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*C**-algebras, approximately proper equivalence relations and thermodynamic formalism

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Abstract. We introduce a non-commutative generalization of the notion of (approximately proper) equivalence relations and propose the construction of a 'quotient space'. We then consider certain one-parameter groups of automorphisms of the resulting *C**-algebra and prove the existence of KMS states at every temperature. In a model originating from thermodynamics we prove that these states are unique as well. We also show a relationship between maximizing measures (the analogue of the Aubry–Mather measures for expanding maps) and ground states. In the last section we explore an interesting example of phase transitions.

1. *Introduction*

An equivalence relation on a compact Hausdorff space is said to be *proper* when the quotient space is Hausdorff, and *approximately proper* when it is the union of an increasing sequence of proper relations. The first major goal of this paper is to extend these concepts to *non-commutative spaces*, that is to *C**-algebras, and to construct the corresponding *quotient space*. This turns out to be another *C**-algebra which is often non-commutative even when the original algebra is commutative. An example of this situation is the *tail-equivalence* relation on Bernoulli's space whose 'quotient space' turns out to be the CAR algebra.

We then introduce the notion of *potentials* and their associated *gauge actions* which are one-parameter groups of automorphisms of the 'quotient space'. A characterization of KMS states is then provided and we use it to show that KMS states exist for all values of the inverse temperature.

Starting with a local homeomorphism T on a compact metric space X we consider the equivalence relation on X under which two points x and y are equivalent if there is a

natural number n such that $T^n(x) = T^n(y)$. This turns out to be an approximately proper equivalence relation and we apply the abstract theory developed in the previous sections, enhanced by the use of Ruelle's Perron–Frobenius Theorem, in order to show uniqueness of KMS states at every temperature.

Ground states are studied next and a general characterization of those states which factor through a certain conditional expectation is obtained in terms of the support of the corresponding measure.

In the last section of the paper we show a relationship between maximizing measures (the analogue of the Aubry–Mather measures for the class of expanding maps) and ground states.

Approximately proper equivalence relations were first defined and studied in [**Re1, Re2**]. Proposition 9.9 and Theorem 11.6 can also be obtained as a particular case of the characterization of KMS states given in [**Re1**, II.5.4]. The existence of KMS states (Theorem 8.2) is also proved in [**Re1**, III.1.5] in a more particular case but with a similar proof.

Our construction of the C^*-algebra for an approximately proper equivalence relation should be viewed as a non-commutative generalization of the groupoid C^*-algebra [**Re1**] for the groupoids treated by Renault in [**Re2, Re3**]. In the special case of approximately proper equivalence relations over commutative algebras, under the assumption that certain conditional expectations are of index-finite type, an assumption which we make from § 6 onwards, our situation actually becomes identical to some situations discussed by Renault in the above-mentioned articles. Unlike Renault, we do not treat these situations by employing groupoid techniques, but there is nevertheless a significant overlap in our conclusions.

2. *Approximately proper equivalence relations*

In order to motivate the construction to be made here consider a compact Hausdorff space X equipped with an equivalence relation R.

When the quotient X/R is a Hausdorff space we say that R is a *proper equivalence relation* in which case the C^*-algebra of continuous complex functions on X/R, which we denote as $C(X/R)$, is canonically *-isomorphic to the subalgebra $C(X; R)$ of $C(X)$ formed by the functions which are constant on each equivalence class.

On the other hand, given any closed unital *-subalgebra $A \subseteq C(X)$, define the equivalence relation R_A on X by

$$(x, y) \in R_A \quad \Leftrightarrow \quad \forall f \in A,\ f(x) = f(y).$$

It is then easy to see that R_A is proper and that $C(X; R_A) = A$. In other words, the correspondence $R \mapsto C(X; R)$ is a bijection between the set of all proper equivalence relations on X and the collection of all closed unital *-subalgebras of $C(X)$.

This could be used to give a definition of 'proper equivalence relations' over a 'non-commutative space', that is, a non-commutative C^*-algebra: such a relation would simply be defined as a closed unital *-subalgebra.

This scenario is undoubtedly very neat, but it ignores some of the most interesting equivalence relations in mathematics, most of which are not proper. Consider, for example,

the tail-equivalence relation on Bernouli's space. The fact that the equivalence classes are dense implies that $C(X; R)$ consists solely of the constant functions. So, in this case the subalgebra $C(X; R)$ says nothing about the equivalence relation we started with.

Fortunately, some badly behaved equivalence relations, such as the example mentioned above, may be described as limits of proper relations, in the following sense.

Definition 2.1. An equivalence relation R on a compact Hausdorff space X is said to be *approximately proper* if there exists an increasing sequence of proper equivalence relations $\{R_n\}_{n\in\mathbb{N}}$ such that $R = \bigcup_{n\in\mathbb{N}} R_n$.

We should perhaps say that we adopt the convention according to which $\mathbb{N} = \{0, 1, 2, \dots\}$. Also, we view equivalence relations in the strict mathematical sense, namely as subsets of $X \times X$, hence the set theoretical union above.

Given $\{R_n\}_{n\in\mathbb{N}}$ as above, consider for each n the subalgebra $\mathcal{R}_n = C(X; R_n)$. Since $R_n \subseteq R_{n+1}$ we have that $\mathcal{R}_n \supseteq \mathcal{R}_{n+1}$. Since each R_n may be recovered from $\mathcal{R}_n$, we conclude that the decreasing sequence $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ encodes all of the information present in the given sequence of equivalence relations. We may then generalize to a non-commutative setting as follows.

Definition 2.2. An *approximately proper equivalence relation* on a unital C^*-algebra A is a decreasing sequence $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ of closed unital *-subalgebras. For convenience we always assume that $\mathcal{R}_0 = A$.

It is our goal in this section to introduce a C^*-algebra which is supposed to be the non-commutative analog of the quotient space by an approximately proper equivalence relation. A special feature of our construction is that the resulting algebra is often non-commutative even when the initial algebra A is commutative.

In order to carry on with our construction it seems that we are required to choose a sequence of faithful conditional expectations $\{E_n\}_{n\in\mathbb{N}}$ defined on A with $E_n(A) = \mathcal{R}_n$ and $E_{n+1} \circ E_n = E_{n+1}$ for every n.

Throughout this section, and for most of this work, we will therefore fix a C^*-algebra A, an approximately proper equivalence relation $\mathcal{R} = \{\mathcal{R}_n\}_{n\in\mathbb{N}}$, and a sequence $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ of conditional expectations as above.

Definition 2.3. The Toeplitz algebra of the pair $(\mathcal{R}, \mathcal{E})$, denoted $\mathcal{T}(\mathcal{R}, \mathcal{E})$, is the universal C^*-algebra generated by A and a sequence $\{\hat{e}_n\}_{n\in\mathbb{N}}$ of projections (self-adjoint idempotents) subject to the relations:

(i) $\hat{e}_0 = 1$;

(ii) $\hat{e}_{n+1}\hat{e}_n = \hat{e}_{n+1}$;

(iii) $\hat{e}_n a \hat{e}_n = E_n(a)\hat{e}_n$;

for all $a \in A$ and $n \in \mathbb{N}$.

When an element $a \in A$ is viewed in $\mathcal{T}(\mathcal{R}, \mathcal{E})$ we will denote it by $\underline{a}$. At first glance it is conceivable that the relations above imply that $\underline{a} = 0$ for some non-zero element $a \in A$. We will soon show that this never happens so that we may identify A with its copy within $\mathcal{T}(\mathcal{R}, \mathcal{E})$, and then we will be allowed to drop the underlining notation.

Note that Definition 2.3(ii) states that the $\hat{e}_n$ form a decreasing sequence of projections. Also, by taking adjoints in Definition 2.3(iii), we conclude that $\hat{e}_n\underline{a}\hat{e}_n = \hat{e}_n\underline{E_n(a)}$ as well. It follows that each $\hat{e}_n$ lies in the commutant of $\underline{\mathcal{R}_n}$.

PROPOSITION 2.4. *Given $n, m \in \mathbb{N}$ and $a, b, c, d \in A$, one has that*

$$(\underline{a}\hat{e}_n\underline{b})(\underline{c}\hat{e}_m\underline{d}) = \begin{cases} \underline{aE_n(bc)}\hat{e}_m\underline{d}, & \text{if } n \leq m, \\ \underline{a}\hat{e}_n\underline{E_m(bc)d}, & \text{if } n \geq m. \end{cases}$$

Proof. If $n \leq m$, we have

$$(\underline{a}\hat{e}_n\underline{b})(\underline{c}\hat{e}_m\underline{d}) = \underline{a}(\hat{e}_n\underline{bc}\hat{e}_n)\hat{e}_m\underline{d} = \underline{aE_n(bc)}\hat{e}_n\hat{e}_m\underline{d} = \underline{aE_n(bc)}\hat{e}_m\underline{d}.$$

If $n \geq m$ the conclusion follows by taking adjoints. □

Definition 2.5. For each $n \in \mathbb{N}$ we denote by $\hat{\mathcal{K}}_n$ the closed linear span of the set $\{\underline{a}\hat{e}_n\underline{b} : a, b \in A\}$.

By Proposition 2.4 we see that for $i \leq n$ one has that both $\hat{\mathcal{K}}_i\hat{\mathcal{K}}_n$ and $\hat{\mathcal{K}}_n\hat{\mathcal{K}}_i$ are contained in $\hat{\mathcal{K}}_n$. In particular, each $\hat{\mathcal{K}}_n$ is a C^*-subalgebra of $\mathcal{T}(\mathcal{R}, \mathcal{E})$.

We now need a concept borrowed from [**E1**, 3.6] and [**E2**, 6.2].

Definition 2.6. Let $n \in \mathbb{N}$. A finite sequence $(k_0, \ldots, k_n) \in \prod_{i=0}^{n}\hat{\mathcal{K}}_i$ such that $\sum_{i=0}^{n} k_i x = 0$ for all $x \in \hat{\mathcal{K}}_n$ will be called an *n-redundancy*. The closed two-sided ideal of $\mathcal{T}(\mathcal{R}, \mathcal{E})$ generated by the elements $k_0 + \cdots + k_n$, for all n-redundancies $(k_0, \ldots, k_n)$, will be called the *redundancy ideal*.

We now arrive at our main new concept.

Definition 2.7. The C^*-algebra of the pair $(\mathcal{R}, \mathcal{E})$, denoted by $C^*(\mathcal{R}, \mathcal{E})$, is defined to be the quotient of $\mathcal{T}(\mathcal{R}, \mathcal{E})$ by the redundancy ideal. Moreover, we will adopt the following notation:

(i) the quotient map from $\mathcal{T}(\mathcal{R}, \mathcal{E})$ to $C^*(\mathcal{R}, \mathcal{E})$ will be denoted by q;
(ii) the image of $\hat{e}_n$ in $C^*(\mathcal{R}, \mathcal{E})$ will be denoted by e_n;
(iii) the image of $\hat{\mathcal{K}}_n$ in $C^*(\mathcal{R}, \mathcal{E})$ will be denoted by $\mathcal{K}_n$.

It is clear that $\mathcal{K}_n$ is the closed linear span of $q(\underline{A})e_nq(\underline{A})$.

3. *A faithful representation*

In this section we provide a faithful representation of $C^*(\mathcal{R}, \mathcal{E})$ which will, among other things, show that the natural maps $A \to \mathcal{T}(\mathcal{R}, \mathcal{E})$ and $A \to C^*(\mathcal{R}, \mathcal{E})$ are injective.

For $n \in \mathbb{N}$ consider the right Hilbert $\mathcal{R}_n$-module M_n obtained by completing A under the $\mathcal{R}_n$-valued inner product

$$\langle a, b\rangle = E_n(a^*b), \quad \forall a, b \in A.$$

The canonical map assigning each $a \in A$ to its class in M_n will be denoted by

$$i_n : A \to M_n.$$

It is obviously a right $\mathcal{R}_n$-module map. For each a in A one may prove that the correspondence

$$i_n(x) \mapsto i_n(ax), \quad \forall x \in A$$

extends to a map $L_a^n \in \mathcal{L}(M_n)$ (adjointable linear operators on M_n). In turn, the correspondence $a \to L_a^n$ may be shown to be an injective *-homomorphism from A to $\mathcal{L}(M_n)$ (recall that the E_n are assumed faithful) and whenever convenient we use it to think of A as a subalgebra of $\mathcal{L}(M_n)$.

We denote by $\check{e}_n$ the projection in $\mathcal{L}(M_n)$ obtained by continuously extending the correspondence $i_n(x) \mapsto i_n(E_n(x))$ to the whole of M_n.

Given any two vectors $\xi, \eta \in M_n$ we denote by $\Omega_{\xi,\eta}$ the 'generalized rank-one compact operator' on M_n given by

$$\Omega_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle, \quad \forall \zeta \in M_n.$$

PROPOSITION 3.1. *Given $a, b \in A$ one has that $a\check{e}_n b^* = \Omega_{i_n(a),i_n(b)}$. Therefore, the closed linear span of the set $\{a\check{e}_n b^* : a, b \in A\}$ is precisely the algebra of generalized compact operators on M_n. This algebra will be denoted by $\check{\mathcal{K}}_n$.*

Proof. For $x \in A$, note that

$$\begin{aligned} a\check{e}_n b^*(i_n(x)) &= i_n(aE_n(b^*x)) = i_n(a)E_n(b^*x) = i_n(a)\langle i_n(b), i_n(x)\rangle \\ &= \Omega_{i_n(a),i_n(b)}(i_n(x)). \end{aligned}$$

□

The following is an important algebraic relation.

PROPOSITION 3.2. *For every $n \in \mathbb{N}$ and every $a \in A$ one has that*

$$\check{e}_n a\check{e}_n = E_n(a)\check{e}_n = \check{e}_n E_n(a).$$

Proof. Given $x \in A$ note that

$$\check{e}_n a\check{e}_n(i_n(x)) = i_n(E_n(aE_n(x))) = i_n(E_n(a)E_n(x)) = E_n(a)\check{e}_n(i_n(x)).$$

So $\check{e}_n a\check{e}_n = E_n(a)\check{e}_n$. That $\check{e}_n a\check{e}_n = \check{e}_n E_n(a)$ follows by taking adjoints. □

We now wish to see how the M_n relate to each other.

PROPOSITION 3.3. *For every $n \in \mathbb{N}$ there exists a continuous $\mathcal{R}_{n+1}$-linear map $j_n : M_n \to M_{n+1}$ such that $j_n(i_n(a)) = i_{n+1}(a)$ for all $a \in A$. Moreover for any $\xi, \eta \in M_n$ one has that*

$$E_{n+1}(\langle\xi, \eta\rangle) = \langle j_n(\xi), j_n(\eta)\rangle.$$

Proof. For every $a \in A$ we claim that $\|i_{n+1}(a)\| \leq \|i_n(a)\|$. In fact,

$$\|i_{n+1}(a)\|^2 = \|E_{n+1}(a^*a)\| = \|E_{n+1}E_n(a^*a)\| \leq \|E_n(a^*a)\| = \|i_n(a)\|^2.$$

Thus, the correspondence $i_n(a) \mapsto i_{n+1}(a)$ is contractive and hence extends to a continuous map $j_n : M_n \to M_{n+1}$ such that $j_n(i_n(a)) = i_{n+1}(a)$. It is elementary to verify that j_n is $\mathcal{R}_{n+1}$-linear. Suppose that $\xi = i_n(a)$ and $\eta = i_n(b)$ where $a, b \in A$. Then

$$\begin{aligned} E_{n+1}(\langle\xi, \eta\rangle) &= E_{n+1}(\langle i_n(a), i_n(b)\rangle) = E_{n+1}(E_n(a^*b)) = E_{n+1}(a^*b) \\ &= \langle i_{n+1}(a), i_{n+1}(b)\rangle = \langle j_n(i_n(a)), j_n(i_n(b))\rangle = \langle j_n(\xi), j_n(\eta)\rangle. \end{aligned}$$

The conclusion now follows because $i_n(A)$ is dense in M_n. □

The preceding result gives a canonical relationship between elements in M_n and M_{n+1}. We now see how to relate operators.

PROPOSITION 3.4. *There exists an injective *-homomorphism*

$$\Phi_n : \mathcal{L}(M_n) \to \mathcal{L}(M_{n+1})$$

such that for $T \in \mathcal{L}(M_n)$ one has that

$$\Phi_n(T)(j_n(\xi)) = j_n(T(\xi)), \quad \forall \xi \in M_n.$$

Proof. Let $T \in \mathcal{L}(M_n)$. Since $T^*T \leq \|T\|^2$ one has for all $\xi \in M_n$ that

$$\langle T(\xi), T(\xi)\rangle = \langle T^*T(\xi), \xi\rangle \leq \|T\|^2\langle \xi, \xi\rangle.$$

Applying E_{n+1} to the above inequality yields

$$E_{n+1}(\langle T(\xi), T(\xi)\rangle) \leq \|T\|^2 E_{n+1}(\langle \xi, \xi\rangle),$$

or

$$\langle j_n(T(\xi)), j_n(T(\xi))\rangle \leq \|T\|^2(\langle j_n(\xi), j_n(\xi)\rangle),$$

which implies that $\|j_n(T(\xi))\| \leq \|T\|\,\|j_n(\xi)\|$. So the correspondence

$$j_n(\xi) \mapsto j_n(T(\xi))$$

extends to a bounded linear map $\Phi_n(T) : M_{n+1} \to M_{n+1}$ such that $\Phi(T)(j_n(\xi)) = j_n(T(\xi))$ for all $\xi \in M_n$.

We claim that $\Phi(T)^* = \Phi(T^*)$ for all $T \in \mathcal{L}(M_n)$. In order to prove this let $\xi, \eta \in M_n$. We have that

$$\begin{aligned}\langle j_n(\xi), \Phi(T)(j_n(\eta))\rangle &= \langle j_n(\xi), j_n(T(\eta))\rangle = E_{n+1}(\langle \xi, T(\eta)\rangle)\\ &= E_{n+1}(\langle T^*(\xi), \eta\rangle) = \langle \Phi(T^*)(j_n(\xi)), j_n(\eta)\rangle,\end{aligned}$$

proving the claim. It is now easy to see that Φ_n is indeed a *-homomorphism from $\mathcal{L}(M_n)$ to $\mathcal{L}(M_{n+1})$.

If T is such that $\Phi_n(T) = 0$ then for every $\xi \in M_n$ one has that

$$0 = \langle \Phi_n(T)(j_n(\xi)), \Phi_n(T)(j_n(\xi))\rangle = \langle j_n(T(\xi)), j_n(T(\xi))\rangle = E_{n+1}(\langle T(\xi), T(\xi)\rangle).$$

Since E_{n+1} is faithful we have that $T(\xi) = 0$. Since ξ is arbitrary we have that $T = 0$. □

Definition 3.5. We denote by $\mathcal{L}_\infty$ the inductive limit of the sequence

$$\mathcal{L}(M_1) \xrightarrow{\Phi_1} \mathcal{L}(M_2) \xrightarrow{\Phi_2} \cdots.$$

Recall that A is viewed as a subalgebra of $\mathcal{L}(M_n)$ via the correspondence $a \mapsto L_a^n$. For $a, x \in A$ note that

$$\begin{aligned}\Phi_n(L_a^n)(i_{n+1}(x)) &= \Phi_n(L_a^n)(j_n(i_n(x)) = j_n(L_a^n(i_n(x))) = j_n(i_n(ax)) = i_{n+1}(ax)\\ &= L_a^{n+1}(i_{n+1}(x)),\end{aligned}$$

so that $\Phi_n(L_a^n) = L_a^{n+1}$. It follows that if we identify $\mathcal{L}(M_n)$ with its image in $\mathcal{L}(M_{n+1})$ under Φ_n, the two corresponding copies of A will be identified with each other via the identity map. Therefore, A sits inside of $\mathcal{L}_\infty$ in a canonical fashion.

We now claim that $\check{e}_{n+1} \leq \Phi_n(\check{e}_n)$ for all $n \in \mathbb{N}$. In fact, for all $a \in A$

$$\begin{aligned}
&\check{e}_{n+1}\Phi_n(\check{e}_n)(i_{n+1}(a)) \\
&\quad = \check{e}_{n+1}\Phi_n(\check{e}_n)(j_n(i_n(a))) = \check{e}_{n+1}(j_n(\check{e}_n(i_n(a)))) = \check{e}_{n+1}(j_n(i_n(E_n(a)))) \\
&\quad = \check{e}_{n+1}(i_{n+1}(E_n(a))) = i_{n+1}(E_{n+1}E_n(a)) = i_{n+1}(E_{n+1}(a)) = \check{e}_{n+1}(i_{n+1}(a)).
\end{aligned}$$

Within $\mathcal{L}_\infty$ we then get a decreasing sequence of projections consisting of the images of $\check{e}_n$ in the inductive limit, which we still denote by $\check{e}_n$.

We are now ready to prove the main result of this section, the main purpose of which is to give a concrete realization of the so far abstractly defined $C^*(\mathcal{R}, \mathcal{E})$.

THEOREM 3.6.

(i) *There exists a unique *-homomorphism $\hat{\pi} : \mathcal{T}(\mathcal{R}, \mathcal{E}) \to \mathcal{L}_\infty$ such that $\hat{\pi}(\underline{a}) = a$ for all a in A and $\hat{\pi}(\hat{e}_n) = \check{e}_n$ for all $n \in \mathbb{N}$.*

(ii) *$\hat{\pi}$ vanishes on the redundancy ideal and so factors through $C^*(\mathcal{R}, \mathcal{E})$ providing a *-homomorphism*

$$\pi : C^*(\mathcal{R}, \mathcal{E}) \to \mathcal{L}_\infty$$

such that $\pi(e_n) = \check{e}_n$ and $\pi(q(\underline{a})) = a$, where q is the quotient map from $\mathcal{T}(\mathcal{R}, \mathcal{E})$ to $C^*(\mathcal{R}, \mathcal{E})$.

(iii) *π is injective and hence $C^*(\mathcal{R}, \mathcal{E})$ is isomorphic to the sub-C*-algebra of $\mathcal{L}_\infty$ generated by A and all of the $\check{e}_n$.*

Proof. The first point follows from Proposition 3.2, the fact that the $\check{e}_n$ are decreasing, and the universal property of $\mathcal{T}(\mathcal{R}, \mathcal{E})$.

Addressing (ii), all we must show is that $\hat{\pi}$ vanishes on any element of the form

$$s = \sum_{i=0}^{n} k_i,$$

where $(k_0, \ldots, k_n)$ is an n-redundancy. Observing that for $i \leq n$ one has that $\hat{\pi}(k_i) \in \mathcal{L}(M_i)$ and that $\mathcal{L}(M_i)$ is contained in $\mathcal{L}(M_n)$ (as subalgebras of the direct limit $\mathcal{L}_\infty$), we see that $\hat{\pi}(s) \in \mathcal{L}(M_n)$. Given $a \in A$, choose $b, c \in A$ such that $E(b^*c) = 1$ (e.g. $b = c = 1$) so that

$$\hat{\pi}(s)|_{i_n(a)} = \hat{\pi}(s)\,\Omega_{i_n(a),i_n(b)}|_{i_n(c)} \overset{\text{Proposition 3.1}}{=} \hat{\pi}(s)\,(a\check{e}_n b^*)|_{i_n(c)} = \hat{\pi}(s\ \underline{a}\hat{e}_n\underline{b}^*)|_{i_n(c)} = 0$$

because $\underline{a}\hat{e}_n\underline{b}$ lies in $\hat{\mathcal{K}}_n$. This shows that $\hat{\pi}(s) = 0$ and hence proves (ii).

In order to proceed we must now prove that the restriction of $\hat{\pi}$ to each $\hat{\mathcal{K}}_n$ is injective. For this purpose, recall from [**Wa**, 2.2.9] that $\check{\mathcal{K}}_n$ is precisely the *unreduced C*-basic construction* relative to E_n and thus possesses the universal property described in [**Wa**, 2.2.7]. The correspondence

$$a \in A \mapsto \underline{a} \in \mathcal{T}(\mathcal{R}, \mathcal{E})$$

together with the idempotent $\hat{e}_n$ gives by Definition 2.3(iii) a *covariant representation* of the conditional expectation E_n, according to Definition 2.2.6 in [**Wa**]. Therefore, there

exists a *-homomorphism $\rho : \check{\mathcal{K}}_n \to \hat{\mathcal{K}}_n$ such that $\rho(a\check{e}_n b) = \underline{a}\hat{e}_n\underline{b}$ for all $a, b \in A$. It follows that the composition $\rho \circ (\hat{\pi}|_{\hat{\mathcal{K}}_n})$ is the identity map, hence proving our claim that $\hat{\pi}|_{\hat{\mathcal{K}}_n}$ is injective.

In order to prove (iii), it suffices to show that for each n, π is injective on the sub-C^*-algebra of $C^*(\mathcal{R}, \mathcal{E})$ given by

$$B_n = \mathcal{K}_0 + \cdots + \mathcal{K}_n,$$

where the $\mathcal{K}_n$ are defined in Definition 2.7(iii) (note that B_n is indeed a sub-C^*-algebra by [**P**, 1.5.8]). In fact, once this is granted, we see that π is isometric on the union of all B_n which is dense in $C^*(\mathcal{R}, \mathcal{E})$. This would prove that π is isometric on all of $C^*(\mathcal{R}, \mathcal{E})$.

Let $b = k_0 + \cdots + k_n \in B_n$, where $k_i \in \mathcal{K}_i$, and suppose that $\pi(b) = 0$. Since $q(\hat{\mathcal{K}}_i) = \mathcal{K}_i$ we may write $k_i = q(\hat{k}_i)$, where $\hat{k}_i \in \hat{\mathcal{K}}_i$. We therefore have that $\hat{\pi}(\hat{k}_0 + \cdots + \hat{k}_n) = 0$.

We now claim that $(\hat{k}_0, \ldots, \hat{k}_n)$ is an n-redundancy. In order to prove this, let $x \in \hat{\mathcal{K}}_n$ and note that $(\hat{k}_0 + \cdots + \hat{k}_n)x \in \hat{\mathcal{K}}_n$ by Proposition 2.4. However, since $\hat{\pi}((\hat{k}_0 + \cdots + \hat{k}_n)x) = 0$ and $\hat{\pi}$ is injective on $\hat{\mathcal{K}}_n$, we have that $(\hat{k}_0 + \cdots + \hat{k}_n)x = 0$ as claimed. So $\hat{k}_0 + \cdots + \hat{k}_n$ lies in the redundancy ideal and hence

$$b = q(\hat{k}_0 + \cdots + \hat{k}_n) = 0. \qquad \square$$

COROLLARY 3.7. *The maps*

$$a \in A \to \underline{a} \in \mathcal{T}(\mathcal{R}, \mathcal{E})$$

and

$$a \in A \to q(\underline{a}) \in C^*(\mathcal{R}, \mathcal{E})$$

are injective.

The proof follows immediately from our last result.

From now on, we will therefore identify A with $\underline{A}$ and also with $q(\underline{A})$.

4. *Stationary equivalence relations*

In this section we study approximately proper equivalence relations which have a special simple description.

Definition 4.1. An approximately proper equivalence relation $\mathcal{R} = \{\mathcal{R}_n\}_{n\in\mathbb{N}}$ over a unital C^*-algebra A is said to be *stationary* if there exists a unital injective *-endomorphism $\alpha : A \to A$ such that $\mathcal{R}_{n+1} = \alpha(\mathcal{R}_n)$ for all n.

In this case observe that $\mathcal{R}_n$ is simply the range of α^n. Throughout this section we fix a stationary approximately proper equivalence relation $\mathcal{R} = \{\mathcal{R}_n\}_{n\in\mathbb{N}}$ over A. We will also fix an endomorphism α as above.

Let E be a given faithful conditional expectation from A to $\mathcal{R}_1$. Define conditional expectations E_n from A to $\mathcal{R}_n$ by

$$E_n = \alpha^{n-1}\underbrace{(E\alpha^{-1})\ldots(E\alpha^{-1})}_{n-1 \text{ times}}E.$$

It is easy to see that $E_{n+1} \circ E_n = E_{n+1}$ for every n.

Definition 4.2. We say that a sequence of conditional expectations $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ is stationary if it is obtained as above from a single faithful conditional expectation $E : A \to \mathcal{R}_1$.

Throughout this section we fix a stationary sequence of conditional expectations as above. Observe that the composition $\mathcal{L} = \alpha^{-1}E$ is a *transfer operator* in the sense of [**E1**, 2.1]. We may then form the crossed-product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ as in [**E1**, 3.7]. Denote by γ the *scalar gauge action* [**E2**, 3.3] on $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

The main result we wish to present in this section is the following.

THEOREM 4.3. *With the hypothesis introduced in this section, $C^*(\mathcal{R}, \mathcal{E})$ is isomorphic to the sub-C*-algebra of the crossed-product algebra $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ formed by the fixed points for the scalar gauge action.*

Proof. The theorem follows immediately from [**E2**, 6.5] since the algebra $\check{\mathcal{U}}$ mentioned there (see also [**E2**, 4.8]) is isomorphic to $C^*(\mathcal{R}, \mathcal{E})$ by Theorem 3.6(iii). For the proof that the fixed point algebra is precisely $\check{\mathcal{U}}$ see [**M**, 4.1]. □

We may now finally give a non-trivial example of our construction. Let A be an $n \times n$ matrix of zeros and ones without any zero rows or columns and let (X, T) be the corresponding Markov sub-shift. Define the endomorphism α of $C(X)$ by $\alpha(f) = f \circ T$, for all f in $C(X)$. Also, let E be the conditional expectation from $C(X)$ to the range of α given by

$$E(f)|_x = \frac{1}{\#\{y : T(y) = x\}} \sum_{T(y)=x} f(y), \quad \forall f \in C(X),\ x \in X.$$

We may then form $\mathcal{R}$ and $\mathcal{E}$ as above. By [**E1**, 6.2], one has that $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is the Cuntz–Krieger algebra $\mathcal{O}_A$. By Theorem 4.3 we then have that $C^*(\mathcal{R}, \mathcal{E})$ is isomorphic to the subalgebra of $\mathcal{O}_A$ formed by the fixed point algebra for the gauge action. When $n = 2$ and

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

we then have that $C^*(\mathcal{R}, \mathcal{E})$ is isomorphic to the CAR algebra.

5. *Gauge automorphisms*

In this section we return to the general case, therefore fixing a C*-algebra A, an approximately proper equivalence relation $\mathcal{R} = \{\mathcal{R}_n\}_{n\in\mathbb{N}}$, and a sequence $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ of compatible conditional expectations as before.

We wish to introduce the notions of *potentials* and their corresponding *gauge automorphisms* which will be the objects of study in later sections. We start with a simple technical fact.

PROPOSITION 5.1. *Let $i \leq n$ and let $b \in \mathcal{Z}(\mathcal{R}_i)$ (meaning the center of $\mathcal{R}_i$) then*

$$E_n(ab) = E_n(ba), \quad \forall a \in A.$$

Proof. We have

$$E_n(ab) = E_n(E_i(ab)) = E_n(E_i(a)b) = E_n(bE_i(a)) = E_n(E_i(ba)) = E_n(ba). \quad \square$$

Definition 5.2. By a *potential* we mean a sequence $z = \{z_n\}_{n\in\mathbb{N}}$ such that z_n belongs to $\mathcal{Z}(\mathcal{R}_n)$ for every $n \in \mathbb{N}$.

Given a potential z observe that every z_n commutes with every other z_m. Therefore, we may set

$$z^{[n]} = z_0 z_1 \dots z_{n-1}, \quad \forall n \in \mathbb{N},$$

without worrying about the order of the factors. We also use the notation $z^{-[n]}$ to mean $(z^{[n]})^{-1}$ when the latter exists.

If $w = \{w_n\}_{n\in\mathbb{N}}$ is another potential, it is clear that $zw := \{z_n w_n\}_{n\in\mathbb{N}}$ is again a potential and that

$$(zw)^{[n]} = z^{[n]} w^{[n]}, \quad \forall n \in \mathbb{N}.$$

Potentials may be used to define automorphisms as follows.

PROPOSITION 5.3. *Let $u = \{u_n\}_{n\in\mathbb{N}}$ be a unitary potential (in the sense that each u_n is a unitary element). Then there is an automorphism $\hat{\varphi}_u$ of $\mathcal{T}(\mathcal{R}, \mathcal{E})$ such that*

$$\hat{\varphi}_u(a) = a, \quad \forall a \in A,$$

and

$$\hat{\varphi}_u(\hat{e}_n) = u^{[n]} \hat{e}_n u^{-[n]}, \quad \forall n \in \mathbb{N}.$$

Moreover, given another unitary potential v, one has that $\hat{\varphi}_{uv} = \hat{\varphi}_u \hat{\varphi}_v$.

Proof. For every $n \in \mathbb{N}$ let $f_n = u^{[n]} \hat{e}_n u^{-[n]}$. Then

$$\begin{aligned} f_n f_{n+1} &= u^{[n]} \hat{e}_n u^{-[n]} u^{[n+1]} \hat{e}_{n+1} u^{-[n+1]} = u^{[n]} \hat{e}_n u_n \hat{e}_{n+1} u^{-[n+1]} \\ &= u^{[n]} u_n \hat{e}_n \hat{e}_{n+1} u^{-[n+1]} = u^{[n+1]} \hat{e}_{n+1} u^{-[n+1]} = f_{n+1}, \end{aligned}$$

so the f_n are decreasing. For $a \in A$ we have

$$\begin{aligned} f_n a f_n &= u^{[n]} \hat{e}_n u^{-[n]} a u^{[n]} \hat{e}_n u^{-[n]} = u^{[n]} E_n(u^{-[n]} a u^{[n]}) \hat{e}_n u^{-[n]} \\ &\overset{\text{Proposition 5.1}}{=} u^{[n]} E_n(a) \hat{e}_n u^{-[n]} = E_n(a) u^{[n]} \hat{e}_n u^{-[n]} = E_n(a) f_n. \end{aligned}$$

By the universal property of $\mathcal{T}(\mathcal{R}, \mathcal{E})$ there exist a *-homomorphism $\hat{\varphi}_u : \mathcal{T}(\mathcal{R}, \mathcal{E}) \to \mathcal{T}(\mathcal{R}, \mathcal{E})$ satisfying the conditions in the statement, except possibly for the fact that $\hat{\varphi}_u$ is an automorphism.

Given another unitary potential v one can easily prove that $\hat{\varphi}_{uv} = \hat{\varphi}_u \hat{\varphi}_v$ by checking on the generators. Plugging $v = u^{-1} := \{u_n^{-1}\}_{n\in\mathbb{N}}$ we then have that $\hat{\varphi}_{u^{-1}}$ and $\hat{\varphi}_u$ are each others inverse and hence $\hat{\varphi}_u$ is an automorphism. $\square$

PROPOSITION 5.4. *For every unitary potential u the automorphism $\hat{\varphi}_u$ leaves the redundancy ideal invariant (in the sense that the image of the redundancy ideal under $\hat{\varphi}_u$ is exactly the redundancy ideal) and hence drops to an automorphism φ_u of $C^*(\mathcal{R}, \mathcal{E})$ which is the identity on A and such that*

$$\varphi_u(e_n) = u^{[n]} e_n u^{-[n]}, \quad \forall n \in \mathbb{N}.$$

Proof. It is elementary to verify that $\hat{\varphi}_u(\hat{\mathcal{K}}_n) = \hat{\mathcal{K}}_n$ for all n. Thus, if $(k_0, \ldots, k_n)$ is a redundancy we have that $(\hat{\varphi}_u(k_0), \ldots, \hat{\varphi}_u(k_n)) \in \prod_{i=0}^{n} \hat{\mathcal{K}}_i$. Moreover, if $x \in \hat{\mathcal{K}}_n$, we have that

$$\sum_{i=0}^{n} \hat{\varphi}_u(k_i)x = \hat{\varphi}_u\left(\sum_{i=0}^{n} k_i \varphi_u^{-1}(x)\right) = 0.$$

Therefore, $(\hat{\varphi}_u(k_0), \ldots, \hat{\varphi}_u(k_n))$ is a redundancy and hence $\hat{\varphi}_u(k_0 + \cdots + k_n)$ lies in the redundancy ideal. So we see that $\hat{\varphi}_u$ sends the redundancy ideal into itself. Since the same holds for $\hat{\varphi}_{u^{-1}} = \hat{\varphi}_u^{-1}$ it follows that the image of the redundancy ideal under $\hat{\varphi}_u$ is precisely the redundancy ideal and hence the proof is concluded. □

So far we have introduced single gauge automorphisms, but now we would like to define one-parameter groups.

Definition 5.5.

(i) A potential $h = \{h_n\}_{n\in\mathbb{N}}$ is said to be strictly positive when for each n there exists a real number $c_n > 0$ such that $h_n \geq c_n$.

(ii) Given a strictly positive potential $h = \{h_n\}_{n\in\mathbb{N}}$ and a complex number z we denote by h^z the potential $\{h_n^z\}_{n\in\mathbb{N}}$, and by $h^{z[n]} = (h^z)^{[n]}$, for $n \in \mathbb{N}$.

(iii) The *gauge action* for a strictly positive potential h is the one-parameter group $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$ of automorphisms of $C^*(\mathcal{R}, \mathcal{E})$ given by $\sigma_t = \varphi_{h^{it}}$ for all $t \in \mathbb{R}$.

Given $a, b \in A$ and $n \in \mathbb{N}$, observe that

$$\sigma_t(ae_nb) = ah^{it[n]}e_nh^{-it[n]}b, \quad \forall a, b \in A, \quad \forall n \in \mathbb{N}. \tag{5.6}$$

It is therefore clear that the gauge action is strongly continuous.

6. *Finite index*

Starting with this section we restrict ourselves to the case in which the E_n are of index-finite type according to [**Wa**, 1.2.2]. We refer the reader to [**Wa**] for the basic definitions and facts about index-finite type conditional expectations, which now acquire a preponderant role in our study.

PROPOSITION 6.1. *If E_m is of index-finite type then its restriction to each $\mathcal{R}_n$, where $n \leq m$, is also of index-finite type. Moreover if $\{u_1, \ldots, u_k\}$ is a quasi-basis for E_m then $\{E_n(u_1), \ldots, E_n(u_k)\}$ is a quasi-basis for the restriction of E_m to $\mathcal{R}_n$.*

Proof. For every $a \in \mathcal{R}_n$ we have that

$$a = E_n(a) = E_n\left(\sum_{i=0}^{k} u_i E_m(u_i^* a)\right) = \sum_{i=0}^{k} E_n(u_i)E_m(E_n(u_i^* a))$$

$$= \sum_{i=0}^{k} E_n(u_i)E_m(E_n(u_i)^* a).$$ □

PROPOSITION 6.2. *Let $n \leq m$. Suppose that the restriction of E_m to $\mathcal{R}_n$ is of index-finite type and let $\{v_1, \ldots, v_k\} \subseteq \mathcal{R}_n$ be a quasi-basis for it. Then:*

(i) $\sum_{i=0}^{k} v_i e_m v_i^* = e_n$;

(ii) $\mathcal{K}_n \subseteq \mathcal{K}_m$.

Proof. Let $a, b \in A$ and observe that

$$\begin{aligned}\left(\hat{e}_n - \sum_{i=0}^{k} v_i \hat{e}_m v_i^*\right) a \hat{e}_m b \\ = \hat{e}_n a \hat{e}_n \hat{e}_m b - \sum_{i=0}^{k} v_i E_m(v_i^* a)\hat{e}_m b = E_n(a)\hat{e}_m b - \sum_{i=0}^{k} v_i E_m(E_n(v_i^* a))\hat{e}_m b \\ = E_n(a)\hat{e}_m b - \sum_{i=0}^{k} v_i E_m(v_i^* E_n(a))\hat{e}_m b = E_n(a)\hat{e}_m b - E_n(a)\hat{e}_m b = 0.\end{aligned}$$

Therefore, the $(m+1)$-tuple

$$\left(0, \ldots, 0, \hat{e}_n, 0, \ldots, 0, -\sum_{i=0}^{k} v_i \hat{e}_m v_i^*\right)$$

is an m-redundancy from which (i) follows. Obviously (ii) follows from (i). □

COROLLARY 6.3. *If all E_n are of index-finite type then $\mathcal{K}_n$ are increasing and $C^*(\mathcal{R}, \mathcal{E})$ is the closure of $\bigcup_{n\in\mathbb{N}} \mathcal{K}_n$.*

Proof. By Proposition 6.1 we have that $E_{n+1}|_{\mathcal{R}_n}$ is of index-finite type. Hence by Proposition 6.2 we have that $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$. Since $A = \mathcal{K}_0$ and for every n we have that $e_n \in \mathcal{K}_n$ the conclusion follows. □

In the finite index case we have the following elementary description of $\mathcal{K}_n$.

PROPOSITION 6.4. *If all E_n are of index-finite type then $M_n = i_n(A)$ and $\mathcal{K}_n = L_{\mathcal{R}_n}(A)$, where $L_{\mathcal{R}_n}(A)$ denotes the set of all (not necessarily adjointable or even continuous) additive right $\mathcal{R}_n$-linear maps on A (where A is identified with M_n via i_n).*

Proof. By [**Wa**, 2.1.5] there exists a constant $\lambda_n > 0$ such that $\|E_n(a^*a)\|^{1/2} \geq \lambda_n \|a\|$, for all a in A. Therefore,

$$\|i_n(a)\| = \|E_n(a^*a)\|^{1/2} \geq \lambda_n \|a\|,$$

so that i_n is a Banach space isomorphism onto its range which is therefore a complete normed space, hence closed. Since $i_n(A)$ is dense in M_n we conclude that $i_n(A) = M_n$. We will therefore identify M_n and A.

It is clear that $\mathcal{K}_n \subseteq L_{\mathcal{R}_n}(A)$. In order to prove the converse inclusion let $\{u_1, \ldots, u_n\}$ be a quasi-basis for E_n. Then, given any additive $\mathcal{R}_n$-linear map T on A and $a \in A$, we have

$$\begin{aligned}T(a) = T\left(\sum_{i=1}^{m} u_i E_n(u_i^* a)\right) = \sum_{i=1}^{m} T(u_i) E_n(u_i^* a) = \sum_{i=1}^{m} T(u_i)\langle u_i, a\rangle \\ = \sum_{i=1}^{m} \Omega_{T(u_i), u_i}(a),\end{aligned}$$

so that $T = \sum_{i=1}^{m} \Omega_{T(u_i),u_i} \in \check{\mathcal{K}}_n$. □

This last result gives a curious description of the dense subalgebra $\bigcup_{n\in\mathbb{N}} \mathcal{K}_n$ of $C^*(\mathcal{R}, \mathcal{E})$, namely that it is formed by the additive operators which are linear with respect to *some* $\mathcal{R}_n$. Observe that this is not quite the same as requiring linearity with respect to the intersection of the $\mathcal{R}_n$!

One of the main tools in our study from now on will be a certain conditional expectation from $C^*(\mathcal{R}, \mathcal{E})$ to A. Unfortunately, we can only show its existence in the finite-index case.

PROPOSITION 6.5. *If all E_n are of index-finite type, then there exists a conditional expectation*

$$G : C^*(\mathcal{R}, \mathcal{E}) \to A,$$

such that for each $n \in \mathbb{N}$ one has that

$$G(e_n) = \lambda_0^{-1} \dots \lambda_{n-1}^{-1},$$

where $\lambda_n = \operatorname{ind}(E_{n+1}|_{\mathcal{R}_n})$. If A is commutative then G is the unique conditional expectation from $C^*(\mathcal{R}, \mathcal{E})$ to A.

Proof. Set $\lambda_n = \operatorname{ind}(E_{n+1}|_{\mathcal{R}_n})$ so that $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is a potential in the sense of Definition 5.2 and the proposed value for $G(e_n)$ above is just $\lambda^{-[n]}$. Observe, moreover, that $\lambda^{-[n]}$ commutes with $\mathcal{R}_{n-1}$.

Let $n \in \mathbb{N}$ be fixed. Observing that $\mathcal{K}_n$ is isomorphic to $\check{\mathcal{K}}_n$ by Theorem 3.6(iii) and arguing exactly as in [**E2**, 8.4], we conclude that there exists a positive A-bimodule map $G_n : \mathcal{K}_n \to A$ such that $G_n(e_n) = \lambda^{-[n]}$.

We claim that G_{n+1} extends G_n. In fact, let $\{u_1, \dots, u_k\}$ be a quasi-basis for E_{n+1}. Then by Proposition 6.1 we have that $\{E_n(u_1), \dots, E_n(u_k)\}$ is a quasi-basis for $E_{n+1}|_{\mathcal{R}_n}$.

By Proposition 6.2(i) we have that $e_n = \sum_{i=1}^k E_n(u_i) e_{n+1} E_n(u_i)^*$, so that

$$\begin{aligned} G_{n+1}(e_n) &= \sum_{i=1}^k E_n(u_i)\lambda^{-[n+1]}E_n(u_i)^* = \lambda^{-[n+1]} \sum_{i=1}^k E_n(u_i)E_n(u_i)^* \\ &= \lambda^{-[n+1]} \operatorname{ind}(E_{n+1}|_{\mathcal{R}_n}) = \lambda^{-[n+1]}\lambda_n = \lambda^{-[n]} = G_n(e_n). \end{aligned}$$

The claim then follows easily from the fact that both G_n and G_{n+1} are A-bimodule maps.

As a consequence we see that each G_n restricts to the identity on A and hence G_n is a conditional expectation from $\mathcal{K}_n$ to A. Conditional expectations are always contractive so there exists a common extension $G : C^*(\mathcal{R}, \mathcal{E}) \to A$ which is the desired map.

Suppose that A is commutative and that G' is another conditional expectation from $C^*(\mathcal{R}, \mathcal{E})$ to A. Given n let $\{u_1, \dots, u_k\}$ be a quasi-basis for E_n and hence by Proposition 6.2(i) we have

$$1 = G'(1) = G'\left(\sum_{i=0}^k u_i e_n u_i^*\right) = \sum_{i=0}^k u_i G'(e_n) u_i^* = G'(e_n) \operatorname{ind}(E_n),$$

so necessarily $G'(e_n) = \operatorname{ind}(E_n)^{-1} = \lambda^{-[n]}$ by [**Wa**, 1.7.1]. Once we know that G and G' coincide on e_n it is easy to see that $G = G'$. □

7. *KMS states*

In this section we begin the general study of KMS states for gauge actions on $C^*(\mathcal{R}, \mathcal{E})$. We refer the reader to [**BR, P**] for the basic theory of KMS states.

Given what are probably limitations in our methods, we will all but have to assume that A is commutative. To be precise, we assume from now on that the conditional expectations E_n satisfy the following trace-like property:

$$E_n(ab) = E_n(ba), \quad \forall a, b \in A, \tag{7.1}$$

which is obviously the case when A is commutative. Unfortunately, we have no interesting non-commutative example of this situation, but since we do not really have to suppose that A is commutative and in the hope that some such example will be found, we proceed without the commutativity of A.

We moreover assume that all E_n are of index-finite type and denote by G the conditional expectation given by Proposition 6.5. Our first result is that any KMS state factors through G.

PROPOSITION 7.2. *Let h be a strictly positive potential, let $\beta > 0$, and let ϕ be a (σ, β)-KMS state (i.e. a KMS state for σ at inverse temperature β) on $C^*(\mathcal{R}, \mathcal{E})$ for the gauge action σ associated to h. Then $\phi = \phi \circ G$.*

Proof. Given $a, b \in A$ and $n \in \mathbb{N}$ it is clear from (5.6) that ae_nb is an analytic element with

$$\sigma_z(ae_nb) = ah^{iz[n]}e_nh^{-iz[n]}b, \quad \forall z \in \mathbb{C}.$$

We claim that

$$\phi(ae_nb) = \phi(h^{\beta[n]}E_n(bah^{-\beta[n]})e_n), \quad \forall a, b \in A, \quad \forall n \in \mathbb{N}. \tag{$*$}$$

In order to prove it we use the KMS condition as follows

$$\begin{aligned}\phi(ae_nb) = \phi(e_nba) = \phi(e_nba\sigma_{i\beta}(e_n)) = \phi(e_nbah^{-\beta[n]}e_nh^{\beta[n]}) \\ = \phi(E_n(bah^{-\beta[n]})e_nh^{\beta[n]}) = \phi(h^{\beta[n]}E_n(bah^{-\beta[n]})e_n),\end{aligned}$$

proving $(*)$. We next claim that

$$\phi(ae_{n+1}) = \phi(\lambda_n^{-1}ae_n), \quad \forall a \in A,$$

where λ_n is defined in Proposition 6.5. In order to prove this claim, let $\{v_1, \ldots, v_k\} \subseteq \mathcal{R}_n$ be a quasi-basis for the restriction of E_{n+1} to $\mathcal{R}_n$. Then by Proposition 6.2(i) we have for all $x \in A$ that

$$\phi(xe_n) = \phi\left(\sum_{i=0}^{k} xv_ie_{n+1}v_i^*\right) \overset{(*)}{=} \sum_{i=0}^{k} \phi(h^{\beta[n+1]}E_{n+1}(v_i^*xv_ih^{-\beta[n+1]})e_{n+1}).$$

Since $v_i \in \mathcal{R}_n$ and since $h^{-\beta[n+1]}$ commutes with $\mathcal{R}_n$, we have that

$$E_{n+1}(v_i^*xv_ih^{-\beta[n+1]}) = E_{n+1}(v_i^*xh^{-\beta[n+1]}v_i) = E_{n+1}(xh^{-\beta[n+1]}v_iv_i^*),$$

by the trace-like property of E_{n+1}. We then conclude that

$$\phi(xe_n) = \phi(h^{\beta[n+1]}E_{n+1}(xh^{-\beta[n+1]}\lambda_n)e_{n+1}).$$

Using $(*)$ once more we have that

$$\phi(ae_{n+1}) = \phi(h^{\beta[n+1]}E_{n+1}(ah^{-\beta[n+1]})e_{n+1}).$$

So when $x = \lambda_n^{-1}a$, we have that $\phi(xe_n) = \phi(ae_{n+1})$, which is precisely the identity we were looking for. By induction, we then have that

$$\phi(ae_n) = \phi(\lambda^{-[n]}a).$$

Therefore, for all $a, b \in A$,

$$\phi(ae_nb) = \phi(bae_n) = \phi(\lambda^{-[n]}ba) = \phi(a\lambda^{-[n]}b) = \phi(G(ae_nb)).$$

As the closed linear span of the set of elements of the form ae_nb is dense in $C^*(\mathcal{R}, \mathcal{E})$ the proof is complete. □

In particular, it follows that every KMS state is determined by its restriction to A. It is therefore useful to know which states on A occur as the restriction of a KMS state.

PROPOSITION 7.3. *Let ϕ be a state on A and let $\beta > 0$. Then the composition $\psi = \phi \circ G$ is a (σ, β)-KMS state if and only if*

$$\phi(a) = \phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}a)), \quad \forall a \in A,\ \forall n \in \mathbb{N},$$

where $\Lambda = \{\Lambda_n\}_{n\in\mathbb{N}}$ is the potential given by $\Lambda_n = h_n^{-\beta}\lambda_n$.

Proof. Suppose that ψ is a (σ, β)-KMS state. Then for all $a, b, c, d \in A$ and all $n \in \mathbb{N}$ we have

$$\psi((ae_nb)\sigma_{i\beta}(ce_nd)) = \psi((ce_nd)(ae_nb)). \tag{$**$}$$

Observe that the left-hand side of $(**)$ equals

$$\begin{aligned}\psi(ae_nbch^{-\beta[n]}e_nh^{\beta[n]}d) &= \psi(aE_n(bch^{-\beta[n]})e_nh^{\beta[n]}d)\\ &= \phi(aE_n(bch^{-\beta[n]})\lambda^{-[n]}h^{\beta[n]}d).\end{aligned}$$

Meanwhile, the right-hand side of $(**)$ equals

$$\psi(cE_n(da)e_nb) = \phi(cE_n(da)\lambda^{-[n]}b).$$

Plugging in $b = 1$, $c = h^{\beta[n]}$, and $d = h^{-\beta[n]}\lambda^{[n]}$ we have that $(**)$ implies that

$$\phi(a) = \phi(h^{\beta[n]}E_n(h^{-\beta[n]}\lambda^{[n]}a)\lambda^{-[n]}) = \phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}a)).$$

In order to prove the converse we first claim that if ϕ satisfies the condition in the statement for $n = 1$, then ϕ must be a trace. In fact, observing that $\Lambda^{[1]} = \Lambda_0 \in \mathcal{Z}(A)$ we have for all $a, b \in A$ that

$$\phi(ab) = \phi(\Lambda^{-[1]}E_1(\Lambda^{[1]}ab)) = \phi(\Lambda^{-[1]}E_1(a\Lambda^{[1]}b)) = \phi(\Lambda^{-[1]}E_1(\Lambda^{[1]}ba)) = \phi(ba),$$

where we have again used the trace-like property of E_1. Supposing now that ϕ satisfies the above condition not only for $n = 1$, but for all $n \in \mathbb{N}$, let us prove that ψ is a KMS state. For this we would like to prove that

$$\psi((ae_nb)\sigma_{i\beta}(ce_md)) = \psi((ce_md)(ae_nb)), \tag{$***$}$$

for all $a, b, c, d \in A$ and $n, m \in \mathbb{N}$. Supposing that $n \leq m$, the left-hand side of $(***)$ equals

$$\begin{aligned}\psi(ae_nbch^{-\beta[m]}e_mh^{\beta[m]}d)&\\ &= \psi(aE_n(bch^{-\beta[m]})e_mh^{\beta[m]}d)\\ &= \phi(aE_n(bch^{-\beta[m]})\lambda^{-[m]}h^{\beta[m]}d) = \phi(E_n(bch^{-\beta[m]})h^{\beta[m]}\lambda^{-[m]}da) = \cdots.\end{aligned}$$

Letting $x = h_n^\beta \ldots h_{m-1}^\beta$, observe that $x \in \mathcal{R}_n$ and $h^{\beta[m]} = xh^{\beta[n]}$, so the above equals

$$\begin{aligned}\cdots &= \phi(E_n(bch^{-\beta[n]}x^{-1})xh^{\beta[n]}\lambda^{-[m]}da) = \phi(E_n(bch^{-\beta[n]})h^{\beta[n]}\lambda^{-[m]}da)\\ &= \phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}E_n(bch^{-\beta[n]})h^{\beta[n]}\lambda^{-[m]}da))\\ &= \phi(\Lambda^{-[n]}E_n(bch^{-\beta[n]})E_n(\lambda^{[n]}\lambda^{-[m]}da)).\end{aligned}$$

Meanwhile, the right-hand side of $(***)$ equals

$$\begin{aligned}\psi(ce_mE_n(da)b) &= \phi(c\lambda^{-[m]}E_n(da)b) = \phi(bc\lambda^{-[m]}E_n(da))\\ &= \phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}bc\lambda^{-[m]}E_n(da)))\\ &= \phi(\Lambda^{-[n]}E_n(bc\lambda^{-[m]}\lambda^{[n]}h^{-\beta[n]})E_n(da)).\end{aligned}$$

Observing that $\lambda^{-[m]}\lambda^{[n]} \in \mathcal{R}_n$ we therefore see that $(***)$ is proved under the hypothesis that $n \leq m$. If, on the other hand, $n \geq m$ the left-hand side of $(***)$ becomes

$$\begin{aligned}\psi(ae_nbch^{-\beta[m]}e_mh^{\beta[m]}d) &= \psi(ae_nE_m(bch^{-\beta[m]})h^{\beta[m]}d)\\ &= \phi(a\lambda^{-[n]}E_m(bch^{-\beta[m]})h^{\beta[m]}d)\\ &= \phi(\Lambda^{-[m]}E_m(\Lambda^{[m]}\lambda^{-[n]}E_m(bch^{-\beta[m]})h^{\beta[m]}da))\\ &= \phi(\Lambda^{-[m]}E_m(bch^{-\beta[m]})E_m(h^{\beta[m]}da\Lambda^{[m]}\lambda^{-[n]}))\\ &= \phi(\Lambda^{-[m]}E_m(bch^{-\beta[m]})E_m(da\lambda^{[m]}\lambda^{-[n]})).\end{aligned}$$

The right-hand side of $(***)$ equals

$$\begin{aligned}\psi(cE_m(da)e_nb) &= \phi(cE_m(da)\lambda^{-[n]}b) = \phi(\lambda^{-[n]}bcE_m(da))\\ &= \phi(\Lambda^{-[m]}E_m(\Lambda^{[m]}\lambda^{-[n]}bcE_m(da)))\\ &= \phi(\Lambda^{-[m]}E_m(bch^{-\beta[m]}\lambda^{[m]}\lambda^{-[n]})E_m(da)).\end{aligned}$$

The conclusion follows once more because $\lambda^{[m]}\lambda^{-[n]} \in \mathcal{R}_m$. □

Putting together our last two results we reach one of our main goals.

THEOREM 7.4. *Let $\mathcal{R}$ be an approximately proper equivalence relation on a C*-algebra A and let $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ be a sequence of conditional expectations of index-finite type defined on A with $E_n(A) = \mathcal{R}_n$ satisfying (7.1) and $E_{n+1} \circ E_n = E_{n+1}$ for every n. Also, let h be any strictly positive potential and denote by σ the associated gauge action on $C^*(\mathcal{R}, \mathcal{E})$. Then for every $\beta > 0$, the correspondence $\psi \mapsto \phi = \psi|_A$ is a bijection from the set of (σ, β)-KMS states ψ on $C^*(\mathcal{R}, \mathcal{E})$ and the set of states ϕ on A satisfying*

$$\phi(a) = \phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}a)), \quad \forall a \in A,\ \forall n \in \mathbb{N},$$

where $\Lambda = \{\Lambda_n\}_{n\in\mathbb{N}}$ is the potential given by $\Lambda_n = h_n^{-\beta}\lambda_n$. The inverse of this correspondence is given by $\phi \mapsto \psi = \phi \circ G$, where G is given in Proposition 6.5.

8. *Existence of KMS states*

Theorem 7.4 gives a precise characterization of the KMS states on $C^*(\mathcal{R}, \mathcal{E})$ in terms of states on A satisfying certain conditions. It does not say, however, if such states exist. We now take up the task of showing the existence of at least one KMS state for each inverse temperature $\beta > 0$. We begin with a technical result which states that the conditions on ϕ required by Proposition 7.3 increase in strength with n.

PROPOSITION 8.1. *Let ϕ be a state on A and suppose that the formula*

$$\phi(a) = \phi(\Lambda^{-[n]} E_n(\Lambda^{[n]} a)), \quad \forall a \in A,$$

holds for $n = k + 1$, where $k \in \mathbb{N}$ is given. Then the formula holds for $n = k$.

Proof. For each $n \in \mathbb{N}$, let F_n be the operator on A given by

$$F_n(a) = \Lambda^{-[n]} E_n(\Lambda^{[n]} a), \quad \forall a \in A.$$

Then the formula in the statement is equivalent to $F_n^*(\phi) = \phi$, where F_n^* refers to the transpose operator on the dual of A.

We claim that for all n one has that $F_{n+1} \circ F_n = F_{n+1}$. In fact, observing that $\Lambda^{[n+1]} \Lambda^{-[n]} = \Lambda_n \in \mathcal{R}_n$, we have

$$\begin{aligned} F_{n+1}(F_n(a)) &= \Lambda^{-[n+1]} E_{n+1}(\Lambda^{[n+1]} \Lambda^{-[n]} E_n(\Lambda^{[n]} a)) \\ &= \Lambda^{-[n+1]} E_{n+1}(E_n(\Lambda^{[n+1]} \Lambda^{-[n]} \Lambda^{[n]} a)) = \Lambda^{-[n+1]} E_{n+1}(\Lambda^{[n+1]} a) \\ &= F_{n+1}(a). \end{aligned}$$

Given that $F_{k+1}^*(\phi) = \phi$ we have

$$F_k^*(\phi) = F_k^*(F_{k+1}^*(\phi)) = (F_{k+1} F_k)^*(\phi) = F_{k+1}^*(\phi) = \phi. \qquad \square$$

We now arrive at the main result of this section.

THEOREM 8.2. *Let $\mathcal{R}$ be an approximately proper equivalence relation on a C*-algebra A and let $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$ be a sequence of conditional expectations of index-finite type defined on A with $E_n(A) = \mathcal{R}_n$ satisfying (7.1) and $E_{n+1} \circ E_n = E_{n+1}$ for every n. Also let h be any strictly positive potential and denote by σ the associated gauge action on $C^*(\mathcal{R}, \mathcal{E})$. Then for every $\beta > 0$ there exists at least one (σ, β)-KMS state on $C^*(\mathcal{R}, \mathcal{E})$.*

Proof. For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be set of all states on A satisfying $F_n^*(\phi) = \phi$, where F_n is the operator defined in the beginning of the proof of Proposition 8.1. It is clear that the $\mathcal{S}_n$ are closed subsets of the state space of A and hence compact.

We claim that $\mathcal{S}_n$ is non-empty for every n. In order to prove this let τ be any trace on A. Observe that traces on A may be obtained by composing any state with E_1. For a given n, let $\phi = F_n^*(\tau)$. Since $F_n^2 = F_n$ it is clear that $F_n^*(\phi) = \phi$. Moreover, ϕ is a positive linear functional because for all $a \in A_+$ we have

$$\phi(a) = \tau(\Lambda^{-[n]} E_n(\Lambda^{[n]} a)) = \tau(\Lambda^{-\frac{1}{2}[n]} E_n(\Lambda^{\frac{1}{2}[n]} a \Lambda^{\frac{1}{2}[n]}) \Lambda^{-\frac{1}{2}[n]}) \geq 0.$$

Thus dividing ϕ by $\phi(1)$ (observe that $\phi(1) \neq 0$ by [**Wa**, 2.1.5]) gives an element of $\mathcal{S}_n$ so that $\mathcal{S}_n \neq \emptyset$. By Proposition 8.1 we have that the $\mathcal{S}_n$ are decreasing so their intersection is non-empty. Any ϕ belonging to that intersection is a state on A satisfying the condition in Proposition 7.3 and hence $\phi \circ G$ is a (σ, β)-KMS state on $C^*(\mathcal{R}, \mathcal{E})$. $\square$

It should be noted that the method employed above may be used to give an iterative process to produce KMS states: start with any state ϕ_0 on A and define

$$\phi_n = \phi_{n-1}(F_n(1))^{-1} F_n^*(\phi_{n-1}).$$

Any weak accumulation point of the sequence $\{\phi_n\}_n$ will be a state ϕ on A satisfying Proposition 7.3 and hence $\phi \circ G$ is the desired KMS state.

In the present level of generality there is not much more we can say about KMS states. In the following sections we discuss an example in which KMS states will be proven to be unique as well.

9. *Thermodynamic formalism and uniquenes of KMS states*

In this section we show a relationship between the KMS states that we have been discussing and the Gibbs states of thermodynamic formalism, as developed by Bowen [**Bo**] and Ruelle [**Ru1, Ru2, Ru3**]. This section should be viewed more as an illustration of the definitions of the previous sections rather than new results. In particular, Proposition 9.9 can also be obtained as a particular case of the characterization of KMS states given in [**Re2**, II.5.4].

Throughout the rest of this section, we fix a compact metric space X and a local homeomorphism $T : X \to X$. We also let α be the endomorphism of $C(X)$ given by

$$\alpha(f) = f \circ T, \quad \forall f \in C(X).$$

Consider the equivalence relation on X given by

$$x \sim y \quad \Leftrightarrow \quad \exists n \in \mathbb{N},\ T^n(x) = T^n(y).$$

In the case of the left shift on Bernouli's space (an example to be kept in the back of one's mind) this equivalence relation turns out to be the tail-equivalence relation which is not proper. However, it is easy to see that it is always approximately proper, and that it is the union of the equivalence relations R_n given by

$$(x, y) \in R_n \quad \Leftrightarrow \quad T^n(x) = T^n(y). \tag{9.1}$$

Clearly each R_n is proper and the algebra $C(X; R_n)$ is precisely the range of α^n. For simplicity we will denote the latter algebra by $\mathcal{R}_n$.

We now need conditional expectations E_n from $C(X)$ onto $\mathcal{R}_n$ and these are obtained as follows. By the assumption that T is a local homeomorphism and that X is compact we see that T is necessarily a covering map. The inverse image under T of each $x \in X$ is therefore a finite set. Given a continuous strictly positive function $p : X \to \mathbb{R}$ consider the associated Ruelle–Perron–Frobenius operator given by

$$\mathcal{L}_p(f)|_x = \sum_{T(z)=x} p(z) f(z), \quad \forall f \in C(X),\ x \in X.$$

We will assume that p is such that $\mathcal{L}_p$ is normalized (meaning that $\mathcal{L}_p(1) = 1$). This means that for every $x \in X$, the association $z \mapsto p(z)$ is a probability distribution on the equivalence class of x relative to R_1.

It is easy to show that $\mathcal{L}_p$ satisfies the identity

$$\mathcal{L}_p(f)g = \mathcal{L}_p(f\alpha(g)), \quad \forall f, g \in C(X). \tag{9.2}$$

For any $n \in \mathbb{N}$ set

$$E_n = \alpha^n \mathcal{L}_p^n. \tag{9.3}$$

Given $f \in C(X)$ one then has that $E_1(f)|_x$ is just the weighted average of f over the equivalence class of x relative to R_1. Therefore, E_1 is a conditional expectation onto $\mathcal{R}_1$. Likewise, E_n is a conditional expectation onto $\mathcal{R}_n$ and because the composition $\mathcal{L}_p \circ \alpha$ is the identity map on $C(X)$, we have that $E_m \circ E_n = E_m$ for $m \geq n$. Setting $\mathcal{R} = \{\mathcal{R}_n\}_{n\in\mathbb{N}}$ and $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ we may then speak of $C^*(\mathcal{R}, \mathcal{E})$.

Observe that the present situation is precisely that of a stationary equivalence relation described in § 4.

Given any $f \in C(X)$ it is clear that $\alpha^n(f) \in \mathcal{R}_n$ for all n and hence the sequence $\{\alpha^n(f)\}_{n\in\mathbb{N}}$ is a potential. Accordingly we will adopt the notation $f^{[n]}$ to mean

$$f^{[n]} = f\alpha(f)\ldots\alpha^{n-1}(f).$$

For later use it is convenient to give an explicit description for $\mathcal{L}_p^n$ as well as E_n.

LEMMA 9.4. *Let $n \in \mathbb{N}$, then for every $f \in C(X)$ and $x \in X$ one has that*

$$\mathcal{L}_p^n(f)|_x = \sum_{T^n(z)=x} p^{[n]}(z) f(z),$$

and

$$E_n(f)|_x = \sum_{(z,x)\in R_n} p^{[n]}(z) f(z),$$

Before giving the proof we should note that in summations of the form $\sum_{(z,x)\in R_n}$, which will be often used from now on, the variable which we mean to sum upon will always be the first mentioned (z in this case) even though equivalence relations are well known to be symmetric.

Proof of Lemma 9.4. In order to prove the first statement we use induction on n observing that the case $n = 1$ follows by definition. Given $n \geq 1$ we have

$$\mathcal{L}_p^{(n+1)}(f)|_x = \mathcal{L}_p^n(\mathcal{L}_p(f))|_x = \sum_{T^n(z)=x} p^{[n]}(z) \sum_{T(w)=z} p(w) f(w) = \cdots.$$

Note that a pair (z, w) is such that $T^n(z) = x$ and $T(w) = z$ if and only if it is of the form $(T(w), w)$ where $T^{n+1}(w) = x$. Therefore, the above equals

$$\cdots = \sum_{T^{n+1}(w)=x} p^{[n]}(T(w)) p(w) f(w) = \sum_{T^{n+1}(w)=x} p^{[n+1]}(w) f(w),$$

proving the first statement. The second statement then follows easily. □

In the following we compute the index of our conditional expectations.

PROPOSITION 9.5. *For each $n \in N$ we have that $E_{n+1}|_{\mathcal{R}_n}$ is of index-finite type and* $\mathrm{ind}(E_{n+1}|_{\mathcal{R}_n}) = \alpha^n(p^{-1})$.

Proof. Let $\{V_i\}_{i=1}^m$ be a finite open covering of X such that the restriction of T to each V_i is one-to-one and let $\{v_i\}_{i=1}^m$ be a partition of unity subordinate to this covering. Set $u_i = (p^{-1}v_i)^{1/2}$ and observe that for every $f \in C(X)$ and $x \in X$ one has that

$$\sum_{i=1}^m u_i E_1(u_i f)|_x = \sum_{i=1}^m u_i(x) \sum_{\substack{z\in X \\ T(z)=T(x)}} p(z)u_i(z)f(z)$$

$$= \sum_{i=1}^m u_i(x)p(x)u_i(x)f(x) = \sum_{i=1}^m v_i(x)f(x) = f(x).$$

Therefore $\{u_1, \ldots, u_m\}$ is a quasi-basis for E_1 so that

$$\mathrm{ind}(E_1) = \sum_{i=1}^m u_i^2 = \sum_{i=1}^m p^{-1}v_i = p^{-1}.$$

Next observe that the diagram

$$\begin{array}{ccc} \mathcal{R}_0 & \xrightarrow{E_1} & R_1 \\ {\scriptstyle \alpha^n}\downarrow & & \downarrow{\scriptstyle \alpha^n} \\ \mathcal{R}_n & \xrightarrow{E_{n+1}} & \mathcal{R}_{n+1} \end{array}$$

is commutative. Therefore, $E_{n+1}|_{\mathcal{R}_n}$ is conjugate to E_1 under α^n and so $\mathrm{ind}(E_{n+1}|_{\mathcal{R}_n}) = \alpha^n(\mathrm{ind}(E_1)) = \alpha^n(p^{-1})$. □

We therefore have that each E_n is of index-finite type. Also, note that in the notation of Proposition 6.5 we have proven that $\lambda_n = \alpha^n(p^{-1})$.

Let H be a strictly positive continuous function on X. Setting $h_n = \alpha^n(H)$ for every $n \in \mathbb{N}$ we have that $h := \{h_n\}_{n\in\mathbb{N}}$ is a strictly positive potential in the sense of Definition 5.5. The corresponding gauge action will be denoted by σ.

We are interested in showing that for every $\beta > 0$ there exists a unique (σ, β)-KMS state on $C^*(\mathcal{R}, \mathcal{E})$, thus improving on Theorem 8.2.

Given $\beta > 0$, consider the Ruelle–Perron–Frobenius operator associated to $H(z)^{-\beta}$, namely

$$\mathcal{L}_{H,\beta}(f)|_x = \sum_{T(z)=x} H(z)^{-\beta} f(z), \quad \forall f \in C(X),\ x \in X.$$

In order to achieve our goal, we need to use the celebrated Ruelle–Perron–Frobenius Theorem whose conclusions are as follows.

THEOREM 9.6. (Conclusions of the Ruelle–Perron–Frobenius Theorem)

(a) *There exists a unique pair $(c_{H,\beta}, \nu_{H,\beta})$ such that $c_{H,\beta}$ is a strictly positive real number, $\nu_{H,\beta}$ is a probability measure on X, and*

$$\mathcal{L}^*_{H,\beta}(\nu_{H,\beta}) = c_{H,\beta}\nu_{H,\beta},$$

where $\mathcal{L}^*_{H,\beta}$ refers to the transpose operator on the dual of $C(X)$, which in turn is identified with the space of finite regular Borel measures on X.

(b) *There exists a strictly positive continuous function $k_{H,\beta}$ on X such that:*

- $\int k_{H,\beta}\, dv_{H,\beta} = 1$;
- $\mathcal{L}_{H,\beta}(k_{H,\beta}) = c_{H,\beta} k_{H,\beta}$; *and*
- $\lim_{n\to\infty} (\mathcal{L}^n_{H,\beta}(f)/c^n_{H,\beta}) = k_{H,\beta} \int f\, dv_{H,\beta}, \quad \forall f \in C(X)$;

where the limit is with respect to the (sup) norm topology of $C(X)$.

Initially proven for the shift on the one-sided Bernouli space and H a Hölder continuous function [**Ru1**, Theorem 3] this theorem has been proved to hold under more general hypotheses: see, for example, [**Ba, Bo, C, F, FJ1, FJ2, K, Ru2, Ru3, W1, W2**].

The reader is referred to the above articles for more details on the various hypotheses under which the Ruelle–Perron–Frobenius Theorem holds, so we will simply assume its conclusions as above.

Definition 9.7. The probability $v_{H,\beta}$ is called the *Gibbs state* associated to $H^{-\beta}$.

In the sequel we show the following elementary relationship between the operators $\mathcal{L}_p$ and $\mathcal{L}_{H,\beta}$.

PROPOSITION 9.8. *Given* $\beta > 0$ *and* $n \in \mathbb{N}$ *we have that*

$$\mathcal{L}^n_{H,\beta}(f) = \mathcal{L}^n_p(\Lambda^{[n]} f), \quad \forall f \in C(X),$$

where the potential $\Lambda = \{\Lambda_n\}_{n\in\mathbb{N}}$ *was defined in Proposition 7.3 by* $\Lambda_n = h_n^{-\beta}\lambda_n$.

Proof. In the present situation we have that $h_n = \alpha^n(H)$ and $\lambda_n = \alpha^n(p^{-1})$ so that

$$\Lambda_n = \alpha^n(H)^{-\beta}\alpha^n(p^{-1}) = \alpha^n(H^{-\beta}p^{-1}).$$

Next, observe that for $f \in C(X)$ we have

$$\mathcal{L}_{H,\beta}(f) = \mathcal{L}_p(H^{-\beta}p^{-1}f) = \mathcal{L}_p(\Lambda_0 f).$$

The conclusion now follows easily by induction using (9.2). □

We now show that the Gibbs states indeed give KMS states on $C^*(\mathcal{R}, \mathcal{E})$.

PROPOSITION 9.9. *For every* $\beta > 0$ *the state* $\phi_{H,\beta}$ *on* $C(X)$ *corresponding via the Riesz representation theorem to the Gibbs state* $v_{H,\beta}$ *satisfies the conditions of Proposition 7.3 and hence the composition* $\psi_{H,\beta} = \phi_{H,\beta} \circ G$ *is a* (σ, β)*-KMS state on* $C^*(\mathcal{R}, \mathcal{E})$.

Proof. The condition that $v_{H,\beta}$ is an eigenmeasure for $\mathcal{L}_{H,\beta}$ gives for every $f \in C(X)$ and any $n \in \mathbb{N}$ that

$$\phi_{H,\beta}(\mathcal{L}^n_p(\Lambda^{[n]} f)) = \phi_{H,\beta}(\mathcal{L}^n_{H,\beta}(f)) = c^n_{H,\beta}\phi_{H,\beta}(f).$$

Plugging $f = \Lambda^{-[n]}\alpha^n(g)$ above, where $g \in C(X)$, we obtain

$$\phi_{H,\beta}(g) = c^n_{H,\beta}\phi_{H,\beta}(\Lambda^{-[n]}\alpha^n(g)).$$

In order to prove the condition in Proposition 7.3 we then compute

$$\phi_{H,\beta}(\Lambda^{-[n]}E_n(\Lambda^{[n]} f)) = \phi_{H,\beta}(\Lambda^{-[n]}\alpha^n\mathcal{L}^n_p(\Lambda^{[n]} f)) = c^{-n}_{H,\beta}\phi_{H,\beta}(\mathcal{L}^n_p(\Lambda^{[n]} f)) = \phi_{H,\beta}(f).$$

This concludes the proof. □

Our next main goal is to show that the state $\psi_{H,\beta}$ given by the above result is the unique (σ, β)-KMS state on $C^*(\mathcal{R}, \mathcal{E})$.

THEOREM 9.10. *Let T be a local homeomorphism on a compact metric space X and consider the approximately proper equivalence relation $\mathcal{R} = \{R_n\}_{n\in\mathbb{N}}$, where each R_n is given by (9.1). Let $p : X \to \mathbb{R}$ be a strictly positive continuous function satisfying $\sum_{T(z)=x} p(z) = 1$ for every $x \in X$ and define the sequence of conditional expectations $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ as in (9.3). Let H be a strictly positive continuous function on X and consider the one parameter automorphism group of $C^*(\mathcal{R}, \mathcal{E})$ given by the potential $h := \{H \circ T^n\}_{n\in\mathbb{N}}$. Assuming Theorem 9.6 we have that for every $\beta > 0$ the state $\psi_{H,\beta}$ given by Proposition 9.9 is the unique (σ, β)-KMS state on $C^*(\mathcal{R}, \mathcal{E})$.*

Proof. Let ψ be a (σ, β)-KMS state on $C^*(\mathcal{R}, \mathcal{E})$ and let ϕ be its restriction to $C(X)$. By Proposition 7.2 we have that $\psi = \phi \circ G$ so it suffices to show that $\phi = \phi_{H,\beta}$. Fix $f \in C(X)$ and note that by Proposition 7.3 we have

$$\begin{aligned}\phi(f) &= \phi(\Lambda^{-[n]} E_n(\Lambda^{[n]} f)) = \phi(\Lambda^{-[n]}\alpha^n \mathcal{L}_p^n(\Lambda^{[n]} f)) = \phi(\Lambda^{-[n]}\alpha^n \mathcal{L}_{H,\beta}^n(f)) \\ &= c_{H,\beta}^n \phi\left(\Lambda^{-[n]}\alpha^n\left(\frac{\mathcal{L}_{H,\beta}^n(f)}{c_{H,\beta}^n}\right)\right). \qquad (\dagger)\end{aligned}$$

We next claim that if we replace the argument of α^n in (†) by its limit, namely $\phi_{H,\beta}(f)k_{H,\beta}$, we will arrive at an expression which converges to $\phi(f)$ as $n \to \infty$. In order to prove this we compute

$$\begin{aligned}&|\phi(f) - c_{H,\beta}^n \phi(\Lambda^{-[n]}\alpha^n(\phi_{H,\beta}(f)k_{H,\beta}))| \\ &\quad = \left|c_{H,\beta}^n \phi\left(\Lambda^{-[n]}\alpha^n\left(\frac{\mathcal{L}_{H,\beta}^n(f)}{c_{H,\beta}^n} - \phi_{H,\beta}(f)k_{H,\beta}\right)\right)\right| \\ &\quad \le c_{H,\beta}^n \phi(\Lambda^{-[n]})\left\|\frac{\mathcal{L}_{H,\beta}^n(f)}{c_{H,\beta}^n} - \phi_{H,\beta}(f)k_{H,\beta}\right\|.\end{aligned}$$

The claim will be proven once we show that the expression $c_{H,\beta}^n \phi(\Lambda^{-[n]})$ is bounded from above with n. In fact, as $k_{H,\beta}$ is strictly positive, there exists $m > 0$ such that $k_{H,\beta} > m$. Therefore, plugging $f := k_{H,\beta}$ into (†) leads to

$$\phi(k_{H,\beta}) = c_{H,\beta}^n \phi(\Lambda^{-[n]}\alpha^n(k_{H,\beta})) \ge c_{H,\beta}^n \phi(\Lambda^{-[n]})m,$$

from where one easily deduces the desired boundedness. Summarizing we have proven that

$$\phi(f) = \phi_{H,\beta}(f) \lim_{n\to\infty} c_{H,\beta}^n \, \phi(\Lambda^{-[n]}\alpha^n(k_{H,\beta})),$$

for every $f \in C(X)$. Since both ϕ and $\phi_{H,\beta}$ evaluate to 1 on the constant function $f = 1$, it follows that $\lim_{n\to\infty} c_{H,\beta}^n \, \phi(\Lambda^{-[n]}\alpha^n(k_{H,\beta})) = 1$ and hence that $\phi = \phi_{H,\beta}$ as desired. □

As a consequence we have the following.

COROLLARY 9.11. *Let X, T, $\mathcal{R}$, p, and $\mathcal{E}$ be as in Theorem 9.10. Then $C^*(\mathcal{R}, \mathcal{E})$ admits a unique trace.*

Proof. Set $H = 1$ in Theorem 9.10 so that the corresponding one parameter automorphism group is the trivial one. Fixing an arbitrary $\beta > 0$ observe that the (σ, β)-KMS states on $C^*(\mathcal{R}, \mathcal{E})$ are precisely the traces. The conclusion then follows from Theorem 9.10. □

10. *Conditional minima and ground states*

So far we have studied KMS states at positive temperature and we have seen how they relate to the Gibbs states of statistical mechanics. We next want to discuss ground states, but before that we need to study the notion of conditional minimum points.

Our discussion in this section may be viewed as a special case of Renault's study of ground-state cocycles over groupoids [**Re1**, §3]. We begin with some notation.

Definition 10.1. Let R be a proper equivalence relation on the compact space X, let h be a continuous real function on X, and let C be a closed subset of X.

(i) We denote by $M_{h,C}$ the set of minimum points for h over C, namely

$$M_{h,C} = \left\{ x \in C : h(x) = \inf_{y \in C} h(y) \right\}.$$

(ii) We denote by M_h^R the union of all $M_{h,C}$ as C runs over the quotient space X/R (observe that each $C \in X/R$ is a closed subset of X).

Observe that a necessary and sufficient condition for x to be in M_h^R is that

$$\forall y \in X, \quad (x, y) \in R \Rightarrow h(x) \leq h(y). \tag{10.2}$$

For this reason the points in M_h^R should be called conditional minimum points of h. Observe also that our hypotheses imply that $M_{h,C}$ is non-empty for every $C \in X/R$ so one sees that M_h^R meets every single equivalence class.

Even though $M_{h,C}$ is closed for every equivalence class C it may be that M_h^R is not closed. However, under suitable conditions we may be sure that M_h^R is closed.

PROPOSITION 10.3. (see [**Re1**, 3.16.iii]) *Let R be a proper equivalence relation on the compact space X and let h be a continuous real function on X. If R is open (recall that an equivalence relation is said to be open when the saturation of each open set is open), then M_h^R is closed.*

The proof is left to the reader.

So far we have been considering a proper equivalence relation R on a compact set X and a continuous real function h on X. From now on we will assume that R is such that the quotient map $\pi : X \to X/R$ is a covering map, which incidentally implies that R is open. We wish to add to this setup a conditional expectation E from $C(X)$ to $\mathcal{R} := C(X; R)$ which will be obtained as follows: fix a strictly positive continuous function p on X and let $E : C(X) \to \mathcal{R}$ be given by

$$E(f)|_x = \sum_{(y,x) \in R} p(y) f(y), \quad \forall f \in C(X),\ x \in X. \tag{10.4}$$

If we assume that

$$\sum_{(y,x) \in R} p(y) = 1, \quad \forall x \in X, \tag{10.5}$$

it is easy to see that E is indeed a conditional expectation onto $\mathcal{R}$.

The following result is a crucial technical tool in our characterization of ground states.

LEMMA 10.6. *Let R be a proper equivalence relation on a compact space X such that the corresponding quotient map is a covering map. Let p be a strictly positive continuous function on X satisfying (10.5) and define the conditional expectation E as in (10.4). If h is another strictly positive continuous function on X, define for each real number $\beta \geq 0$ the operator E^β on $C(X)$ by*

$$E^\beta(f) = h^\beta E(h^{-\beta} f), \quad \forall f \in C(X).$$

Then for every probability measure μ on X the following conditions are equivalent:

(i) *the support of μ is contained in M_h^R;*

(ii) *for every $f, g \in C(X)$ one has that*

$$\sup_{\beta \geq 0} \left| \int_X f E^\beta(g)\, d\mu \right| \leq \|f\|\, \|g\|;$$

(iii) *for every $f, g \in C(X)$ one has that*

$$\sup_{\beta \geq 0} \left| \int_X f E^\beta(g)\, d\mu \right| < \infty;$$

(iv) *the inequality in (iii) holds for $f = g = 1$.*

The proof is left to the reader.

Now we apply the conclusions reached above to study ground states on $C^*(\mathcal{R}, \mathcal{E})$. The setup for now will be as follows: X will be a compact Hausdorff space and $\mathcal{R} = \{R_n\}_{n \in \mathbb{N}}$ an approximately proper equivalence relation on X. We also fix a real potential $h = \{h_n\}_{n \in \mathbb{N}}$. Recall from Definition 5.2 that this means that each h_n is a continuous real function in $\mathcal{R}_n := C(X; R_n)$.

PROPOSITION 10.7. *For every $n \in \mathbb{N}$ let M_n be the set of conditional minimum points of $h^{[n]}$ relative to R_n, namely*

$$M_n = M_{h^{[n]}}^{R_n}$$

in the notation of Definition 10.1(ii). Then $M_{n+1} \subseteq M_n$.

Proof. Let $x \in M_{n+1}$. In order to show that $x \in M_n$ we employ the characterization given in (10.2). So let y be such that $(x, y) \in R_n$. Since the R_k are increasing we have that $(x, y) \in R_{n+1}$ and hence

$$h^{[n+1]}(x) \leq h^{[n+1]}(y). \qquad (\ddagger)$$

Observe that because h_n belongs to $C(X; R_n)$ we have that $h_n(x) = h_n(y)$. Dividing both sides of (‡) by this common value leads to $h^{[n]}(x) \leq h^{[n]}(y)$, completing the proof. □

If one tries to apply the definition of conditional minimum points for the relation $R = \bigcup_{n \in \mathbb{N}} R_n$, which we are attempting to approximate by the sequence $\{R_n\}_{n \in \mathbb{N}}$, one is likely to run into some trouble, not least because equivalence classes need not always be closed (in fact, they are often dense). An alternative approach is to look at points which are conditional minima for all R_n.

Definition 10.8. Given an approximately proper equivalence relation $\mathcal{R} = \{R_n\}_{n\in\mathbb{N}}$ on a compact space X and a real potential $h = \{h_n\}_{n\in\mathbb{N}}$, we denote by $M_h^{\mathcal{R}}$ the intersection of $M_{h^{[n]}}^{R_n}$ as n range in $\mathbb{N}$.

Observe that if all of the R_n are open equivalence relations, it follows from Propositions 10.3 and 10.7 that $M_h^{\mathcal{R}}$ is a non-empty compact subset of X.

From this point on we assume that R_n are not only open but also that the quotient maps are covering maps as in Lemma 10.6. In addition to this we will fix a strictly positive potential $p = \{p_n\}_{n\in\mathbb{N}}$. Following Lemma 9.4 and 10.4 we define maps $E_n : C(X) \to \mathcal{R}_n$ by

$$E_n(f)|_x = \sum_{(y,x)\in R_n} p^{[n]}(y) f(y), \quad \forall f \in C(X),\ x \in X.$$

LEMMA 10.9. *Suppose that for every n and every R_{n+1}-equivalence class C one has that*

$$\sum_D p_n(D) = 1,$$

where the sum extends over all R_n-equivalence classes D contained in C, and for each such D one interprets $p_n(D)$ as the common value of $p_n(x)$ for any $x \in D$. Then each E_n is a conditional expectation of index-finite type onto $\mathcal{R}_n$ and $E_{n+1} \circ E_n = E_{n+1}$.

Proof. We first claim that for every $n \in \mathbb{N}$ and every $x \in X$ one has that $\sum_{(y,x)\in R_n} p^{[n]}(y) = 1$. In order to prove this we use induction observing that the case '$n = 1$' follows from the hypothesis. Assuming that $n \geq 1$ we have

$$\sum_{(y,x)\in R_{n+1}} p^{[n+1]}(y) = \sum_{i=1}^{n} \sum_{y\in C_i} p^{[n+1]}(y) = \cdots$$

where $\{C_1, \ldots, C_n\}$ is the decomposition of the R_{n+1}-equivalence class of x into R_n-equivalence classes. The above then equals

$$\cdots = \sum_{i=1}^{n} \sum_{y\in C_i} p_n(y) p^{[n]}(y) = \sum_{i=1}^{n} p_n(C_i) \sum_{y\in C_i} p^{[n]}(y) = \sum_{i=1}^{n} p_n(C_i) = 1,$$

where the penultimate equality follows from the induction hypothesis and the last equality is a consequence of our hypothesis. It immediately follows that E_n is in fact a conditional expectation onto $\mathcal{R}_n$. The proof that E_n is of index-finite type is a simple modification of Proposition 9.5 and hence will be omitted.

With respect to the last part of the statement, let $f \in C(X)$ so that for $x \in X$, we have

$$E_{n+1}(E_n(f))|_x = \sum_{(y,x)\in R_{n+1}} p^{[n+1]}(y) \sum_{(w,y)\in R_n} p^{[n]}(w) f(w) = \cdots.$$

Letting $\{C_1, \ldots, C_n\}$ be as in the first part of the proof we have that the above equals

$$\begin{aligned}
\ldots &= \sum_{i=1}^{n} \sum_{y\in C_i} p^{[n+1]}(y) \sum_{w\in C_i} p^{[n]}(w) f(w) = \sum_{i=1}^{n} \sum_{y,w\in C_i} p_n(y) p^{[n]}(y) p^{[n]}(w) f(w) \\
&= \sum_{i=1}^{n} \sum_{y,w\in C_i} p_n(w) p^{[n]}(y) p^{[n]}(w) f(w) = \sum_{i=1}^{n} \sum_{y\in C_i} p^{[n]}(y) \sum_{w\in C_i} p^{[n+1]}(w) f(w) \\
&= E_{n+1}(f)|_x.
\end{aligned}$$

□

We are now ready to present our main theorem on ground states. Unlike Proposition 7.2 one cannot prove that all ground states factor through the conditional expectation G of Proposition 6.5. For example, if we choose the potential h given by $h_n \equiv 1$, then the dynamics are trivial and hence any state is a ground state, regardless of whether it factors through G or not. Our result will therefore be restricted to the characterization of the ground states of the form $\phi \circ G$, where ϕ is a state on $C(X)$.

THEOREM 10.10. (see [**Re1**, 5.4]) *Let X be a compact Hausdorff space and $\mathcal{R} = \{R_n\}_{n\in\mathbb{N}}$ an approximately proper equivalence relation on X such that the quotient map relative to each R_n is a covering map. Fix a strictly positive potential $p = \{p_n\}_{n\in\mathbb{N}}$ satisfying Lemma 10.9 and let E_n be the conditional expectations provided by Lemma 10.9. Also, let σ be a one-parameter group of automorphisms of $C^*(\mathcal{R}, \mathcal{E})$ obtained from a strictly positive potential h. Given a measure μ on X let ϕ be the state on $C(X)$ given by integration against μ. Then the composition $\psi = \phi \circ G$ is a ground state on $C^*(\mathcal{R}, \mathcal{E})$ if and only if the support of μ is contained in $M_h^{\mathcal{R}}$.*

Proof. Let $a, b, c, d \in C(X)$, let $n, m \in \mathbb{N}$, and let $z = \alpha + i\beta$. If $n \leq m$ we have by Proposition 2.4 that

$$\begin{aligned}\psi((ae_nb)\sigma_z(ce_md)) &= \psi(aE_n(bch^{i\alpha[n]}h^{-\beta[n]})e_mh^{-i\alpha[n]}h^{\beta[n]}d) \\ &= \int aE_n(bch^{i\alpha[n]}h^{-\beta[n]})\lambda^{-[m]}h^{-i\alpha[n]}h^{\beta[n]}d\,d\mu = \int fE_n^{\beta}(g)\,d\mu,\end{aligned} \tag{$\sharp$}$$

where $f = a\lambda^{-[m]}h^{-i\alpha[n]}d$, $g = bch^{i\alpha[n]}$, and E_n^{β} is defined as in Lemma 10.6 in terms of $h^{[n]}$.

If $n \geq m$ we instead have

$$\psi((ae_nb)\sigma_z(ce_md)) = \int fE_m^{\beta}(g)\,d\mu, \tag{$\sharp'$}$$

where g is as above and f is now $a\lambda^{-[n]}h^{-i\alpha[n]}d$.

Assuming that the support of μ is contained in $M_h^{\mathcal{R}}$ it follows from Lemma 10.6(ii) that both ($\sharp$) and ($\sharp'$) are bounded as z runs in the upper half plane and hence that ψ is a ground state. The converse also follows easily from Lemma 10.6. □

11. *Ground states and maximizing measures*

In a similar way as in § 9, in the present section we want to obtain (in an interesting particular case) a characterization of ground states of C^*-algebras by means of maximizing measures in the sense of [**CLT**] (or, in other words, by means of zero temperature Gibbs measures in the sense of [**RF**, Appendix B]).

Let (X, d) be a compact metric space and $T : X \to X$ be an expanding transformation (see [**Ba**] for a definition and properties). In order to simplify our proof we assume in this section that the transformation T has the property that each point $x \in X$ has $k > 1$ distinct preimages and take $p = 1/k = 1/\Lambda$, where p, Λ are defined as in § 9. Similar results can be obtained for p Hölder.

We denote by μ_0 the maximal entropy measure because we are considering here $p = 1/k = 1/\Lambda$. Then μ_0 is the eigenmeasure for $\mathcal{L}_p^*$ associated to the eigenvalue 1.

We consider the associated C^*-algebra $C^*(\mathcal{R}, \mathcal{E})$ as before.

Consider a fixed Hölder real function $H : X \to \mathbb{R}$. We say $\tilde{H} > 0$ is cohomologous to H if there exists a real function V and real constant c such that $\log \tilde{H} = \log H - [(V \circ T) - V] + c$.

We denote as usual by $\mathcal{M}(T)$ the set of invariant probabilities for T.

An important point in § 9 is that for a given real β the measure $\nu_{H,\beta}$ is an eigenmeasure (not necessarily invariant) for the Ruelle operator $\mathcal{L}_{H,\beta}$. Given H there exists, however, another potential $\tilde{H}$, cohomologous to H such that the eigenmeasure $\nu_{\tilde{H},\beta}$ for $\mathcal{L}_{\tilde{H},\beta}$ is an invariant measure (see [**Bo**]).

We would like to investigate similar properties for the ground state problem. In principle, it can happen that for a certain H there is no invariant measure μ with support inside M_H of Theorem 10.10.

Given H we define a certain $\tilde{H}$ cohomologous to H. Consider ϕ a ground state for σ_z associated to such $h = \tilde{H}$ (defined as before in § 5). It follows from the reasoning of this section that there exist a measure ν, which is an invariant maximizing measure in the sense of [**CLT**], such that for any continuous function f we have $\phi(f) = \int f d\nu$. These measures ν are the discrete time analogs (for the case of expanding maps) of the Aubry–Mather measures of Lagrangian mechanics. In the case of the geodesic flow in compact surfaces of negative curvature, they exactly correspond to each other under the action of the discrete group of Moebius transformations in the boundary of the Poincare disk (see [**BS**] and [**LT**]).

First we will recall some general results for maximizing measures.

Definition 11.1. Given an α-Hölder function B we denote

$$\mathrm{Hol}_\alpha(B) = \sup_{d(x,y)>0} \left\{ \frac{|B(x) - B(y)|}{d(x, y)^\alpha} \right\}.$$

If we denote by $\|B\|_\infty$ the uniform norm, then we define the α-Hölder norm of B by $\|B\|_\alpha = \mathrm{Hol}_\alpha(B) + \|B\|_\infty$. We also let $\mathcal{H}_\alpha$ be the set of α-Hölder functions.

Definition 11.2. Given $\log H \in \mathcal{H}_\alpha$ we define

$$m(-\log H) = \sup \left\{ -\int \log H(x)\, d\rho(x) \mid \rho \in \mathcal{M}(T) \right\}$$

and

$$\mathcal{M}_H(T) = \left\{ \rho \in \mathcal{M}(T) : -\int \log H(x)\, d\rho(x) = m(-\log H) \right\}.$$

We call any $\rho \in \mathcal{M}_H(T)$ a maximizing measure for H and it will be generically denoted by μ_H.

The maximizing measure is not necessarily unique.

It was shown in [**CLT**, Proposition 15, p. 1387] that a measure μ is maximizing if and only if its support is contained in the $\Omega(-\log H, T)$ set (see [**CLT**, p. 1386] for a definition). This result is the version of Theorem 10.10 above for the case of invariant

measures. We refer the reader to [**CLT**] for general references on the topics considered in the present section.

Consider $\mathcal{F}_\alpha^+ = \bigcup_{\gamma>\alpha} \mathcal{H}_\gamma$ equipped with the α-norm.

THEOREM 11.3. ([**CLT**, p. 1382]) *For an open and dense set $\mathcal{G}$ contained in $\mathcal{F}_\alpha^+$, when $-\log H \in \mathcal{G}$, the measure $\mu_H \in \mathcal{M}_H(T)$ is unique and has support in a unique periodic orbit.*

It can be shown that for any H, the omega-limit set of points in M_H (of Theorem 10.10) is contained in the support of the maximizing measure μ_H. Note that M_H is not necessarily a forward invariant set for T.

In [**CLT**, p. 1394] examples were shown of H where μ_H is uniquely ergodic and has positive entropy. Denote by $S : \mathcal{L}^2(\mu_0) \to \mathcal{L}^2(\mu_0)$ the Koopman operator where for $\eta \in \mathcal{L}^2(\mu_0)$ we define $(S\eta)(x) = \eta(T(x))$. Such S defines a linear operator in $\mathcal{L}^2(\mu_0)$. It is well known that $S^* = \mathcal{L}_p$ acting on $\mathcal{L}^2(\mu_0)$. We consider the same C^*-algebra as in the previous sections associated to $p = 1/\Lambda = 1/k$. We assume that $\tilde{H} > 0$ is Hölder and consider the corresponding $\sigma_z = e^{-iz\tilde{H}}\, S$ as before.

By Proposition 11 of [**CLT**], there exist $V : X \to \mathbb{R}$, Hölder continuous strictly positive and satisfying for all x the inequality

$$V(T(x)) - V(x) \geq -\log H(x) - m(-\log H).$$

This inequality is called a sub-cohomological equation. The inequality is an equality for x in the support of μ_H.

The function V is defined by

$$V(x) = \sup\left\{ \sum_{j=0}^{n-1} (-\log H - m(-\log H))(T^j(y)) \mid T^n(y) = x, n \in \mathbb{N} \right\}.$$

Note that $m(-\log H + V - V \circ T) = \sup\left\{ \int -\log H d\rho \mid \rho \in \mathcal{M}(T) \right\} = m(-\log H)$, because we are considering ρ an invariant measure.

We say that a probability measure ψ is a *ground measure* when the state on $C^*(\mathcal{R}, \mathcal{E})$ given by $\phi = \psi \circ G$ is a ground state, as in Theorem 10.10.

Consider the C^*-algebra described in § § 7 and 8 in the particular case we consider here. We say that a certain state ϕ is a ground-state for σ_z if for any pair a and analytic b

$$\sup_{\mathrm{Im} z \geq 0} |\phi(a\sigma_z(b))| < \infty.$$

Note that a measure is maximizing for H Hölder if and only if it is maximizing for $\tilde{H}$ Hölder, where $-\log \tilde{H}$ is cohomologous to $-\log H$.

Given H, we would like to associate a T-invariant measure $\mu_{\tilde{H}}$ to the ground state ϕ of the automorphism associated to $\tilde{H}$, where $-\log \tilde{H} = -\log H + V - V \circ T$, and V was defined before.

Given a ground state ϕ for $\tilde{H}$, the action of ϕ over the continuous functions identifies a probability ν such that $\phi(f) = \int f \, d\nu$, for all continuous functions f.

We will show later that ν is a maximizing measure for $\tilde{H}$ (or for H).

Using the same procedure as §§ 7 and 8 for such ν, one can easily show that an equivalent characterization of the ground state ϕ for $\tilde{H}$ is as follows: for all $f, g \in C(X)$, all m and all complex $\beta = -iz$, such that $\mathrm{Re}(\beta) \geq 0$, we have

$$\begin{aligned}
&|\phi(M_g \sigma_\beta (S^m (S^*)^m M_f))| \\
&\quad \leq \int |g\alpha^m(\mathcal{L}_p^m(f H^{-\beta[m]} e^{(V[m]-V\circ T[m])\beta})) H^{\beta[m]} e^{(-V[m]+V\circ T[m])\beta}| \, d\nu \\
&\quad \leq \|f\|_\infty \|g\|_\infty < \infty.
\end{aligned}$$

We show that such ν exists, is invariant and is a maximizing measure for $\tilde{H}$ (or for H). For a generic H it will follow from Theorem 11.3 that ν has support in a unique periodic orbit.

For $z \in X$, and $n \in \mathbb{N}$, denote by $x_n^i(z)$, $i \in \{1, 2, \ldots, k^n\}$, the k^n solutions of $T^n(z) = x$.

Fix a point x from now on. We are going to define a sequence of points y_n inductively. We set $y_0 = x$, and for y_1, we choose a point over the set $\{z | T(z) = y_0\}$ such that $V(T(y_1)) - V(y_1) = -\log H(x) - m(-\log H)$. From the definition of V one can easily show that there is always such a point y_1. Inductively, given y_i, for y_{i+1}, we choose a point over the set $\{z | T(z) = y_i\}$ such that $V(T(y_{i+1})) - V(y_{i+1}) = -\log H(y_{i+1}) - m(-\log H)$. Note that $T(y_{i+1}) = y_i$, for all i. Consider $\mu_n = (1/n) \sum_{l=0}^{n-1} \delta_{y_l}$, and by compactness a measure ρ such that is a weak limit $\rho = \lim_{r\to\infty} \mu_{n_r}$. The measure ρ is invariant for T. Indeed, for any continuous function

$$\int F \circ T \, d\rho = \lim_{r\to\infty} \int f \circ T \mu_{n_r} = \lim_{r\to\infty} \int f \mu_{n_r} = \int f \, d\rho,$$

because f is bounded. This ρ is our candidate for being a ground measure for $\tilde{H} = H e^{-V+V\circ T}$.

We assume from now on that H is such that μ_H is unique (and so uniquely ergodic by Theorem 6 of [**CLT**]). This is not really necessary, but for the sake of simplicity we assume this.

PROPOSITION 11.4. *We have $\rho = \mu_H$.*

Proof. $V(x)$ is Hölder continuous on x, and therefore bounded, so

$$\begin{aligned}
-\int \log H \, d\rho &= -\lim_{r\to\infty} \int \log H \, d\mu_{n_r} = -\lim_{r\to\infty} \frac{1}{n_r} \sum_{j=0}^{n_r-1} (\log H(y_j)) \\
&= \lim_{r\to\infty} \frac{1}{n_r} \sum_{j=0}^{n_r-1} (V(y_{j-1}) - V(y_j) + m(-\log H)) \\
&= \lim_{r\to\infty} \frac{1}{n_r} (V(x) - V(y_{n_r-1}) + n_r m(-\log H)) = m(-\log H).
\end{aligned}$$

Therefore, $\rho = \mu_H$ and does not depend on x. □

We denote such ρ by ν_∞. This measure is invariant.

PROPOSITION 11.5. *For any* $\mathrm{Re}(\beta) \geq 0$, $m \in \mathbb{N}$ *and* $f, g \in C(X)$

$$\int |g\alpha^m \mathcal{L}_p^m (f H^{-\beta[m]} e^{(V[m]-V\circ T[m])\beta}) H^{\beta[m]} e^{(-V[m]+V(T))[m])\beta}| \, dv_\infty \leq \|g\|_\infty \|f\|_\infty.$$

Proof. The proof is similar to Lemma 10.6.

Consider $m > 0$ and $k > 1$ fixed. Consider the transformation T^m, which has degree k^m and the function $-\log \overline{H}(x) = -\log H(x) - \log(H(T(x)) - \cdots - \log(H(T^{m-1}(x))) = -\log H^{[m]}$.

From the fact that we consider just invariant probabilities ρ:

$$m(-\log \overline{H}) = m(-\log H(x) - \log(H(T(x)) - \cdots - \log(H(T^{m-1}(x)))) = mm(-\log H).$$

It is easy to see that the maximizing measure v_∞ for $-\log H$ is also the maximizing measure for $-\log \overline{H}$.

Note that the previously defined V is such that

$$-\log \overline{H}(x) - mm(-\log H) \leq V(T^m(x)) - V(x).$$

We are going to define a sequence of points x_n inductively. We set $x_0 = x$, and for x_1, we choose a point over the set $\{z \mid T^m(z) = x_0\}$ such that $V(T^m(x_1)) - V(x_1) = -\log \overline{H}(x_1) - mm(-\log H)$. There is always such a point from the definition of V (see [**CLT**, p. 1384]). Inductively, given x_i, for x_{i+1}, we choose one over the set $\{z \mid T^m(z) = x_i\}$ such that $V(T^m(x_{i+1})) - V(x_{i+1}) = -\log \overline{H}(x_{i+1}) - mm(-\log H)$. Note that $T^m(x_{i+1}) = x_i$, for all i.

Consider $\mu_n = (1/n) \sum_{l=0}^{n-1} \delta_{x_n}$, and by compactness a measure ρ that is a weak limit $\rho = \lim_{r\to\infty} \mu_{n_r}$. From

$$m(-\log H(x) - \log(H(T(x)) - \cdots - \log(T^{m-1}(x))) = mm(-\log H),$$

and in the same way as before one can show that $-\int \log \overline{H} \, d\rho = mm(-\log H)$, and finally that $\rho = v_\infty$.

The important relation that follows from the above ([**CLT**, Proposition 11, p. 1384]) is

$$-\log \overline{H}(x) + V(x) = -\log H(x)[m] + V(x) \leq V(T^m(x)) + mm(-\log H).$$

Note that for any fixed j and any $x_i^m(T^m(x_j))$ with $i \in \{1, 2, \ldots, k^m\}$, we have

$$\begin{aligned} &V(T^m(x_i^m(T^m(x_j)))) - V(x_i^m(T^m(x_j))) \\ &\quad = V(T(x_i^m(T^m(x_j))))[m] - V(x_i^m(T^m(x_j)))[m] \\ &\quad \geq (-\log H(x_i^m(T^m(x_j))))[m] - m(-\log H) \end{aligned}$$

and

$$V(T^m(x_j)) - V(x_j) = V(T(x_j))[m] - V(x_j)[m] = (-\log H(x_j))[m] - mm(-\log H),$$

therefore,

$$\begin{aligned} &-(\log H(x_j))[m] - mm(-\log H) + V(x_j) \\ &\quad = V(T^m(x_j)) = V(T^m(x_i^m(T^m(x_j)))) \\ &\quad \geq -(\log H(x_i^m(T^m(x_j))))[m] - mm(-\log H) + V(x_i^m(T^m(x_j))). \end{aligned}$$

Therefore, from the way we choose x_j we have

$$\left| \frac{H(x_j)^{\beta[m]} e^{-V(x_j)\beta}}{H(x_i^m(T^m(x_j)))^{\beta[m]} e^{-V(x_i^m(T^m(x_j)))\beta}} \right| \leq 1.$$

Now, by the triangle inequality

$$\begin{aligned}
&\int |g\alpha^m \mathcal{L}_p^m(fH^{-\beta[m]} e^{(V-V\circ T)[m]\beta}) H^{\beta[m]} e^{(-V+V\circ T)[m]\beta}| \, dv_\infty \\
&\quad = \lim_{r\to\infty} \int |g\alpha^m \mathcal{L}_p^m(fH^{-\beta[m]} e^{(V-V\circ T)[m]\beta}) H^{\beta[m]} e^{(-V+V\circ T)[m]\beta}| \, d\mu_{n_r} \\
&\quad = \lim_{r\to\infty} \left| \frac{1}{n_r} \sum_{j=0}^{n_r-1} g(x_j) \frac{1}{k^m} \sum_{i=1}^{k^m} (f(x_i^m(T^m(x_j)) H(x_i^m(T^m(x_j)))^{-\beta[m]} \right. \\
&\qquad \left. \times e^{(V-(V\circ T))(x_i^m(T^m(x_j)))[m]\beta}) H(x_j)^{\beta[m]} e^{(-V(x_j)+(V\circ T)(x_j))[m]\beta} \right| \\
&\quad \leq \|f\|_\infty \|g\|_\infty < \infty.
\end{aligned}$$

The conclusion is that the minimizing measure $\mu_H = \nu_\infty$ determines the ground state ϕ_{ν_∞}. □

It follows from this proposition that we have the next theorem.

THEOREM 11.6. *Given $H > 0$ Hölder, there is $V > 0$ Hölder, such that if ν_∞ is the maximizing measure for $-\log H$, then the state ϕ defined by*

$$\phi(M_f S^m (S^*)^m) = \int \frac{f}{\Lambda^{[m]}} \, d\nu_\infty,$$

for all $m \in \mathbb{N}$, $f \in C(X)$, is a ground-state for the potential $\tilde{H} = He^{-V+V\circ T}$.

The conclusion is that, if one considers $p = 1/k$ and $H > 0$ Hölder, then the state ϕ_{ν_∞} associated to a minimizing measure ν_∞ for H is a ground-state for some $\tilde{H}$ (such that $\log \tilde{H}$ is cohomologous to $\log H$).

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