$

Lemma 1 is used to achieve stability conditions in a regional context. However, when the open-loop system is not strictly unstable, global stability conditions can be potentially achieved. In order to do that, Lemma 1 needs to be verified for $\mathcal{S}_0 = \mathbb{R}^m$ which can be satisfied by letting $w = v$. This is equivalent to model $\psi(\cdot\,,\cdot)$ as a generic sector-bounded nonlinearity that verifies $\psi^\top(v[k], u_0)J[\psi(v[k], u_0) - v[k]] \leq 0$.

## 2.2 Synchronization Problem

The objective of the MSS problem is to ensure that the slave states track those of master and remain in synchrony with each other (PECORA; CARROLL, 1990). In this sense, the master can be interpreted as a reference generator since it has to generate the trajectories for the slave system to follow. This is achieved, for example, when the states of the slave system follow those of the master such that the error between them converges to zero.

In this work, the following discrete-time master ($\mathcal{M}$) and slave systems ($\mathcal{S}$) are considered:

$$\mathcal{M} : x_M[k+1] = Ax_M[k] + B\sigma(Cx_M[k]) \tag{9}$$

$$\mathcal{S} : x_S[k+1] = Ax_S[k] + B\sigma(Cx_S[k]) + Eu[k] \tag{10}$$

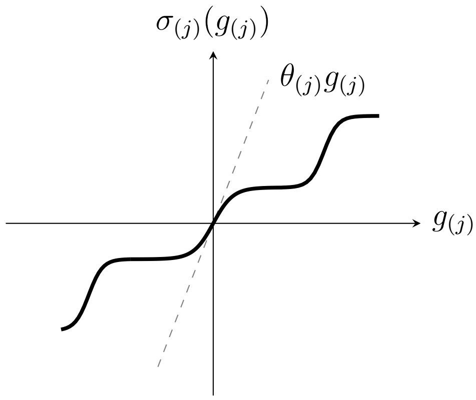
where $x_M[k]$, $x_S[k] \in \mathbb{R}^n$ are the respective state vectors, and $u[k] \in \mathbb{R}^n$ is the control input; $A$, $B$, $C$ and $E$ are constant real-valued matrices of appropriate dimensions that determine the master's and the slave's dynamics; $\sigma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a state-dependent slope-restricted memoryless decentralized nonlinear vector-valued static function whose components $\sigma_j(\cdot)$, $j \in \mathbb{N}_m$, verify the following properties:

(P1) $\sigma_j(g_{(j)})g_{(j)} \geq 0, \forall g_{(j)} \in \mathbb{R}$ and $\sigma_j(0) = 0$.

(P2) $0 \leq \frac{\sigma_j(g_{(j)})-\sigma_j(\hat{g}_{(j)})}{g_{(j)}-\hat{g}_{(j)}} \leq \theta_j, \forall g_{(j)} \neq \hat{g}_{(j)}, g_{(j)}, \hat{g}_{(j)} \in \mathbb{R}$ and $\theta_{(j)}$ is a positive real scalar.

Note that the fulfillment of P1-P2 implies that $\sigma_j(\cdot)$ globally belongs to the sector $[0, \theta_{(j)}]$. Figure 1 illustrates an example of a nonlinear function that satisfies properties P1-P2.

Figure 1 – Slope-restricted nonlinear function.




Source: the author.

Defining the synchronization error as the difference between the states of both systems, i.e., $e[k] = x_M[k] - x_S[k]$, then it follows from (9)-(10) that the error dynamics is given by:

$$e[k+1] = Ae[k] + B\rho(Ce[k], Cx_S[k]) - Eu[k] \tag{11}$$

where $\rho(\cdot\,,\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a memoryless decentralized nonlinear vector-valued static function defined as:

$$\rho(Ce[k], Cx_S[k]) = \sigma(Ce[k] + Cx_S[k]) - \sigma(Cx_S[k]). \tag{12}$$

Taking into account the deadzone in (3) and considering a time-triggered synchronization error feedback controller as follows

$$v[k] = Ke[k], \tag{13}$$

the synchronization error dynamics (11) can be rewritten as

$$e[k+1] = (A - EK)e[k] + B\rho(Ce[k], Cx_S[k]) + E\psi(Ke[k], u_0). \tag{14}$$

At this point, it is clear that the problem of MSS can be implicitly solved by the stabilization of origin of the nonlinear system (14), i.e., if $e[k] \to 0$ then $x_S[k] \to x_M[k]$ and, consequently, synchronization is achieved. It is important to note that due to the nonlinear nature of the closed-loop error, the global stability of the origin (for all $e[0] \in \mathbb{R}^n$), in general, may no longer be achievable. In this case, the MSS will be ensured only for a set of admissible initial synchronization errors $\mathcal{E}_0$ included in the region of attraction (RA) of the origin of (14), which is denoted by $\mathcal{R}_a$. A formal definition of RA is presented in Appendix B. It is important to highlight that only the initial difference between the states of the master and slave needs to belong to the region $\mathcal{E}_0$; the actual initial states do not matter. Hence, the following problem can be formally stated:

**Problem 1.** Compute a gain matrix $K$ such that all trajectories of the synchronization error (14) starting within a given suitable set $\mathcal{E}_0$ converge asymptotically to the origin.

For this class of nonlinear systems, this problem has already been addressed in the synchronization literature by ZANI (2018), not considering event-triggered implementations. Latter, it will become clear that this is a relevant result particularly to design controllers for the emulation-based approach. The Appendix C is dedicated to synthesize synchronization error feedback controllers under a TTC implementation.

An interesting property that has been formally proved by ZANI (2018) is that under the slope-restriction of $\sigma(\cdot)$, the composed nonlinear function $\rho(\cdot\,,\cdot)$ satisfies an inequality similar to a sector condition. In this context, the purpose of developing this inequality, which can be cast into a LMI-based constraint, is to include particular characteristics of the class of nonlinear systems under consideration and, therefore, to derive potentially less conservative synchronization conditions. Thus, the following lemma was developed:

**Lemma 2** (Sector-boundness of $\rho(\cdot\,,\cdot)$ - ZANI, 2018)**.** *If $\sigma(g)$ satisfies properties P1-P2, then the nonlinear function $\rho(g_1, g_0) = \sigma(g_1 + g_0) - \sigma(g_0)$ is such that relation*

$$\Upsilon_\rho = \rho^\top(g_1, g_0) J[\rho(g_1, g_0) - \Theta g_1] \leq 0, \quad \forall g_1, g_0 \in \mathbb{R}^m \tag{15}$$

*with* $\Theta = diag\{\theta_{(1)}, \dots, \theta_{(m)}\}$ *is globally verified for any diagonal positive definite matrix* $J \in \mathbb{R}^{m \times m}$.

Besides generic slope-restricted Lur'e systems, this work addresses the problem of MSS of PWA systems, i.e., synchronization for the particular case where the nonlinearity $\sigma(\cdot)$ is a PWA function. This is motivated by many systems of practical interest that feature state-dependent nonlinearities that can be conveniently described or approximated by this class of functions. Although the PWA functions can be considered as sector-bounded nonlinearities, it is important to take advantage of their particular structure to obtain synchronization conditions that are less conservative. On this topic, FISCHMANN (2017) worked on the MSS of continuous-time systems considering PWA functions, and ZANI (2018) worked on the discrete-time counterpart. In this scenario, by employing the PWA description proposed by HU; HUANG; LIN (2004) a specialized sector-based inequality, which takes into account the slope of the line segments instead of only its maximal and minimal slopes, was also derived.

An odd symmetric vector-valued PWA function can be described elementwise as follows:

$$\sigma_{(j)}(g) = \begin{cases} \lambda_{1_{(j)}} g_{(j)}, & g_{(j)} \in [0, b_{1_{(j)}}] \\ \lambda_{2_{(j)}} g_{(j)} + c_{2_{(j)}}, & g_{(j)} \in (b_{1_{(j)}}, b_{2_{(j)}}] \\ \quad \vdots & \\ \lambda_{N_{(j)}} g_{(j)} + c_{N_{(j)}}, & g_{(j)} \in (b_{N-1_{(j)}}, \infty) \end{cases} \tag{16}$$

with $g_{(j)} > 0$, $c_{l+1_{(j)}} = b_{l(j)}(\lambda_{l_{(j)}} - \lambda_{l+1_{(j)}}) + c_{l_{(j)}}$, $\lambda_{l_{(j)}} > 0$, $j \in \mathbb{N}_n$, $l \in \mathbb{N}_{N-1}$ and $c_{1(j)} = 0$ where $N$ is the number of segments, and $N-1$ is the number of bends of the nonlinearity. Note that thanks to the odd symmetry property, it is only required to define $\sigma(g)$ for $g > 0$. Figure 2 graphically illustrates a PWA function with $N = 3$.

Note that $\sigma(\cdot)$ as defined in (16) satisfies properties P1-P2. On the other hand, as stated by HU; HUANG; LIN (2004), it can be described as a sum of a linear part and saturation functions as follows:

$$\sigma(g) = \Lambda_N g + \sum_{l=1}^{N-1} \bar{\Lambda}_l sat(g, b_l) \tag{17}$$

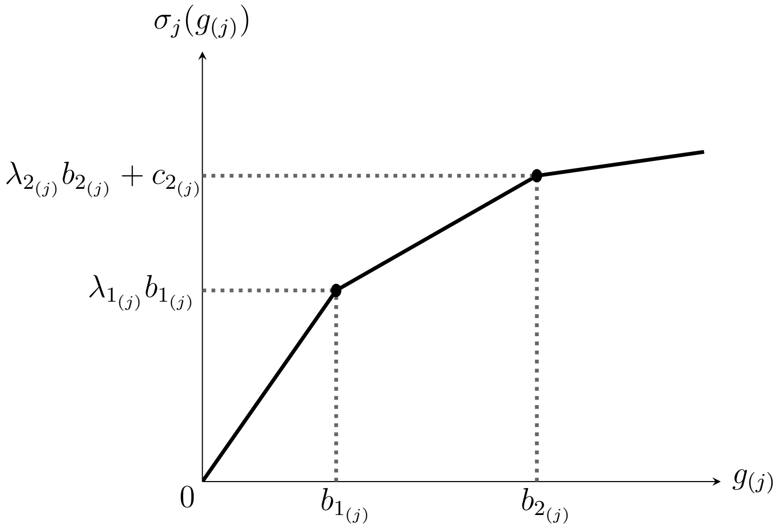
with $\bar{\Lambda}_l = \Lambda_l - \Lambda_{l+1}$ and $\Lambda_l = diag\{\lambda_{l_{(1)}}, \dots, \lambda_{l_{(m)}}\}$. Equivalently, replacing the saturation terms in (17) by vector-value decentralized deadzone functions as in (3), then $\sigma(g)$ can be rewritten as:

$$\sigma(g) = \Lambda_1 g - \sum_{l=1}^{N-1} \bar{\Lambda}_l \psi(g, b_l) \tag{18}$$

where $\psi(g, b_l) = g - sat(g, b_l)$.

Therefore, based on the description of $\sigma(\cdot)$ given in (18), the nonlinearity $\rho(\cdot, \cdot)$ can

Figure 2 – PWA function with two bends.




Source: the author.

be written as:

$$\rho(Ce[k], Cx_S[k]) = \Lambda_1 Ce[k] - \sum_{l=1}^{N-1} \bar{\Lambda}_l \Phi_l(Ce[k], Cx_S[k]) \tag{19}$$

with

$$\Phi_l(Ce[k], Cx_S[k]) = \psi(Ce[k] + Cx_S[k], b_l) - \psi(Cx_S[k], b_l). \tag{20}$$

The synchronization error dynamics (14) for $\rho(Ce[k], Cx_S[k])$ described as in (19) is now given by:

$$e[k+1] = \bar{A}e[k] - B\sum_{l=1}^{N-1} \bar{\Lambda}_l \Phi_l(Ce[k], Cx_S[k]) + E\psi(Ke[k], u_0) \tag{21}$$

where $\bar{A} = A - EK + B\Lambda_1 C$.

Similarly to the case when $\sigma(\cdot)$ is a generic slope-restricted Lur'e nonlinearity, from this point the synchronization problem can be solved by the stabilization of the origin of (21). In this context, a specialized sector-based inequality was derived by ZANI (2018) as can be seen in the following lemma:

**Lemma 3** (ZANI (2018)). *The nonlinear function $\Phi_l(g_1, g_0) = \psi(g_1 + g_0, b_l) - \psi(g_0, b_l)$ is such that relation*

$$\Upsilon_\Phi = \sum_{l=1}^{N-1} \Phi_l^\top(g_1, g_0) U_l [\Phi_l(g_1, g_0) - g_1] \leq 0, \quad \forall g_1, g_0 \in \mathbb{R}^m \tag{22}$$

*is globally verified for any diagonal matrix $U_l > 0 \in \mathbb{R}^{m\times m}$.*

## 2.3 Event-Triggered Control Systems

In the traditional digital implementation schemes, the control signal is updated based on time, causing redundant data transmission since, usually, many of them are not necessary to achieve the desired stability and performance indexes (DE SOUZA, 2021). By monitoring the evolution of important variables of the system (inputs, states and outputs), the ETC can evaluate whether or not to update the control signal. Algorithm 1 illustrates, in a convenient way, how a generic ETC policy is implemented:

**Algorithm 1** Event-triggering strategy

**if** $f(\delta[k], x[k]) \geq 0$ **then**
  $i \leftarrow i + 1$
  $k_i \leftarrow k$
  $v[k_i] \leftarrow v[k]$   (*update the control signal*)
**else**
  $v[k] \leftarrow v[k_i]$   (*keep the control signal unchanged*)
**end if**

where $k$ is the current discrete-time instant, $k_i$ is the event instant with $i \in \mathbb{N}$, $f(\delta[k], x[k])$ is the triggering function and $\delta[k]$ is a state degradation variable defined as the deviation between the state value in the last event and in the current instant, that is,

$$\delta[k] = x[k_i] - x[k]. \tag{23}$$

Essentially, the current state of the triggering function $f(\delta[k], x[k])$ is computed at each instant $k$ and, if it is positive, then an event occurs and the control signal is updated. Note, however, that when the triggering criterion is satisfied, $x[k_i] = x[k]$ such that the variable $\delta[k]$ is reset to $0$.

In this context, one of the main technical challenges is choosing a suitable triggering criterion that is able to guarantee the desired performance requirements, closed-loop stability, and also an effective reduction in the triggering activity. Some examples of ETMs that have been successfully used in the ETC literature are presented in the next sections.

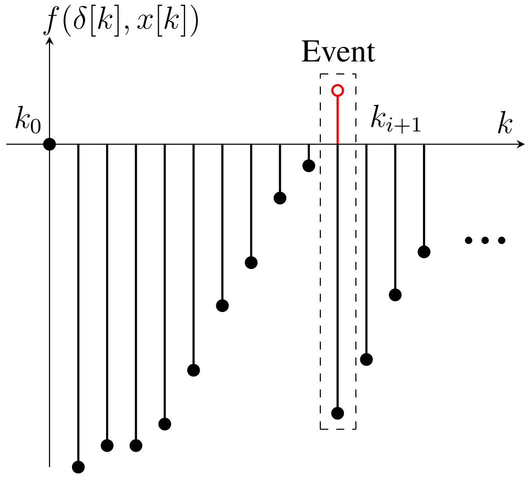
### 2.3.1 Static ETM

Static event-triggering strategies consist of generating an event whenever the triggering function reaches or exceeds the threshold $0$. Consequently, in the time interval between two successive events, $f(\delta[k], x[k])$ is negative. In this context, the class of quadratic triggering functions can establish different types of event-triggering criteria depending on the structure that is used. Mathematically, the static ETM can be generically described as:

$$k_0 = 0, \quad k_{i+1} = \min\{k > k_i : f(\delta[k], x[k]) \geq 0\}. \tag{24}$$

Figure 2.3.1 illustrates the evolution of the triggering function in conjunction with the static ETM. The black stems are values of $f(\delta[k], x[k])$ and the red dot represents the value that $f(\delta[k], x[k])$ would reach at that instant if no event were generated at that instant. However, as the triggering criterion is fulfilled, i.e., $f(\delta[k], x[k])$ would become positive, then the ETM triggers an event and $f(\delta[k], x[k])$ is reset to a negative value.

Figure 3 – Static ETM.




Source: the author.

Another important feature that is illustrated by Figure 2.3.1 is the notion of less communication resources usage provided by the ETC policy. In the TTC, the control signal would be updated for all $k \in \mathbb{N}$, in particular in the $14$ instants illustrated in Figure 2.3.1. On the other hand, the ETC strategy in this example updates the control signal only $2$ times in this interval. Although there is a certain loss of performance in the closed-loop system, how to manage this trade-off between performance and reduction in the number events is one of the main challenges of this research area. To answer this question, several methods have been proposed in the ETC literature.

TABUADA (2007), for example, proposed an ETM based on the relative error threshold, that is, the state degradation variable is normalized with respect to the current state. This ETM is able to achieve asymptotic stability of the origin for a broad class of systems and has the following triggering function:

$$f(\delta, x) = \|\delta(t)\| - \sigma_0 \|x(t)\|, \tag{25}$$

where $\sigma_0$ is a positive real scalar that can be seen as a design parameter. Thus, substituting (25) in the triggering criterion (24), an event will occur only when $\|\delta(t)\| \geq \sigma_0 \|x(t)\|$ holds. As a result, one can see that greater values of $\sigma_0$ allow larger state degradation before an event occurs and, consequently, less events are expected.

An extension of this criterion was introduced by MOREIRA *et al.* (2016), where weights for the different variables were added in the triggering function, leading to a weighted relative error condition. In this case, the triggering function is defined as

$$f(\delta, x) = \delta^\top(t) Q_\delta \delta(t) - x^\top(t) Q_x x(t), \tag{26}$$

where $Q_\delta$ and $Q_x$ are symmetric positive definite constant matrices of appropriate dimensions that act as free tuning parameters in the triggering function (26). Therefore, (26) can be seen as a generalization of the triggering function proposed in (TABUADA, 2007) with additional degrees of freedom provided by considering full matrices $Q_\delta$ and $Q_x$. In particular, the "larger" $Q_x$ and the "smaller" $Q_\delta$, the more the current state can deviate from the last event one before a new event is triggered and, therefore, less events are expected with this triggering function.

### 2.3.2 Relaxed ETM

The relaxed ETM, originally proposed in (SBARBARO; GOMES DA SILVA JR.; MOREIRA, 2020) to avoid the periodic updating that occurs in some cases when dealing with tracking and rejection of non constant signals using ETC, aims to further reduce the triggering activity by adding a threshold (positive scalar) to the triggering function. Mathematically, it can be described as:

$$k_0 = 0, \quad k_{i+1} = \min\{k > k_i : f(\delta[k], x[k]) \geq \kappa\}. \tag{27}$$

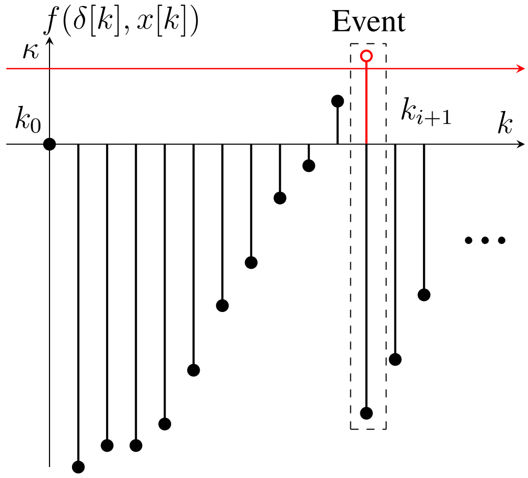
Note that the static criterion (24) forces $f(\delta[k], x[k])$ to be strictly negative in the time interval between two consecutive events. On the other hand, the relaxed criterion described in (27) allows $f(\delta[k], x[k])$ to assume positive values smaller than $\kappa$ in this interval. It will become clear later that, as consequence of this fact, the closed-loop system trajectories present a residual steady-state error which depends on the threshold $\kappa$, leading to a trade-off between triggering activity and residual error. In this case, the asymptotic stability of the origin is no longer achievable, and one can only reach practical stabilization, that is, the state trajectories will be UUB to a set near the origin (KHALIL, 1992).

Figure 2.3.2 illustrates the evolution of the triggering function in conjunction with the relaxed ETM, where the red line indicates the limit $\kappa$. Note that $f(\delta[k], x[k])$ becomes greater than zero before an event happens. This behavior allows the state trajectory to deviate more when compared to static ETM and, therefore, the number of events can be reduced at the expense of a certain steady-state error. However, besides the determination of the triggering function parameters, an appropriate choice of $\kappa$ needs to be done.

### 2.3.3 Dynamic ETM

In the dynamic ETM, an additional internal dynamic variable is introduced to act as the positive threshold present in the relaxed ETM. However, since this threshold evolves according to a stable dynamic system, the asymptotic stability can still be guaranteed,

Figure 4 – Relaxed ETM.




Source: the author.

differently from the relaxed approach. The dynamic criterion is mathematically described as follows:

$$k_0 = 0, \quad k_{i+1} = \min\{k > k_i : f(\delta[k], x[k]) \geq \frac{1}{\nu}\eta[k]\}, \tag{28}$$

where $\nu \in \mathbb{R}^+$ is a design parameter and $\eta[k]$ is the solution to the following initial value problem:

$$\eta[k+1] = \epsilon\eta[k] - f(\delta[k], x[k]), \quad \eta[0] = \eta_0 \tag{29}$$

where $\epsilon \in (0, 1)$ is an additional parameter and $\eta_0 \in \mathbb{R}^+$ is the initial condition. Note that the static ETM (24) can be seen as a limit case of the dynamic ETM (28) when $\nu$ grows large.

Originally, the dynamic ETC strategy was proposed by GIRARD (2015) in the context of nonlinear continuous-time systems. An advantage of dynamic ETMs over the static counterpart is that the same level of performance can be achieved potentially with less number of events. The purpose of adding a dynamic variable that converges to zero when $k$ tends to infinity is that the closed-loop stability can be ensured even if the triggering function assumes now positive values before an event occurs. In this context, it is sufficient for the triggering function to be negative on average (most of time), which is a less-stringent requirement that can be achieved by triggering events in such a way that $\eta[k]$ is always non-negative.

In order to ensure that the dynamic threshold remains non-negative for all time, the following lemma is presented:

**Lemma 4** (Non-negative $\eta[k]$ dynamic - GE; HAN; WANG (2019))**.** *Considering the dynamic of $\eta[k]$ given by* (29)*, if there exist scalars satisfying $\eta_0 \geq 0$, $0 < \epsilon < 1$ and $\epsilon\nu \geq 1$, then*

$$\eta[k] \geq 0 \tag{30}$$

*holds for all $k \in \mathbb{N}$.*

Another relevant feature with respect to the dynamic ETM is that the next event instant $k^d_{i+1}$ will not be smaller than that computed by the static ETM $k^s_{i+1}$ when the previous event instant $k_i$ occurs at the same time for both event-based strategies. This idea was summarized in the following lemma.

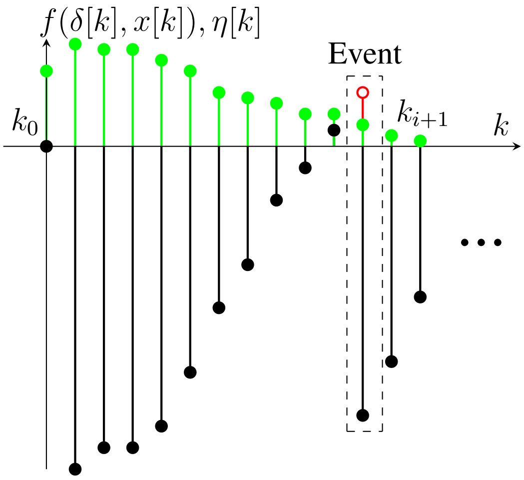
**Lemma 5** (GE; HAN; WANG (2019))**.** *Supposing that when $k = k_i$ both the static* (24) *and the dynamic ETM* (28) *generate an event, and assuming the satisfaction of Lemma 4, then the following relation holds*

$$k^s_{i+1} \leq k^d_{i+1}. \tag{31}$$

However, it should be pointed out that the satisfaction of this lemma does not guarantee anything on further event instants. It means that there is no theoretical proof that the total number of events generated by the dynamic ETM will always be smaller than those of the static ETM, even though the simulation results in the ETC literature typically point in this direction. Recently, dynamic ETC techniques have also been applied to solve the synchronization problem as can be seen in (RUAN *et al.*, 2022; LI *et al.*, 2019) and references therein.

Figure 2.3.3 illustrates the evolution of the triggering function in conjunction with the dynamic ETM, where the green stems are values of $\eta[k]/\nu$. Observe that, in this case, $f(\delta[k], x[k])$ can assume positive values before an event happens, similarly to the relaxed approach. However, as $\eta[k] \to 0$ when $k \to \infty$, the asymptotic stability can still be ensured, differently from the relaxed case.

Figure 5 – Dynamic ETM.




Source: the author.

# Part I

# Slope-restricted Lur’e systems

# Part II

# Piecewise-affine systems

# 7 CONCLUSION

The main purpose of this dissertation was to propose efficient design methodologies to ensure the MSS of generic slope-restricted Lur'e systems subject to input saturation via ETC techniques. A discrete-time framework was considered, as it is suitable for digital implementations, avoiding the continuous evaluation of the triggering function and also the discretization process, which may not be trivial for nonlinear systems. The problem of designing event-based synchronization error feedback controllers has been addressed in emulation and in co-design contexts for slope-restricted Lur'e systems and for the particular case of PWA systems. For each case, different types of ETMs were investigated and Lyapunov Stability theory arguments were used to derive constructive conditions that ensure the asymptotic or the practical stability of the origin of the closed-loop system. These conditions were then cast into optimization problems which allow to tune the ETC system parameters aiming to reduce the number of events with respect to a TTC implementation. In view of the primary goal, some theoretical results have been obtained as presented throughout the development of the work.

Regarding the MSS problem, it was solved taking into account a Control Systems framework, in particular, through the stabilization of the error between the master's and the slave's states. In Chapter 3, LMI-based sufficient conditions were proposed to guarantee that the origin is a locally asymptotically stable equilibrium point of the closed-loop system. In order to deal with the nonlinear effects of saturation in the closed-loop system, a deadzone model was employed in conjunction with a generalized sector condition. Furthermore, the results in this chapter were derived considering static and also dynamic event generators. From these conditions, optimization problems were then formulated to systematically compute the parameters of the event-based strategy with the aim of reducing the number of events. This design methodology is exactly main the contribution stated in this chapter. Finally, numerical simulations have been performed to illustrate the application and effectiveness of the proposed methods.

Subsequently, in Chapter 4 the asymptotic approach was extended to the practical MSS by imposing a relaxation in the triggering function. The motivation for developing this approach is that less events can be generated when compared to the triggering criterion without relaxation at the expense of not achieving asymptotic MSS. By virtue of that, an ultimately bounded stabilization method was developed, now ensuring that the synchronization error trajectories are UUB to a set near the origin. In this sense, the slave states initiated in a certain vicinity of the master states converge in finite-time to a closer neighborhood of the master trajectories, meaning that the synchronization occurs with residual error in steady-state. The main contribution reported in this chapter, similarly to the asymptotic approach in the previous case, is the systematic design method carried out by the solution of optimization problems to tune the parameters of the relaxed ETC strategy.

Afterwards, in chapters 5 and 6, the previous developments were particularized to the case where the system nonlinearity is a PWA function. The interest of studying this class of systems comes from the fact that several systems feature nonlinearities that can be conveniently described or approximated by PWA functions. The stability conditions take into account the particular structure of PWA nonlinearities through the application of Lemma 3 and, by the simulation results, one can see that the PWA formulation tends to generate, indeed, less events when compared to the generic slope-restricted one. In this case, equivalent optimization problems need to be used in the sense that the set of constraints is now provided by the results in Part II, which has analogous innovations of those in Part I.

Facing all that, one can highlight that the proposed work has provided efficient design techniques to achieve synchronization with different ETC strategies. When compared to the current literature on event-triggered synchronization, for instance, (JIN; TANG, 2018) and (LI; ZHANG; FAN, 2021), where the designer needs to arbitrate the event-triggering parameters *a priori*, the main advantage is the systematic synthesis of the ETC system parameters by the solution of optimization problems. In this sense, the numerical results allow to conclude that the proposed techniques are effective in reducing the number of events not only in the emulation context, but also in the co-design context.

The contributions of this research were first presented in the following articles:

- LISBÔA, C.; FLORES, J. V.; MOREIRA, L. G.; GOMES DA SILVA JR, J. M. Event-Triggered Synchronization of Saturated Lur'e-type Systems. *In*: (CDC) CONFERENCE ON DECISION AND CONTROL, 2021, Austin, USA. **Proceedings [. . . ]** IEEE, 2021. p. 5416-5421.

- LISBÔA, C.; FLORES, J. V.; MOREIRA, L. G. Sincronização de Redes Dinâmicas Complexas com Controle Saturado Baseado em Eventos. *In*: (SBAI) SIMPÓSIO BRASILEIRO DE AUTOMAÇÃO INTELIGENTE, 2021, Rio Grande, BRA. **Proceedings [. . . ]** SBA, 2021. v. 1, p. 564-569.

- LISBÔA, C.; FLORES, J. V.; MOREIRA, L. G.; GOMES DA SILVA JR, J. M. Condições relaxadas para sincronização baseada em eventos de sistemas Lur'e com saturação (aceito). *In*: (CBA) CONGRESSO BRASILEIRO DE AUTOMÁTICA, 2022, Fortaleza, BRA. **Proceedings [. . . ]** SBA, 2022.

- LISBÔA, C.; FLORES, J. V.; MOREIRA, L. G.; GOMES DA SILVA JR, J. M. Event-Triggered Saturating Control for Practical Synchronization of Lur'e Systems (accepted). *In*. (CDC) CONFERENCE ON DECISION AND CONTROL, 2022, Cancun, MEX. **Proceedings [. . . ]** IEEE, 2022.

At this point, in view of the theoretical results presented in this dissertation, some potentialities of future investigation are stated as follows:

- Extend Lemma 2 to the local context. In this case, the technical challenge is to provide conditions under which the composed nonlinear function $\rho(\cdot\,,\cdot)$ locally satisfies an inequality similar to the sector condition. When $\sigma(\cdot)$ is a PWA function, a specialized generalized sector condition which takes into account the description by deadzone functions can be derived in order to extend Lemma 3 to the local context.
- Generalize Lemma 3 for all slope-restricted state-dependent nonlinearities below a given PWA function. This can be achieved by using similar arguments as in (HU; HUANG; LIN, 2004), however the theoretical proof may not be trivial.
- Assuming that only the output of the master and the slave system is available, propose a nonlinear dynamic event-based controller with anti-windup action to ensure the asymptotic MSS. A similar approach as in (FISCHMANN, 2017) can be employed.
- Considering that the master's and the slave's trajectories evolve in continuous-time, propose a periodic ETC strategy to ensure the MSS. Similarly to (ZANETTE *et al.*, 2021), a looped-functional approach can be used to deal with the sampling effects.
- Aiming to obtain further reductions on the number of events when compared to ETMs considered in this work, the parameters $Q_\delta$ and $Q_e$ can be matrix functions of the states of closed-loop system. Relaxing the current constraint of being constant matrices could potentially lead to less events generation at the expense, however, of a more complex design of the triggering parameters.
- Study the ETC synchronization problem of multi-agent systems. In this scenario, the graph theory formalist would be used to represent how the information is exchanged between agents and a distributed control algorithm, which takes into account only local data, can be employed to achieve synchronization.

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# APPENDIX A COMPLEMENTARY MATERIAL

This appendix addresses a supplementary material to the main results of the dissertation. In particular, a summary on the main ideas about linear matrix inequalities in the control systems theory and some useful related lemmas are presented.

## A.1 Linear Matrix Inequalities

Thanks to the development of interior point methods, back to the end of twentieth century, the use of LMIs in the analysis and synthesis of control systems became a very important tool. In this context, one of the technical challenges faced consists in how to describe the conditions for solving the problem of interest in terms of LMIs, since typically there are nonlinear forms involved in the resulting formulation. An LMI can be written in a standard way as

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{115}$$

where $x \in \mathbb{R}^m$ is a vector of decision variables and $F_i \in \mathbb{R}^{n \times n}$ are symmetric matrices for all $i \in \mathbb{N}_m$ that are known. An important property related to LMIs is that the set of solutions $\mathcal{F} = \{x \in \mathbb{R}^n : F(x) > 0\}$ is convex, meaning that a feasibility problem or an optimization problem can be formulated. In the latter case, the optimization criterion is

$$\min_{x} \quad c^\top x \quad s.t. \quad F(x) > 0 \tag{116}$$

where $c \in \mathbb{R}^m$, $c^\top x$ is a linear objective function and $x \in \mathcal{F}$ is a convex constraint of the optimization problem (BOYD *et al.*, 1994). Now, some important lemmas and instrumental tools are presented to cast problems in the form of LMI constraints.

## A.2 Schur's Complement

The Schur's Complement is an algebraic tool used to convert nonlinear matrix inequalities involving decision variables into LMIs. It is presented in the following lemma.

**Lemma 6.** *Consider matrices $P = P^\top \in \mathbb{R}^{n\times n}$, $Q = Q^\top \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{n\times m}$ such that $Q > 0$. Then it follows that:*

$$P - RQ^{-1}R^\top > 0 \quad \Leftrightarrow \quad \begin{bmatrix} P & R \\ R^\top & Q \end{bmatrix} > 0. \tag{117}$$

*Proof.* First, note that the right-hand side of (117) can be rewritten as

$$\begin{bmatrix} P - RQ^{-1}R^\top & 0 \\ 0 & Q \end{bmatrix} > 0, \tag{118}$$

by applying a congruence transformation with matrix

$$\begin{bmatrix} I & RQ^{-1} \\ 0 & I \end{bmatrix} > 0. \tag{119}$$

Since this is a non-singular matrix, it follows that the satisfaction of the right-hand side of (117) holds if and only if (118) is verified. Observe that relation (118) is satisfied if and only if $P - RQ^{-1}R^\top > 0$ holds with $Q > 0$, which is in turn the left-hand side of (117). Therefore, one concludes that (117) presents equivalent conditions provided that $Q > 0$ is verified. □

## A.3 S-Procedure

The S-Procedure is an instrumental tool, which can be seen as relaxation technique, used to verify the satisfaction of a quadratic inequality only over a specified region in the space-state described by a set of quadratic inequalities. The S-Procedure for strict inequalities is formally defined in the next lemma.

**Lemma 7.** *Consider symmetric matrices $P_0, \ldots, P_m \in \mathbb{R}^{n\times n}$ where*

$$\mathcal{Q} = \{x \in \mathbb{R}^n : x^\top P_i x \le 0, \quad i \in \mathbb{N}_m\}. \tag{120}$$

*If there exist non-negative real scalars $\tau_1, \ldots, \tau_m$ such that*

$$P_0 + \sum_{i=1}^{m} \tau_i P_i > 0, \tag{121}$$

*then $x^\top P_0 x > 0$, for all $x \in \mathcal{Q}$, $x \neq 0$.*

*Proof.* First, assume that $x \in \mathcal{Q}$ meaning that $x^\top P_i x \le 0$, $i \in \mathbb{N}_m$. If for some scalars $\tau_1$, $\ldots$, $\tau_m$ relation

$$x^\top P_0 x + \tau_1 x^\top P_1 x + \cdots + \tau_m x^\top P_m x > 0 \tag{122}$$

is satisfied, then the quadratic form $x^\top P_0 x > 0$ is verified. Therefore, if the inequality (121) is satisfied, it follows that (122) holds for all $x \neq 0$ and one conclude that $x^\top P_0 x > 0$ is satisfied for all $x \in \mathcal{Q}$, $x \neq 0$. □

## A.4 Congruence Transformation

One of the main uses of the congruence transformation is to convexify multiplications between the decision variables, which is a typical problem that arises in control systems design.

**Lemma 8.** *Let $P$, $Q \in \mathbb{R}^{n\times n}$ be symmetric matrices such that $Q$ is non-singular. Then*

$$P > 0 \quad \Leftrightarrow \quad QPQ^\top > 0. \tag{123}$$

*Proof.* First, pre- and post-multiply $P > 0$ by matrix $Q$ and its transpose, respectively, to get $QPQ^\top > 0$. On the other hand, pre- and post-multiply $QPQ^\top > 0$ by matrix $Q^{-1}$ and its transpose, respectively, to get $P > 0$ again. Note that these conditions are equivalent provided that $Q$ is a non-singular (invertible) matrix. □

# APPENDIX B DYNAMICAL SYSTEMS STABILITY

This appendix presents a brief introduction on the concepts related to the stability analysis of nonlinear dynamical systems based on the Lyapunov theory arguments.

## B.1 Lyapunov Stability Theory

The Lyapunov theory is a powerful tool to analyze the stability of nonlinear systems (KHALIL, 1992) and has also been conveniently applied to ensure the internal stability of ETC systems. Although initially conceived for differential equations, it has already extended to difference equations and, therefore, to discrete-time systems (ÅSTRÖM; WITTENMARK, 1984). The main idea of the second method of Lyapunov (also know as the direct method) consists in determining a LF, which in some sense represents the energy of the system, such that it is zero when the system trajectories achieve the equilibrium point and positive anywhere else in the space-state. Thus, consider the following nonlinear discrete-time dynamical system:

$$x[k+1] = h(x[k]), \quad x_0 = x[0] \tag{124}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state of the autonomous system (124) and $h : \mathcal{X} \to \mathbb{R}^n$ is a locally Lipschitz map from $\mathcal{X}$, containing the origin, into $\mathbb{R}^n$. Without loss of generality, the origin is assumed to be an equilibrium point of (124), that is, $h(0) = 0$. When this is not the case, a change of coordinates (variables) can be applied and the analysis of stability is accomplished in the translated system for which $0$ is now an equilibrium point (KHALIL, 1992).

**Definition 1** (Equilibrium Point)**.** *An state $\bar{x}$ is an equilibrium point of system* (124) *if, once the trajectory $x[k]$ verifies $x[k] = \bar{x}$ in $k = \bar{k}$, then $x[k]$ remains equal to $\bar{x}$ for all $k \geq \bar{k}$, i.e., $h(\bar{x}) = \bar{k}$ for all $k \geq \bar{k}$.*

Now, the basic concepts of stability and asymptotic stability are formally defined as:

**Definition 2** (Stability in sense of Lyapunov)**.** *Let the origin be an equilibrium point of the system* (124)*, with $h : \mathcal{X} \to \mathbb{R}^n$ locally Lipschitz in a domain $\mathcal{X} \subset \mathbb{R}^n$ containing the origin. The origin is said to be:*

- *stable if, for each* $\epsilon > 0$*, there exists some* $\delta > 0$ *such that* $\|x[0]\| < \delta \implies \|x[0]\| < \epsilon$ *for all* $k \in \mathbb{N}$*.*
- *asymptotically stable if it is stable and the positive scalar* $\delta$ *can be chosen such that* $\|x[0]\| < \delta \implies \lim_{k\to\infty} \|x[k]\| = 0$*.*
- *unstable if it is not stable.*

When the origin is asymptotically stable, an important theoretical aspect that needs to be analyzed is the following: how far the trajectories of (124) can start such that they converge to the origin. From this idea, the concept of region of attraction is formally defined as:

**Definition 3** (KHALIL, 1992)**.** *The region of attraction of the origin is the set of all initial states* $x_0 \in \mathbb{R}^n$ *for which* $x[0] = x_0 \implies x[k] \to 0$ *when* $k \to \infty$*, i.e., the corresponding trajectory converges asymptotically to the origin.*

It is well known that the exact characterization of the RA is, in general, a hark task (and even impossible for some systems). Therefore, a fundamental aspect is the estimate of the RA by sets with a well-defined analytical representation like polyhedrons and ellipses. In this context, regions in the state space in which the convergence of the trajectories to the origin is guaranteed are formally defined as:

**Definition 4** (TARBOURIECH *et al.*, 2011)**.** *A Region of Asymptotic Stability (RAS) of an equilibrium point is a region of the state space that contains the equilibrium and is a subset of the region of attraction of that equilibrium.*

A fundamental result on the stability of nonlinear systems is summarized in the following theorem:

**Theorem 7** (KHALIL, 1992)**.** *Let the origin be an equilibrium point of the system* (124)*, with* $h : \mathcal{X} \to \mathbb{R}^n$ *locally Lipschitz in a domain* $\mathcal{X} \subset \mathbb{R}^n$ *containing the origin. If there exists a scalar function* $V : \mathcal{X} \to \mathbb{R}$ *continuous in* $x$ *such that:*

$i)$ $V(0) = 0$.

$ii)$ $V(x) > 0$, $\forall x \in \mathcal{X} - \{0\}$.

$iii)$ $\Delta V(x) = V(h(x)) - V(x) \leq 0$, $\forall x \in \mathcal{X}$.

*then the origin of the system* (124) *is a stable equilibrium point. Furthermore, if* $\Delta V(x) < 0$ *for all* $x \in \mathcal{X} - \{0\}$*, then the origin is an asymptotically stable equilibrium point. Moreover, the origin can also be a globally asymptotically stable equilibrium point (its region of attraction is the entire state space) if* $\mathcal{X} = \mathbb{R}^n$ *and* $V(x)$ *is radially unbounded, that is,* $V(x) \to \infty$ *when* $\|x\| \to \infty$*.*

Any function that fulfills the conditions of Theorem (124) is called a LF of the system (124). It is important to mention that Theorem (124) only provides sufficient conditions for the stability of the origin, meaning that nothing can be asserted when a candidate LF does not satisfy these conditions. Furthermore, many LFs can prove the stability for the same system each with their degree of conservativeness in the estimate of the RA. Thus, finding an adequate LF is precisely on of the main challenges in the second method of Lyapunov. In order to provide an estimate of the RA the following Theorem can be used:

**Theorem 8** (KHALIL, 1992)**.** *Let $V(x)$ be a LF of the system* (124)*. Then any bounded surface level of $V(x)$ such as $\mathcal{D} = \{x \in \mathbb{R}^n : V(x) \leq \epsilon\} \subset \mathcal{X}$ is a RAS of the system* (124)*.*

# APPENDIX C CONTROLLER DESIGN WITH TTC

This appendix focuses on the synthesis of controllers to achieve asymptotic MSS with TTC. The results in this chapter are useful only in the emulation-based method, since it requires a previously designed controller ensuring closed-loop stability and performance.

## C.1 Controller Design for Generic Lur'e Systems

In order to compute the gain matrix $K$ that ensures closed-loop stability in the context of generic Lur'e systems, ZANI (2018) provided LMI-based sufficient conditions as in the following lemma.

**Lemma 9.** *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, diagonal positive definite matrices $F_1 \in \mathbb{R}^{m\times m}$, $F_2 \in \mathbb{R}^{n\times n}$, matrices $Y$, $\bar{G}_1 \in \mathbb{R}^{n\times n}$ satisfying the relations*

$$\begin{bmatrix} W & \star & \star & \star \\ -\Theta CW & 2F_1 & \star & \star \\ -\bar{G}_1 & 0 & 2F_2 & \star \\ AW - EY & BF_1 & EF_2 & W \end{bmatrix} > 0 \tag{125}$$

$$\begin{bmatrix} W & \star \\ Y_{(r)} - \bar{G}_{1_{(r)}} & u^2_{0_{(r)}} \end{bmatrix} > 0 \quad r \in \mathbb{N}_n \tag{126}$$

*with $K = YW^{-1}$, then all trajectories of the system* (11) *starting at the set $\mathcal{E}_1(P)$ defined in* (41)*, with $P = W^{-1}$, converge asymptotically to the origin.*

## C.2 Controller Design for PWA Systems

Now, aiming to compute the gain matrix $K$ for the particular case of PWA functions, ZANI (2018) provided the following lemma.

**Lemma 10.** *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, diagonal positive definite matrices $R_l \in \mathbb{R}^{m\times m}$, $l \in \mathbb{N}_{N-1}$, $F_2 \in \mathbb{R}^{n\times n}$, matrices $Y$ and $\bar{G}_1 \in \mathbb{R}^{n\times n}$*

*satisfying the relations* (126) *and*

$$\begin{bmatrix} W & \star & \star & \star & \star & \star \\ -CW & 2R_1 & \star & \star & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -CW & 0 & \cdots & 2R_{N-1} & \star & \star \\ -\bar{G}_1 & 0 & \cdots & 0 & 2F_2 & \star \\ \tilde{A} & \Xi_1 & \cdots & \Xi_{N-1} & EF_2 & W \end{bmatrix} > 0 \tag{127}$$

*with* $\tilde{A} = (A + B\Lambda_1 C)W - EY$, $\Xi_l = -B\bar{\Lambda}_l R_l$ *and* $K = YW^{-1}$, *then all trajectories of system* (21) *starting at the set* $\mathcal{E}_1(P)$ *defined in* (41)*, with* $P = W^{-1}$*, converge asymptotically to the origin.*

Based on the lemmas presented on this chapter, the controller gain matrix $K$ can be computed by the solution of an optimization problem subject to the LMI constraints (125) and (126) in the case of generic Lur'e systems, or (127) and (126) in the case of PWA functions. In order to solve Problem 1, ZANI (2018) proposed the following optimization problem aiming to find a matrix $K$ that leads to the maximization of the volume of the ellipsoidal set $\mathcal{E}_0 = \mathcal{E}_1(P)$:

$$\begin{aligned} \max \quad & \text{tr}(W) \\ \text{subject to:} \quad & (125),\ (126)\ (\text{or}\ (127),\ (126)) \end{aligned} \tag{128}$$