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THREE STUDIES ON RISK MEASURES: A FOCUS ON THE COMONOTONIC ADDITIVITY PROPERTY

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Tese de Doutorado apresentada junto ao Curso de Economia Aplicada do Programa de Pós-Graduação em Economia da Universidade Federal do Rio Grande do Sul, como requisito parcial à obtenção do título de Doutor em Economia.

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1 INTRODUCTION TO THE THESIS

Professors Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath published in 1999 what became the cornerstone of a new paradigm in the study of risk measures (Artzner et al., 1999). In this new paradigm, a **risk measure** is a functional ρ satisfying certain axioms and that assigns to each financial position, say X, a certain real-valued risk, $\rho(X)$. Most of the literature that grew since then—our work being no exception—has focused on studying the properties that risk measures ought to satisfy to assist risk managers and regulatory authorities to determine regulatory capital, i.e, the amount of capital a financial institution must hold as conservative and liquid investments to be liquidated and used to cover losses whenever necessary. Such properties are called axioms and their main functions are to endow a *mathematical structure* to risk measures and to capture *intuitive aspects* of the notion of risk.

A common *modus operandi* of the research in the axiomatic theory of risk measures includes, usually as first steps, the definition of a set of axioms that risk measures should satisfy and, afterward, the investigation of additional properties implied by the axioms. The axioms, therefore, determine the suitability of risk measures for the task of regulatory capital determination. On the other hand, the axioms also imply the limitations of risk measures satisfying them, in terms of desirable properties these risk measures cannot fulfill.

Chapter 2 is dedicated to the limitations of risk measures satisfying the axiom of comonotonic additivity, which requires ρ to satisfy $\rho(X_1 + X_2) = \rho(X_1) + \rho(X_2)$ whenever the random variables X_1 and X_2 are such that $(X_1(\omega') - X_1(\omega))(X_2(\omega') - X_2(\omega)) \geqslant 0$ for $\mathbf{P} \otimes \mathbf{P}$-almost all $(\omega, \omega') \in \Omega \times \Omega$, i.e., whenever the random variables X and Y always vary in the same direction. Since its principle in the theory of risk measures and in actuarial mathematics, the axiom of comonotonic additivity has occupied a distinguished place in the theory (see, for instance, Wang (1996), Goovaerts and Dhaene (1997), Kusuoka (2001), and Acerbi and Tasche (2002)). More recently, however, several authors have shown that the axiom of comonotonic additivity implies the absence of several desirable properties. Most of these incompatibilities were discovered in the past decade, and to the best of our knowledge, there exists no published paper presenting all the properties that are absent in the comonotonic additive framework. The goal of Chapter 2 is to fill this gap. In that chapter, we provide an extensive review of the incompatibilities between desirable properties and the axiom of comonotonic additivity. As a secondary contribution of this chapter, we point out that comonotonic additive risk measures cannot fulfill the property of excess invariance. As a consequence, if a comonotonic additive risk measure is used for regulatory capital determination, then the potential profits of the financial firms may influence their compulsory reserves of capital.

An acceptance set represents a criterion, according to which a financial regulator separates the positions that financial firms are allowed to hold from the positions they are not. Acceptance sets induce risk measures, which then associate with each financial position a real number representing the minimal quantity of the cash asset that makes the positions acceptable. As discussed in Chapter 3 the properties that a risk measure fulfills are determined by the properties of the acceptance set associated with it. There are several classical results relating the basic properties of the acceptance sets to the basic properties of their induced risk measures. However, no previous paper obtained simple conditions on the acceptance sets that guarantee its induced risk measures to be comonotonic additive. This was a gap in the elementary theory that we believe to fill in Chapter 3. In that chapter, we show that to induce a comonotonic additive risk measure, one must consider acceptance sets that are convex for comonotonic random variables, and such that the acceptance set's complements satisfy the same property. These convexity properties have a natural interpretation in terms of the absence of benefits and the absence of any deleterious effects from diversification between comonotonic random variables. Also, the approach we develop can be used to study risk measures that are additive for many other specific classes of random variables, in particular, risk measures that are additive for independent random variables (Borch, 1962; Bühlmann, 1985; Gerber, 1974; Gerber and Goovaerts, 1981; Goovaerts, Kaas, Dhaene and Tang, 2004; Goovaerts et al., 2010; Goovaerts, Kaas, Laeven and Tang, 2004) and for uncorrelated random variables (Heijnen and Goovaerts, 1986).

In the third essay, we depart from the purely theoretical study of risk measures and consider the more applied problem faced by risk managers that need to understand how the inclusion of an incremental asset would affect the risk of a current portfolio. If the risk manager has a decisive view on how risk should be measured, then this problem is trivially solved by comparing the risks of the current portfolio with that of the incremented portfolio. However, there are instances in which considering a unique risk measure is not enough. For instance, the risk manager might be in charge of aggregating the attitudes towards risk of multiple stakeholders or, even if the portfolio belongs to a single investor, it might be that this investor's risk attitude is only partially observed.

We propose a tool that gives risk managers decisive and conservative conclusions about the effect of an additional asset on the risk of a current portfolio. This tool is a monetary risk measure that allows the risk manager to identify financial positions that reduce the risk of the current portfolio, according to all monetary risk measures that are consistent with second-degree stochastic dominance. Also, the risk measure we propose provides the smallest amount of money (the cost) necessary to turn the financial positions into risk reducers for the original portfolio. We characterize the cost of robust risk reduction through a monetary risk measure, a monetary acceptance set, the family of average values at risk, and through the infimum of the certainty equivalents of risk-averse agents with random initial wealth.

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The limitations of comonotonic additive risk measures: a literature review

2 THE LIMITATIONS OF COMONOTONIC ADDITIVE RISK MEASURES: A LITERATURE REVIEW

Abstract

The theory of risk measures has grown enormously in the last twenty years. In particular, risk measures satisfying the axiom of comonotonic additivity were extensively studied, arguably because of the affluence of results indicating interesting aspects of such risk measures. Recent research, however, has shown that this axiom is incompatible with properties that are central in specific contexts. In this paper we present a literature review of these incompatibilities. As a secondary contribution, we show that the comonotonic additivity axiom conflicts with the property of excess invariance for risk measures and, in a milder form, with the property of surplus invariance for acceptance sets.

Key-words: Comonotonic additive risk measures. Regulatory capital. Excess invariance. Risky eligible assets.

2.1 INTRODUCTION

Financial institutions and investors use risk measures to quantify potential losses and variability. Beyond these internal concerns, financial regulators use risk measures to determine the regulatory capital that financial institutions—notably banks, investment funds, and insurance companies—must hold as a buffer against their potential losses. These risk measures are called *monetary*, arguably because the values they assume represent monetary units.

Monetary risk measures are a pillar of external risk management, and, for such relevance, they have been studied thoroughly during the last two decades. The cornerstones of the theoretical research on risk measures were set by Artzner, Delbaen, Eber and Heath (1999), which were followed-up by several fundamental building blocks of the theory (see, for instance, Delbaen (2002), Föllmer and Schied (2002), Frittelli and Gianin (2002), Kusuoka (2001), and Acerbi (2002)). According to this paradigm of research, a risk measure is a functional ρ satisfying certain axioms and that assigns to each financial position, say a random variable X, a certain real number $\rho(X)$, which is interpreted as the financial risk of X. As most of the literature, we give special attention to the operational interpretation of $\rho(X)$, namely the amount of regulatory capital that financial institutions holding the position X must hold to cover potential losses from X.

The axioms play a central role in the theory. They capture intuitive aspects of the notion of financial risk, define basic properties that turn risk measures into the invaluable tools they are and, from a technical point of view, the axioms provide risk measures a base mathematical structure. In this paper, we focus on measures of financial risk satisfying the axiom of comonotonic additivity, namely, risk measures such that $\rho(X_1+X_2)=\rho(X_1)+\rho(X_2)$ whenever X_1 and X_2 are comonotonic, i.e., whenever the assets X_1 and X_2 are increasing functions of a common underlying asset.

The axiom of comonotonic additivity has occupied a distinguished place in the theory of risk measures (see, for instance, Kusuoka (2001), Acerbi (2002), Dhaene et al. (2003), Dhaene et al. (2004), Deelstra et al. (2011), Ekeland et al. (2012), Kou and Peng (2016), Rieger (2017), Koch-Medina et al. (2018), Wang et al. (2020)). As we could appraise, such imminence comes from the strong intuition behind the axiom: first, when comonotonic random variables vary, they do so in the same direction and, therefore, comonotonic random variables do not hedge each other. As a reasonable extension of this fact, one could say—and, in fact, the traditional view is that—"there is no benefit in pooling comonotonic random variables together". For risk measures, this statement translates into "the risk of the sum should be the sum of the risks", which provides a strong basis for the axiom. This intuition transcends the theory of risk measures and, in fact, was embraced earlier in non-expected utility theory (Schmeidler, 1986, 1989; Yaari, 1987) and in the theory of premium principles (Goovaerts and Dhaene, 1997; Wang, 1996; Wang et al., 1997).

Axioms also determine which additional properties—these may assume central relevance in certain applications —are necessarily fulfilled by a given risk measure, as well as the properties that are necessarily absent. The present paper is a (tentatively) comprehensive literature review reporting the properties that are necessarily absent for central classes of comonotonic additive risk measures. A such instance, where we say that comonotonic additivity "conflicts" with a given property, may take two forms (logically equivalent but with rather different intuitions). First, there are some (possibly) desirable properties that are not fulfilled by any reasonable comonotonic additive risk measure (see sections, 2.2, 2.3, 2.4, and 2.5). A second form of conflict is characterized when all reasonable comonotonic additive risk measures present an idiossyncratic and possibly troublesome feature (sections 2.6, and 2.7).

2.1.1 Roadmap

In the Appendix (Section 2.9), we present the elementary on comonotonic random variables and risk measures, focusing on comonotonic additive risk measures and their integral representations. The goal is to equip the general audience with the basilar theory underlying the results we discuss in our literature review. The *connoisseur* will find no novelty in the appendix but may want to give it a quick overview to get familiarized with the definitions we adopt. For the readers skipping the appendix, we should mention that, as it is standard in the literature, we work with an atomless probability space $(\Omega, \mathcal{F}, \mathbf{P})$, except when explicitly stated otherwise. Also, the net present value of the financial positions are represented by random variables in $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, since it allows us to transition smoothly between the papers in the literature. Also, we use the term **risk measure** to refer to any functional $\rho : \mathcal{X} \to \mathbb{R}$, and the term **acceptance set** to refer to any non-empty set $\mathcal{A} \subsetneq \mathcal{X}$. Exceptions to these definitions are explicitly mentioned. Most results presented in this paper are not ours, but collected from the literature. In these cases, the statements always begin with the respective citation. Any formal statement not beginning with a citation is new.

We begin our bibliographical review in Section 2.2, where we discuss the difficulties of using comonotonic additive risk measures to determine regulatory capital in the context of limited liability. The view presented by the papers in this topic is that, when financial firms have limited liability, the regulatory capital should be determined solely by the potential losses incurred by the financial institutions, being therefore insensitive to the size and probability of surpluses (Cont et al., 2013; He and Peng, 2018; Koch-Medina et al., 2015, 2017; Staum, 2013). This insensitivity can be captured by different axioms that reflect the notion of *excess/surplus invariance*. In Section 2.2, we present new results indicating the incompatibility of the comonotonic additivity axioms with two different notions of excess invariance proposed in the literature. In particular, we show that, if a risk measure is monotone and excess invariant—the later definition being employed in Staum (2013) and Cont et al. (2013)—then it cannot be comonotonic additive. In addition to that, we show that the property of excess invariance is stronger than that of surplus invariance—the later being employed in Koch-Medina et al. (2017), He and Peng (2018), and Gao and Munari (2020). To conclude Section 2.2 (where virtually all new results appear) we show that, as a corollary of a theorem of He and Peng (2018), there is also a conflict between comonotonic additivity and surplus invariance, although this conflict is milder than the one previously mentioned. The incompatibility of comonotonic additive risk measures with the excess/surplus invariant framework was never mentioned in the literature and, therefore, we believe these results are worth being brought to light.

In Section 2.3, we discuss the findings of Koch-Medina et al. (2018) showing that the regulator must choose between using a comonotonic additive risk measure or allowing banks to use risky assets to compose their regulatory reserves (which should be allowed according to the Basel Committee's guidelines (BCBS, 2019)). Koch-Medina et al. (2018) showed (under mild conditions) that the cost of insisting on both properties is prohibitive, as the regulator would be forced to accept arbitrarily large fully-leveraged positions. As a consequence, when risky eligible assets are considered, spectral risk measures must lose their comonotonic additivity (even the value at risk and the average value at risk).

In Section 2.4, we discuss the lack of elicitability of comonotonic additive risk measures. The property of elicitability for risk measures lies at the heart of the recent literature studying the basic principles that allow one to compare different risk forecasting procedures. Therefore, the lack of elicitability imposes additional difficulties in comparing different risk forecasting procedures. Essentially, a risk measure is elicitable if it minimizes a specific expected score, which allows us to rank risk forecasting procedures in a meaningful manner. As noticed in Gneiting (2011), the lack of elicitability or the usage of inadequate score functions can lead to misleading evaluations of the forecasting procedures' relative performances. Elicitable risk measures, however, are scarce (Bellini and Bignozzi, 2015; Gneiting, 2011; Weber, 2006; Ziegel, 2016). In particular, Ziegel (2016) showed that, if a coherent risk measure is elicitable and law invariant, then it is an expectile. Also, if in addition the risk measure is comonotonic additive, then it corresponds to the expected loss w.r.t. the physical probability (Ziegel, 2016). For the non-coherent case, Kou and Peng (2016) showed that there is no law invariant monetary risk measure that is comonotonic additive and elicitable, except the expected loss and the value at risk. Taken together, these results tell us that, if the property of elicitability is of utmost importance and coherency is desirable, then the cost of requiring a risk measure to be comonotonic additive is that we would end up confined to the expected loss, which is inadequate to measure tail risk; if coherence is dropped, we can still employ the value at risk.

All the properties and papers mentioned so far were developed in a one-period framework. Despite being remarkably useful, this framework's potential to measure risk in a dynamic setting is limited, as it does not allow the risk to depend on new information. In Section 2.5, we discuss comonotonic risk measures in the dynamic framework, where the risk of a position is measured at each period. A central topic in this context is the property of time-consistency, which defines how the risk at different periods should relate. Loosely speaking, this property requires that, if the risk of X is greater than that of Y at the period $t+1$ with probability one, then the same must hold at t. We discuss two incompatibilities between time-consistency and the comonotonic additivity axiom. First, Kupper and Schachermayer (2009) showed that the the entropic is the unique monetary dynamic risk measure that is law invariant, relevant, and time-consistent. Since the static entropic risk measures are not comonotonic additive, the findings of Kupper and Schachermayer (2009) imply that one cannot construct a time-consistent dynamic risk measure with comonotonic additive components. Second, we present the conflict between time-consistency and comonotonic additivity discovered by Delbaen (2021). As he showed, the unique (not necessarily law invariant) risk measure that is coherent, time-consistent, and comonotonic additive is an expected loss with respect to an absolutely continuous probability measure.

A major appeal of comonotonic additive risk measures is their spectral representations, which are intuitive and allow us to explicitly manipulate general risk measures in this class (see Kusuoka (2001), Acerbi (2002), Föllmer and Schied (2016), and Wang et al. (2020) for details). Because of these representations, one could expect them to be valuable tools in applied problems, in particular, in portfolio selection problems. As shown in Brandtner (2013), however, such application of spectral risk measures leads to two (possible problematic) idiosyncrasies that we discuss in Section 2.6. First, recall that in the mean–variance framework, the optimal weights of a portfolio can be found by maximizing expected returns subject to a certain level of variance or, equivalently, by minimizing the variance subject to a certain level of expected return. As a first quirk in the context of portfolio optimization, Brandtner (2013) showed that those problems are no longer equivalent when the variance is replaced by a spectral risk measure. A second complication emerging in such problems—also brought about by Brandtner (2013)—is that when a spectral risk measure is used, the solution tends to be at the corners. As a consequence, when short sales are allowed, the investors either invest an infinite amount in the tangency portfolio (by short selling the risk-free asset) or invest zero in the risky portfolio. If short sales are restricted, the investor invests all or nothing of her money in the risk-free asset.

The usage of comonotonic additive functionals is far from being restricted to the field of risk measures. In fact, these functionals' early roots lie in non-expected utility theory (Schmeidler, 1989; Yaari, 1987), where the notion of *risk aversion* of Arrow (1965) and Pratt (1964) has fundamental importance. In classical utility theory, we can compare the risk aversion of two agents through their certainty equivalents or, equivalently, through their Arrow-Pratt coefficients of risk aversion (and these comparisons coincide). In Section 2.7, we discuss the findings of Brandtner and Kürsten (2015) showing that, if the agents' preferences are represented by a spectral risk measure, their relative risk aversion may be different depending on if it is measured through the certainty equivalents or the Arrow-Pratt coefficient. As argued in Brandtner and Kürsten (2015), this lack of consistency makes the usage of the Arrow-Pratt coefficient troublesome, because the relative risk aversion of two individuals can be different when, instead, it is measured by the certainty equivalent. Brandtner and Kürsten (2015) extended their analysis to the framework of Ross (1981), which considers a random level of initial wealth. As an extension of the previously mentioned inconsistency, they showed that, for agents whose preferences are represented by spectral risk measures, the ordering based on the Arrow-Pratt coefficient does not necessarily coincide with the ordering based on the coefficient of risk aversion of Ross (1981). We summarize and conclude the paper in Section 2.8.

2.2 EXCESS INVARIANCE

Creditors do not benefit from the banks' profits, nonetheless, they bear the risk of their bank falling short with its liabilities due to the occurrence of losses exceeding the banks' capital. The same can be said regarding the gains and extreme losses of insurance companies. As argued in Staum (2013), Koch-Medina et al. (2015), Koch-Medina et al. (2017), and He and Peng (2018), the social justification of a regulator should be to secure the banks' creditors against the risk of default. Therefore, it makes sense to determine the risk of a financial institution—and its regulatory capital—based exclusively on the negative part of the banks' financial positions.

Definition 2.1. *(Cont et al., 2013; Gao and Munari, 2020; Staum, 2013) A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is* **excess invariant** *if $\rho(X) = \rho(-X^-)$ for all $X \in \mathcal{X}$*[1].

Remark 2.1. Koch-Medina et al. (2015), and Koch-Medina et al. (2017) studied a relaxed version of excess invariance, according to which ρ must satisfy $\rho(X) = \rho(-X^-)$ only for $X \in \mathcal{X}$ such that $\rho(X) \geqslant 0$.

Excess invariant risk measures—let us temporally denote them as $\tilde{\rho}$—can always be constructed from traditional risk measures, say ρ, through $\tilde{\rho}(X) := \rho(-X^-)$, which is real-valued for all $X \in \mathcal{X}$. Excess invariant risk measures have some distinctive features, as it conflicts with cash additivity and are proned to be non-negative (see Staum (2013), Cont et al. (2013), and Koch-Medina et al. (2015)). In this framework, the acceptance sets take the form $\mathcal{A}(\beta) := \{X \in \mathcal{X} : \tilde{\rho}(X) \leqslant \beta\}$ for some $\beta > 0$, which represents a level of risk tolerance. This interpretation also fits into the traditional framework, although if $\tilde{\rho}$ is excess invariant, the surplus of the positions $X \in \mathcal{X}$ does not play a role in determining if it is tolerable or not. Also, notice that when $\tilde{\rho}$ is non-negative, using $\beta = 0$ as in the traditional framework could be too restrictive.

Proposition 2.1. *Consider a risk measure ρ satisfying monotonicity and excess invariance.*

1. *If ρ is normalized, then it is non-negative.*

2. *If ρ is non-zero, then it is not comonotonic additive.*

Proof. The first item follows directly from Proposition 3.1 of Staum (2013). The proof of item 2 goes by contradiction, so lets begin assuming that ρ satisfies all the properties mentioned in the statement. By Proposition 2.5 of Koch-Medina et al. (2018), non-zero monotone and comonotonic additive risk measures are positive homogeneous and, therefore, normalized. Hence, item 1 implies that ρ is non-negative. Therefore, the non-zero property implies the existence of $X \in \mathcal{X}$ such that $\rho(X) > 0$. Also, notice that $X + \|X\|_\infty \geqslant 0$ and, therefore, $-(X + \|X\|_\infty)^- = 0$. Excess invariance implies $\rho(X + \|X\|_\infty) = \rho(-(X + \|X\|_\infty)^-)$, so normalization implies $\rho(X + ||X||_\infty) = 0$. On the other hand, comonotonic additivity and positive homogeneity imply that $\rho(X + ||X||_\infty) = \rho(X) + ||X||_\infty \rho(1)$. But excess invariance implies $\rho(c) = \rho(0)$ for all $c \geqslant 0$, and normalization implies $\rho(0) = 0$. Therefore, we conclude that $\rho(X + ||X||_\infty) = \rho(X) > 0$, which is absurd. ☐

Remark 2.2. Item 2 of Proposition 2.1 unveils a decisive incompatibility between the properties of excess invariance and comonotonic additivity. As an intuitive explanation of the contradiction in the previous proof, notice that, on the one hand, the monotonicity, normalization, and excess invariance of ρ allow us to "neutralize" the risk of X (assuming $\rho(X) > 0$) by adding $||X||_\infty$ to it and using excess invariance to obtain $\rho(X + ||X||_\infty) = 0$. On the other hand, when comonotonic additivity and excess invariance are taken together one can no longer neutralize the risk of X by adding a large constant to it—for instance $||X||_\infty$—because in this case we would end up with $\rho(X + ||X||_\infty) = \rho(X) + ||X||_\infty \rho(1) = \rho(X) > 0$.

As explained in the above remark, the conflict presented in Proposition 2.1 arises as the property of comonotonic additivity can be used to get $\rho(X + \|X\|_\infty) = \rho(X) + \|X\|_\infty \rho(1)$, which is similar to what one

[1] We adopt the concise notation $-X^- = \min\{X, 0\}$.

would obtain with the property of cash additivity. In this context, therefore, the properties of comonotonic additivity and cash additivity play a similar role and, as shown in the next proposition, lead to similar conflicts with excess invariance.

Proposition 2.2. *(Staum (2013) - Proposition 3.2) There is no risk measure that is both excess invariant and cash additive.*

In Proposition 2.1 we unveiled a new conflict between comonotonic additivity and the property of excess invariance for risk measures. The notion of excess invariance can also be introduced through acceptance sets and, as a second new result, we will show that a milder conflict with comonotonic additivity remains.

Definition 2.2. *(Gao and Munari, 2020; He and Peng, 2018; Koch-Medina et al., 2017) An acceptance set* $\mathcal{A}$ *is* **surplus invariant** *if whenever* $X \in \mathcal{A}$ *and* $Y \in \mathcal{X}$ *are such that* $Y^- \leqslant X^-$ *we have that* $Y \in \mathcal{A}$.

Remark 2.3. For monotone acceptance sets, surplus invariance can be equivalently defined by exchanging the condition $Y^- \leqslant X^-$ by $Y^- = X^-$. This second definition is used in Staum (2013) and Koch-Medina et al. (2015).

Compared to Staum (2013) and Cont et al. (2013), He and Peng (2018) and Koch-Medina et al. (2017) focus on acceptance sets rather than risk measures. According to the axiom of surplus invariance, if a position $X \in \mathcal{X}$ passes the regulator's criteria, i.e., if X belongs to the regulator's acceptance set $\mathcal{A}$, and if $Y \in \mathcal{X}$ is another financial position whose *option to default*, i.e., its negative part, is smaller than that of X, then the regulator should also accept Y. In the sense of the following proposition, excess invariance is stronger than surplus invariance. For the proof, it will be useful to recall that the value at risk at level $p \in [0,1]$ is defined as $\mathrm{VaR}_p(X) = \inf\{m \in \mathbb{R} : \mathbf{P}(X + m < 0) \leqslant p\}$.

Proposition 2.3. *If a risk measure* ρ *is excess invariant, then* $\mathcal{A}_\rho$ *is surplus invariant. An acceptance set* $\mathcal{A}$ *being surplus invariant does not imply that* $\rho_{\mathcal{A}}$ *is excess invariant.*

Proof. The first assertion follows from Proposition 4.1 of Staum (2013). For the second assertion, we rely on a counter-example. For $p \in (0,1)$ take $\mathcal{A} = \{X \in \mathcal{X} : \mathbf{P}(X < 0) \leqslant p\}$. The surplus invariance of $\mathcal{A}$ was proved in Proposition 1.ii of He and Peng (2018). We replicate their argument here for the sake of completeness.

To see that $\mathcal{A} = \{X \in \mathcal{X} : \mathbf{P}(X < 0) \leqslant p\}$ is surplus invariant, take $X \in \mathcal{A}$ and let $Y \in \mathcal{X}$ be such that $Y^- \leqslant X^-$. Then we have

$$\mathbf{P}(Y < 0) = \mathbf{P}(-Y^- < 0) \leqslant \mathbf{P}(-X^- < 0) = \mathbf{P}(X < 0) \leqslant p.$$

This, in turn, implies that $Y \in \mathcal{A}$ and that $\mathcal{A}$ is surplus invariant.

It is well-known that $\rho_{\mathcal{A}} = \mathrm{VaR}_p$. Therefore, to conclude the second item it remains to show that VaR_p is not excess invariant. This follows from the VaR's cash additivity and Proposition 2.2. □

In the context of He and Peng (2018)'s discussion, the next result conveys the message that the surplus invariance axiom is quite restrictive, even if not restrictive enough to necessarily generate excess invariant risk measures, as shown in Proposition 2.3.

Theorem 2.1. *(He and Peng (2018) - Theorem 2) A non-empty acceptance set* $\mathcal{A}$ *is surplus invariant, law invariant, conic, and closed with respect to convergence in probability if and only if there exists* $p \in [0,1]$ *such that* $\mathcal{A} = \{X \in \mathcal{X} : \mathbf{P}(X < 0) \leqslant p\}$.

As a direct corollary of Theorem 2.1, the next result shows that, even if ρ is not excess invariant, if we require it to be comonotonic additive and its induced acceptance set $\mathcal{A}_\rho$ to be surplus invariant, then ρ is necessarily the value at risk.

Corollary 2.1. *Let $\rho : \mathcal{X} \to \mathbb{R}$ be a monetary law invariant comonotonic additive risk measure satisfying lower-semicontinuity with respect to convergence in probability. If $\mathcal{A}_\rho$ is surplus invariant, then $\rho = \mathrm{VaR}_p$ for some $p \in [0,1]$.*

Proof. Since ρ is law invariant, so is $\mathcal{A}_\rho$. Also, since ρ is positive homogeneous and non-zero, the acceptance set $\mathcal{A}_\rho$ is conic (see Theorem 2.7). Additionally, since ρ is lower-semicontinuous w.r.t. the convergence in probability topology, we conclude that $\mathcal{A}_\rho$ is closed in that topology. Since $\mathcal{A}_\rho$ is surplus invariant, Theorem 2.1 implies that $\mathcal{A}_\rho = \mathcal{A}_{\mathrm{VaR}_p}$ for some $p \in [0,1]$. Since ρ is monetary, Theorem 2.7 implies that $\rho = \mathrm{VaR}_p$ for some $p \in [0,1]$. □

2.3 RISK ELIGIBLE ASSETS

Most of the literature assumes, if only for the sake of simplicity, the existence of a risk-free asset in which a financial institution bearing the risk $X \in \mathcal{X}$ can invest the regulatory capital $\rho(X)$ in order to meet the regulator's criteria of acceptability. Assuming the existence of a unique risk–free asset is helpful, but not realistic in some contexts. For instance, assuming the existence of a risk–free asset during a period of crisis is, to our minds, controversial. In the absence of such asset, one is led to work with random or ambiguous interest rates (El Karoui and Ravanelli, 2009). On the other hand, if there exists more than one risk-free asset (possibly for a financial institution with assets and liabilities denominated in different currencies) one is left with the problem of deciding between the two (Artzner et al., 2009).

Circumstances of this sort motivate the study of capital regulation in contexts where financial institutions are allowed to compose their regulatory capital with risky assets. This generalization is in accordance with the regulatory framework proposed by the Basel Committee on Banking Supervision (BCBS, 2019), which allows the banks to compose their regulatory reserves with assets in different classes of risk.

An **eligible asset** is a couple $S = (S_0,\, S_1) \in (0,\, \infty) \times L^\infty_+(\Omega, \mathcal{F}, \mathbf{P})$. The S_0 component is a constant representing the time $t = 0$—today's—value of the asset, and the component S_1 represents its terminal payoff. If S_1 is non-constant, we say that S is a risky eligible asset; otherwise S is riskless. We also assume that S_1 is bounded away from zero, i.e., $S_1 \geqslant \epsilon$ for some $\epsilon > 0$. Throughout this section, we follow Koch-Medina et al. (2018) and assume that all acceptance sets are closed and monotone. The combination of an acceptance set $\mathcal{A}$ and an eligible asset S defines a risk measure through

$$\rho_{\mathcal{A},S}(X) = \inf \left\{ m \in \mathbb{R} : X + \frac{m}{S_0} S_1 \in \mathcal{A} \right\}. \tag{2.1}$$

Remark 2.4. In the zero interest rate framework (see Section 2.9.2), there is no difference between taking the random variables $X \in \mathcal{X}$ as terminal or discounted payoffs. For most of this paper, we work with discounted payoffs, however, when risky eligible assets are considered, it is convenient to let the random variables $X \in \mathcal{X}$ represent terminal payoffs. In view of this remark, notice that both X and $(m/S_0)S_1$ stand for financial positions expressed in the same (terminal) monetary unit. The intuition behind eq. (2.1) is that the financial institution will invest m dollars to acquire m/S_0 units of the eligible asset S which has terminal payoff S_1.

Remark 2.5. As showed in Proposition 2.12 of Farkas et al. (2014), the properties we assume for $\mathcal{A}$ and S implies that $\rho_{\mathcal{A},S}$ is Lipschitz continuous and finite. Under these hypothesis, the functional $\rho_{\mathcal{A},S}$ satisfies

- *(S-additivity)* $\rho_{\mathcal{A},S}(X + \lambda S_1) = \rho_{\mathcal{A},S}(X) - \lambda S_0,\ \forall X \in \mathcal{X},\ \forall \lambda \in \mathbb{R}$,

meaning that the risk is equivariant with respect to the amount invested in the eligible asset. In the special case of $S = (1, 1)$, we recover the traditional cash additivity property.

2.3.1 Comonotonicity and Risky Eligible Assets

To our review of comonotonic additive risk measures, the main contribution of Koch-Medina et al. (2018) is to have obtained necessary and sufficient conditions for $\rho_{\mathcal{A},S}$ to be comonotonic additive. As the following proposition shows, these conditions are quite restrictive as they require the regulator to deem highly leveraged financial positions acceptable. Notice that, when specialized to a slightly less general case, the result of Koch-Medina et al. (2018) shows that the usage of comonotonic additive risk measures is incompatible with the usage of risky eligible assets, so that the financial firms must held their regulatory capital in the form of a risk-free asset.

Proposition 2.4. *(Koch-Medina et al. (2018) - Proposition 2.18, Corollary 2.20) Assume that $\rho_{\mathcal{A}}$ is comonotonic. Then, the following statements are equivalent:*

1. *$\rho_{\mathcal{A},S}$ is comonotonic.*
2. *$\mathcal{A} \pm \left(1 + \frac{\rho_{\mathcal{A},S}(1)}{S_0} S_1\right) \subset \mathcal{A}$.*

Moreover, if $\mathcal{A}$ is pointed and S is a risky eligible asset, then $\rho_{\mathcal{A},S}$ is not comonotonic.

Remark 2.6. The condition of pointedness amounts to $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$. As showed in Koch-Medina et al. (2018), this condition holds for the VaR, AVaR, and the spectral risk measures. These are main representatives of the class of comonotonic additive risk measures. However, the authors showed that, as a consequence of the pointedness condition, the comonotonic additivity of those representative risk measures is lost once risky eligible assets are considered.

As suggested in Koch-Medina et al. (2018), we can grasp a stronger intuition for the condition in the second item of Proposition 2.4 by assuming that $\rho_{\mathcal{A},S}(1) = -1$. In this case we have

$$\mathcal{A} \pm \left(1 - \frac{S_1}{S_0}\right) \subset \mathcal{A}. \tag{2.2}$$

If $0 \in \mathcal{A}$—let's assume that $\mathcal{A}$ is not pointed—then eq. (2.2) implies that

$$\pm \left(1 - \frac{S_1}{S_0}\right) \subset \mathcal{A}. \tag{2.3}$$

The random variable $1 - (S_1/S_0)$ in Equation (2.3) represents the position of a bank that financed one unit of the risk-free asset by short-selling $1/S_0$ units of S[2]. This position realizes losses exactly when $S_1 > S_0$. Therefore, if the eligible asset pays positive interests with probability one—which amounts to $\mathbf{P}(S_1 > S_0) = 1$—the position $1 - (S_1/S_0)$ realize losses with probability one and, nonetheless, is acceptable.

Corollary 2.2. *Let $\mathcal{A}$ be a monetary acceptance set such that $0 \in \mathcal{A}$. If $\rho_{\mathcal{A},S}$ is comonotonic additive with $\rho_{\mathcal{A},S}(1) = -1$, then*

$$span \left(1 - \frac{S_1}{S_0}\right) \subset \mathcal{A}. \tag{2.4}$$

[2]Ignoring transaction costs.

Proof. In view of eq. (2.3), it suffices to show that $\mathcal{A}$ is conic. To see this is the case, notice that the comonotonicity of $\rho_{\mathcal{A},S}$ implies that of $\rho_{\mathcal{A}}$ (see Proposition 2.15 of Koch-Medina et al. (2018)). The monetarity of $\mathcal{A}$ implies that of $\rho_{\mathcal{A}}$ (see item 2 of Theorem 2.7). Also, the cash additivity of $\rho_{\mathcal{A}}$ implies that it is non-zero. Therefore we can apply Proposition 2.5 of Koch-Medina et al. (2018) to conclude that $\rho_{\mathcal{A}}$ is positive homogeneous. Then $\mathcal{A}_{\rho_{\mathcal{A}}}$ is conic (see item 5 of Theorem 2.7) and, therefore, $\mathcal{A}$ is conic (see item 3 of Theorem 2.7). □

Remark 2.7. The above corollary summarizes the discussion Koch-Medina et al. (2018) presented after their Proposition 2.18. Essentially, it says that, if an acceptance set $\mathcal{A}$ and a risky eligible asset S induces a comonotonic additive risk measure, then $\mathcal{A}$ must contain arbitrarily large fully leveraged positions. In the following subsection, we illustrate a consequence of this result by comparing the risk of $1 - (S_1/S_0)$ as measured by the traditional value at risk (which is comonotonic additive), and by the value at risk based on a risky eligible asset (which turns out not being comonotonic additive).

2.3.2 Examples

In this section, we review the findings of Koch-Medina et al. (2018) regarding the lack of comonotonic additivity of particular risk measures based on risky eligible assets. Koch-Medina et al. (2018) constructed "risky eligible" counter-parts of VaR, ES, and of the class of distortion risk measures. All these risk measures are, originally, comonotonic additive. However, their counterparts inherit the property of comonotonic additivity if and only if the eligible asset being used is risk-free.

Recall that the value at risk at level $p \in (0,1)$ of a position $X \in \mathcal{X}$ is defined as the following real number:

$$\mathrm{VaR}_p(X) = \inf\{m \in \mathbb{R} : \mathbf{P}(X + m < 0) \leqslant p\}.$$

Koch-Medina et al. (2018) defined the counter-part of the value at risk with respect to an eligible asset $S = (S_0, S_1)$ as

$$\begin{aligned} \text{S-VaR}_p(X) \coloneqq \rho_{\mathcal{A}_{\mathrm{VaR}_p},S}(X) &= \inf\left\{m \in \mathbb{R} : X + \frac{m}{S_0} S_1 \in \mathcal{A}_{\mathrm{VaR}_p}\right\} \\ &= \inf\left\{m \in \mathbb{R} : \mathrm{VaR}_p\left(X + \frac{m}{S_0}S_1\right) \leqslant 0\right\} \\ &= \inf\left\{m \in \mathbb{R} : \mathbf{P}\left(X + \frac{m}{S_0}S_1 < 0\right) \leqslant p\right\}. \end{aligned}$$

Proposition 2.5. *(Koch-Medina et al. (2018) - Proposition 3.4) The risk measure* S-VaR$_p$ *is comonotonic if and only if S is risk-free.*

Therefore, if the regulatory authority insists on a comonotonic additive risk measure, then it cannot determine the banks' regulatory capital through a rule of the type "banks should invest in S until $\mathbf{P}(X + mS_1/S_0 < 0) \leqslant p$", where p is usually taken as 0.01.

Results of the same nature as Proposition 2.5 were also obtained for the AVaR and the class of distortion risk measures. In the following definitions, $p \in (0,1]$ and, for $X \in \mathcal{X}$, $\mathrm{DR}_\mu(X) \coloneqq \int_0^1 \mathrm{AVaR}_p(X)\mu(dp)$, where μ is a probability measure of the Borel sets of $[0,1]$ (see Theorem 2.8 for more details). Koch-Medina et al. (2018) defined the following risk measures:

$$\begin{aligned}\text{S-AVaR}_p(X) &:= \rho_{\mathcal{A}_{\text{AVaR}_p},S}(X) = \inf\left\{m \in \mathbb{R} : X + \frac{m}{S_0}S_1 \in \mathcal{A}_{\text{AVaR}_p}\right\} \\ &= \inf\left\{m \in \mathbb{R} : \text{AVaR}_p\left(X + \frac{m}{S_0}S_1\right) \leqslant 0\right\}, \text{ and}\end{aligned}$$

$$\begin{aligned}\text{S-DR}_\mu(X) &:= \rho_{\mathcal{A}_{\text{DR}_\mu},S}(X) = \inf\left\{m \in \mathbb{R} : X + \frac{m}{S_0}S_1 \in \mathcal{A}_{\text{DR}_\mu}\right\} \\ &= \inf\left\{m \in \mathbb{R} : \text{DR}_\mu\left(X + \frac{m}{S_0}S_1\right) \leqslant 0\right\}.\end{aligned}$$

Proposition 2.6. *(Koch-Medina et al. (2018) - Propositions 3.7 and 3.10) The risk measure* S-AVaR$_p$ *is comonotonic additive if and only if S is risk-free. Also, the risk measure* S-DR$_\mu$ *is comonotonic additive if and only if one of the following conditions holds:*

1. $\mu(\{1\}) = 1$ *(so that* $\text{DR}_\mu(X) = -\mathbf{E}[X]$ *for all* $X \in \mathcal{X}$*).*
2. *S is risk-free.*

Corollary 2.3. *Let $\mathcal{A}$ be a monetary acceptance set and S a risky eligible asset such that $\rho_{\mathcal{A},S}(1) = -1$.*

1. *If $\rho_{\mathcal{A},S}$ is comonotonic additive and $0 \in \mathcal{A}$, then*

$$\rho_{\mathcal{A},S}\left(\lambda\left(1 - \frac{S_1}{S_0}\right)\right) \leqslant 0, \forall \lambda \geqslant 0. \tag{2.5}$$

2. *If* $\mathbf{P}(S_0 < S_1) > p$*, for* $p \in (0, 1)$*, then*

$$\lim_{\lambda\to\infty} \text{S-VaR}_p\left(\lambda\left(1 - \frac{S_1}{S_0}\right)\right) = \lim_{\lambda\to\infty} \text{S-AVaR}_p\left(\lambda\left(1 - \frac{S_1}{S_0}\right)\right) = \infty. \tag{2.6}$$

Proof. The first item is a direct consequence of Equation (2.3). The second item follows for $\mathbf{P}(S_0 < S_1) > p$ implies $\mathbf{P}(1 - (S_1/S_0) < 0) > p$ and, therefore, $\text{S-VaR}_p(1 - (S_1/S_0)) > 0$. The conclusion follows by the positive homogeneity of S-VaR and the fact that $\text{S-VaR}(X) \leqslant \text{S-AVaR}(X)$ for all $X \in \mathcal{X}$. □

Remark 2.8. The above corollary shows that, if the eligible asset is risky, the risk measurements obtained through a comonotonic additive risk measure in the form $\rho_{\mathcal{A},S}$ can be drastically different from those obtained through the more traditional (non-comonotonic additive) S-VaR and S-AVaR.

2.4 ELICITABILITY

In the last decade, the issue of elicitability has become a research agenda in the literature on risk measures (Acerbi and Szekely, 2017; Bellini and Bignozzi, 2015; Fissler and Ziegel, 2021; Kou and Peng, 2016; Ziegel, 2016). Arguably, the reason for such interest is that the elicitability of a risk measure allows a meaningful comparison of the predictive performance of competing forecasting procedures, if not only to provide meaningful inference procedures (Gneiting, 2011). Comparing risk forecasting procedures is especially important for risk management because the tails of the distributions are particularly difficult to estimate (Kou and Peng, 2016). Compelling evidence for the importance of elicitability was presented in Gneiting (2011), Patton (2011), in the supplementary material of Nolde and Ziegel (2017), and in Fissler and Ziegel (2021). In this section, we review some results of Bellini and Bignozzi (2015), Kou and Peng (2016), and Ziegel (2016) unveiling the scarcity of comonotonic additive elicitable risk measures.

Determining the regulatory capital for a position $X \in \mathcal{X}$—which, in theory, is denoted as $\rho(X)$—usually involves estimating X's distribution[3]. For such applied purposes, requiring the risk measures ρ to be law invariant is of primal importance, and therefore we assume it for this entire section. Law invariant risk measures on $\mathcal{X}$ induce risk measures mapping the set of probability distributions with bounded support, denote it by $\mathcal{P} := \{F_X : X \in \mathcal{X}\}$, into risk measurements[4]. It is convenient not to change the symbol used to denote the induced "statistical" risk measures and, therefore, these are defined as

$$\rho(F) := \rho(X) \text{ if and only if } F_X = F, \ \forall F \in \mathcal{P}. \tag{2.7}$$

The criterion to rank two alternative forecasting procedures, say A and B, that produce theoretical forecasts x^A and x^B for the true value $\rho(F_X)$, follows rules of the type:

$$\text{“}A \text{ is better than } B \text{ if and only if } \mathbf{E}[S(x^A, X)] \leqslant \mathbf{E}[S(x^B, X)]\text{”} \tag{2.8}$$

where $S : \mathbb{R}^2 \to \mathbb{R}_+$ is a non-negative function such that $S(x, y)$ is (usually) increasing in the difference $|x - y|$. There are two main assumptions behind such rules:

1. $\rho(F_X)$ minimizes $\mathbf{E}[S(x, X)]$ w.r.t. $x \in \mathbb{R}$, and;
2. If $x^A, x^B \in \mathbb{R}$ are such that $\rho(F_X) < x^A < x^B$ or $x^B < x^A < \rho(F_X)$, then $\mathbf{E}[S(x^A, X)] \leqslant \mathbf{E}[S(x^B, X)]$.

The first condition guarantees that the criterion in eq. (2.8) ranks an estimation procedure producing the true value $\rho(F_X)$ above any other procedure. The second condition guarantees that the criterion in eq. (2.8) is meaningful even if neither of the estimation procedures being compared was able to produce the true value $\rho(F_X)$. This second condition received special attention in Bellini and Bignozzi (2015).

There are slight differences in the literature regarding the formal definition of elicitability. The following definition, for instance, does not require the second condition.

Definition 2.3. *(Kou and Peng (2016)) A single-valued statistical functional $\rho\colon \mathcal{P} \to \mathbb{R}$ is* **general elicitable** *with respect to a class of distributions $\mathcal{M} \subseteq \mathcal{P}$ if there exists a scoring function $S : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\rho(F) = -\min\Big\{x \in \mathbb{R} \,|\, x \in \arg\min_x \int S(x, y) dF(y)\Big\}, \quad \forall F \in \mathcal{M}$$

In this case, we say that S is **consistent** *for ρ with respect to the class $\mathcal{M}$.*

Definition 2.3 draws from the intuition that, when it comes to the estimation of a given (elicitable) functional and, more specifically, to the evaluation of estimation procedures, there should be a match between the functional being estimated, on the one hand, and the corresponding score function being used, on the other. In this regard, Gneiting (2011) presented a compelling argument showing that a mismatch between S and ρ can lead to twisted decisions regarding the relative performance of alternative estimation procedures.

Example 2.1. The squared deviation score $S(x, y) = (x - y)^2$ is consistent for $\rho(\cdot) = -\mathbb{E}[\cdot]$ with respect to the class of distributions with finite first moment. This happens because, for F with finite first moment, it

[3] Alternatively, in bayesian/subjectivist approaches one consider *personal credences* to determine the regulatory capital.

[4] The results of Kou and Peng (2016) were obtained for more general domains. We keep with $\mathcal{X} = L^\infty$ for the sake of unity with the rest of the paper.

follows that $-\mathbf{E}[F] = -\arg\min_{x\in\mathbb{R}} \mathbf{E}[(x-Y)^2]$ whenever $Y \sim F$. As documented in Gneiting (2011), the squared deviation function is, by far, the most used in academia and industry.

Example 2.2. The function

$$S(x,y) = (1(x \geqslant y) - p)(g(x) - g(y)),$$

where $1(\cdot)$ denotes the indicator functions and $g : \mathbb{R} \to \mathbb{R}$ is increasing, is consistent for the value at risk with respect to the class of distributions with finite first moment.

Remark 2.9. A necessary condition for a statistical functional ρ to be elicitable with respect to a given class $\mathcal{P}$ is that, for all $F_1, F_2 \in \mathcal{P}$ such that $\lambda F_1 + (1-\lambda)F_2 \in \mathcal{P}$ for all $\lambda \in [0,1]$, it must hold that, if $\rho(F_1) = \rho(F_2)$, then $\rho(F_1) = \rho(\lambda F_1 + (1-\lambda)F_2)$, $\forall \lambda \in [0,1]$. A functional satisfying this is said to have **convex levels sets**. Also, it is valid to observe that not being elicitable with respect to a class $\mathcal{P}_0 \subseteq \mathcal{P}$ implies not being elicitable with respect to $\mathcal{P}$. This fact follows for, if the level sets of ρ are not convex within $\mathcal{P}_0$, then they are not convex when the larger class $\mathcal{P}$ is considered. In some cases, this observation allows one to restrict attention to elicitability with respect to very simple classes of probability distributions. For instance, Kou and Peng (2016) studied elicitability with respect to the class of discrete distributions $F = \sum_{i=1}^{n} p_i \delta_{x_i}$, where δ_{x_i} is the Dirac measure at the point $x_i \in \mathbb{R}$, $0 \leqslant x_1 < x_2 < \cdots < x_n$, $p_i > 0$, $i = 1, \cdots, n$, and $\sum_{i=1}^{n} p_i = 1$.

In line with the elicitability's relevance for risk management, several authors have put a great amount of effort to understand which risk measures are elicitable. In this regard, Weber (2006), Gneiting (2011), Bellini and Bignozzi (2015), Ziegel (2016), and Kou and Peng (2016) were invaluable contributions. As it turns out, elicitability for risk measures is the exception rather than the rule. For instance, AVaR is not elicitable (Gneiting, 2011; Weber, 2006). Keeping the focus on coherent risk measures, Bellini and Bignozzi (2015) and Ziegel (2016) showed (independently) that the class of coherent elicitable risk measures consists of expectiles.

Definition 2.4. *(Newey and Powell (1987), Ziegel (2016)) For $\tau \in (0,1)$ and $X \in \mathcal{X}$, the τ-expectile of X, denoted $\mu_\tau(X)$, is the unique solution to the following equation:*

$$\tau \int_x^\infty (y-x)dF_X(y) = (1-\tau)\int_{-\infty}^x (x-y)dF_X(y). \tag{2.9}$$

Theorem 2.2. *(Ziegel (2016) - Corollaries 4.3 and 4.6) Let $\rho : \mathcal{X} \to \mathbb{R}$ be a monetary law invariant risk measure whose statistical counter-part is elicitable with respect to any class of probability distributions that contains the two-point distributions. Then*

1. *ρ is coherent if and only if $\rho(X) = \mu_\tau(X)$ for some $\tau \in (0, 1/2]$ and all $X \in \mathcal{X}$.*
2. *ρ is coherent and comonotonic additive if and only if $\rho(X) = -\mathbf{E}[X]$ for all $X \in \mathcal{X}$.*

Remark 2.10. Theorem 2.2 tells us that, for applications in which elicitability is essential and coherence is desirable, one must adopt an expectile as the risk measure (or must give up coherence). Moreover, by further requiring comonotonic additivity, the set of coherent elicitable risk measures collapses to the expected loss, which is the expectile for $\tau = 1/2$.

Theorem 2.3. *(Kou and Peng (2016) - Theorem 1) Let $\rho : \mathcal{X} \to \mathbb{R}$ be a monetary law invariant comonotonic additive risk measure (see Lemma 2.1). Then the statistical counter-part of ρ is elicitable with respect to the class of discrete distributions if and only if one of the following holds:*

1. *$\rho(X) = \mathrm{VaR}_p(X)$ for some $p \in [0,1]$ and all $X \in \mathcal{X}$.*

2. *$\rho(X) = \mathbf{E}[-X]$ for all $X \in \mathcal{X}$.*

Remark 2.11. Theorem 2.3 complements Theorem 2.2 by showing that, even outside the coherent framework, the axiom of comonotonic additivity considerably narrows the class of elicitable risk measures (this result was corroborated in Wang and Ziegel (2015)).

Remark 2.12. Before concluding, we must mention the work of Fissler and Ziegel (2016). They generalized the concept of elicitability, extending it to vector-valued functionals. In this case, one says that the components of the vector-valued functional are jointly-elicitable. As much as for real-valued functionals, jointly elicitability gives us a method to compare the performance of alternative forecast procedures. Remarkably, this can be done even if the components of the vector-valued functional are not individually elicitable. A prominent example of this is the functional $T(X) = (\mathrm{AVaR}_p(X), \mathrm{VaR}_p(X))$, which is jointly-elicitable, even if AVaR_p is not elicitable individually. Also, Fissler and Ziegel (2016) showed that any (finite) convex combination of AVaR_p (for significance levels $0 < p_0 < p_1 < \cdots < p_n \leqslant 1$) is jointly-elicitable with the quantiles $p_0 < p_1 < \cdots < p_n$. These convex combinations are coherent risk measures, which form a (narrow) subclass of spectral risk measures (see Theorem 2.8).

2.5 TIME-CONSISTENCY

The most prominent benefit of generalizing risk measures to the dynamic context is to allow the risk to depend on new information. For a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the information flow is modeled through a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$. The σ-algebra $\mathcal{F}_t$ represents the information available at time $t \in \mathcal{T}$, and the time horizon $\mathcal{T}$ may be discrete ($\mathcal{T} := \{0, 1, 2, \ldots, T\}$) or continuous ($\mathcal{T} = [0, T]$), and might be finite ($T \in \mathbb{R}$) or infinite ($T = \infty$). To simplify the exposition we restrict our attention to the discrete finite case. We denote $L^\infty_t := L^\infty(\Omega, \mathcal{F}_t, \mathbf{P})$, and assume $\mathcal{F}_T = \mathcal{F}$. Therefore we have $L^\infty_T = L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. To measure the risk of a financial position $X \in L^\infty$ conditional on the information available at $t \in \mathcal{T}$ one usually relies on the following tools:

Definition 2.5. *(Föllmer and Schied (2016), Delbaen (2021)) For $t \in \mathcal{T}$ we call a map $\rho_t : L^\infty \to L^\infty_t$ a* **conditional risk measure**. *Also, we call ρ_t a* **monetary conditional risk measure** *if it satisfy the following properties:*

1. *(Conditional Cash Additivity) ρ_t is* **conditionally cash additive** *if $\rho_t(X + Z) = \rho_t(X) - Z$ for any $X \in L^\infty$ and $Z \in L^\infty_t$.*

2. *(Monotonicity) ρ_t is* **monotone** *if $X \leqslant Y$ implies $\rho_t(Y) \leqslant \rho_t(X)$ for all $X, Y \in L^\infty$.*

3. *(Normalization) ρ_t is* **normalized** *if $\rho_t(0) = 0$.*

In addition, a conditional risk measure might satisfy

4. *(Conditional Comonotonicity) ρ_t is* **conditionally comonotonic** *if $\rho_t(X + Y) = \rho_t(X) + \rho_t(Y)$ for all comonotonic $(X, Y) \in \mathcal{X}^2$.*

Notice that the static framework is recovered by the risk measure $\rho_0 : L^\infty \to L^\infty_0 = \mathbb{R}$. Conditional risk measures generalize this static perspective on risk, allowing us to measure, at $t = 0$, the abstract notion of the "risk of $X \in \mathcal{X}$ at $t > 0$". In the same vein as conditional expectations, these conditional risk measurements are random variables whose distribution depends on the filtration. To illustrate this analogy, notice that conditional cash additivity and normalization implies that $\rho_T(X) = -X$ for all $X \in L^\infty$, which is (up to the sign) the result of taking expectation w.r.t. $\mathcal{F}$.

The traditional properties of convexity, positive homogeneity, and subadditivity have also counterparts for conditional risk measures, and most of the basic theory presented in the Appendix's section 2.9.2 can be immediately adapted to conditional risk measures (see Acciaio and Penner (2011) and Föllmer and Schied (2016) for details).

Definition 2.6. *A collection $(\rho_t)_{t\in\mathcal{T}}$ is called a* **dynamic risk measure** *if ρ_t is a conditional risk measure for each $t \in \mathcal{T}$. The following are properties that a dynamic risk measure might satisfy:*

1. *(Time-consistency) $(\rho_t)_{t\in\mathcal{T}}$ is* **time-consistent** *if*

$$\rho_{t+1}(X) \geqslant \rho_{t+1}(Y) \Rightarrow \rho_t(X) \geqslant \rho_t(Y)$$

for any $X, Y \in L^\infty$ and for all $t \in \{0, 1, \ldots, T-1\}$.

2. *(Relevance) $(\rho_t)_{t\in\mathcal{T}}$ is* **relevant** *if $\rho_0(-\epsilon 1_A) > 0$ for all $A \in \mathcal{F}$ and all $\epsilon > 0$.*

Remark 2.13. We say that a dynamic risk measure $(\rho_t)_{t\in\mathcal{T}}$ satisfy a property presented in Definition 2.5 if the respective property holds for ρ_t for all $t \in \mathcal{T}$. In particular, $(\rho_t)_{t\in\mathcal{T}}$ is monetary if each ρ_t is monetary.

Arguably, the main concern about dynamic risk measures is to define how the risks in different periods should relate. For instance, consider two financial positions X, Y in L^∞ such that $X \leqslant Y$. In this case, an investor in $t = 0$ knows with certainty that at $t = T$ the result of X will be worse than that of Y. With this in mind, the investor would know, at $t = 0$, that, irrespectively of what might happens between $t = 0$ and $t = T$, the risk of X will be greater than that of Y at $t = T$, i.e., $\rho_T(X) \geqslant \rho_T(Y)$. This follows by assuming that ρ_T is monotone, which is a minimal assumption for risk measures. Now, if the investor knows that, at the end of the game, the risk of X is greater than that of Y, then it would be "reasonable" to use this information when comparing the risk of the positions at $t = T - 1$. An iteration of this argument leads to time-consistency.

Remark 2.14. The time-consistency property can be equivalently defined in a manner similar to the "tower property" of conditional expectation: $\rho_t = \rho_t(-\rho_{t+1})$ for all $t \in \{0, 1, \ldots, T-1\}$. This condition illustrates that, in the time-consistent framework, the time t risk of an $\mathcal{F}_T$ measurable random variable X is fully determined by the random variable $\rho_{t+1}(X)$, which is measurable with respect to $\mathcal{F}_{t+1}$ (see, for instance, Acciaio and Penner (2011) and Föllmer and Schied (2016) for details).

Theorem 2.4. *(Kupper and Schachermayer (2009) - Theorem 1.10) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbf{P})$ be a standard filtered probability space. A monetary dynamic risk measure $(\rho_t)_{t\in\mathcal{T}}$ is time-consistent, relevant, and law invariant if and only if there is $\beta \in (-\infty, \infty]$ such that*

$$\rho_t(X) = \frac{1}{\beta} \ln \mathbf{E}[e^{-\beta X} | \mathcal{F}_t], \text{ for all } t \in \mathcal{T}. \tag{2.10}$$

Theorem 2.4 illustrates the scarcity of time-consistent dynamic risk measures. The above risk measure is called **entropic**. Notice, however, that the static counter-parts of entropic risk measures are not comonotonic additive and, therefore, one cannot obtain a time-consistent dynamic risk measure whose conditional components are comonotonic additive.

The conflict between comonotonicity and time-consistency also appears in Delbaen (2021). In his three period framework $\mathcal{T} = \{0, 1, 2\}$, a dynamic risk measure is a pair (ρ_0, ρ_1), with ρ_2 existing only implicitly since its conditional cash additivity would imply $\rho_2(X) = -X$ for all $X \in L_2^\infty$. In this setting, the tower property requirement for time-consistency boils down to $\rho_0(X) = \rho_0(-\rho_1(X))$ for all $X \in L_2^\infty$.

Definition 2.7. *Consider the following definitions:*

1. *Let ρ_t be a conditional risk measure for some $t \in \mathcal{T}$. We say ρ_t is* **Lebesgue continuous** *if, whenever $(X_n) \subseteq L^\infty$ is uniformly bounded and $X_n \to X$ in probability, we have $\rho_t(X_n) \to \rho_t(X)$ in probability.*
2. *We say that $\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$ if for every $A \in \mathcal{F}_2$, there exists a set $B \subseteq A$, $B \in \mathcal{F}_2$, such that $0 < \mathbf{E}[1_B|\mathcal{F}_1] < \mathbf{E}[1_A|\mathcal{F}_1]$ on the set $\{\mathbf{E}[1_A|\mathcal{F}_1] > 0\}$.*

Theorem 2.5. *(Delbaen (2021) - Theorem 6.1) Assume that $\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$ and let $(\rho_t)_{t\in\mathcal{T}}$ be a time-consistent dynamic risk measure. Also, assume that ρ_0 is coherent, relevant, comonotonic additive, and Lebesgue continuous. Then there is a probability $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that*

$$\rho_0(X) = \mathbf{E}_{\mathbf{Q}}[-X] \text{ for all } X \in L^\infty(\mathcal{F}_1). \tag{2.11}$$

Remark 2.15. Theorem 2.5 shows that, by insisting in both comonotonic additivity and time-consistency, the set of coherent risk measures (satisfying the additional hypothesis of the theorem) collapses to an expected value. In comparison to Theorem 2.4, the conflict between comonotonic additivity and time-consistency presented in Theorem 2.5 is more direct. Also, Theorem 2.5 does not rely on law invariance, which called for different proof techniques and juxtapose Delbaen (2021) with the recent research on law invariant risk measures that collapses to the mean (Bellini et al., 2021; Liebrich and Munari, 2022).

2.6 APPLICATION: PORTFOLIO RISK ANALYSIS

In the realm of Finance, the main justification to study risk measures is their potential as a tool for regulatory capital determination. With this application in mind, it makes sense to study risk measures on their own, as in the previous sections, so we can better understand the risk measures potentials and limitations for regulatory capital determination.

The notion of risk, however, is pervasive in Finance, Actuarial Science, and Decision Theory. Therefore, risk measures may be used as an element of other problems, even beyond the scope of determining regulatory capital. For instance, risk is a central element of portfolio analysis.

In this section, we depart from the study of risk measures in their own right, and summarizes some findings of Brandtner (2013) regarding the usage of spectral risk measures in portfolio selection problems. This leads to two quirks, which are not present in the mean variance framework and that may impose extra difficulties to portfolio optimization: first, Brandtner (2013) showed that, the traditionally equivalent problems of, on the one hand, minimizing risk subject to a prespecified level of expected return and, on the other hand, maximizing a utility function that balances the trade-off between risk and return are no longer equivalent. A second quirk that comes with the usage of spectral risk measures in portfolio selection is that the solutions lie at the corners. In particular, when the risk-free asset is included in the analysis—which is the case we focus—these corners solutions correspond to invest all or nothing in the risk-free asset or in the tangency portfolio. Therefore, if short-sales are allowed, using spectral risk measures for portfolio selection may involve assuming extremely leveraged positions.

We consider two risky assets, $X_1, X_2 \in \mathcal{X}$, and a risk-free asset, $X_0 \in \mathbb{R}$. We refer to Brandtner (2013) for the extension to the case of a finite general number of risky assets. The set of possible portfolios is $\mathcal{X}^\star = \{\beta(\gamma X_1 + (1-\gamma)X_2) + (1-\beta)X_0 : \beta \geqslant 0, \gamma \in \mathbb{R}\}$. A typical element of $\mathcal{X}^\star$ is denoted as $X_{\beta,\gamma}$.

2.6.1 Mean-variance portfolio analysis

Let $\mathbf{V}$ be the variance operator and consider the two following problems:

1. Limited analysis:

$$\min_{\beta \geqslant 0, \gamma \in \mathbb{R}} \mathbf{V}(X_{\beta,\gamma}) \tag{2.12}$$

$$s.t. : \mathbf{E}[X_{\beta,\gamma}] = \mu. \tag{2.13}$$

2. Trade-off analysis:

$$\max_{\beta \geqslant 0, \gamma \in \mathbb{R}} \mathbf{E}[X_{\beta,\gamma}] - \frac{\lambda}{2} \mathbf{V}(X_{\beta,\gamma}). \tag{2.14}$$

Remark 2.16. The level $\mu \in \mathbb{R}$ in eq. (2.13) is usually required to be greater than the expected return of the minimum variance portfolio. In the absence of this restriction, the minimum variance portfolio is the obvious solution.

Remark 2.17. The trade-off analysis given in eq. (2.14) has a strong theoretical basis for the case where the investor's absolute risk aversion is constant (and equal to λ) and the return of the risky assets is normally distributed (Bamberg, 1986). The limited analysis, on the other hand, might be more adequate for applications where the return level, μ, is determined at a higher hierarchical level of the financial analysis, so that the portfolio manager is restricted to portfolios with a mean return equal to μ.

For the next proposition, γ_{MVP} denotes the weight for the minimum variance portfolio, and X_{T,σ^2} denotes the tangency portfolio of the (μ, σ^2)-analysis. To focus on the main message, we will refer the readers interested in the exact expressions for γ_{MVP} and X_{T,σ^2} to the original article.

Proposition 2.7. *(Brandtner (2013)-Proposition 4.2) The solution to the (μ, σ^2) trade-off analysis (eq. 2.14) is given by*

$$\gamma^\star = \gamma_{MVP} - \frac{\mathbf{E}[X_2 - X_1]}{\lambda(\mathbf{V}(X_1) + \mathbf{V}(X_2) - 2Cov(X_1, X_2))} \text{ and} \tag{2.15}$$

$$\beta^\star = \frac{\mathbf{E}[X_{T,\sigma^2} - X_0]}{\lambda \mathbf{V}(X_{T,\sigma^2})}. \tag{2.16}$$

The limited and trade-off analysis approaches are equivalent in the mean–variance framework: there exists a one-to-one correspondence between the parameters μ and λ such that the problems (2.12-2.13) and (2.14) generate the same solution whenever μ is chosen as $\mu(\lambda)$ or, equivalently, $\lambda = \lambda(\mu)$[5]. This equivalence, however, does not hold if the variance is replaced by a spectral risk measure.

Also, notice that item 1 above shows that the solution to the mean–variance problem does not lie in the corner, i.e., $\gamma^\star$ and $\beta^\star$ are finite and are different from 0, except for very specific cases. Also, notice that the optimal proportion invested in risky assets, the $\beta^\star$ in eq. (2.16), is proportional to the risk-adjusted return of the tangency portfolio.

2.6.2 Portfolio analysis with spectral utilities

Let us denote the spectral risk measures of Corollary 2.4 by ρ_ϕ, where ϕ stands for a non-negative decreasing function $\phi : [0, 1] \to \mathbb{R}_+$ satisfying $\int_0^1 \phi(t)dt = 1$. Also, let's consider the following counter-parts of the mean–variance problems where the variance is replaced by ρ_ϕ:

[5] The specific form of the correspondence $\mu \leftrightarrow \lambda$ can be found in Brandtner (2013) and Steinbach (2001).

1. Limited analysis

$$\min_{\beta\geqslant 0,\gamma\in\mathbb{R}} \rho_\phi(X_{\beta,\gamma}) \tag{2.17}$$

$$s.t.: \mathbf{E}[X_{\beta,\gamma}] = \mu \tag{2.18}$$

2. Trade-off analysis

$$\max_{\beta\geqslant 0,\gamma\in\mathbb{R}} (1-\lambda)\,\mathbf{E}[X_{\beta,\gamma}] - \lambda\rho_\phi(X_{\beta,\gamma}), \quad \lambda\in[0,1] \tag{2.19}$$

Proposition 2.8. *(Brandtner (2013)-Proposition 4.3) The following items give the solutions for the problem eq.* (2.19) *when short-sales are allowed and restricted, respectively.*

1. *The solution to the* (μ,ρ_ϕ) *trade-off analysis (eq. 2.19) is given by*

$$\beta^\star = \begin{cases} 0, & \text{if } \frac{\mathbf{E}[X_{T,\rho_\phi}-X_0]}{\rho_\phi(X_{T,\rho_\phi})-\rho_\phi(X_0)} \leqslant \frac{\lambda}{1-\lambda} \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.20}$$

2. *The solution to the* (μ,ρ_ϕ) *trade-off analysis (eq. 2.19) when* β *is restricted to* $[0,1]$ *is given by*

$$\beta^\star = \begin{cases} 0, & \text{if } \frac{\mathbf{E}[X_{T,\rho_\phi}-X_0]}{\rho_\phi(X_{T,\rho_\phi})-\rho_\phi(X_0)} \leqslant \frac{\lambda}{1-\lambda} \\ 1, & \text{otherwise.} \end{cases} \tag{2.21}$$

As for the mean-variance framework, the risk-adjusted return of the tangency portfolio also plays a major role in the definition of the optimal β^* in the (μ,ρ_ϕ) framework (see equations 2.20 and 2.21). In this case, however, we have $\beta^\star\in\{0,+\infty\}$ when short-sales are allowed, and $\beta^\star\in\{0,1\}$ when short-sales are restricted. Moreover, differently from what happens in the mean–variance framework, the solutions to the mean-spectral problems do not vary continuously with respect to the risk aversion λ.

Definition 2.8. *A portfolio* $X_{\beta,\gamma}$ *belongs to the* $(\boldsymbol{\mu},\boldsymbol{\rho_\phi})$**-efficient frontier** *if there is no portfolio* $X_{\beta',\gamma'}$ *with* $\mathbf{E}[X_{\beta',\gamma'}]\geqslant\mathbf{E}[X_{\beta,\gamma}]$ *and* $\rho_\phi(X_{\beta',\gamma'})\leqslant\rho_\phi(X_\gamma)$*, with at least one of the two inequalities being strict.*

The problems (2.17-2.18) and (2.19) induce the same (μ,ρ_ϕ)-efficient frontier. As in the mean–variance framework, the (μ,ρ_ϕ)-efficient frontier consists of the linear combinations of the risk-free asset and the tangency portfolio (if short-sales are not allowed, only convex combinations are considered). Therefore, the solutions given in eq. (2.20) and eq. (2.21) show that the set of optimal solutions does not coincide with the set of portfolios in the efficient frontier. Moreover, differently from what happens in the mean–variance framework, the problems (2.17-2.18) and (2.19) are not equivalent in the sense that there is no one-to-one correspondence between the parameters μ and λ such that, once these parameters are chosen appropriately, they induce the same solution.

2.7 COMPARATIVE RISK AVERSION

The comonotonic additive risk measures of Theorem 2.8 are defined through the weights attributed to the surpluses and losses. The possibility of explicitly studying these weighting functions makes comonotonic additive risk measures interesting candidates to represent preferences. In Brandtner and Kürsten (2015), the authors study preferences represented through coherent comonotonic additive risk measures. These preferences on $\mathcal{X}$ are denoted by $\preceq$ and, for any $X, Y \in \mathcal{X}$, are defined as $Y \preceq X$ if and only if $\rho_\phi(X) \leqslant \rho_\phi(Y)$.

In the Arrow-Pratt (AP) setting (Arrow, 1965; Pratt, 1964), the risk aversion of two agents can be compared through their certainty equivalents. The certainty equivalent $c_\phi : \mathcal{X} \to \mathbb{R}$ associated with ρ_ϕ is defined implicitly as $\rho_\phi(c_\phi(X)) = \rho_\phi(X)$ for $X \in \mathcal{X}$. By cash additivity and normalization of ρ_ϕ one can always find such a c_ϕ, which is given by $c_\phi(X) = -\rho_\phi(X)$ for all $X \in \mathcal{X}$. Following Brandtner and Kürsten (2015), we say that an agent whose preferences are represented by ρ_{ϕ_1} is more **AP risk-averse** than another agent with preferences represented by ρ_{ϕ_2} if $\rho_{\phi_1}(X) \geqslant \rho_{\phi_2}(X)$ for all $X \in \mathcal{X}$. Equivalently, ρ_{ϕ_1} is more AP risk-averse than ρ_{ϕ_2} if $c_{\phi_1}(X) \leqslant c_{\phi_2}(X)$ for all $X \in \mathcal{X}$. The intuition for this last definition is that more risk-averse agents require less money in exchange for lotteries. In most of the relevant literature, the **AP coefficient of risk aversion** for spectral preferences is defined as

$$R_\phi(p) = -\frac{\phi'(p)}{\phi(p)}, \quad \forall p \in [0, 1]. \tag{2.22}$$

In this regard, Brandtner and Kürsten (2015) proved that $R_{\phi_1}(p) \geqslant R_{\phi_2}(p)$ for all $p \in [0, 1]$ implies ϕ_1 is more AP risk-averse than ϕ_2, that is, $\rho_{\phi_1}(X) \geqslant \rho_{\phi_2}(X)$ for all $X \in \mathcal{X}$. However, they also proved that the converse is not true, i.e., it is possible that the AP coefficient does not correctly reflect the relative risk aversion of two spectral preferences. Therefore, the usage of the AP coefficient of risk aversion in eq. (2.22)—which is a classical tool to order the risk aversion of different agents—is incompatible with spectral preferences, in the sense that the rank based on the AP coefficient does not necessarily match the rank based on the certainty equivalents.

Another inconsistency in comparative risk aversion for spectral preferences is that the AP risk aversion ordering between two agents is not necessarily the same as the risk aversion ordering based on the Ross (R) criterion (Ross, 1981). Ross generalized the framework of Arrow and Pratt by considering uncertain levels of wealth, which will be represented by a random variable $X \in \mathcal{X}$. He defined the incremental risk premium as the amount an agent is willing to pay to avoid changing her wealth from X to $X + Y$, where $Y \in \mathcal{X}$. In the spectral framework of Brandtner and Kürsten (2015), the **incremental risk premium** induced by a spectral risk measure ρ_ϕ is defined as $R_\phi(X, Y) \coloneqq \rho_\phi(X + Y) - \rho_\phi(X)$, for $X, Y \in \mathcal{X}$ being two non-constant random variables satisfying $\mathbf{E}[Y|X] = 0$. The hypothesis of zero conditional expectation is aligned with the interpretation of Y as a random variable adding noise to X, without being correlated with it. In this framework, an agent whose preferences are represented by a spectral risk measure ϕ is **R risk-averse** if $R_\phi(X, Y) \geqslant 0$ for all $X, Y \in \mathcal{X}$ satisfying the previously mentioned conditions. Accordingly, an agent whose preferences are represented by ρ_{ϕ_1} is more R risk-averse than another agent with preferences represented by ρ_{ϕ_2} if $R_{\phi_1}(X, Y) \geqslant R_{\phi_2}(X, Y)$ for all $X, Y \in \mathcal{X}$ satisfying the previously mentioned conditions.

Notice the change in the criterion for risk aversion: in the AP framework, the criterion depends only on the levels of ρ_ϕ, while in the R framework the criteria also involve the increment in the risk. Propositions 3.3 and 4.3 of Brandtner and Kürsten (2015) show that if an agent with preferences ρ_{ϕ_1} is more R risk-averse than another agent with preferences ρ_{ϕ_2}, then the same holds for their relative AP risk aversion. However, they also showed that the converse is not true, meaning that the ranking of preferences based on the AP risk aversion might not coincide with the ranking based on the R risk aversion, for spectral preferences. As a practical consequence for the AVaR, this implies that $p_1 < p_2$ does not imply that the agents with preferences AVaR_{p_1} is more R risk-averse than an agent with preferences AVaR_{p_2}. Also, Brandtner and Kürsten (2015) showed that similar inconsistencies hold for the exponential and power families of spectral risk measures.

2.8 CONCLUDING REMARKS

There are several reasons for which a risk manager may choose to measure financial risk with a comonotonic additive risk measure. First, the property of comonotonic additivity may be desirable for the application at hand, let it be, for instance, internal or external risk management. Also, the Kusuoka, spectral, and Choquet representations of comonotonic additive risk measures allow the risk manager to specify, very explicitly, how each level of the potential losses affects the final risk measurement.

Recent research, however, unveiled the fact that comonotonic additive risk measures cannot satisfy some other properties which, in some contexts, may be of utmost importance. In this paper, we present a comprehensive literature review focused on the properties that are necessarily absent for main classes of comonotonic additive risk measures. In total, we found four of such properties. In addition, we found two issues related to the usage of spectral risk measures in portfolio selection problems, and two other issues in comparing risk aversion when spectral risk measures are used as monetary utility functions.

We present these issues in self-contained separate sections, where we motivate the application at hand and discuss the respective conflict with comonotonic additivity. Also, we provide an appendix presenting the elementary of comonotonic random variables and comonotonic additive risk measures. Therefore, in addition to this paper's potential to serve as a reference guide for the experienced reader, the main content of our paper is also accessible to the audience not familiar with the theory of risk measures.

In addition to the literature review, which is this paper's main contribution, we present original results showing that the comonotonic additivity axiom conflicts with the excess invariance and surplus invariance properties. First, we show that there is no monotone risk measure that is comonotonic additive and that satisfies the property of excess invariance, as employed in Staum (2013), Cont et al. (2013), and Gao and Munari (2020). We prove that this property of excess invariance is stronger than the surplus invariance property for acceptance sets, employed in Koch-Medina et al. (2017) and He and Peng (2018). The conflict with comonotonic additivity, however, remains even when one considers the more general class of risk measures that induce surplus invariant acceptance set, although this second conflict is milder, as the value at risks are comonotonic additive and have surplus invariant acceptance sets.

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2.9 APPENDIX A - BACKGROUND

The basic components of our setup are an atomless probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the space of essentially bounded random variables $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, which will serve as the domain of the risk measures considered. Also, we consider $\mathbb{R}$ as the sub-space of $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ containing the $\mathbf{P}$-almost surely constant random variables. The random variables $X \in \mathcal{X}$ represent the discounted net value of a financial position at the end of the trading period. Accordingly, $X(\omega) > 0$ stands for a gain and $X(\omega) < 0$ represents a loss. Inequalities (and equalities) of the type $X > 0$ should be understood in the $\mathbf{P}$-almost sure sense, unless otherwise specified. The notation $X \sim F_X$ stands for $F_X(x) \equiv \mathbf{P}(X \leqslant x)\ \forall x \in \mathbb{R}$, and $X \overset{\mathrm{d}}{=} Y$ means that $F_X = F_Y$ point-wise. We denote the expectation and variance of $X \in \mathcal{X}$ as $\mathbf{E}[X] = \int X d\mathbf{P}$ and $\mathbf{V}[X] = \int (X - \mathbf{E}[X])^2 d\mathbf{P}$, respectively. For $X \in \mathcal{X}$ and $p \in [0, 1]$, the left p-quantile of X is defined as $q_X(p) := \inf\{x \in \mathbb{R} : F_X(x) \geqslant p\}$. Therefore we have $q_X(0) = -\infty$ and $q_X(1) = \operatorname{ess\,sup} X$. Although we use it only rarely, it is also convenient to mention that, for $X \in \mathcal{X}$ and $p \in (0, 1)$, the right p-quantile of X is defined as $q^+_X(p) := \inf\{x \in \mathbb{R} : F_X(x) > p\}$. For $p \in \{0, 1\}$ we have $q^+_X(0) = \lim_{t\downarrow 0} q_X(t) = \operatorname{ess\,inf} X$ and $q^+_X(1) = +\infty$ (see He and Peng (2018)). Also, we denote $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. The terms "increasing" and "decreasing" are employed in the weak sense.

2.9.1 Characterization of comonotonic random variables

The concept of comonotonicity dates back at least to Theorem 236 in Hardy et al. (1934), where it appears under the name of "similarly ordered functions". In more recent treatments, especially in the literature of risk measures and premium principles, the concept is usually defined as follows:

Definition 2.9. *Consider the following definitions for* $\mathbf{X} = (X_1, X_2, \ldots, X_n) \in \mathcal{X}^n$.

1. *The random vector* $\mathbf{X}$ *is* **comonotonic** *if*

$$(X_i(\omega) - X_i(\omega'))(X_j(\omega) - X_j(\omega')) \geqslant 0 \quad \forall\, i, j \in \{1, 2 \ldots, n\}\ \mathbf{P} \otimes \mathbf{P}\text{-}a.s. \tag{2.23}$$

   In this case, we also say that the random variables $X_1, X_2, \ldots, X_n$ *are comonotonic.*

2. *With* $n = 2$, $\mathbf{X}$ *is said to be* **counter-comonotonic** *if* $(X_1, -X_2)$ *is comonotonic. In this case we also say that the random variables* X_1 *and* X_2 *are counter-comonotonic.*

The distinctive feature of comonotonic random vectors is that if one of its components varies, the others do not vary in the opposite direction. This property has a clear financial meaning: comonotonic random variables do not hedge each other. On the contrary, counter-comonotonic random couples are such that, whenever one of its components varies, the other does not vary in the same direction. Roughly speaking, the property of comonotonicity (counter-comonotonicity, respectively) implies a non-negative (non-positive, respectively) dependence between the random variables. Additionally, notice that constant random variables are comonotonic and counter-comonotonic with each other and with every random variable. Also, if X_1 and X_2 are comonotonic, and $f, g : \mathbb{R} \to \mathbb{R}$ are both increasing or both decreasing, then $f(X_1)$ and $g(X_2)$ are comonotonic. On the other hand, if f is increasing and g is decreasing (or vice-versa), then $f(X_1)$ and $g(X_2)$ are counter-comonotonic. The following classical Theorem gives alternative characterizations of comonotonicity.

Theorem 2.6. *(Rüschendorf (2013) - Theorem 2.14; Dhaene et al. (2020) - Theorem 4) Consider a random vector* $\mathbf{X} = (X_1, X_2, \ldots, X_n) \in \mathcal{X}^n$ *with marginal distributions* $(F_{X_1}, F_{X_2}, \ldots, F_{X_n})$ *and joint distribution* F*. The following statements are equivalent:*

1. *The random vector* $\mathbf{X}$ *is comonotonic.*

2. *The random vectors in* $\{(X_i, X_j) : i, j \in \{1, 2, \ldots, n\}\}$ *are comonotonic.*

3. $F(x_1, x_2, \ldots, x_n) = \min\{F_{X_i}(x_i) : i \in \{1, 2, \ldots, n\}\},\ \forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

4. $F(x_1, x_2, \ldots, x_n) \geqslant \tilde{F}(x_1, x_2, \ldots, x_n)$ *whenever* $\tilde{F}$ *is a joint distribution of a random vector whose marginals are given by* $(F_{X_1}, F_{X_2}, \ldots, F_{X_n})$.

5. *For* $U \sim \mathsf{Uniform}(0, 1]$*, we have*

$$\mathbf{X} \stackrel{d}{=} (q_{X_1}(U), q_{X_2}(U), \ldots, q_{X_n}(U)). \tag{2.24}$$

Moreover, if $\mathbf{X}$ *is comonotonic and* X_1 *is continuously distributed, then there exist increasing functions* $f_2, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ *such that*

$$(X_1, X_2, \ldots, X_n) = (X_1, f_2(X_1), \ldots, f_n(X_1)).$$

The second item of the Theorem shows us that the essence of comonotonicity is captured by pairs of random variables. In this light, for simplicity, we focus on comonotonicity for pairs of random variables. Item 3 says that the dependence structure of comonotonic random vectors are captured by a copula $C : [0, 1]^n \to [0, 1]$ which is given by $C(u_1, u_2, \ldots, u_n) = \min\{u_i : i \in \{1, 2, \ldots, n\}\}$. It is valid to mention that the joint distribution of a random vector is usually harder to estimate than its marginal distributions. For comonotonic random vectors though, item 3 says that the joint distribution can be readily recovered from the marginals. Item 4 shows that once the marginal distributions of a random vector are fixed, the comonotonic structure of dependence leads to the highest probability of joint losses. Random vectors are written as in eq. (2.24) are remarkably useful in the theory of comonotonic random variables. Also, item 5 is often taken as the definition of comonotonicity (see Rüschendorf (2013), for instance). Notice that item 5 says nothing about the marginal distribution of the vector $(q_{X_1}(U), \ldots, q_{X_n}(U))$. Therefore, it is valid to mention that $q_{X_i}(U) \sim F_{X_i}$ for all $i \in \{1, 2, \ldots, n\}$ (see Lemma A.23 of Föllmer and Schied (2016)). This fact, taken with the equivalence between items 5 and 3, implies that one can obtain customized comonotonic random vectors in the sense that, for any n-tuple of marginal distributions $(F_{X_1}, F_{X_2}, \ldots, F_{X_n})$, any random vector written as in eq. (2.24) is comonotonic with marginals given by $(F_{X_1}, F_{X_2}, \ldots, F_{X_n})$. The final part of the Theorem indicates that the components of comonotonic random vectors share the same source of variability.

Definition 2.10. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n) \in \mathcal{X}^n$ *be any random vector with quantile functions* $(q_{X_1}, \ldots, q_{X_n})$ *and let* $U \sim \mathsf{Uniform}(0, 1]$*. A* ***comonotonic counter-part*** *of* $\mathbf{X}$*, denoted by* $\mathbf{X}^c = (X_1^c, X_2^c, \ldots, X_n^c)$*, is any random vector written as in eq.* (2.24).

Notice that item 5 of Theorem 2.6 implies that every random vector $\mathbf{X} \in \mathcal{X}$ admits a comonotonic counter-part. Additionally, item 3 of the Theorem specifies the joint distribution of any comonotonic counter-part.

Examples of comonotonic and counter-comonotonic random variables are abundant in finance and actuarial science. Here we give just two examples, and refer to Kass et al. (2001) and Denuit et al. (2006) for comprehensive treatments.

Example 2.3. The payoff of derivative securities, in particular call (resp., put) options, forms a comonotonic (resp., counter-comonotonic) pair when combined with the price of the underlying asset. Consider, for instance, a European call option with underlying asset X and strike price $K > 0$. Its payoff at the expiration date is given by $(X - K)^+$, which is a increasing function of X. The opposite holds between the underlying asset X and the cash-flow of those underwriting the call options or buying European put options. The payoff of these positions (at the expiration date) are decreasing functions of X and are respectively given by $-(X - K)^+$ and $(K - X)^+$.

Example 2.4. In actuarial science, comonotonic random variables appear, for instance, as the layers of a given loss $X \in \mathcal{X}^+$. These are contracts where the policy holder must pay the insurer a franchise of $a > 0$ in exchange for the insurer to face the loss X up to a limit $h > a$ (see Denuit et al. (2006), for more details). The loss faced by the insurer is then

$$X_{[a,h]} = \begin{cases} 0, & \text{if } X < a, \\ X - a, & \text{if } a \leqslant X \leqslant h, \\ h - a, & \text{if } h < X \end{cases}$$

Notice that any two layers $X_{(a_1,h_1]}$ and $X_{(a_2,h_2]}$ are increasing functions of the same random variable X and, therefore, are comonotonic. For more applications of comonotonic random variables in finance and actuarial science see, for instance, Dhaene et al. (2002a), Deelstra et al. (2011), Denuit and Dhaene (2012), Cheung et al. (2014), Chen et al. (2015), and Dhaene et al. (2020).

2.9.2 The basics of risk measures

Since the landmark work of Artzner, Delbaen, Eber and Heath (1999), the theory of risk measures grew in symbiosis with that of insurance premium principles (Cai and Mao, 2020; Kass et al., 2001) and choice theory (Delbaen, 2011; Tsanakas and Desli, 2003). Below we follow the traditional heuristic of interpreting risk measures as tools that help determine regulatory capital requirements for financial institutions. For a given financial position $X \in \mathcal{X}$, the real number $\rho(X)$ represents the minimum amount of capital, in terms of $t = 0$ numéraire, that the financial institution must prudently invest (in liquid and stable assets) to have a "reasonable" buffer against potential losses from X.

Remark 2.18. The interpretation of $\rho(X)$ as a quantity expressed in the $t = 0$ numéraire is in line with the convention we adopted that the random variables in $\mathcal{X}$ represents discounted payoffs. This section would follow unchanged if $\rho(X)$ and the random variables in $\mathcal{X}$ were expressed in the numéraire of the terminal date.

The following axioms can be motivated with this application in mind.

Definition 2.11. *We call any functional $\rho\colon \mathcal{X} \to \mathbb{R}$ a* **risk measure**. *Also, we say that*

1. *(Monotonicity) ρ is* **monotone** *if $\rho(X) \geqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \leqslant Y$.*
2. *(Cash Additivity) ρ is* **cash additive** *if $\rho(X - b) = \rho(X) + b$ for all $b \in \mathbb{R}$ and $X \in \mathcal{X}$.*
3. *(Positive Homogeneity) ρ is* **positive homogeneous** *if $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \geqslant 0$ and $X \in \mathcal{X}$.*

4. *(Subadditivity) ρ is* **subadditive** *if $\rho(X+Y) \leqslant \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.*

5. *(Convexity) ρ is* **convex** *if $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0,1]$ and $X, Y \in \mathcal{X}$.*

6. *(Normalization) ρ is* **normalized** *if $\rho(0) = 0$.*

7. *(Law invariance) ρ is* **law invariant** *if $\rho(X) = \rho(Y)$ whenever $F_X = F_Y$ pointwise.*

8. *(Comonotonic additivity) ρ is* **comonotonic additive** *if $\rho(X+Y) = \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$ such that (X, Y) is a comonotonic random vector.*

The axiom of monotonicity requires that if the payout of a financial position X is always smaller than that of Y, then the risk of X—and therefore its regulatory capital—must be greater than that of Y. Notice that, as is prevalent in the literature, we are assuming a unitary discount factor. In this light, the property of cash additivity says that if the future (unknown) result X is depleted by a known amount b, becoming $X - b$, then the same amount, $b \in \mathbb{R}$, must be added to the original regulatory capital to maintain the same level of risk. Notice that cash additivity yields $\rho(X + \rho(X)) = 0$, which in turn implies that no capital reserves need to be made after $\rho(X)$ has been added to X. Risk measures satisfying monotonicity and cash additivity are called **monetary**.

Positive homogeneity requires the risk to vary "linearly" with respect to variations in the financial position's size. The axiom of subadditivity reflects the notion that a merge does not create extra risks. Risk measures satisfying axioms 1 to 4 are called **coherent** and were first studied in Artzner et al. (1999) for discrete probability spaces and in Delbaen (2002) for general probability spaces. Although the class of coherent risk measures is one of the most widely employed in both theory and practice, some authors consider the axiom of positive homogeneity too restrictive. For instance, Frittelli and Gianin (2002) and Föllmer and Schied (2002) argue that scaling up a financial position may create extra liquidity risks, which are not accounted for if the risk measure is positive homogeneous. This consideration has motivated the study of a less restrictive class of risk measures, which is defined by axioms 1, 2, and 5 and is called the class of **monetary convex** risk measures (Föllmer and Schied, 2002; Frittelli and Gianin, 2002; Heath, 2000). Similar to subadditivity, the axiom of convexity aims to reflect diversification benefits. The axiom of normalization is implied by positive homogeneity and is usually introduced to ease the notation. For normalized risk measures, convexity, subadditivity, and positive homogeneity are linked as each pair of these axioms implies the remaining one.

The axiom of law invariance was introduced for risk measures and non-expected utility theory in the seminal contributions of Kusuoka (2001) and Yaari (1987), respectively. This axiom was embraced by most of the literature because it is necessary for empirical applications where one only observes a set of data points (say, $X_1(\omega), \ldots, X_n(\omega)$) drawn from an unknown probability distribution.

The axiom of comonotonic additivity says that the risk of a comonotonic sum equals the sum of the individual risks. This axiom was introduced in decision theory by Schmeidler (1986) and Yaari (1987), in premium principles in Wang (1996) and Wang and Dhaene (1998), and for risk measures in Kusuoka (2001) and Acerbi (2002). Of course, many others have contributed to the development of the theory of comonotonic additive risk measures. For details on the theory and applications of comonotonic additive risk measures see Dhaene et al. (2002a,b). The rationale behind this axiom is based on the strong dependence structure between comonotonic random variables, as mentioned in the preceding section. Such degree of dependence forbids hedging between comonotonic pairs, in the sense that their variations never "compensate" each other.

Therefore, it is argued that if X and Y are comonotonic, then the risk of the position $X+Y$ should be equal to the sum of the risks of X and Y.

Example 2.5. The widely used **value at risk** is a monetary positive homogeneous law invariant comonotonic risk measure. The value at risk of $X \in \mathcal{X}$ at the significance level $p \in [0,1]$ is defined as

$$\text{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbf{P}(X + x < 0) \leqslant p\} = q_{-X}(1-p). \tag{2.25}$$

Remark 2.19. It is valid to mention that for $p \in [0,1]$ and $X \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ the equality $q_{-X}(1-p) = -q_X(p)$ holds for almost all p.

Despite being widely employed in practice, the use of VaR for the determination of regulatory capital has been extensively criticized for two main reasons: first, it does not account for the size of the position below the p-quantile, allowing for instance that $\text{VaR}_p(X) = \text{VaR}_p(Y)$ even if the tail of X below its p-quantile is heavier than that of Y below its respective p-quantile. The second main critique is that VaR_p does not satisfy the axiom of subadditivity and, therefore, the value at risk does not capture the financial intuition behind this axiom.

Example 2.6. The **average value at risk** (AVaR) circumvents both drawbacks of the VaR. First, the average value at risk is coherent and, second, it takes into account all level of losses below the significance level being used. The average value at risk of $X \in \mathcal{X}$ at the significance level $p \in (0,1]$, denoted as $\text{AVaR}_p(X)$, is defined as

$$\text{AVaR}_p(X) = \frac{1}{p}\int_0^p \text{VaR}_q(X)dq. \tag{2.26}$$

For $p = 0$ it is defined as $\text{AVaR}_0(X) = -\operatorname{ess\,inf} X$. For $p \in (0,1]$, the $\text{AVaR}_p(X)$ is a type of average of the $p(100)\%$ smaller values X. In particular, $\int_0^1 \text{VaR}_p(X)dp = E[-X]$ holds even if X is not continuously distributed. If X is continuously distributed, AVaR_p can be expressed as the expectation of the loss, $-X$, conditional on X being no greater than $q_X(p)$, that is,

$$\text{AVaR}_p(X) = E[-X | X \leqslant q_X(p)], \quad \text{where} \quad q_X(p) \text{ is any } p\text{-quantile of } X.$$

Risk measures related to the average value at risk can be found, for instance, in Acerbi and Tasche (2002) and Dhaene et al. (2006).

The average value at risk has gained the acceptability of practitioners, being included in the Basel accord for banking regulation and in the Swiss Solvency Test for insurance companies (BCBS, 2019; Keller and Luder, 2004). The AVaR's mathematical properties were scrutinized in Acerbi and Tasche (2002). Since then, the AVaR was studied comprehensively by several authors and was shown to have a solid economic foundation in Wang and Zitikis (2021). Also, in the next section we will see that AVaR plays a central role in the theory of comonotonic risk measures. In short, all comonotonic additive risk measures satisfying certain additional properties can be represented as a mixture of AVaRs at different significance levels.

Regardless of the risk measure being employed, a fundamental task of the regulator is to decide between accepting or not the position of the financial institutions. Once a risk measure, say ρ, has been chosen, this task can be accomplished through the set

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \leqslant 0\},$$

which can be viewed as a gauge according to which the financial institutions' position are appraised: a financial institution with a position represented by X is deemed acceptable if and only if $X \in \mathcal{A}_\rho$.

Sets used to define the theoretical acceptability of financial positions are called **acceptance sets**. These sets are of fundamental importance and can be taken as the primal concept in the theory of risk measures. Similar to risk measures, acceptance sets have an "axiomatic menu" of their own:

Definition 2.12. *We call any non-empty set $\mathcal{A} \subsetneq \mathcal{X}$ an* **acceptance set***. Also, we say that*

1. *(Monotonicity) $\mathcal{A}$ is* **monotone** *if $X \leqslant Y$ and $X \in \mathcal{A}$, imply $Y \in \mathcal{A}$.*

2. *(Boundedness on constants) $\mathcal{A}$ is* **bounded on constants** *if $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$.*

3. *(Convexity) $\mathcal{A}$ is* **convex** *if $\lambda\mathcal{A} + (1-\lambda)\mathcal{A} \subseteq \mathcal{A}$ whenever $\lambda \in [0,1]$.*

4. *(Conicity) $\mathcal{A}$ is* **conic** *if $\lambda\mathcal{A} \subseteq \mathcal{A}$ for all $\lambda \geqslant 0$.*

5. *(Normalization) $\mathcal{A}$ is* **normalized** *if $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} = 0$.*

The property of monotonicity is a basic requirement for acceptance sets: if the regulator accepts X while Y pays more than X with probability one, then Y should also be acceptable. The property of boundedness on constants says that there is a lower bound on the size of certain losses that are acceptable. In fact, for much of the theory, the stronger property of normalization holds, which means that no certain loss is deemed acceptable. Acceptance sets satisfying monotonicity and boundedness on constants are called **monetary**. The property of convexity corresponds to the requirement that diversification does not increase the risk. Conicity implies that the acceptability of a position should never be affected by changes in its scale.

An acceptance set $\mathcal{A}$ induces a real-valued functional through $\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$. The next Proposition shows that there is a correspondence between the properties of acceptance sets and those of risk measures.

Theorem 2.7. *The following illustrates the relation between acceptance sets and risk measures.*

1. *If ρ is monotone, so is $\mathcal{A}_\rho$. Reciprocally, if $\mathcal{A}$ is monotone, so is $\rho_{\mathcal{A}}$.*

2. *If ρ is monetary, then it is Lipschitz continuous w.r.t. $\|\cdot\|_\infty$ and with unitary Lipschitz constant. In this case, $\mathcal{A}_\rho$ is non-empty, monetary, and closed w.r.t. $\|\cdot\|_\infty$. Reciprocally, if $\mathcal{A}$ is monetary, then $\rho_{\mathcal{A}}$ is monetary and, as a consequence, Lipschitz continuous w.r.t. $\|\cdot\|_\infty$.*

3. *If ρ is monetary, then $\rho_{\mathcal{A}_\rho} = \rho$. Reciprocally, if $\mathcal{A}$ is monetary then $\mathcal{A}_{\rho_{\mathcal{A}}}$ corresponds to the $\|\cdot\|_\infty$ closure of $\mathcal{A}$.*

4. *If ρ is convex, then $\mathcal{A}_\rho$ is convex. Also, if $\mathcal{A}$ is convex and monetary, then $\rho_{\mathcal{A}}$ is convex and monetary.*

5. *If ρ is positive homogeneous, then $\mathcal{A}_\rho$ is conic. Reciprocally, if $\mathcal{A}$ is conic, then $\rho_{\mathcal{A}}$ is positive homogeneous.*

6. *If ρ is non-zero monotone and comonotonic additive, then it is positive homogeneous and Lipschitz continuous w.r.t. $\|\cdot\|_\infty$.*

7. *If ρ is cash additive and normalized, then $\mathcal{A}_\rho$ is normalized. Reciprocally, if $\mathcal{A}$ is normalized, then $\rho_{\mathcal{A}}$ is normalized.*

Proof. For items 1-5 see Föllmer and Schied (2016). Item 6 is proved in Proposition 2.5 of Koch-Medina et al. (2018). To prove the first assertion of item 7, notice that the cash additivity of ρ implies $\inf\{m \in \mathbb{R} : \rho(m) \leqslant 0\} = \inf\{m \in \mathbb{R} : \rho(0) \leqslant m\} = \rho(0) = 0$, where the last equality follows by the normalization hypothesis on ρ. Conversely, if $\mathcal{A}$ is normalized one has $0 = \inf\{m \in \mathbb{R} : m \in \mathcal{A}\} = \rho_{\mathcal{A}}(0)$, which implies that $\rho_{\mathcal{A}}$ is normalized. □

2.9.3 Representation Theorems

A major theoretical appeal of comonotonic additive risk measures is that they can be represented as certain integrals. These representations are simpler than, for instance, those of coherent risk measures provided in Artzner et al. (1999) and Delbaen (2002), and of monetary convex risk measure provided in Frittelli and Gianin (2002) and Föllmer and Schied (2002). They are also important in clarifying the incompatibilities existing between comonotonic additivity and the desirable properties we discuss. The following definition is necessary to this section's main Theorem.

Definition 2.13. *Consider the following continuity properties:*

1. *A risk measure* $\rho\colon \mathcal{X} \to \mathbb{R}$ *is* **continuous from above** *if* $\rho(X_n) \to \rho(X)$ *whenever* $X_n \downarrow X$ $\mathbf{P}$*-a.s.*

2. *A risk measure* $\rho\colon \mathcal{X} \to \mathbb{R}$ *is* **continuous from below** *if* $\rho(X_n) \to \rho(X)$ *whenever* $X_n \uparrow X$ $\mathbf{P}$*-a.s.*

For the next definition, denote by Ψ the set of increasing concave functions $\psi : [0,1] \to [0,1]$ satisfying $\psi(0) = 0$ and $\psi(1) = 1$. For the reasons put forward in remarks 2.24, 2.25, and 2.26, we refer to the elements of Ψ as **concave distortions**. Also, for $\psi \in \Psi$, define the set function $c_\psi : \mathcal{F} \to [0,1]$ as $c_\psi(A) = \psi(\mathbf{P}(A))$ for all $A \in \mathcal{F}$.

Definition 2.14. *The* **Choquet integral** *of* $X \in \mathcal{X}$ *with respect to* ψ *is given by*

$$\int X dc_\psi = \int_{-\infty}^{0} (c_\psi(X > x) - 1)dx + \int_0^{+\infty} c_\psi(X > x)dx.$$

The following Theorem dates back to Kusuoka (2001) and Acerbi (2002). We denote by $\mathcal{M}([0,1])$ the set of probability measures on the Borel sets of $[0,1]$.

Theorem 2.8. *(Föllmer and Schied (2016)) Consider a risk measure* $\rho\colon \mathcal{X} \to \mathbb{R}$*. Then the following are equivalent:*

1. ρ *is coherent, comonotonic, law invariant, and continuous from above.*

2. ρ *has the following* **Kusuoka** *representation*

$$\rho(X) = \int \mathrm{AVaR}_t(X)\mu(dt),\ X \in \mathcal{X},\ \textit{for some } \mu \in \mathcal{M}([0,1]). \tag{2.27}$$

3. ρ *has the following* **Choquet** *representation*

$$\rho(X) = \int -X dc_\psi, X \in \mathcal{X},\ \textit{for some } \psi \in \Psi. \tag{2.28}$$

4. ρ *has the following* **spectral** *representation*

$$\rho(X) = \psi(0+)\,\mathrm{AVaR}_0(X) + \int_0^1 q_{-X}(t)\psi'(1-t)dt, X \in \mathcal{X},\ \textit{for some } \psi \in \Psi. \tag{2.29}$$

Proof. All assertions are proved in Föllmer and Schied (2016). The equivalence between items 1 and 2 was proved in their Theorem 4.93. The equivalence between items 2 and 3 follows by their Corollary 4.77. The equivalence between items 3 and 4 was given in their Theorem 4.70. □

Remark 2.20. The equivalence between items 2, 3, and 4 is possible because there exists a one-to-one correspondence between $\mathcal{M}([0,1])$ and Ψ (see Föllmer and Schied (2016), Acerbi (2002), or Dhaene et al. (2012) for details).

Remark 2.21. Since AVaR is coherent, continuous from above, and law invariant, any risk measure in the form given in eq. (2.27) has the same properties (see Föllmer and Schied (2016) sec. 4.6, p.246). Also, the convexity of the risk measures in eq. (2.27) is a consequence of AVaR being convex and the fact that convex combinations of convex function are convex (see Proposition 2 in Acerbi (2002)).

Denote by Φ the set of non-negative decreasing functions $\phi : [0,1] \to \mathbb{R}_+$ such that $\int_0^1 \phi(t)dt = 1$.

Corollary 2.4. *A risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$ is a coherent comonotonic additive law invariant continuous from above and continuous from below if and only if*

$$\rho(X) = -\int_0^1 q_X(t)\phi(t)dt, X \in \mathcal{X}, \text{ for some } \phi \in \Phi. \tag{2.30}$$

Remark 2.22. The above corollary gives us a simplified version of the spectral representation for continuous from below risk measures. It assumes particular importance in Section 2.6.

Remark 2.23. Notice that if ψ is not concave in the Choquet representation, then the risk measure associated with it is not subadditive. In this case, the map $t \mapsto \psi'(1-t)$ is not decreasing and, as a consequence, the associated spectral representation will not be coherent.

Remark 2.24. Notice that risk measures represented as in eq. (2.28) can be regarded as a generalization of the famous "expectation formula" for $-\mathbf{E}[\cdot]$, namely

$$\mathbf{E}[-X] = \int_{-\infty}^0 (\mathbf{P}(-X \geqslant x) - 1)dx + \int_0^{+\infty} \mathbf{P}(-X > x)dx. \tag{2.31}$$

In fact, one can obtain $\rho(X) = \mathbf{E}[-X]$ through the representations given in eq. (2.28) and eq. (2.30) by taking $\psi(t) = t$.

Remark 2.25. Notice that, for $\psi \in \Psi$, we have

$$\begin{aligned}\psi(\mathbf{P}(-X > x)) &= \psi(\mathbf{P}(-X > x) + 0(1 - \mathbf{P}(-X > x))) \\ &\geqslant \mathbf{P}(-X > x)\psi(1) + (1 - \mathbf{P}(-X > x))\psi(0) \\ &= \mathbf{P}(-X > x) \quad \forall\, x \in \mathbb{R}\end{aligned}$$

for all $x \in \mathbb{R}$. Therefore, one can compare eq. (2.31) and eq. (2.28) to conclude that $\rho(X) \geqslant \mathbf{E}[-X]$ for all $X \in \mathcal{X}$ and ρ as defined in Theorem 2.8.

Remark 2.26. Most distortion functions used in practice are continuous at zero, which implies

$$\lim_{x \to +\infty} \psi(\mathbf{P}(-X > x)) = 0.$$

In view of the representation given in eq. (2.28) this means that, as the losses' size grows and $\mathbf{P}(-X > x) \to 0$, the distorted probability also goes to zero. Nonetheless, the relative distortion is greater for higher losses, in the sense that for $x_1 < x_2$, the concavity of ψ implies

$$\frac{\psi(\mathbf{P}(-X > x_2))}{\mathbf{P}(-X > x_2)} \geqslant \frac{\psi(\mathbf{P}(-X > x_1))}{\mathbf{P}(-X > x_1)},$$

which captures the idea that high losses should be more penalized.

For the following lemma, denote by Ψ^* the set of increasing functions $\psi : [0,1] \to [0,1]$ satisfying $\psi(0) = 0$ and $\psi(1) = 1$.

Lemma 2.1. *(Föllmer and Schied (2016) - Theorem 4.88; Kou and Peng (2016) - Lemma 1) A monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is comonotonic and law invariant if and only if there exists $\psi \in \Psi^*$ such that*

$$\rho(X) = \int (-X) dc_\psi, \; \forall X \in \mathcal{X}. \tag{2.32}$$

The class of risk measures defined above clearly contains the class put forward in Theorem 2.8 and Corollary 2.4. The larger class in Lemma 2.1 assumes particular importance in Section 2.4. The above risk measures are not necessarily convex but are monotone. For the interested readers, non-monotone Choquet integrals were studied in Wang et al. (2020).

Definition 2.15. *(Acerbi, 2002) A function $\delta : \mathbb{R} \to \mathbb{R}$ is called the* **Dirac delta function** *if*

$$\int_a^b f(x)\delta(x-c)dx = f(c), \; \forall c \in (a,b). \tag{2.33}$$

The function $\delta' : \mathbb{R} \to \mathbb{R}$ is the first derivative of the Dirac delta function δ if

$$\int_a^b f(x)\delta'(x-c)dx = -f'(c), \; \forall c \in (a,b). \tag{2.34}$$

Remark 2.27. Equation (2.33) is, in fact, an abuse of notation. Arguably, in the case of Acerbi (2002), it was used to avoid long detours from the innovative ideas that were being presented. We believe that, to focus on the main elements of the theory, it is reasonable to make use of the same definition.

Proposition 2.9. *The following are examples of risk measures in their Kusuoka, Choquet, and spectral representations:*

1. *For $p \in [0,1]$ the risk measure VaR_p can be recovered from the Kusuoka representation by using $\mu(dt) = -t\delta'(t-p)dt$. From the Choquet representation by using $\psi(t) = 1_{(t>p)}$, and from the spectral representations by using $\phi(t) = \delta(t-p)$.*

2. *For $p \in [0,1]$ the risk measure AVaR_p can be recovered from the Kusuoka representation by using*

$$\mu(A) = 1_p(A) = \begin{cases} 1 & \text{if} \quad p \in A, \\ 0 & \text{otherwise.} \end{cases}$$

   From the Choquet representation by using $\psi(t) = (1/p)(\min\{t,p\})$, and from the spectral representation by using $\phi(t) = (1/p)1_{(t \leqslant p)}$.

3. *The risk of $X \in \mathcal{X}$ as measured by the MinVaR (Cherny and Madan, 2009) is given by*

$$\mathrm{MinVaR}(X) = -\,\mathbf{E}[\min(X_1, \ldots, X_n)]$$

where $\{X_i\}_{i=1}^n$ *are* $n \in \mathbb{N}$ *independent copies of* X*. It can be recovered from the Kusuoka representation by using* μ *such that*

$$n(1-t)^{n-1} = \int_{(t,1]} s^{-1}\mu(ds).$$

It can be recovered from the Choquet representation by using $\psi(t) = 1-(1-t)^n$*, and it can be recovered from the spectral representation by using* $\phi(t) = n(1-t)^{n-1}$*.*

3 INDUCING COMONOTONIC ADDITIVE RISK MEASURES FROM ACCEPTANCE SETS

Abstract

An elementary fact in the theory of risk measures is that acceptance sets induce risk measures and vice-versa. We present simple and yet general conditions on the acceptance sets under which their induced risk measures are comonotonic additive. With this result, we believe to fill a gap in the literature linking the properties of acceptance sets and risk measures: we show that acceptance sets induce comonotonic additive risk measures if the acceptance sets and their complements are stable under convex combinations of comonotonic random variables. As an extension of our results, we obtain a set of axioms on acceptance sets that allows one to induce risk measures that are additive for *a priori* chosen classes of random variables. Examples of such classes that were previously considered in the literature are independent random variables, uncorrelated random variables, and notably, comonotonic random variables.

Key-words: Comonotonic risk measures. Acceptance sets. Comonotonic convex acceptance sets.

3.1 INTRODUCTION

An acceptance set represents a criterion, according to which a financial regulator separates the positions that financial firms are allowed to hold from the positions they are not. Acceptance sets *per se* do not provide direct guidance in how to make non-acceptable positions acceptable. For this task, they induce risk measures, which then gives us a monetary value for the risk of the financial positions. The risk of non-acceptable financial positions are real numbers representing the minimal quantity of the cash-asset that makes the positions acceptable. For acceptable positions, risk measures give the amount of the cash-asset that can be withdraw from the positions without compromising their acceptability[1].

[1] The usage of the "cash-asset" is a non-essential simplification. See, for instance, Farkas et al. (2014) and Koch-Medina et al. (2018) for detailed discussion on this topic.

The properties that a risk measure fulfills are determined by the properties of the acceptance set associated with it. For instance, convex acceptance sets induce convex risk measures (Föllmer and Schied, 2002; Frittelli and Gianin, 2002). For other examples and a more detailed discussion, we refer to Chapter 4 of Föllmer and Schied (2016) and to Artzner et al. (1999), in particular to their propositions 2.1 and 2.2.

In the present paper, we focus on acceptance sets and risk measures that satisfy certain properties related to *comonotonic* random variables. Roughly speaking, two random variables are comonotonic if the variability of the one never off-sets the variability of the other, i.e., they move in the same direction (se Definition 3.1 for a precise definition). In Finance, canonical examples of comonotonic pairs are those formed by call options and their underlying assets. Additionally, a variety of papers have concluded that the Pareto optimal allocation of an economy's risk are comonotonic (Chateauneuf et al., 2000; Landsberger and Meilijson, 1994; Ludkovski and Rüschendorf, 2008). For other applications of the concept of comonotonicity, see Dhaene et al. (2002a), Deelstra et al. (2011), and the references therein.

The main properties in our study are comonotonic convexity for acceptance sets—meaning that one does not compromise the acceptability of comonotonic positions by taking convex combinations of them—and comonotonic convexity for the acceptance sets' complements—meaning that one cannot turn non-acceptable comonotonic positions into acceptable by taking convex combinations of them.

We will show that these properties are tightly linked to the axiom of comonotonic additivity for risk measures, which occupies a central place in the theory (seminal papers in this regard are Wang et al. (1997), Yaari (1987), Kusuoka (2001), and Acerbi (2002)). A major appeal of comonotonic additive risk measures is that they are robust, in the sense of accommodating model uncertainty, model misspecification, and are robust with respect to small changes in the data (in this respect, see Kou et al. (2013), Huber and Ronchetti (2009), Ahmed et al. (2008), Cont et al. (2010), Tian and Suo (2012), Krätschmer et al. (2014), and Santos et al. (2022)). Despite the literature has devoted such a great deal of attention to comonotonic additive risk measures (see, for instance Rieger (2017), Wang, Wei and Willmot (2020), Wang, Wang and Wei (2020)), no previous paper have established a simple manner to induce comonotonic additive risk measures from acceptance sets. This was a gap in the elementary theory, which we believe to fill with the following result:

Theorem. (Informal) Let $\mathcal{A} \subset L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ be a normalized acceptance set. Then, its induced risk measure,

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\},\ \forall X \in L^\infty(\Omega, \mathcal{F}, \mathbf{P}), \tag{3.1}$$

is comonotonic additive if $\mathcal{A}$ and $\mathcal{A}^c$ are convex for comonotonic random variables.

The property of comonotonic convexity for $\mathcal{A}$ is tightly linked to the homonymous property for risk measures, a fact that have been explored, for instance, by Kou et al. (2013) and Jia et al. (2020). Also, Jia et al. (2020) showed that law-invariant monetary risk measures are the lower envelope of a family of law-invariant comonotonic convex monetary risk measures. Risk measures satisfying comonotonic convexity (or comonotonic subadditivity) were also studied in Song et al. (2006), Song and Yan (2009a), and Song and Yan (2009b), where several representation results were provided.

The property of comonotonic convexity for $\mathcal{A}^c$ is tightly linked to the comonotonic concavity of the risk measures $\rho_{\mathcal{A}}$. In this regard, our work illustrates the potential of imposing axioms on the complements of acceptance sets, a practice that have been overlooked in the literature. Also, we are not aware of previous research considering the property of comonotonic concavity for risk measures. Arguably, this is because concavity represents aversion to diversification, contrasting with the subjective perception that diversification decreases variability and extreme losses.

Rieger (2017) and Koch-Medina et al. (2018) also contributed to understand the relation between acceptance sets and comonotonic additive risk measures. Working in a finite state space, Rieger (2017) characterized the acceptance sets that induce convex comonotonic additive risk measures as certain convex polygons. Compared to the present paper, Rieger (2017) relied on discrete mathematics to provide a more detailed geometric description of the acceptance sets that induce comonotonic additive risk measures. We, on the other hand, provide a general characterization of those acceptance sets, which answers two research questions raised by Rieger (2017): first, our characterization does not rely on the assumption of finite state space and, second, we characterize the class of acceptance sets that induce comonotonic additive risk measures that are not convex.

Koch-Medina et al. (2018) studied risk measures in the more general framework where one does not assume the existence of a risk-free asset. The focus of Koch-Medina et al. (2018) is to show that, in a market without a risk-free asset, a regulatory authority can adopt a comonotonic additive risk measure if and only if it is willing to accept certain highly leveraged positions. In their more general framework, another necessary condition for an acceptance set to induce a comonotonic additive risk measure is that it does so in the simplified framework, where the existence of a risk-free asset is assumed (see Proposition 2.15 in Koch-Medina et al. (2018)). Therefore, the conditions we provide in the present paper—for the more restricted framework—also must hold for any acceptance set inducing comonotonic additive risk measures in the framework without the existence of a risk free asset.

Generalization: The property of comonotonic additivity is well established in the literature for more than two decades and, for this reason, we devote the first part of the paper to studying comonotonic additive risk measures and their associated acceptance sets. However, the approach we develop is not restricted to comonotonic convex acceptance sets with comonotonic convex complements. We generalize our results to acceptance sets and their complements when they are convex for some *a priori* chosen class of random variables. By choosing the class of comonotonic random variables, we recover the framework of comonotonic convexity.

Acceptance sets whose complements are convex for an *a priori* chosen class of random variables give the regulatory authority the ability to separate, on the one hand, non-acceptable financial positions that, when combined, may become acceptable; from, on the other hand, financial positions that, if not acceptable, cannot become acceptable by convex combinations. This method provides extra flexibility for modeling the regulator's criterion of acceptability and, to the best of our knowledge, was not explored before.

As in our main theorem (see the "informal" theorem above), these properties of restricted convexity for $\mathcal{A}$ and $\mathcal{A}^c$ imply their induced risk measures to be additive for the class of random variables for which $\mathcal{A}$ and $\mathcal{A}^c$ are convex. Therefore, our study of restricted convexity relates to the literature on risk measures and actuarial premium principles that are additive for specific classes of random variables.

Notably, premium principles that are additive for independent random variables (these are simply called *additive* premium principles) have been extensively studied in the realm of Actuarial Mathematics. Borch (1962) argued that the property of additivity for independent random variables is desirable from a practical point of view because it is natural that insurance companies receive the same amount whether they accept two independent portfolios in a single transaction or separately. This perspective is corroborated by Gerber (1974). The literature on additive premium principles focuses on the functional form that a premium principle (or risk measure) takes when they satisfy the property of additivity for independent risks (see, for instance, Borch (1962), Gerber (1974), Gerber and Goovaerts (1981), Bühlmann (1985), Goovaerts, Kaas, Dhaene and Tang (2004), Goovaerts, Kaas, Laeven and Tang (2004), and Goovaerts et al. (2010)). Through our general approach, we show that, if an acceptance set and its complement are convex for independent random variables, then the risk measure they induce is additive. We believe this is a valid contribution to the literature on premium principles because, to the best of our knowledge, no previous work explicitly studies the acceptance sets associated with risk measures that are additive for independent risks.

Our work is structured as follows: in Section 3.2 we present basic definitions and preliminary results on comonotonic convex and comonotonic concave risk measures; in Section 3.3, we prove our main results, showing a strong link between comonotonic additive risk measures and comonotonic convex acceptance sets that have comonotonic convex complements; in Section 3.4, we present our main result related to acceptance sets and complements of acceptance sets that are convex for *a priori* chosen classes of random variables; in Section 3.5, we present a summary of the paper; we conclude the paper in Appendix 3.6, where we provide a complete presentation of our generalized approach for restricted convexity.

3.2 BASIC FRAMEWORK

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $L^0 := L^0(\Omega, \mathcal{F}, \mathbf{P})$ the space of equivalence classes of random variables (under the $\mathbf{P}$-a.s. relation). Equalities and inequalities must be understood in the $\mathbf{P}$-a.s. sense, unless otherwise specified. For the sake of conciseness, we chose to restrict our analysis to random variables in $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, \mathbf{P}) = \{X \in L^0 : \|X\|_\infty < +\infty\}$, where $\|X\|_\infty = \inf\{m \in \mathbb{R} : |X| < m\}$ for all $X \in L^0$. All topological concepts mentioned in the text should be understood with respect to the topology induced by the norm $\|\cdot\|_\infty$. The elements $X \in \mathcal{X}$ represent discounted net financial payoffs. We identify $\mathbb{R} \equiv \{X \in \mathcal{X} : X = c \text{ for some } c \in \mathbb{R}\}$. We denote vectors and random vectors as $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{X} = (X_1, \cdots, X_n) \in \mathcal{X}^n$, respectively.

Definition 3.1. *A random vector $\boldsymbol{X} \in \mathcal{X}^n$ is* **comonotonic** *if*

$$(X_i(\omega) - X_i(\omega'))(X_j(\omega) - X_j(\omega')) \geqslant 0 \quad \forall i, j \in \{1, 2 \ldots, n\} \; \mathbf{P} \otimes \mathbf{P}\text{-}a.s. \tag{3.2}$$

Equivalently, we refer to the comonotonicity of $\boldsymbol{X} \in \mathcal{X}^n$ by saying that the random variables $X_1, X_2, \cdots, X_n$ are comonotonic. As is customary in the literature, we will restrict our attention to comonotonic pairs of random variables.

Definition 3.2. *A nonempty set $\mathcal{A} \subsetneq \mathcal{X}$ is called an* **acceptance set** *if it satisfies:*

1. *(Monotonicity) $\mathcal{A}$ is* **monotone** *if $X \in \mathcal{A}$ and $X \leqslant Y$ implies $Y \in \mathcal{A}$;*
2. *(Boundedness from below on constants) $\mathcal{A}$ is* **bounded from below on constants** *if $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$.*

In addition, an acceptance set may fulfill

3. *(Normalization) $\mathcal{A}$ is* **normalized** *if $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} = 0$.*
4. *(Comonotonic Convexity) $\mathcal{A}$ is* **comonotonic convex** *if, for all comonotonic pairs (X, Y) such that $X, Y \in \mathcal{A}$ and $\lambda \in [0, 1]$, it holds that $\lambda X + (1 - \lambda)Y \in \mathcal{A}$.*

An acceptance set represents a regulator's criterion of acceptability as follows: $X \in \mathcal{A}$ if and only if X is deemed acceptable. The adequacy of axioms 1 and 2 is virtually consensual, and their interpretations were presented, for instance, in Artzner et al. (1999), McNeil et al. (2015), and Föllmer and Schied (2016). The property of comonotonic convexity embeds into $\mathcal{A}$ the notion that diversification among comonotonic random variables does not compromise acceptability. Comonotonic convex acceptance sets were also studied in Kou et al. (2013) and Jia et al. (2020).

Definition 3.3. *A functional $\rho : \mathcal{X} \to \mathbb{R}$ is called a* **risk measure** *if it satisfies:*

1. *(Monotonicity) ρ is* **monotone** *if $\rho(Y) \leqslant \rho(X)$ whenever $X \leqslant Y$ for $X, Y \in \mathcal{X}$.*

2. *(Cash invariance) ρ is* **cash invariant** *if $\rho(X - m) = \rho(X) + m$ for any $X \in \mathcal{X}$ and $m \in \mathbb{R}$.*

In addition, a risk measure may fulfill the following:

3. *(Normalization) ρ is* **normalized** *if $\rho(0) = 0$.*

4. *(Positive homogeneity) ρ is* **positive homogeneous** *if $\rho(aX) = a\rho(X)$ for any $X \in \mathcal{X}$ and any $a \geqslant 0$.*

5. *(Comonotonic Convexity) ρ is* **comonotonic convex** *if, for any comonotonic random vector $(X, Y) \in \mathcal{X}^2$ and any $\lambda \in [0, 1]$, it holds that $\rho(\lambda X + (1 - \lambda)Y) \leqslant \lambda\rho(X) + (1 - \lambda)\rho(Y)$.*

6. *(Comonotonic Concavity) ρ is* **comonotonic concave** *if, for any comonotonic random vector $(X, Y) \in \mathcal{X}^2$ and any $\lambda \in [0, 1]$, it holds that $\rho(\lambda X + (1 - \lambda)Y) \geqslant \lambda\rho(X) + (1 - \lambda)\rho(Y)$.*

7. *(Comonotonic Additivity) ρ is* **comonotonic additive** *if, for any comonotonic random vector $(X, Y) \in \mathcal{X}^2$, $\rho(X + Y) = \rho(X) + \rho(Y)$.*

The axioms of monotonicity and cash invariance are standard, and their interpretations were provided, for instance, in Föllmer and Schied (2016). The property of comonotonic convexity is discussed, for instance, in Kou et al. (2013) and Bignozzi et al. (2019). We provide conditions on the acceptance sets that are equivalent to comonotonic convexity, comonotonic concavity, and comonotonic additivity for risk measures (see Proposition 3.1 and Theorem 3.1). The property of comonotonic concavity—as will be shown—is tightly related to acceptance sets with comonotonic convex complements. For a thorough discussion of the property of comonotonic additivity, see Dhaene et al. (2002b), Dhaene et al. (2002a), and the references therein. It is also valid noticing that the property of cash invariance implies that any risk measure is non-constant.

Remark 3.1. Although we focus on the axioms presented in Definition 3.2 and Definition 3.3, several other sets of axioms were proposed in the literature. See, for instance, Bignozzi et al. (2019), Righi (2019), Mao and Wang (2020), and Castagnoli et al. (2021).

The following results are essentially known. We will use them extensively through the paper and, for this reason, we provide proofs.

Lemma 3.1. *Let ρ be a normalized risk measure. Then we have the following:*

1. *(Föllmer and Schied (2016) - Lemma 4.83) If ρ is comonotonic additive, then it is positive homogeneous.*

2. *If ρ is comonotonic convex and comonotonic concave, then it is positive homogeneous.*

3. *ρ is comonotonic additive if and only if it is comonotonic convex and comonotonic concave.*

Proof. For item 2, pick $X \in \mathcal{X}$. We must show that $\rho(aX) = a\rho(X)$ for any $a \geqslant 0$. This fact is immediately true if $a = 1$. Also, the normalization property implies that $\rho(0X) = 0\rho(X)$, which concludes the case of

$a = 0$. For the case of $a \in (0,1)$, notice that comonotonic convexity and comonotonic concavity imply that, for all comonotonic pairs $(X,Y) \in \mathcal{X}^2$ and all $a \in (0,1)$ it holds that

$$\rho(aX + (1-a)Y) = a\rho(X) + (1-a)\rho(Y). \tag{3.3}$$

Therefore, if $a \in (0,1)$ it holds that

$$\begin{aligned}\rho(aX) &= \rho(aX + (1-a)0)\\ &= a\rho(X) + (1-a)\rho(0) = a\rho(X),\end{aligned}$$

where the second equality follows for X and 0 are comonotonic.

For $a > 1$, we have

$$\begin{aligned}a\rho(X) &= a\rho\left(\frac{aX}{a} + \frac{(a-1)0}{a}\right)\\ &= a\left(\frac{1}{a}\rho(aX) + \frac{(a-1)}{a}\rho(0)\right)\\ &= \rho(aX),\end{aligned}$$

where we used the fact that aX and 0 are comonotonic and that $\rho(0) = 0$.

For the "only if" part of item 3, take a comonotonic pair $(X,Y) \in \mathcal{X}^2$ and $\lambda \in [0,1]$. Since λX and $(1-\lambda)Y$ are comonotonic and ρ is comonotonic additive, it holds that

$$\begin{aligned}\rho(\lambda X + (1-\lambda)Y) &= \rho(\lambda X) + \rho((1-\lambda)Y)\\ &= \lambda\rho(X) + (1-\lambda)\rho(Y),\end{aligned}$$

where the last equality is true for ρ is positive homogeneous, according to item 1.

For the "if" part, let $(X,Y) \in \mathcal{X}^2$ be comonotonic and pick $\lambda \in (0,1)$. It follows that the random variables $X' := X/\lambda$ and $Y' := Y/(1-\lambda)$ are comonotonic and that $X + Y = \lambda X' + (1-\lambda)Y'$. Therefore, it holds that

$$\begin{aligned}\rho(X+Y) &= \rho(\lambda X' + (1-\lambda)Y')\\ &= \lambda\rho\left(\frac{X}{\lambda}\right) + (1-\lambda)\rho\left(\frac{Y}{1-\lambda}\right)\\ &= \rho(X) + \rho(Y),\end{aligned}$$

where the last inequality follows from the positive homogeneity of ρ, which was established in item 2. □

Remark 3.2. Notice that a simple adaptation of item 2's proof shows that, if ρ is comonotonic convex, then $\rho(aX) \leqslant a\rho(X)$ for $a \in [0,1]$ and $\rho(aX) \geqslant a\rho(X)$ for $a > 1$. Risk measures satisfying this property are called *star-shaped*. For theory and applications of star-shaped risk measures, see Castagnoli et al. (2021), Righi (2021), Righi and Moresco (2022), and Moresco and Righi (2022).

Definition 3.4. *Let ρ be a risk measure and $\mathcal{A}$ an acceptance set.*

1. *The acceptance set induced by ρ is defined as*

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leqslant 0\}. \tag{3.4}$$

2. *The risk measure induced $\mathcal{A}$ is defined as*

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}, \quad \forall X \in \mathcal{X}. \tag{3.5}$$

Remark 3.3. In their propositions 4.6 and 4.7, Föllmer and Schied (2016) proved that, if ρ is a risk measure and $\mathcal{A}$ is an acceptance set, then $\mathcal{A}_\rho$ is an acceptance set and $\rho_{\mathcal{A}}$ is a risk measure. For this reason, the above definitions of $\mathcal{A}_\rho$ and $\rho_{\mathcal{A}}$ are consistent.

As shown, for instance, in Artzner et al. (1999), Cheridito and Li (2009), and Kaina and Rüschendorf (2009), there exist direct links between acceptance sets and risk measures. The following relations between risk measures and acceptance sets will be used throughout the paper:

Lemma 3.2. *Let ρ be a risk measure and let $\mathcal{A}$ be an acceptance set. Then we have the following:*

1. *(Föllmer and Schied (2016) - Proposition 4.6) $\rho(X) = \rho_{\mathcal{A}_\rho}(X)$ for all $X \in \mathcal{X}$.*
2. *(Föllmer and Schied (2016) - Proposition 4.7) $\mathcal{A}_{\rho_{\mathcal{A}}}$ equals the closure of $\mathcal{A}$.*
3. *If $\mathcal{A}$ is closed, then $\rho_{\mathcal{A}}(X) > 0$ for all $X \in \mathcal{A}^c$.*
4. *If $\mathcal{A}$ is normalized, then $\rho_{\mathcal{A}}$ is normalized. If ρ is normalized, $\mathcal{A}_\rho$ is normalized.*

Proof. To prove item 3, notice that if $\mathcal{A}$ is closed, $\mathcal{A}^c$ is open. Therefore, for all $X \in \mathcal{A}^c$ there exists $\epsilon > 0$ such that $\{Y \in \mathcal{X} : \|X - Y\|_\infty < \epsilon\} \subset \mathcal{A}^c$. In particular, notice that $\|X - (X + \eta)\|_\infty < \epsilon$ whenever $\eta \in (0, \epsilon)$. Therefore,

$$\eta \in \{m \in \mathbb{R} : X + m \in \mathcal{A}^c\} \tag{3.6}$$

and, as a direct consequence,

$$\eta \notin \{m \in \mathbb{R} : X + m \in \mathcal{A}\}. \tag{3.7}$$

Since the set $\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$ is unbounded from above, eq. (3.7) implies that $\eta \leqslant \rho_{\mathcal{A}}(X)$. Since $0 < \eta$, it holds that $0 < \rho_{\mathcal{A}}(X)$, from which we conclude the proof of item 3.

We begin to prove item 4 by showing that if $\mathcal{A}$ is normalized, then $\rho_{\mathcal{A}}$ is normalized. To this end, we must show that $\rho_{\mathcal{A}}(0) = 0$. By definition, this is the same as $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} = 0$, which corresponds exactly to the property of normalization for acceptance sets.

To conclude the proof of item 4, we must show that, if ρ is normalized, then

$$\inf\{m \in \mathbb{R} : \rho(m) \leqslant 0\} = 0. \tag{3.8}$$

By the cash invariance and the normalization of ρ, it holds that $\rho(m) = -m$ for all $m \in \mathbb{R}$. Therefore, the condition in eq. (3.8) can be rewritten as

$$\inf\{m \in \mathbb{R} : m \geqslant 0\} = 0, \tag{3.9}$$

which is evidently true. □

3.3 MAIN THEOREM

In this section, we prove our main result. See Theorem 3.1.

Definition 3.5. *Let $\mathcal{A}$ be an acceptance set. Let the functional $\psi_{\mathcal{A}^c} : \mathcal{X} \to \mathbb{R}$ induced by $\mathcal{A}^c$ be defined as*

$$\psi_{\mathcal{A}^c}(X) := \sup\{m \in \mathbb{R} : X + m \in \mathcal{A}^c\}. \tag{3.10}$$

Notice that, for an acceptance set $\mathcal{A}$ and $X \in \mathcal{X}$, the quantity $\psi_{\mathcal{A}^c}(X)$ corresponds to the smallest upper bound for the amount of cash that can be added to X without making it acceptable.

Lemma 3.3 gives us an alternative view on how to induce risk measures from acceptance sets. For a similar result in the context of deviation measures, see Proposition 3.17 in Moresco et al. (2020).

Lemma 3.3. *Let $\mathcal{A}$ be an acceptance set. Then $\rho_{\mathcal{A}}(X) = \psi_{\mathcal{A}^c}(X)$ for all $X \in \mathcal{X}$.*

Proof. Take $X \in \mathcal{X}$ and lets us prove that $\rho_{\mathcal{A}}(X) \leqslant \psi_{\mathcal{A}^c}(X)$. To this end, assume that $\psi_{\mathcal{A}^c}(X) < \rho_{\mathcal{A}}(X)$ and pick $m_0 \in (\psi_{\mathcal{A}^c}(X), \rho_{\mathcal{A}}(X))$. Since $m_0 < \rho_{\mathcal{A}}(X)$, it holds that $X + m_0 \notin \mathcal{A}$. On the other hand, since $\psi_{\mathcal{A}^c}(X) < m_0$, it follows that $X + m_0 \notin \mathcal{A}^c$, which is absurd.

Now, to show that $\rho_{\mathcal{A}}(X) \geqslant \psi_{\mathcal{A}^c}(X)$, assume that $\rho_{\mathcal{A}}(X) < \psi_{\mathcal{A}^c}(X)$ and pick $m_1 \in (\rho_{\mathcal{A}}(X), \psi_{\mathcal{A}^c}(X))$. Since $\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$ is an interval containing $(\rho_{\mathcal{A}}(X), +\infty)$, the fact that $\rho_{\mathcal{A}}(X) < m_1$ implies that $X + m_1 \in \mathcal{A}$. Analogously, since $\{m \in \mathbb{R} : X + m \in \mathcal{A}^c\}$ is an interval containing $(-\infty, \psi_{\mathcal{A}^c}(X))$, the fact that $m_1 < \psi_{\mathcal{A}^c}(X)$ implies that $X + m_1 \in \mathcal{A}^c$, which is absurd. □

The next result gives us sufficient conditions to induce comonotonic convex or comonotonic concave risk measures.

Proposition 3.1. *Let $\mathcal{A}$ be an acceptance set.*

1. *If $\mathcal{A}$ is comonotonic convex, then $\rho_{\mathcal{A}}$ is comonotonic convex.*

2. *If $\mathcal{A}^c$ is comonotonic convex, then $\rho_{\mathcal{A}}$ is comonotonic concave.*

Proof. To prove item 2, take a comonotonic pair $(X, Y) \in \mathcal{X}^2$ and two constants $x, y \in \mathbb{R}$ such that

$$X + x \in \mathcal{A}^c \quad \text{and} \quad Y + y \in \mathcal{A}^c. \tag{3.11}$$

Notice that such x and y always exist because X and Y are essentially bounded and $\mathcal{A}$ is bounded on constants. Since $(X + x, Y + y)$ is comonotonic and $\mathcal{A}^c$ is comonotonic convex, it follows that

$$\lambda(X + x) + (1 - \lambda)(Y + y) \in \mathcal{A}^c, \quad \forall \lambda \in [0, 1]. \tag{3.12}$$

This implies that

$$0 \notin \{m \in \mathbb{R} : \lambda(X + x) + (1 - \lambda)(Y + y) + m \in \mathcal{A}\}. \tag{3.13}$$

But notice that

$$\{m \in \mathbb{R} : \lambda(X + x) + (1 - \lambda)(Y + y) + m \in \mathcal{A}\} \supseteq (\rho_{\mathcal{A}}(\lambda(X + x) + (1 - \lambda)(Y + y)), +\infty), \tag{3.14}$$

which implies that $0 \notin (\rho_{\mathcal{A}}(\lambda(X + x) + (1 - \lambda)(Y + y)), +\infty)$. Therefore, it immediately follows that

$$0 \leqslant \rho_{\mathcal{A}}(\lambda(X + x) + (1 - \lambda)(Y + y)). \tag{3.15}$$

Now, the cash invariance of $\rho_{\mathcal{A}}$ implies that

$$\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) \geqslant \lambda x + (1-\lambda)y. \tag{3.16}$$

By taking the supremum on the right-hand side we obtain:

$$\begin{aligned}\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) &\geqslant \lambda \sup\{x \in \mathbb{R} : X + x \in \mathcal{A}^c\} + (1-\lambda)\sup\{y \in \mathbb{R} : Y + y \in \mathcal{A}^c\} \\ &= \lambda \psi_{\mathcal{A}^c}(X) + (1-\lambda)\psi_{\mathcal{A}^c}(Y) \\ &= \lambda \rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y),\end{aligned}$$

which concludes the proof of the second statement.

Item 1 can be proved similarly. The main adjustments are: to exchange $\mathcal{A}^c$ for $\mathcal{A}$ in eq. (3.11); revert the inequalities in eq. (3.15) and eq. (3.16); and take the infimum instead of the supremum in eq. (3.16) (once it was adapted). □

Remark 3.4. The above proof is an adaptation of the proof of Proposition 4 in Föllmer and Schied (2002). There, the authors proved that, if an acceptance set $\mathcal{A}$ is convex, then the risk measure $\rho_{\mathcal{A}}$ is convex[2].

The property of comonotonic additivity for risk measures captures the notion that the risk of a sum of comonotonic financial positions should be the same, whether these positions are held jointly or separately. Theorem 3.1 tells us that this notion can be equivalently captured by the property of comonotonic convexity for acceptance sets and their complements.

Theorem 3.1. *Let $\mathcal{A}$ be a normalized acceptance set and ρ a normalized risk measure. Then we have the following:*

1. *$\rho_{\mathcal{A}}$ is comonotonic additive if $\mathcal{A}$ and $\mathcal{A}^c$ are comonotonic convex.*

2. *Assume that $\mathcal{A}$ is closed. Then $\rho_{\mathcal{A}}$ is comonotonic additive if and only if $\mathcal{A}$ and $\mathcal{A}^c$ are comonotonic convex.*

3. *ρ is comonotonic additive if and only if $\mathcal{A}_\rho$ and $\mathcal{A}_\rho^c$ are comonotonic convex.*

Proof. Item 1 follows from Proposition 3.1 and item 3 of Lemma 3.1. The "if" part of item 2 follows directly from item 1. The "only if" part goes by a contra-positive argument. Assume that $\mathcal{A}$ is a closed acceptance set which is not comonotonic convex. Then, there exist comonotonic random variables $X, Y \in \mathcal{A}$ such that $\lambda X + (1-\lambda)Y \in \mathcal{A}^c$ for some $\lambda \in (0,1)$. Since $X, Y \in \mathcal{A}$, we have $\rho_{\mathcal{A}}(X), \rho_{\mathcal{A}}(Y) \leqslant 0$ and, therefore,

$$\lambda \rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y) \leqslant 0. \tag{3.17}$$

If ρ is comonotonic additive, then it is also comonotonic convex and comonotonic concave, according to Lemma 3.1. This fact, taken with eq. (3.17), implies that $\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) \leqslant 0$. But this is absurd because, according to item 3 of Lemma 3.2, whenever $\mathcal{A}$ is closed, $\lambda X + (1-\lambda)Y \in \mathcal{A}^c$ implies $\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) > 0$. By a similar argument, one proves that $\mathcal{A}^c$ must be comonotonic convex, which concludes the proof of item 1

For the "only if" part of item 3, assume that ρ is comonotonic additive and let us prove that $\mathcal{A}_\rho^c$ is comonotonic convex. Pick any two comonotonic random variables $X, Y \in \mathcal{A}_\rho^c$ and notice that, by definition,

[2]An acceptance set $\mathcal{A}$ is convex if $\lambda X + (1-\lambda)Y \in \mathcal{A}$ whenever $X, Y \in \mathcal{A}$ and $\lambda \in [0,1]$. A risk measure ρ is convex if $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$, $\forall X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$.

it holds that $\rho(X), \rho(Y) > 0$. Also, notice that, for any $\lambda \in [0,1]$, the random vector $(\lambda X, (1-\lambda)Y)$ is comonotonic. Therefore, the comonotonic additivity of ρ implies that

$$\begin{aligned}\rho(\lambda X + (1-\lambda)Y) &= \rho(\lambda X) + \rho((1-\lambda)Y) \\ &= \lambda\rho(X) + (1-\lambda)\rho(Y) > 0,\end{aligned}$$

where the second equality follows from the positive homogeneity of ρ (see Lemma 3.1). Therefore, $\lambda X + (1-\lambda)Y \in \mathcal{A}_\rho^c$ and we conclude that $\mathcal{A}_\rho^c$ is comonotonic convex. The same reasoning proves the comonotonic convexity of $\mathcal{A}$. The "if" direction follows from item 1 of Lemma 3.2—which asserts that $\rho = \rho_{\mathcal{A}_\rho}$—and item 1, which implies that $\rho_{\mathcal{A}_\rho}$ is comonotonic additive. □

3.4 GENERALIZATION

The generalization we develop in this section gives a manner of using acceptance sets to generate risk measures that are convex and/or concave for prespecified classes of random vectors. As in the preceding section, we will see that if a risk measure is convex *and* concave for a prespecified class of random vectors, then it is additive for random vectors in that class. The concepts of convexity, concavity and additivity for specific classes of random vectors are defined as follows:

Definition 3.6. *For $\mathcal{P} \subset \mathcal{X}^2$, a risk measure ρ may fulfill the following properties:*

1. *($\mathcal{P}$-convexity) ρ is* **$\boldsymbol{\mathcal{P}}$-convex** *if $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y),\ \forall (X,Y) \in \mathcal{P},\ \forall \lambda \in [0,1]$.*
2. *($\mathcal{P}$-concavity) ρ is* **$\boldsymbol{\mathcal{P}}$-concave** *if $\rho(\lambda X + (1-\lambda)Y) \geqslant \lambda\rho(X) + (1-\lambda)\rho(Y),\ \forall (X,Y) \in \mathcal{P},\ \forall \lambda \in [0,1]$.*
3. *($\mathcal{P}$-additivity) ρ is* **$\boldsymbol{\mathcal{P}}$-additive** *if $\rho(X+Y) = \rho(X) + \rho(Y),\ \forall (X,Y) \in \mathcal{P}$.*

Also, the following property may hold for a non-empty set $A \subsetneq \mathcal{X}$:

1. *($\mathcal{P}$-convexity) A is* **$\boldsymbol{\mathcal{P}}$-convex** *if $\forall (X,Y) \in \mathcal{P}$ such that $X, Y \in A$ it follows that $\lambda X + (1-\lambda)Y \in A,\ \forall \lambda \in [0,1]$.*

Remark 3.5. The above concepts are generalizations of the properties of convexity, concavity, and additivity for risk measures and convexity for acceptance sets. Despite the huge literature on convex risk measures and convex acceptance sets (see Föllmer and Schied (2002) and Frittelli and Gianin (2002), for instance), we are not aware of previous works studying concepts similar to those presented in Definition 3.6.

We shall add the following extra notation: for any non-empty subset $\mathcal{P} \subseteq \mathcal{X}^2$, we denote $\mathcal{P} + \mathbb{R}^2 := \{(X+x, Y+y) \in \mathcal{X}^2 : (X,Y) \in \mathcal{P},\ (x,y) \in \mathbb{R}^2\}$. The following is this section's main theorem. It tells us how to generate risk measures that are additive for specific classes of random variables.

Theorem (A.1)**.** *Let $\mathcal{A}$ be a normalized acceptance set and ρ a normalized risk measure. Also, consider a set $\mathcal{P} \subset \mathcal{X}^2$ fulfilling the following properties:*

1. *$(0, X) \in \mathcal{P}$ for all $X \in \mathcal{X}$.*
2. *If $(X,Y) \in \mathcal{P}$, then $(aX, bY) \in \mathcal{P}$ for all $(a,b) \in \mathbb{R}^2$.*
3. *$\mathcal{P} = \mathcal{P} + \mathbb{R}^2$.*

Under the above conditions, we have the following:

1. *$\rho_{\mathcal{A}}$ is positive homogeneous and $\mathcal{P}$-additive if $\mathcal{A}$ and $\mathcal{A}^c$ are $\mathcal{P}$-convex.*

2. *Assume that $\mathcal{A}$ is closed. Then $\rho_{\mathcal{A}}$ is positive homogeneous and $\mathcal{P}$-additive if and only if $\mathcal{A}$ and $\mathcal{A}^c$ are $\mathcal{P}$-convex.*

3. *ρ is $\mathcal{P}$-additive and positive homogeneous if and only if $\mathcal{A}_\rho$ and $\mathcal{A}_\rho^c$ are $\mathcal{P}$-convex.*

Remark 3.6. Proving the above theorem requires a lengthy construction and, for this reason, we opted to postpone it to Theorem 3.2 at the Appendix 3.6. In this section, we will refer to the above result as the "general theorem". Clearly, the general theorem specializes to Theorem 3.1 if $\mathcal{P} = \{(X,Y) \in \mathcal{X}^2 : (X,Y) \text{ is comonotonic}\}$.

Notice that the class of independent random pairs, namely $\mathcal{P} = \{(X,Y) \in \mathcal{X}^2 : X \text{ and } Y \text{ are independent}\}$, satisfies the assumptions of the general theorem. Therefore, this theorem relates to the literature studying risk measures and premium principles that are additive for independent random variables (see, for instance, Borch (1962), Gerber (1974), Gerber and Goovaerts (1981), Bühlmann (1985), Goovaerts, Kaas, Dhaene and Tang (2004), Goovaerts, Kaas, Laeven and Tang (2004), and Goovaerts et al. (2010)). In addition to this, Heijnen and Goovaerts (1986) studied premium principles that are additive for random variables with zero covariance. Since the class $\mathcal{P} = \{(X,Y) \in \mathcal{X}^2 : \text{Cov}(X,Y) = 0\}$ fulfills the requirements outlined in the statement of the general theorem, we also contribute to the study of the premium principles proposed in Heijnen and Goovaerts (1986).

3.5 SUMMARY

In this paper we believe to fill a long-standing gap in the elementary theory of risk measures. We obtain a simple and yet general link between acceptance sets and comonotonic additive risk measures: a sufficient condition for an acceptance set to induce a comonotonic additive risk measure is that the acceptance set and its complement are convex for combinations of comonotonic random variables.

The paper that closely relates to ours is Rieger (2017). In a finite state-space, he showed that, for an acceptance set to induce a coherent comomonotonic risk measure, the acceptance set must consist of certain polygons. In the present paper, we advance the knowledge on this topic in two directions suggested by Rieger (2017): first, we drop the assumption of finite state space and, second, our results are not restricted to coherent risk measures.

As a second contribution, we generalize our results about comonotonic additive risk measures, providing a basic theory of risk measures that are additive for an *a priori* chosen class of random variables. Of course, from this more general framework, one can recover the case of comonotonic additivity.

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3.6 APPENDIX A - PROOF OF THE GENERAL THEOREM

In this appendix, we study the link between risk measures and acceptane sets in the $\mathcal{P}$-convex and $\mathcal{P}$-additive frameworks. Also, we provide a proof for the general theorem presented in Section 3.4.

Lemma 3.4. *Let ρ be a normalized risk measure and consider a non-empty set $\mathcal{P} \subsetneq \mathcal{X}^2$ such that*

1. *$(0, X) \in \mathcal{P}$ for all $X \in \mathcal{X}$, and*
2. *If $(X, Y) \in \mathcal{P}$, then $(aX, bY) \in \mathcal{P}$ for all $(a, b) \in \mathbb{R}^2_+$.*

It holds that, ρ is $\mathcal{P}$-convex and $\mathcal{P}$-concave if and only if ρ is positive homogeneous and $\mathcal{P}$-additive.

Proof. For the "if" part, take $(X, Y) \in \mathcal{P}$ and notice that $(\lambda X, (1-\lambda)Y) \in \mathcal{P}$ for any $\lambda \in [0, 1]$. Therefore, the $\mathcal{P}$-additivity and the positive homogeneity of ρ implies that

$$\begin{aligned}\rho(\lambda X + (1-\lambda)Y) &= \rho(\lambda X) + \rho((1-\lambda)Y) \\ &= \lambda\rho(X) + (1-\lambda)\rho(Y), \quad \forall (X,Y) \in \mathcal{P},\ \forall \lambda \in [0,1],\end{aligned}$$

which implies that ρ is $\mathcal{P}$-convex and $\mathcal{P}$-concave.

For the "only if" part let us start by showing that ρ is positive homogeneous, i.e., we must show that $\rho(aX) = a\rho(X)$ for any $a \geqslant 0$ and $X \in \mathcal{X}$. This fact is immediately true if $a = 1$. Also, the normalization property implies that $\rho(0X) = 0\rho(X)$, which concludes the case of $a = 0$. For the case of $a \in (0, 1)$, notice that $\mathcal{P}$-convexity and $\mathcal{P}$-concavity imply that, for all $(X, Y) \in \mathcal{P}$ and all $a \in (0, 1)$ it holds that

$$\rho(aX + (1-a)Y) = a\rho(X) + (1-a)\rho(Y). \tag{3.18}$$

Since $(0, X) \in \mathcal{P}$, eq. (3.18) implies that

$$\begin{aligned}\rho(aX) &= \rho(aX + (1-a)0) \\ &= a\rho(X) + (1-a)\rho(0) = a\rho(X), \quad \forall a \in (0,1).\end{aligned}$$

For $a > 1$, we have

$$\begin{aligned}a\rho(X) &= a\rho\left(\frac{aX}{a} + \frac{(a-1)0}{a}\right) \\ &= a\left(\frac{1}{a}\rho(aX) + \frac{(a-1)}{a}\rho(0)\right) \\ &= \rho(aX),\end{aligned}$$

where we used the fact that $(0, aX) \in \mathcal{P}$ and that $\rho(0) = 0$. This establishes the positive homogeneity of ρ.

To prove $\mathcal{P}$-additivity, pick $(X, Y) \in \mathcal{P}$ and $\lambda \in (0, 1)$. It follows that the random variables $X' := X/\lambda$

and $Y' := Y/(1-\lambda)$ belong to $\mathcal{P}$ and that $X+Y = \lambda X' + (1-\lambda)Y'$. Therefore, it holds that

$$\begin{aligned}\rho(X+Y) &= \rho(\lambda X' + (1-\lambda)Y') \\ &= \lambda\rho(X') + (1-\lambda)\rho(Y') \\ &= \lambda\rho\left(\frac{X}{\lambda}\right) + (1-\lambda)\rho\left(\frac{Y}{1-\lambda}\right) \\ &= \rho(X) + \rho(Y),\end{aligned}$$

where the last inequality follows from the positive homogeneity of ρ. □

Remark 3.7. Lemma 3.4 is similar to—although more parsimonious than—Lemma 3.1. We abstain from extending Lemma 3.4 for the sake of conciseness.

Lemma 3.5. *Consider an acceptance set $\mathcal{A}$ and let a nonempty set $\mathcal{P} \subset \mathcal{X}^2$ be such that $\mathcal{P} = \mathcal{P} + \mathbb{R}^2$. If $\mathcal{A}$ is $\mathcal{P}$-convex, then $\rho_{\mathcal{A}}$ is $\mathcal{P}$-convex. Also, if $\mathcal{A}^c$ is $\mathcal{P}$-convex, then $\rho_{\mathcal{A}}$ is $\mathcal{P}$-concave.*

Proof. To prove the first statement, take a pair $(X,Y) \in \mathcal{P}$ and two constants $x, y \in \mathbb{R}$ such that $X + x \in \mathcal{A}$ and $Y + y \in \mathcal{A}$. Notice that such x and y always exist because $\mathcal{A}$ is non-empty and monotone. Since $\mathcal{P} = \mathcal{P} + \mathbb{R}^2$, it holds that $(X+x, Y+y) \in \mathcal{P}$. Since $\mathcal{A}$ is $\mathcal{P}$-convex, it follows that

$$\lambda(X+x) + (1-\lambda)(Y+y) \in \mathcal{A}, \tag{3.19}$$

and, therefore,

$$\rho_{\mathcal{A}}(\lambda(X+x) + (1-\lambda)(Y+y)) \leqslant 0. \tag{3.20}$$

In addition, the cash invariance of $\rho_{\mathcal{A}}$ implies that

$$\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) \leqslant \lambda x + (1-\lambda)y. \tag{3.21}$$

The above inequality also holds if we take the infimum on the right-hand side, i.e., it holds that

$$\begin{aligned}\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) &\leqslant \lambda \inf\{x \in \mathbb{R} : X + x \in \mathcal{A}\} + (1-\lambda)\inf\{y \in \mathbb{R} : Y + y \in \mathcal{A}\} \\ &= \lambda\rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y),\end{aligned}$$

from which we conclude that $\rho_{\mathcal{A}}$ is $\mathcal{P}$-convex. For the second statement, it suffices to exchange $\mathcal{A}$ for $\mathcal{A}^c$, revert all inequalities, and take the supremum instead of the infimum in eq. (3.21). □

Next, we restate and prove our main result regarding $\mathcal{P}$-additive risk measures.

Theorem 3.2 (general)**.** *Let $\mathcal{A}$ be a normalized acceptance set and ρ a normalized risk measure. Also, consider a set $\mathcal{P} \subset \mathcal{X}^2$ fulfilling the following properties:*

1. *$(0, X) \in \mathcal{P}$ for all $X \in \mathcal{X}$.*
2. *If $(X,Y) \in \mathcal{P}$, then $(aX, bY) \in \mathcal{P}$ for all $(a,b) \in \mathbb{R}^2$.*
3. *$\mathcal{P} = \mathcal{P} + \mathbb{R}^2$.*

Under the above conditions, we have the following:

1. *$\rho_{\mathcal{A}}$ is positive homogeneous and $\mathcal{P}$-additive if $\mathcal{A}$ and $\mathcal{A}^c$ are $\mathcal{P}$-convex.*

2. *Assume that $\mathcal{A}$ is closed. Then $\rho_{\mathcal{A}}$ is positive homogeneous and $\mathcal{P}$-additive if and only if $\mathcal{A}$ and $\mathcal{A}^c$ are $\mathcal{P}$-convex.*

3. *ρ is $\mathcal{P}$-additive and positive homogeneous if and only if $\mathcal{A}_\rho$ and $\mathcal{A}_\rho^c$ are $\mathcal{P}$-convex.*

Proof. We start by proving item 1. Notice that, under the conditions of Theorem 3.2's statements, Lemma 3.5 holds. Therefore, the $\mathcal{P}$-convexity of $\mathcal{A}$ and $\mathcal{A}^c$ imply that ρ is $\mathcal{P}$-convex and $\mathcal{P}$-concave. Also, the normalization of $\mathcal{A}$ implies the homonymous property for ρ. Then, Lemma 3.4 implies that $\rho_{\mathcal{A}}$ is positive homogeneous and $\mathcal{P}$-additive.

The "if" part of item 2 follows immediately from item 1. For the "only if" part, we go by a contrapositive argument. Assume that $\mathcal{A}$ is not $\mathcal{P}$-convex. Then, there exist $(X, Y) \in \mathcal{P}$ such that $X, Y \in \mathcal{A}$ and such that $\lambda X + (1-\lambda)Y \in \mathcal{A}^c$ for some $\lambda \in (0, 1)$. Since $X, Y \in \mathcal{A}$, we have $\rho_{\mathcal{A}}(X), \rho_{\mathcal{A}}(Y) \leqslant 0$ and, therefore,

$$\lambda\rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y) \leqslant 0. \tag{3.22}$$

If $\rho_{\mathcal{A}}$ was to be $\mathcal{P}$-additive and positive homogeneous, then it would also be $\mathcal{P}$-convex and $\mathcal{P}$-concave, according to Lemma 3.4. This fact, taken with eq. (3.22), implies that $\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) \leqslant 0$. But this is an absurd, for $\lambda X + (1-\lambda)Y \in \mathcal{A}^c$ implies $\rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) > 0$. By a similar argument, one proves that $\mathcal{A}^c$ must be $\mathcal{P}$-convex, which concludes the proof foritem 2.

For the "only if" part of item 3, assume that ρ is $\mathcal{P}$-additive and positive homogeneous and lets us prove that $\mathcal{A}_\rho^c$ is $\mathcal{P}$-convex. Pick any $(X, Y) \in \mathcal{P}$ such that $X, Y \in \mathcal{A}_\rho^c$ and notice that, by definition, it holds that $\rho(X), \rho(Y) > 0$. Also, notice that, for any $\lambda \in [0, 1]$, it holds that $(\lambda X, (1-\lambda)Y) \in \mathcal{P}$. Therefore, the $\mathcal{P}$-additivity of ρ implies that

$$\begin{aligned}\rho(\lambda X + (1-\lambda)Y) &= \rho(\lambda X) + \rho((1-\lambda)Y) \\ &= \lambda\rho(X) + (1-\lambda)\rho(Y) > 0,\end{aligned}$$

where the second equality follows from the positive homogeneity of ρ. Therefore, $\lambda X + (1-\lambda)Y \in \mathcal{A}^c$ and we conclude that $\mathcal{A}_\rho^c$ is $\mathcal{P}$-convex. The same reasoning proves the $\mathcal{P}$-convexity of $\mathcal{A}$.

For the "if" direction, assume that $\mathcal{A}_\rho$ and $\mathcal{A}_\rho^c$ are $\mathcal{P}$-convex. Then, Lemma 3.5 implies that $\rho_{\mathcal{A}_\rho}$ is $\mathcal{P}$-convex and $\mathcal{P}$-concave. Since we are assuming that ρ is normalized, Lemma 3.4 implies that $\rho_{\mathcal{A}_\rho}$ is $\mathcal{P}$-additive and positive homogeneous. But we know that $\rho = \rho_{\mathcal{A}_\rho}$ (see item 1 of Lemma 3.2), therefore we conclude that ρ is $\mathcal{P}$-additive and positive homogeneous. $\square$

4 COST OF ROBUST RISK REDUCTION

Abstract

Taking investment decisions requires managers to consider how the current portfolio would be affected by the inclusion of other assets. In particular, it is of interest to know if adding a given asset would increase or decrease the risk of the current portfolio. However, this addition may reduce or increase the risk, depending on the risk measure being used. Arguably, risk sub-estimation is a major concern to regulatory agencies, and possibly to the financial firms themselves. To provide a more decisive and conservative conclusion about the effect of an additional asset on the risk of the current portfolio, we propose to assess this effect through the family of monetary risk measures that are consistent with second-degree stochastic dominance (SSD-consistent risk measures). This criterion provides a tool to identify financial positions that reduce the risk of the current portfolio, according to all monetary SSD-consistent risk measures. Also, this tool measures the smallest amount of money (the cost) necessary to turn the financial positions into risk reducers for the original portfolio. We characterize the cost of robust risk reduction through a monetary risk measure, a monetary acceptance set, the family of average values at risk, and through the infimum of the certainty equivalents of risk-averse agents with random initial wealth.

Key-words: Robust risk reduction. Robust certainty equivalents. Preference robust optimization. SSD-consistent risk measures.

4.1 INTRODUCTION

A main challenge for portfolio managers is to understand how the inclusion of additional assets would affect the risk of a current portfolio. If the risk manager has a decisive view on how the risk of the portfolio should be measured, then no more than a single risk measure must be considered to assess the effect of adding a new position to the portfolio. There are several mathematical tools that can be used to analyze such situations, for instance, the measures of systemic risk studied in Chen et al. (2013), Kromer et al. (2016), Biagini et al. (2019), and Arduca et al. (2021); the measures for portfolio vectors studied in Jouini and Napp (2004), Burgert and Rüschendorf (2006), Cai et al. (2022), and the references therein, and;

notably, the capital allocation rules and risk contribution rules studied, for instance, in Kalkbrener (2005), Wei and Hu (2022), Guan et al. (2022), Canna et al. (2020), Canna et al. (2020), and Canna et al. (2021).

However, there are instances in which considering a unique risk measure is not enough. For instance, the risk manager might be in charge of aggregating the attitudes towards risk of multiple stakeholders or, even if the portfolio belongs to a single investor, it might be that this investor's risk attitude is only partially observed. This lack of information may become particularly troublesome in portfolio optimization problems. In fact, if the solution to the problem is very sensitive to the choice of the risk measure, then a minor mistake in the elicitation of the investor's attitude towards risk may lead the portfolio manager to assume more risk than what would be in the investor's best interest.

As a remedy for these drawbacks from the lack of information in financial decision making, Armbruster and Delage (2015) proposed the *preference robust optimization* paradigm (PRO), through which we can make use of partial information about individuals' preferences (or risk attitudes) to obtain measures of utilities, certainty equivalents, and risk measurements that conform to what is known about the individuals' preferences and risk attitudes. As developments of this approach, Delage and Li (2018), Wang and Xu (2020), and Li (2021) proposed that risk managers should use, instead of a single risk measure, a family of risk measures whose axioms conform to what is known about the investor's attitude towards risk. One can consider, for instance, the family of average values at risk (considering all significance levels), the family of coherent risk measures (Artzner et al., 1999), convex risk measures (Föllmer and Schied, 2002; Frittelli and Gianin, 2002), risk measures consistent with the second-degree stochastic dominance (consistent risk measures) (Mao and Wang, 2020), or monetary risk measures (Jia et al., 2020). The PRO paradigm can also be used to provide assessments of the desirability of the financial options based on the partial knowledge of an agent's preferences for risky prospects. In the realm of theories of choice, one can consider, for instance, the class of agents whose preferences conform to strictly increasing strictly concave utility functions (see, for instance, Chapter 2 of Föllmer and Schied (2016)), to the Yaari's theory of choice (Yaari, 1987), or to the S-shaped value functions proposed in Kahneman and Tversky (1979).

Based on the PRO approach, we propose a functional that identifies financial positions that reduces the risk of an original portfolio, according to any risk measure or individual conforming to a pre-established robust criterion. Concerning risk measures, we focus on the family of consistent risk measures or—what is equivalent—the family of average values at risk (considering all significance levels). On the side of theories of choice, we consider agents whose preferences conform to strictly increasing strictly concave utility functions or, equivalently, to the certainty equivalents associated with these utility functions. It is well-known that these families of functionals (consistent risk measures and strictly increasing strictly concave) characterize the second-degree stochastic order and that, therefore, these families are linked to each other. What have not been mentioned in the literature (although is essentially known) is the fact that the link among these families—which has the second-degree stochastic order in the center—allows one to equivalently approach robust risk assessment (in the spirit of PRO) based on each of these families of functionals.

Our main contribution goes a step beyond this observation, as we extend the link between those classes of functionals, showing that they generate equivalent robust criteria to identify which positions decrease the risk of a given initial portfolio. We adopt the concept of risk reducers proposed in Cheung et al. (2014), with a slight adaptation in the definition and in the denomination, as we refer to them as *robust risk reducers*. The framework we develop is based on the standard notions of acceptance sets and risk measures (Artzner et al., 1999). As a consequence, in addition to identify robust risk reducers, the functional we propose measures the cost to make an incremental position a robust risk reducer for any given initial portfolio.

In analogy with Artzner et al. (1999), an incremental position is identified as a risk reducer for the existing portfolio if its *cost to robust risk reduction* is negative.

The concept of robust risk reducers has a mirrored concept of robust risk increasers, which consists of positions that increase the risk of the initial portfolio, according to all consistent risk measures. However, these concepts are not complement of each other, in the sense that they do not cover all the existing financial positions. Although our focus lies on the cost to robust risk reduction, studying risk increasers is also relevant, at least to the extent that comparing these concepts helps to better understand them. In fact, these concepts leads to two different robust certainty equivalents.

Based on the concept of robust risk reduction, we define the *robust upper certainty equivalent*, which is a possible robust version of the standard certainty equivalent. This functional comes, at a first step, by allowing the individuals to hold an initial financial portfolio, which can be interpreted as a random initial wealth. Then, we make this certainty equivalent *robust*, in the sense that it identifies the financial positions that are robust risk reducers or, equivalently, robust utility increasers. In analogy to the cost to robust risk reduction, it holds that the robust upper certainty equivalent of a financial position is positive if and only if it is a risk reducer for the portfolio, according to all individuals with preferences represented by strictly increasing strictly concave utility functions. In fact, we show that the robust upper certainty equivalent corresponds to the cost to risk reduction, up to a sign conversion.

The paper is structured as follows: in Section 4.2 we set the mathematical framework of the paper, and provide basic definitions and facts about stochastic orders and comonotonic random variables. In Section 4.3, we present the axioms for risk measures and acceptance sets that will be relevant in the text. Additionally, we provide an introductory discussion of robust risk reducers and robust risk increasers. In Section 4.4, we present the concepts of upper and lower robust certainty equivalents. In Section 4.5 we present a robust criterion to identify risk reducers and to measure the minimum cost of turning financial positions into risk reducers for an original portfolio.

4.2 PRELIMINARY DEFINITIONS

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be an atomless probability space. Random variables $X \in L^0(\Omega, \mathcal{F}, \mathbf{P})$ represent the net present value of financial positions. Accordingly, $X(\omega) > 0$ represents a certain gain, while $X(\omega) < 0$ stands for a certain loss. We assume that all financial positions lie in $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, \mathbf{P})$. This assumption possibly entails a loss of generality, and extending the theory to larger spaces is a possibility for future work. We identify $\mathbf{P}$-a.s. constant random variables with constants, i.e. $\mathbb{R} \equiv \{X \in \mathcal{X} : \mathbf{P}(X = c) = 1, \text{ for some } c \in \mathbb{R}\}$. For a random variable $X \in \mathcal{X}$, we denote its (marginal) probability distribution and its quantile function as $F_X(x) := \mathbf{P}(X \leqslant x)$, $\forall x \in \mathbb{R}$, and $q_X(p) := \inf\{x \in \mathbb{R} : F_X(x) \geqslant p\}$, $\forall p \in [0, 1]$. We write $X \overset{\mathrm{d}}{=} Y$ if $\mathbf{P}(X \leqslant x) = \mathbf{P}(Y \leqslant x)$ for all $x \in \mathbb{R}$. Also, we adopt the notation $X^+ = \max\{X, 0\}$, and $X^- = \max\{-X, 0\}$ for all $X \in \mathcal{X}$.The terms "increasing" and "decreasing" refer to non-decreasing and to non-increasing functions, respectively.

A random variable X_1 is said to be **first-order stochastic dominated** by another random variable X_2, which is denoted by $X_1 \preceq_{st} X_2$, if and only if, $\mathbf{E}[f \circ X_1] \leqslant \mathbf{E}[f \circ X_2]$ for all increasing functions $f : \mathbb{R} \to \mathbb{R}$ for which the expectations exist. A random variable X_1 is **second-order stochastic dominated** by another random variable X_2, which is denoted by $X_1 \preceq_{sd} X_2$, if and only if $\mathbf{E}[f \circ X_1] \leqslant \mathbf{E}[f \circ X_2]$ for all increasing concave functions $f : \mathbb{R} \to \mathbb{R}$ for which the expectations exist. We will employ the acronym **SSD** to refer to second-order stochastic dominance. The **convex order**, denoted as $\preceq_{cx}$, is defined as $X_1 \preceq_{cx} X_2$ if and only if $E[f \circ X_1] \leqslant E[f \circ X_2]$ for all convex functions $f : \mathbb{R} \to \mathbb{R}$ for which the expectation exists. For applications and alternative characterizations of these stochastic orderings, see Denuit et al. (2006), Marshall et al. (1979), Shaked and Shanthikumar (2007), and Föllmer and Schied (2016).

Definition 4.1. *A random vector* $(X, Y) \in \mathcal{X}^2$ *is* **comonotonic** *if and only if*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0 \quad \mathbf{P} \otimes \mathbf{P}\text{-}a.s. \tag{4.1}$$

The concept of comonotonicity generalizes immediately to random vectors in $\mathcal{X}^n$. The essence of our discussion, however, is captured by random pairs. For alternative characterizations of comonotonicity, see for instance Theorem 2.14 of Rüschendorf (2013) and Theorem 4 of Dhaene et al. (2020).

Proposition 4.1. *Consider the following results:*

1. *(Shaked and Shanthikumar (2007) - Theorem 4.A.8 [adapted]) Consider two random variables* $X, Y \in \mathcal{X}$ *and a sequence* $\{X_i : i \geqslant 1\} \subset \mathcal{X}$*. If* $X_i \to X$ *in the* $\|\cdot\|_\infty$ *topology and* $Y \preceq_{sd} X_i$ *for all* $i \geqslant 1$*, then* $Y \preceq_{sd} X$*.*

2. *(Denuit et al. (2006) - Corollary 3.4.30) For any* $X, Y, X^c, Y^c \in \mathcal{X}$ *such that* (X^c, Y^c) *is comonotonic,* $X \overset{d}{=} X^c$*, and* $Y \overset{d}{=} Y^c$ *it follows that* $X^c + Y^c \preceq_{sd} X + Y$*.*

Remark 4.1. Item 1 shows that the relation $\preceq_{sd}$ is closed with respect to uniform convergence. A financial interpretation of this property is that, if Y is riskier than every term X_i of an uniformly convergence sequence $X_i \to X$, then Y is also riskier than the limit, X. Item 2 shows that the $\preceq_{sd}$ order reflects the absence of hedging among comonotonic random variables. It captures the fact that a portfolio composed of comonotonic financial positions has the highest risk among all portfolios whose components have the same marginal distribution.

4.3 RISK MEASURES

Definition 4.2. *A* **risk measure** *is a functional* $\rho : \mathcal{X} \to \mathbb{R}$*. A risk measure is called* **monetary** *if it satisfies the following properties:*

1. *(Cash-additivity) A risk measure* ρ *is* **cash-additive** *if, for all* $X \in \mathcal{X}$ *and* $m \in \mathbb{R}$*, it holds that* $\rho(X + m) = \rho(X) - m$*.*

2. *(Monotonicity) A risk measure* ρ *is* **monotone** *if, for all* $X, Y \in \mathcal{X}$*, it holds that* $\rho(X) \geqslant \rho(Y)$ *if* $X \leqslant Y$*.*

In addition, a risk measure may satisfy the following properties:

3. *(Law-invariance) A risk measure* ρ *is* **law-invariant** *if, for all* $X, X' \in \mathcal{X}$*, it holds that* $\rho(X) = \rho(X')$ *if* $F_X(x) = F_{X'}(x)$ *for all* $x \in \mathbb{R}$*.*

4. *(SSD-consistency) A risk measure is* **SSD-consistent** *if, for all* $X, Y \in \mathcal{X}$*, it holds that* $\rho(X) \geqslant \rho(Y)$ *if* $X \preceq_{sd} Y$*.*

5. *(Convexity) A risk measure* ρ *is* **convex** *if, for all* $X, Y \in \mathcal{X}$ *and* $\alpha \in [0, 1]$*, it holds* $\rho(\alpha X + (1-\alpha)Y) \leqslant \alpha\rho(X) + (1-\alpha)\rho(Y)$*.*

6. *(Subadditivity) A risk measure* ρ *is* **subadditive** *if, for all* $X, Y \in \mathcal{X}$*, it holds that* $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$

7. *(Comonotonic additivity) A risk measure ρ is* **comonotonic additive** *if, for all $X, Y \in \mathcal{X}$ such that (X, Y) is comonotonic, it holds that $\rho(X+Y) = \rho(X) + \rho(Y)$.*

8. *(Diversification consistency) A risk measure ρ is* **diversification consistent** *if, for all $X, Y, X^c, Y^c \in \mathcal{X}$ such that $X \overset{d}{=} X^c$, $Y \overset{d}{=} Y^c$ and (X^c, Y^c) is comonotonic, it holds that $\rho(X+Y) \leqslant \rho(X^c + Y^c)$.*

The property of cash-additivity tells us how the risk of a position can be reduced as one adds extra capital to it. By being monotone, a risk measure attributes higher risk to financial positions that will realize the worst results with probability one. As shown in Lemma 4.3 of Föllmer and Schied (2016), all monetary risk measure is Lipschitz continuous with respect to the $\|\cdot\|_\infty$ norm. Law-invariance requires the risk of the positions to be fully determined by their distribution, which is especially important for data-based applications. The properties of SSD-consistency, convexity, subadditivity, and diversification consistency are designed to reflect risk aversion and gains from diversification. The property of comonotonic additivity, one the other hand, reflects the perception that the risk of a portfolio composed of comonotonic position is exactly equal to the sum of the risk of the portfolio's components. In the same spirit, diversification consistency implies that the portfolio with comonotonic components is riskier than all portfolios composed of assets with given marginal distributions. The next proposition illustrates the links between these properties.

Proposition 4.2. *Consider the following results:*

1. *(Mao and Wang (2020) - Proposition 3.2) In an atomless probability space, any monetary law-invariant convex risk measure is SSD-consistent.*

2. *In an atomless probability space, any monetary SSD-consistent risk measure is law-invariant.*

Proof. To prove item 2, notice that $X \overset{\mathrm{d}}{=} X'$ implies either $X \preceq_{sd} X'$ and $X' \preceq_{sd} X$. If ρ is SSD-consistency, it holds that $\rho(X) \leqslant \rho(X')$ and $\rho(X') \leqslant \rho(X)$, which implies that $\rho(X) = \rho(X')$ and concludes the proof. □

Remark 4.2. As shown in Mao and Wang (2020), the properties of SSD consistency and diversification consistency are equivalent to monetary risk measures. These properties have several other different forms, each one reflecting risk aversion through alternative perspectives.

SSD-consistent risk measures are central in our study, for this reason, we adopt the following notation:

$$\Theta = \{\rho : \mathcal{X} \to \mathbb{R} : \rho \text{ is monetary and SSD-consistent}\}. \tag{4.2}$$

Definition 4.3. *Consider the following risk measures:*

1. *The* **value at risk** *of $X \in \mathcal{X}$ at the significance level $p \in [0,1]$, denoted as $\mathrm{VaR}_p(X)$, is defined as*

$$\mathrm{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbf{P}(X + x < 0) \leqslant p\} = q_{-X}(1-p). \tag{4.3}$$

2. *The* **average value at risk** *of $X \in \mathcal{X}$ at the significance level $p \in (0,1]$, denoted as $\mathrm{AVaR}_p(X)$, is defined as*

$$\mathrm{AVaR}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)dq. \tag{4.4}$$

3. *The* **expected loss** *of $X \in \mathcal{X}$, denoted as $EL(X)$, is defined as $EL(X) = \mathbf{E}[-X]$.*

4. *The* **maximum loss** *of $X \in \mathcal{X}$, denoted as $ML[X]$, is defined as $ML(X) = -\operatorname{ess\,inf} X$.*

Remark 4.3. The average value at risk is of central importance in the financial industry and is a major tool for regulators to determine regulatory capital. It will be used extensively through the paper. Therefore, it is worth noticing that $\lim_{p\to 0} \mathrm{AVaR}_p(X) = ML(X)$[1]. In addition, since $p \mapsto \mathrm{AVaR}_p(\cdot)$ is decreasing, it holds that $\sup_{p\in(0,1]} \mathrm{AVaR}_p(X) = ML(X)$. An additional well-known fact is that $\mathrm{AVaR}_1 = EL(X)$.

The risk measures presented in Definition 4.3 are monetary, law-invariant, and convex except for the value at risk, which is not SSD-consistent. The second-order stochastic dominance is fully characterized by the average value at risk, in the sense of the following result.

Proposition 4.3. *For any two random variables $X, Y \in \mathcal{X}$ the following conditions are equivalent to $X \preceq_{sd} Y$:*

1. *$\rho(Y) \leqslant \rho(X)$ for any $\rho \in \Theta$.*
2. *(Föllmer and Schied (2016) - Theorem 2.57) $\mathrm{AVaR}_p(Y) \leqslant \mathrm{AVaR}_p(X)$ for all $p \in (0,1]$.*

Proof. The condition $X \preceq_{sd} Y$ implies item 1 from the definition of SSD-consistency. Since $\mathrm{AVaR}_p \in \Theta$ for all $p \in (0,1]$, item 1 implies item 2. The implication from item 2 to $X \preceq_{sd} Y$ is proved in Föllmer and Schied (2016). □

Remark 4.4. In the spirit of item 1 of Proposition 4.3, Wang et al. (2020) provided a characterization of the $\preceq_{cx}$ order based on comonotonic additive functionals.

Stochastic orders are used to compare risks; risk measures are used to quantify risks; and acceptance sets are used by regulatory authorities to determine which positions can be held by the financial institution being considered.

Definition 4.4. *An* **acceptance set** *is any non-empty set $\mathcal{A} \subsetneq \mathcal{X}$. An acceptance set is called* **monetary** *if it satisfies the following properties:*

1. *(Monotonicity) $\mathcal{A}$ is* **monotone** *if $X \leqslant Y$ and $X \in \mathcal{A}$, imply $Y \in \mathcal{A}$.*
2. *(Boundedness from below on constants) $\mathcal{A}$ is* **bounded from below on constants** *if $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$.*

In addition, an acceptance set may satisfy the following properties:

3. *(Normalization) $\mathcal{A}$ is* **normalized** *if $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} = 0$.*
4. *(Convexity) $\mathcal{A}$ is* **convex** *if $\lambda\mathcal{A} + (1-\lambda)\mathcal{A} \subseteq \mathcal{A}$ whenever $\lambda \in [0,1]$.*
5. *(SSD-consistency) $\mathcal{A}$ is* **SSD-consistent** *if $X \preceq_{sd} Y$ and $X \in \mathcal{A}$, then $Y \in \mathcal{A}$.*

The property of monotonicity gives a sufficient condition for acceptability, namely, if the regulatory agency accepts X while Y pays more than X with probability one, then Y should also be acceptable. The property of boundedness on constants says that there is a lower bound on the size of certain losses that are acceptable. In fact, for much of the theory, the stronger property of normalization holds, which means that no certain loss is deemed acceptable. The property of convexity corresponds to the requirement that diversification does not increase the risk. As for risk measures, the property of SSD-consistency indicates that the risk assessment must agree with the $\preceq_{sd}$ order.

As the next result shows, monetary risk measures and monetary acceptance sets are tightly linked.

[1]To see this, notice that $\mathrm{VaR}_p(X) \leqslant \mathrm{AVaR}_p(X) \leqslant ML(X)$, for all $p \in (0,1]$ and all $X \in \mathcal{X}$. Therefore, since $\lim_{p\to 0} \mathrm{VaR}_p = ML(X)$, it also holds that $\lim_{p\to 0} \mathrm{AVaR}_p(X) = ML(X)$ for all $X \in \mathcal{X}$.

Theorem 4.1. *(Föllmer and Schied (2016) - Proposition 4.6 and 4.7) If ρ is a monetary risk measure, then*

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \leqslant 0\}\} \tag{4.5}$$

is a monetary acceptance set. Analogously, if $\mathcal{A}$ is a monetary acceptance set, then

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\} \tag{4.6}$$

is a monetary risk measure. Moreover, we have the following:

1. $\rho_{\mathcal{A}_\rho}(X) = \rho(X), \quad \forall X \in \mathcal{X}$.
2. $\mathcal{A}_{\rho_{\mathcal{A}}}$ *corresponds to the* $\|\cdot\|_\infty$*-closure of* $\mathcal{A}$.

As was shown in Chapter 3, one can derive risk measures through acceptance sets' complements:

Proposition 4.4. *Let $\mathcal{A}$ be a monetary acceptance set. Then it follows that*

$$\begin{aligned}\rho_A(X) &= \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}\\ &= \sup\{m \in \mathbb{R} : X + m \in \mathcal{A}^c\}, \quad \forall X \in \mathcal{X}.\end{aligned}$$

Proof. See Lemma 3.3 of Chapter 3. □

Proposition 4.4 reveals an equivalence between, on the one hand, how "distant" a certain position is from acceptability[2]—which is represented in eq. (4.6)—and, on the other hand, how persistently non-acceptable that given position is—which is represented in Proposition 4.4. This equivalence holds for all monetary acceptance sets and their complements.

Proposition 4.5. *Consider the acceptance set $\mathcal{A} = \{X \in \mathcal{X} : 0 \preceq_{sd} X\}$ and the set $\mathcal{W} = \{X \in \mathcal{X} : X \preceq_{sd} 0\}$. Then it holds that*

1. $\rho_{\mathcal{A}}(X) = ML(X)$, $\forall X \in \mathcal{X}$.
2. $\psi_{\mathcal{W}}(X) := \sup\{m \in \mathbb{R} : X + m \in \mathcal{W}\} = EL(X)$, $\forall X \in \mathcal{X}$.

Proof. To prove item 1, notice that we have:

$$\begin{aligned}\rho_{\mathcal{A}}(X) &= \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}\\ &= \inf\{m \in \mathbb{R} : 0 \preceq_{sd} X + m\}\\ &= \inf\left\{m \in \mathbb{R} : m \geqslant \sup_{p\in(0,1]} \text{AVaR}_p(X)\right\} = ML(X),\end{aligned}$$

where we used the AVaR characterization of $\preceq_{sd}$ (item 2 of Proposition 4.3) and the fact that $\sup_{p\in(0,1]} \text{AVaR}_p(X) = ML(X)$.

Similarly for item 2, it holds that

$$\begin{aligned}\psi_{\mathcal{W}}(X) &= \sup\{m \in \mathbb{R} : X + m \preceq_{sd} 0\}\\ &= \sup\left\{m \in \mathbb{R} : m \leqslant \inf_{p\in(0,1]} \text{AVaR}_p(X)\right\} = EL(X).\end{aligned}$$ □

[2]We use the word "distant" in quotation marks to highlight that the function $\rho_{\mathcal{A}}$ is not a *bona fide* metric and, therefore, does not provide proper measures of distance.

Remark 4.5. The fact that for most $X \in \mathcal{X}$, the magnitudes of $\rho_{\mathcal{A}}(X)$ and $\psi_{\mathcal{W}}(X)$ are substantially different will be discussed under the light of certain equivalents in remark 4.11. At this point, however, it is valid to highlight that $\rho_{\mathcal{A}}$ and $\psi_{\mathcal{W}}$ are different because the set $\mathcal{W}$ does not correspond to $\mathcal{A}^c$. In fact, it holds that $\mathcal{W} \subsetneq \mathcal{A}^c$ and, for this reason, it is natural that

$$\begin{aligned}\rho_{\mathcal{A}}(X) &= \sup\{m \in \mathbb{R} : X + m \in \mathcal{A}^c\} \\ &\geqslant \sup\{m \in \mathbb{R} : X + m \in \mathcal{W}\} = \psi_{\mathcal{W}}(X).\end{aligned}$$

Our goal is to present robust assessments of the effect of adding a position $X \in \mathcal{X}$ to an initial portfolio $Y \in \mathcal{X}$. Notice that the acceptance set $\mathcal{A} = \{X \in \mathcal{X} : 0 \preceq_{sd} X\}$ is robust in the sense that, if $X \in \mathcal{A}$, then $\rho(X) \leqslant 0$ for all $\rho \in \Theta$. Analogously, the risk measure $ML(\cdot)$ is robust in the sense that $\rho(X + ML(X)) \leqslant \rho(0)$ for all $\rho \in \Theta$. If we consider the initial portfolio as being $Y = 0$, then $\mathcal{A}$ specifies all the positions X such that $X + Y$ is less risky than Y considering all $\rho \in \Theta$. Analogously, the fact that $ML(X) = \inf\{m \in \mathbb{R} : \rho(X+Y+m) \leqslant \rho(Y),\ \forall \rho \in \Theta\}$ (see item 1 of Proposition 4.5) tells us that $ML(X)$ is the smallest amount of money (m) that makes $Y+X+m$ less risk then the initial portfolio $Y \in \mathcal{X}$, according to all $\rho \in \Theta$. Therefore, we can say that $ML(X)$ is the cost of making X a robust risk reducer of Y according to the class Θ.

The analogous reasoning holds for $\mathcal{W}$, which is also a robust criterion. First, notice that $X \in \mathcal{W}$ if and only if $\rho(X) \geqslant 0$ for all $\rho \in \Theta$. In addition, we also have that the risk measure $EL(\cdot)$ is robust in the sense that $\rho(X + EL(X)) \geqslant \rho(0)$ for all $\rho \in \Theta$[3]. If we consider the initial portfolio as being $Y = 0$, then $\mathcal{W}$ specifies all the positions X such that $X + Y$ is riskier than Y considering all $\rho \in \Theta$. Moreover, the fact that $EL(X) = \sup\{m \in \mathbb{R} : \rho(Y) \leqslant \rho(Y + X + m),\ \forall \rho \in \Theta\}$ tells us that $EL(X)$ is the larger amount of money (m) that makes $Y + X + m$ more risky than the initial portfolio, Y, considering all $\rho \in \Theta$. Therefore, we can say that $EL(X)$ is the threshold above which X is not a robust risk increaser of Y according to the class Θ.

4.4 ROBUST CERTAINTY EQUIVALENTS

Definition 4.5. *(Föllmer and Schied (2016) - Definition 2.35) A function $u : \mathbb{R} \to \mathbb{R}$ is called a* **utility function** *if it is strictly concave and strictly increasing. We denote the set of utility functions as $\mathcal{U}$.*

Remark 4.6. There are several links between utility functions and risk measures. For instance, an utility function $u \in \mathcal{U}$ induces a monetary acceptance set as $\mathcal{A} = \{X \in \mathcal{X} : \mathbf{E}[u(X)] \geqslant x_0\}$, for some $x_0 \in \mathbb{R}$. Therefore, $u \in \mathcal{U}$ also induces a risk measure $\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : \mathbf{E}[u(X+m)] \geqslant x_0\}$. For a detailed study of this risk, measure see Föllmer and Schied (2002). Moreover, one will find additional interesting connections between risk measures and utility functions in Delbaen (2011), Ben-Tal and Teboulle (2007), and Tsanakas and Desli (2003).

Proposition 4.6. *The following holds for any utility function $u \in \mathcal{U}$:*

1. *If $X \leqslant Y$ and $\mathbf{P}(X < Y) > 0$, then $\mathbf{E}[u(X)] < \mathbf{E}[u(Y)]$ for all $X, Y \in \mathcal{X}$.*
2. *It holds that $\mathbf{E}[u(\alpha X + (1-\alpha)Y)] > \alpha E[u(X)] + (1-\alpha)\,\mathbf{E}[u(Y)]$ for all $X, Y \in \mathcal{X}$ and $\alpha \in (0,1)$.*

The above results are direct consequences of utility functions being strictly increasing and strictly concave. These are analogs of the property of monotonicity and convexity, respectively, for risk measures. In

[3]To see this is true notice that, according to Proposition 4.3 it holds that $EL(X) = \inf_{p\in(0,1]} \mathrm{AVaR}_p(X) \leqslant \inf_{\rho\in\Theta} \rho(X)$. In turn, this implies that $\rho(X + EL(X)) = \rho(X) - EL(X) \geqslant 0$.

addition, since utility functions $u \in \mathcal{U}$ are strictly increasing and continuous, the Intermediate Value Theorem implies that, for any $X \in \mathcal{X}$, there exists $m \in \mathbb{R}$ such that $u(m) = \mathbf{E}[u(X)]$.

Definition 4.6. *Let $u \in \mathcal{U}$ and $X \in \mathcal{X}$. The* **certainty equivalent** *of X according to an agent with utility function u and endowed with wealth $w \in \mathbb{R}$ is given by $c_u(X, w) = u^{-1}(\mathbf{E}[u(X + w)]) - w$.*

Remark 4.7. The certainty equivalent $c_u(X, w)$ comes from comparing the well-being of an agent in two different circumstances, each of which keeps the individual's initial wealth. An alternative approach would be to consider the certainty equivalent of a position X from the perspective of an agent that must choose between, on the one hand, keeping its initial wealth and an additional amount of money (the last corresponding to the alternative certainty equivalent) and, on the other hand, having only the financial position X. In this case, the certainty equivalent would be defined as the quantity $m_c(X, w)$ such that $u(w + m_c(X, w)) = \mathbf{E}[u(X)]$. Notice, however, that $m_c(X, w)$ is just a translation away from the certainty equivalent with respect to individuals with no initial wealth, because $m_c(X, w) = c_u(X, 0) - w$.

The following results will be used as a benchmark for the developments we propose. For more details see, for instance, Hennessy and Lapan (2006), Chapter 2 of Föllmer and Schied (2016), and Chapter 6 of Mas-Colell et al. (1995).

Proposition 4.7. *Let $X, Y \in \mathcal{X}$, $u \in \mathcal{U}$, and $w \in \mathbb{R}$. Then it holds that*

1. *It holds that $c_u(b, w) = b$ for all $b, w \in \mathbb{R}$.*
2. *If $X \leqslant Y$ and $\mathbf{P}(X < Y) > 0$, then $c_u(X, w) < c_u(Y, w)$ for all $w \in \mathbb{R}$.*
3. *Assume that u and u^{-1} are differentiable, and the map $w \mapsto \mathbf{E}[u(X+w)]$ is differentiable for all $X \in \mathcal{X}$. Then, the map $w \mapsto c_u(X, w)$ is differentiable and, for all $X \in \mathcal{X}$ and $w \in \mathbb{R}$, it holds that*
$$\frac{\partial c_u(X, w)}{\partial w} \geqslant 0 \quad \textit{if and only if} \quad \frac{\partial\, \mathbf{E}[u(X + w)]}{\partial w} \geqslant u'(c_u(X, w) + w). \tag{4.7}$$
4. *If there exists $\lambda_0 \in [0, 1]$ and $m_0 \in \mathbb{R}$ such that $\mathbf{E}[u(\lambda_0 X + (1 - \lambda_0)Y) + m_0] \geqslant \max\{\mathbf{E}[u(X + m_0)], \mathbf{E}[u(Y + m_0)]\}$, then*
$$c_u(\lambda_0 X + (1 - \lambda_0)Y, w_0) \geqslant \lambda c_u(X, w_0) + (1 - \lambda_0)c_u(Y, w_0). \tag{4.8}$$

In addition, $c_u(X, w)$ admits the following representations:
$$c_u(X, w) = \inf\{m \in \mathbb{R} : u(m + w) \geqslant \mathbf{E}[u(X + w)]\} = \sup\{m \in \mathbb{R} : u(m + w) \leqslant \mathbf{E}[u(X + w)]\}. \tag{4.9}$$

Proof. Items 1 and 2 follow from Definition 4.6. To prove Item 3, notice that
$$\frac{\partial c_u(X, w)}{\partial w} = \frac{\partial u^{-1}(\mathbf{E}[u(X + w)])}{\partial w} - 1 = \frac{\frac{\partial}{\partial w}\, \mathbf{E}[u(X + w)]}{u'(c_u(X, w) + w)} - 1. \tag{4.10}$$

Then, item 3 follows for $u'(x) > 0$ for all $x \in \mathbb{R}$ and because $(\partial\, \mathbf{E}[u(X + w)]/\partial w) > 0$ for all $X \in \mathcal{X}$ and $w \in \mathbb{R}$. To prove item 4, notice that we have (under the assumptions of the proposition) $u^{-1}(\mathbf{E}[u(\lambda_0 X + (1 - \lambda_0)Y + m_0)]) \geqslant \max\{u^{-1}(\mathbf{E}[u(X + m_0)]), u^{-1}(\mathbf{E}[u(Y + m_0)])\}$. In turn, this implies that
$$u^{-1}(\mathbf{E}[u(\lambda_0 X + (1 - \lambda_0)Y + m_0)]) \geqslant \lambda_0 u^{-1}(\mathbf{E}[u(X + m_0)]) + (1 - \lambda_0)u^{-1}(\mathbf{E}[u(Y + m_0)]), \tag{4.11}$$

To prove eq. (4.9), it suffices to notice that

$$c_u(X,w) = \inf\{m \in \mathbb{R} : m \geqslant u^{-1}(\mathbf{E}[u(X+w)]) - w\} = \sup\{m \in \mathbb{R} : m \leqslant u^{-1}(\mathbf{E}[u(X+w)]) - w\}. \qquad \square$$

Remark 4.8. Individuals with preferences such that $w \mapsto c_u(X,w)$ is increasing become willing to take more risk when their initial wealth increases. When this happens, they fear less the bad outcomes from X and, therefore, only will be willing to give up the possibility of the good outcomes for a high certainty equivalent.

Remark 4.9. Item 4 gives a very strong condition under which we can compare the certainty equivalent of a convex combination to the convex combination of the certainty equivalents. For a deeper study of the curvature of c_u see Hennessy and Lapan (2006).

Remark 4.10. Notice that the equivalence of eq. (4.9) is similar (in spirit) to that in Proposition 4.4.

Proposition 4.8. *For any two random variables $X, Y \in \mathcal{X}$ the following conditions are equivalent to $X \preceq_{sd} Y$:*

1. *(Föllmer and Schied (2016) - Theorem 2.57)* $\mathbf{E}[u(X)] \leqslant \mathbf{E}[u(Y)]$ *for all* $u \in \mathcal{U}$.
2. $c_u(X,0) \leqslant c_u(Y,0)$ *for all* $u \in \mathcal{U}$.

In addition, it holds that:

3. $\inf\{m \in \mathbb{R} : 0 \preceq_{sd} X + m\} = -\inf_{u \in \mathcal{U}} c_u(X,0) = -\operatorname{ess\,inf} X$.
4. $\sup\{m \in \mathbb{R} : X + m \preceq_{sd} 0\} = -\sup_{u \in \mathcal{U}} c_u(X,0) = \mathbf{E}[-X]$.

Proof. The condition $X \preceq_{sd} Y$ is equivalent to item 1 according to Föllmer and Schied (2016). Then, item 1 implies item 2 because $u^{-1}(\cdot)$ is strictly increasing. Item 2 implies item 1 by a similar argument, which concludes the proof for the first part of the proposition.

Proposition 4.3 tells us that $\inf\{m \in \mathbb{R} : 0 \preceq_{sd} X + m\} = -\operatorname{ess\,inf} X$. Therefore, it remains to show that $\inf_{u \in \mathcal{U}} c_u(X,0) = \operatorname{ess\,inf} X$. But notice that these are two representations of $\sup\{m \in \mathbb{R} : m \preceq_{sd} X\}$. On the one hand, we can use Proposition 4.3 to write

$$\begin{aligned} \sup\{m \in \mathbb{R} : m \preceq_{sd} X\} &= \sup\{m \in \mathbb{R} : m \leqslant -\operatorname{AVaR}_p(X),\ \forall p \in (0,1]\} \\ &= -\sup_{p \in (0,1]} \operatorname{AVaR}_p(X) = \operatorname{ess\,inf}(X). \end{aligned}$$

On the other, we have

$$\begin{aligned} \sup\{m \in \mathbb{R} : m \preceq_{sd} X\} &= \sup\{m \in \mathbb{R} : u(m) \leqslant \mathbf{E}[u(X)],\ \forall u \in \mathcal{U}\} \\ &= \sup\left\{m \in \mathbb{R} : m \leqslant \inf_{u \in \mathcal{U}} u^{-1}(E[u(X)])\right\} \\ &= \inf_{u \in \mathcal{U}} c_u(X), \end{aligned}$$

which concludes the proof of item 3. To prove item 4, notice that Proposition 4.3 implies that $\sup\{m \in \mathbb{R} : X + m \preceq_{sd} 0\} = \mathbf{E}[-X],\ \forall X \in \mathcal{X}$. Therefore, it remains to show that $\sup_{u \in \mathcal{U}} c_u(X,0) = \mathbf{E}[X]$. On the one hand, Jensen's inequality implies that $c_u(X,0) \leqslant \mathbf{E}[X]$ and; on the other hand, notice that $\mathbf{E}[X] = c_u(X,0)$ when $u(x) = x$ for $x \in \mathbb{R}$, which concludes the proof for item 4. $\square$

In accordance with the right-hand side of eq. (4.9) of Proposition 4.7, we propose the following definition:

Definition 4.7. *Let $u \in \mathcal{U}$ and $X \in \mathcal{X}$. The* **certainty equivalent** *of X according to an agent with utility function u and initial financial position $Y \in \mathcal{X}$ is given by*

$$C_u(X;Y) = \sup\{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \leqslant \mathbf{E}[u(X+Y)]\}. \tag{4.12}$$

The quantity $C_u(X;Y)$ establishes how much money (m) is necessary to make $\mathbf{E}[u(Y+m)] = \mathbf{E}[u(X+Y)]$. If $C_u(X;Y) > 0$, then the composite portfolio $X+Y$ is preferred to Y by individuals with utility u, and the quantity $C_u(X;Y)$ gives us a "monetary" measure of how much—the maximum value, actually—these individuals would be willing to pay to add X into their initial position Y. If $C_u(X;Y) < 0$, then $\mathbf{E}[u(X+Y)] < \mathbf{E}[u(Y)]$, so that individuals with utility u would demand a compensation to add X to their initial portfolio Y. Therefore, when $C_u(X;Y) < 0$ it can be interpreted as the smallest compensation that would make an individual with utility function u willing to add X to its initial position Y.

Proposition 4.9. *The certainty equivalent of X according to an agent with utility function u and initial financial position $Y \in \mathcal{X}$ admits the following representation:*

$$C_u(X,Y) = \inf\{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \geqslant \mathbf{E}[u(X+Y)]\}. \tag{4.13}$$

Proof. To prove eq. (4.13), let's denote

$$C_u^{\inf}(X,Y) = \inf\{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \geqslant \mathbf{E}[u(X+Y)]\}. \tag{4.14}$$

Notice that, if $C_u(X,Y) < C_u^{\inf}(X,Y)$, then any $m_0 \in (C_u(X,Y) < C_u^{\inf}(X,Y))$ satisfies $\mathbf{E}[u(Y+m_0)] > \mathbf{E}[u(Y+X)]$—because $m_0 > C_u(X,Y)$)—and $\mathbf{E}[u(Y+m_0)] < \mathbf{E}[u(Y+X)]$—because $m_0 < C_u^{\inf}(X,Y)$. This is absurd and, therefore, $C_u(X,Y) \geqslant C_u^{\inf}(X,Y)$. Now, assume that $C_u(X,Y) > C_u^{\inf}(X,Y)$. Then, for any $m_0 \in (C_u^{\inf}(X,Y), C_u(X,Y))$ there exists

$$\begin{aligned} &m_1 \in \{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \geqslant \mathbf{E}[u(X+Y)]\} \quad \text{and} \\ &m_2 \in \{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \leqslant \mathbf{E}[u(X+Y)]\} \end{aligned}$$

such that $m_1 < m_0 < m_2$. But this is absurd because the strict increasingness of u implies that $m_1 \geqslant m_2$. □

Notice that the result in Proposition 4.9 (as well as that in eq. (4.9)) is similar to that established in Proposition 4.4. We should notice, however, that this kind of equivalence does not hold in the robust framework, as illustrated in Proposition 4.5 and in the discussion thereafter.

The next result establishes basic facts that will be used freely throughout the text.

Proposition 4.10. *For $u \in \mathcal{U}$ and $X, Y \in \mathcal{X}$, consider the following set*

$$\mathcal{M}_u := \{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \leqslant \mathbf{E}[u(X+Y)]\}. \tag{4.15}$$

The following facts hold:

1. *$\mathcal{M}_u$ is a non-empty interval unbounded from below.*

2. $\sup \mathcal{M}_u \in \mathbb{R}$.

3. $C_u(X,Y) \in \mathcal{M}_u$.

As a consequence of the above items, it holds that $\mathcal{M}_u = (-\infty, \sup \mathcal{M}_u]$.

Proof. Let's begin by proving item 1. To show that $\mathcal{M}_u$ is non-empty, notice that $Y + m \leqslant X + Y$ for any $m \leqslant -\|X\|_\infty \in \mathbb{R}$. For such m it holds that $\mathbf{E}[u(X+Y)] - \mathbf{E}[u(Y+m)] \geqslant 0$, which implies that $m \in \mathcal{M}_u$. To show that $\mathcal{M}_u$ is an interval unbounded from below, take $m \in \mathcal{M}_u$ and notice that, since the map $m \mapsto \mathbf{E}[u(X+Y)] - \mathbf{E}[u(Y+m)]$ is decreasing, it holds that $m' \in \mathcal{M}_u$ whenever $m' < m$.

For item 2, it suffices to show that $\mathcal{M}_u$ is bounded above. For that end, notice that, if $m_0 > \|X\|_\infty$, then $\mathbf{E}[u(X+Y)] - \mathbf{E}[u(m_0+Y)] < 0$, i.e., $m_0 \notin \mathcal{M}_u$. Since $\mathcal{M}_u$ is an interval unbounded from below, we conclude that m_0 is an upper bound of $\mathcal{M}_u$.

For item 3, it suffices to show that the map $m \mapsto \mathbf{E}[u(Y+m)]$ is continuous. To that end, notice that the functions $u \in \mathcal{U}$ are continuous because they are real-valued and strictly concave. Therefore, for any sequence $\{m_n\}_{n\geqslant 1} \subset \mathbb{R}$ such that $m_n \to m$, it holds that $u(Y+m_n) \to u(Y+m)$ $\mathbf{P}$-a.s. Since $|u(Y+m_n)| \leqslant |u(\|Y\|_\infty + \sup_{n\geqslant 1} m_n)|$, we can apply the Dominated Convergence Theorem to conclude that $\mathbf{E}[u(Y+m_n)] \to \mathbf{E}[u(Y+m)]$, which concludes the proof. □

The next result is the counterpart of Proposition 4.7 for agents with initial financial positions.

Proposition 4.11. *For* $u \in \mathcal{U}$ *and* $X, Y \in \mathcal{X}$. *Then* $C_u(X,Y)$ *admits the following properties:*

1. $C_u(X,w) = c_u(X,w)$ *for all* $X, \in \mathcal{X}$ *and* $w \in \mathbb{R}$. *In particular, it holds that* $C_u(b,w) = b$ *for all* $b, w \in \mathbb{R}$.

2. *If* $X \leqslant Z$ *and* $\mathbf{P}(X < Z) > 0$, *then* $C_u(X,Y) < C_u(Z,Y)$.

3.

Proof. Item 1 follows from the definitions, and item 2 follows for, under the hypothesis of the statement, it holds that

$$\{m \in \mathbb{R} : \mathbf{E}[u(m+Y)] \leqslant \mathbf{E}[u(Z+Y)]\} \supsetneq \{m \in \mathbb{R} : \mathbf{E}[u(m+Y)] \leqslant \mathbf{E}[u(X+Y)]\}.$$ □

Definition 4.8. *Consider the following definitions:*

1. *The* **robust upper certainty equivalent** *of* X *considering an initial financial position* $Y \in \mathcal{X}$ *is given by*
$$\widetilde{C}(X,Y) = \sup\{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \leqslant \mathbf{E}[u(X+Y)],\ \forall u \in \mathcal{U}\}. \tag{4.16}$$

2. *The* **robust lower certainty equivalent** *of* X *considering an initial financial position* $Y \in \mathcal{X}$ *is given by*
$$\underset{\sim}{C}(X,Y) = \inf\{m \in \mathbb{R} : \mathbf{E}[u(Y+m)] \geqslant \mathbf{E}[u(X+Y)],\ \forall u \in \mathcal{U}\}. \tag{4.17}$$

The criteria embodied in the sets of eq. (4.16) and eq. (4.17) are robust in the sense they are agreed upon by all agents whose preferences are represented by a utility function $u \in \mathcal{U}$. The link between the class $\mathcal{U}$ and $\preceq_{sd}$ allows us to recover the characterizations provided in Proposition 4.12.

Proposition 4.12. *The following representations hold:*

1. $\widetilde{C}(X,Y) = \sup\{m \in \mathbb{R} : Y + m \preceq_{sd} X + Y\}$.

2. $\underset{\sim}{C}(X,Y) = \inf\{m \in \mathbb{R} : X + Y \preceq_{sd} Y + m\}$.

In addition, if we consider $\mathcal{A}$ and $\mathcal{W}$ as defined in Proposition 4.5 and set $Y = 0$, then the following holds:

3. $\widetilde{C}(X,0) = -\rho_{\mathcal{A}}(X) = \operatorname{ess\,inf}(X)$.

4. $\underset{\sim}{C}(X,0) = -\psi_{\mathcal{W}}(X) = \mathbf{E}[X]$.

Proof. Items 1 and 2 follow directly from Definition 4.8 and item 1 of Proposition 4.8. For item 3, notice that, according to item 1 of the current result, it holds that

$$\begin{aligned}\widetilde{C}(X,0) &= \sup\{m \in \mathbb{R} : m \preceq_{sd} X\} = \sup\{m \in \mathbb{R} : 0 \preceq_{sd} X - m\} \\ &= -\inf\{-m \in \mathbb{R} : 0 \preceq_{sd} X - m\} = -\inf\{m \in \mathbb{R} : 0 \preceq_{sd} X + m\} \\ &= -\rho_{\mathcal{A}}(X).\end{aligned}$$

The proof of item 4 follows the same lines. □

Hereafter we give more focus to the robust upper certainty equivalent because it is directly related to the risk measure $\rho_{\mathcal{A}}$ of Proposition 4.5 and to the cost to robust risk reduction that we will study in the next section. However, it is worth comparing $\widetilde{C}(X,0)$ and $\underset{\sim}{C}(X,0)$ under the light of Proposition 4.12.

Remark 4.11. Notice that $\widetilde{C}(X,0)$ is defined through the set

$$\begin{aligned}\{m \in \mathbb{R} : m \preceq_{sd} X\} &= \{m \in \mathbb{R} : u(m) \leqslant \mathbf{E}[u(X)],\ \forall u \in \mathcal{U}\} \\ &= (-\infty, \inf_{u\in\mathcal{U}} c_u(X)].\end{aligned}$$

It happens, however, that there is no limit to the degree of risk aversion that one can find within the set of risk-averse agents. In fact, we have shown in item 3 fo Proposition 4.8 that $\inf_{u\in\mathcal{U}} c_u(X) = \operatorname{ess\,inf}(X)$ and, for this reason, it follows that

$$\widetilde{C}(X,0) = \sup \inf_{u\in\mathcal{U}} c_u(X) = \operatorname{ess\,inf}(X). \tag{4.18}$$

Therefore, it is exactly because of the possibility of extreme risk aversion that the robust upper certainty equivalents are related to the robust risk measure $\rho_{\mathcal{A}}$, as defined in Proposition 4.5.

We should highlight that the relation between $\underset{\sim}{C}(X,0)$ and $\psi_{\mathcal{W}}(X)$ does not pass through agents with extreme risk aversion. It passes through the agents with the smallest risk aversion within the class of risk-averse agents. We have

$$\begin{aligned}\underset{\sim}{C}(X,0) &= \inf\{m \in \mathbb{R} : u(m) \geqslant \mathbf{E}[u(X)],\ \forall u \in \mathcal{U}\} \\ &= \sup_{u\in\mathcal{U}} c_u(X).\end{aligned}$$

It happens that, inside the class of risk-averse individuals, the larger certainty equivalent is the expected value. In this sense, it is because risk-averse agents have small certainty equivalents (in comparison to risk seeker agents) that the robust lower certainty equivalent does not attain "extreme values" and, therefore, does not relate to measures of tail risk[4].

[4]Notice that the reasoning conducted in this remark is similar to that of remark 4.5.

Proposition 4.13. *The following properties hold for robust certainty equivalents*

1. $\widetilde{C}(b, Y) = b$ *for any* $b \in \mathbb{R}$

2. *If* $X \leqslant Z$ *and* $\mathbf{P}(X < Z) > 0$*, then* $\widetilde{C}(X, Y) \leqslant \widetilde{C}(Z, Y)$*, for all* $Y \in \mathcal{X}$*.*

In addition, $\widetilde{C}$ *admits the following alternative representation:*

$$\widetilde{C}(X, Y) = \inf_{u \in \mathcal{U}} C_u(X, Y), \quad \forall X, Y \in \mathcal{X}. \tag{4.19}$$

Proof. The first statement is a direct consequence of the definitions. To establish the last claim, let's define the function $f_u(m) = \mathbf{E}[u(X + Y)] - \mathbf{E}[u(Y + m)]$, which is strictly decreasing and continuous. Also, notice that

$$\begin{aligned} \widetilde{C}(X, Y) &= \sup\{m \in \mathbb{R} : \inf_{u \in \mathcal{U}} f_u(m) \geqslant 0\} \\ &= \sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \end{aligned}$$

Therefore, we must show that

$$\sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\} = \inf_{u \in \mathcal{U}} C_u(X, Y), \quad \forall X, Y \in \mathcal{X}. \tag{4.20}$$

Notice that

$$\cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\} \subseteq \{m \in \mathbb{R} : f_u(m) \geqslant 0\}, \quad \forall u \in \mathcal{U}. \tag{4.21}$$

Therefore,

$$\sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\} \leqslant \sup\{m \in \mathbb{R} : f_u(m) \geqslant 0\}, \quad \forall u \in \mathcal{U}. \tag{4.22}$$

In turn, this implies that

$$\sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\} \leqslant \inf_{u \in \mathcal{U}} \sup\{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \tag{4.23}$$

To establish the converse inequality, notice that eq. (4.23) implies that $\sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\}$ is a lower bound for

$$\{\sup\{m \in \mathbb{R} : f_u(m) \geqslant 0\} : u \in \mathcal{U}\} = \{C_u(X, Y) : u \in \mathcal{U}\}. \tag{4.24}$$

Therefore, it remains to show that $\sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\}$ is the largest lower bound of $\{C_u(X, Y) : u \in \mathcal{U}\}$. According to Proposition 4.10, it holds that $\{m \in \mathbb{R} : f_u(m) \geqslant 0\} = (-\infty, c_u(X, Y)]$. Therefore, if $m_0 \leqslant C_u(X, Y)$, it holds that $f_u(m_0) \geqslant 0$. Therefore, if m_0 is a lower bound of $\{C_u(X, Y) : u \in \mathcal{U}\}$, it follows that

$$\begin{aligned} m_0 &\in \{m \in \mathbb{R} : f_u(m) \geqslant 0,\ u \in \mathcal{U}\} \\ &= \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \end{aligned}$$

In turn, it holds that $m_0 \leqslant \sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\}$, from which we conclude that

$$\sup \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : f_u(m) \geqslant 0\} = \inf_{u \in \mathcal{U}} \sup\{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \tag{4.25}$$

□

Remark 4.12. The above representations show that for any $m > \widetilde{C}(X,Y)$ there is a risk-averse agent that prefers $Y + m$ to $X + Y$. Analogously, for any $m < \underset{\sim}{C}(X,Y)$, there is a risk-averse agent that prefers $X + Y$ to $Y + m$.

4.5 ROBUST RISK REDUCERS

In this section, we generalize the notion of robust risk reducers that we introduced in Section 4.3 and show how this generalized notion relates to that of robust certainty equivalents of Section 4.4.

Definition 4.9. *We say that a position X is a* **robust risk reducer** *of Y if $Y \preceq_{sd} X + Y$. The set of robust risk reducers of Y is given by*

$$\mathcal{A}_Y = \{X \in \mathcal{X} : Y \preceq_{sd} X + Y\}. \tag{4.26}$$

Notice that the above sets represent an acceptability criterion that is consistent with the preferences of all risk-averse agents. Also, a similar concept of risk reduction was considered in Cheung et al. (2014). In that paper, the authors study conditions for risk reduction in the case that $(-X, Y)$ is comonotonic.

Proposition 4.14. *The set $\mathcal{A}_Y$ is monotone, normalized, convex, and $\|\cdot\|_\infty$-closed. In addition, $\mathcal{A}_Y$ fulfills the following properties:*

1. *If (X^c, Y) and (Z^c, Y) are comonotonic and $X^c \in \mathcal{A}_Y$, then $Z^c \in \mathcal{A}_Y$ if $X^c \preceq_{sd} Z^c$.*
2. *If (X, Y) is comonotonic and $X \in \mathcal{A}_Y$, then $X \geqslant 0$.*

Proof. The monotonicity property follows for, if $X \leqslant Z$, it holds that $X + Y \leqslant Z + Y$. In turn, this implies that $X + Y \preceq_{sd} Z + Y$ and, therefore, the transitivity of $\preceq_{sd}$ implies that, if $X \in \mathcal{A}_Y$, then $Z \in \mathcal{A}_Y$. The normalization of $\mathcal{A}_Y$ comes directly from its definition. To see that $\mathcal{A}_Y$ is convex, take $X, Z \in \mathcal{A}_Y$ and notice that the strict concavity of utility functions implies that

$$\begin{aligned} \mathbf{E}[u(\lambda X + (1-\lambda)Z) + Y] &> \lambda\, \mathbf{E}[u(X+Y)] + (1-\lambda)\, \mathbf{E}[u(Z+Y)] \\ &\geqslant \mathbf{E}[u(Y)], \end{aligned}$$

where the last inequality follows for $X, Z \in \mathcal{A}_Y$.

To prove that $\mathcal{A}_Y$ is $\|\cdot\|_\infty$-closed, take a sequence $\{X_i : i \geqslant 1\} \subset \mathcal{A}_Y$ such that $X_i \to X$ in the $\|\cdot\|_\infty$ topology. Then we have $Y \preceq_{sd} X_i + Y$ for all $i \geqslant 1$ and $X_i + Y \xrightarrow{p} X + Y$. Therefore, Proposition 4.1 implies that $Y \preceq_{sd} X + Y$, from which we conclude that $X \in \mathcal{A}_Y$ and that $\mathcal{A}_Y$ is closed with respect to the $\|\cdot\|_\infty$ topology.

For the property in item 1, it suffices to show that if (X, Y) and (Z, Y) are comonotonic, then $X \preceq_{sd} Z$ implies $X + Y \preceq_{sd} Z + Y$. This follows from the characterization of $\preceq_{sd}$ through AVaR_p (see item 2 of Proposition 4.3) and the fact that AVaR_p is comonotonic additive, for all $p \in (0, 1]$.

To prove item 2 notice that, if (X,Y) is comonotonic, then Proposition 4.3 implies that

$$\begin{aligned} X \in \mathcal{A}_Y &\Leftrightarrow Y \preceq_{sd} X+Y \\ &\Leftrightarrow \mathrm{AVaR}_p(X+Y) \leqslant \mathrm{AVaR}_p(Y), \quad \forall p \in (0,1] \\ &\Leftrightarrow \mathrm{AVaR}_p(X) \leqslant 0, \quad \forall p \in (0,1] \\ &\Leftrightarrow \sup_{p\in(0,1]} \mathrm{AVaR}_p(X) \leqslant 0 \\ &\Leftrightarrow \operatorname{ess\,inf}(X) \geqslant 0, \end{aligned}$$

where the last equivalence relation follows from remark 4.3. □

Remark 4.13. Since the acceptability of a given position X according to $\mathcal{A}_Y$ is influenced by the dependence structure between X and Y, the acceptance set $\mathcal{A}_Y$ is not SSD consistent in general (see Definition 2.12). In this regard, Proposition 4.14 illustrates the fact that, if we can fix the dependence structure between the original portfolio and the incremental positions, then the set $\mathcal{A}_Y$ becomes SSD-consistent. This is the case of item 1 of Proposition 4.14, where we consider two incremental random variables X^c and Z^c that have the same dependence with Y—more specifically, (X^c,Y) and (Z^c,Y) must be comonotonic—then the set $\mathcal{A}_Y$ becomes SSD-consistent for comonotonic random variables.

Remark 4.14. In studying the family $\{\mathcal{A}_Y\}_{Y\in\mathcal{X}}$, one could consider the adequacy of requiring that $Y \in \mathcal{A}_Y$ for all $Y \in \mathcal{X}$. This was considered, for instance, in Canna et al. (2020) in the context of capital allocation and risk contribution rules. In the present context, however, the set $\mathcal{A}_Y$ represents a quite restrictive criterion and, as a consequence, the property $Y \in \mathcal{A}_Y$ for all $Y \in \mathcal{X}$ would imply, in view of item 2 of Proposition 4.14, that $\mathcal{A}_Y = \mathcal{X}^+$.

Remark 4.15. As another robust criterion, one could consider the set

$$\mathcal{W}_Y = \{X \in \mathcal{X} : X+Y \preceq_{sd} Y\}, \tag{4.27}$$

which contains the positions $X \in \mathcal{X}$ for which all risk averse agents agree that X increases the risk of the initial position, Y. We will confine our comments on $\mathcal{W}_Y$ to a few remarks because our focus is not on robust risk increase. Nonetheless, it is worth pointing that $\mathcal{W}_Y$ is also normalized and $\|\cdot\|_\infty$-closed. In addition, $\mathcal{W}_Y$ is decreasing as $X \leqslant Y$ and $Y \in \mathcal{W}_Y$ implies $X \in \mathcal{W}_Y$. Moreover, considering the characterization of $\preceq_{sd}$ through utility functions as in Proposition 4.8, it is easy to see that $\mathcal{W}_Y$ is not convex. Also, since $\preceq_{sd}$ is not a complete order on $\mathcal{X}$, it holds that $\mathcal{W}_Y$ and $\mathcal{A}_Y$ does not form a partition of $\mathbb{R}$.

Definition 4.10. *We define the* **cost to robust risk reduction** *as*

$$\widetilde{\Lambda}(X,Y) = \sup_{\rho\in\Theta} \rho(X+Y) - \rho(Y), \quad \forall X,Y \in \mathcal{X}. \tag{4.28}$$

The above functional takes a financial position $Y \in \mathcal{X}$ as a starting point and provides a very conservative measure of how much the risk would vary if one adds to Y an additional financial position $X \in \mathcal{X}$. Notice that, if $\widetilde{\Lambda}(X,Y) < 0$, then the portfolio $X+Y$ is less risky than the original portfolio Y according to all monetary SSD-consistent risk measures. If, on the other hand, it holds that $\widetilde{\Lambda}(X,Y) > 0$, then there is at least one risk measure $\rho \in \Theta$ such that $\rho(X+Y) > \rho(Y)$.

Proposition 4.15. *Let $\widetilde{\Lambda}(X,Y)$ be defined as in Definition 4.10 for $X,Y \in \mathcal{X}$. This functional fulfills the following properties:*

1. *If $X \leqslant Z$, then $\widetilde{\Lambda}(Z,Y) \leqslant \widetilde{\Lambda}(X,Y)$ for all $X,Z \in \mathcal{X}$.*
2. *It holds that $\widetilde{\Lambda}(X+m,Y) = \widetilde{\Lambda}(X,Y) - m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$.*
3. *The map $X \mapsto \widetilde{\Lambda}(X,Y)$ is Lipschitz continuous with respect to $\|\cdot\|_\infty$ norm.*
4.

Proof. Items 1 and 2 are direct consequences of the respective properties being fulfilled by the risk measures $\rho \in \Theta$. Item 3 follows from Lemma 4.3 in Föllmer and Schied (2016). ☐

Proposition 4.14 and 4.15 imply that the set $\mathcal{A}_Y$ and the functional $\widetilde{\Lambda}(\cdot,Y)$ are monetary, for all $Y \in \mathcal{X}$. Therefore, Theorem 4.1 implies that the functional $\rho_{\mathcal{A}_Y}(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\}$ induced by $\mathcal{A}_Y$ and the acceptance set $\{X \in \mathcal{X} : \widetilde{\Lambda}(X,Y) \leqslant 0\}$ induced by $\widetilde{\Lambda}(\cdot,Y)$ are also monetary, for all $Y \in \mathcal{X}$. In the next result, we show that $\mathcal{A}_Y$ and $\widetilde{\Lambda}(\cdot,Y)$ are linked as each of them is induced by the other.

Theorem 4.2. *Let $\mathcal{A}_Y$ and $\widetilde{\Lambda}$ be defined as in Definition 4.9 and Definition 4.10. It holds that*

1. *$\widetilde{\Lambda}(X,Y) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\}$.*
2. *$\mathcal{A}_Y = \{X \in \mathcal{X} : \widetilde{\Lambda}(X,Y) \leqslant 0\}$.*

Proof. For item 1, notice that $X + m \in \mathcal{A}_Y$ if and only if $Y \preceq_{sd} X + Y + m$. By Proposition 4.3, this condition is equivalent to $m \geqslant \rho(X+Y) - \rho(Y)$ for all $\rho \in \Theta$. Therefore,

$$\inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\} = \inf\left\{m \in \mathbb{R} : m \geqslant \sup_{\rho\in\Theta} \rho(X+Y) - \rho(Y)\right\} = \widetilde{\Lambda}(X,Y).$$

For item 2, it is immediate that $\widetilde{\Lambda}(X,Y) \leqslant 0$ implies $\rho(X+Y) \leqslant \rho(Y)$ for all $\rho \in \Theta$. In turn, this implies that $Y \preceq_{sd} X + Y$, from which we conclude that $\{X \in \mathcal{X}\, \widetilde{\Lambda}(X,Y) \leqslant 0\} \subseteq \mathcal{A}_Y$. Conversely, if $X \in \mathcal{A}_Y$, i.e., $Y \preceq_{sd} X + Y$, then $\widetilde{\Lambda}(X,Y) \leqslant 0$ by definition. This concludes the proof of item 2. ☐

The sign of $\widetilde{\Lambda}(X,Y) = \sup_{\rho\in\Theta} \rho(X+Y) - \rho(Y)$ provides a very decisive assessment of the effect of adding positions into an original portfolio. In addition, item 1 of Theorem 4.2 shows that the quantity $\widetilde{\Lambda}(X,Y)$ gives the smallest amount of money that makes X a robust risk reducer for Y, meaning that $\rho(X + \widetilde{\Lambda}(X,Y) + Y) \leqslant \rho(Y)$ for all $\rho \in \Theta$.

Remark 4.16. It is in order to compare $\widetilde{\Lambda}(\cdot,Y)$ with the marginal rule for capital allocation studied in Canna et al. (2020). The marginal rule is based on a single monetary and normalized risk measure, let's denote it by ρ, and on an initial position Y. The marginal rule is defined as $\rho_Y(X) := \rho(Y) - \rho(Y - X)$, for all $X \in \mathcal{X}$. This functional is designed to measure how much of the risk of a portfolio Y is due to X, where X is seen as a sub-portfolio of Y. This is a distinctive problem from what we are studying in the present paper. Hence, the functional $\rho_Y(\cdot)$ studied in Canna et al. (2020) and $\widetilde{\Lambda}(\cdot,Y)$ studied here should be seen as complementary.

Theorem 4.3. *The functional $\widetilde{\Lambda}$ as defined in Definition 4.10 admits the following representations:*

1. *$\widetilde{\Lambda}(X,Y) = \sup_{p\in(0,1]} \mathrm{AVaR}_p(X+Y) - \mathrm{AVaR}_p(Y)$.*

2. $\widetilde{\Lambda}(X,Y) = -\widetilde{C}(X,Y)$.

Proof. In view of item 1 of Theorem 4.2, we can prove item 1 of the present result by showing that

$$\sup_{p\in(0,1]} \text{AVaR}_p(X+Y) - \text{AVaR}_p(Y) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\}. \tag{4.29}$$

The proof is essentially the same as that used in Theorem 4.2. Notice that, by Proposition 4.3 we have $X + m \in \mathcal{A}_Y$ if and only if $m \geqslant \text{AVaR}_p(X+Y) - \text{AVaR}_p(Y)$, $\forall p \in (0,1]$. Therefore,

$$\begin{aligned}\inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\} &= \inf\left\{m \in \mathbb{R} : m \geqslant \sup_{p\in(0,1]} \text{AVaR}_p(X+Y) - \text{AVaR}_p(Y)\right\}\\ &= \sup_{p\in(0,1]} \text{AVaR}_p(X+Y) - \text{AVaR}_p(Y).\end{aligned}$$

To prove item 2, just notice that

$$\begin{aligned}-\widetilde{\Lambda}(X,Y) &= -\sup_{\rho\in\Theta} \rho(X+Y) - \rho(Y)\\ &= \inf_{\rho\in\Theta} \rho(Y) - \rho(X+Y)\\ &= \sup\{m \in \mathbb{R} : m \leqslant \rho(Y) - \rho(X+Y),\ \forall \rho \in \Theta\}\\ &= \sup\{m \in \mathbb{R} : \rho(Y+m) \geqslant \rho(X+Y),\ \forall \rho \in \Theta\}\\ &= \sup\{m \in \mathbb{R} : Y + m \preceq_{sd} X + Y\}\\ &= \widetilde{C}(X,Y),\end{aligned}$$

where the last equality comes from Proposition 4.12. □

The set of SSD-consistent monetary risk measures, Θ, is much larger then $\{\text{AVaR}_p : p \in (0,1]\}$. However, Theorem 4.3 shows that optimizing $\rho(X+Y) - \rho(Y)$ restricted to $\rho \in \Theta$ is equivalent to restrict $\rho \in \{\text{AVaR}_p : p \in (0,1]\}$. Therefore, one can express $\widetilde{\Lambda}(\cdot, Y)$ through any family of risk measures Θ' contained in Θ and containing $\{\text{AVaR}_p : p \in (0,1]\}$.

Proposition 4.16. *Let Θ' be any family of risk measures such that $\{\text{AVaR}_p : p \in (0,1]\} \subseteq \Theta' \subseteq \Theta$. Then it holds that*

$$\widetilde{\Lambda}(X,Y) = \sup_{\rho\in\Theta'} \rho(X+Y) - \rho(Y). \tag{4.30}$$

Proof. It suffices to notice that, under the conditions of the statement, we have

$$\begin{aligned}\sup_{p\in(0,1]} \text{AVaR}_p(X+Y) - \text{AVaR}_p(Y) &\leqslant \sup_{\rho\in\Theta'} \rho(X+Y) - \rho(Y)\\ &\leqslant \sup_{\rho\in\Theta} \rho(X+Y) - \rho(Y)\\ &= \sup_{p\in(0,1]} \text{AVaR}_p(X+Y) - \text{AVaR}_p(Y).\end{aligned}$$

□

Remark 4.17. The above result tells us that the *value* of the optimization problem in eq. (4.30) is the same regardless of ρ being restricted to Θ, Θ', or $\{\text{AVaR}_p : p \in (0,1]\}$. However, the problems with these different restrictions need not be equivalent to one another in terms of the existence, uniqueness, or form of their solution.

Proposition 4.17. *Let $\widetilde{\Lambda}(X,Y)$ be defined as in Definition 4.10 for $X,Y \in \mathcal{X}$. This functional fulfills the following properties:*

1. *$\widetilde{\Lambda}(\lambda X + (1-\lambda)Z, Y) \leqslant \lambda\widetilde{\Lambda}(X,Y) + (1-\lambda)\widetilde{\Lambda}(Z,Y)$ for all $X,Z \in \mathcal{X}$ and $\lambda \in [0,1]$.*

2. *If (X,Y) is comonotonic, then $\widetilde{\Lambda}(X,Y) = ML(X)$.*

Proof. To prove item 1, notice that for any $X,Z \in \mathcal{X}$ and $\lambda \in [0,1]$ it holds that

$$\begin{aligned}
\widetilde{\Lambda}(\lambda X + (1-\lambda)Z, Y) &= \sup_{p\in(0,1]} \mathrm{AVaR}_p(\lambda(X+Y) + (1-\lambda)(Z+Y) - \rho(Y)) \\
&\leqslant \sup_{p\in(0,1]} \lambda\, \mathrm{AVaR}_p(X+Y) + (1-\lambda)\, \mathrm{AVaR}_p(Z+Y) - \mathrm{AVaR}_p(Y) \\
&\leqslant \lambda \sup_{p\in(0,1]} \mathrm{AVaR}_p(X+Y) - \mathrm{AVaR}_p(Y) + (1-\lambda) \sup_{p\in(0,1]} \mathrm{AVaR}_p(Z+Y) - \mathrm{AVaR}_p(Y) \\
&= \lambda\widetilde{\Lambda}(X,Y) + (1-\lambda)\widetilde{\Lambda}(Z,Y).
\end{aligned}$$

To prove item 2 it suffices to notice that, if (X,Y) is comonotonic, then

$$\sup_{p\in(0,1]} \mathrm{AVaR}_p(X+Y) - \mathrm{AVaR}_p(Y) = \sup_{p\in(0,1]} \mathrm{AVaR}_p(X) = ML(X), \tag{4.31}$$

where the last equality follows from remark 4.3. □

Notice that the convexity property of $\widetilde{\Lambda}$ comes from its AVaR representation, which further highlights the usefulness of this characterization.

Remark 4.18. The counter-part of $\widetilde{\Lambda}$ with respect to the set $\mathcal{W}_Y$, which was discussed in remark 4.15, is given by

$$\underset{\sim}{\Lambda}(X,Y) = \inf_{\rho\in\Theta} \rho(X+Y) - \rho(Y). \tag{4.32}$$

This functional is associated to $\mathcal{W}_Y$ as $\underset{\sim}{\Lambda}(X,Y) = \sup\{m \in \mathbb{R} : X + m \in \mathcal{W}_Y\}$. Analogously, it holds that $\mathcal{W}_Y = \{X \in \mathcal{X} : \underset{\sim}{\Lambda}(X,Y) \geqslant 0\}$. We will abstain from proving it to avoid making this detour too long. The reason for which the set $\mathcal{W}_Y$ induces the smallest risk variation—namely, $\underset{\sim}{\Lambda}(X,Y)$—while $\mathcal{A}_Y$ induces the largest risk variation—i.e. $\widetilde{\Lambda}(X,Y)$—is similar to that discussed in Proposition 4.5.

The $\widetilde{\Lambda}(\cdot,Y)$ functional is essentially characterized by the $\preceq_{sd}$ order. Moreover, notice that the $\preceq_{sd}$ order satisfy the following "cancellation law": $Y + m \preceq_{sd} X \Leftrightarrow Y \preceq_{sd} X - m$ for any $X,Y \in \mathcal{X}$ and $m \in \mathbb{R}$. Joining these facts, it holds that the functional $\widetilde{\Lambda}(\cdot,Y)$ admits several other representations. To illustrate the range of these possibilities, consider the fact that

$$\widetilde{\Lambda}(X,Y) = \inf\{m \in \mathbb{R} : Y \preceq_{sd} X + Y + m\} = \inf\{m \in \mathbb{R} : Y - m \preceq_{sd} X + Y\}. \tag{4.33}$$

As the following proposition shows, different representations can be obtained through eq. (4.33)

Proposition 4.18. *The function $\widetilde{\Lambda}$ as defined in Definition 4.10 admits the following representations:*

1. *$\widetilde{\Lambda}(X,Y) = \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : \mathbf{E}[u(X+Y+m)] \geqslant \mathbf{E}[u(Y)]\}$.*

2. $\widetilde{\Lambda}(X,Y) = \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : \mathbf{E}[u(X+Y)] \geqslant \mathbf{E}[u(Y-m)]\}$.

Proof. To prove item 1, notice that Proposition 4.8 gives us that $\widetilde{\Lambda}(X,Y) = \inf\{m \in \mathbb{R} : \mathbf{E}[u(X+Y+m)] \geqslant \mathbf{E}[u(Y)],\ \forall u \in \mathcal{U}\}$. Moreover, notice that

$$\{m \in \mathbb{R} : \mathbf{E}[u(X+Y+m)] \geqslant \mathbf{E}[u(Y)],\ \forall u \in \mathcal{U}\} = \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : \mathbf{E}[u(X+Y+m)] \geqslant \mathbf{E}[u(Y)]\}. \quad (4.34)$$

Therefore, we can prove item 1 by showing that

$$\inf \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : \mathbf{E}[u(X+Y+m)] \geqslant \mathbf{E}[u(Y)]\} = \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : \mathbf{E}[u(X+Y+m)] \geqslant \mathbf{E}[u(Y)]\}. \quad (4.35)$$

To ease notation, let's define, for each utility function $u \in \mathcal{U}$, the auxiliary function $f_u(m) = \mathbf{E}[u(X+Y+m)] - \mathbf{E}[u(Y)]$, for all $m \in \mathbb{R}$. We should remark that this function is strictly increasing and continuous. With this new notation, eq. (4.35) can be rewritten as:

$$\inf \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : f_u(m) \geqslant 0\} = \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \quad (4.36)$$

First, notice that if $m_0 \in \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$, then for $u \in \mathcal{U}$ it holds that $m_0 \in \{m \in \mathbb{R} : f_u(m) \geqslant 0\}$. As a consequence, it holds that $m_0 \geqslant \inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$ for all $u \in \mathcal{U}$, which is the same as saying that m_0 is an upper-bound of the set $\{\inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\} : u \in \mathcal{U}\}$. Therefore, $m_0 \geqslant \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$. Since this holds for all $m_0 \in \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$, we conclude that

$$\inf \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : f_u(m) \geqslant 0\} \geqslant \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \quad (4.37)$$

This inequality implies that $\inf \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$ is an upper-bound for $\{\inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\} : u \in \mathcal{U}\}$. To conclude the proof we will show that it is, in fact, the smallest upper bound. Let's begin by noticing that, since f_u is strictly increasing and continuous, it holds that

$$\{m \in \mathbb{R} : f_u(m) \geqslant 0\} = [\inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}, +\infty), \quad \forall u \in \mathcal{U}. \quad (4.38)$$

Therefore, if $m_0 \geqslant \inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$, it holds that $f_u(m_0) \geqslant 0$. Therefore, if m_0 is an upper-bound of $\{\inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\} : u \in \mathcal{U}\}$, it follows that

$$m_0 \in \cap_{u\in\mathcal{U}}\{m \in \mathbb{R} : f_u(m) \geqslant 0\}. \quad (4.39)$$

From this, we conclude that $m_0 \geqslant \sup_{u\in\mathcal{U}} \inf\{m \in \mathbb{R} : f_u(m) \geqslant 0\}$.

To prove item 2, notice that

$$\begin{aligned}\widetilde{\Lambda}(X,Y) &= \inf\{m \in \mathbb{R} : Y \preceq_{sd} X+Y+m\} \\ &= \inf\{m \in \mathbb{R} : Y-m \preceq_{sd} X+Y\}.\end{aligned}$$

Let's define for each utility function $u \in \mathcal{U}$, the auxiliary function $g_u(m) = \mathbf{E}[u(X+Y)] - \mathbf{E}[u(Y-m)]$, for

$m \in \mathbb{R}$. Then, it holds that

$$\begin{aligned} \{m \in \mathbb{R} : Y - m \preceq_{sd} X + Y\} &= \{m \in \mathbb{R} : g_u(m) \geqslant 0,\ \forall u \in \mathcal{U}\} \\ &= \{m \in \mathbb{R} : \inf_{u \in \mathcal{U}} g_u(m) \geqslant 0\} \\ &= \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : g_u(m) \geqslant 0\}. \end{aligned}$$

Following the same lines as we did to prove item 1, one concludes that

$$\inf \cap_{u \in \mathcal{U}} \{m \in \mathbb{R} : g_u(m) \geqslant 0\} = \sup_{u \in \mathcal{U}} \inf \{m \in \mathbb{R} : g_u(m) \geqslant 0\}. \tag{4.40}$$

Therefore, once we recover the meaning of g_u we obtain

$$\widetilde{\Lambda}(X, Y) = \sup_{u \in \mathcal{U}} \inf \{m \in \mathbb{R} : \mathbf{E}[u(X + Y)] - \mathbf{E}[u(Y - m)] \geqslant 0\}, \tag{4.41}$$

which concludes the proof. □

4.6 CONCLUDING REMARKS

This paper proposes a functional to identify financial positions that reduce the risk of an original portfolio according to any monetary risk measure consistent with second-degree stochastic dominance. We call these financial positions robust risk reducers of the original portfolio, and show that they can be equivalently identified through what we called robust certainty equivalents. Our framework is based on the standard notions of acceptance sets and risk measures, and the functional proposed measures the cost of making an incremental position a robust risk reducer for a given initial portfolio.

The proposed framework addresses the limitations of considering a single risk measure and can be used in cases where there is incomplete information about the risk attitude of investors, or when the portfolio manager needs to aggregate the attitudes towards risk of multiple stakeholders. The approach is based on the preference robust optimization paradigm, which makes use of partial information about individuals' preferences to obtain measures of utilities, certainty equivalents, and risk measurements that conform to what is known about their preferences and risk attitudes. The paper contributes to the literature by providing a practical tool that portfolio managers can use to comply with the risk attitude of the more conservative among a considerable set of investors.

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