Eduardo Tondo Pian

Efficiency-based Index Tracking Optimization Model

Brasil

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Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Administração da Universidade Federal do Rio Grande do Sul, como requisito parcial para obtenção do título de Mestre em Administração.

Universidade Federal do Rio Grande do Sul

Programa de Pós-Graduação em Administração

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Brasil

2023

CIP - Catalogação na Publicação

Pian, Eduardo Tondo
Efficiency-based index tracking optimization model / Eduardo Tondo Pian. -- 2023.
56 f.
Orientador: Leonardo Riegel Sant'Anna.

Dissertação (Mestrado) -- Universidade Federal do Rio Grande do Sul, Escola de Administração, Programa de Pós-Graduação em Administração, Porto Alegre, BR-RS, 2023.

1. Portfolio optimization. 2. Market efficiency. 3. Non-linear programming. I. Sant'Anna, Leonardo Riegel, orient. II. Título.

Elaborada pelo Sistema de Geração Automática de Ficha Catalográfica da UFRGS com os dados fornecidos pelo(a) autor(a).

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Acknowledgements

Primeiramente, agradeço a meu Senhor e Salvador, Jesus Cristo, pela sua graça que me salva da morte todos os dias, e por ser o responsável de tudo aquilo que há de bom em mim. Para Ele, seja dada toda honra e toda glória, até mesmo este pequeno trabalho. Agradeço à minha esposa, Tanara, por todo o apoio, incentivo, amor e dedicação durante todo este tempo, que me auxiliaram a continuar nesta caminhada. À minha família, que sempre esteve me ajudando incondicionalmente.

Agradeço ao professor e grande amigo Leonardo Sant'Anna, por todo seu tempo e todo o esforço empregado na elaboração deste trabalho. Com certeza, este trabalho é resultado de todo o seu esforço e contribuição! Agradeço aos professores Tiago Filomena e João Caldeira, que contribuíram imensamente quando da estruturação do projeto. Agradeço aos membros da banca de dissertação, Prof. Tiago Pascoal Filomena, Prof. Guilherme Kirch e Prof. Henrique Castro Martins, cujas contribuições para uma boa construção deste trabalho, bem como sua evolução, são de extrema importância. Agradeço aos meus colegas do mestrado em finanças, Boris Barcelos e Henrique Rodenbusch, por todo o companheirismo e auxílio ao longo desta jornada. Agradeço, também, aos professores da linha de Finanças do Programa de Pós-Graduação em Administração da Universidade Federal do Rio Grande do Sul, em especial Prof. Leonardo Sant'Anna, Prof. Guilherme Kirch e Prof. Tiago Pascoal Filomena, que me auxiliaram de maneira inestimável com seus conhecimentos durante as aulas.

Agradeço à toda a equipe de Finanças da Vertical de Serviços Financeiros e Digitais das Empresas Randon, em especial ao Luis Felipe Oliveira e ao Pablo Luis Krummenauer. Todo o apoio, incentivo e auxílio durante os cada momento exerceram um papel fundamental no andamento deste trabalho. Muitas de nossas conversas e construções estão refletidas ao longo desta dissertação.

"No passado, todas essas coisas valiam muito para mim; mas agora, por causa de Cristo, considero que não têm nenhum valor. E não somente essas coisas, mas considero tudo uma completa perda, comparado com aquilo que tem muito mais valor, isto é, conhecer completamente Cristo Jesus, o meu Senhor."

Carta de Paulo aos Filipenses (3:7)
Bíblia Sagrada

Resumo

Utilizamos restrições de eficiência no nível dos ativos, medidas pela *Multifractal-Detrended Fluctuation Analysis* (MF-DFA), em um modelo clássico de *index tracking*, obtendo um problema de programação quadrática (PQ). Formamos carteiras que buscam replicar os índices S&P500, Nikkei 225 e Ibovespa, dos mercados norte-americano, japonês e brasileiro, respectivamente, de 2012 a 2021. Nossos resultados indicam que as restrições de eficiência atuam de forma semelhante a uma restrição de cardinalidade, reduzindo o número de ativos à medida que a restrição se torna mais severa, e esse movimento resulta em maior *tracking error*. Evidenciamos que a utilização de restrição de eficiência é uma forma adequada de reduzir o número de ativos sem a necessidade de formular um problema com maior complexidade computacional, comparando com outros métodos já utilizados na literatura, principalmente em mercados mais desenvolvidos, onde os níveis gerais de eficiência são maiores e, portanto, as restrições geram um custo menor em termos de *tracking error*. Também, mostramos brevemente que as restrições de eficiência e de liquidez podem ter efeitos diferentes em carteiras de *index tracking*.

Keywords: Otimização de carteiras, Eficiência de mercado, Programação não-linear.

Abstract

We used asset efficiency constraints, measured by the Multifractal-Detrended Fluctuation Analysis (MF-DFA), in a classic index tracking model, obtaining a quadratic programming problem (QP). We form portfolios that seek to replicate the S&P500, Nikkei 225 and Ibovespa index, from the US, Japanese and Brazilian markets, respectively, from 2012 to 2021. Our results indicate that efficiency constraints act in a similar way to a cardinality constraint, reducing the number of assets as the constraint becomes more severe, and this movement results in greater tracking error. We evidenced that using efficiency constraint is an adequate way to reduce the number of assets without having to formulate a problem with greater computational complexity, comparing with other methods already used in the literature, especially in more developed markets, where the general levels of efficiency are higher and, therefore, constraints generate a lower cost in terms of tracking error. We also show that efficiency and liquidity constraints could have different effects on tracking portfolios.

Keywords: Portfolio optimization, Market efficiency, Non-linear programming.

List of Figures

List of Tables

Contents

1 Introduction

Index tracking (IT) is a passive portfolio selection strategy that aims to replicate the performance of some market index. The main justification for passive investment strategies such as IT is in the Efficiency Market Hypothesis (EMH), proposed by Fama (1970). In an efficient market, assets would be priced correctly, with no arbitrage opportunities. Thus, active portfolio selection strategies would not make sense, as they would not be able to outperform market returns in the long run. Empirical evidence of market superiority over active portfolio selection strategies is demonstrated by Frino and Gallagher (2001) and Fama and French (2010), for example, as well as several other empirical studies.

In line, Tiwari, Aye and Gupta (2019) demonstrate, using a long range of data from developed and emerging markets, that the efficiency of equity markets varies over time and that markets are more efficient in the long term than in the short term. The authors argue that the lack of efficiency can be justified by the lack of liquidity in the markets, and that the level of efficiency can be improved with greater transparency of information to investors, greater activity of arbitrage strategies based on variations in efficiency in the markets, and better trading technologies, for example. Arshad et al. (2016) demonstrate that several markets show improvements in their efficiency over time. Such results are also obtained by several other empirical studies that tested the EMH.

With equity markets becoming increasingly efficient, including emerging markets, looking for better ways to replicate the performance of a market index becomes increasingly relevant. One of the possible ways is the so-called full replication, where the exact composition of the index and the exact weight of the assets are used to perform the tracking. The main penalties for this strategy are the high management costs that arise from monitoring large portfolios, especially for broader indices, such as the S&P500 or the Russell 1000. Also, this approach might become expensive due to increasing transaction costs, when it is necessary to trade a high volume of assets. Alternatively, a widely used formulation is to minimize the variance of the difference of portfolio and index returns (the tracking error), including a cardinality constraint to control the number of assets in the portfolio. However, in this case, such constraint includes a binary variable, generating a mixed-integer quadratic programming problem (MIQP), which significantly increases the complexity of the model from a computational point of view. In this sense, several ways to perform tracking with a good performance have already been proposed. Use of quadratic programming (quadratic programming - QP) with genetic algorithms (Sant'Anna et al., 2017), the cointegration approach (Dunis; Ho, 2005; Alexander; Dimitriu, 2005), and use of heuristics (Beasley; Meade; Chang, 2003; Scozzari et al., 2013) are just some examples among the various studies that test different methods for solving the IT problem, in the search for reducing solution time and maintaining a good tracking performance.

Concerning market efficiency, Maciel (2021) (which we will refer to as LM) makes use of efficiency considerations in the minimum variance problem, using an exogenous approach, optimizing portfolios with a set of more efficient assets and another set of less efficient ones. Their results demonstrate a good performance for the most efficient set, when compared to the less efficient. Similarly, the present study proposes to employ efficiency constraints in the index tracking (IT), as an alternative approach to perform portfolio selection. By including asset-level efficiency constraints, the solution space is restricted to a certain efficiency level, reducing the number of assets that are part of the set of feasible solutions to the problem. In this way, the efficiency acts as a cardinality constraint, reducing the size of the portfolio as the efficiency constraint in the model becomes more severe. Therefore, tracking portfolios are created with a limited number of assets without the use of a cardinality constraint in the optimization model. The optimization model can be defined as a regular quadratic programming (QP) model, which generates solutions instantly. As in LM, the Multifractal-Detrended Fluctuation Analysis (MF-DFA) – proposed by Kantelhardt et al. (2002) – was used to compute the Market Deficiency Measure (MDM).

To analyze the effects of efficiency constraints on IT models, we go beyond simply including them in the models. Our tests are carried out in different markets, with different levels of development (United States, Japan and Brazil), which should show important results and internationalization for the role of asset efficiency in IT models. We chose the US and Japan market as these are classified as developed markets, and have one of the highest trading volumes among their class. Similarly, among markets classified as advanced emerging, Brazil stands out as one of the main ones, in terms of trading volume. We use a long range of portfolio projection data, from 2012 to 2021, which includes both bull and bear markets, allowing us to investigate the performance of these portfolios under different market conditions. Our results indicate a good performance of tracking portfolios with efficiency constraints at the asset level. We identified that there is a trade-off between efficiency and tracking error: as we insert efficiency restrictions into the model, and, consequently, restrict the average number of assets in the portfolio, the tracking error increases. This effect is more pronounced for emerging markets, as is the case of Brazil in our sample. Additionally, the results of tracking portfolios with efficiency constraints are compared with the results obtained by other approaches studied in the literature, which are concerned with improving the tracking performance of portfolios, but which are more complex from a computational point of view. Our results are similar to those obtained using more complex methods, demonstrating the relevance of using efficiency constraints.

This dissertation contribute to the literature concerning portfolio optimization and index tracking in different ways. First, the results show that the models based on efficiency for index tracking generate portfolios with an acceptable number of assets, obtaining results instantly, since the problems are formulated in the form of quadratic programming (QP). As a second contribution, from an empirical point of view, the time window chosen and the three financial markets selected for the empirical analysis resulted in distinct findings for markets with different levels of financial development, both in periods of strong stability and high volatility (such as during Covid-19 pandemic). Finally, as a third one, we believe that this study can, together with that of LM, open the field for further research on the consideration of asset efficiency in portfolio optimization.

The dissertation is structured as follows. The 2 section reviews the literature relevant to the problem of index tracking and market efficiency; the 3 section demonstrates the efficiency measurement method and formulates the optimization problems used; the 4 section presents the empirical results procedures; and, finally, the 5 section concludes the dissertation.

2 Related Literature

2.1 The efficienct market hypothesis

The efficient markets hypothesis (EMH), formally developed by Samuelson (1965) and Fama (1970), changed the way investors look at their investments, raising doubts about the effectiveness of active portfolio management. An efficient market is one in which prices always reflect all the information available to agents, thus there is no arbitrage opportunities. The implications of EMH go beyond the performance of portfolios, having an effect on real decisions in the economy, i.e., inefficient markets, where prices do not reflect the fair value of assets, can be responsible for misallocations of resources in the economy, given that financial assets also guide capital allocation decisions.

Sharpe (1991) demonstrates what he calls the "arithmetic of active management", where, using only simple arithmetic, it is possible to see that active investment strategies do not make sense, and present worst performance than passive strategies after costs (i.e. , $\alpha < 0$), considering that the costs of actively managed portfolios are considerably higher than those of passive portfolios (French, 2008). In line, several empirical studies demonstrate the inability of active strategies to overcome passive returns (market returns) in the long term (Fama; French, 2010; Frino; Gallagher, 2001; Malkiel, 1995; Busse; Goyal; Wahal, 2010).

This is the basis on which passive investment strategies, such as index tracking (IT) strategies, are supported. In an efficient market, where asset prices reflect all the information available to agents, i.e., equilibrium prices reflect the fair value of assets, where arbitrage opportunities would not exist. In this sense, the popularity and growth of index funds (funds that seek to replicate market performance) and ETFs (exchange-traded funds) is justified (Appel; Gormley; Keim, 2016; Hshieh; Li; Tang, 2021).

2.2 The index tracking problem

A relatively simple way to replicate a market index would be to form a portfolio with the same assets and their proportions as the objective index. However, holding a large number of assets in broad indices, such as the Nikkei 225, S&P500 and Russell 1000, for example, would cause wealth to be allocated in a very dispersed way, generating high monitoring costs, due to the need to monitor the weights of a large number of assets. With this, the need arises to seek the yield of an index, but with a smaller number of assets. This is the essence of the index tracking (IT) approach. In short, the objective is to obtain a return on the formed portfolio that is as close as possible to the return of the market index, with an acceptable number of assets. In this sense, several methods were developed, and trade-offs were observed, aiming to improve the tracking error (TE) of the portfolios formed, i.e., the difference between the portfolio return and the index return.

In order to try to improve the results of IT portfolios, Beasley, Meade and Chang (2003) uses genetic algorithms to solve the problem, including transaction costs, limit on the number of assets and rebalancing controls in the model. The authors observe interesting trade-offs, as in the case of transaction costs, where the TE reduces with higher limits of transaction costs, and also for the case of return in excess of the index (enhanced index tracking problem), where, when this increases, the TE also increases.

Gaivoronski, Krylov and Wijst (2005) investigate the role of the number of assets in IT portfolios, as well as the impact of adjustments to new information available in the market (the rebalancing of these portfolios), analyzing static and dynamic IT strategies. The authors identify a trade-off between tracking error and transaction costs: when portfolios are rebalanced more frequently, TE decreases, but transaction costs increase, given the greater volume of transactions. As for the number of assets, it is observed that larger portfolios have lower TE, and, consequently, less need for rebalancing. The opposite occurs for portfolios with a low number of assets.

Due to the complexity of solving the IT problem, especially in cases of large indices, several heuristics were developed, with the objective of obtaining good results in an acceptable computational time, such as Guastaroba and Speranza (2012), which use the Kernel Search heuristic to solve the problem, comparing with the performance of a commercial solver; Scozzari et al. (2013), which use Differential Evolution; Sant'Anna et al. (2017) (henceforth, SFGB), which use a hybrid solution method, combining a genetic algorithm and nonlinear mathematical programming, obtaining good results with low computational time. Also, the authors employ tests in poorly studied markets, such as the Brazilian market, which is marked by high volatility when compared to developed markets. In that market, the authors replicate an index composed of 69 stocks, in the period studied, with only 5 and 10 assets.

Studies also address metaheuristics to solve the problem. Gnägi and Strub (2020), compare the performance of portfolios formed for the enhanced index tracking problem (i.e., portfolios that seek to obtain a slightly higher performance than the market), using different objective functions, and formulating a mixed-integer linear programming problem (MILP) and mixed-integer quadratic programming (MIQP), aggregating metaheuristics, applying to large indices, such as indices of up to 9,000 stocks.

Several other methods were used, in search of improvements in the performance of the portfolios, as in Dunis and Ho (2005) and Sant'Anna, Filomena and Caldeira (2017), which use the cointegration approach to form the portfolios; Corielli and Marcellino

(2006) and Jiang and Perez (2021), which use factor models to the IT model; Sant'Anna, Caldeira and Filomena (2020) uses a method derived from lasso regression, comparing with cointegration strategies.

Another extremely relevant consideration in IT models is the rebalancing strategy. New information available to the markets may alter the prices of the assets in the portfolio, causing a distance from the index. In line, adverse market conditions can make tracking the index difficult, generating the need to review the composition of the portfolios. Strub and Baumann (2018) use a MILP formulation, considering rebalancing, and highlight the trade-off between a static approach (with little or no rebalancing) and portfolio performance (tracking error).

2.3 EMH and index tracking strategy

A common objective in all the studies cited is the search for improvements in index tracking models. Improvements are often accompanied by costs, either in the form of complex numerical methods or empirical trade-offs. This article focuses on another factor that we consider fundamental to obtain a portfolio that effectively replicates an index: the quality of the price time series considered in the optimization period (in-sample period, i.e., the past data of the assets). By quality, we refer to the degree to which asset prices reflect all available information, i.e., whether equilibrium prices reflect the fair value of the stocks. As discussed in the 3.1 section, different methods were applied to measure efficiency, in different forms of EMH, and several results and implications were obtained, justifying the extensive amount of tests in the literature (Cajueiro; Tabak, 2004; Tran; Leirvik, 2019, among many others).

LM employs efficiency considerations in the classic portfolio optimization problem, the minimum-variance (MV) model, in the Brazilian market. The author uses an exogenous approach, selecting the set of assets, based on the efficiency levels, to compose the set used in the optimization process. The performance of portfolios formed considering a set of more efficient assets is compared with the portfolios formed by the set of assets with lower levels of efficiency. The results demonstrate better performance of portfolios formed by more efficient assets in terms of risk and return.

The main insight that can be extracted from the study of LM is: portfolios with more efficient assets can generate a better estimated covariance matrix, precisely because the asset price series more adequately reflects the risk-return relationship of the assets, i.e., because prices have a good quality of information efficiency. As assets are, in fact, more efficient, models based on the CAPM (Sharpe, 1964), for example, will be able to provide better information for a more efficient set of assets than for less efficient ones. Moving on to passive management, we can imagine that the set of assets available to us is made up of more efficient assets that have or do not have a good relationship with the index, and less efficient assets that have a good relationship with the index or not. In this way, it would be preferable, in an ex-ante thinking, to select assets with a good relationship with the index and that are more efficient, as their good relationship would be more consistent. However, when we restrict the space of feasible solutions to just the most efficient assets, this subgroup also includes assets that do not have a good relationship with the index. Thus, when we demand more efficient assets, and consequently reduce the number of assets that are part of the portfolios, we are faced with the trade-off between efficiency and tracking error. This trade-off will be discussed in the 4 section.

3 Method

In this section, we will present the structure of the optimization models that will be used in the empirical tests described in the section 4. First, in the subsection 3.1, we discuss the measurement of the inefficiency measure, the Market Deficiency Measure (MDM), through the Multifractal-Detrended Fluctuation Analysis (MF-DFA) approach; later, in the subsection 3.2, we present the formulation of the optimization models used for the formation of portfolios, whose performance will be analyzed in the section 4.

3.1 Efficiency measurement

Since the structuring of the EMH by Fama (1970), several studies have focused on testing the three proposed forms: weak, semistrong and strong forms. In particular, tests related to the last two forms may present several empirical difficulties, as they require information in addition to stock trading data, such as other financial information and internal information. Keown and Pinkerton (1981) analyze the stock price response of companies targeted by mergers and acquisitions; Patell and Wolfson (1984) analyze, using intraday data, the response of stock prices to information on earnings and dividends disclosed; Bardos (2011) analyzes the effect of the quality of firms' financial statements on the liquidity of their shares. The findings indicate the existence of a positive relationship between the quality of financial statements and liquidity. These are some of the many examples of testing the incorporation of financial information, which do not include past trading data, into asset prices.

Such empirical tests of the semi-strong or strong form are extremely laborious, as they seek to analyze informational efficiency with higher levels of information, unlike the weak form (Holderness; Sheehan, 1985; Lin; Howe, 1990; Brio; Miguel; Perote, 2002). The weak form, in turn, has been extensively tested in the literature, as it focuses only on the reflection of past trading data, seeking to answer whether or not stock prices follow a random walk.

Basically, testing the weak form of EMH is testing whether there is predictability of asset prices, using only past trading data as inputs. We can say that this test consists of verifying whether future prices depend on past prices, so that future prices can be predicted. There are numerous methods for testing the weak form of EMH, such as serial correlation, variance ratio, unit root and spectral analysis, for example, which have been extensively employed in the EMH literature (Lim, 2007).

A method that has gained notoriety in the literature is the Multifractal-Detrended

Fluctuation Analysis (MF-DFA), proposed by Kantelhardt et al. (2002). In summary, the measure seeks to identify the persistence of asset returns, verifying the dependence of future asset returns on their past returns, within the scope of the weak form of the EMH. In summary, the measure seeks to identify the persistence of asset returns, verifying the dependence of future asset returns on past returns. Al-Yahyaee et al. (2020) use the aforementioned method to verify efficiency in the cryptocurrency market, as well as search for the determinants of efficiency. The results indicate that higher liquidity combined with lower volatility help arbitrageurs to eliminate existing arbitrage opportunities, raising efficiency levels. In another study, Al-Yahyaee, Mensi and Yoon (2018) compares the efficiency, measured through the MF-DFA, of Bitcoin with the stock market and the gold market, indicating that Bitcoin is the most inefficient among the analyzed assets.

Several other studies use the MF-DFA to analyze the efficiency of markets. Choi (2021) analyzes the efficiency of different sectors of the economy that are part of the S&P500 during periods of instability. Zhu and Bao (2019) compare the efficiency of the largest ETFs traded in the US market: SPY, DIA and QQQ, which seek to replicate the S&P500, Dow Jones Industrial Average and NASDAQ 100, respectively. Their results indicate that the QQQ is the most efficient ETF in the sample, and that the 2008 financial crisis negatively affected ETFs in terms of efficiency. Tiwari, Aye and Gupta (2019) uses almost a century of data from eight markets, both developed and emerging, seeking answers about the efficiency of these markets. The results indicate that markets are more efficient in the long run than in the short run, and that efficiency varies over time.

Unlike the literature that seeks to test the weak form of the EMH by the MF-DFA, the present study only makes use of this efficiency measure, aiming to verify the impact of the inclusion of efficiency constraints, measured by the MF-DFA, on index tracking portfolios. Basically, MF-DFA collects the volatility of the time series in each time interval, as a statistical point that is used to calculate volatility functions. Then the Hurst exponents are determined based on the power law of volatility functions. According to Kantelhardt et al. (2002), the estimate follows the following steps. Let $x(i)$, $i = 1, ..., N$ be a time series of log asset returns, where N is its length. The first step is to determine the profile function, $y(i)$, which can be obtained by the difference between $x(i)$ and its mean, $\bar{x}(i)$, for $i = 1, .., N$:

Step 1. Profile Function:

$$y(i) = \sum_{k=1}^{i} [x(k) - \bar{x}], \tag{3.1}$$

where $\bar{x}$ comprises the mean of the time series.

Step 2: the profile function $(y(i))$ is divided into $N_s \equiv int(N/s)$ non-overlapping segments, of equal length s. The number of N segments will not necessarily be an integer that is a multiple of the segments s. Thus, a small part at the end of the series can be "left over". In order not to disregard this part of the time series, we repeated the same procedure, this time starting from the opposite end of the series, until its beginning. The result is two N_s segments, so we have $2N_s$.

Step 3: the local trend is calculated for each of the $2N_s$ segments by a least squares fit of the series. From there, the variance is obtained:

$$F^2(s,v) = \frac{1}{s}\sum_{i=1}^{s}\{y[(v-1)s+i] - y_v(i)\}^2, \tag{3.2}$$

for each segment v, $v = 1, ..., N_s$, and

$$F^2(s,v) = \frac{1}{s}\sum_{i=1}^{s}\{y[N-(v-N_s)s+i] - y_v(i)\}^2 \tag{3.3}$$

for each segment v, $v = 1, ..., 2N_s$. Here, $y_v(i)$ is the polynomial fitted in the segment v.

Step 4: the q-th order fluctuation function $F_q(s)$ is obtained by averaging all segments (subsets):

$$F_q(s) = \left[\frac{1}{2N_s}\sum_{v=1}^{2N_s}[F^2(s,v)]^{q/2}\right]^{\frac{1}{q}} \tag{3.4}$$

where $q \neq 0$. For $q = 0$, the value $h(0)$ cannot be determined directly because of the divergent exponent. Instead, a logarithmic averaging procedure should be employed. For $q = 2$, we have a standard DFA procedure (Tiwari; Aye; Gupta, 2019).

Step 5: Determine the scaling behavior of the fluctuation functions by analyzing log-log plots of $F_q(s)$ versus each value of q. If the series $x(i)$ is correlated to the power law over a long interval, $F_q(s)$ increases to large values of s, like a power law:

$$F_q(s)s^{h(q)} \tag{3.5}$$

In general, the Hurst exponent, $h(q)$, will depend on q. If $h(q)$ does not depend on q, the time series is monofractal, otherwise it is multifractal, meaning that the behavior of scaling of small fluctuations ($q < 0$) is different from that of large variations ($q > 0$). We adopt a range from -4 to 4 for the q, to capture short-term and long-term properties, respectively, in the same way as LM, as well as in several other studies in the literature that test the weak form of EMH using MF-DFA (Tiwari; Albulescu; Yoon, 2017; Zhu; Bao, 2019; Tiwari; Aye; Gupta, 2019). If $0 < h(q) < 0.5$, the series has anti-persistence. If $0.5 < h(q) < 1$, the time series has persistence. If $h(q) = 0.5$, the stochastic process corresponds to an uncorrelated geometric Brownian motion - a random walk. To determine the asset deficiency level, we used the Market Deficiency Measure (MDM), according to Tiwari, Aye and Gupta (2019) and LM:

$$MDM = \frac{1}{2}(|h(q_{min}) - 0.5| + |h(q_{max}) - 0.5|) \tag{3.6}$$

In this study, $q_{min} = -4$ and $q_{max} = 4$. Thus, the interpretation of the MDM is as follows. The higher (lower) the MDM value, the less (more) efficient the asset. If MDM $= 0$, the asset or market in question can be considered efficient.

3.2 Index tracking optimization models

In this subsection, we present the mathematical formulation of the optimization models used in the study. First, in the 3.2.1 subsection, we present the unrestricted model, which we call the benchmark model, which does not have any cardinality constraint, which will be used as a basis for comparison for the efficiency constrained model, in turn, presented in the 3.2.2 section.

3.2.1 Benchmark model

For all models, we will use a classic objective function, which consists of minimizing the distance between the return on the portfolio formed and the return on the market index. We formulate a benchmark model, which will be the basis of performance comparison for the results obtained by the other models formulated in this work. We call this model M1-B.

Let I be a set of assets $i = 1 : N$ that are part of the composition of the market index that we are trying to replicate. Let R_t be the return on the market index in period t, and $x_{i,t}$ be the weight of asset i in the portfolio in period t. Let X_t^* be the portfolio used to replicate the market index in the period t. The objective is to form a portfolio $X_t^* = \{x_{i,t}, i \in I\}$, in each period t, that minimizes the average difference of return between the index and the portfolio. Let ψ be the set of portfolio projection periods. Then, we must build portfolios X_t^*, $\forall\ t \in \psi$, such that the return distance in relation to the index is minimized. The frequency by which we restate the weights of assets in X_t^* is determined by the rebalancing interval.

The objective function used is associated with the formulation made by Gaivoronski, Krylov and Wijst (2005), and used in several other studies, such as in SFGB, and consists of minimizing the mean squared difference of the return of the portfolio and the index, in a given estimation interval (in-sample period), to be projected after this process (out-of-sample period). Let $r_{i,t}$ be the return on asset $i \in I$ in the period t, we formulate the benchmark model:

$$\min_{x} \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N} x_i r_{it} - R_t\right)^2 \tag{3.7}$$

s.t.

$$\sum_{i=1}^{N} x_i = 1 \tag{3.8}$$

$$x_i \geq 0 \qquad \forall\, i \in \{1, \cdots, N\} \tag{3.9}$$

where the constraint 3.8 establishes that the total wealth must be allocated to the portfolio, and 3.9 comprises the constraint of non-negativity, where short positions are not allowed. In this problem, we have a quadratic objective function and a set of linear constraints, thus having a quadratic programming problem (QP).

3.2.2 Efficiency-constrained model

$$\min_{x} \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N} x_i r_{it} - R_t\right)^2 \tag{3.10}$$

s.t.

$$\sum_{i=1}^{N} x_i = 1 \tag{3.11}$$

$$x_i \gamma_i \leq x_i \pi \qquad \forall\, i \in \{1, \cdots, N\} \tag{3.12}$$

$$x_i \geq 0 \qquad \forall\, i \in \{1, \cdots, N\} \tag{3.13}$$

where γ_i is the Market Deficiency Measure (MDM) of asset i, and π is the applicable inefficiency threshold at the asset level. We will use three different values for the efficiency limit, π, for each rebalancing: (i) the median of the MDM distribution; (ii) the 35th percentile of the MDM distribution; and (iii) the first quartile of the MDM distribution. So, the M2-E model can take three different forms, which vary with the efficiency limit constraint imposed on each asset $i \in I$: model with efficiency constraint that comprises the median of the MDM distribution (we call this model M2-E-M); (ii) an efficiency-constrained model that comprises the 35th percentile of the MDM distribution (we call it the M2-E-P35 model); and (iii) an efficiency-constrained model that comprises the first quartile of the MDM distribution (we call it the M2-E-1Q model). This constraint is represented by the expression (3.12).

As mentioned in the 3.1 section, the lower the MDM for a given asset, the more efficient it is. Thus, selecting assets that have an MDM below the median of the MDM distribution, comprises selecting the most efficient half of the distribution. When the constraint changes to the 35th percentile, in the M2-E-P35 model, we are being more rigorous in relation to the level of MDM that the assets need to present to be part of the set of feasible solutions. The M2-E-1Q model is the most demanding, as it only considers assets that have MDMs that are in the first quartile of the MDM distribution.

We believe that the effect of π on the optimized portfolios should be the following: as we make the efficiency constraint more severe in the model, i.e., we move from the median (M2-E-M model) to the first quartile (M2-E-1Q model), the number of assets is reduced considerably, and the tracking error (TE) increases. This cost of maintaining a tracking portfolio with a low number of assets, the TE, should be more pronounced in emerging markets, where overall efficiency levels tend to be lower. Tracking portolios in markets with lower efficiency levels would be more penalized by the efficiency constraint, presenting a more restricted set of assets, which do not necessarily present a good relationship with the target index.

4 Results

This section is concerned with presenting the results of the dissertation. In the section 4.1.1, the data used are described, and the procedures conducted in the empirical tests are presented in the section 4.1.2; in 4.1.3, we comment on the results obtained in measuring the efficiency of assets in the different markets analyzed; the 4.2 section presents the main results of the optimized portfolios; in the section 4.3, we compare our results with the results of another methods used in the literature; in the section 4.4, we compare the results of efficiency-constrained portfolios with liquidity-constrained portfolios; in the subsection 4.5, we discuss the results as a whole; and finally, in 4.6, we present the results of a statistical test of difference in means of the projected portfolios.

4.1 Data and empirical strategy

4.1.1 Data

To carry out the empirical research procedures, we selected three markets: the US market, the Japanese market and the Brazilian market. Thus, we have two developed markets (USA and Japan) and an emerging one (Brazil), which will help to identify differences in the behavior of projected portfolios in markets with different patterns of efficiency, liquidity and volatility. Table 1 presents the FTSE's 2022 annual classification of equity markets, showing the classes in which the mentioned markets fall.

We chose, among the markets in the range of markets considered developed, the US and Japanese markets, as they are one of the largest markets in the world, one of the most explored in the literature, with great availability of historical data, and which have a total traded value[1] in 2019 of USD 23 and USD 5 trillion, respectively. In the emerging class, we selected the Brazilian market, which is classified as an advanced emerging market, being one of the most representative in its classification, with the largest trading volume in 2019 (about USD 1 trillion).

For the US market, the portfolios' target index is the S&P500, which is one of the most famous indices in the world. For this market, our dataset comprises the daily returns of the 505 assets that were part of this index in February 2022, plus the index itself, from January 2010 to December 2021. For the Japanese market, we selected as a target index the Nikkei 225, which is one of the most popular in that market, and the sample of its components in February 2022, comprising 224 assets, in the same data range (Jan/10 to Dec/21). Finally, for the Brazilian market, the Ibovespa was selected as the target index

[1] Information obtained from the World Bank database <https://data.worldbank.org/>

for the portfolios, which is the most popular index in this market. As a dataset, we selected the assets that make up the Ibovespa in February 2022, comprising 93 assets, for the same mentioned interval (Jan/10 to Dec/21). The daily returns of the datasets are adjusted for splits, mergers and dividend payments. Therefore, our tests are conducted considering a large range of data, covering periods of bull and bear markets, and in portfolios that seek to replicate both smaller indices, such as the Ibovespa, and broader indices, such as the S&P500.

All daily return data was obtained from Yahoo Finance, using the yfR (Perlin, 2022) package. We estimate the Hurst exponents, $h(q)$, using the MFDFA package (Laib; Telesca; Kanevski, 2019). This stage of data collection and estimation was conducted in R. For the optimization process, we used the AMPL software, with the Gurobi solver. The tests were conducted on a computer with an AMD Ryzen 5 3600 processor, 3.60 GHz, and 16GB of RAM. As will be commented in the following sections, the results obtained in the optimization process are instantaneous.

4.1.2 Empirical strategy

The portfolio projection approach used in this study is the rolling window, where, for the formation of the initial portfolio, for example, we use a data interval of $t = 120$ trading days immediately prior to the initial portfolio projection date, as used by Filomena and Lejeune (2012) and Filomena and Lejeune (2014), to perform the optimization, period that we call in-sample. After the portfolio is formed, it is projected from $t = 121$ until the next rebalancing period, which we call out-of-sample, and so on. We use an out-of-sample portfolio projection range from January 2012 to December 2021 for all markets covered, comprising a 10-year projection. In this interval, we use a dynamic approach, where the portfolios are updated every 120 (semestrally), 240 (annually) and every 480 days (two years). For each rebalance, we then carry out a new optimization process, using the daily return data from the 120 days immediately prior to the rebalance date (in-sample period).

With regard to efficiency, we also adopt a dynamic approach, differently from LM, which performs the estimation of efficiency only before the beginning of the out-of-sample period (projection period for the portfolios), and considers that the assets have a static efficiency throughout rebalancing. LM makes use of three years of data before the out-of-sample period to estimate asset efficiency. In a different way, our study uses one year of daily returns immediately prior to each rebalancing to estimate efficiency, instead of three years. This means that, for each rebalancing, we use a new, more up-to-date set of asset efficiency data, unlike the one used by LM, which measures asset efficiency only once for portfolio projection. We also dynamically estimate asset efficiency using three years of data immediately prior to each rebalancing, and the results are not affected when compared to portfolios formed with efficiency restrictions formed by one year of past data (these results are available upon request). Considering all markets and the entire range, we have a total of 798 portfolios formed in this study, demonstrating the extent of the empirical procedures that will be dealt with in this section.

To measure the tracking performance of the portfolios in the projection interval, we use the tracking error (TE), that we define as the variance of the difference between index and portfolio daily tracking returns, as in Beasley, Meade and Chang (2003):

$$TE^* = \frac{1}{T^*}\left[\sum_{i=1}^{T^*} |r_t^p - R_t^*|^2\right]^{\frac{1}{2}} \tag{4.1}$$

where T^* represents the total range of out-of-sample periods; $r_{i,t}^*$ represents the return on asset i for the out-of-sample period $t \in T^*$; therefore, $r_t^{p^*} = \sum_{i=1}^{I} x_i r_{i,t}^*$ represents the return of the portfolio formed in the out-of-sample interval; and R_t^* is the index return in $t \in T^*$.

The turnover of portfolios cannot be ignored either, being extremely important when choosing suitable models. Turnover is a proxy for the amount of trades carried out by the portfolios over time. Thus, the greater the turnover, the greater the amount of asset trades in the time interval. In this way, we measure the average monthly turnover of the portfolios based on the following formulation, used by Sant'Anna, Caldeira and Filomena (2020):

$$\left[\sum_{p=2}^{np}\left(\frac{\sum_{i=n}^{N}|x_i^p - x_i^{p-1}|}{2}\right)\right] \times \frac{1}{f} \tag{4.2}$$

where np is the number of portfolios formed in each model; p and $p-1$ are the rebalancing time instants; and $f = 6$ for the semiannual rebalancing, 12 for the annual rebalancing, and 24 for de two years interval of rebalancing.

4.1.3 Efficiency levels

The level of market efficiency could have a significant influence on the performance of tracking portfolios. This section is concerned with discussing a little about the difference between these levels in the markets used in this study. The FTSE Russell classifies capital markets according to their level of development, as can be seen in the 1 table, which shows the Equity Country Classification's report of September 2022. The level of efficiency also varies across markets, where emerging markets tend to be less efficient in terms of risk-return pricing than developed markets.

The figure 1 demonstrates the trajectory of the minimum of the Market Deficiency Measure (MDM) of the assets, i.e., the MDM of the most efficient asset, for each market studied, along the rebalancing intervals. For each market, and for the portfolios that are rebalanced at an interval of 120 days (semi-annual), efficiency was measured 21 times, comprising the 21 rebalancing of these portfolios, described in the previous subsection.

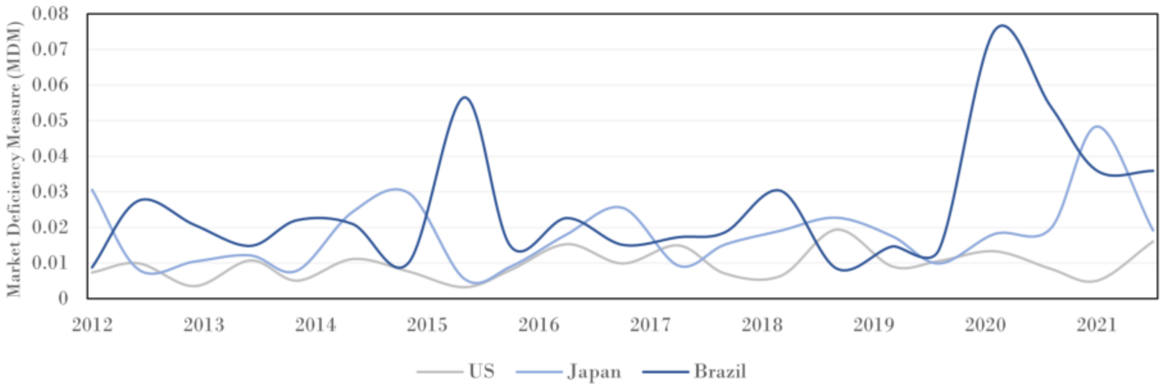

Figure 1 – Minimum of the Market Efficiency Measure (MDM), during the rebalancing intervals (semi-annual, comprising 21 rebalances), for each market. We have a total of 21 observations for each market.

The MDM is a measure of inefficiency, in this way, the higher, the less efficient the asset or the market, and the lower, the more efficient the asset or the market.

The figure 1 demonstrates the predominance of the US market in terms of efficiency, followed by Japan and, finally, Brazil. The Japanese market has shown a trajectory of volatile efficiency over time, in the same way as the Brazilian market, where the latter is the most affected in periods of high volatility, such as the period of the covid-19 crisis. In this period, we observed an increase in MDM, demonstrating a loss of efficiency for all markets, especially for Brazil. We also observe a loss of efficiency for the Brazilian market during the period 2014-2016, which coincides with a period of national crisis in the Brazilian economy.

4.2 Optimization results

In this subsection, we will address the main results obtained with the empirical procedures. After performing the optimization of the portfolios, using the models listed in the section 3, and the projection of the portfolios, according to the aspects mentioned in the sections 4.1.1 and 4.1, we calculate the descriptive statistics of the projection of each model, for each type of rebalancing. Tables 2, 3 and 4 present the results described for the portfolios projected for the US, Japanese and Brazilian markets, respectively.

For the US market, in the 120-day (semi-annual) rebalancing strategy, we observed that all models, including the benchmark model (M1-B), have an accumulated return in the out-of-sample period that is higher than the accumulated return of the S&P500 index. In terms of correlation, we observe that this is slightly lower for the models with efficiency constraints (M2 models), when compared to the benchmark model, and the correlation decreases as the efficiency constraint becomes more severe, i.e., for the M2-E-P35 and M2-E-1Q models, which have an efficiency constraint at the asset level that comprises

Table 1 – FTSE Equity Country Classification - September 2022.

Developed	**Advanced Emerging**	**Secondary Emerging**	**Frontier**
Australia	Brazil	Chile	Bahrain
Austria	Czech Republic	China	Bangladesh
Belgium/Luxemburg	Greece	Colombia	Botswana
Canada	Hungary	Egypt	Bulgaria
Denmark	Malaysia	Icenland	Côte d'Ivoire
Finland	Mexico	India	Croatia
France	South Africa	Indonesia	Cyprus
Germany	Taiwan	Kuwait	Estonia
Hong Kong	Thailand	Pakistan	Ghana
Ireland	Turkey	Phillipines	Jordan
Israel		Watar	Kazakhstan
Italy		Romania	Kenya
Japan		Saudi Arabia	Latvia
Netherlands		United Arab Emirates	Lithuania
New Zeland			Malta
Norway			Mauritius
Poland			Morocco
Portugal			Nigeria
Singapore			Oman
South Korea			Palestine
Spain			Peru
Sweden			Republic of North Macedonia
Switzerland			Serbia
UK			Slovak Republic
USA			Slovenia
			Sri Lanka
			Tanzania
			Vietnam

the 35th percentile and first quartile of the efficiency distribution in each rebalancing, respectively (remembering that, the lower the MDM, more efficient is the asset).

However, it is worth mentioning that the M2 models also show a considerable reduction in the number of assets that are part of the portfolio, having, on average, 119 assets in the M2-E-1Q model, with a more severe restriction of efficiency, compared to 474 in the benchmark model (M1-B). As the model requires that most efficient assets be part of the feasible solutions space, we expect fewer assets to compose the portfolios of these models. As will be seen below, when we look at the tracking error (TE), we expect it to increase as we reduce the number of assets that are part of the portfolios.

For the Japanese market, in a slightly different way from the US market, we observe a greater proximity of the accumulated return in the period of the models with efficiency restrictions to the accumulated return of the Nikkei 225 index. For example, in the case of rebalancing in an interval of 120 days, for the M2-E-P35 model, we have a cumulative return of 267%, while the Nikkei's cumulative return was 241%, and that of the M1-B model was 335%. It is also worth mentioning the difference between the average number

Table 2 – Descriptive results for the US market.

US (developed)		120d				240d				480d			
Descriptive Stats	**S&P500**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**
Min	-11.98%	-11.78%	-12.16%	-12.11%	-13.50%	-11.74%	-13.05%	-12.89%	-13.08%	-11.75%	-13.05%	-12.89%	-13.08%
Max	9.38%	10.37%	10.99%	11.34%	11.81%	9.98%	12.13%	11.43%	11.27%	9.99%	12.13%	11.43%	11.27%
Annual Volatility	16.35%	16.35%	16.53%	16.72%	17.09%	16.32%	17.01%	17.05%	17.16%	16.36%	16.98%	17.03%	17.34%
Cumulative Return	280%	397%	379%	344%	406%	430%	416%	375%	454%	451%	465%	415%	451%
Correlation	1.000	0.994	0.987	0.981	0.976	0.994	0.981	0.976	0.973	0.990	0.975	0.968	0.964
Avg. Number of Assets		474.19	237.24	166.29	118.90	476.27	237.64	166.64	118.91	478.67	238.33	167.17	119.67
Monthly Avg. Turnover		3.53%	6.80%	6.97%	7.24%	3.45%	6.95%	7.21%	7.59%	1.73%	3.53%	3.61%	3.80%

Annual volatility (σ_a) is calculated as follows: $\sigma_a = \sigma_d \sqrt{252)}$, where σ_d comprises the standard deviation of daily returns. We use the convention of 252 trading days for one year.

Table 3 – Descriptive results for Japanese market.

Japan (developed)		120d				240d				480d			
Descriptive Stats	**Nikkei 225**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**
Min	-7.92%	-7.99%	-7.90%	-8.30%	-8.07%	-8.25%	-7.94%	-7.93%	-7.93%	-8.27%	-8.19%	-7.93%	-7.93%
Max	8.04%	7.87%	7.99%	7.49%	7.69%	8.07%	8.06%	7.31%	7.16%	8.28%	8.06%	7.31%	7.16%
Annual Volatility	20.66%	20.68%	20.61%	20.67%	20.73%	20.84%	20.74%	20.81%	21.02%	21.31%	21.29%	21.37%	21.39%
Cumulative Return	241%	335%	279%	267%	231%	330%	338%	316%	245%	348%	262%	212%	212%
Correlation	1.000	0.989	0.977	0.969	0.961	0.987	0.977	0.968	0.955	0.988	0.977	0.966	0.949
Avg. Number of Assets		201.19	107.57	75.43	54.05	204.36	109.18	76.45	55.00	209.17	111.50	78.33	56.00
Monthly Avg. Turnover		8.83%	12.21%	13.08%	13.95%	9.33%	12.81%	13.97%	14.58%	2.22%	3.28%	3.62%	3.76%

Annual volatility (σ_a) is calculated as follows: $\sigma_a = \sigma_d \sqrt{252)}$, where σ_d comprises the standard deviation of daily returns. We use the convention of 252 trading days for one year.

Table 4 – Descriptive results for Brazilian market

Brazil (emerging)		120d				240d				480d			
Descriptive Stats	**Ibovespa**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**
Min	-14.78%	-14.75%	-14.72%	-16.03%	-14.07%	-14.75%	-14.72%	-16.03%	-14.07%	-15.27%	-14.92%	-16.32%	-14.07%
Max	13.91%	14.40%	14.44%	15.19%	12.93%	14.40%	14.44%	15.19%	12.93%	14.82%	14.55%	15.35%	12.93%
Annual Volatility	25.24%	22.04%	22.48%	23.10%	22.93%	22.49%	23.33%	24.20%	24.40%	23.15%	24.06%	25.23%	24.85%
Cumulative Return	85%	120%	223%	169%	76%	326%	496%	407%	177%	373%	410%	375%	164%
Correlation	1.000	0.887	0.870	0.851	0.829	0.895	0.878	0.857	0.832	0.905	0.885	0.879	0.852
Avg. Number of Assets		70.19	35.52	25.00	17.76	71.55	35.91	25.27	17.82	73.50	36.83	25.67	17.67
Monthly Avg. Turnover		9.02%	12.12%	12.71%	13.29%	4.79%	6.95%	7.64%	7.53%	2.55%	3.56%	3.84%	4.08%

Annual volatility (σ_a) is calculated as follows: $\sigma_a = \sigma_d \sqrt{252)}$, where σ_d comprises the standard deviation of daily returns. We use the convention of 252 trading days for one year.

of assets that make up the portfolios of each model. In the 120-day rebalancing strategy, while the M2-E-P35 model uses, on average, 75 assets, the M1-B model uses, on average, 201. Correlation of returns with the return of the index presents, in the same way as the US market, a downward trend as the models demand more efficient assets in their composition.

For the Brazilian market, we noticed some differences from the other markets analyzed, precisely because of the difference in efficiency and volatility standards existing between developed and emerging markets. In Brazil, dealing with an emerging market and with higher volatility patterns, we have lower correlations between the returns of the Ibovespa index and the models, including for the benchmark model (M1-B). The correlation of the model returns with the index return also decreases as the number of assets decreases, i.e., moving from the M1-B model (unrestricted model) to the M2-E-1Q model (with severe efficiency restriction).

The tables 5, 6 and 6 show the average annual tracking error (TE), and the overall average, for the US, Japanese and Brazilian markets, respectively. To measure the tracking error, we use the definition of the equation 4.1. For the US market, we observed that the best average TE is always the M1-B model, which presents, for all rebalancing strategies, a lower TE than the models with efficiency constraints. The results show that, as the efficiency constraint becomes more severe in the model, i.e., we require more efficient assets, the average number of assets that are part of the portfolios decreases, and the TE increases slightly. For example, considering the semiannual rebalancing strategy in the US market, the benchmark model (M1-B) has an average number of assets of 474, and an average TE of 0.10% over the entire period. On the other hand, the model with efficiency constraint on the median of the Market Deficiency Measure (MDM) distribution, the M2-E-M, has an average number of assets in the period of 237, about half the number used by the M1-B model, with an average TE of 0.16%. Thus, the M2-E-M model demonstrates that the cost of reducing the average amount of assets by 50% is a 0.06% increment in the average tracking error.

Also, comparing the same two models, and looking at the year 2020, period of the covid-19 shock in the markets, we notice that the two models have the same TE. This means that, in this specific case, portfolios formed by the M2-E-M model showed the same level of vulnerability to crisis, in terms of TE, as the portfolios formed by the M1-B model.

For the Japanese market, we also see the same movement in terms of the average number of assets and TE. The efficiency constraint helps to reduce the average number of assets that are part of the portfolios, making them more manageable than larger portfolios, however, this reduction comes with a cost, which is the TE. In the 120-day rebalancing strategy, for example, the average TE for the entire out-of-sample period (projection period) was 0.18%, 0.26%, 0.31% and 0.35%, for the M1-B, M2-E-M, M2-E-P35 and M2-E-1Q models, respectively, while the average number of assets was 201, 108, 75 and 54, respectively. A highlight for the volatility of the models with efficiency restriction, which does not present a substantial increase in relation to the benchmark model and the Nikkei 225 index.

Regarding the Brazilian market, which is the emerging market in our set of analyzed markets, we observed a general level of tracking error (TE) higher than the other markets, precisely because of the difference in terms of volatility and structure in these markets. Looking at the 120-day (semi-annual) rebalancing strategy, the Brazilian market has an average TE for all models of 0.74%, compared to 0.17% for the US market and 0.28% for the Japanese market. As evidenced by several studies, as in SFGB, high volatility environments make tracking strategies difficult, increasing the tracking error of models more severely than in more developed markets, where the level of volatility tends to be lower. In the same way as for the US and Japanese markets, we observed a lower TE for the unrestricted model in all rebalancing strategies, which is expected, given that it has

Table 5 – Tracking Error results for the US market.

US (developed)												
	120d				**240d**				**480d**			
Annual Tracking Error	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**
2012	0.08%	0.13%	0.14%	0.20%	0.08%	0.14%	0.13%	0.20%	0.08%	0.14%	0.13%	0.19%
2013	0.07%	0.13%	0.15%	0.18%	0.07%	0.13%	0.16%	0.19%	0.07%	0.15%	0.16%	0.21%
2014	0.07%	0.14%	0.16%	0.17%	0.07%	0.13%	0.14%	0.15%	0.07%	0.13%	0.14%	0.16%
2015	0.08%	0.15%	0.17%	0.18%	0.09%	0.14%	0.16%	0.17%	0.09%	0.16%	0.18%	0.18%
2016	0.09%	0.16%	0.19%	0.20%	0.11%	0.16%	0.20%	0.21%	0.12%	0.17%	0.20%	0.20%
2017	0.07%	0.14%	0.16%	0.17%	0.07%	0.15%	0.17%	0.18%	0.08%	0.14%	0.18%	0.19%
2018	0.10%	0.17%	0.21%	0.22%	0.12%	0.16%	0.20%	0.22%	0.14%	0.17%	0.23%	0.23%
2019	0.07%	0.17%	0.20%	0.19%	0.07%	0.16%	0.17%	0.19%	0.09%	0.14%	0.17%	0.18%
2020	0.26%	0.26%	0.34%	0.42%	0.25%	0.48%	0.50%	0.53%	0.33%	0.55%	0.60%	0.65%
2021	0.11%	0.19%	0.26%	0.28%	0.13%	0.18%	0.25%	0.25%	0.17%	0.28%	0.34%	0.36%
Average	**0.10%**	**0.16%**	**0.20%**	**0.22%**	**0.11%**	**0.18%**	**0.21%**	**0.23%**	**0.12%**	**0.20%**	**0.23%**	**0.25%**

Table 6 – Tracking Error results for the Japanese market.

Japan (developed)												
	120d				**240d**				**480d**			
Annual Tracking Error	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**
2012	0.18%	0.28%	0.34%	0.36%	0.17%	0.25%	0.36%	0.39%	0.18%	0.27%	0.39%	0.44%
2013	0.20%	0.38%	0.41%	0.44%	0.21%	0.38%	0.42%	0.47%	0.19%	0.27%	0.41%	0.47%
2014	0.12%	0.17%	0.19%	0.26%	0.12%	0.18%	0.19%	0.28%	0.12%	0.18%	0.22%	0.28%
2015	0.13%	0.18%	0.26%	0.31%	0.14%	0.21%	0.31%	0.37%	0.15%	0.27%	0.31%	0.34%
2016	0.40%	0.44%	0.49%	0.49%	0.44%	0.45%	0.50%	0.49%	0.43%	0.46%	0.50%	0.48%
2017	0.18%	0.24%	0.26%	0.29%	0.24%	0.25%	0.26%	0.28%	0.12%	0.19%	0.22%	0.24%
2018	0.14%	0.20%	0.27%	0.30%	0.14%	0.24%	0.26%	0.31%	0.14%	0.27%	0.26%	0.34%
2019	0.15%	0.18%	0.25%	0.26%	0.14%	0.19%	0.30%	0.32%	0.15%	0.28%	0.31%	0.36%
2020	0.18%	0.32%	0.35%	0.46%	0.21%	0.27%	0.34%	0.58%	0.26%	0.31%	0.39%	0.66%
2021	0.13%	0.24%	0.30%	0.38%	0.15%	0.23%	0.28%	0.37%	0.19%	0.30%	0.36%	0.49%
Average	**0.18%**	**0.26%**	**0.31%**	**0.35%**	**0.20%**	**0.27%**	**0.32%**	**0.38%**	**0.19%**	**0.28%**	**0.34%**	**0.41%**

Table 7 – Tracking Error results for the Brazilian market.

Brazil (emerging)												
	120d				**240d**				**480d**			
Annual Tracking Error	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**
2012	0.96%	0.95%	1.07%	1.20%	0.57%	0.64%	0.70%	0.80%	0.62%	0.64%	0.69%	0.79%
2013	0.97%	0.87%	0.94%	0.89%	0.86%	0.80%	0.89%	0.88%	0.64%	0.72%	0.72%	0.74%
2014	1.00%	1.07%	1.07%	1.08%	1.05%	1.07%	0.99%	0.91%	0.99%	1.04%	0.95%	0.90%
2015	1.07%	1.14%	1.28%	1.34%	1.10%	1.21%	1.46%	1.58%	0.98%	1.07%	1.06%	1.13%
2016	1.04%	1.12%	1.11%	1.15%	1.06%	1.13%	1.12%	1.24%	1.08%	1.17%	1.18%	1.29%
2017	0.44%	0.45%	0.46%	0.53%	0.64%	0.63%	0.63%	0.70%	0.68%	0.73%	0.74%	0.85%
2018	0.15%	0.43%	0.46%	0.61%	0.16%	0.31%	0.46%	0.64%	0.16%	0.29%	0.48%	0.64%
2019	0.11%	0.38%	0.41%	0.50%	0.13%	0.43%	0.44%	0.63%	0.15%	0.28%	0.40%	0.63%
2020	0.22%	0.41%	0.58%	0.57%	0.25%	0.42%	0.61%	0.67%	0.30%	0.47%	0.68%	0.72%
2021	0.19%	0.31%	0.37%	0.56%	0.17%	0.28%	0.39%	0.50%	0.26%	0.40%	0.56%	0.63%
Average	**0.62%**	**0.71%**	**0.77%**	**0.84%**	**0.60%**	**0.69%**	**0.77%**	**0.85%**	**0.59%**	**0.68%**	**0.75%**	**0.83%**

an average number of assets considerably higher than the efficiency-constrained models. Looking specifically at the 120-day rebalancing strategy, the average TE of the benchmark model (M1-B) is 0.62% in the period, with an average number of assets of 70. Meanwhile, the model with the most severe efficiency constraint (M2-E-1Q), has an average TE of 0.84%, with an average of 18 assets. The figure 2 illustrates the trajectory of nominal $ 1 USD invested in the S&P500, M1-B and M2-E-M portfolios formed for the US market, with the semiannual update strategy, at the beginning of the projection period.

4.3 Method comparison

In this section, we compare the results of models with efficiency restriction with results obtained by other methods considered in the literature. We begin, in the 4.3.1

section, comparing the results of our model with the hybrid SFGB approach. In the 4.3.2 section, we make a comparison with the cointegration approach performed in Sant'Anna, Filomena and Caldeira (2017).

4.3.1 Heuristic method

As we mentioned in the 2.3 section, the nature of the index tracking problem is to replicate a market index with a limited number of assets. The mathematical formulation of this problem is often treated as an mixed-integer quadratic programming problem (MIQP), making it an NP-hard problem (Coleman; Li; Henniger, 2004). Thus, in order to find a solution to the problem in a reasonable time, several methods have been formulated, however, many end up with high computational complexity. SFGB uses a hybrid approach with a genetic algorithm and nonlinear mathematical programming, obtaining good tracking results for the Brazilian market with a very small number of assets – 5 and 10 assets, out of a set of 67 assets, and also for developed markets (USA, UK and Germany). These results are achieved in less than 10 minutes of computational processing.

In order to compare the performance of models with efficiency constraint with some method already used in the literature, we selected the study by SFGB for this purpose. To obtain an adequate basis for comparison, we need to use the same number of assets in the portfolios obtained by SFGB. Then, we raise the model's efficiency constraint until the number of assets in that study is reached, i.e., 5 and 10 assets. The portfolio projection range (out-of-sample) starts in January 2010 and runs until July 2012, using daily returns from the 67 assets that make up the Ibovespa index (target index), during the same period. This study made use of rebalancing strategies of 20, 60, 120 and 240 trading days (monthly, quarterly, semiannually and annually, respectively). We use the same database used in SFGB.

Thus, in order to make the comparison with the aforementioned study, first, we performed the efficiency estimations of each of the 67 assets that are part of the assets set, using a year to estimate the Market Deficiency Measure (MDM) immediately prior to the initial date of portfolio projection. As for the portfolios designed and analyzed in the 4.2 section, we use a dynamic efficiency approach, measuring the MDM at each rebalancing, in order to have an updated measure to be used in the optimization process.

To obtain the same number of assets as the portfolios obtained by SFGB, we analyze, at each rebalancing, the distribution of the MDM of the assets, in order to place an efficiency constraint that limits the space of feasible solutions to just the number of assets from the compared portfolios, i.e., 5 and 10 assets. Thus, we have extremely severe efficiency constraints in this approach, limiting the solution space to the 5 and 10 most efficient assets in each rebalancing interval.

We list the comparison of the results obtained by the hybrid approach of SFGB, which makes use of a genetic algorithm and nonlinear mathematical programming, with the results of the efficiency-constrained models addressed in the present study, which, in turn, only make use of nonlinear mathematical programming, being characterized as quadratic programming problem (QP), obtaining an instant solution to the problem. Thus, obtaining similar or better results than those generated by the approach of SFGB, we would be achieving similar or better results using a method of low computational complexity, as opposed to the one used by SFGB. The 8 table shows the results obtained by the study of SFGB, the results obtained by the models with efficiency restrictions, and the difference between the average TE obtained between the aforementioned study and the model formulated in the present research.

The results demonstrate a slight superiority of the hybrid approach between genetic algorithm and nonlinear mathematical programming, in terms of average tracking error (TE), where the hybrid solution approach has a slightly lower average TE than the model with efficiency constraints. However, as mentioned, this approach has high computational complexity, and its results, despite taking a low computational processing time (less than 10 minutes), are not instantaneous. Although the efficiency constrained model presents a slightly worse result in terms of average TE, this problem is simply formulated through quadratic programming (QP), where efficiency constraints play the role of a cardinality constraint, limiting the number of assets that are part of the set of feasible solutions as the efficiency requirement becomes more severe. Since this is a QP problem, the efficiency constrained approach presents an instantaneous result and relatively low computational complexity. Thus, we obtained similar results, in terms of average TE, using a relatively simpler approach, through a QP formulation, when comparing the results of a more worked method, which involves high computational complexity and does not provide an instantaneous solution (solution with less than 10 minutes of processing time).

It is worth noting that when we limit the space of feasible solutions to only extremely efficient assets, as in the case of 5 assets, for example, we are not necessarily choosing the assets with the best relation to the index over time, but rather limiting our set of solutions to the most efficient assets in the period. Thus, the benefit of considering efficiency constraints (reduction in the number of assets) for cases of severe constraints, may end up being the main disadvantage in terms of TE, thus showing the trade-off between efficiency and tracking error.

4.3.2 Cointegration

Another alternative for the formation of tracking portfolios is through the cointegration approach. This approach aims to find long-term relationships between the set of assets that will compose the tracking portfolios and the index that will be replicated

Table 8 – Comparison of the hybrid approach used in SFGB and the efficiency-constrained models.

Sant'Anna et al (2017) results								
	10 Assets				**5 Assets**			
Tracking Error	**20**	**60**	**120**	**240**	**20**	**60**	**120**	**240**
Average	0.055%	0.032%	0.024%	0.017%	0.078%	0.049%	0.034%	0.023%
Minimum	0.037%	0.025%	0.014%	0.015%	0.046%	0.036%	0.025%	0.022%
Maximum	0.088%	0.045%	0.037%	0.021%	0.131%	0.071%	0.052%	0.027%
SD	0.014%	0.006%	0.007%	0.003%	0.023%	0.010%	0.009%	0.003%
Efficiency-constrained								
	10 Assets				**5 Assets**			
Tracking Error	**20**	**60**	**120**	**240**	**20**	**60**	**120**	**240**
Average	0.111%	0.065%	0.040%	0.028%	0.149%	0.079%	0.049%	0.033%
Minimum	0.003%	0.042%	0.029%	0.025%	0.001%	0.060%	0.031%	0.030%
Maximum	0.189%	0.093%	0.055%	0.030%	0.294%	0.122%	0.064%	0.039%
SD	0.042%	0.016%	0.011%	0.002%	0.059%	0.018%	0.010%	0.004%
Difference in Average TE (Performance Gain/Loss)	**0.056%**	**0.033%**	**0.016%**	**0.011%**	**0.071%**	**0.030%**	**0.015%**	**0.010%**

Difference in Average tracking error (TE) comprises the difference between the average TE of the portfolios generated by the model with efficiency constraints and the average TE of the portfolios formed by the hybrid approach of SFGB.

(Alexander; Dimitriu, 2005; Dunis; Ho, 2005). This method has been shown to be an alternative to the classic approach to solving the index tracking problem, especially when the objective is to form portfolios with a very small number of assets, where, by inserting binary variables in the problem, we significantly increase its computational complexity. In order to compare the results of optimization models with efficiency constraints, we selected the study by Sant'Anna, Filomena and Caldeira (2017) (henceforth SFC), which compares the performance of tracking portfolios formed by the cointegration approach and the optimization approach (correlation) in two markets with different levels of development: Brazilian market and the US market.

For the formation of portfolios using the cointegration method for the US market, the index to be replicated is the S&P100, with a sample of 97 assets. SFC performs 35,000 random regressions, where each one uses a combination of only 10 assets, which is the number of assets that will compose the tracking portfolios. Within the 35,000 regressions, the combination of assets that satisfy the cointegration requirements are pre-selected as candidates. Among the pre-selected portfolios, the portfolio that presents the lowest value in the sum of squares of the residuals is chosen. Then, SFC builds portfolios to replicate the S&P100 index with just 10 assets.

For comparison purposes, we selected the portfolios of the aforementioned study formed by the cointegration method to track the S&P100 index, and we use the same database as SFC. Specifically, we select portfolios denoted C1y.6m, C1y.1y, C2y.6m and C2y.1y, which represent portfolios formed with an in-sample interval of one and two years (terms "1y"and "2y"at the beginning of the portfolio name), and with semi-annual and annual rebalancing ("6m"and "1y"at the end of the portfolio nomenclature). As mentioned,

Table 9 – Descriptive results for portfolios formed by the cointegration method and comparable portfolios formed by the efficiency-constrained optimization method.

		Efficiency-constrained		**Cointegration**			
Descriptive Stats	**S&P100**	**EC.6m**	**EC.1y**	**C1y.6m**	**C1y.1y**	**C2y.6m**	**C2y.1y**
Annual Volatility	20.79%	21.15%	21.54%	22.72%	22.65%	24.20%	24.17%
Cumulative Return	53.63%	136.65%	127.84%	61.44%	70.19%	90.53%	115.89%
Correlation	1.000	0.936	0.927	0.958	0.966	0.951	0.948
Avg. Number of Assets		10	10	10	10	10	10
Monthly Avg. Turnover		13.10%	7.82%	15.03%	7.78%	13.86%	7.45%

The descriptive results presented for the portfolios formed by the cointegration method were obtained from the SFC paper.

these portfolios are made up of 10 assets. So, in order to form comparable portfolios through the efficiency-constrained model, we perform the same approach applied to the comparison with the hybrid approach of SFGB: we identified the level of Market Deficiency Measure (MDM) that makes the space of feasible solutions composed of only 10 assets, in each rebalancing. In this way, we set up a portfolio with efficiency restriction composed of the 10 most efficient assets from the set of 97 assets used by SFGB, in each rebalancing, adopting comparable rebalancing approaches, i.e., semi-annual and annual. As for the in-sample range of daily returns for the optimization process, we remain using the 120 trading days immediately preceding the optimization (rebalancing) date.

The out-of-sample period selected for comparison (portfolio projection period) starts in January 2006, running until August 2014. The 9 table presents the descriptive statistics of the projected portfolios. We denote EC.6m and EC.1y for the portfolios formed by the efficiency-constrained optimization model, with semi-annual and annual rebalancing intervals, respectively. Initially, we noticed a higher correlation with the reference index (in this case the S&P100) for the portfolios formed by the cointegration approach, presenting an average of the 4 models (C1y.6m, C1y.1y, C2y.6m and C2y.1y) of 0.956, while the average of the two models with efficiency restriction (EC.6m and EC.1y) was 0.932. In terms of volatility and cumulative annual return, efficiency-restricted portfolios performed better, with 21.34% and 132.24% of annual volatility and cumulative annual return, respectively, compared to an average of 23.44% and 84.51% for portfolios formed by the cointegration model. In terms of average monthly turnover, efficiency-constrained portfolios also performed slightly better, averaging 10.46%, compared to 11.03% for cointegration portfolios.

The 10 table presents the tracking error accumulated in each portfolio projection year, this time represented by the equation 4.3:

Table 10 – Comparison of the results obtained by the cointegration approach with efficiency-constrained models.

	Efficiency-constrained		**Cointegration**			
Year	**EC6M**	**EC12M**	**C1y.6m**	**C1y.12m**	**C2y.6m**	**C2y.12m**
2006	11.37%	5.51%	9.18%	1.87%	2.41%	0.25%
2007	-5.91%	9.36%	-8.41%	-1.43%	3.03%	9.24%
2008	20.13%	5.27%	-6.56%	6.24%	2.57%	14.97%
2009	14.43%	8.91%	7.79%	7.10%	11.21%	19.98%
2010	-2.55%	4.31%	-0.74%	-4.57%	8.65%	3.98%
2011	-1.75%	-7.03%	5.32%	2.83%	4.87%	8.34%
2012	1.78%	-0.57%	3.30%	2.80%	0.39%	1.91%
2013	5.24%	11.89%	4.76%	5.56%	4.92%	4.88%
2014	1.99%	3.11%	0.20%	3.20%	5.87%	5.75%
Average	**4.97%**	**4.53%**	**1.65%**	**2.62%**	**4.88%**	**7.70%**

$$TE^b = \sum_{i=1}^{T} x_i r_{i,t} - R_t \tag{4.3}$$

where $x_i r_{i,t}$ represents the portfolio return in time t, for $t = 1, ..., T$, and R_t represents the return of the market index at time t. We observed that, in general, portfolios formed by the cointegration method have a lower average annual tracking error (measured by the 4.3 equation) than portfolios formed by the efficiency-constrained optimization method. Clearly, when we measure the performance by this variable, the C1y.6m portfolio shows the best performance, which is formed by the cointegration method, with one year of data for its estimation and with semi-annual rebalancing. However, when we analyze the portfolios formed by the cointegration method using two years of data for estimation (i.e., portfolios C2y.6m and C2y.1y), we observe that they present a higher average annual tracking error. The average tracking error of these two portfolios comprises 6.29%, while the average annual tracking error of the portfolios formed by the efficiency-constrained optimization method comprises 4.75%.

As a result of the results obtained in this comparison, we argue for the relevance of the method addressed in this research, which is capable of presenting satisfactory performance, when compared with the cointegration method, obtaining an instantaneous solution to the problem, and with low computational complexity. According to SFC, the portfolios formed by the cointegration method showed a limit on the size of random simulations, where, for the S&P100 index, the number of simulations used was 35,000. This can clearly be a limiting aspect when we seek to form portfolios to replicate larger indices, such as the S&P500 or Russell 1000, for example. In addition, the authors mention that the model does not instantly generate a solution, despite having a low computational processing time, which is up to 5 minutes.

4.4 Efficiency and liquidity

A question that naturally arises when observing our results is whether there is an equivalence in considering efficiency and liquidity constraints in the index tracking (IT) models. Vieira et al. (2021) incorporates liquidity restrictions in index tracking models for the Brazilian market, however, at the portfolio level, demonstrating the existing trade-off between portfolio liquidity and tracking error, when using Amihud's illiquidity. Studies, such as Chordia, Roll and Subrahmanyam (2008), demonstrate that liquidity is essential to eliminate arbitrage opportunities. The authors verify, observing intraday returns, that the predictability of returns reduces considerably during periods of greater liquidity in the markets, where prices behave more similarly to a random walk. It is argued that the evidence obtained is consistent with the hypothesis that an increase in arbitrage activity during periods of high liquidity improves market efficiency. So, the authors suggest that an increase in liquidity can help improve market efficiency.

To try to verify if there is some kind of equivalence of results between portfolios with efficiency constraints and portfolios with liquidity constraints, we make a brief comparison between the results obtained with these two types of constraints. A problem immediately arises when formulating a liquidity-constrained index tracking (IT) model to make a comparison with efficiency-constrained models: the measure of liquidity to be used. The liquidity literature emphasizes this difficulty, given that liquidity is a multidimensional concept, involving quantity, costs and time (Goyenko; Holden; Trzcinka, 2009). As a measure of liquidity, we will use the illiquidity of Amihud (2002), ξ_i, represented by the equation 4.4, which is one of the most used in the literature, and already used in IT models by Vieira et al. (2021):

$$\xi_i = \frac{1}{D_i} \sum_{t=1}^{D_i} \frac{|r_{i,t}|}{VOL_{i,t}} \tag{4.4}$$

where D_i comprises the number of observations of asset i; $r_{i,t}$ comprises the return on the asset i in the period t; and $VOL_{i,t}$ comprises the traded volume of the asset i in the period t.

Then, we will structure a new model, with the same form as the M2-E model, but which, this time, incorporates liquidity constraint at the asset level, instead of an efficiency constraint. We call this model M2-L. The results obtained with this model may suggest an answer about the existence or not of an equivalence in considering efficiency and liquidity constraints in index tracking (IT) models. To obtain the M2 model, we only modify the efficiency constraint (expression (3.12)), replacing it with the liquidity constraint shown below. This simple constraint replacement is possible, since the Market Deficiency Measure (MDM) and the Amihud Illiquidity (ξ_i) grow with inefficiency and illiquidity, respectively. So, instead of using the constraint (3.12), we use the following constraint, to create IT models with liquidity constraints. Thus, to form the M2-L model, we have:

$$x_i \xi_i \leq x_i \lambda, \ \forall i \in I \tag{4.5}$$

where x_i represents the weight allocated to asset i in the portfolio; ξ_i represents the illiquidity of asset i; and λ represents, in this case, the illiquidity threshold at the asset level. Thus, assets $i \in I$ with liquidity levels $0 < \xi_i \leq \lambda$ are part of the set of feasible solutions. Then, the constraint (4.5) replaces the constraint (3.12) in the M2-E model.

To try to show whether considering liquidity constraints (represented by the measure (4.4)) and efficiency constraints have some equivalence, we compared the results already obtained for the M1-B and M2-E models with the results obtained by the model M2-L. We obtained portfolios with liquidity restrictions for the same time interval and in the same way as the tests performed for the models with efficiency restrictions, where our projection interval (out-of-sample) was from Jan/2012 to Dec/2021. To measure liquidity, we also use a dynamic approach, where, at each rebalancing, we measure ξ_i of the set of assets, with a data range of one year of daily returns immediately prior to the projection date of the aforementioned rebalancing portfolio. In this way, we have models with liquidity constraints that are updated over time. We designed the liquidity-constrained portfolios considering only a semi-annual rebalancing strategy, i.e., 120 trading days. And, regarding restrictions, we use the same level of restrictions as in the M2-E models, i.e., we consider the distribution of ξ_i of assets at each rebalancing interval, using the median, 35th percentile and first quartile as liquidity constraint, forming the M2-L-M, M2-L-P35 and M2-L-1Q models, respectively. So, we have models with a slightly more relaxed liquidity constraint (M2-L-M), going up to more severe liquidity constraints (M2-L-1Q), where only extremely liquid assets, according to the measure of Amihud (2002), are part of the set of feasible solutions.

Tables 11, 12 and 13 show the results obtained for the US, Japanese and Brazilian markets, respectively, both descriptive and tracking error (TE). For the US market, we did not observe large differences in volatility and cumulative return between the efficiency and liquidity constrained models. As expected, liquidity constraints also decrease the average number of assets that are part of the portfolios as the constraint becomes more severe, i.e., requiring only more liquid assets. For example, for the M2-L-M model (with a more relaxed liquidity constraint – distribution median), we have an average number of assets of 237, while for the M2-L-1Q model (with a more severe liquidity constraint – first quartile of the distribution), we have an average number of assets of 120. This reduction in the number of assets is similar to the reduction caused by efficiency constraints, where we went from 237 assets, in the M2-E-M model, to 119 assets, in the M2-E-1Q model. In terms of TE, we observe that both models show a tendency for TE to increase as the efficiency and liquidity constraints become more severe, which was expected, given that

Table 11 – Results of optimization models with efficiency and liquidity constraints for the US market, with semiannual rebalancing (120 trading days).

US								
		120d						
Descriptive Stats	**S&P500**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
Min	-11.98%	-11.78%	-12.16%	-12.11%	-13.50%	-11.67%	-12.00%	-12.81%
Max	9.38%	10.37%	10.99%	11.34%	11.81%	10.88%	10.55%	10.72%
Annual Volatility	16.35%	16.35%	16.53%	16.72%	17.09%	16.42%	16.33%	16.63%
Cumulative Return	280%	397%	379%	344%	406%	364%	363%	358%
Correlation	1.000	0.994	0.987	0.981	0.976	0.992	0.991	0.988
Avg. Number of Assets		474.19	237.24	166.29	118.90	237.48	167.57	120.33
Monthly Avg. Turnover		3.53%	6.80%	6.97%	7.24%	4.38%	4.78%	4.15%

Tracking Error	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
2012	0.08%	0.13%	0.14%	0.20%	0.10%	0.09%	0.10%
2013	0.07%	0.13%	0.15%	0.18%	0.09%	0.11%	0.10%
2014	0.07%	0.14%	0.16%	0.17%	0.07%	0.08%	0.10%
2015	0.08%	0.15%	0.17%	0.18%	0.09%	0.11%	0.13%
2016	0.09%	0.16%	0.19%	0.20%	0.09%	0.12%	0.13%
2017	0.07%	0.14%	0.16%	0.17%	0.08%	0.13%	0.12%
2018	0.10%	0.17%	0.21%	0.22%	0.12%	0.13%	0.14%
2019	0.07%	0.17%	0.20%	0.19%	0.10%	0.11%	0.13%
2020	0.26%	0.26%	0.34%	0.42%	0.30%	0.27%	0.36%
2021	0.11%	0.19%	0.26%	0.28%	0.13%	0.15%	0.17%
Average	**0.10%**	**0.16%**	**0.20%**	**0.22%**	**0.12%**	**0.13%**	**0.15%**

Table 12 – Results of optimization models with efficiency and liquidity constraints for the Japanese market, with semiannual rebalancing (120 trading days).

Japan								
		120d						
Descriptive Stats	**Nikkei 225**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
Min	-7.92%	-7.99%	-7.90%	-8.30%	-8.07%	-7.71%	-7.52%	-7.50%
Max	8.04%	7.87%	7.99%	7.49%	7.69%	7.96%	8.06%	9.56%
Annual Volatility	20.66%	20.68%	20.61%	20.67%	20.73%	20.49%	20.37%	20.51%
Cumulative Return	241%	335%	279%	267%	231%	271%	318%	305%
Correlation	1.000	0.989	0.977	0.969	0.961	0.968	0.965	0.953
Avg. Number of Assets		201.19	107.57	75.43	54.05	108.33	76.14	54.52
Monthly Avg. Turnover		8.83%	12.21%	13.08%	13.95%	9.77%	9.15%	8.98%

Tracking Error	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
2012	0.18%	0.28%	0.34%	0.36%	0.29%	0.31%	0.32%
2013	0.20%	0.38%	0.41%	0.44%	0.34%	0.39%	0.43%
2014	0.12%	0.17%	0.19%	0.26%	0.24%	0.26%	0.34%
2015	0.13%	0.18%	0.26%	0.31%	0.27%	0.29%	0.37%
2016	0.40%	0.44%	0.49%	0.49%	0.46%	0.46%	0.49%
2017	0.18%	0.24%	0.26%	0.29%	0.24%	0.25%	0.28%
2018	0.14%	0.20%	0.27%	0.30%	0.30%	0.30%	0.37%
2019	0.15%	0.18%	0.25%	0.26%	0.30%	0.30%	0.36%
2020	0.18%	0.32%	0.35%	0.46%	0.44%	0.43%	0.50%
2021	0.13%	0.24%	0.30%	0.38%	0.31%	0.35%	0.45%
Average	**0.18%**	**0.26%**	**0.31%**	**0.35%**	**0.32%**	**0.33%**	**0.39%**

Table 13 – Results of optimization models with efficiency and liquidity constraints for the Brazilian market, with semiannual rebalancing (120 trading days).

Brazil								
		120d						
Descriptive Stats	**Ibovespa**	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
Min	-14.78%	-14.75%	-14.72%	-16.03%	-14.07%	-14.79%	-14.78%	-14.73%
Max	13.91%	14.40%	14.44%	15.19%	12.93%	14.29%	14.92%	15.19%
Annual Volatility	25.24%	22.04%	22.48%	23.10%	22.93%	22.65%	23.35%	23.79%
Cumulative Return	85%	120%	223%	169%	76%	314%	360%	314%
Correlation	1.000	0.887	0.870	0.851	0.829	0.899	0.886	0.883
Avg. Number of Assets		70.19	35.52	25.00	17.76	37.00	26.10	18.67
Monthly Avg. Turnover		9.02%	12.12%	12.71%	13.29%	8.48%	7.78%	7.81%

Tracking Error	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
2012	0.96%	0.95%	1.07%	1.20%	0.96%	0.90%	0.88%
2013	0.97%	0.87%	0.94%	0.89%	0.81%	0.86%	0.89%
2014	1.00%	1.07%	1.07%	1.08%	0.99%	1.10%	1.01%
2015	1.07%	1.14%	1.28%	1.34%	0.95%	1.07%	1.06%
2016	1.04%	1.12%	1.11%	1.15%	1.04%	1.06%	1.12%
2017	0.44%	0.45%	0.46%	0.53%	0.44%	0.52%	0.57%
2018	0.15%	0.43%	0.46%	0.61%	0.16%	0.20%	0.26%
2019	0.11%	0.38%	0.41%	0.50%	0.14%	0.17%	0.23%
2020	0.22%	0.41%	0.58%	0.57%	0.25%	0.29%	0.37%
2021	0.19%	0.31%	0.37%	0.56%	0.22%	0.27%	0.38%
Average	**0.62%**	**0.71%**	**0.77%**	**0.84%**	**0.60%**	**0.64%**	**0.68%**

we are limiting the space for solutions to the most efficient and more liquid assets, and these do not necessarily have a better relationship with the S&P500 index. In this market, the average TE for models with liquidity constraint is slightly lower than the average TE for models with efficiency constraint.

For the Japanese market, we observed a slight superiority of the efficiency-constrained models in relation to the liquidity-constrained models. The average tracking error of the three M2-E models is 0.31%, while the average TE of the three M2-L models is 0.35%. The average number of assets in the three efficiency and liquidity models is around 79 assets. In terms of correlation of daily returns with the Nikkei 225, we also observed a slight superiority for the efficiency constrained models, where the mean of these models was 0.969, while the mean of the liquidity constrained models was 0.962. In short, the results for this market point to a similar performance between the two approaches, with an slight advantage for the models with efficiency constraint.

For the Brazilian market, which is the emerging market in our sample, we observed slightly different results between the two approaches. Regarding correlation, the mean of the models with efficiency constraint was 0.85, while the mean of the models with liquidity constraint was 0.889. The average number of assets remains close, being 26 for M2-E models and 28 for M2-L models. Analyzing the TE, we see a superiority of the models with liquidity restriction, which presented an average of TE of 0.64%, compared to the models with restriction of efficiency, which presented an average of TE of 0.78%. This

Table 14 – Cost of reducing the number of assets.

	Efficiency-constrained			**Liquidity-constained**		
	US	**Japan**	**Brazil**	**US**	**Japan**	**Brazil**
Asset reduction	355	147	52	354	147	52
TE Gap	0.1213%	0.1733%	0.2281%	0.0485%	0.2087%	0.0600%
TE Cost	0.0003%	0.0012%	0.0044%	0.0001%	0.0014%	0.0012%

Asset reduction represents the reduction in the average number of assets generated by the more restricted models (1Q models), compared to the M1-B model. TE Gap comprises the difference between the average tracking error of the more restricted models and the average tracking error of the M1-B model. TE Cost comprises the ratio between TE Gap and Asset reduction.

effect may occur considering that the Ibovespa index is based on liquidity, so that more liquid assets are more likely to compose the index.

Efficiency and liquidity are different concepts and measured in different ways. Our results of the brief empirical procedures applied to this comparison indicate that the efficiency and liquidity constraints, measured by the Market Deficiency Measure (MDM) and Amihud's Illiquidity, respectively, present different similarities when placed in the index tracking (IT) portfolio optimization problem, reducing the average number of assets in portfolios.

4.5 Discussion

A different way of analyzing the impact of restrictions in different markets, with different structures, is to analyze the reduction in the average number of assets from the unrestricted model (M1-B) to the more restricted models (M2-E-1Q and M2-L-1Q), and the cost of this reduction, i.e., the increase in tracking error (TE). This comparison highlights the cost of maintaining a portfolio with a manageable number of assets in terms of TE. The 14 table shows the cost, in terms of TE, caused by the reduction in the number of assets for the models with efficiency and liquidity restrictions, dealing with the 120-day (semi-annual) rebalancing strategy. Asset reduction comprises the reduction in the average number of assets from the unrestricted model (M1-B) to the more restricted models (with restriction in the first quartile – 1Q). The TE Gap comprises the difference between the average TE of the more restricted models (1Q models) and the average TE of the unrestricted model (M1-B). And, finally, the TE Cost comprises the ratio between the TE Gap and the Asset reduction, thus demonstrating what the cost is, in terms of TE, for each asset reduction in the average number of assets.

Addressing efficiency constraints, in the US market, we have a reduction in the average number of assets of 355 (from 474 to 119), and an TE Gap of 0.12%. For the Japanese market, we have a reduction in the average number of assets of 147 (from 201 to 54), with an TE Gap of 0.17%. Finally, in the Brazilian market, we have a reduction in the average number of assets of 52 (from 70 to 18), resulting in a TE GAP of 0.23%. So, for the efficiency constrained approach, we have a TE Cost of 0.0003%, 0.0012% and 0.0044%, for the US, Japanese and Brazilian markets, respectively. We observe that the TE Cost increases for emerging markets, such as the Brazilian market, making efficiency constraints more costly in terms of TE.

On the other hand, when we look at the results of the liquidity constrained approach (measured by the illiquidity of Amihud (2002)), we do not see the same movement observed in the efficiency constrained approach. We have a relatively low TE Cost for the US and Brazilian market, and a higher one for the Japanese market. These results also corroborate the idea presented in the 4.4 section, where we argue that, with our results, there is apparently no equivalence of effects on tracking portfolios when we use efficiency and liquidity constraints, measured by the Market Deficiency Measure (MDM) and the illiquidity of Amihud (2002), respectively.

Thus, the results from the table 14 show us that efficiency constraints appear to be less costly (in terms of tracking error) in more efficient markets. As markets become less efficient, it is expected that we will have a limited set of assets that are able to meet the constraints, harming portfolios formed in terms of tracking error, resulting in a higher TE Cost. On the other hand, in markets with higher overall efficiency levels, we expect the TE Cost to be lower when compared to emerging markets, as we would have a less restrictive set of assets that satisfy the efficiency conditions of the models. It is worth remembering that more efficient assets do not necessarily have a better relationship with the target index. So, the results seem to suggest that efficiency constraints, despite being able to make the average number of assets of the formed portfolios more manageable with low computational cost, would be even more effective in more developed markets, where the general level of efficiency tends to be higher. In these markets, the cost of incorporating efficiency constraints (TE Cost), in terms of TE, is lower than in emerging markets.

4.6 Statistical difference in means

As a way of validating our results, we tested the difference in the average of daily returns and the tracking error (represented by the equation 4.1) of the projected portfolios. For the case of daily returns, we performed the test for the difference in the average of the portfolio's returns with the returns of the index of the respective market. For the case of tracking error, we tested the difference in the average of the daily TE between the portfolios with restrictions (for efficiency and liquidity) and the unrestricted model (benchmark model).

We performed the mean difference test using the bootstrap method, both for daily

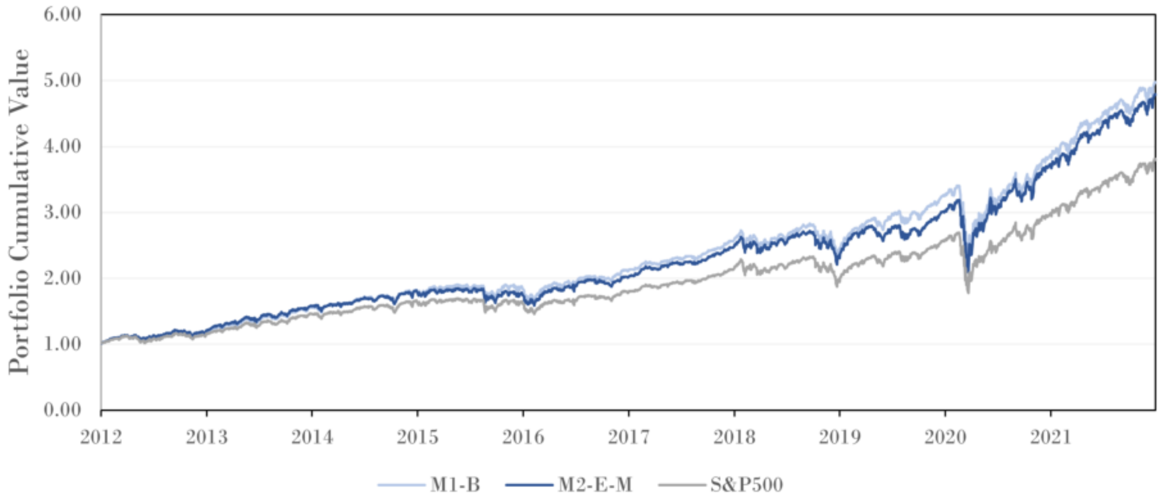

Figure 2 – Accumulated values of the portfolios invested in the S&P500 index and in the portfolios formed by the M1-B and M2-E-M models, with a 120-day rebalancing (the value of the portfolios starts at $1 in January 2012).

returns and for daily tracking error, in the same way as performed in Sant'Anna et al. (2019) and Vieira et al. (2021). Given two time series, $y_{1,t}$ and $y_{2,t}$, $t = 1, 2, ..., T$, we perform the t-test to test the null hypothesis $H_0 : \mu_{y_1} = \mu_{y_2}$. From the two time series, we select V values, where $V \in T$, forming two new subsets, $y^s_{1,t}$ and $y^s_{2,t}$, where $v = 1, 2, ..., V$. For these subsets, then, we test the null hypothesis $H_0 : \mu_{y^s_{2,t}} = \mu_{y^s_{2,t}}$. This random selection procedure is performed S times for each year of the portfolio projection interval, and, for each test, we perform the calculation of the statistic $z_s = \mu_{y^s_{1,t}} - \mu_{y^s_{2,t}}$, thus obtaining a set z_s, $s = 1, 2, ..., S$; with this set, and considering a confidence interval $1 - \alpha$, (where we consider $\alpha = 5\%$), we compute the lower and upper limits, CI- and CI+. If 0 (zero) is between the limits, i.e., CI- $\leq 0 \leq$ CI+, we do not reject H_0; otherwise, H_0 is rejected. For the tests, we used $V = 50$ and $S = 1000$.

Tables 15, 16 and 17 present the results for the average difference tests of daily returns for the US, Japanese and Brazilian markets, respectively. The results indicate the failure to reject H_0, demonstrating the lack of statistically significant difference between the daily returns of the optimized portfolios (both unrestricted and restricted) and the daily returns of the reference index, being a favorable aspect in relation to the similarity of the returns daily portfolios to their benchmark. Other favorable results for the restricted models are the tests of the mean difference between the daily tracking error (TE) of the unrestricted model (benchmark model) and the restricted models (models with efficiency and liquidity constraints). The tables 18, 19 and 20 present the results for the daily TE mean difference test between the unrestricted model and the restricted models, for the US market, Japanese and Brazilian, respectively. Not rejecting the null hypothesis for this test means saying that the daily tracking error of the constrained portfolios and the portfolios formed by the unrestricted model is similar, i.e., the tracking quality between the portfolios is similar. It is exactly these results that we can verify in the mentioned tables: the failure to reject H_0 for all cases. So, we argue in favor of the tracking quality of the constrained models, especially the efficiency-constrained models, in relation to the unconstrained model. This means that, while the unrestricted model does not present restrictions on the number of component assets, always presenting portfolios with a high number of assets, the models with efficiency and liquidity restrictions, which present smaller portfolios, especially when such restrictions are more severe, even so these constrained models present tracking quality similar to the unrestricted model, maintaining a smaller number of assets. This becomes even more relevant when dealing with larger indices, as is the case of the S&P500, where the unrestricted model makes use of a large number of assets to track the index, while models with efficiency constraints perform tracking of similar quality, but with a reduced number of assets.

Table 15 – Test results for difference in means of daily returns in the portfolio projection period (out-of-sample period) for the US market.

2012	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0000568	-0.0001344	-0.0001653	-0.0002294	-0.0001753	-0.0001328	-0.0001086
CI+	0.0003220	0.0004758	0.0005344	0.0007397	0.0003164	0.0003139	0.0003535
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2013	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0000406	-0.0001856	-0.0002633	-0.0002024	-0.0001112	-0.0001327	-0.0000776
CI+	0.0002911	0.0004943	0.0004745	0.0006373	0.0003132	0.0004097	0.0004183
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2014	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0000388	-0.0003110	-0.0003749	-0.0003178	-0.0001386	-0.0001474	-0.0002089
CI+	0.0002830	0.0003490	0.0004056	0.0005511	0.0002415	0.0002684	0.0002977
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2015	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0000678	-0.0004289	-0.0004230	-0.0004912	-0.0001241	-0.0001242	-0.0001997
CI+	0.0003367	0.0003135	0.0004041	0.0003867	0.0003382	0.0004192	0.0004237
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2016	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0000855	-0.0002145	-0.0003032	-0.0002962	-0.0000674	-0.0000206	-0.0000806
CI+	0.0003770	0.0005642	0.0006550	0.0006631	0.0003799	0.0005337	0.0005650
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2017	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0001007	-0.0002428	-0.0003188	-0.0002680	-0.0001551	-0.0002980	-0.0002924
CI+	0.0002316	0.0004531	0.0004853	0.0005528	0.0002324	0.0003475	0.0002992
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2018	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0001684	-0.0003057	-0.0006588	-0.0005675	-0.0002400	-0.0002541	-0.0003557
CI+	0.0003337	0.0004751	0.0003402	0.0005506	0.0003262	0.0003384	0.0003531
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2019	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0000419	-0.0004392	-0.0005964	-0.0005412	-0.0001175	-0.0002757	-0.0001987
CI+	0.0003085	0.0003626	0.0003450	0.0004399	0.0003804	0.0002791	0.0004322
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2020	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005446	-0.0003637	-0.0006772	-0.0010859	-0.0006868	-0.0006760	-0.0008377
CI+	0.0007122	0.0008598	0.0009474	0.0010202	0.0007886	0.0006496	0.0008841
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2021	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0002278	-0.0004173	-0.0004556	-0.0003844	-0.0002654	-0.0003689	-0.0004668
CI+	0.0002893	0.0004725	0.0008283	0.0009349	0.0003444	0.0003914	0.0003767
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

Table 16 – Test results for difference in means of daily returns in the portfolio projection period (out-of-sample period) for the Japanese market.

2012	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005074	-0.0005875	-0.0008447	-0.0007639	-0.0010171	-0.0010374
CI+	0.0006800	0.0009005	0.0007834	0.0005830	0.0005203	0.0006301
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2013	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0008528	-0.0011152	-0.0012968	-0.0007283	-0.0007498	-0.0009538
CI+	0.0007133	0.0006256	0.0005549	0.0006536	0.0008954	0.0010298
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2014	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0004122	-0.0004157	-0.0005129	-0.0006332	-0.0005606	-0.0008572
CI+	0.0004980	0.0005850	0.0008560	0.0006188	0.0008088	0.0009221
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2015	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003536	-0.0006275	-0.0007771	-0.0007117	-0.0005856	-0.0008003
CI+	0.0004679	0.0005221	0.0007751	0.0005775	0.0007971	0.0009856
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2016	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0006660	-0.0008143	-0.0008259	-0.0010175	-0.0010010	-0.0010483
CI+	0.0006335	0.0006711	0.0008101	0.0003777	0.0006424	0.0006134
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2017	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0008867	-0.0008015	-0.0008208	-0.0005193	-0.0006287	-0.0007485
CI+	0.0002961	0.0005144	0.0006197	0.0004885	0.0004365	0.0004987
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2018	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005690	-0.0009067	-0.0010069	-0.0008436	-0.0007685	-0.0008483
CI+	0.0003273	0.0004861	0.0005419	0.0005445	0.0006939	0.0009117
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2019	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005802	-0.0007350	-0.0008165	-0.0005943	-0.0005282	-0.0006882
CI+	0.0002731	0.0004046	0.0004694	0.0006624	0.0007239	0.0007822
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2020	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0009065	-0.0010467	-0.0015864	-0.0011348	-0.0013475	-0.0016300
CI+	0.0004686	0.0005553	0.0006123	0.0007977	0.0006001	0.0004357
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2021	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003989	-0.0005824	-0.0008260	-0.0005509	-0.0004574	-0.0005132
CI+	0.0007628	0.0008470	0.0010177	0.0008512	0.0011810	0.0015247
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

Table 17 – Test results for difference in means of daily returns in the portfolio projection period (out-of-sample period) for the Brazilian market.

2012	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0025687	-0.0024039	-0.0026467	-0.0037450	-0.0024462	-0.0020384	-0.0023762
CI+	0.0020939	0.0022673	0.0026175	0.0023192	0.0025914	0.0024280	0.0020327
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2013	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0022750	-0.0016638	-0.0016617	-0.0015447	-0.0006574	-0.0008582	-0.0007470
CI+	0.0023247	0.0025827	0.0030224	0.0028308	0.0031564	0.0033057	0.0034895
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2014	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0022564	-0.0025633	-0.0022613	-0.0024775	-0.0016618	-0.0017450	-0.0013066
CI+	0.0029401	0.0030032	0.0033268	0.0029151	0.0034981	0.0040821	0.0037518
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2015	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0019391	-0.0017997	-0.0024748	-0.0024539	-0.0012742	-0.0014835	-0.0018880
CI+	0.0030253	0.0035215	0.0036125	0.0040240	0.0031327	0.0033128	0.0030567
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2016	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0033167	-0.0033191	-0.0038101	-0.0041881	-0.0032671	-0.0033554	-0.0032575
CI+	0.0018335	0.0021968	0.0017834	0.0014083	0.0016978	0.0020152	0.0022916
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2017	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0009464	-0.0009412	-0.0009924	-0.0011963	-0.0009060	-0.0008723	-0.0010003
CI+	0.0013188	0.0012439	0.0013207	0.0015214	0.0013509	0.0017196	0.0019588
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2018	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0004797	-0.0011405	-0.0011889	-0.0017202	-0.0003754	-0.0004625	-0.0004739
CI+	0.0002582	0.0010594	0.0010543	0.0012987	0.0004221	0.0005712	0.0008792
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2019	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0001896	-0.0005966	-0.0006028	-0.0009389	-0.0003033	-0.0001963	-0.0003860
CI+	0.0003290	0.0011941	0.0014119	0.0015072	0.0003848	0.0006556	0.0007103
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2020	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0004130	-0.0007389	-0.0016802	-0.0016088	-0.0004262	-0.0004635	-0.0006641
CI+	0.0007128	0.0012796	0.0011385	0.0013115	0.0007400	0.0009146	0.0011020
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2021	**M1-B**	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0002929	-0.0004983	-0.0006994	-0.0017285	-0.0003274	-0.0005291	-0.0008840
CI+	0.0006287	0.0010157	0.0011637	0.0011130	0.0007259	0.0007574	0.0008802
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

Table 18 – Test results for difference in means of daily Beasley TE in the portfolio projection period (out-of-sample period) for the US market.

2012	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0002839	-0.0003000	-0.0003481	-0.0002786	-0.0002771	-0.0003106
CI+	0.0003372	0.0004224	0.0005471	0.0001253	0.0001727	0.0002512
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2013	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003023	-0.0003982	-0.0003239	-0.0002035	-0.0002176	-0.0002358
CI+	0.0003679	0.0003741	0.0005537	0.0001908	0.0002778	0.0003123
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2014	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003909	-0.0004502	-0.0003746	-0.0002343	-0.0003035	-0.0003614
CI+	0.0002385	0.0002800	0.0004084	0.0000888	0.0001593	0.0001841
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2015	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005038	-0.0005353	-0.0006014	-0.0002346	-0.0002410	-0.0003306
CI+	0.0001404	0.0002781	0.0002492	0.0001544	0.0002703	0.0002494
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2016	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003303	-0.0004078	-0.0004008	-0.0001824	-0.0002290	-0.0002355
CI+	0.0003703	0.0004202	0.0004805	0.0001977	0.0004198	0.0004748
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2017	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0002795	-0.0003989	-0.0003549	-0.0001807	-0.0002783	-0.0003249
CI+	0.0003493	0.0003846	0.0004623	0.0001376	0.0002079	0.0001988
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2018	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003701	-0.0006795	-0.0005278	-0.0002594	-0.0003131	-0.0004227
CI+	0.0003548	0.0002207	0.0004263	0.0001993	0.0002304	0.0002259
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2019	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005272	-0.0006625	-0.0006232	-0.0002279	-0.0003817	-0.0003046
CI+	0.0002197	0.0002136	0.0003344	0.0002146	0.0001384	0.0003276
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2020	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0001887	-0.0004433	-0.0007252	-0.0004080	-0.0005091	-0.0005664
CI+	0.0005467	0.0005473	0.0005986	0.0002802	0.0003150	0.0003942
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2021	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003909	-0.0004480	-0.0003617	-0.0002027	-0.0002589	-0.0004231
CI+	0.0003826	0.0007784	0.0009600	0.0001977	0.0002193	0.0002262
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

Table 19 – Test results for difference in means of daily Beasley TE in the portfolio projection period (out-of-sample period) for the Japanese market.

2012	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005074	-0.0005875	-0.0008447	-0.0007639	-0.0010171	-0.0010374
CI+	0.0006800	0.0009005	0.0007834	0.0005830	0.0005203	0.0006301
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2013	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0008528	-0.0011152	-0.0012968	-0.0007283	-0.0007498	-0.0009538
CI+	0.0007133	0.0006256	0.0005549	0.0006536	0.0008954	0.0010298
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2014	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0004122	-0.0004157	-0.0005129	-0.0006332	-0.0005606	-0.0008572
CI+	0.0004980	0.0005850	0.0008560	0.0006188	0.0008088	0.0009221
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2015	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003536	-0.0006275	-0.0007771	-0.0007117	-0.0005856	-0.0008003
CI+	0.0004679	0.0005221	0.0007751	0.0005775	0.0007971	0.0009856
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2016	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0006660	-0.0008143	-0.0008259	-0.0010175	-0.0010010	-0.0010483
CI+	0.0006335	0.0006711	0.0008101	0.0003777	0.0006424	0.0006134
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2017	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0008867	-0.0008015	-0.0008208	-0.0005193	-0.0006287	-0.0007485
CI+	0.0002961	0.0005144	0.0006197	0.0004885	0.0004365	0.0004987
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2018	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005690	-0.0009067	-0.0010069	-0.0008436	-0.0007685	-0.0008483
CI+	0.0003273	0.0004861	0.0005419	0.0005445	0.0006939	0.0009117
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2019	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005802	-0.0007350	-0.0008165	-0.0005943	-0.0005282	-0.0006882
CI+	0.0002731	0.0004046	0.0004694	0.0006624	0.0007239	0.0007822
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2020	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0009065	-0.0010467	-0.0015864	-0.0011348	-0.0013475	-0.0016300
CI+	0.0004686	0.0005553	0.0006123	0.0007977	0.0006001	0.0004357
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2021	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0003989	-0.0005824	-0.0008260	-0.0005509	-0.0004574	-0.0005132
CI+	0.0007628	0.0008470	0.0010177	0.0008512	0.0011810	0.0015247
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

Table 20 – Test results for difference in means of daily Beasley TE in the portfolio projection period (out-of-sample period) for the Brazilian market.

2012	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0009167	-0.0013108	-0.0020549	-0.0009045	-0.0012025	-0.0017436
CI+	0.0013392	0.0016979	0.0013995	0.0013406	0.0019104	0.0016505
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2013	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0006842	-0.0007533	-0.0012761	-0.0005421	-0.0006518	-0.0007082
CI+	0.0016367	0.0019727	0.0025646	0.0030676	0.0032012	0.0033334
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2014	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0012774	-0.0014151	-0.0022530	-0.0008501	-0.0011352	-0.0012404
CI+	0.0012533	0.0017697	0.0020090	0.0019811	0.0026353	0.0030864
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2015	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0007626	-0.0011640	-0.0014146	-0.0012123	-0.0018467	-0.0024762
CI+	0.0013731	0.0015200	0.0023281	0.0021017	0.0023903	0.0021879
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2016	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0007941	-0.0013777	-0.0022989	-0.0014940	-0.0016315	-0.0016009
CI+	0.0011559	0.0010953	0.0008745	0.0012998	0.0013532	0.0018560
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2017	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005871	-0.0006330	-0.0008606	-0.0002688	-0.0006083	-0.0006690
CI+	0.0005230	0.0006056	0.0008773	0.0003025	0.0009845	0.0011133
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2018	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0010694	-0.0011682	-0.0016297	-0.0001431	-0.0002400	-0.0002831
CI+	0.0011068	0.0012554	0.0013517	0.0003793	0.0005580	0.0008410
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2019	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0006450	-0.0006358	-0.0009354	-0.0002281	-0.0001801	-0.0003588
CI+	0.0011226	0.0012721	0.0013111	0.0001812	0.0004935	0.0006157
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2020	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0008875	-0.0017086	-0.0016035	-0.0003898	-0.0004763	-0.0007015
CI+	0.0009580	0.0007980	0.0010368	0.0004069	0.0006796	0.0009132
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

2021	**M2-E-M**	**M2-E-P35**	**M2-E-1Q**	**M2-L-M**	**M2-L-P35**	**M2-L-1Q**
CI-	-0.0005117	-0.0007128	-0.0017894	-0.0003632	-0.0006105	-0.0009765
CI+	0.0008716	0.0009684	0.0007744	0.0004179	0.0004676	0.0005995
	Fail to Reject	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**	**Fail to Reject**

5 Final remarks

In general, given the results presented in the previous subsections, we argue for the relevance of considering efficiency constraints in the formation of tracking portfolios. As mentioned, the index tracking (IT) problem consists of replicating the returns of a market index with a limited amount of assets. Normally, it is formulated through a problem of minimizing the quadratic difference between the return of the portfolio and the index in a given time interval. And, to limit the amount of assets, integer constraints are normally used, having, finally, a mixed-integer quadratic programming problem (MIQP), which has high computational complexity, and normally requires considerable processing time, also being characterized as an NP-hard problem. We developed a formulation to solve the index tracking (IT) problem, reducing the level of computational complexity and obtaining an instant solution to the problem. In particular, we have included an asset-level efficiency constraint, the Market Deficiency Measure (MDM), measured by the Multifractal-Detrended Fluctuation Analysis (MF-DFA). We project portfolios with efficiency constraints comparing the results with an unrestricted model (benchmark) in three markets with different structures: US (developed), Japanese (developed) and Brazilian (emerging), from January 2012 to December 2021, comprising 10 years of projection.

The results demonstrate the existence of a trade-off between tracking error (TE) and the efficiency of the portfolios: insofar as we demand more efficient assets to compose the portfolios, and, consequently, we generate portfolios with a smaller number of assets, the TE increases. We also demonstrate the results obtained by some concepts that are related in the literature: efficiency and liquidity. Our results indicate that these restrictions have similar effects on index tracking portfolios, reducing the average number of assets when we require portfolios with more efficient assets or with more liquid assets.

We compare our results with methods already used in the literature, one that uses a hybrid approach with genetic algorithm and nonlinear mathematical programming, and another that uses the cointegration approach. We demonstrate that, although our results are not absolutely better, they are very close to those obtained by the methods mentioned above, which are more complex from a computational point of view, and, despite having a very low processing time, do not generate an instantaneous solution. Thus, with efficiency constraints, we achieved good results and instantaneous solution, with a problem of low computational complexity.

We also note that efficiency constraints have a greater impact on the tracking error (TE) of portfolios in emerging markets, which tend to have lower levels of efficiency than developed markets, where the inclusion of restrictions penalizes the TE more severely than in developed markets with high levels of efficiency. Thus, the use of efficiency constraints appears to be an interesting alternative to reduce the size of tracking portfolios with a lower cost in terms of tracking error (TE cost), specially in developed markets.

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