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Some L^p estimates for rotationally invariant systems of viscous conservation laws

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Abstract We study the Cauchy problem for a system of one-dimensional viscous conservation laws with rotational invariance, $\mathbf{u}_t + \left[|\mathbf{u}|^2 \mathbf{u} \right]_x = \mathbf{u}_{xx}$,with the aim of deriving some a priori estimates for various L^p norms of its solutions through a direct method.

Keywords Rotationally invariant systems · L^p estimates · Viscous conservation laws

Mathematics Subject Classification (2010) 35B40 (primary) · 35B45 · 35K15

1 Introduction

In this work, we will derive some estimates for bounded solutions $\mathbf{u}(\cdot, t)$ of the Cauchy problem for the system of advection–diffusion equations

$$\mathbf{u}_t(x,t) + \left[|\mathbf{u}(x,t)|^2 \mathbf{u}(x,t) \right]_x = \mathbf{u}_{xx}(x,t), \qquad \mathbf{u}(\cdot, 0) = \mathbf{u}_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{1}$$

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where $x \in \mathbb{R}$, $t \in [0, T]$, $1 \leq p_0 < \infty$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. The initial condition is satisfied in the sense of $L^1_{\mathrm{loc}}(\mathbb{R})$, i.e., $\lim_{t\to 0} \|\mathbf{u}(\cdot,t) - \mathbf{u}_0\|_{L^1(\mathbb{K})} = 0$ for each compact set $\mathbb{K} \subset \mathbb{R}$. Standard results (see e.g., Serre (1999), Ch. 6) assure the existence of a smooth classical solution $\mathbf{u}(\cdot,t)$ defined for $0 < t \leq T$, for some $T > 0$, and satisfying

$$|\mathbf{u}(x,t)|^2 \leq B(T), \qquad \forall\, x \in \mathbb{R},\ 0 < t \leq T. \tag{2}$$

It follows from Theorem 2.1 in Braz e Silva et al. (submitted) and Theorem 3 in Sect. 2 below that this solution $\mathbf{u}(\cdot,t)$ is uniquely defined for $t \in [0, T]$ and we actually have $\mathbf{u}(\cdot,t) \in C^0([0,T], L^{p_0}(\mathbb{R}))$, so that, in particular,

$$\| \mathbf{u}(\cdot,t) - \mathbf{u}_0 \|_{L^{p_0}(\mathbb{R})} \to 0 \quad \text{as } t \to 0.$$

We will also prove in Sect. 2 that, for each $p_0 \leq p < \infty$, we have

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \quad \forall\, t \in [0,T],$$

where $K(t) = K(t, T, p) = \exp\left\{\frac{B(T)^2(p-1)}{2}\, t\right\}$, and

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq C_\gamma t^{-\frac{3}{4p}}, \quad \forall\, t \in (0,T],$$

where $C_\gamma > 0$ is a constant depending only on the parameters

$$\gamma = \left\{n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\right\}.$$

Finally, in Sect. 3 we will derive the supnorm estimate

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa t^{-\frac{1}{2p}}, \quad \forall\, t \in (0,T],$$

where C_κ is a positive constant depending on the parameters given in the list $\kappa = \{n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\}$. Similar results can be obtained using our approach for solutions $\mathbf{u}(\cdot,t)$ of the more general equations given in (3) below, but for simplicity we will restrict our attention to solutions of (1) only.

The estimates above are derived through a direct method, extending the treatment in Schütz (2008), Zingano (1999) to rotationally invariant systems. Let us briefly review the main results found in these references.

In Zingano (1999), the system considered is

$$\mathbf{u}_t(x,t) + [\varphi(|\mathbf{u}(x,t)|)\mathbf{u}(x,t)]_x = (B(\mathbf{u})\mathbf{u}_x(x,t))_x, \tag{3}$$

where φ is a smooth real function, $x \in \mathbb{R}$, $t > 0$, and $B(\mathbf{u}(x,t)) = (b_i(\mathbf{u}(x,t)))_{n\times n}$ is an $n \times n$ diagonal matrix with real smooth entries such that $b_i(\mathbf{u}(x,t)) \geq \mu$, $\forall\, i = 1, 2, ..., n$, where μ is a positive constant. The author establishes the inequality

$$\|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R})} \leq K_\Lambda(1+t)^{-\frac{1}{4}}, \quad \forall\, t > 0,$$

where $\mathbf{u}_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and K_Λ is a constant depending only on the parameters

$$\Lambda = \{\|\mathbf{u}(\cdot,0)\|_{L^1(\mathbb{R})}, \|\mathbf{u}(\cdot,0)\|_{L^2(\mathbb{R})}, n, \mu\}.$$

In Schütz (2008), the author considers the multidimensional scalar equation

$$u_t(\mathbf{x},t) + \mathrm{div}[\mathbf{b}(\mathbf{x},t,u(\mathbf{x},t))u(\mathbf{x},t)] = \mathrm{div}[A(\mathbf{x},t,u(\mathbf{x},t))\nabla u(\mathbf{x},t)] + c(\mathbf{x},t,u(\mathbf{x},t))u(\mathbf{x},t),$$

where $\mathbf{x} \in \mathbb{R}^m$, $t \in (0, T]$ e $\mathbf{b}$, c are smooth functions such that

$$|\mathbf{b}(\mathbf{x}, t, u)| \leq B(T), \quad |c(\mathbf{x}, t, u)| \leq C(T),$$

and $A(\mathbf{x}, t, u) = (a_{ij}(\mathbf{x}, t, u))_{m\times m}$ is an $m \times m$ real positive definite matrix with smooth entries a_{ij} such that

$$\sum_{i,j=1}^{m} a_{ij}(\mathbf{x}, t, u)\xi_i\xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2,$$

where μ, $B(T)$, $C(T)$ are positive constants and $\xi_i \in \mathbb{R}$, $\forall\, i = 1, 2, ..., m$. The main results obtained therein for a solution $u(\mathbf{x}, t)$ are the bounds

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^m)} \leq K_p \|u_0\|_{L^p(\mathbb{R}^m)}, \quad \forall\, t \in [0, T],$$

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^m)} \leq K_\infty \|u_0\|_{L^p(\mathbb{R}^m)} t^{-\frac{m}{2p}}, \quad \forall\, t \in (0, T],$$

where $K_p = K_p(p, T, \mu)$ and $K_\infty = K_\infty(m, p, T, \mu)$.

There is a vast literature on systems of advection–diffusion equations in various settings using different techniques from those used here, see, for example, Braz e Silva and Zingano (2006), Brio and Hunter (1990), Carlen and Loss (1996), Chern (1991), Chern and Liu (1987), Duro and Carpio (2001), Duro and Zuazua (1999), Escobedo et al. (1993), Escobedo and Zuazua (1991), Freistühler (1990), Freistühler and Serre (2001), Harabetian (1988), Ilin et al. (1962), Kawashima (1987), Keyfitz and Kranzer (1979/1980), Schonbek (1980), Schonbek (1986). Our aim here is to use a direct method to derive bounds for the solutions easily, without the need to use very technical arguments.

Throughout this work we use the notation

$$\|\mathbf{u}(\cdot, t)\|^p_{L^p(\mathbb{R})} := \sum_{i=1}^{n} \|u_i(\cdot, t)\|^p_{L^p(\mathbb{R})} \tag{4}$$

and

$$\|\mathbf{u}(\cdot, t)\|_{L^\infty(\mathbb{R})} := \max_{i=1,\dots,n} \left\{\|u_i(\cdot, t)\|_{L^\infty(\mathbb{R})}\right\}. \tag{5}$$

2 L^p estimates

Let $p \in [\,p_0, \infty)$ be given (fixed but arbitrary). We begin by showing that there is a constant $C_\gamma > 0$, depending on $\gamma = \gamma\left(n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\right)$ such that

$$\|\mathbf{u}(\cdot, t)\|_{L^{2p}(\mathbb{R})} \leq C_\gamma t^{-\frac{3}{4p}}, \quad \forall\, t \in (0, T]. \tag{6}$$

To prove inequality (6), we first show that a solution of system (1) is bounded in L^p. More precisely,

Theorem 1 (L^p estimate I) *Given $T > 0$, let $\mathbf{u}(\cdot, t) \in C^0([0, T], L^p(\mathbb{R}))$ be a solution for system (1) satisfying the hypothesis (2). Then*

$$\|\mathbf{u}(\cdot, t)\|_{L^p(\mathbb{R})} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \quad \forall\, t \in [0, T], \tag{7}$$

where $K(t) = K(t, T, p) = \exp\left\{\frac{B(T)^2(p-1)}{2}\, t\right\}$.

Proof We begin by defining what we call a regularized sign function (see also Kreiss and Lorenz (1989), Lemma 4.3.1): Let $\mathrm{Sgn} \in C^\infty(\mathbb{R})$ be a fixed function such that

$$\begin{cases} \mathrm{Sgn}(y) = -1, & \text{for } y \leq -1, \\ \mathrm{Sgn}(y) = 1, & \text{for } y \geq 1, \\ \mathrm{Sgn}(0) = 0, \end{cases}$$

and

$$\mathrm{Sgn}'(y) \geq 0, \quad \text{for all } y \in \mathbb{R}.$$

For each $\delta > 0$, define $\mathrm{Sgn}_\delta \in C^\infty(\mathbb{R})$ by

$$\mathrm{Sgn}_\delta(y) = \mathrm{Sgn}(y/\delta), \quad \text{for all } y \in \mathbb{R}.$$

The function L_δ given by

$$L_\delta(y) = \int_0^y \mathrm{Sgn}_\delta(x)\mathrm{d}x, \quad \text{for all } y \in \mathbb{R}, \tag{8}$$

is called a regularized sign function. Note that

$$\lim_{\delta \to 0} L_\delta(y) = |y|, \quad \text{uniformly on } \mathbb{R},$$

$$\lim_{\delta \to 0} L'_\delta(y) = \mathrm{sgn}(y), \quad \text{for each } y \in \mathbb{R},$$

$$L''_\delta(y) \geq 0, \quad \text{and} \quad |yL''_\delta(y)| \leq C, \text{ for all } \delta > 0,\ y \in \mathbb{R},$$

where C is a positive constant and sgn is the sign function.

We now proceed to prove the theorem. Consider $\Phi_\delta(\cdot) = L_\delta(\cdot)^p$ and $T > 0$. We show that the desired estimate is valid for each component of a solution for (1), (2), i.e.,

$$\|u_i(\cdot, t)\|_{L^p(\mathbb{R})} \leq K(t)\|u_{i0}\|_{L^p(\mathbb{R})}, \quad \forall\, t \in [0, T],$$

for each $i = 1, \ldots n$. To this end, let f be a $C_0^\infty(\mathbb{R})$ function such that

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \geq 2, \end{cases}$$

with $0 < f(x) < 1$ for $1 < |x| < 2$. For a given $R > 0$, define the function $f_R \in C_0^\infty(\mathbb{R})$ by $f_R(x) = f(x/R)$, for all $x \in \mathbb{R}$. For each $i = 1, \ldots, n$, multiply the equation

$$u_{it} + \left[|\mathbf{u}|^2 u_i\right]_x = u_{ixx}$$

by $f_R(x)\Phi'_\delta(u_i)$ and integrate over $\mathbb{R} \times [t_0, t]$, where $t_0 \in (0, t)$ and $t \in (0, T]$. One gets

$$\int_{t_0}^{t}\int_{\mathbb{R}} f_R(x)\Phi'_\delta(u_i)u_{i\tau}\,\mathrm{d}x\mathrm{d}\tau + \int_{t_0}^{t}\int_{\mathbb{R}} f_R(x)\Phi'_\delta(u_i)\left[|\mathbf{u}|^2 u_i\right]_x\mathrm{d}x\mathrm{d}\tau$$
$$= \int_{t_0}^{t}\int_{\mathbb{R}} f_R(x)\Phi'_\delta(u_i)u_{ixx}\mathrm{d}x\mathrm{d}\tau. \tag{9}$$

Let us examine each term in the above equation. The first term in the left hand side of equation (9) can be written, using the Fundamental Theorem of Calculus, as

$$\int_{t_0}^{t}\int_{-2R}^{2R} f(x/R)\Phi_\delta'(u_i)u_{i\tau}\mathrm{d}x\mathrm{d}\tau = \int_{-2R}^{2R} f(x/R)\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}[\Phi_\delta(u_i)]\mathrm{d}\tau\mathrm{d}x$$
$$= \int_{-2R}^{2R} f(x/R)\Phi_\delta(u_i(\cdot,t))\mathrm{d}x - \int_{-2R}^{2R} f(x/R)\Phi_\delta(u_i(\cdot,t_0))\mathrm{d}x.$$

Taking the limit as $R\to\infty$, one has

$$\int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta'(u_i)u_{i\tau}\mathrm{d}\tau = \int_{\mathbb{R}}\Phi_\delta(u_i(\cdot,t))\mathrm{d}x - \int_{\mathbb{R}}\Phi_\delta(u_i(\cdot,t_0))\mathrm{d}x,$$

since $f(0)=1$. For the second term on the left hand side of Eq. (9), one gets through integration by parts that

$$\int_{-2R}^{2R} f(x/R)\Phi_\delta'(u_i)\left[|\mathbf{u}|^2u_i\right]_x\mathrm{d}x = \left[f(x/R)\Phi_\delta'(u_i)|\mathbf{u}|^2u_i\right]\Big|_{-2R}^{2R}$$
$$-\int_{-2R}^{2R} f(x/R)\Phi_\delta''(u_i)|\mathbf{u}|^2u_iu_{ix}\mathrm{d}x - \frac{1}{R}\int_{-2R}^{2R} f'(x/R)\Phi_\delta'(u_i)|\mathbf{u}|^2u_iu_{ix}\mathrm{d}x$$
$$= -\int_{-2R}^{2R} f(x/R)\Phi_\delta''(u_i)|\mathbf{u}|^2u_iu_{ix}\mathrm{d}x - \frac{1}{R}\int_{-2R}^{2R} f'(x/R)\Phi_\delta'(u_i)|\mathbf{u}|^2u_iu_{ix}\mathrm{d}x,$$

since $f(-2)=f(2)=0$. Passing to the limit as $R\to\infty$, one finds

$$\int_{\mathbb{R}}\Phi_\delta'(u_i)\left[|\mathbf{u}|^2u_i\right]_x\mathrm{d}x = -\int_{\mathbb{R}}\Phi_\delta''(u_i)|\mathbf{u}|^2u_iu_{ix}\mathrm{d}x.$$

Then, using that $\Phi_\delta''(\cdot)\geq 0$ and condition (2), one has

$$\int_{\mathbb{R}}\Phi_\delta'(u_i)\left[|\mathbf{u}|^2u_i\right]_x\mathrm{d}x \geq -\int_{\mathbb{R}}\Phi_\delta''(u_i)|\mathbf{u}|^2|u_i||u_{ix}|\mathrm{d}x$$
$$\geq -\frac{1}{2}\int_{\mathbb{R}}\Phi_\delta''(u_i)|\mathbf{u}|^4u_i^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\Phi_\delta''(u_i)u_{ix}^2\mathrm{d}x$$
$$\geq -\frac{B(T)^2}{2}\int_{\mathbb{R}}\Phi_\delta''(u_i)u_i^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\Phi_\delta''(\mathrm{u_i})\mathrm{u_{ix}^2}\mathrm{d}x. \qquad (10)$$

Lastly, for the right hand side of (9), integrate by parts to write

$$\int_{-2R}^{2R} f(x/R)\Phi_\delta'(u_i)u_{ixx}\mathrm{d}x = -\int_{-2R}^{2R} f(x/R)\Phi_\delta''(u_i)u_{ix}^2\mathrm{d}x - \frac{1}{R}\int_{-2R}^{2R} f'(x/R)\Phi_\delta'(u_i)u_{ix}\mathrm{d}x.$$

Taking the limit as $R \to \infty$, one gets

$$\int_{\mathbb{R}} \Phi'_\delta(u_i) u_{ixx} \mathrm{d}x = -\int_{\mathbb{R}} \Phi''_\delta(u_i) u_{ix}^2 \mathrm{d}x. \tag{11}$$

Therefore, from identity (9), one obtains

$$\begin{aligned}\int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t))\mathrm{d}x &\leq \int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t_0))\mathrm{d}x + \frac{B(T)^2}{2}\int_{t_0}^{t}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2 \mathrm{d}x\mathrm{d}\tau \\ &\quad - \frac{1}{2}\int_{t_0}^{t}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2\mathrm{d}x\mathrm{d}\tau \\ &\leq \int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t_0))\mathrm{d}x + \frac{B(T)^2}{2}\int_{t_0}^{t}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2 \mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Passing to the limit as $\delta \to 0$ and using Lebesgue's Dominated Convergence Theorem, one obtains

$$\int_{\mathbb{R}} |u_i(\cdot,t)|^p \mathrm{d}x \leq \int_{\mathbb{R}} |u_i(\cdot,t_0)|^p\mathrm{d}x + \frac{B(T)^2 p(p-1)}{2}\int_{t_0}^{t}\int_{\mathbb{R}} |u_i|^p \mathrm{d}x\mathrm{d}\tau,$$

i.e.,

$$\|u_i(\cdot,t)\|^p_{L^p(\mathbb{R})} \leq \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R})} + \frac{B(T)^2 p(p-1)}{2}\int_{t_0}^{t} \|u_i\|^p_{L^p(\mathbb{R})}\mathrm{d}\tau.$$

By Gronwall's Lemma, it follows that

$$\|u_i(\cdot,t)\|^p_{L^p(\mathbb{R})} \leq \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R})} \exp\left\{\frac{B(T)^2 p(p-1)}{2}(t-t_0)\right\}, \text{ for all } t \in (0,T].$$

Thus, taking the pth root and passing to the limit as $t_0 \to 0$,

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R})} \leq \|u_{i0}\|_{L^p(\mathbb{R})} \exp\left\{\frac{B(T)^2(p-1)}{2}t\right\}, \text{ for all } t \in (0,T]. \tag{12}$$

Consequently, summing on i from 1 to n [see (4)], one gets the desired bound

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \text{ for all } t \in [0,T],$$

where $\mathbf{u}_0 = (u_{10}, u_{20}, ..., u_{i0}, ..., u_{n0})$ and $K(t) = K(t,T,p) = \exp\left\{\frac{B(T)^2(p-1)}{2}t\right\}$. □

The following theorem gives the desired estimate (6).

Theorem 2 (L^p estimate II) *Given $T > 0$, let $\mathbf{u}(\cdot,t) \in C^0([0,T], L^p(\mathbb{R}))$ be a solution of system (1) under the hypothesis (2). Then*

$$\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} \leq C^3\|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R})} F(t)^{\frac{3}{2}}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} t^{-\frac{3}{2}}, \quad \forall\, t \in (0,T],$$

where $C = C(n)$ is a positive constant and

$$F(t) = F(t,T,p) = \frac{3}{2}\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2}{2}\left(\int_0^t \tau^{\frac{3}{2}} K(\tau)^{2p}\mathrm{d}x\right)^{\frac{2}{3}}.$$

Proof Consider $\Phi_\delta(\cdot) = L_\delta(\cdot)^{2p}$. For each $i = 1,\ldots,n$, multiply the equation

$$u_{it} + \left[|\mathbf{u}|^2 u_i\right]_x = u_{ixx}$$

by $(t-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)$ and integrate over $\mathbb{R}\times[t_0,t]$, where $t_0\in(0,t)$ and $t\in(0,T]$, to get

$$\int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)u_{i\tau}\,\mathrm{d}x\mathrm{d}\tau + \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)\left[|\mathbf{u}|^2u_i\right]_x\,\mathrm{d}x\mathrm{d}\tau$$
$$= \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)u_{ixx}\,\mathrm{d}x\mathrm{d}\tau. \tag{13}$$

Note that

$$\int_{t_0}^t(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)u_{i\tau}\mathrm{d}\tau = \int_{t_0}^t(\tau-t_0)^{\frac{3}{2}}\frac{d}{d\tau}[\Phi_\delta(u_i)]\mathrm{d}\tau$$
$$= (\tau-t_0)^{\frac{3}{2}}\Phi_\delta(u_i(\cdot,\tau))|_{t_0}^t - \frac{3}{2}\int_{t_0}^t(\tau-t_0)^{\frac{1}{2}}\Phi_\delta(u_i)\mathrm{d}\tau$$
$$= (t-t_0)^{\frac{3}{2}}\Phi_\delta(u_i(\cdot,t)) - \frac{3}{2}\int_{t_0}^t(\tau-t_0)^{\frac{1}{2}}\Phi_\delta(u_i)\mathrm{d}\tau. \tag{14}$$

Recalling that [see (10) and (11)]

$$\int_{\mathbb{R}}\Phi'_\delta(u_i)\left[|\mathbf{u}|^2u_i\right]_x\mathrm{d}x \geq -\frac{B(T)^2}{2}\int_{\mathbb{R}}\Phi''_\delta(u_i)u_i^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\Phi''_\delta(u_i)u_{ix}^2\mathrm{d}x, \tag{15}$$

and

$$\int_{\mathbb{R}}\Phi'_\delta(u_i)u_{ixx}\mathrm{d}x = -\int_{\mathbb{R}}\Phi''_\delta(u_i)u_{ix}^2\mathrm{d}x, \tag{16}$$

we get, using (14), (15), (16) in (13) and passing to the limit when $\delta\to 0$, that

$$(t-t_0)^{\frac{3}{2}}\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2}\int_{t_0}^t(\tau-t_0)^{\frac{3}{2}}\int_{\mathbb{R}}|u_i|^{2p-2}u_{ix}^2\mathrm{d}x\mathrm{d}\tau$$
$$\leq \frac{3}{2}\int_{t_0}^t(\tau-t_0)^{\frac{1}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p}\mathrm{d}\tau + \frac{2p(2p-1)B(T)^2}{2}\int_{t_0}^t(\tau-t_0)^{\frac{3}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p}\mathrm{d}\tau.$$

Therefore, letting $t_0 \to 0$, we obtain

$$t^{\frac{3}{2}}\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2}\int_0^t \tau^{\frac{3}{2}}\int_{\mathbb{R}} |u_i|^{2p-2}u_{ix}^2\,\mathrm{d}x\mathrm{d}\tau$$
$$\le \frac{3}{2}\int_0^t \tau^{\frac{1}{2}}\|u_i\|^{2p}_{L^{2p}(\mathbb{R})}\mathrm{d}\tau + \frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}}\|u_i\|^{2p}_{L^{2p}(\mathbb{R})}\mathrm{d}\tau.$$

Now, define the function

$$w_i(x,t) = \begin{cases} u_i(x,t), & \text{if } p = 1,\\ |u_i(x,t)|^p, & \text{if } p > 1,\end{cases}$$

where $x \in \mathbb{R}$ and $t \in [0,T]$. In particular, $w_{ix}^2 = p^2|u_i|^{2p-2}u_{ix}^2$ and $\|w_i\|^2_{L^2(\mathbb{R})} = \|u_i\|^{2p}_{L^{2p}(\mathbb{R})}$. Define

$$X(t) := t^{\frac{3}{2}}\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2}\int_0^t \tau^{\frac{3}{2}}\int_{\mathbb{R}} |u_i|^{2p-2}u_{ix}^2\,\mathrm{d}x\mathrm{d}\tau$$
$$= t^{\frac{3}{2}}\|w_i(\cdot,t)\|^{2}_{L^{2}(\mathbb{R})} + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^{\frac{3}{2}}\int_{\mathbb{R}} w_{ix}^2\,\mathrm{d}x\mathrm{d}\tau$$
$$= t^{\frac{3}{2}}\|w_i(\cdot,t)\|^{2}_{L^{2}(\mathbb{R})} + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^{\frac{3}{2}}\|w_{ix}\|^2_{L^2(\mathbb{R})}\mathrm{d}\tau. \tag{17}$$

[Note that $X(t) = X(t,p)$]. Therefore,

$$X(t) \le \frac{3}{2}\int_0^t \tau^{\frac{1}{2}}\|u_i\|^{2p}_{L^{2p}(\mathbb{R})}\mathrm{d}\tau + \frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}}\|u_i\|^{2p}_{L^{2p}(\mathbb{R})}\mathrm{d}\tau$$
$$= \frac{3}{2}\int_0^t \tau^{\frac{1}{2}}\|w_i\|^{2}_{L^{2}(\mathbb{R})}\mathrm{d}\tau + \frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}}\|w_i\|^{2}_{L^{2}(\mathbb{R})}\mathrm{d}\tau.$$

Using the Sobolev inequality

$$\|v\|_{L^2(\mathbb{R})} \le C\|v\|^{\frac{2}{3}}_{L^1(\mathbb{R})}\|v_x\|^{\frac{1}{3}}_{L^2(\mathbb{R})}, \quad \forall\, v \in C_0^1(\mathbb{R}), \tag{18}$$

one obtains

$$X(t) \le \frac{3}{2}\int_0^t \tau^{\frac{1}{2}}C^2\|w_i\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}\mathrm{d}\tau$$
$$+\frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}}C^2\|w_i\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}\mathrm{d}\tau$$

$$\leq \frac{3}{2}C^2 \int_0^t \tau^{\frac{1}{2}} \|u_i\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} \mathrm{d}\tau$$

$$+\frac{2p(2p-1)B(T)^2}{2} C^2 \int_0^t \tau^{\frac{3}{2}} \|u_i\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} \mathrm{d}\tau$$

$$\leq \frac{3}{2}C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \int_0^t \tau^{\frac{1}{2}} K(\tau)^{\frac{4p}{3}} \|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} \mathrm{d}\tau$$

$$+\frac{2p(2p-1)B(T)^2}{2} C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \int_0^t \tau^{\frac{3}{2}} K(\tau)^{\frac{4p}{3}} \|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} \mathrm{d}\tau$$

$$\leq \frac{3}{2}C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \left(\int_0^t K(\tau)^{2p} \mathrm{d}\tau \right)^{\frac{2}{3}} \left(\int_0^t \tau^{\frac{3}{2}} \|w_{ix}\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau \right)^{\frac{1}{3}}$$

$$+\frac{2p(2p-1)B(T)^2}{2} C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \left(\int_0^t \tau^{\frac{3}{2}} K(\tau)^{2p} \mathrm{d}\tau \right)^{\frac{2}{3}} \left(\int_0^t \tau^{\frac{3}{2}} \|w_{ix}\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau \right)^{\frac{1}{3}}$$

$$\leq C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} F(t) \left(\int_0^t \tau^{\frac{3}{2}} \|w_{ix}\|_{L^2(\mathbb{R})}^2 \mathrm{d}\tau \right)^{\frac{1}{3}}, \quad \forall\, t \in [0, T],$$

where $F(t) = F(t, T, p) = \left[\frac{3}{2} \left(\int_0^t K(\tau)^{2p} \mathrm{d}\tau \right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2}{2} \left(\int_0^t \tau^{\frac{3}{2}} K(\tau)^{2p} \mathrm{d}\tau \right)^{\frac{2}{3}} \right]$.

Therefore, using the definition (17) of $X(t)$, one gets

$$X(t) \leq C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} F(t) \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{3}} X(t)^{\frac{1}{3}}.$$

Thus,

$$X(t)^{\frac{2}{3}} \leq C^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} F(t) \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{3}},$$

that is,

$$X(t) \leq C^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}}.$$

Using definition (17), one concludes that

$$t^{\frac{3}{2}} \|u_i(\cdot, t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2} \int_0^t \tau^{\frac{3}{2}} \int_{\mathbb{R}} |u_i|^{2p-2} u_{ix}^2 \mathrm{d}x \mathrm{d}\tau$$

$$\leq C^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}},$$

for all $t \in [0, T]$. Therefore,

$$\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \le C^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} t^{-\frac{3}{2}}$$
$$\le C^3 \|\mathbf{u}_0\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} t^{-\frac{3}{2}}, \quad \forall\, t \in (0, T]. \qquad (19)$$

Finally, summing $i = 1, \ldots, n$, [see (4)], one gets the desired result

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \le C^3 \|\mathbf{u}_0\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} t^{-\frac{3}{2}}, \quad \forall\, t \in (0, T],$$

where $C = C(n)$ is a positive constant. □

We close this section by observing that bounded solutions of (1) are uniquely defined by their initial data, as shown by the following result.

Theorem 3 (well posedness) *Given $\mathbf{u}_0, \mathbf{v}_0 \in L^\infty(\mathbb{R})$, let $\mathbf{u}(\cdot,t)$, $\mathbf{v}(\cdot,t)$ be any pair of (bounded) solutions of (1) with initial states $\mathbf{u}(\cdot,0) = \mathbf{u}_0$, $\mathbf{v}(\cdot,0) = \mathbf{v}_0$, respectively, both defined for $0 \le t \le T$ and such that $\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le B(T)$ and $\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le B(T)$ for all $t \in [0, T]$. Then, for every $p \ge 2$, we have*

$$\|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot,t)\|_{L^p(\mathbb{R})} \le \|\mathbf{u}_0 - \mathbf{v}_0\|_{L^p(\mathbb{R})} \exp\left\{(2n+1)(p-1)B(T)^2\, t\right\}$$

for all $t \in [0, T]$.

This result can be shown in the same way as Theorem 1 above.

3 Estimates for the sup norm

In this section we will derive bounds for the supnorm of solutions of (1), (2). Again, let $p \in [p_0, \infty)$ be given. We will prove that

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C t^{-\frac{1}{2p}}, \quad \forall\, t \in (0, T],$$

where C is a positive constant depending on the parameters $C = C(n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})})$.

Theorem 4 (Estimate for the Supnorm) *Let $T > 0$ be given and $\mathbf{u}(\cdot,t) \in C^0([0,T], L^p(\mathbb{R}))$ be the solution of system (1), under the hypothesis in (2).*

Then

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C_n^{\frac{3}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R})} 2^{\frac{8}{p}} 3^{-\frac{1}{2p}} p^{\frac{3}{p}} C_T^{\frac{3}{2p}} \exp\left\{\frac{B(T)^2}{4} 3pt\right\} t^{-\frac{1}{2p}}$$

for all $t \in (0, T]$, where C_n is a positive constant depending on n only and $C_T = \left[\frac{1}{2} + \frac{B(T)^2 T}{2}\right]$.

Proof Firstly, note that

$$\begin{aligned} F(t) &= \frac{3}{2}\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2}{2}\left(\int_0^t \tau^{\frac{3}{2}}K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} \\ &\leq \frac{3}{2}\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2T}{2}\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} \\ &\leq \left[\frac{3}{2} + \frac{2p(2p-1)B(T)^2T}{2}\right]\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} \\ &\leq (2p)^2\left[\frac{1}{2} + \frac{B(T)^2T}{2}\right]\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} \\ &\leq (2p)^2 C_T\left(\int_0^t K(\tau)^{2p}\mathrm{d}\tau\right)^{\frac{2}{3}} \\ &\leq (2p)^2 C_T K(t)^{2p\frac{2}{3}}t^{\frac{2}{3}}, \end{aligned}$$

for all $t \in (0, T]$, where $C_T = \left[\frac{1}{2} + \frac{B(T)^2T}{2}\right]$. Therefore,

$$F(t)^{\frac{3}{2}} \leq (2p)^3 C_T^{\frac{3}{2}} K(t)^{2p} t, \quad \forall\, t \in (0, T].$$

By computations analogous to the ones used to derive inequality (19), one can establish

$$\begin{aligned} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} &\leq C^3\|u_i(\cdot,s)\|_{L^p(\mathbb{R})}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} \\ &\times\left[(2p)^3 C_T^{\frac{3}{2}}\exp\left\{\frac{B(T)^2}{2}2p(p-1)(t-s)\right\}\right]\cdot(t-s)^{-\frac{1}{2}}, \quad \text{where } 0 \leq s < t \leq T. \end{aligned}$$

Now, let $k \in \mathbb{N}$ be arbitrary. Define recursively $t_0 := \dfrac{t}{4^k}$, and $t_j := t_{j-1} + \dfrac{3t}{4^j}$, for $j \in \mathbb{N}$ and $1 \leq j \leq k$. One has $t_k = t$, and

$$\begin{aligned} \|u_i(\cdot,t_j)\|_{L^{2^j p}(\mathbb{R})}^{2^j p} &\leq C^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^j p}\left(\frac{2^j p(2^j p-1)}{2^{2j-1}p^2}\right)^{-\frac{1}{2}}(2^j p)^3 C_T^{\frac{3}{2}} \\ &\cdot\exp\left\{\frac{B(T)^2}{2}2^j p(2^{j-1}p-1)\left(\frac{3t}{4^j}\right)\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le C^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}\left(\frac{2(2^jp-1)}{2^jp}\right)^{-\frac{1}{2}}(2^jp)^3C_T^{\frac{3}{2}}\\
&\quad\cdot\exp\left\{\frac{B(T)^2}{2}p(2^{j-1}p-1)\left(\frac{3t}{2^j}\right)\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}}\\
&\le C^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}2^{-\frac{1}{2}}\left(\frac{2^jp-1}{2^jp}\right)^{-\frac{1}{2}}(2^jp)^3C_T^{\frac{3}{2}}\\
&\quad\cdot\exp\left\{\frac{B(T)^2}{2}p(2^{j-1}p)\left(\frac{3t}{2^j}\right)\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}}\\
&\le C^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}(2^jp)^3C_T^{\frac{3}{2}}\exp\left\{\frac{B(T)^2}{4}p^23t\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}}.
\end{aligned}
$$

Taking $j=k,k-1,\ldots,1$ successively, one then obtains

$$
\begin{aligned}
&\|u_i(\cdot,t)\|_{L^{2^kp}(\mathbb{R})}\\
&\le C^{\frac{3}{2^kp}}\|u_i(\cdot,t_{k-1})\|_{L^{2^{k-1}p}(\mathbb{R})}(2^kp)^{\frac{3}{2^kp}}C_T^{\frac{3}{2^{k+1}p}}\exp\left\{\frac{B(T)^2}{4}p3t\frac{1}{2^k}\right\}\left(\frac{3t}{4^k}\right)^{-\frac{1}{2^{k+1}p}}\\
&\le\prod_{j=1}^kC^{\frac{3}{2^jp}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}\prod_{j=1}^k(2^jp)^{\frac{3}{2^jp}}\prod_{j=1}^kC_T^{\frac{3}{2^{j+1}p}}\prod_{j=1}^k\exp\left\{\frac{B(T)^2}{4}p3t\frac{1}{2^j}\right\}\\
&\quad\cdot\prod_{j=1}^k\left(\frac{3t}{4^j}\right)^{-\frac{1}{2^{j+1}p}}\\
&\le C^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}\prod_{j=1}^k(2^jp)^{\frac{3}{2^jp}}C_T^{\frac{3}{2p}\sum_{j=1}^k\frac{1}{2^j}}\exp\left\{\frac{B(T)^2}{4}p3t\sum_{j=1}^k\frac{1}{2^j}\right\}\\
&\quad\cdot\prod_{j=1}^k\left(\frac{3t}{4^j}\right)^{-\frac{1}{2^{j+1}p}}\\
&\le C^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}2^{\frac{3}{p}\sum_{j=1}^k\frac{j}{2^j}}p^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}C_T^{\frac{3}{2p}\sum_{j=1}^k\frac{1}{2^j}}\\
&\quad\times\exp\left\{\frac{B(T)^2}{4}p3t\sum_{j=1}^k\frac{1}{2^j}\right\}\\
&\quad\cdot(3t)^{-\frac{1}{2p}\sum_{j=1}^k\frac{1}{2^j}}2^{\frac{1}{p}\sum_{j=1}^k\frac{j}{2^j}}\\
&\le C^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}2^{\frac{4}{p}\sum_{j=1}^k\frac{j}{2^j}}p^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}C_T^{\frac{3}{2p}\sum_{j=1}^k\frac{1}{2^j}}\\
&\quad\times\exp\left\{\frac{B(T)^2}{4}p3t\sum_{j=1}^k\frac{1}{2^j}\right\}
\end{aligned}
$$

$$\cdot\,(3t)^{-\frac{1}{2p}\sum_{j=1}^{k}\frac{1}{2^j}},\ \forall\, t\in(0,T].$$

Therefore,

$$\begin{aligned}\|u_i(\cdot,t)\|_{L^{2^k p}(\mathbb{R})} &\le C^{\frac{3}{p}\left(1-\frac{1}{2^k}\right)}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}2^{\frac{4}{p}\sum_{j=1}^{k}\frac{j}{2^j}}p^{\frac{3}{p}\left(1-\frac{1}{2^k}\right)}C_T^{\frac{3}{2p}\left(1-\frac{1}{2^k}\right)}\\ &\quad\cdot\exp\left\{\frac{B(T)^2}{4}3pt\left(1-\frac{1}{2^k}\right)\right\}(3t)^{-\frac{1}{2p}\left(1-\frac{1}{2^k}\right)},\quad \forall\, t\in(0,T].\end{aligned}$$

Passing to the limit when $k\to\infty$ and using that, by hypothesis, $\mathbf{u}(\cdot,t)\in C^0([0,T],L^p(\mathbb{R}))$, one obtains

$$\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C^{\frac{3}{p}}\|u_{i0}\|_{L^p(\mathbb{R})}2^{\frac{8}{p}}3^{-\frac{1}{2p}}p^{\frac{3}{p}}C_T^{\frac{3}{2p}}\exp\left\{\frac{B(T)^2}{4}3pt\right\}t^{-\frac{1}{2p}},$$

for all $t\in(0,T]$. Therefore, using definition (5), one gets the desired inequality

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C^{\frac{3}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R})}2^{\frac{8}{p}}3^{-\frac{1}{2p}}p^{\frac{3}{p}}C_T^{\frac{3}{2p}}\exp\left\{\frac{B(T)^2}{4}3pt\right\}t^{-\frac{1}{2p}},\ \forall\, t\in(0,T],$$

where $C=C(n)$ is a positive constant. □

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