Matemática Contemporânea
Vol. 50, 76–138
http://doi.org/10.21711/231766362022/rmc505

Geometrization in Geometry

Izabella de Freitas[1] and **Álvaro K. Ramos**[2]

[1]Programa de Pós Graduação em Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre/Brazil
[2]Departamento de Matemática Pura e Aplicada, Universidade Federal do Rio Grande do Sul, Porto Alegre/Brazil

Dedicated to Professor Renato Tribuzy on the occasion of his 75th birthday

Abstract. So far, the most magnificent breakthrough in mathematics in the 21st century is the Geometrization Theorem, a bold conjecture by William Thurston (generalizing Poincaré's Conjecture) and proved by Grigory Perelman, based on the program suggested by Richard Hamilton. In this survey article, we will explain the statement of this result, also presenting some examples of how it can be used to obtain interesting results in differential geometry.

Keywords: geometrization, hyperbolization, Poincaré's conjecture, orbifolds, Seifert fibered spaces, geometric topology.

2020 Mathematics Subject Classification: Primary: 51H25. Secondary: 53A10, 57K32, 53C42.

1 Introduction

In certain sense, a *mathematical object* is an idealistic representation of the reality, and in several contexts this idealization suffices to give us

The second author is partially supported by CNPq/Brazil, e-mail:alvaro.ramos@ufrgs.br

insight about nature and, more broadly, the universe. However, in the past few centuries, mathematics have developed to a degree in which it escapes reality and immerses into the world of imagination. Astonishing is the fact that, even in its retreat from reality, mathematics has the ability to provide tools that later can be used to explain natural effects and develop science and technology as a whole.

In this survey paper, dedicated to Professor Renato Tribuzy, we are interested in the shapes of certain mathematical objects called *3-manifolds*. More specifically, we present what is arguably the most significant development on mathematics in the 21st century, which is the *geometrization theorem*. Rather than presenting its proof, which was given in a series of papers [41–43] by Grigory Perelman, we will focus on its precise statement and on some of the background material necessary for its understanding.

Although the main topic of this article is topology of 3-manifolds, it is actually a paper in *geometric topology*, but we would like to mention that it was written with a special care for differential geometers readers, which sometimes may have trouble (as we did) for translating the concepts of topology to a more geometric perspective. We also notice that the main results concerning basic aspects of geometrization of 3-manifolds that we present may be found in a deeper level in the books of M. Aschenbrenner, S. Friedl, H. Wilton [6], B. Martelli [32] or P. Scott [46].

Next, we explain the organization of the manuscript. In Section 2, we present the intuitive concept of topology and how it is related to geometry. We also explain the concept of an orbifold, with emphasis for dimension 2, which is necessary for geometrization. Section 3 is where we present the geometrization theorem, after making precise the definition and classification of Seifert fibered spaces. In Section 4, we particularize to the hyperbolic geometry, presenting Thurston's hyperbolization criterion, several examples and some applications.

Acknowledgments

The authors would like to thank Colin Adams and Vanderson Lima for useful discussions in the preparation of this manuscript and the referees for several suggestions and comments that improved the paper. The first author was partially supported by CAPES/Brazil, and second author was partially supported by CNPq/Brazil, grant number 312101/2020-1.

2 Topology and Geometry

In an intuitive manner, we may say that *Topology* is the study of shapes. If two topological spaces can be obtained one from the other by means of a homeomorphism, then Topology regards both objects as the same. On the other hand, *Riemannian Geometry* is the study of certain smooth objects, called manifolds, together with their intrinsic distances, measured by (Riemannian) metrics, and two Riemannian manifolds will be thought as being the same if there is an isometry (which is a metric preserving diffeomorphism) between them, see Figure 2.1.

At first glance these two concepts may seem to be disjoint, but there are several connections between them, which make both theories richer. First of all, every differentiable manifold M has an intrinsic topology associated to it[1], hence we are allowed to regard M as a topological space and to use the expression *the topology of* M to refer to such topological space structure. For details, see [10, Chapter 2] (which contains a topological introduction) or [14, Chapter 0] (for a more geometric point of view).

Although there are several topological spaces that are not manifolds

[1]This intrinsic topology is obtained by stating that a subset $U \subset M$ is open if $x_\alpha^{-1}(U \cap x_\alpha(U_\alpha))$ is an open set of $\mathbb{R}^n$ for every local chart $x_\alpha \colon U_\alpha \subset \mathbb{R}^n \to M$. It is important to mention that we assume that this topology is *Hausdorff*, i.e., any two points can be separated by disjoint neighborhoods, and that M can be covered by a countable number of charts. When M is connected, these assumptions are equivalent to the existence of a differentiable partition of unity on M, which is an essential tool for the study of several questions on manifolds.

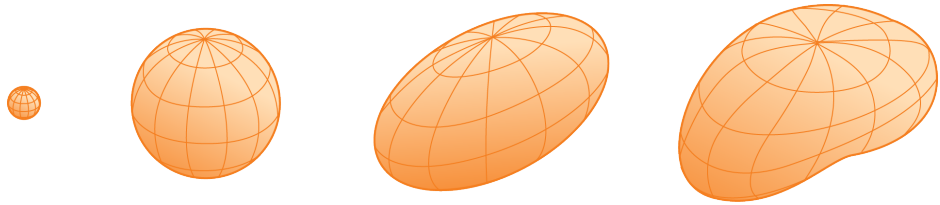


Figure 2.1: As topological spaces, all these 2-spheres are equivalent, since they are homeomorphic. However, from a geometric point of view, they are distinguishable from each other by their respective metrics.

and even homeomorphic manifolds that are not diffeomorphic [35], a striking observation is that it is possible to relate geometry and topology in elegant and deep manners. For instance, if S is a closed (compact, without boundary) surface endowed with a Riemannian metric, then Gauss-Bonnet Theorem implies that its total Gaussian curvature $\int_S K_S$ (a geometric quantity) is related with its Euler characteristic (a topological invariant) $\chi(S)$ by

$$\int_S K_S = 2\pi\chi(S).$$

In particular, independently on the metric considered on the 2-sphere (for instance, see the spheres depicted in Figure 2.1), its total curvature will always equals 4π. We cannot *bend a sphere* to increase or decrease its total curvature without tearing it apart, thus changing its topology.

Other interesting results relating topology and geometry are the Bonnet-Myers Theorem and the Cartan-Hadamard Theorem, which restrict the topology of a given geometry:

Theorem 2.1 (Bonnet-Myers [9, 39])**.** *Let M be a complete Riemannian manifold with sectional curvature $K_M \geq \delta > 0$. Then, M is compact and $\pi_1(M)$ is finite.*

Theorem 2.2 (Cartan-Hadamard, see [25] or [14])**.** *Let M be a complete Riemannian manifold with sectional curvature $K_M \leq 0$. Then, the exponential map of M at any point p is a covering transformation. In*

particular, the universal covering of M is diffeomorphic to $\mathbb{R}^n$, where $n = \dim(M)$.

But perhaps even more interestingly than the aforementioned results, geometry can be used to answer an old, natural topology question: *what are all the possible shapes of surfaces?* This is going to be discussed in the next section.

2.1 Surfaces.

In this section, we will discuss what are the possible topologies for a closed, orientable surface, and we will also present the geometrization[2] theorem for surfaces, which will be a starting point for the reader to understand the geometrization of a closed, orientable 3-manifold.

Definition 2.3. Let S be a closed, orientable surface. Then, the *genus* of S is the maximal number of pairwise disjoint simple closed curves in S whose collection do not separate S.

Example 2.4. Since the sphere $\mathbb{S}^2$ is simply connected, every simple closed curve in $\mathbb{S}^2$ separates. Thus, its genus is zero.
The genus of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is positive, since there exists a nonseparating simple closed curve (for instance, $\mathbb{S}^1 \times \{p\}$ for any $p \in \mathbb{S}^1$) in $\mathbb{T}^2$. However, if γ_1 and γ_2 are pairwise disjoint simple closed curves that individually do not separate, then $\mathbb{T}^2 \setminus \gamma_1$ has the topology of $\mathbb{S}^1 \times (0,1)$, and we may see γ_2 as a nontrivial simple closed curve in $\mathbb{S}^1 \times (0,1)$, so γ_2 is parallel to $\mathbb{S}^1 \times \{1/2\}$, thus separating. This proves that the collection $\gamma_1 \cup \gamma_2$ separates $\mathbb{T}^2$, showing that its genus is equal to one.

More than an interesting topological invariant, the genus of a closed, orientable surface is sufficient to completely classify its topology. This follows from the following *Classification Theorem*, whose original proof dates

[2]By geometrization of a manifold M we mean finding a decomposition of M such that each component admits a geometric structure, which is a complete metric locally isometric to a given simply connected homogeneous manifold; a detailed description will be given in Section 3.3.

back from the 1860's, and a modern approach presented by J. Conway as the *Zero Irrelevancy Proof* (or ZIP proof, for short) can be found in [18].

Theorem 2.5 (Classification Theorem)**.** *Let S be a connected, orientable, closed surface. Then, S is diffeomorphic to a connected sum (see Definition 2.6 below) of the 2-sphere $\mathbb{S}^2$ with $g \geq 0$ copies of the 2-torus $\mathbb{T}^2$.*

We notice that the statement of the Classification Theorem is more general than the presented above, which was a restriction to the orientable case. We also used the concept of connected sum, depicted in Figure 2.2 and whose definition we present next in dimensions 2 and 3.

Definition 2.6. Let M_1 and M_2 be two oriented manifolds of the same dimension $n \in \{2, 3\}$[3]. Then, the *connected sum* of M_1 and M_2 is the manifold $M_1 \# M_2$ obtained by the following operation: let B_1, B_2 be respective $n-$balls in M_1, M_2 with boundaries S_1, S_2. Let f be an orientation-reversing diffeomorphism from S_1 to S_2. Then $M_1 \# M_2 = (M_1 \setminus B_1) \cup (M_2 \setminus B_2)/\sim$, where we identify $S_1 \ni x \sim f(x) \in S_2$.

As previously explained, the Classification Theorem allows us to list all closed, orientable surfaces in terms of their genus, since such a surface has genus g if and only if it is the connected sum of $\mathbb{S}^2$ with g tori. Having this construction in mind, we may now present the Uniformization Theorem, which was conjectured by Felix Klein and Henri Poincaré in the 1880's and proved independently by Poincaré [45] and Koebe [30] in 1907.

Theorem 2.7 (Uniformization theorem)**.** *Let S be an orientable, closed surface. Then, S admits a metric of constant curvature k. Furthermore, if g is the genus of S, the following hold:*

1. *$k > 0$ if and only if $g = 0$.*

[3]The definition of connected sum of M_1 and M_2 can be generalized for any $n > 3$. However, in dimensions higher than 3, the differentiable structure of the connected sum $M_1 \# M_2$ can depend not only on the chosen orientations on M_1 and M_2 but also on the choice of the gluing map f, while if $n \in \{2, 3\}$ this does not occur.

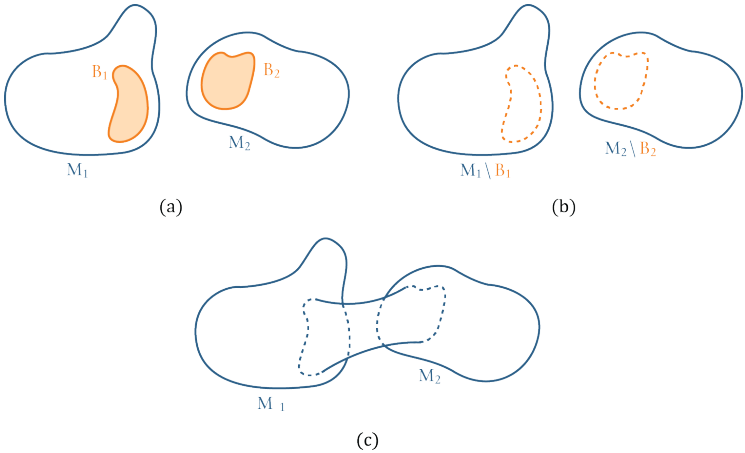


Figure 2.2: The connected sum of M_1 and M_2, as in Definition 2.6, removes an n-ball B_1 from M_1 and an n-ball B_2, and then joins the resulting manifolds with spherical boundary by an orientation-reversing diffeomorphism, which has the visual effect of gluing them by a cylindrical neck.

2. $k = 0$ *if and only if* $g = 1$.

3. $k < 0$ *if and only if* $g \geq 2$.

In Section 3.3 (more precisely in Theorem 3.25), we will revisit Theorem 2.7, justifying the fact that it is also known by the name of *geometrization for surfaces*. At the present moment, we will restrain ourselves to observe that the uniformization theorem shows that a connected, closed and orientable surface has a special metric of constant curvature, and its Riemannian universal covering is isometric (after a homothety) to one and only one of the space forms $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{H}^2$, which will be called the *model geometry* for S. In particular, if X is the model geometry for a surface S and $G = \mathrm{ISO}(X)$, there exists a subgroup H of G such that S is diffeomorphic to the quotient X/H. We also point out that the original statement of the Uniformization Theorem (which is well-known to be equivalent to the one presented above) makes use of the theory of Riemann surfaces, stating that that any simply connected Riemann surface is either conformal to the

open unit disk, to the complex plane or to the Riemann sphere.

Example 2.8. Let $S = \mathbb{T}^2$ be the torus. Then, the model geometry for S is $\mathbb{R}^2$ and if we let H be the group generated by two linearly independent translations in $\mathbb{R}^2$, it follows that $S = \mathbb{R}^2/H$.

Generalizing the concept of surfaces, we will next introduce the concept of *orbifolds*, which are certain not-so-well behaved quotients of manifolds. Orbifolds play a major role in the geometrization of 3-manifolds, and this nomenclature was introduced in the 1970's by William Thurston, after a vote by his students, but they appeared earlier in the literature under the name of V-manifolds.

2.2 Orbifolds

In some sense, orbifolds are structures used to understand group actions over manifolds, but differently from surfaces, where the group action is properly discontinuous[4] and free[5], on orbifolds the groups will act uniquely in a properly discontinuous manner. We also recall that the quotient of a topological space X by a group G, denoted by X/G, is the set of orbits, together with the quotient topology.

Before giving the precise definition of an orbifold, we will present some examples to bring up some intuition to the reader.

Example 2.9. Let $R \subset \mathbb{R}^2$ be a rectangle and G the group of isometries of $\mathbb{R}^2$ generated by the reflections along the four lines containing the sides of R.

[4]The action of a group G over a manifold X is properly discontinuous if for every compact $K \subset X$ the number of elements g of G that satisfy $gK \cap K \neq \emptyset$ is finite.

[5]The action of a group G over a set X is called *free* if it has no fixed points unless it is the action of the identity element.

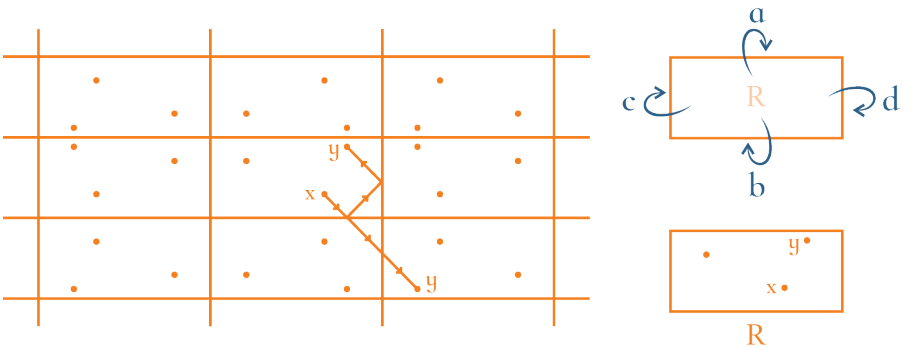


Figure 2.3: Tiling of $\mathbb{R}^2$ generated by the reflections a, b, c, d over the sides of R.

The reflections a, b (and also c, d) over two parallel sides give rise to the free product group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$. Thus, in this case we have $G = D_\infty \times D_\infty$ and the quotient space $\mathbb{R}^2/G$ is the rectangle R.

This space is called the *rectangular billiard*, because if we think the points of R as pool balls, to hit a certain point y from another point x, it suffices to aim in any other copy of y into a reflected image of R, as shown in Figure 2.3.

Example 2.10. Let $G = D_\infty \times D_\infty$ be the group as in Example 2.9 and let H be the index-2 subgroup of G given by the orientation preserving isometries. Then, $\mathbb{R}^2/H$ is the *pillowcase space*, which is topologically a 2-sphere with 4 singularities, at the points corresponding to the vertexes of R (see Figure 2.4).

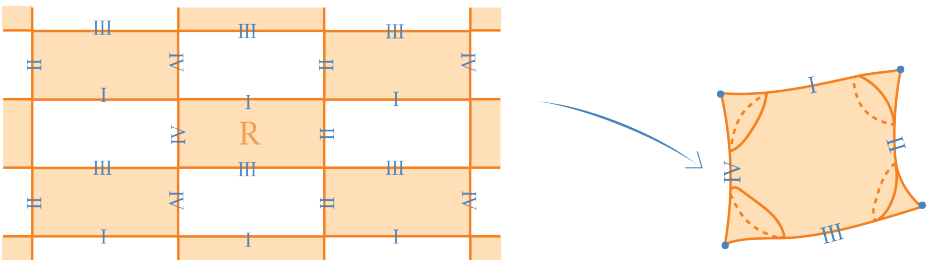


Figure 2.4: Quotient of $\mathbb{R}^2$ by H. Note that H maps the filled rectangles in the tiling into other filled rectangles with the depicted orientation.

Taking into consideration the two examples above, we say that an **orbifold** $\mathcal{O}$ is a topological space locally modelled in the quotient of $\mathbb{R}^n$ by finite group actions with some additional structure. We will first present the rigorous definition given by Thurston [49], but we notice that the orbifolds that appear in order to understand geometrization of 3-manifolds are simpler, and most of the times the intuition behind the definition will suffice.

Definition 2.11 (Thurston)**.** An n-dimensional orbifold $\mathcal{O}$ is a Hausdorff space $X_{\mathcal{O}}$ (called the *base space*) endowed with the following additional structure. There exists a covering of $X_{\mathcal{O}}$ by a collection of open sets $\{U_i\}$ that satisfy:

- The collection $\{U_i\}$ is closed with respect to finite intersections;
- For each U_i there is a finite group Γ_i together with an action of Γ_i over an open set $\widetilde{U}_i \subset \mathbb{R}^n$ and a homeomorphism $\varphi_i \colon U_i \to \widetilde{U}_i/\Gamma_i$;
- If $U_i \subset U_j$ for some i, j, there exists an injective homomorphism $f_{ij} \colon \Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\widetilde{\varphi}_{ij} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j$ that is equivariant with respect to f_{ij}, in the sense that $\widetilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\widetilde{\varphi}_{ij}(x)$ for all $\gamma \in \Gamma_i$, $x \in U_i$, so the diagram below commutes

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\varphi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i"] & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_j \arrow[uu] \arrow[r, phantom, "\subset"] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}$$

We regard $\tilde{\varphi}_{ij}$ as being defined only up to composition with elements of Γ_j, and f_{ij} as being defined up to conjugation by elements of Γ_j. It is not

generally true that $\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should exist an element $\gamma \in \Gamma_k$ such that $\gamma\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$.

If $\mathcal{O}$ is an orbifold and $x \in \mathcal{O}$, we let Γ_x denote the *isotropy group of the point* x, i.e.,

$$\Gamma_x = \{g \in \Gamma_i \mid x \in U_i \text{ and } gx = x\},$$

which is well-defined up to conjugation. In other words, in a neighborhood $U_i = \widetilde{U}_i/\Gamma_i$ of x, Γ_x is the subgroup of Γ_i that acts over $\widetilde{U}_i$ leaving $\widetilde{x}$ invariant, where $\widetilde{x} \in \widetilde{U}_i$ projects over x.

Definition 2.12. The *singular locus* of an orbifold $\mathcal{O}$ is the set $\Sigma_{\mathcal{O}} = \{x \in \mathcal{O} \mid \Gamma_x \neq \{1\}\}$, which means that $\Sigma_{\mathcal{O}}$ is the subset of $\mathcal{O}$ for which there exists a correspondent neighborhood $\widetilde{U}$ with nontrivial isotropy group. If $p \in \Sigma_{\mathcal{O}}$, we say that p is a *singular* point. Otherwise, p is called *regular*. Note that the singular locus is a closed subset of $\mathcal{O}$.

In this article, we are interested in 2-dimensional orbifolds, where the *singular locus* is simpler than in the general situation, admitting a simple description as we next present.

Proposition 2.13 (Thurston [49]). *The* singular locus *of a 2-dimensional orbifold has one of the three local models below:*

1. **Lines of reflection**. $\mathbb{R}^2/\mathbb{Z}_2$, *where* $\mathbb{Z}_2$ *acts by reflection along a line in* $\mathbb{R}^2$*;*

2. **Elliptic points of order** $n \geq 2$. $\mathbb{R}^2/\mathbb{Z}_n$, *where* $\mathbb{Z}_n$ *acts on* $\mathbb{R}^2$ *by a rotation of* $2\pi/n$*;*

3. **Corner reflectors of order** n. $\mathbb{R}^2/D_n$, *where* D_n *is the dihedral group of order* $2n$*, with representation* $\langle a, b \ : \ a^2 = b^2 = (ab)^n = 1\rangle$*, and the generators* a *and* b *correspond to reflections along lines intersecting themselves at an angle* π/n*.*

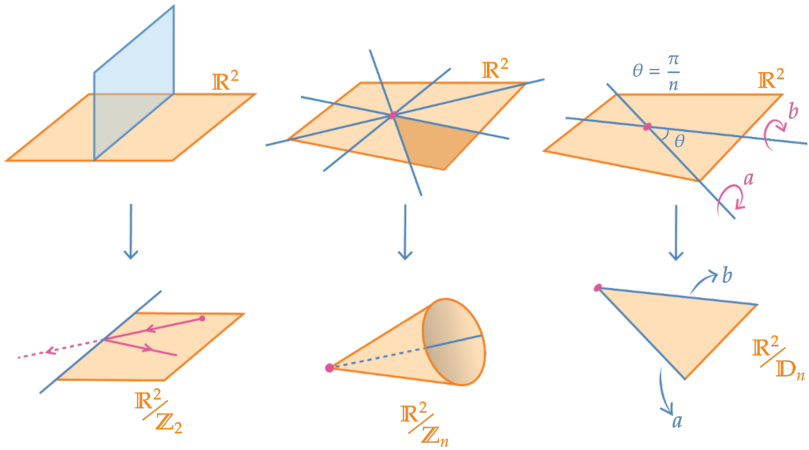


Figure 2.5: Quotients of $\mathbb{R}^2$ by the groups $\mathbb{Z}_2, \mathbb{Z}_n$ and D_n, giving the respective singular loci for a 2-dimensional orbifold: a line of reflection, an elliptic point of order n and a corner reflector of order n.

The proof of the proposition uses that the only finite subgroups of the orthogonal group $\mathrm{O}(2)$ are the ones described above, together with the fact that given a local coordinate system $U = \widetilde{U}/\Gamma$ to an orbifold $\mathcal{O}$, there exists a homeomorphism between a neighborhood of U and a neighborhood of the origin in the orbifold $\mathbb{R}^2/\Gamma$, where $\Gamma \subset \mathrm{O}(2)$ is a finite subgroup. For more details, see Thurston [49, Proposition 13.3.1]

Example 2.14. Recall Example 2.9, where R is a rectangle in $\mathbb{R}^2$ and G is the group of isometries generated by reflections a, b, c, d. Now that we have the definition of orbifold and singular locus, we may explore the example with more depth.

Points in R can be inside the rectangle, or on the edge, or a vertex. It is not hard to notice that points inside the rectangle are not fixed by any element of G. Given $x \in int(R)$ and U_i a neighborhood of x, there is a homeomorphism $\varphi_i : U_i \to \tilde{U}_i$, where $\tilde{U}_i \subset \mathbb{R}^2$.

If $x \in R$ is a point in the edge labeled as d, but not a vertex, for each neighborhood U_i of x we have $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$, where $\tilde{U}_i \subset \mathbb{R}^2$. Γ_i is the group generated by the reflection d and $d^2 = id$, therefore, $G \supset \Gamma_i \cong \mathbb{Z}_2$

and d is a line of reflection.

If x is the vertex of the edges a and d, for each neighborhood U_i of x we have $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$, where $\tilde{U}_i \subset \mathbb{R}^2$. Γ_i is generated by a and d according to the relations $a^2 = id$, $d^2 = id$, $(ad)^2 = id$. By definition $G \supset \Gamma_i \cong D_2$ and the vertex is a corner reflector. In fact, all vertices of R are corner reflectors.

Example 2.15. In Example 2.10 we have $H \subset G$ the group of orientation-preserving isometries generated by reflections a, b, c, d over the lines containing the four sides of R. It is straightforward to see that $d \circ a$, $a \circ c$, $c \circ b$ and $b \circ d$ are all elements of H, each of which leaves one of the four vertexes P_1, P_2, P_3, P_4 of the original rectangle R invariant. Thus, in the pillowcase orbifold $\mathcal{O} = \mathbb{R}^2/H$, $\{P_1, P_2, P_3, P_4\} \subset \Sigma_{\mathcal{O}}$. In fact, these are all the singular points of this orbifold.

If x is the vertex of the edges a and d, for each neighborhood U_i of x we have $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$, where $\tilde{U}_i \subset \mathbb{R}^2$. Γ_i is generated by $d \circ a$, a rotation by π, therefore we have that $\Gamma_i \cong \mathbb{Z}_2$ and x is an elliptic point of order 2, and the same holds for all vertices of R.

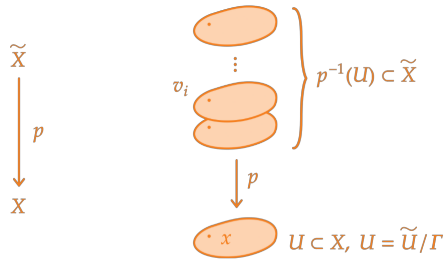
The next proposition is of great importance to the study of orbifolds in the context of geometrization. It may even be used as an intuitive (local) definition of an orbifold.

Proposition 2.16 (Thurston [49, Proposition 13.2.1])**.** *If $\mathcal{M}$ is a manifold and Γ is a group acting properly discontinuously on $\mathcal{M}$, then $\mathcal{M}/\Gamma$ has an orbifold structure.*

The above proposition shows that the structure of an orbifold can be reasonably wild. Next, we present a definition that will be used to introduce the concept of a *good orbifold*.

Definition 2.17. Let $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ be two orbifolds with respective base spaces X, $\widetilde{X}$. We say that $\widetilde{\mathcal{O}}$ is a *covering orbifold* of $\mathcal{O}$ if there is a projection $p\colon \widetilde{X} \longrightarrow X$ such that each $x \in X$ admits a neighborhood $U = \tilde{U}/\Gamma$ ($\tilde{U}$ is an open subset of $\mathbb{R}^n$) for which each component v_i of $p^{-1}(U)$ is

isomorphic to $\tilde{U}/\Gamma_i$, where Γ_i is a subgroup of Γ and the isomorphism respect the projections



Remark 2.18. The main distinction between the concept of covering orbifolds and the usual notion of covering spaces of topological spaces is that when an orbifold $\widetilde{\mathcal{O}}$ covers another orbifold $\mathcal{O}$, the components of the inverse images of an open set U of $\mathcal{O}$ may not be isomorphic to themselves or even to U. This is because each component v_i that projects onto U is a quotient of an open set of $\mathbb{R}^n$ by a subgroup $\Gamma_i \subset \Gamma$, see the example below.

Example 2.19. It is not difficult to see that the *pillowcase orbifold* $\mathbb{R}^2/H$ of Example 2.10 is a covering orbifold of the rectangular billiard $\mathbb{R}^2/G$ of Example 2.9. However, each component of the inverse images of neighborhoods of each of the four vertexes P_1, P_2, P_3, P_4 of $\mathbb{R}^2/G$ are not isomorphic to the image of the projection.

Definition 2.20. An orbifold $\mathcal{O}$ is called a *good* orbifold if $\mathcal{O}$ admits a covering orbifold that is a differentiable manifold. Otherwise, $\mathcal{O}$ is called a *bad* orbifold.

Since the pillowcase and the billiard orbifolds are both covered by $\mathbb{R}^2$, they are both good orbifolds. In fact, in dimension two there are only a few bad orbifolds and their classification can be seen in [49, Theorem 13.3.6].

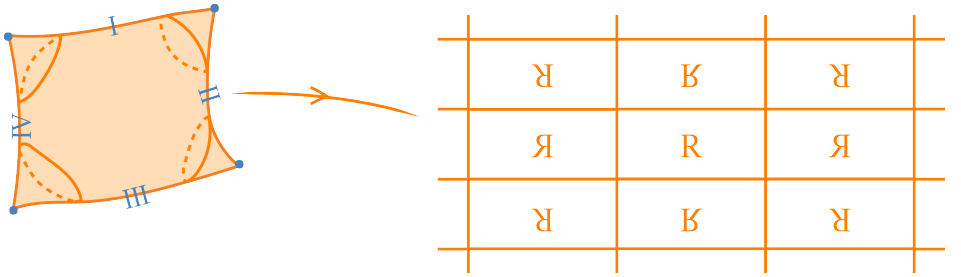


Figure 2.6: A rectangular billiard with the pillowcase orbifold as a covering orbifold.

3 3-manifolds

After noticing that geometry and topology may be deeply related and presenting some initial concepts such as orbifolds, we will now start the study of the geometric topology of 3-manifolds, with the fundamental goal of presenting the *geometrization theorem* (for orientable 3-manifolds).

Recall Theorem 2.7, where given an orientable, closed surface S, we could find a *special metric* for S, as being a metric of constant curvature k being $-1, 0$ or 1, the number k depending uniquely on the topology of S. The attempt to generalize this result to the class of orientable closed 3-manifolds is quite natural. However, (presently) it is easy to see that the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is an orientable, closed 3-manifold that does not admit a metric of constant curvature. Indeed if that was the case, its Riemannian universal cover would be $\mathbb{S}^2 \times \mathbb{R}$ with a metric of constant curvature. Since $\mathbb{S}^2 \times \mathbb{R}$ is simply connected, it would be isometric to a space form and diffeomorphic to either $\mathbb{S}^3$ or to $\mathbb{R}^3$.

This creates a great difficulty on understanding what could be *the best metric* for a 3-manifold. And even Poincaré, after proving the Uniformization Theorem for surfaces, struggled with this question. After trying to generalize this to 3-manifolds, he noticed it wouldn't be an easy task. To get closer to this geometric classification, Poincaré developed several topological concepts, such as the homology groups and the fundamental group

of a manifold, arriving at the following question [44, Page 110].

Consider a compact 3-dimensional manifold V without boundary. Is it possible that the fundamental group of V could be trivial, even though V is not homeomorphic to the 3-dimensional sphere?

After adding that *cette question nous entraînerait trop loin* (this question would take us too far) and trying to prove this result, his question became known as the *Poincaré's Conjecture*, a widely known problem that, in the year 2000, was deemed by the Clay Mathematics Institute, one of the seven *millennium problems* (presently, the only one with a complete solution).

Over the next sections, we will present the theory of the topology and geometry of 3-manifolds, showing the notion of a *model geometry* and explaining the geometrization theorem, which proves (and generalizes) Poincaré's conjecture by decomposing any orientable, closed 3-manifold into components, each of which admits a model geometry that depends uniquely on its topology.

3.1 Seifert fibered spaces

The next step towards geometrization is to present the concept of a *Seifert fibered space*. There are several ways of introducing those spaces, and we chose to present them from a geometric point of view, as 3-manifolds that admit a decomposition by circles (or fibers) with a certain structure called a *Seifert fibration*. In particular, we must distinguish these two nomenclatures: a Seifert fibered space will be the total space of a Seifert fibration (and a Seifert fibered space may have distinct Seifert fibrations related to it).

As suggested by the denomination of the manifolds studied in this section, Seifert fibered spaces were first studied by H. Seifert in the 1930's, with the intention of getting closer to the topological classification of closed 3-manifolds (or the "*homeomorphism problem for 3-dimensional closed manifolds*"). In [47] (see the book [48], which contains a geometric introduction to Topology and also an English translation of [47]), Seifert was

able to completely classify, up to fiber-preserving homeomorphisms, his fibered spaces (*gefaserte Räume*), and his research was a fundamental step towards geometrization, as it will become clear in the JSJ decomposition presented in Theorem 3.22 below.

Before we present the precise definition of a Seifert fibered space, we will introduce the concept of a *fibered solid torus*. Consider the (closed) unit disk of dimension two, $\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$ and let the *trivial fibered solid torus* be the product $\mathbb{D}^2 \times \mathbb{S}^1$, endowed with the product foliation by circles, so that, for each $y \in \mathbb{D}^2$, $\{y\} \times \mathbb{S}^1$ is a fiber.

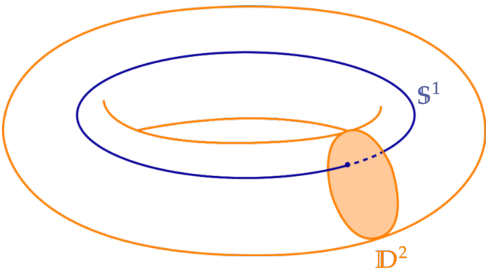


Figure 3.1: Trivial fibered solid torus: each fiber is of the type $\{y\} \times \mathbb{S}^1$, for $y \in \mathbb{D}^2$.

This trivial decomposition of $\mathbb{D}^2 \times \mathbb{S}^1$ by circles is a particular case of the next more general construction, which, in the context of orientable Seifert fibered spaces, provides the local picture.

Definition 3.1. Given a pair p, q of co-prime integers with $p > 0$, the *standard fibered solid torus* $T(p, q)$ is obtained from the trivial fibered solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ by cutting along a disk $\mathbb{D}^2 \times \{y\}$ and gluing it back together after a twist of $\frac{2q\pi}{p}$ in one of its sides, i.e.,

$$T(p, q) = \frac{\mathbb{D}^2 \times [0, 1]}{\{(z, 0) \sim (\psi_{p,q}(z), 1)\}},$$

where $\psi_{p,q} : \mathbb{D}^2 \to \mathbb{D}^2$ is defined by $\psi_{p,q}(z) = e^{\frac{2\pi q}{p} i} z$. Then, $T(p, q)$ is a solid torus, naturally endowed with a fibration by circles where we have a *central fiber* (or *core fiber*) that comes from $\{0\} \times [0, 1]$ and any other fiber

rotates p times around the generator of the fundamental group of the solid torus and q times around the central fiber.

Example 3.2. Let $p = 8$, $q = 3$ and let $T(8,3)$ be the corresponding standard fibered solid torus, given by the identification $(\mathbb{D}^2 \times [0,1])/\sim$ where $\mathbb{D}^2 \times \{0\} \ni (z,0) \sim (e^{\frac{3\pi}{4}i}z, 1) \in \mathbb{D}^2 \times \{1\}$.

Let $x \in \mathbb{D}^2$ be given. If $x = 0$, then the line $\{0\} \times [0,1]$ in $\mathbb{D}^2 \times [0,1]$ descends to the central fiber of $T(8,3)$. Otherwise, let $x_1 = x$ and let $x_{j+1} = e^{\frac{3\pi}{4}i}x_j$, for $j \in \mathbb{N}$. Then $x_9 = x_1$ and the union of the eight lines $\{x_j\} \times [0,1]$, $j = 1, 2, \ldots, 8$, descend to $T(8,3)$ as one fiber, that intersects any of the meridianal disks $\mathbb{D}^2 \times \{t\}$ in $T(8,3)$ eight times.

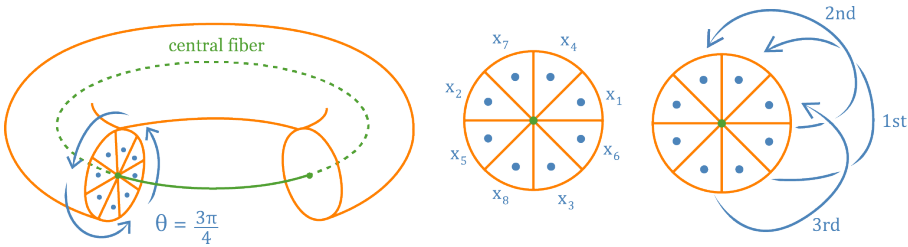


Figure 3.2: x_j is the intersection of the fiber with the disk after $j-1$ turns.

In the torus $T(p,q)$, p is called the *multiplicity of the central fiber*. When $p > 1$, we say that the central fiber is *singular*, because this fiber goes around the solid torus one time, while all the other fibers go around p times. Otherwise, the central fiber is called *regular*.

Having defined the structure of the standard fibered solid torus, we will next present the concept of a Seifert Fibered space in the context of orientable 3-manifolds.

Definition 3.3. A *Seifert fibered space* is an orientable 3-manifold $\mathcal{M}$ that admits a decomposition into circles (called *fibers*) such that each fiber admits a neighborhood U that is the union of other fibers and U is isomorphic (as a fibered space) to a standard fibered solid torus. A fiber of a Seifert fibered space is called *regular* if it admits a neighborhood isomorphic (as a fibration) to the trivial fibered solid torus. Otherwise, it is called *singular*.

We notice that the same manifold $\mathcal{M}$ may admit more than one distinct decomposition by circles. Hence, whenever we say that $\mathcal{M}$ is a Seifert fibered space, we are assuming that the circle decomposition is fixed. We also observe that there are manifolds which do not admit any such decomposition, such as open simply connected spaces, see [47, 48].

Example 3.4. Let $C = [0,1] \times [0,1] \times [0,1]$ be a solid cube and let $\mathcal{C}$ be the 3-manifold obtained by identifying the opposite sides of C as follows (see Figure 3.3):

$$(0,y,z) \sim (1,y,z) \text{ (left to right)}, \quad (x,0,z) \sim (x,1,z) \text{ (front to back)},$$

$$(x,y,0) \sim (1-x,1-y,1) \text{ (top to a } 180^\circ \text{ rotation of the bottom)}.$$

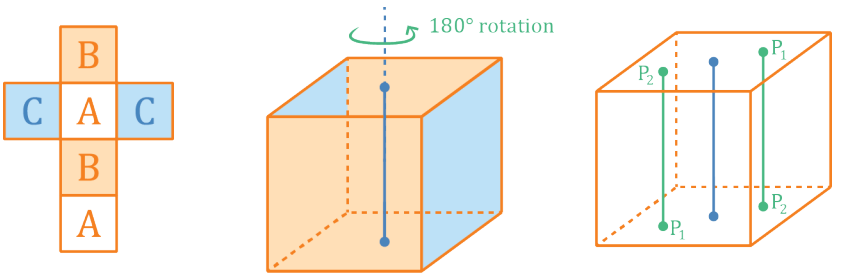


Figure 3.3: Identifications of the cube constructing the Seifert fibered space of Example 3.4.

We may see that the closed, orientable 3-manifold $\mathcal{C}$ is a Seifert fibered space. Indeed, $\mathcal{C}$ admits a decomposition in circles that arise from the vertical lines in C, joining the top side to the bottom side of C. In the remainder of this example, we will let $I_{(x,y)} \subset \mathcal{C}$ denote the equivalence class of the vertical segment $\{(x,y,t) \mid t \in [0,1]\} \subset C$ and we will let $\gamma_{(x,y)}$ denote the respective fiber in $\mathcal{C}$ that contains $I_{(x,y)}$. We have that, for all $(x,y) \in [0,1] \times [0,1]$, $\gamma_{(x,y)} = I_{(x,y)} \cup I_{(1-x,1-y)} = \gamma_{(1-x,1-y)}$ is a regular fiber with the following four exceptions, each of which has a neighborhood isomorphic to $T(2,1)$ (see Figure 3.4).

- $\gamma_{(0,0)} = \gamma_{(1,0)} = \gamma_{(1,1)} = \gamma_{(0,1)}$, which comes from $I_{(0,0)} = I_{(0,1)} = I_{(1,1)} = I_{(1,0)}$;
- $\gamma_{(0,1/2)} = \gamma_{(1,1/2)}$, which comes from $I_{(0,1/2)} = I_{(1,1/2)}$;
- $\gamma_{(1/2,0)} = \gamma_{(1/2,1)}$, which comes from $I_{(1/2,0)} = I_{(1/2,1)}$;
- $\gamma_{(1/2,1/2)}$, which comes from $I_{(1/2,1/2)}$.

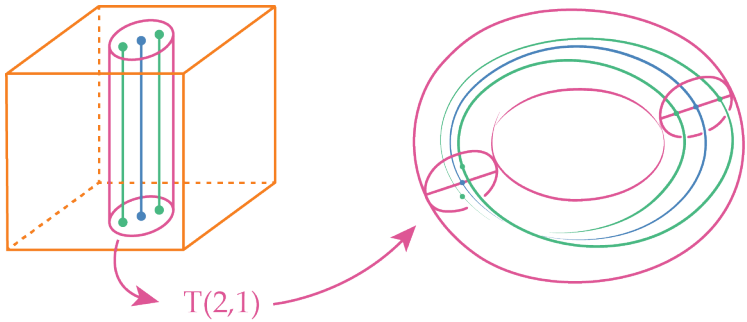


Figure 3.4: Identification of a neighborhood of $\gamma_{(1/2,1/2)}$ in $\mathcal{C}$ and $T(2,1)$.

A fundamental result by Epstein [17] is that in the context of 3-manifolds, a foliation by circles is equivalent to being a Seifert fibered space.

Theorem 3.5 (Epstein)**.** *If $\mathcal{M}$ is a compact 3-manifold that admits a foliation by circles, then $\mathcal{M}$ has the structure of a Seifert fibered space.*

The topology of a Seifert fibered space can be understood in terms of the way the fibers lie in the manifold. In this sense, we introduce the definition of a *Seifert fibration* and of the *base space* of a Seifert fibered manifold.

Definition 3.6 (Seifert fibration)**.** Let $\mathcal{M}$ be a 3-dimensional Seifert fibered space and let X be the topological space obtained by collapsing each fiber of $\mathcal{M}$ to a point. More precisely, X is the quotient of $\mathcal{M}$ by the equivalence relation $x \sim y$ if and only if x and y are in the same fiber of

$\mathcal{M}$. We say that X is the *base space* for $\mathcal{M}$. Furthermore, if π is the map $\pi\colon \mathcal{M} \to X$ that maps a point x in a given fiber of $\mathcal{M}$ to the equivalence class $[x] \in X$, we say that the triple $(\mathcal{M}, X, \pi)$ is the *Seifert fibration* of $\mathcal{M}$.

One of the most important properties of X as above is that it admits a structure of a good orbifold, where each singular fiber of $\mathcal{M}$ becomes a cone point in X and $\pi\colon \mathcal{M} \to X$ is a fibration in the orbifold sense. With that in mind, we will say that X is the *base orbifold* of $\mathcal{M}$.

Proposition 3.7. *Let $\mathcal{M}$ be a 3-dimensional Seifert fibered space and let $(\mathcal{M}, X, \pi)$ be the Seifert fibration of $\mathcal{M}$. Then, X admits the structure of a good orbifold.*

A rigorous proof of Proposition 3.7 can be found in [46, Chapter 3]. We next present a pictorial description of this good orbifold structure.

Idea of the proof of Proposition 3.7. Let $\mathcal{M}$ be a Seifert fibered space with respective Seifert fibration $(\mathcal{M}, X, \pi)$. Let x be a point in X with respective fiber $S = \pi^{-1}(\{x\}) \subset \mathcal{M}$. Then, S admits a neighborhood V, composed by fibers, that descends to a a neighborhood U of x in X. First, assume that V is isomorphic to a standard fibered solid torus $T(p, q)$. Then, if $p = 1$, U is diffeomorphic to the disk $\mathbb{D}$ and if $p > 1$, U is isomorphic to a neighborhood of the cone point in the orbifold $\mathbb{R}^2/\mathbb{Z}_p$. On the other hand, if V is isomorphic to a standard fibered Klein bottle, X will be an orbifold with nonempty boundary and U is double-covered by $\mathbb{D}$ with a $\mathbb{Z}^2-$action, being a neighborhood of a line of reflection. $\square$

Example 3.8. Let $\mathcal{C}$ be the Seifert fibered space given by Example 3.4. Next, we will show that the base orbifold X of $\mathcal{C}$ is the pillowcase orbifold of Example 2.10. The fibers of $\mathcal{C}$ are parameterized by $(x, y) \in [0, 1] \times [0, 1]$, so the quotient of $[0, 1] \times [0, 1]$ by the identifications $(x, y) \sim (1 - x, 1 - y)$, $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ give the structure of X. It is not difficult to see that this structure is the same as $D = [0, 1] \times [0, 1/2]$, glued along its boundaries by $(0, y) \sim (1, y)$, $(x, 0) \sim (1 - x, 0)$ and $(x, 1/2) \sim (1 - x, 1/2)$

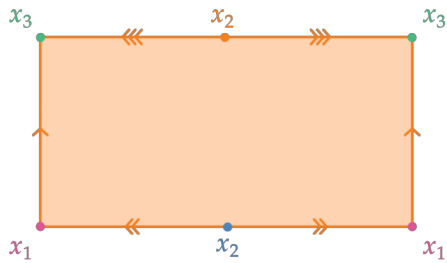


Figure 3.5: The domain D with the respective identifications along its boundary. The points x_1 (representing $\gamma_{(0,0)}$), x_2 (representing $\gamma_{(1/2,0)}$), x_3 (representing $\gamma_{(0,1/2)}$) and x_4 (representing $\gamma_{(1/2,1/2)}$) are the four cone points of the orbifold X, each of which comes from a singular fiber of $\mathcal{C}$.

(see Figure 3.5), showing that X is topologically a sphere with four cone points which is the pillowcase orbifold.

3.1.1 Classification and the Euler number

As seen previously, we may interpret a Seifert fibered space as a fibration over an orbifold where the fibers are all diffeomorphic to $\mathbb{S}^1$. Along this section, we will follow the construction of A. Hatcher [24, Section 2.1] to present a classification (up to isomorphisms[6]) of Seifert fibered spaces and introduce a topological invariant called the Euler number. This invariant will be useful both for distinction of Seifert fibered spaces and also for identifying existence of horizontal surfaces (see Definition 3.13) on closed Seifert fibered spaces.

We start this discussion by presenting a general construction of a Seifert fibered space.

Let S be a compact surface (S is either orientable or non orientable, and has possibly nonempty boundary). For a given $n \in \mathbb{N}$, let D_1, D_2, $\ldots$, D_n

[6]We say that two Seifert fibrations $(\mathcal{M}_1, X_1, \pi_1)$ and $(\mathcal{M}_2, X_2, \pi_2)$ are isomorphic if there exists a diffeomorphism $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ that carries the fibers of π_1 to the fibers of π_2.

be a pairwise disjoint collection of closed disks in the interior of S and let S' be the closure of $S \setminus (D_1 \cup \ldots \cup D_n)$, so S' is a compact surface with nonempty boundary. Let M' be the total space of the orientable circle bundle over S'. Specifically, if S' is orientable, $M' = S' \times \mathbb{S}^1$. Otherwise, $M' = S' \widetilde{\times} \mathbb{S}^1$ can be defined as follows. Let S' be given by an identification of pairs a_i, b_i of oriented arcs in the boundary of a topological closed disk D (see the example in Figure 3.6). Then, M' can be obtained from $D \times \mathbb{S}^1$ by identifying the surfaces $a_i \times \mathbb{S}^1$ and $b_i \times \mathbb{S}^1$ via the product of the given identification $a_i \sim b_i$ with either the identity or a reflection in the $\mathbb{S}^1$ factor, whichever makes M' orientable. In particular, M' is compact and each boundary component of M' has the topology of a 2-torus.

The construction presented above allows us to see the circle bundle $\pi\colon M' \to S'$ as the double of an $[0,1]$-bundle, thus, there exists a well defined global cross section $\sigma\colon S' \to M'$, i.e., σ is a continuous function and $\pi(\sigma(x)) = x$ for all $x \in S'$. Next, we will make use of σ, together with an orientation on M', to obtain a well-defined notion of *slopes*[7] for nontrivial simple closed curves in the boundary components of M'. Fix T a component of $\partial M'$ and let $m = \sigma(\sigma^{-1}(T))$. Since $\sigma^{-1}(T)$ is a boundary component of S', m is a nontrivial closed curve in T. Now, choose any $p \in m$ and consider the fiber over p, $l = \pi^{-1}(\{p\})$. Once again, l is a nontrivial closed curve in T and, since σ is a cross section of π, $m \cap l = \{p\}$. Moreover, the curves m and l (which are also known as *natural curves* for T, see Figure 3.7) generate $\pi_1(T, p)$. Then, there exists a diffeomorphism $\varphi\colon T \to \mathbb{S}^1 \times \mathbb{S}^1$ such that φ maps m to $\mathbb{S}^1 \times \{0\}$ (a slope 0 curve) and l to $\{0\} \times \mathbb{S}^1$ (a slope ∞ curve). For simplicity, when we talk about the slope $\frac{r}{s}$ of a given curve, we are assuming that r and s are co-primes.

The next step in our construction of a Seifert fibered 3-manifold is to *fill*

[7]Recall that the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ has universal covering map defined by $(x, y) \in \mathbb{R}^2 \mapsto (e^{2\pi x i}, e^{2\pi y i})$, and that a line $\{y = \alpha x\} \subset \mathbb{R}^2$ descends to $\mathbb{T}^2$ as a simple closed curve $c_\alpha \subset \mathbb{T}^2$ if and only if $\alpha \in \mathbb{Q}$. We may also extrapolate this definition to allow the curve $\{x = 0\}$ to be seen as the $\alpha = \infty$ case. Moreover, using this model, a nontrivial simple closed curve in $\mathbb{T}^2$ is always isotopic to a unique curve c_α as defined above, for some $\alpha \in \mathbb{Q} \cup \{\infty\}$, and the number α is defined as the *slope* of the curve.

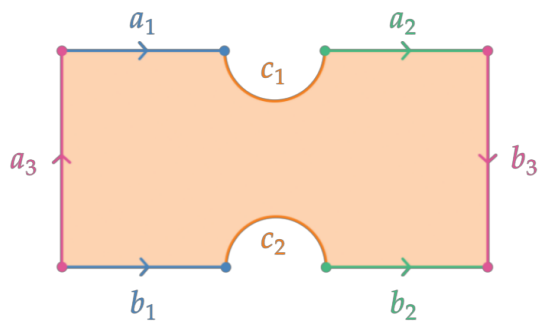


Figure 3.6: Let S' be the Klein bottle with one disk removed. Then, S' can be constructed from a topological disk D (highlighted in the above figure) by identifying the oriented arcs $a_1 \sim b_1$, $a_2 \sim b_2$ and $a_3 \sim b_3$. Note that c_1 and c_2 are not identified other than by its shared endpoints with the arcs a_1, b_1, a_2 and b_2.

the boundary components of M' that are generated from $\partial D_1, \ldots, \partial D_n$ by attaching solid tori to them. By an abuse of notation, in the remainder of this construction when we choose *a boundary component* of M' we will assume, without further comments, that it is one of the n components we just described. Note that for each such component, there are infinitely many ways of doing such a gluing, which is called a *Dehn filling*. Let T be a boundary component of M' with natural curves m and l as defined in the previous discussion. Then, after choosing orientations, (m, l) provides a positively oriented basis for the first homology group $H_1(T, \mathbb{Z})$ and the curve with slope $\frac{r}{s}$, defined as $\gamma = rl + sm$ is prime, in the sense that it is not a multiple $k\gamma'$ (in homology) of another curve γ' unless $|k| = 1$ and $\gamma' = \pm\gamma$.

Definition 3.9. The Dehn filling of T generated by $\frac{r}{s} \in \mathbb{Q}$ is the unique (up to homeomorphism) manifold generated by gluing a solid torus $\mathbb{D} \times \mathbb{S}^1$ to T by its boundary in such a way that the boundary of the *meridianal disk* $\mathbb{D} \times \{0\}$ is glued (by a diffeomorphism) to a curve of slope $\frac{r}{s}$ in T. In other words, the Dehn filling of T determined by $\frac{r}{s}$ glues a solid torus to T, making the curve $rl + sm$ (and therefore any of its multiples) trivial in

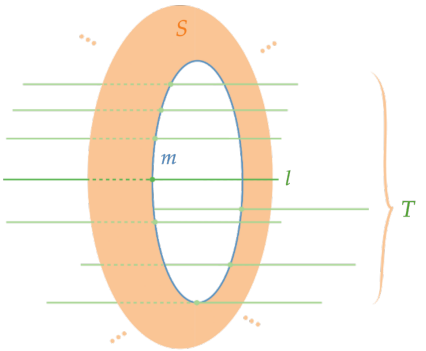


Figure 3.7: The natural curves m and l on a torus boundary component T of a Seifert fibered space.

homology.

Let M^1 be the manifold obtained by performing a Dehn filling generated by a slope $\frac{r_1}{s_1}$ in a torus boundary of M'. Then, the circle bundle over M' extends naturally to a circle bundle over M^1, since the fibers (slope ∞) are not isotopic to meridian circles in the attached $\mathbb{D} \times \mathbb{S}^1$. Intuitively, the Dehn-filling as above glues a neighborhood with the structure of a $T(s_1, r_1)$ fibered solid torus to the original Seifert fibration of M'. In particular, the base space of the new fibration has the structure of an orbifold (possibly with boundary) which has one cone point of multiplicity s_1 (if $s_1 = 1$, the fiber is regular).

The above observation that the original circle bundle $M' \to S'$ extended after performing one Dehn filling in one boundary component of M' allows us to repeat the process, generating the following resulting manifold.

Definition 3.10. Let S be a compact surface and let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n} \in \mathbb{Q}$. Then, the 3-manifold

$$M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$$

is the resulting Seifert-fibered space after performing n Dehn fillings with slopes $\frac{r_i}{s_i}$ on the boundary components of M' as previously described.

Note that, by construction, $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$ has a Seifert fibration over the orbifold $(S, x_1, ..., x_n)$, where each x_i is a cone point in S with multiplicity s_i. There are several questions regarding this construction, and we expect the next proposition to answer many, if not all of them. A proof of it can be found in Hatcher [24, Proposition 2.1].

Proposition 3.11. *Using the notation introduced by Definition 3.10, the following hold:*

1. *Every compact, orientable Seifert fibered 3-manifold is isomorphic to one of the models* $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$.
2. $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$ *is isomorphic to* $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n}, 0)$.
3. $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$ *and* $M(S, -\frac{r_1}{s_1}, -\frac{r_2}{s_2}, \ldots, -\frac{r_n}{s_n})$ *are related by a change of orientation.*
4. $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$ *is isomorphic to* $M(S, \frac{r'_1}{s'_1}, \frac{r'_2}{s'_2}, \ldots, \frac{r'_n}{s'_n})$ *by an orientation preserving diffeomorphism if and only if the following two conditions hold:*
   (a) *After a permutation of indices, it holds, for all* $i \in \{1, 2, \ldots, n\}$, $\frac{r_i}{s_i} \equiv \frac{r'_i}{s'_i} \bmod 1$.
   (b) *If* $\partial S = \emptyset$, $\displaystyle\sum_{i=1}^{n} \frac{r_i}{s_i} = \sum_{i=1}^{n} \frac{r'_i}{s'_i}$.

Proposition 3.11, together with the construction of Definition 3.10, gives a complete classification of Seifert fibrations, up to isomorphisms. We note that, in order to obtain a classification of Seifert fibered spaces up to diffeomorphisms, more work still needs to be done, since there are diffeomorphic Seifert fibered spaces which are not isomorphic as Seifert fibrations (in other words, there are manifolds with more than one Seifert

fibration structure). For such a classification, we suggest [32, Chapter 10] or [24, Theorem 2.3].

We are now ready to define an invariant of a Seifert fibration, which is the *Euler number*. Note that this invariant is well defined by item 4b of Proposition 3.11.

Definition 3.12 (Euler number)**.** Let η be a Seifert fibration given by $M(S, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_n}{s_n})$ with a closed, orientable total Seifert fibered space $\mathcal{M}$. Then, the *Euler number* of η is

$$e(\eta) = \sum_{i=1}^{n} \frac{r_i}{s_i}.$$

Intuitively, other than being helpful for distinguishing Seifert fibrations, the Euler number measures how far we are from obtaining a section for the respective fiber bundle. The next definition (and the following Proposition 3.14) make this intuition precise.

Definition 3.13. Let $\mathcal{M}$ be a compact Seifert fibered space (together with a Seifert fibration) and let $\Sigma \subset \mathcal{M}$ be a closed surface embedded in $\mathcal{M}$. We say that Σ is *vertical* if it is a union of regular fibers (in this case, Σ is either a torus or a Klein bottle whose projection over the base orbifold X is a simple closed curve in the complement of the cone points of X). On the other hand, we say that Σ is *horizontal* if Σ is everywhere transverse to the fibers.

A proof to Proposition 3.14 can be found in [24, Proposition 2.2].

Proposition 3.14. *Let $\mathcal{M}$ be a compact, orientable Seifert fibered space with respective Seifert fibration η. Then:*

1. *If $\partial\mathcal{M} \neq \emptyset$, then there exists a horizontal surface in $\mathcal{M}$.*

2. *If $\partial\mathcal{M} = \emptyset$, then there exists a horizontal surface in $\mathcal{M}$ if and only if $e(\eta) = 0$.*

3.2 Decomposition of 3-manifolds

Having defined the concept of a Seifert fibered space, we are able to present some of the main developments of the theory of 3-manifolds before geometrization. The natural path for understanding any mathematical object is to try to break it up into simpler pieces, and that was firstly obtained using the concept of connected sum (see Definition 2.6). Note that the 3-sphere $\mathbb{S}^3$ acts as the *neutral element* for the connected sum of 3-manifolds, since for any 3-manifold M, the connected sum $M\#\mathbb{S}^3$ is diffeomorphic to M.

Definition 3.15. A 3-manifold $\mathcal{M}$ is called *prime* if any connected sum $\mathcal{M} = \mathcal{M}_1\#\mathcal{M}_2$ is trivial in the sense that either $\mathcal{M}_1$ or $\mathcal{M}_2$ is the 3-sphere $\mathbb{S}^3$.

Note that if a 3-manifold $\mathcal{M}$ is not prime, then there exists a decomposition of $\mathcal{M}$ in a nontrivial connected sum $\mathcal{M} = \mathcal{M}_1\#\mathcal{M}_2$. In particular, there is an embedded topological 2-sphere $S \subset \mathcal{M}$ that separates $\mathcal{M}$ into two regions, one diffeomorphic to $\mathcal{M}_1 \setminus B^3$ and another diffeomorphic to $\mathcal{M}_2 \setminus B^3$, where B^3 represents the 3-ball. Thus, we may introduce the closely related notion of an irreducible manifold as follows:

Definition 3.16. We say that a 3-manifold $\mathcal{M}$ is *irreducible* if any embedded 2-sphere in $\mathcal{M}$ is the boundary of a 3-ball in $\mathcal{M}$.

It is straightforward to see that if $\mathcal{M}$ is irreducible, then $\mathcal{M}$ is prime. The converse does not hold, since for a given $p \in \mathbb{S}^1$, the sphere $\mathbb{S}^2 \times \{p\}$ does not bound any 3-ball in the prime 3-manifold $\mathbb{S}^2 \times \mathbb{S}^1$. But in fact, the only closed, orientable prime 3-manifold that is not irreducible is $\mathbb{S}^2 \times \mathbb{S}^1$.

The next result, due to Kneser [29] and Milnor [36], establishes that any closed orientable 3-manifold admits a *unique* decomposition by prime factors.

Theorem 3.17 (Kneser-Milnor)**.** *Let $\mathcal{M}$ be a closed, oriented 3-manifold. Then, there are closed, oriented, prime 3-manifolds $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$*

such that $\mathcal{M}$ is homeomorphic to the connected sum $\mathcal{M}_1\#\mathcal{M}_2\#\ldots\#\mathcal{M}_k$. Furthermore, the nontrivial factors in this decomposition are unique up to reordering and orientation-preserving diffeomorphisms.

Although Theorem 3.17 cuts a closed, oriented 3-manifold along spheres, providing a standard decomposition over prime factors (and thus along irreducible and $\mathbb{S}^2 \times \mathbb{S}^1$ factors), it is not sufficient to understand the topology of 3-manifolds, since even restricting to irreducible, closed, oriented 3-manifold, this classification is not an easy task. Another step towards this goal was to obtain another standard decomposition, by the (in some sense) second simplest topology of a surface, which is the decomposition of any irreducible, closed and orientable 3-manifold along tori. This decomposition (which will be presented in Theorem 3.22 below) is called the JSJ decomposition, an acronym to the names of the researchers that proved its existence: Jaco-Shalen [26] and Johannson [27]. Before presenting the statement of this decomposition, we need a few extra definitions.

Definition 3.18. Let $\mathcal{M}$ be a compact 3-manifold and let S be a surface properly embedded[8] in $\mathcal{M}$. A *compression disk* D for S is an embedded disk in $\mathcal{M}$ which intersects S transversely, and such that $\partial D = D \cap S$ does not bound a disk in S. Furthermore, if S admits a compression disk, we say that S is *compressible*, and if S is not compressible and not a 2-sphere, we say that S is *incompressible*.

In some sense, the next definition gives the equivalent definition of a prime 3-manifold in the context of a torus decomposition.

Definition 3.19 (Atoroidal manifold)**.** Let $\mathcal{M}$ be a compact 3-manifold with empty or toroidal boundary. We say that $\mathcal{M}$ is *atoroidal* (or homotopically atoroidal) if any (immersed) incompressible torus is homotopic to a component of $\partial\mathcal{M}$.

Closely related to the notion of atoroidal manifolds is the following definition.

[8]In this setting, we say that S is proper if S is compact and $\partial S = S \cap \partial M$.

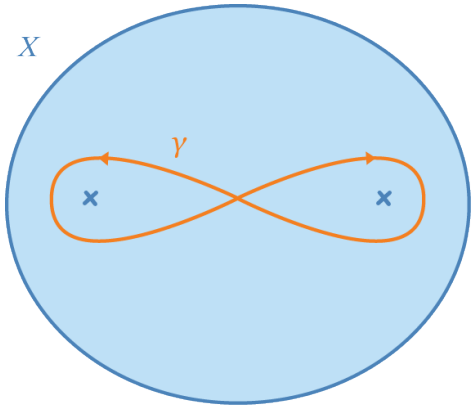


Figure 3.8: Let X be a *pair of pants*, i.e. $X = \mathbb{D} \setminus \{p_1, p_2\}$ is the open disk, punctured twice. Then, the manifold $\mathcal{M} = X \times \mathbb{S}^1$ is geometrically atoroidal (since any torus in $\mathbb{D} \times \mathbb{S}^1$ separates) but not homotopically atoroidal, since the torus $\gamma \times \mathbb{S}^1$, where γ is the curve depicted above, is incompressible and not boundary-parallel. However, $\mathcal{M}$ is a small Seifert fibered space.

Definition 3.20 (Geometrically atoroidal manifold)**.** Let $\mathcal{M}$ be a compact 3-manifold with (possibly empty) toroidal boundary. We say that $\mathcal{M}$ is *geometrically atoroidal* if any embedded, incompressible torus is isotopic to a component of $\partial\mathcal{M}$.

Remark 3.21. Any atoroidal manifold is geometrically atoroidal. However, the converse does not hold, as it is easy to see by the example of Figure 3.8. Nonetheless, it is true that if $\mathcal{M}$ is a compact 3-manifold with (possibly empty) toroidal boundary and $\mathcal{M}$ is geometrically atoroidal, then $\mathcal{M}$ is atoroidal unless it is a *small Seifert fibered space*, in the sense that it is a Seifert fibered space and the base orbifold has genus zero and the number of cone points, together with the number of boundary components, is at most three.

Having defined the concepts of a Seifert fibered space and of an atoroidal manifold, we may now present the JSJ decomposition theorem.

Theorem 3.22 (JSJ decomposition)**.** *Let $\mathcal{M}$ be an irreducible, compact and orientable 3-manifold with empty or toroidal boundary. Then, $\mathcal{M}$ admits a (possibly empty) pairwise disjoint collection $T_1, T_2, \ldots, T_n$ of embedded, incompressible tori that separate $\mathcal{M}$ into components, each of which is either atoroidal or Seifert fibered. Furthermore, any collection of such tori with the minimal number of elements is unique, up to isotopy.*

Theorem 3.22 gives a good reason for we to work with incompressible tori, since using this concept avoids artificial decompositions such as cutting $\mathbb{S}^3$ by some *knotted torus*, separating it into a component that is a solid torus and another component a knot complement (we will revisit knot complements in Section 4).

After the JSJ decomposition, the next step in order to classify the topology of 3-manifolds was to understand atoroidal and Seifert fibered compact 3-manifolds with (possibly empty) toroidal boundary that appeared in a finer version of the JSJ decomposition (see Remark 3.27 in Section 3.3). However, this was not an easy task, as it is easy to assume because the Poincaré Conjecture, which arguably dealt with the simplest possible topology among 3-manifolds, was still open and would play a definite role on this subject. The next step towards this goal was given by Thurston, which put the Poincaré Conjecture as a particular case in a broad context, by understanding that that geometrization could be achieved for toroidal decompositions for 3-manifolds. We remark that there are several equivalent (but perhaps not so easily seen as being equivalent) statements of the geometrization in the literature, and we will focus in presenting the ones that appear more geometric.

3.3 Geometrization of 3-manifolds

The history of geometrization actually starts with Henri Poincaré, which, after proving the Uniformization theorem (Theorem 2.7, also known as the geometrization for surfaces), developing several important tools for geometry and topology, such as homology, homeomorphism and fundamen-

tal group, started to wonder whether homology could be sufficient to characterize a topology (the answer is no, as Poincaré himself answered with his example of a *homology sphere*), generating, in this process, the Poincaré Conjecture. But it was William Thurston that was able to tackle the problem of bringing the geometrization for 3-manifolds. In fact, Thurston proved several results concerning geometrization of 3-manifolds, including the fact that there are 8 maximal *model geometries* (see Definition 3.23 below) of dimension 3 that admit compact quotients, and the geometrization itself for a broad class of 3-manifolds. For his groundbreaking work, he was awarded with a Fields medal in the ICM held in Warsaw, Poland, 1982[9].

Definition 3.23 (Model geometry)**.** A *geometry* $\mathbb{X}$ is a pair (X, G) where X is a simply connected manifold and G is a Lie group acting on X transitively, via diffeomorphisms, and with compact isotropy[10] groups. When the group G is maximal with respect to all Lie groups acting on X transitively, via diffeomorphisms and with compact isotropy groups, $\mathbb{X} = (X, G)$ is called a *model geometry*[11]. We say that two geometries (X, G) and (X', G') are equivalent if there exists a diffeomorphism $X \to X'$ that maps the action of G to the action of G'.

Geometrically, a model geometry can be seen as a simply connected homogeneous manifold X, together with the action of its full isometry group $\mathrm{ISO}(X)$, and two geometries are equivalent if the corresponding Riemannian manifolds are isometric. Using this point of view, we say that a manifold $\mathcal{M}$ has a *geometric structure* modelled over a geometry $\mathbb{X} = (X, \mathrm{ISO}(X))$ if $\mathcal{M}$ admits a Riemannian metric such that its Riemannian universal covering is isometric to X, and we say that such a geometric

[9] Actually, due to the introduction of the Martial Law in Poland in December 1981, the conference was postponed until 1983.

[10] If G is a group acting on a manifold X, for each $x \in X$ the isotropy group over x is the stabilizer subgroup $G_x = \{g \in G \mid gx = x\}$.

[11] Note that if (X, G) is a geometry that is not maximal, it can always be extended to a model geometry (X, G').

structure is *complete* if this metric over $\mathcal{M}$ is complete. Next, we present a more rigorous, group-theoretical, definition for a geometric structure.

Definition 3.24 (Geometric structure)**.** Let $\mathcal{M}$ be a manifold. We say that $\mathcal{M}$ admits a *geometric structure* over a geometry $\mathbb{X} = (X, G)$ if there exists a diffeomorphism between $\mathcal{M}$ and X/Π, where Π is a discrete subgroup of G acting freely on X.

Using the concepts of a model geometry and of a geometric structure, we notice that Theorem 2.7 has the following interpretation.

Theorem 3.25 (Geometrization of surfaces)**.** *Let S be an orientable, closed surface. Then, S admits a geometric structure over a model geometry $\mathbb{X}$. Furthermore, if g is the genus of S, it holds that $g = 0$ if and only if $\mathbb{X} = \mathbb{S}^2$, $g = 1$ if and only if $\mathbb{X} = \mathbb{R}^2$ and $g \geq 2$ if and only if $\mathbb{X} = \mathbb{H}^2$.*

A consequence of Theorem 3.25 is that *any closed, orientable surface is geometrizable* and that there are 3 possible model geometries (in fact, if a manifold admits a geometric structure over a model geometry $\mathbb{X}$, although the geometric structure is not unique, the geometry is). In the context of 3-manifolds, it is not difficult to find examples of closed, orientable 3-manifolds that do not admit any geometric structure; for instance, a nontrivial connected sum $M_1 \# M_2$ does not admit a geometric structure, with the unique exception of $\mathbb{RP}^3 \# \mathbb{RP}^3$.

But even among orientable, closed irreducible 3-manifolds we may find examples that do not admit a geometric structure[12]. Therefore, we need to consider a further decomposition to obtain geometrization. The natural context to do so is to attempt to show that the pieces that appear in the JSJ decomposition of an irreducible closed 3-manifolds (Theorem 3.22),

[12]Let M_1 and M_2 be compact, irreducible 3-manifolds with both ∂M_1 and ∂M_2 diffeomorphic to a 2-torus T. In comparison to the connected sum, we may choose a gluing diffeomorphism $\varphi\colon \partial M_1 \to \partial M_2$ to obtain a 3-manifold $M = (M_1 \cup M_2)/(\partial M_1 \ni x\ \varphi(x) \in \partial M_2)$. Depending on φ and on M_1, M_2, it holds that M will be irreducible and not geometrizable.

which are compact manifolds with toroidal boundary, are geometric. More precisely, we have the following definition.

Definition 3.26. Let $\mathcal{M}$ be a compact 3-manifold with (possibly empty) toroidal boundary. We say that $\mathcal{M}$ is *geometric* if $\mathrm{int}(\mathcal{M})$ admits a complete geometric structure of finite volume.

As explained above, the main goal of the geometrization is to obtain a standard decomposition of an orientable, irreducible, closed 3-manifold such that any component is geometric, and the JSJ decomposition is the natural decomposition to start the analysis. However, there exists one final obstacle to avert.

Remark 3.27 (Geometric decomposition)**.** There is one *special* compact 3-manifold with toroidal boundary that may appear as a component of the JSJ decomposition, which is the total space of the oriented twisted $[0,1]$-bundle over the Klein bottle[13], denoted by $\mathbb{K}\widetilde{\times}[0,1]$. This manifold is not geometric, although its interior admits a complete geometric flat (i.e., modelled over $\mathbb{R}^3$) structure, but of infinite volume. In order to avoid the issue of a nongeometric component appearing in a *good* decomposition by tori, if $\mathcal{M} \neq \mathbb{K}^2\widetilde{\times}[0,1]$ is as in Theorem 3.22 and U is one of the components of a minimal JSJ decomposition which is diffeomorphic to $\mathbb{K}\widetilde{\times}[0,1]$, we may replace the torus boundary $T = \partial U$ in the decomposition by the central Klein bottle of U, thus generating a different decomposition for $\mathcal{M}$. After performing this procedure on each component diffeomorphic to $\mathbb{K}\widetilde{\times}[0,1]$, we obtain a new decomposition for $\mathcal{M}$, where each component is now a compact manifold with (possibly empty) boundary composed of tori and Klein bottles. This decomposition (which, when minimal, is also unique up to isotopy) is called the *geometric decomposition* of $\mathcal{M}$, for more details see [32, Section 11.5.3].

[13] Let $f\colon \mathbb{T}^2 \to \mathbb{K}$ denote the oriented double cover of the Klein bottle $\mathbb{K}$ by the 2-torus $\mathbb{T}^2$ and let $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the nontrivial covering transformation related to f. Then, $\mathbb{K}\widetilde{\times}[0,1]$ is diffeomorphic to the quotient $(\mathbb{T}^2 \times [-1,1])/\varphi$, where $\varphi\colon \mathbb{T}^2 \times [-1,1] \to \mathbb{T}^2 \times [-1,1]$ is defined as $\varphi(x,t) = (\sigma(x), -t)$. Then, $\mathbb{K}\widetilde{\times}[0,1]$ is orientable and has a central Klein bottle identified with $\mathbb{T}^2 \times \{0\}/\varphi$.

Having defined what we mean by a *geometric* manifold and by the *geometric decomposition* of a closed, orientable 3-manifold, we will next present Thurston's geometrization conjecture.

Conjecture 3.28 (Thurston, 1982 [51]). *Let $\mathcal{M}$ be a closed, orientable and irreducible 3-manifold. Then, the geometric decomposition of $\mathcal{M}$ is in such a way that each resulting component is geometric.*

The geometrization conjecture (which implies the Poincaré's conjecture, as we will see later on this section) was proved by G. Perelman on a series of articles [41–43] posted on the ArXiv but never officially published. For proving the Geometrization conjecture (and, consequently, the Poincaré's conjecture), Perelman was awarded with a Fields Medal in 2006 and with a one million dollars prize given by the Clay Institute for Mathematics for solving one of the so-called Millennium Problems. He refused both prizes, and later he explained

The Fields Medal was completely irrelevant for me. Everybody understood that if the proof is correct then no other recognition is needed.

After Perelman's refusal on the prize, the Clay Institute used the one million dollars dedicated for the prize to fund the *Poincaré Chair* at the Paris Institut Henri Poincaré. The arguments that Perelman used in order to prove the geometrization conjecture were deeply analytic, based on the program proposed by R. Hamilton [21–23] using the Ricci Flow. A more detailed version of Perelman's arguments can be found in articles such as B. Kleiner and J. Lott [28] or in the monographs by J. Morgan and G. Tian [37, 38].

Although Thurston was not able to prove Conjecture 3.28 in its full generality, his work completely revolutionized 3-dimensional topology. First, we mention his classification of all possible maximal geometries which admit compact quotients.

Theorem 3.29 (Thurston [50]). *Let (X, G) be a model geometry that admits a compact quotient. Then, (X, G) is equivalent to one of the eight*

geometries $(\mathbb{X}, \mathrm{ISO}(\mathbb{X}))$*, where* $\mathbb{X}$ *is one of the following Riemannian manifolds:*

$$\mathbb{R}^3,\ \mathbb{H}^3,\ \mathbb{S}^3,\ \mathbb{H}^2 \times \mathbb{R},\ \mathbb{S}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}}_2(\mathbb{R}),\ \mathrm{Nil}_3,\ \mathrm{Sol}_3. \tag{3.1}$$

Remark 3.30. The geometries given by $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ are well-known. For completeness, we will briefly introduce the geometries $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, Nil_3, Sol_3. They are all defined as Lie groups endowed with a left-invariant metric.

- $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the universal covering of the special linear group $\mathrm{SL}(2, \mathbb{R})$, which is the group of 2×2 real matrices with determinant equal to 1. The family of non-isometric left invariant metrics on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ has three parameters (which are the three nonzero *structure constants* when we regard $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ as a unimodular Lie group, for more details see [33, Section 2.7]). Inside this family, there is a two-parameter family of metrics for which the isometry group has dimension four, and when we think as $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ as a model geometry, we think it is endowed with any metric in this family (for a general metric, the isometry group will have dimension three and will not be maximal).

- The Lie group Nil_3 is easily defined as the group of upper triangular 3×3 real matrices with diagonal entries equal to one:

$$\mathrm{Nil}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

  Up to homotheties, there is only one left-invariant metric on Nil_3, and its isometry group has dimension four.

- The *solvable group* Sol_3 is defined as a matrix group as

$$\mathrm{Sol}_3 = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Sol_3 is the *least symmetric* model geometry of all, since any left-invariant metric (it admits a 2-parameter family of them) on it gives rise to a 3-dimensional isometry group. The left-invariant metrics that makes this geometry maximal are the ones that admit some planar reflections, where the full isometry group has 8 connected components.

The eight model geometries that appear in Theorem 3.29 are presently known as *Thurston's geometries*. Note that there are infinitely many non-equivalent model geometries that do not admit any compact quotient, but, as we will see next, they are not relevant for geometrization of 3-manifolds.

Other than classifying 3-dimensional model geometries that admit compact quotients, Thurston was able to prove Conjecture 3.28 for the following class of manifolds.

Definition 3.31 (Haken manifold)**.** A compact, orientable 3-manifold $\mathcal{M}$ is called *Haken* if it is irreducible and it contains an embedded, two-sided, incompressible surface Σ of genus $g \geq 1$.

A simple consequence of Definition 3.31 is that if $\mathcal{M}$ is a compact, orientable and irreducible manifold with toroidal boundary, it is Haken. In particular, although geometrization is a result for closed 3-manifolds, a great part of Thurston's work was dedicated on understanding geometrization of noncompact manifolds, seen as the interior of compact manifolds with toroidal boundaries. The geometrization theorem proved by Thurston can be stated as follows.

Theorem 3.32 (Thurston's geometrization theorem)**.** *Let $\mathcal{M}$ be a compact, orientable, Haken 3-manifold with either empty or toroidal boundary that is not diffeomorphic to $\mathbb{D} \times \mathbb{S}^1$, $\mathbb{T}^2 \times [0,1]$ or to $\mathbb{K}\widetilde{\times}[0,1]$. Then, $\mathcal{M}$ admits a geometric decomposition such that each resulting component is geometric and modelled by one of the eight model geometries that admit a compact quotient.*

The three exceptions in Theorem 3.32 are necessary for two reasons. First, they do not appear as components in the geometric decomposition of

any closed, orientable 3-manifold. Secondly, they are not geometric in the sense that any complete geometric structure in their interiors is of infinite volume. Also, if a closed manifold $\mathcal{M}$ has finite fundamental group, then it is never Haken, so Theorem 3.32 does not apply to them (and does not help to solve the Poincaré conjecture). However, it is worthwhile mentioning that Thurston's geometrization theorem (and his geometrization conjecture) put the Poincaré conjecture in a broad perspective, and changed the point of view that the topologist community had on the subject. We next quote the words of John Morgan, in his talk at the 2018 Clay Research Conference.

[Before Thurston's work,] there was no strong reason to believe that Poincaré's conjecture was either true or false. [...] But the fact that you put the Poincaré conjecture, which was about one particular 3-manifold, in a vast conjecture that is supposed to classify all 3-manifolds, and has some positive evidence for it, makes you believe that you shouldn't spend your time looking for a counterexample.

As we already mentioned, the full geometrization theorem was proved by Perelman, and its statement, unifying both closed (Conjecture 3.28) and compact 3-manifolds with toroidal boundary (Theorem 3.32) in the same result, can be read as follows.

Theorem 3.33 (Geometrization theorem, Perelman)**.** *Let $\mathcal{M}$ be a compact, orientable, irreducible 3-manifold with either empty or toroidal boundary that is not diffeomorphic to $\mathbb{D} \times \mathbb{S}^1$, $\mathbb{T}^2 \times [0,1]$ or to $\mathbb{K}\widetilde{\times}[0,1]$. Then, we can cut $\mathcal{M}$ along a finite, possibly empty collection of incompressible, disjointly embedded surfaces, each of which either is a torus or a Klein bottle, such that each resulting component is geometric and modelled by one of the eight Thurston's geometries.*

While Thurston's proof of Theorem 3.32 is mostly topological, the proof of Theorem 3.33 when $\mathcal{M}$ is a closed, non-Haken manifold carried out by Perelman is radically different, being deeply analytic. Both proofs

are out of the scope of this manuscript, but we next present an intuitive and superficial account of Perelman's proof. This approach was first suggested by Richard Hamilton, which defined the Ricci flow and thought that it could be used to prove the geometrization conjecture. This flow may intuitively be regarded as a heat flow for manifolds, but instead of distributing temperature uniformly, it changes the metric of a manifold, distributing curvature uniformly, hopefully converging to a stationary state where the underlying smooth manifold now has a geometric metric. The main difficulty on this argument is that the Ricci flow generates certain singularities in finite time, preventing one to obtain a geometric limit, but Hamilton visualized that the singularities, if controlled, could actually provide the geometric decomposition of the original manifold[14]. Perelman, after a deep analysis that (under some technical assumptions) classified all possible singularities on the Ricci flow, could then perform the *Ricci flow with surgeries*, controlling the topology of the original manifold where a singularity appeared and continuing the flow past this singularity, doing so only a finite number of times and obtaining a convergence as Hamilton envisaged.

When a closed 3-manifold $\mathcal{M}$ is geometrizable, its underlying geometry is unique (see [46, Theorem 5.2]). Moreover, it is possible to determine the underlying geometry of $\mathcal{M}$ in terms of its topology. The next two theorems provide this description. The first result deals with the 6 geometries that give rise to Seifert fibered spaces, and classify the geometry in terms of the Euler characteristic χ of the base orbifold and the Euler number e of the Seifert fibration, while the second treats with the Sol_3 geometry.

Theorem 3.34 (P. Scott [46, Theorem 5.3 (ii)]). *A closed 3-manifold $\mathcal{M}$ admits a geometric structure modelled on $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ or Nil_3 if and only if $\mathcal{M}$ is a Seifert fibered space. In this case, the relation*

[14] In fact, the geometric decomposition is not provided by the singularities themselves, but by components that even after rescaling are collapsing, in the Gromov-Hausdorff sense, to a lower dimensional space with curvature bounded from below by -1.

between the underlying geometry of $\mathcal{M}$ and the Euler number e and the Euler characteristic χ of the Seifert fibration is given by the following table.

	$\chi < 0$	$\chi = 0$	$\chi > 0$
$e = 0$	$\mathbb{H}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{S}^2 \times \mathbb{R}$
$e \neq 0$	$\widetilde{SL}_2(\mathbb{R})$	Nil_3	$\mathbb{S}^3$

Theorem 3.35 (P. Scott [46, Theorem 5.3 (i)]). *A closed 3-manifold $\mathcal{M}$ admits a geometric structure modelled in* Sol_3 *if and only if $\mathcal{M}$ is finitely covered by a* torus bundle[15] *over $\mathbb{S}^1$ with hyperbolic identification map.*

Together, Theorems 3.34 and 3.35 provide a complete classification of a closed, geometric 3-manifold $\mathcal{M}$ in terms of its topology: if $\mathcal{M}$ is a Seifert-fibered space, the underlying geometry is one of the *Seifert fibered geometries* of Theorem 3.34. If $\mathcal{M}$ is finitely covered by a torus bundle with hyperbolic identification map, it is modelled by Sol_3. Otherwise, the underlying geometry is $\mathbb{H}^3$.

Another result that classify the underlying geometry of a geometric 3-manifold in terms of its topology (in some sense extending Theorems 3.34 and 3.35) is the following Theorem 3.36, which can be found in [6, Section 1.8] or as [32, Proposition 12.8.3]. It makes use of the following nomenclature: if P is a certain group property, we say that a group G is *virtually* P if G admits a finite index subgroup that satisfies the property P.

Theorem 3.36. *Let $\mathcal{M}$ be a closed, orientable 3-manifold modelled by a geometry $\mathbb{X}$. Then, the following hold:*

- *If $\pi_1(\mathcal{M})$ is finite, then $\mathbb{X} = \mathbb{S}^3$. In this case, $\mathcal{M}$ is finitely covered by $\mathbb{S}^3$ and has a structure of a Seifert fibered space with $\chi > 0$ and $e \neq 0$.*

[15] Let $\mathbb{T}^2$ be the 2-torus and let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a orientation preserving homeomorphism. The torus bundle generated by f is the 3-manifold $M(f) = (\mathbb{T}^2 \times [0,1])/\sim$, where $\sim$ is the identification $\mathbb{T}^2 \times \{0\} \ni (x,0) \sim (f(x),1) \in \mathbb{T}^2 \times \{1\}$.

- *Otherwise, if* $\pi_1(\mathcal{M}) = \mathbb{Z}$ *or* D_∞*, then* $\mathbb{X} = \mathbb{S}^2 \times \mathbb{R}$*. In this case,* $\mathcal{M}$ *is either* $\mathbb{S}^2 \times \mathbb{S}^1$ *or* $\mathbb{RP}^3 \# \mathbb{RP}^3$*, so it has a structure of a Seifert fibered space with* $\chi > 0$ *and* $e = 0$*.*

- *Otherwise, if* $\pi_1(\mathcal{M})$ *is virtually* $\mathbb{Z}^3$*, then* $\mathbb{X} = \mathbb{R}^3$*. In this case,* $\mathcal{M}$ *is finitely covered by* $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ *and has a structure of a Seifert fibered space with* $\chi = 0$ *and* $e = 0$*.*

- *Otherwise, if* $\pi_1(\mathcal{M})$ *is virtually nilpotent, then* $\mathbb{X} = \mathrm{Nil}_3$*. In this case,* $\mathcal{M}$ *is finitely covered by a torus bundle with nilpotent monodromy, and has a Seifert fibered structure with* $\chi = 0$ *and* $e \neq 0$*.*

- *Otherwise, if* $\pi_1(\mathcal{M})$ *is solvable, then* $\mathbb{X} = \mathrm{Sol}_3$*. In this case,* $\mathcal{M}$ *(or a double cover) is a torus bundle with Anosov monodromy and* $\mathcal{M}$ *does not admit a Seifert fibered structure.*

- *Otherwise, if* $\pi_1(\mathcal{M})$ *is virtually a product* $\mathbb{Z} \times F$*, where* F *is noncyclic and free, then* $\mathbb{X} = \mathbb{H}^2 \times \mathbb{R}$*. In this case,* $\mathcal{M}$ *is finitely covered by* $\Sigma \times \mathbb{S}^1$*,* Σ *a surface with* $\chi(\Sigma) < 0$ *and* $\mathcal{M}$ *admits a Seifert fibered structure with* $\chi < 0$ *and* $e = 0$*.*

- *Otherwise, if* $\pi_1(\mathcal{M})$ *is a nonseparable extension of a noncyclic free group* F *by* $\mathbb{Z}$*, then* $\mathbb{X} = \widetilde{\mathrm{SL}}(2, \mathbb{R})$*. In this case,* $\mathcal{M}$ *is finitely covered by an* $\mathbb{S}^1$ *bundle over a surface* Σ *with* $\chi(\Sigma) < 0$ *and* $\mathcal{M}$ *admits a Seifert fibered structure with* $\chi < 0$ *and* $e \neq 0$*.*

- *Otherwise, then* $\mathbb{X} = \mathbb{H}^3$*. In this case,* $\mathcal{M}$ *is atoroidal and does not admit a Seifert fibered structure.*

An important observation is that when we are classifying the underlying geometry of a geometric 3-manifold, the hyperbolic case is always the *otherwise* case. In fact, the two most difficult steps into proving geometrization in its full generality were the *elliptization conjecture* and the *hyperbolization conjecture*, that dealt with the respective $\mathbb{S}^3$ and $\mathbb{H}^3$ geometries.

Theorem 3.37 (Elliptization Theorem)**.** *Let $\mathcal{M}$ be a closed, orientable 3-manifold with finite fundamental group. Then $\mathcal{M}$ is* elliptic*, i.e., $\mathcal{M}$ admits a geometric structure modelled by the 3-sphere $\mathbb{S}^3$.*

Theorem 3.38 (Hyperbolization Theorem)**.** *Let $\mathcal{M}$ be a compact, orientable and irreducible 3-manifold with (possibly empty) toroidal boundary, $M \neq \mathbb{D} \times \mathbb{S}^1$, $M \neq \mathbb{T}^2 \times [0,1]$, $M \neq \mathbb{K}\widetilde{\times}[0,1]$. If $\mathcal{M}$ is atoroidal and $\pi_1(\mathcal{M})$ is infinite, then $\mathcal{M}$ is* hyperbolic*, i.e., $\mathcal{M}$ admits a geometric structure of finite volume modelled by $\mathbb{H}^3$.*

As already mentioned, Thurston proved the geometrization in the class of Haken manifolds, and the most crucial step in his proof was to prove Theorem 3.38 for this class of manifolds. Next, we will show how both the hyperbolization and elliptization theorems follow from the geometrization theorem.

Sketch of the proof of Theorems 3.37 and 3.38. Let $\mathcal{M}$ be a compact, orientable 3-manifold that satisfies either the hypothesis of Theorem 3.37 or of Theorem 3.38. Note that if $\mathcal{M}$ is closed and $\pi_1(\mathcal{M})$ is finite, $\mathcal{M}$ is irreducible, and in both cases there is no $\mathbb{Z} \times \mathbb{Z}$ subgroup in $\pi_1(\mathcal{M})$, so the JSJ decomposition of $\mathcal{M}$, given by Theorem 3.22, must be trivial. Hence, Theorem 3.33 gives that $\mathcal{M}$ is itself geometric and admits a model geometry $\mathbb{E}$. The fact that $\pi_1(\mathcal{M})$ does not contain any $\mathbb{Z} \times \mathbb{Z}$ subgroup implies directly that $\mathbb{E}$ is not one of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, Nil_3 and Sol_3 (whose quotients always have such a subgroup). Hence, either $\mathbb{E} = \mathbb{S}^3$ or $\mathbb{E} = \mathbb{H}^3$. Since a quotient of $\mathbb{S}^3$ has finite fundamental group while no (nontrivial) quotient of $\mathbb{H}^3$ does so, this proves both theorems. □

At this point, it is almost irrelevant to present (or prove) the next statement. However, due to its beautiful and old history, we chose to do so.

Corollary 3.39. *The Poincaré conjecture is true.*

Proof. Let $\mathcal{M}$ be a simply connected, closed, orientable 3-manifold. Then, Theorem 3.37 implies that the total space of its universal covering is $\mathbb{S}^3$.

But since $\pi_1(\mathcal{M})$ is trivial, the covering map $\pi\colon \mathbb{S}^3 \to \mathcal{M}$ must be a diffeomorphism. $\square$

To finish this section, we note that the Geometrization Theorem has several deep applications not only to topology but also to differential geometry, and next we present just a few of them, in order to illustrate the advances it made possible. First, we observe that geometrization allowed for the solution of the so called *homeomorphism problem*, which asks for an algorithm to decide if two given compact 3-manifolds are homeomorphic, for details see [8, Section 1.4.1]. Also the complete proof of the Poincaré conjecture, together with previous results by Gromov-Lawson and Schoen-Yau, made possible to obtain the topological classification of closed 3-manifolds that admit a metric of positive scalar curvature (see, for instance, [12] or [28]). Moreover, it also was used in the classification of 3-manifolds with non-negative Ricci curvature [31].

4 Hyperbolization of noncompact 3-manifolds

As seen in Section 3, the richest topology of all is the one of the hyperbolic 3-manifolds, and a great part in the work of Thurston was to obtain a deep understanding of the topology of such manifolds. In this section, we will focus our attention to the hyperbolization of noncompact 3-manifolds. We will present an algorithmic characterization, equivalent to the hyperbolization theorem, that allows us to decide whether the interior of a compact, orientable 3-manifold with nonempty toroidal boundary admits a complete hyperbolic metric of finite volume. We will also present a few applications of this criterion. We would like to point out that several recent developments in the theory of hyperbolic 3-manifolds were achieved by the works of Agol, Kahn, Markovic, Wise and many others and we suggest the book [6] by Aschenbrenner, Friedl and Wilton and its extensive list of references for aspects of hyperbolic 3-manifolds not covered in this manuscript. We also suggest the book [7] by Benedetti and Petronio for several classical results for hyperbolic 3-manifolds.

4.1 Thurston's hyperbolicity criterion

Although presently we know that among the geometries appearing on the geometric decomposition of closed 3-manifolds, the hyperbolic geometry is the most prominent, until the work of Thurston very few explicit examples of noncompact hyperbolic 3-manifolds of finite volume were known. In the words of Thurston [52], we quote:

To most topologists at the time, hyperbolic geometry was an arcane side branch of mathematics, although there were other groups of mathematicians such as differential geometers who did understand it from certain points of view.

We start this section by stating a celebrated result in knot theory, proved by Thurston, that provided infinitely many examples of hyperbolic 3-manifolds as complements of knots in $\mathbb{S}^3$ (recall that a *knot* in a manifold P is the image of an embedding $f\colon \mathbb{S}^1 \to P$). Along this section, by a *hyperbolic 3-manifold* we mean a noncompact, orientable 3-manifold endowed with a complete hyperbolic metric of finite volume. Also, when K is a subset of a 3-manifold, $N(K)$ will denote a small, open, regular tubular neighborhood around K and $\overline{N}(K)$ will denote its closure, when these concepts make sense.

Theorem 4.1 (Classification of knots in $\mathbb{S}^3$)**.** *Let K be a knot in $\mathbb{S}^3$. Then, one of the following holds (see Figure 4.1):*

- *K is a* torus knot*, i.e., there exists an ambient isotopy that maps K to the boundary of a standard*[16] *solid torus in $\mathbb{S}^3$;*
- *K is a* satellite knot*, i.e., there exists a knot $\widetilde{K}$ in $\mathbb{S}^3$ such that K is contained in a regular, tubular neighborhood around $\widetilde{K}$, and K is not isotopic to $\widetilde{K}$;*

[16] Using the model $\mathbb{S}^3 \subset \mathbb{R}^4$, a *standard* torus (or a unknotted torus) is any torus that is isotopic to $\{(x,y,z,w) \mid x^2+y^2=\frac{1}{2},\ z^2+y^2=\frac{1}{2}\} \sim \mathbb{S}^1 \times \mathbb{S}^1$.

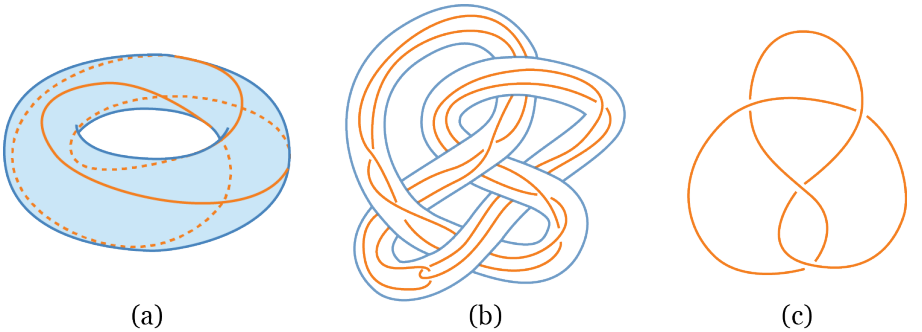


Figure 4.1: (a) The torus knot $(2,3)$; (b) A satellite knot of the knot 5_2; (c) An example of a hyperbolic knot, 4_1

- *K is a* hyperbolic knot*, i.e., the open manifold $\mathbb{S}^3 \setminus \overline{N}(K)$ admits a complete hyperbolic metric of finite volume.*

Remark 4.2. The first known example of a noncompact, complete hyperbolic 3-manifold of finite volume was given by H. Gieseking [19] in 1912. Such a manifold is actually nonorientable and C. Adams [1] proved that it is the noncompact hyperbolic 3-manifold with the smallest possible volume[17] $\mathcal{V} \simeq 1.0149$, where $\mathcal{V}$ is the volume of the ideal regular tetrahedron in $\mathbb{H}^3$. The first description of a hyperbolic 3-manifold as a knot complement in $\mathbb{S}^3$ is due to R. Riley, and it is the complement of the *Figure-eight knot*, which is the double cover of the Gieseking manifold. It was proven by Cao and Meyerhoff [11] that the Figure-eight knot complement and its *sibling manifold* (which is not a knot complement in $\mathbb{S}^3$ but can be described as $(5,1)$ Dehn surgery on the right-handed Whitehead Link), are the two unique orientable, noncompact hyperbolic 3-manifolds with the

[17] Among several results about hyperbolic 3-manifolds of finite volume we are omitting in this article is the *Mostow-Prasad rigidity theorem*, which states that the geometry of hyperbolic 3-manifolds is rigid. In contrast with the case of hyperbolic surfaces, where the same topology may admit infinitely many hyperbolic metrics, in dimension three any diffeomorphism between two hyperbolic 3-manifolds is isotopic to an isometry. Thus, the *hyperbolic volume* of a given hyperbolic 3-manifold is a well-defined topological invariant.

minimum volume $2\mathcal{V} \simeq 2.0298$.

Theorem 4.1 provides a great intuition of how to find noncompact hyperbolic 3-manifolds. Let P be a closed 3-manifold and let L be a link in P (i.e. a finite, pairwise disjoint collection of knots). Then, the manifold $\mathcal{M} = P \setminus N(L)$ is a compact 3-manifold with toroidal boundary. Each boundary component comes from a component of the original link L and gives rise to an *end* (with the topology of $\mathbb{T}^2 \times [0, \infty)$) of the open manifold $P \setminus \overline{N}(L)$. Moreover, Theorem 3.38 gives us the intuition that if the link L is sufficiently *complicated*, the interior of $\mathcal{M}$ will admit a complete hyperbolic metric of finite volume. This intuition will be put in a rigorous form in Theorem 4.4 below, right after a definition necessary for its statement. Here, we will let $\mathcal{M}$ denote a connected, orientable, compact 3-manifold with toroidal boundary and, once again, we will use the nomenclature *a surface* Σ *in* $\mathcal{M}$ to represent a properly embedded surface $\Sigma \subset \mathcal{M}$, i.e., Σ is compact, embedded in $\mathcal{M}$ and $\partial\Sigma = \Sigma \cap \partial\mathcal{M}$.

Definition 4.3.

1. A sphere S in $\mathcal{M}$ is *essential* if S does not bound a ball in $\mathcal{M}$. If $\mathcal{M}$ does not admit any essential sphere, $\mathcal{M}$ is *irreducible* (as in Definition 3.16).

2. A disk D in $\mathcal{M}$ is *essential* if ∂D is homotopically nontrivial in $\partial\mathcal{M}$. If $\mathcal{M}$ does not admit any essential disks, $\mathcal{M}$ is called *boundary-irreducible.*

3. A torus T in $\mathcal{M}$ is *essential* if T is incompressible (as in Definition 3.18) and not boundary parallel, in the sense that it is not isotopic to a component of $\partial\mathcal{M}$. If $\mathcal{M}$ does not admit any essential torus, $\mathcal{M}$ is *geometrically atoroidal* (as in Definition 3.20).

4. An annulus A is *essential* in $\mathcal{M}$ if A is incompressible, boundary-incompressible[18] and not boundary parallel, in the sense that it is

[18] If Σ is a surface in $\mathcal{M}$, a *boundary-compression disk* for Σ is a disk D in $\mathcal{M}$ with

not isotopic, relative to ∂A, to an annulus $A' \subset \partial\mathcal{M}$. If $\mathcal{M}$ does not admit any essential annuli, we say $\mathcal{M}$ is *acylindrical*.

5. If $\mathcal{M}$ is irreducible, boundary-irreducible, geometrically atoroidal and acylindrical, we say that $\mathcal{M}$ is *simple*.

The above definition allows us to obtain a criterion for hyperbolicity that generates noncompact, complete hyperbolic 3-manifolds of finite volume. It follows directly from Theorem 3.38, so it is also commonly known as the *Hyperbolization theorem*.

Theorem 4.4 (Thurston's hyperbolization criterion)**.** *Let $\mathcal{M}$ be an orientable, compact 3-manifold with nonempty toroidal boundary. Then, $\mathcal{M}$ is hyperbolic if and only if $\mathcal{M}$ is simple.*

Proof. Let $\mathcal{M}$ be as stated and assume that $\mathcal{M}$ is simple. We will show that $\mathcal{M}$ satisfies the hypothesis of Theorem 3.38. First, $\mathcal{M}$ is irreducible by the definition of being simple. Also, since $\partial\mathcal{M}$ is nonempty and toroidal, $\pi_1(\mathcal{M})$ is infinite. Furthermore, the fact that there are no essential disks implies that $\mathcal{M} \neq \mathbb{D}\times\mathbb{S}^1$. Since both $\mathbb{T}^2\times[0,1]$ and $\mathbb{K}\widetilde{\times}[0,1]$ contain essential annuli, $\mathcal{M}$ is neither of them. It remains to show that $\mathcal{M}$ is atoroidal. By hypothesis, $\mathcal{M}$ is geometrically atoroidal, so the other option (see Remark 3.21) is that $\mathcal{M}$ is a small Seifert fibered space. But since $\partial\mathcal{M} \neq \emptyset$, we may use Proposition 3.11 to see that if the number of components in $\partial\mathcal{M}$ is one, then $\mathcal{M}$ admits an essential disk and if it is two or three, it admits an essential annulus, a contradiction since $\mathcal{M}$ is simple.

On the other hand, assume that $\mathcal{M}$ is hyperbolic. Then, there exists a complete hyperbolic metric of finite volume in $\mathrm{int}(\mathcal{M})$ and a standard minimization argument (such as in [20] or in [16]) shows that any essential sphere, disk, torus or annulus in $\mathcal{M}$ would provide a properly embedded minimal surface in the hyperbolic metric of $\mathrm{int}(\mathcal{M})$ with nonnegative Euler

$\partial D = D \cap (\Sigma \cup \partial\mathcal{M})$ such that $\partial D = \alpha \cup \beta$, where $\alpha = D \cap \Sigma$ and $\beta = D \cap \partial\mathcal{M}$ are arcs intersecting only in their endpoints, and α does not cut a disk from Σ. If Σ does not admit any boundary-compression disk, we say that Σ is *boundary-incompressible*.

characteristic. Since there are no such a surface in a hyperbolic 3-manifold of finite volume (see, for instance, [15, Theorem 2.1] or [34, Corollary 4.7]), $\mathcal{M}$ is simple. $\square$

To finish this section, we will present some examples of link complements in $\mathbb{S}^3$. Sometimes, we shall talk about *projections* of links, which is an intuitive concept, but we suggest [2] as a first reference to the reader interested in the topic. As in Theorem 4.1, we say that a link L in $\mathbb{S}^3$ is hyperbolic if $\mathbb{S}^3 \setminus N(L)$ admits a complete hyperbolic metric of finite volume.

Example 4.5 (The unknot, Figure 4.2 (a))**.** Let $K \subset \mathbb{S}^3$ be the trivial knot (i.e., K is isotopic in $\mathbb{S}^3$ to $\{(x, y, z, w) \in \mathbb{S}^3 \mid x^2 + y^2 = 1, z = w = 0\}$). Via stereographic projection, we may see that $\mathbb{S}^3 \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus Z$, where Z is the z-axis. Since $\mathbb{R}^3 \setminus Z$ admits a product structure $\mathbb{S}^1 \times (0, \infty) \times \mathbb{R}$, we can see that $\mathbb{S}^3 \setminus K$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ and to $\mathbb{S}^1 \times \mathbb{D}$. In particular, using the coordinates of $\mathbb{S}^1 \times \mathbb{D}$, we may see that for any given $p \in \mathbb{S}^1$, $\{p\} \times \mathbb{D}$ is an essential disk in $\mathbb{S}^3 \setminus K$, hence K is not hyperbolic.

Example 4.6 (The trefoil knot, Figure 4.2 (b))**.** If K is the trefoil knot, K is a torus knot, hence it is not hyperbolic. Note that $\mathbb{S}^3 \setminus N(K)$ admits an essential annulus. The complement of the trefoil knot was among the first 3-manifolds (since the trefoil is the *simplest* nontrivial knot, in the sense that it has the projection with the fewest possible crossings) which Thurston attempted to endow with a complete hyperbolic metric of finite volume, before he developed his criterion. He didn't succeeded because it was not possible, although he still wasn't aware of that.

Example 4.7 (The figure-eight knot, Figure 4.2 (c))**.** If K is the figure-eight knot, $\mathbb{S}^3 \setminus N(K)$ is hyperbolic and its complement has a hyperbolic volume of approximately 2.0298. Although its hyperbolicity was proved first by R. Riley, this was the first noncompact 3-manifold in which Thurston could find a hyperbolic structure of finite volume.

Example 4.8 (The Hopf link, Figure 4.2 (d))**.** The Hopf link L is the union of two trivial knots C_1, C_2. It is not hyperbolic, since its complement admits an essential annulus. Using the stereographic projection about a point (say in C_1) turns $\mathbb{S}^3 \setminus C_1$ diffeomorphically into $\mathbb{R}^3 \setminus Z$, where Z denotes the $z-$axis. In particular, $\mathbb{S}^3 \setminus L$ is diffeomorphic to $\mathbb{R}^3 \setminus (Z \cup C)$, where $C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, which is easily seen as diffeomorphic to $\mathbb{T}^2 \times (0, \infty)$ since it admits a foliation by tori having C as their core curves. In particular, the manifold $\mathbb{S}^3 \setminus N(L)$ is diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, which can also be seen as non-hyperbolic, but this time we go further and notice it admits not one (or two, which are easy to find in $\mathbb{S}^3 \setminus L$) but infinitely many non-isotopic essential annuli, just take a nontrivial curve γ in $\mathbb{T}^2$ and look at the annulus $\gamma \times [0, 1]$ in $\mathbb{T}^2 \times [0, 1]$.

Example 4.9 (Borromean rings, Figure 4.2 (e))**.** The Borromean rings with three components are three disjoint, trivially embedded circles such that each two of them are not linked (in the sense that there exists a sphere which separates one from the other), but the three components together are linked. It is hyperbolic, and its complement has a hyperbolic volume of approximately 7.3277.

Example 4.10 (The (n, s)-chain, Figure 4.2 (f))**.** For given $n \geq 3$ and $s \in \mathbb{Z}$, a (n, s)-chain is a link with n trivial components C_1, C_2, $\ldots$, C_n in such a way that for each $i \in \{1, \ldots, n\}$ the component C_i is linked only with C_{i-1} and with C_{i+1} (where we extend our notation to allow $C_0 = C_n$ and $C_{n+1} = C_1$), and each pair C_i and C_{i+1} is linked as in the Hopf link. Also, we add s left half twists to one of the components (if $s \geq 0$, the link is alternating, otherwise we add $-s$ right twists to one component and the projection of the link will no longer be alternating). It was proven by W. Neumann and A. Reid [40] that the (n, s)-chain is hyperbolic if and only if $\{|n + s|, |s|\} \not\subset \{0, 1, 2\}$. In particular, a chain with 3 components is not hyperbolic if and only if $s = -1$ or $s = -2$, a chain with 4 components is hyperbolic unless $s = -2$ and any chain with 5 or more components is hyperbolic.

Example 4.11 (Composite knots, Figure 4.2 (g))**.** There is a notion of composition for oriented knots, and a knot is called a *composite* knot if it can be obtained from such an operation. A knot is called *prime* if it cannot be obtained from two nontrivial knots by composition. There is a simple characterization to identify if a knot K in $\mathbb{S}^3$ is a composite knot. Let S be an embedded sphere in $\mathbb{S}^3$, separating $\mathbb{S}^3$ into two ball regions B_1, B_2. If $S \cap K$ consists of a transversal intersection in two points and the resulting components $\gamma_1 = K \cap B_1$ and $\gamma_2 = K \cap B_2$ are nontrivial, in the sense that there is no isotopy in B_i that fixes the endpoints of γ_i and maps γ_i to S (in other words, the two pieces of K in each of B_1, B_2 are themselves knotted while fixing their endpoints in S), then L is a composite. Note that a composite knot is never hyperbolic, since the sphere S with the two points removed provides an essential annuls in $\mathbb{S}^3 \setminus N(L)$.

Example 4.12 (Unlinked knots, Figure 4.2 (h))**.** Let K_1, K_2 be two distinct knots in $\mathbb{S}^3$. If there is an embedded sphere S that separates K_1 and K_2, then the knots are not linked. Since, in this case, S is an essential sphere to $\mathbb{S}^3 \setminus N(K_1 \cup K_2)$, the link $K_1 \cup K_2$ is not hyperbolic.

A useful tool for deciding whether a given link in $\mathbb{S}^3$ is hyperbolic or not is the software SnapPy [13], which allows the user to draw a projection of a knot or a link and computes several topological invariants (including hyperbolic volume, if the link is hyperbolic). According to the developers, *SnapPy is a program for studying the topology and geometry of 3-manifolds, with a focus on hyperbolic structures.* It was written using the kernel of a previous program, SnapPea, by Jeff Weeks.

4.2 Applications to knot and link complements

With a little effort, it is not difficult to obtain Theorem 4.1 from Theorem 4.4. In fact, any complement of a torus knot will admit an essential disk (if the knot is the unknot) or an essential annulus (the projection of the knot to the standard torus separates it into annuli), and any satellite knot admits an essential torus. In this section, we will present some recent

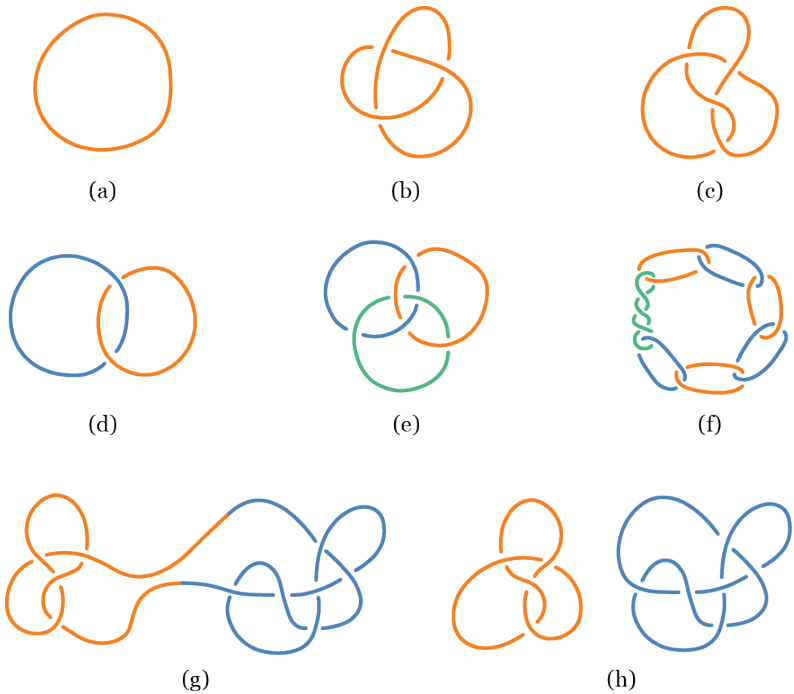


Figure 4.2: (a) The Unknot; (b) The Trefoil knot; (c) The Figure-eight knot; (d) the Hopf link; (e) The Borromean rings with three components; (f) An (n, s)-chain with $n = 7$ and $s = 3$; (g) A composite knot, obtained by the composition of a figure-eight knot and the knot 6_3; (h) Two knots (figure-eight knot and 6_3) which are unlinked.

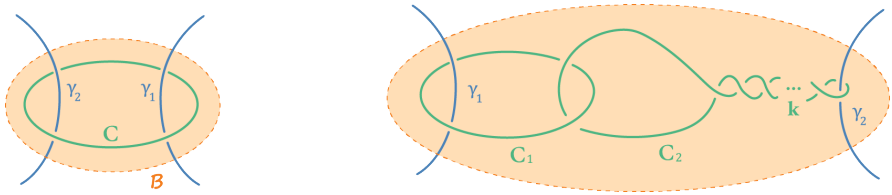


Figure 4.3: Inside the 3-ball $\mathcal{B}$, replacing the component C by the two components C_1, C_2 preserves hyperbolicity of the complement, provided that $(P \setminus \mathcal{B}) \cap L$ is not a rational tangle in a 3-ball.

results concerning hyperbolicity of link complements in 3-manifolds that use Thurston's hyperbolicity criterion in their proofs. These results are based on the works of Adams, Meeks and the second author [4, 5] and allowed the construction of new examples of hyperbolic 3-manifolds of finite volume, containing totally umbilic surfaces for any admissible topology and mean curvature.

The setting to be considered is the following. Let P be a closed 3-manifold and let L be a link in P such that the compact manifold with nonempty toroidal boundary $M = P \setminus N(L)$ is hyperbolic (this will be assumed throughout all the statements that follow). We consider two moves that one can perform on L to obtain a new link L' in P such that the corresponding manifold $M' = P \setminus N(L')$ will also be hyperbolic.

The first move we consider is called the *chain move* [4, Theorem 3.1]. Here, we start with a trivial component bounding a twice-punctured disk in a ball $\mathcal{B} \subset P$ as in Figure 4.3, and we replace the tangle on the left with the tangle on the right in Figure 4.3, where k is any integer. Assuming that the $(P \setminus \mathcal{B}) \cap M$ is not the complement of a rational tangle in a 3-ball (in particular this is trivially satisfied if $P \neq \mathbb{S}^3$, see [2, Chapter 2]), the result is hyperbolic. We note that there are counterexamples to extending the result to the case where $P = \mathbb{S}^3$ and $(P \setminus \mathcal{B}) \cap M$ is a rational tangle complement in a 3-ball, but they are completely classified in the appendix of [4].

The second move is called the *switch move* [4, Theorem 4.1]. Suppose

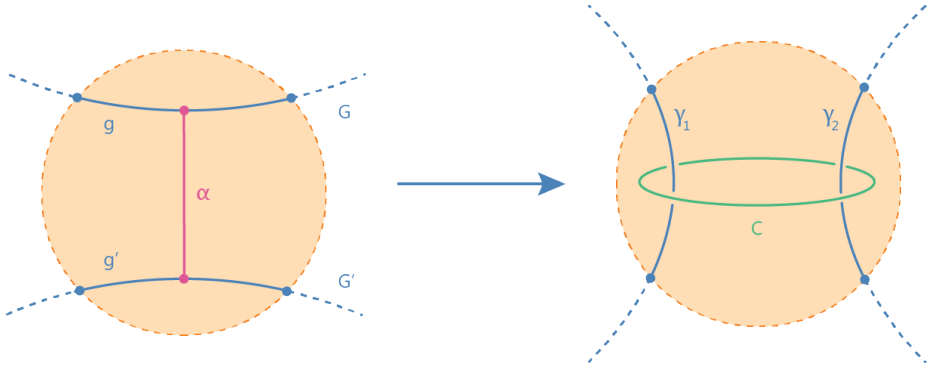


Figure 4.4: The switch move replaces the arcs g and g' in a neighborhood of a complete geodesic by the tangle $\gamma_1 \cup \gamma_2 \cup C$.

that α is an embedded arc in P that intersects L only on its two distinct endpoints and with interior that is isotopic (in $P \setminus L$) to an embedded geodesic in the hyperbolic metric of M. Let $\mathcal{B}$ be a small neighborhood of α in P. Then $\mathcal{B}$ intersects L in two arcs, as in Figure 4.4 (a). The switch move allows us to surger L and add in a trivial component as in Figure 4.4 (b) while preserving hyperbolicity.

The proofs of the moves above follow step by step Thurston's criterion presented in Theorem 4.4 above. Namely, the authors analyze all possibilities for an essential surface that would prevent M' from being hyperbolic and show that whenever such obstruction exists, there is an obstruction for the hyperbolicity of M as well. Together, they allowed the following construction, which is one of the main results of [5].

Theorem 4.13 (Adams-Meeks-Ramos [5, Theorem 1.2])**.** *A connected surface S appears as a properly embedded totally umbilic surface with constant mean curvature $H \in [0, 1)$ in some hyperbolic 3-manifold of finite volume if and only if S has finite negative Euler characteristic.*

Remark 4.14. In fact, Theorem 1.1 of [5] implies that any embedded totally umbilic surface in a hyperbolic 3-manifold of finite volume is proper. Also, the same theorem proves that if Σ is a totally umbilic surface properly

embedded in a hyperbolic 3-manifold of finite volume, then $\chi(\Sigma) < 0$ if and only if the mean curvature of Σ satisfies $|H_\Sigma| < 1$. Thus, in some sense, Theorem 4.13 is sharp, since it shows that any possible pair of topological type and mean curvature $H \in [0, 1)$ actually exists.

Main steps in the proof of Theorem 4.13. First, we briefly sketch the complete proof of the theorem. Let S be a connected surface with finite, negative Euler characteristic. For simplicity, we will assume that S is orientable, hence S is an orientable surface of genus g with n open disks removed, so that $\chi(S) = 2 - 2g - n < 0$. We will construct an explicit compact 3-manifold M with toroidal boundary that satisfies Thurston's conditions and where there exists an order-two diffeomorphism R with fixed point set a properly embedded, possibly disconnected, separating surface $\mathcal{S} \subset M$, where one connected component Σ of $\mathcal{S}$ is diffeomorphic to S (and this part of the proof is divided in cases depending on g and n).

Since the interior of M admits a complete hyperbolic metric of finite volume, Mostow-Prasad rigidity Theorem implies that R is isotopic to an isometry φ, which will also be an order-two diffeomorphism (since φ^2 is an isometry isotopic to the identity, hence equal to it). In particular, the fixed point set of φ, which is itself isotopic to $\mathcal{S}$ and contains a component diffeomorphic to S, is totally geodesic. After producing this explicit totally geodesic example, we will fix $H \in (0, 1)$ and use a property of the fundamental groups of hyperbolic 3-manifolds of finite volume to construct finite covers of $\mathrm{int}(M)$ where Σ lifts and there is a totally umbilic surface Σ_H, isotopic to the lift of Σ and with constant mean curvature H.

Case 1. *S is an n-punctured sphere ($g = 0$) for $n \geq 3$.*

In this case, let $P = \mathbb{S}^3$ and let $\widehat{S}$ be an equatorial sphere $\mathbb{S}^2$ in $\mathbb{S}^3$. Then, there exists a reflection $R\colon \mathbb{S}^3 \to \mathbb{S}^3$ with $\widehat{S} = \mathrm{Fix}(R)$. Consider the *daisy chain* link L_{2n} in $\mathbb{S}^3$ with $2n$ components, where every other component lies in $\widehat{S}$ and the other components are *perpendicular* to $\widehat{S}$ in the sense that they are invariant under R. Since $2n \geq 6$, L_{2n} is hyperbolic by Example 4.10. In particular, as explained above, the restriction of R to

$N = \mathbb{S}^3 \setminus L_{2n}$ is an isometry of the hyperbolic metric in N. The fixed point set of such isometry contains n components diffeomorphic to 3-punctured spheres and one component diffeomorphic to S.

Case 2. *S is closed, i.e., $g \geq 2$ and $n = 0$.*

Consider $P = S \times \mathbb{S}^1$ and, after identifying $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, let $P^+ = P \cap \{y \geq 0\}$, $P^- = P \cap \{y \leq 0\}$ and let $R\colon P \to P$ be the reflection that interchanges P^+ and P^-, leaving $P^+ \cap P^- = S \times \{(-1, 0), (1, 0)\}$ fixed. Both P^+ and P^- have the topology of $S \times [0, 1]$, so the work of Adams *et al.* [3, Theorem 1.1] imply that there exists a hyperbolic link L_1 in P^+ such that $P^+ \setminus L_1$ admits a complete hyperbolic metric of finite volume with totally geodesic boundary. After letting $L_2 = R(L_1) \subset P^-$ and $L = L_1 \cup L_2$, it follows that the manifold $P \setminus L$ is hyperbolic, the reflection R restricts to an isometry with fixed point set equal to $S \times \{(-1, 0), (1, 0)\}$, providing two totally geodesic surfaces diffeomorphic to S.

Case 3. *$n = 1$ and $g \geq 2$, i.e., S is a one-time-punctured surface of genus $g \geq 2$.*

Let S_g denote the closed surface of genus g and let $P = S_g \times \mathbb{S}^1$ and L be as in Case 2. In the hyperbolic metric of $P \setminus L$, let γ^+ be a minimizing geodesic ray joining a point in $S_g \times \{(-1, 0)\}$ to a point (at infinity) in L_1. Then, γ^+ is orthogonal to $S_g \times \{(-1, 0)\}$ and we may use R to reflect γ^+ and obtain an arc $\gamma = \gamma^+ \cup R(\gamma^+)$ which is a complete geodesic in the hyperbolic metric of $P \setminus L$. But a small neighborhood in P of γ is a 3-ball $\mathcal{B}$ which intercepts L in two arcs $g_1 \subset L_1$ and $g_2 \subset L_2$. We can choose $\mathcal{B}$ so that $R(\mathcal{B}) = \mathcal{B}$ and $R(g_1) = g_2$. Then, we can use the Switch Move as described previously to replace the arcs g_1 and g_2 by a tangle such as in Figure 4.4, and do so in an equivariant manner so the trivial circle C added lies (and bounds a disk in) $S_g \times \{(-1, 0)\}$. This creates a new link L' which is hyperbolic in P and satisfies $R(L') = L'$. Furthermore, the reflection R once again restricts to an isometry of this hyperbolic metric and the fixed point set of such isometry has three components: $S_g \times \{(1, 0)\}$, a 3-punctured sphere (more easily seen as a twice-punctured disk) bounded

by C in $S_g \times \{(-1,0)\}$ and the other component of $(S_g \times \{(-1,0)\}) \setminus C$, which is diffeomorphic to S.

Case 4. *$g \geq 2$ and $n \geq 2$.*

This case once again follows from the previous one. Just note that in Case 3 we created a hyperbolic link L' in $P = S_g \times \mathbb{S}^1$ in the ball $\mathcal{B}$, and the tangle $\mathcal{B} \cap L'$ is such that we can apply the Chain Move, again in an equivariant manner with respect to R, and add punctures to one of the totally geodesic surfaces in $(S_g \times \{(-1,0)\}) \setminus C$.

Case 5. *$g = 1$ and $n \geq 1$, so S is a torus punctured at least one time.*

Since this case deals with a surface of genus 1 in a hyperbolic 3-manifold, it is probably the most difficult to solve. Again by [3], there exists a link L_1 in the interior of the compact 3-manifold $P^+ = \mathbb{T}^2 \times [0,1]$ such that $P^+ \setminus N(L_1)$ is hyperbolic, but this time, in a distinction from Case casebase1, the resulting hyperbolic manifold is complete, so it has no boundary and the Chain Move does not apply directly. However, it is possible to adapt the proof of the Chain move, together with the fact that the only obstacle for hyperbolicity of the manifold $M = (\mathbb{T}^2 \times [-1,1]) \setminus N(L_1 \cup R(L_1))$, where $R\colon \mathbb{T}^2 \times [-1,1] \to \mathbb{T}^2 \times [-1,1]$ is $R(x,t) = (x,-t)$, is the existence of the essential torus $\mathbb{T}^2 \times \{0\}$, to prove that the ad hoc analogous of the Switch move to this specific setting applies. From here, the proof follows analogously as in Cases 3 and 4, firstly obtaining a once-punctured totally geodesic torus and then adding punctures to it using the Chain move.

The arguments in the above cases show that any admissible orientable topology for a totally geodesic surface in a hyperbolic 3-manifold of finite volume actually can be realized as such. Similar arguments can be done for nonorientable topologies, see [5]. To finish the sketch of the proof of Theorem 4.13, we need the following result, which was obtained in a series of recent works (see [6] for an appropriate list of references).

Theorem 4.15. *Let N be a noncompact hyperbolic 3-manifold of finite volume. Then, $\pi_1(N)$ is Locally Extendable Residually Finite (for short, LERF), i.e., for every finitely generated subgroup K of $\pi_1(N)$ and any finite set $\mathcal{F} \subset \pi_1(N)$, $\mathcal{F} \cap K = \emptyset$, there exists a representation $\sigma\colon \pi_1(N) \to F$ to a finite group F such that $\sigma(\mathcal{F}) \cap \sigma(K) = \emptyset$.*

The main idea to produce a totally umbilic example from a totally geodesic one is to use the notion of a *t-parallel surface*. Specifically, if Σ is a two-sided totally geodesic surface in a hyperbolic 3-manifold of finite volume N as produced before, and Σ is oriented with respect to a unitary normal vector field η, for any $t > 0$ we let

$$\Sigma_t = \{\exp_p(t\eta(p)) \mid p \in \Sigma\}.$$

Then, Σ_t is totally umbilic and has mean curvature $H_t = \tanh(t)$. Moreover, it is not difficult to use the ambient geometry of the ends of N to see that Σ_t is proper and that for small values of $t > 0$, Σ_t is embedded. With this in mind, one can look at the first time $t_0 > 0$ where Σ_{t_0} is not embedded. Such t_0 exists (so the first intersection point of the family $\{\Sigma_t\}$ is not at infinity) and, for some $p \in \Sigma$, the set $\{\exp_p(s\eta(p)) \mid s \in [0, 4t_0]\}$ is a closed geodesic of length $4t_0$, orthogonal to Σ in exactly two distinct points (see [5, Lemma 5.2]).

The next (and last) step in the proof is to assume that $\tanh(t_0) < H$ (otherwise the theorem follows) and use that $\pi_1(N)$ is LERF to construct a finite cover $\Pi\colon \widehat{N} \to N$ where Σ lifts to a totally geodesic surface $\widehat{\Sigma}$ and there is no closed geodesic in $\widehat{N}$ with length less than or equal to $5\tanh^{-1}(H)$. In such a manifold, for $t = \tanh^{-1}(H)$ the corresponding t-parallel surface to $\widehat{\Sigma}$, $\widehat{\Sigma}_t$, will be a properly embedded totally umbilic surface as promised. □

References

[1] C. Adams. *The Noncompact Hyperbolic 3-Manifold of Minimal Volume*, Proc. Amer. Math. Soc. **100**, no. 4 (1987): 601–606. https:

//doi.org/10.2307/2046691.

[2] C. Adams. *The knot book*, American Mathematical Society, Providence, RI, 2004. ISBN:0-8218-3678-1, MR:2079925.

[3] C. Adams, C. Albors-Riera, B. Haddock, Z. Li, D. Nishida, B. Reinoso, and L. Wang. *Hyperbolicity of links in thickened surfaces*. Topology Appl., **256**:262–278, (2019). DOI 10.1016/j.topol.2019.01.022, MR3916014.

[4] C. Adams, W. Meeks III and A. Ramos, *Modifications Preserving Hyperbolicity of Link Complements*, to appear in Journal of Algebraic and Geometric Topology, preprint available at arXiv:2007.01739 [math.DG]

[5] C. Adams, W. Meeks III and A. Ramos, *Totally umbilic surfaces in hyperbolic 3-manifolds of finite volume*. Preprint at arXiv:2007.03166 [math.DG]

[6] M. Aschenbrenner, S. Friedl, H. Wilton. *3-manifold groups*, EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015. xiv+215 pp. ISBN: 978-3-03719-154-5.

[7] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992. xiv+330 pp. ISBN: 3-540-55534-X.

[8] L. Bessières, G. Besson, S. Maillot, M. Boileau, and J. Porti. *Geometrisation of 3–manifolds*. EMS Tracts in Mathematics, 13. European Mathematical Society (EMS), Zürich, 2010.

[9] O. Bonnet. *Sur quelques propriétés des lignes géodésiques*, C.R. Ac. Sc. Paris 40 (1855) 1311-1313.

[10] G. Bredon. *Topology and geometry*. Graduate Texts in Mathematics, 139. Springer-Verlag, New York, 1993. ISBN: 0-387-97926-3

[11] C. Cao and G. Meyerhoff. *The orientable cusped hyperbolic 3-manifolds of minimum volume*. Invent. math. **146**, 451–478 (2001). https://doi.org/10.1007/s002220100167

[12] A. Carlotto and C. Li. *Constrained deformations of positive scalar curvature metrics*. To appear in Journal of Differential Geometry, preprint available at arXiv:1903.11772 [math.DG]

[13] M. Culler, N. Dunfield, M. Goerner, and J. Weeks. *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, available at http://snappy.computop.org

[14] M. P. do Carmo. *Riemannian geometry*. Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. xiv+300 pp. ISBN: 0-8176-3490-8 MR1138207.

[15] P. Collin, L. Hauswirth and H. Rosenberg. *Minimal surfaces in finite volume hyperbolic 3-manifolds N and in $M \times \mathbb{S}^1$, M a finite area hyperbolic surface*, Amer. J. Mat., **140**, 4, (2018) 1075-1112. https://doi.org/10.1353/ajm.2018.0024.

[16] P. Collin, L. Hauswirth, L. Mazet and H. Rosenberg. *Minimal surfaces in finite volume noncompact hyperbolic 3-manifolds*. Trans. Amer. Math. Soc. **369** (2017), no. 6, 4293–4309.

[17] D.B.A. Epstein. *Periodic flows on 3-manifolds*, Ann. of Math., 95 (1972), 66-82.

[18] G. Francis and J. Weeks. *Conway's ZIP Proof*, The American Mathematical Monthly, 106:5, 393-399, (1999).

[19] H. Gieseking. *Analytische untersuchungen über topologische gruppen*, PhD Thesis, Münster (1912).

[20] J. Hass, J. H. Rubinstein and S. Wang. *Boundary slopes of immersed surfaces in 3-manifolds*. J. Differential Geom. **52** (1999), no. 2, 303–325.

[21] R. Hamilton. *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255–306.

[22] R. Hamilton. *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, (1995).

[23] R. Hamilton. *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom. 7 (1999), no. 4, 695–729.

[24] A. Hatcher. *Notes on Basic 3-Manifold Topology*. Available at https://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html

[25] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001.ISBN: 0-8218-2848-7

[26] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.

[27] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, vol. 761, Springer-Verlag, Berlin, (1979).

[28] B. Kleiner and J. Lott. *Notes on Perelman's papers*. Geom. Topol. **12**(5): 2587-2855 (2008). DOI: 10.2140/gt.2008.12.2587

[29] H. Kneser. *Geschlossene Flachen in dreidimensionale Mannigfaltigkeiten*, Jahresber. Deutsch. Math.-Verein., 38 (1929), 248–260.

[30] P. Koebe. *Uber die Uniformisierung beliebiger analytischer Kurven.*Nachr. Ges. Wiss. Gottingen (1907), 191–210 and 633–649.

[31] G. Liu. *3-manifolds with nonnegative Ricci curvature*. Invent. Math. **193** (2013), no. 2, 367–375.

[32] B. Martelli. *An Introduction to Geometric Topology*. CreateSpace Independent Publishing Platform (2016). Available at arXiv:1610.02592 [math.GT]

[33] W. H. Meeks III and J. Pérez, *Constant mean curvature surfaces in metric Lie groups*. Contemp. Math., **570** (2012) 25-110.

[34] W. Meeks III and A. Ramos. *Properly immersed surfaces in hyperbolic 3-manifolds*, J. Differential Geom., **112**, 2 (2019), 233-261. https://doi.org/10.4310/jdg/1559786424

[35] J.W. Milnor. *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. (2), **64**, N 2 (1956), 399-405.

[36] J.W. Milnor. *A unique factorisation theorem for 3-manifolds*, Amer. J. Math., 84 (1962), 1-7.

[37] J. Morgan and G. Tian. *Ricci flow and the Poincaré Conjecture*, Clay Mathematics Monographs, vol. 3, Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA (2007).

[38] J. Morgan and G. Tian. *The Geometrization Conjecture*, Clay Mathematics Monographs, vol. 5, Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA (2014).

[39] S. Myers. *Riemannian manifolds with positive mean curvature*, Duke Math. J. 8 (1941), 401-404.

[40] W. Neumann and A. Reid. *Arithmetic of hyperbolic manifolds*. TOPOLOGY 90, Proceedings of the Research Semester in Low Dimensional Topology at Ohio State University, 273–310, (1992).

[41] G. Perelman. *The entropy formula for the Ricci flow and its geometric applications*, preprint (2002), available at arXiv:math/0211159 [math.DG].

[42] G. Perelman. *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint (2003), available at arXiv:math/0307245 [math.DG].

[43] G. Perelman. *Ricci flow with surgery on three-manifolds*, preprint (2003), available at arXiv:math/0303109 [math.DG].

[44] H. Poincaré. *Cinquième complément à l'Analysis situs.* Rendiconti del Circolo Matematico di Palermo (1884-1940), **18**, (1904) 45–110.

[45] H. Poincaré. *Sur l'uniformisation des fonctions analytiques.* Acta Math. 31 (1907), 1–64.

[46] P. Scott. *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), no. 5, 401–487.

[47] H. Seifert. *Topologie Dreidimensionaler Gefaserter Räume*, Acta Math. 60 (1933), no. 1, 147–238.

[48] H. Seifert and W. Threlfall. *Seifert and Threlfall: a textbook of topology.* Translated from the German edition of 1934 by Michael A. Goldman. With a preface by Joan S. Birman. With "Topology of 3-dimensional fibered spaces" by Seifert. Translated from the German by Wolfgang Heil. Pure and Applied Mathematics, 89. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. xvi+437 pp. ISBN: 0-12-634850-2.

[49] W. P. Thurston. *The geometry and topology of 3-manifolds*, Princeton Lecture Notes (1979), available at http://www.msri.org/publications/books/gt3m/

[50] W. P. Thurston. *Three-dimensional geometry and topology.* Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp. ISBN 0-691-08304-5

[51] W. P. Thurston. *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc., New Ser. 6 (1982), 357–379.

[52] W. Thurston. *On proof and progress in mathematics*, Bull. Amer. Math. Soc. (N.S.), **30** (1994), 161–177.