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Sobre a existência global e limitação uniforme de soluções da equação dos meios porosos com termos advectivos arbitrários

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Resumo

Neste trabalho estudamos algumas propriedades básicas de soluções não negativas da equação de advecção-difusão

$$u_t + (\mathbf{b}(x,t,u)u)_x = \mu(t)(u^\alpha u_x)_x, \qquad x \in \mathbb{R}\ t > 0$$

correspondentes a estados iniciais

$$u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \qquad (u_0 \geq 0)$$

para algum $1 \leq p_0 < \infty$, onde $\alpha > 0$ é uma constante dada e $\mu \in C^0([0,+\infty[)$ é positiva. Como velocidade de advecção $\mathbf{b}(x,t,u)$ tomamos

$$\mathbf{b}(x,t,u) = b(x,t)u^\kappa, \qquad \forall\, x \in \mathbb{R}, \quad t \geq 0, \quad u \geq 0$$

para algum $\kappa \geq 0$ dado (constante), onde $b : \mathbb{R} \times [0,+\infty[\longrightarrow \mathbb{R}$ é uma função contínua (com b_x contínua) e limitada em cada faixa $S_T = \mathbb{R} \times [0,T]$, para cada $T > 0$ dado.

Como propriedades básicas podemos citar a conservação de massa, a permanência da solução em L^q e a continuidade dos dados iniciais.

Com uma combinação de estimativas de energia e o princípio da comparação apropriado para o problema, mostramos o seguinte resultado geral para $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: para cada $p \geq p_0$ finito (com ademais $p > \kappa - \alpha$ se $\kappa \geq \alpha + p_0$), tem-se,

$$u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^p(\mathbb{R})) \qquad \Longleftrightarrow \qquad u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R})).$$

Abstract

In this work we study some basic properties of solution nonnegative of the diffusion-advection equation

$$u_t + (\mathbf{b}(x,t,u)u)_x = \mu(t)(u^\alpha u_x)_x, \qquad x \in \mathbb{R}\ t > 0$$

corresponding to initial states

$$u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \qquad (u_0 \geq 0)$$

for some $1 \leq p_0 < \infty$, where $\alpha > 0$ is a given constant and $\mu \in C^0([0,+\infty[)$ is positive. As speed of advection $\mathbf{b}(x,t,u)$ we take

$$\mathbf{b}(x,t,u) = b(x,t)u^\kappa, \qquad \forall\, x \in \mathbb{R}, \quad t \geq 0, \quad u \geq 0$$

for any given $\kappa \geq 0$ (constant), where $b : \mathbb{R} \times [0,+\infty[\longrightarrow \mathbb{R}$ is a continuous function (with b_x continuous) and limited in each strip $S_T = \mathbb{R} \times [0,T]$, for each given $T > 0$.

As basic properties we mention the conservation of mass, the permanence of the solution in L^q and continuity of the initial data.

With a combination of energy estimates and the comparison principle suitable for the problem, show the following general result for $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: for each $p \geq p_0$ finite (with addition $p > \kappa - \alpha$ if $\kappa \geq \alpha + p_0$), we have

$$u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^p(\mathbb{R})) \qquad \Longleftrightarrow \qquad u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R})).$$

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1 Introdução

Neste trabalho investigamos algumas propriedades básicas de soluções não negativas da equação de advecção-difusão

$$u_t + (\mathbf{b}(x,t,u)u)_x = \mu(t)(u^\alpha u_x)_x, \qquad x \in \mathbb{R}\ t > 0 \tag{1.1}$$

correspondentes a estados iniciais

$$u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \qquad (u_0 \geq 0) \tag{1.2}$$

para algum $1 \leq p_0 < \infty$, onde $\alpha > 0$ é uma constante dada e $\mu \in C^0([0,+\infty[)$ é positiva. Na equação (1.1) acima, supõe-se que a velocidade de advecção $\mathbf{b}(x,t,u)$ tem a forma

$$\mathbf{b}(x,t,u) = b(x,t)u^\kappa, \qquad \forall\, x \in \mathbb{R}, \quad t \geq 0, \quad u \geq 0 \tag{1.3}$$

para algum $\kappa \geq 0$ dado (constante), onde $b : \mathbb{R} \times [0,+\infty[\longrightarrow \mathbb{R}$ é uma função contínua (com b_x contínua) e limitada em cada faixa $S_T = \mathbb{R} \times [0,T]$, para cada $T > 0$ dado. Nestas condições, o problema acima possui solução não negativa $u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$, $0 < T_* \leq \infty$, satisfazendo (1.2) no sentido de $L^1_{loc}(\mathbb{R})$, ou seja,

$$\|u(\cdot,t) - u_0\|_{L^1(\mathbb{K})} \longrightarrow 0 \qquad \text{ao} \qquad t \longrightarrow 0 \tag{1.4}$$

para cada $\mathbb{K} \subset \mathbb{R}$ compacto dado. Ver [14], [5], [13].

Esta solução $u(x,t) \geq 0$ satisfaz (1.1) no sentido fraco, ou seja,

$$\int_0^{T_*}\int_{\mathbb{R}} u\Psi_t dxdt + \int_0^{T_*}\int_{\mathbb{R}} \mathbf{b}(x,t,u)u\Psi_x dxdt + \frac{1}{\alpha+1}\int_0^{T_*}\mu(t)\int_{\mathbb{R}} u^{\alpha+1}\Psi_{xx}dxdt = 0 \tag{1.5}$$

para toda Ψ suave de suporte compacto em $\mathbb{R}\times]0,T_*[$; ademais, tem-se $u \in C^0(\mathbb{R}\times]0,T_*[)$, com $u(x,t)$ satisfazendo (1.1) no sentido clássico em todo ponto (x,t) tal que $u(x,t) > 0$, ver [14], [5], [13].

No Capítulo 2, mostramos várias propriedades básicas das soluções $u(\cdot,t)$ acima. Em particular é mostrado que se tem

$$u(\cdot,t) \in C^0([0,T_*[, L^q(\mathbb{R})) \tag{1.6}$$

para cada $p_0 \leq q < \infty$, de modo que (1.2) na verdade fica satisfeita no sentido mais forte

$$\|u(\cdot,t) - u_0\|_{L^{p_0}(\mathbb{R})} \quad \longrightarrow \quad 0 \text{ ao } \quad t \searrow 0 \tag{1.7}$$

(e também $\|u(\cdot,t) - u_0\|_{L^q(\mathbb{R})} \to 0$ ao $t \searrow 0$, para cada $p_0 \leq q < \infty$). A questão fundamental de que trata este trabalho é estabelecer condições garantindo a existência global (isto é, $T_* = \infty$) das soluções, obtendo-se ademais certas estimativas para os valores de $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ para $t \gg 1$. Como supomos $b(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$ geral, este problema torna-se bastante complicado, como mostra o argumento a seguir.

Escrevendo (1.1) e (1.3) na forma

$$u_t + a(x,t)u_x = \mu(t)(u^\alpha u_x)_x - b_x(x,t)u(\cdot,t)^{1+\kappa} \tag{1.8}$$

vemos que nas regiões onde $b_x(x,t) < 0$ o termo $-b_x(x,t)u^{1+\kappa}$ tende a fazer $u(x,t)$ crescer, especialmente se $\kappa > 0$ (onde existe a possibilidade de explosão (*blow-up*) em tempo finito[1], isto é, ter-se

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \longrightarrow \quad \infty \text{ ao } t \to T_f,$$

para certo $T_f < \infty$), ainda mais onde se tenham $b_x(x,t) \gg 1$. Observa-se também que $-b_x$ pode assumir valores arbitrariamente grandes, como por exemplo no caso $b(x) = \sin(x^2)$. Afinal, temos suposto até o momento apenas que $b(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$; parece claro que algo mais terá de ser suposto sobre $b(\cdot,t)$. Nossos resultados requerem que se tenha

$$\limsup_{t\to+\infty} \frac{B(t)}{\mu(t)} \equiv B_\mu < \infty, \tag{1.9}$$

onde $2B(t)$ é a oscilação de $b(\cdot,t)$ em $\mathbb{R}$, ou seja,

$$\begin{aligned} B(t) &= \frac{1}{2}\left(\sup_{x\in\mathbb{R}} b(x,t) - \inf_{x\in\mathbb{R}} b(x,t)\right) \\ &= \frac{1}{2}\sup_{x,y\in\mathbb{R}} |b(x,t) - b(y,t)| \end{aligned} \tag{1.10}$$

Uma característica importante da equação (1.1) é que se trata de uma lei de conservação (isto é, todos os termos são derivadas). Em particular, no caso fundamental $p_0 = 1$, tem-se a conservação de massa,

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx, \qquad \forall\, 0 \le t \le T_f. \tag{1.11}$$

Assim, caso haja crescimento persistente de $u(x,t)$ numa certa região (onde $-b_x > 0$), este crescimento resultará na formação de pulsos longos e finos como mostrado na figura (1) abaixo.

Estas estruturas alongadas, que correspondem a uma alta frequência, tendem a ser eficientemente dissipadas pelo termo difusivo $u^\alpha u_{xx}$ (reforçado pelo fato de o coeficiente de difusão $u^\alpha(x,t)$ tornar-se maior onde $u(x,t)$ for grande, aumentando a capacidade de dissipar estes pulsos). Tem-se, portanto, uma complicada competição entre o termo advectivo o termo difusivo, cujo efeito final sobre a solução

[1] Na verdade, segue dos resultados dos Capítulos 3 e 5 que só pode ocorrer blow-up de $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ em tempo finito para $\kappa > \frac{\alpha}{2}$.

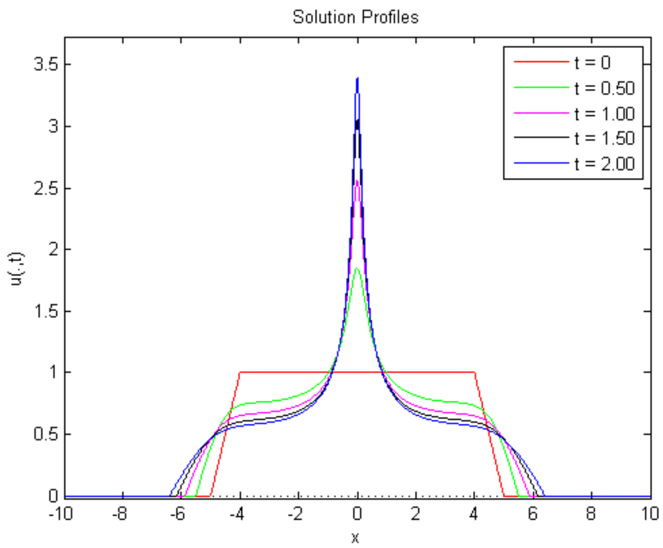

Figura 1: Perfil das Soluções

$u(\cdot,t)$ é difícil de provar. Esta competição só não acontece quando se tem $b_x \geq 0$ sempre, caso em que $u(\cdot,t)$ é globalmente definida e

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, 0 \leq t_0 < t \leq +\infty \tag{1.12}$$

para cada $p_0 \leq q \leq \infty$, tendo-se o seguinte decaimento de $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ ao $t \to \infty$:

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathrm{K}(\alpha,p_0)\,\|u_0\|_{L^{p_0}(\mathbb{R})}^{\widetilde{\delta}} \cdot t^{-\widetilde{\gamma}}, \qquad \forall\, t > 0 \tag{1.13}$$

onde $\widetilde{\delta} = \dfrac{2p_0}{2p_0+\alpha}$, $\widetilde{\gamma} = \dfrac{1}{2p_0+\alpha}$ e $\mathrm{K}(\alpha,p_0) > 0$ é uma constante que depende apenas de α, p_0 (e não depende de $u_0, u(\cdot,t)$, κ ou t). A prova destes resultados no caso $b_x \geq 0$ é apresentado na seção 2.9.

Voltando ao caso geral $b(\cdot,t) \in L^\infty_{loc}([0,+\infty[, L^\infty(\mathbb{R}))$ (com b_x qualquer), vamos conseguir mostrar o seguinte resultado geral para $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: para cada $p \geq p_0$ finito (com ademais $p > \kappa - \alpha$ se $\kappa \geq \alpha + p_0$), tem-se,

$$u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^p(\mathbb{R})) \qquad \Longleftrightarrow \qquad u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R})) \tag{1.14}$$

com

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathrm{K}_1(\alpha,\kappa;p)\cdot\max\left\{\|u_0\|_{L^\infty(\mathbb{R})}, \mathbb{B}_\mu(0,t)^\delta \cdot \mathbb{U}_p(0,t)^\gamma\right\} \qquad \forall\, 0 < t < T_*, \tag{1.15}$$

onde $\mathrm{K}_1(\alpha,\kappa;p) > 0$ é uma constante absoluta que depende somente dos parâmetros α, κ e p, com $\mathrm{K}_1(\alpha,\kappa;p) \to 1$ ao $p \to +\infty$, e $\mathbb{B}_\mu(0,t)$, $\mathbb{U}_p(0,t)$, δ e γ são dados por

$$\mathbb{B}_\mu(0,t) = \sup\left\{\frac{B(\tau)}{\mu(\tau)};\quad 0 \leq \tau \leq t\right\} \tag{1.16}$$

$$\mathbb{U}_p(0,t) = \sup\Big\{ \|u(\cdot,t)\|_{L^p(\mathbb{R})}\,;\quad 0 \le \tau \le t \Big\} \tag{1.17}$$

$$\delta = \frac{1}{p+\alpha-\kappa}, \qquad\qquad \gamma = \frac{p}{p+\alpha-\kappa}, \tag{1.18}$$

(ver Teoremas (3.8), (4.8) e (5.11)). Estas estimativas substituem (1.13) no caso geral em que $b_x(\cdot,t)$ é arbitrária. Quando $\mathbb{B}_\mu(0,+\infty)$ e $\mathbb{U}_p(0,+\infty)$ são ambos finitos, resulta de (1.14) que a solução u é limitada em $\mathbb{R}\times[0,+\infty[$, sendo de interesse estimar mais finamente os valores de $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ para $t \gg 1$. É mostrado (nos Capítulos 3, 4 e 5) que se tem, neste caso,

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \mathbb{K}(\alpha,\kappa;p)\cdot \mathrm{B}_\mu^{\delta}\cdot \mathrm{U}_p^{\gamma}, \tag{1.19}$$

onde $\delta = \dfrac{1}{p+\alpha-\kappa}$, $\gamma = \dfrac{p}{p+\alpha-\kappa}$ são dadas em (1.19) acima, e

$$\mathrm{B}_\mu = \limsup_{t\to+\infty} \frac{B(\tau)}{\mu(\tau)} \tag{1.20}$$

$$\mathrm{U}_p = \limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^p(\mathbb{R})} \tag{1.21}$$

e

$$\mathbb{K}(\alpha,\kappa;p) = \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\alpha-\kappa}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p + 2\alpha}{3\cdot 2^j p + 2\alpha}\right)^{\frac{1}{2^j p+\alpha-\kappa}} \tag{1.22}$$

se $\kappa \ge \dfrac{\alpha}{2}$. Se $0 \le \kappa < \dfrac{\alpha}{2}$,

$$\mathbb{K}(\alpha,\kappa;p) = \left(\frac{3\cdot(p-\gamma)+\alpha}{4}\right)^{\frac{1}{p+\alpha-\kappa}} \cdot \left[\prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j (p-\gamma) + 2\alpha}{3\cdot 2^j (p-\gamma) + 2\alpha}\right)^{\frac{p-\gamma+\alpha-\kappa}{2^j(p-\gamma)+\alpha-\kappa}}\right]^{\frac{1}{p+\alpha-\kappa}}, \tag{1.23}$$

onde $\gamma = 2\kappa-\alpha$. (Ver Teoremas (3.13), (4.14) e (5.12)). Convém observar que o valor optimal (isto é, mínimo) para as constantes $K(\alpha,p_0)$, $K_1(\alpha,\kappa;p)$ e $\mathbb{K}(\alpha,\kappa;p)$, dadas em (1.13), (1.15) e (1.19) acima não é conhecido; os valores obtidos para estas contantes neste trabalho (como por exemplo em (1.22) e (1.23) acima) não são optimais. Quanto aos expoentes δ e γ nas estimativas (1.13), (1.15) e (1.19) acima, são coasistentes com análise de escalas, como mostrado na Seção 2.8.

Para obter (1.19), o resultado fundamental é a estimativa (1.15), como mostraremos a seguir. Aplicando (1.15) num instante inicial $t_0 \ge 0$ qualquer, temos, supondo $\mathbb{B}_\mu(0,\infty) < \infty$ e $\mathbb{U}_p(0,\infty) < \infty$,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \mathrm{K}_1(\alpha,\kappa;p)\max\Big\{\|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})}, \mathrm{B}_\mu(t_0,t)^{\delta}\cdot \mathrm{U}_p(t_0,t)^{\gamma}\Big\} \tag{1.24}$$

para todo $t > t_0$, onde

$$\mathbb{B}_\mu(t_0,t) = \sup_{t_0 \le \tau \le t} \frac{B(\tau)}{\mu(\tau)} \tag{1.25}$$

$$\mathbb{U}_p(t_0,t) = \sup_{t_0 \le \tau \le t} \|u(\cdot,\tau)\|_{L^p(\mathbb{R})}\,. \tag{1.26}$$

e onde $K_1(\alpha,\kappa;p)$ é a constante dada em (1.15) acima. Em particular, sendo

$$\begin{array}{lrcl} (a) & \mathbb{B}_\mu(t_0) & = & \displaystyle\sup_{\tau \ge t_0} \frac{B(\tau)}{\mu(\tau)} \\ (b) & \mathbb{U}_p(t_0) & = & \displaystyle\sup_{\tau \ge t_0} \|u(\cdot,\tau)\|_{L^p(\mathbb{R})} \\ (c) & \mathbb{U}_\infty(t_0) & = & \displaystyle\sup_{\tau \ge t_0} \|u(\cdot,\tau)\|_{L^\infty(\mathbb{R})} \end{array} \tag{1.27}$$

obtemos

$$\mathbb{U}_\infty(t_0,t) \le \mathrm{K}_1(\alpha,\kappa;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})}\,, \mathbb{B}_\mu(t_0)^\delta \cdot \mathbb{U}_p(t_0)^\gamma \right\}, \tag{1.28}$$

onde $t_0 \ge 0$ é arbitrário.

Tomando $t_0 \nearrow \infty$ de modo a se ter

$$\|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} \longrightarrow \liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}\,, \tag{1.29}$$

obtemos então, de (1.28) acima, fazendo $t_0 \nearrow \infty$ como em (1.29),

$$\mathrm{U}_\infty \le \mathrm{K}_1(\alpha,\kappa;p) \max\left\{ \liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}\,, \mathrm{B}_\mu^\delta \cdot \mathrm{U}_p^\gamma \right\}, \tag{1.30}$$

onde $\mathrm{K}_1(\alpha,\kappa;p) > 1$ é a constante dada em (1.15), visto que, ao $t \nearrow \infty$, tem-se

$$\mathbb{U}_\infty(t_0) \searrow \mathrm{U}_\infty, \qquad \mathbb{B}_\mu(t_0) \searrow \mathrm{B}_\mu, \qquad \mathbb{U}_p(t_0) \searrow \mathrm{U}_p, \tag{1.31}$$

onde U_∞, B_μ e U_p são as quantidades dadas por

$$\begin{array}{llclcl} (a) & \mathrm{B}_\mu & = & \displaystyle\lim_{t\to\infty} \mathbb{B}_\mu(t) & = & \displaystyle\limsup_{t\to\infty} \frac{B(t)}{\mu(t)} \\ (b) & \mathrm{U}_p & = & \displaystyle\lim_{t\to\infty} \mathbb{U}_p(t) & = & \displaystyle\limsup_{t\to\infty} \|u(\cdot,\tau)\|_{L^p(\mathbb{R})} \\ (c) & \mathrm{U}_\infty & = & \displaystyle\lim_{t\to\infty} \mathbb{U}_\infty(t) & = & \displaystyle\limsup_{t\to\infty} \|u(\cdot,\tau)\|_{L^\infty(\mathbb{R})} \end{array} \tag{1.32}$$

O passo seguinte requer que estimemos $\liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$ em termos de B_μ e U_p. Isso envolve considerável trabalho (ver Teoremas (3.12), (4.12) e (5.12)) tendo-se aqui estendido para o problema (1.1), (1.2) ideias desenvolvidas no caso $\kappa = 0$, $\alpha = 0$ em [10], [7], que utilizam a desigualdade de Sobolev unidimensional

$$\|\,\mathsf{u}\,\|_{L^\infty(\mathbb{R})} \le \|\,\mathsf{u}\,\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \,\|\,\mathsf{u}_x\,\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\,, \qquad \forall\, \mathsf{u} \in H^1(\mathbb{R}). \tag{1.33}$$

O resultado obtido (derivado em detalhe nas seções 3.3, 4.3 e 5.3) tem a forma

$$\liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathrm{K}_2(\alpha,\kappa,p)\cdot \mathrm{B}_\mu^\delta \cdot \mathrm{U}_p^\gamma \tag{1.34}$$

onde $\delta = \dfrac{1}{p+\alpha-\kappa}$, $\gamma = \dfrac{p}{p+\alpha-\kappa}$, como em (1.19), e $\mathrm{K}_2(\alpha,\kappa,p) > 0$ é certa constante cujo valor depende apenas de α, κ, p e que satisfaz $\mathrm{K}_2(\alpha,\kappa,p) \to 1$ ao $p \to +\infty$. Resulta de (1.30) e (1.34)

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathrm{K}(\alpha,\kappa;p)\cdot \mathrm{B}_\mu^\delta \cdot \mathrm{U}_p^\gamma \tag{1.35}$$

onde δ, γ são os expoentes dados em (1.19) acima e

$$\mathrm{K}(\alpha,\kappa;p) = \mathrm{K}_1(\alpha,\kappa;p)\cdot \max\{1, \mathrm{K}_2(\alpha,\kappa;p)\}, \tag{1.36}$$

sendo K_1, K_2 as constantes dadas em (1.15) e (1.34) acima. A parte final do argumento consiste em mostrar que, uma vez obtido (1.35) com $\mathrm{K}(\alpha,\kappa;p)$ dada em (1.36) acima, pode-se na verdade oter (1.35) com $\mathrm{K}(\alpha,\kappa;p)$ dada em (1.23). Este aperfeiçoamento final é feito nos Teoremas (3.12), (4.12) e (5.10).

No caso de $b_x \geq 0$, segue de (1.13) que

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \longrightarrow 0 \qquad \text{ao} \qquad t \longrightarrow +\infty \qquad (\text{se } b_x \geq 0), \tag{1.37}$$

um resultado bem conhecido para a equação $u_t = (u^\alpha u_x)_x$. No caso geral $b_x \not\geq 0$, que é o interesse principal deste trabalho, (1.37) não vale em geral. Um contra-exemplo simples pode ser obtido no caso da equação

$$u_t + (b(x)u^{\kappa+1})_x = (u^\alpha u_x)_x \tag{1.38}$$

quando ela possui soluções estacionárias (ou seja, $u(x,t)$ não depende de t).

Se $\alpha \geq \kappa$, existem infinitas funções $b \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ para as quais a equação $u_t + (b(x)u^{\kappa+1})_x = (u^\alpha u_x)_x$ admite soluções estacionárias.

Por exemplo, se $\alpha = \kappa$, e $b \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfaz $b(+\infty) \in [-\infty, 0[$, $b(-\infty) \in]0, +\infty]$, tem-se que

$$\mathrm{U}(x) = A \cdot e^{\int_0^x b(s)\,ds}, \qquad x \in \mathbb{R}$$

é solução da equação $u_t + (b(x)u^{\alpha+1})_x = (u^\alpha u_x)_x$ para qualquer $A \geq 0$ dado com $\mathrm{U} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Se $\alpha > \kappa$, e $b \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfaz

$$\int_{\mathbb{R}} b(s)\,ds = 0$$

de tal modo a se ter $\displaystyle\int_{-\infty}^x b(s)\,ds \geq 0, \forall\, x \in \mathbb{R}$ e $\displaystyle\int_{-\infty}^x b(s)\,ds \in L^{\frac{q}{\alpha-\kappa}}(\mathbb{R})$ para algum $q \in [1, +\infty[$, então

$$\mathrm{U}(x) = \left((\alpha-\kappa)\int_{-\infty}^x b(s)\,ds\right)^{\frac{1}{\alpha-\kappa}}$$

satisfaz $u_t + (b(x)u^{\kappa+1})_x = (u^\alpha u_x)_x$, tendo-de U $\in L^q(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Mesmo tendo-se obtido vários resultados sobre soluções $u(\cdot, t)$ de (1.1), (1.2), muitas questões permanecem não investigadas. Alguns exemplos são:

(*i*) identificar os campos de velocidade $\mathbf{b}(x, t, u)$ da forma (1.3) para os quais todas as soluções de (1.1), (1.2) sejam globalmente definidas (isto é, definidas $\forall t > 0$);

(*ii*) identificar os campos de velocidade $\mathbf{b}(x, t, u)$ da forma (1.3) para os quais se tenha $\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \to 0$ ao $t \to +\infty$ (para todas as soluções (1.1), (1.2));

(*iii*) identificar os campos de velocidade $\mathbf{b}(x, t, u)$ da forma (1.3) para os quais se tenha $u \in L^\infty(\mathbb{R} \times [0, +\infty[)$, para qualquer solução de (1.1), (1.2);

(*iv*) se $p_0 > 1$, identificar os campos $\mathbf{b}(x, t, u)$ da forma (1.3) para os quais se tenha $\|u(\cdot, t)\|_{L^{p_0}(\mathbb{R})} \to 0$ ao $t \to +\infty$[2] (para qualquer solução de (1.1), (1.2));

(*v*) no caso de soluções estacionárias, examinar suas propriedades de estabilidade com respeito a perturbações nos valores iniciais.

[2]por exemplo, se $\mathbf{b}(x, t, u)$ não depender explicitamente de x, (*iv*) é verdadeira, ver [14].

2 Resultados Preliminares

2.1 Introdução

Iniciamos nosso estudo fixando algumas das notações, decorrências e funções usadas no decorrer do texto, como funções de corte e regiões de integração. Definimos as hipóteses necessárias para a obtenção dos resultados propostos neste trablaho.

2.2 Notações e Hipóteses Gerais

Como descrito na seção anterior, consideramos $[0, T_*[$, com $0 < T_* \leq \infty$, o intervalo maximal de existência de solução (a existência deste T_* segue da teoria clássica de equações parabólicas, veja [5], [14]). A solução de (1.1) existe para cada $0 < T < T_*$ e é limitada na faixa espaço-tempo $S_T := \mathbb{R} \times [0, T]$. Damos a seguir algumas definições e considerações sobre as funções suavizadoras e de corte utilizadas no texto. Para $0 < \epsilon \leq 1$ e $R > 0$ quaisquer, definimos

$$\zeta_R(x) = \begin{cases} 0, & |x| > R \\ \\ e^{-\epsilon\sqrt{1+x^2}} - e^{-\epsilon\sqrt{1+R^2}}, & |x| \leq R. \end{cases} \tag{2.1}$$

Observe que a função $\zeta_R(x)$ anula-se em $|x| = R$, mas suas derivadas primeira e segunda não. Temos

$$\begin{aligned} &\zeta_R'(x) = -\frac{\epsilon x}{\sqrt{1+x^2}} e^{-\epsilon\sqrt{1+x^2}}, \ |x| \leq R \\ &|\zeta_R'(x)| \leq \epsilon e^{-\epsilon\sqrt{1+x^2}}, \ |x| \leq R \\ &\zeta_R''(x) = -\frac{\epsilon}{\sqrt{1+x^2}} e^{-\epsilon\sqrt{1+x^2}} + \frac{\epsilon x^2}{\sqrt{(1+x^2)^3}} e^{-\epsilon\sqrt{1+x^2}} + \frac{\epsilon^2 x^2}{1+x^2} e^{-\epsilon\sqrt{1+x^2}}, \\ &|\zeta_R''(x)| \leq 3\epsilon^2 e^{-\epsilon\sqrt{1+x^2}}. \end{aligned} \tag{2.2}$$

Para $R > 0$ qualquer, definimos $\psi : \mathbb{R} \to \mathbb{R}$ uma função positiva, de Classe C^2 e tal que

$$\psi(x) = \begin{cases} 1, & \text{se } |x| \leq 1 \\ 0, & \text{se } |x| \geq 2 \end{cases} \tag{2.3}$$

A partir da função de corte $\psi(x)$, definimos a função de corte $\psi_R(x)$ por

$$\psi_R(x) = \psi\left(\frac{x}{R}\right) \tag{2.4}$$

e, como podemos observar, existem constantes positivas C_1 e C_2 tais que

$$\left|\psi_R'(x)\right| = C_1 \qquad \text{e} \qquad \left|\psi_R''(x)\right| = C_2 \tag{2.5}$$

Construiremos, agora, a função de corte $\Phi \in C^0([0, +\infty[)$ satisfazendo

$$\Phi(x) = \begin{cases} 0, & \text{se} \quad x \leq 1 \\ 1, & \text{se} \quad x \in [2, 3] \\ 0, & \text{se} \quad x \geq 4 \end{cases} \tag{2.6}$$

Sejam $R_1 > 0$ e $R_2 > 1$ constantes dadas e seja $\xi_{R_1,R_2} \in C^2(\mathbb{R})$ dada por

$$\xi_{R_1,R_2}(x) = \begin{cases} 0 \text{ se } |x| \leq R_1 \\ \Phi(1 + |x| - R_1) \text{ se } R_1 \leq |x| \leq R_1 + 1 \\ 1 \text{ se } R_1 + 1 \leq |x| \leq R_1 + R_2 \\ \Phi(3 + |x| - R_1 - R_2) \text{ se } R_1 + R_2 \leq |x| \leq R_1 + R_2 + 1 \\ 0 \text{ se } |x| \geq R_1 + R_2 + 1 \end{cases} \tag{2.7}$$

Assim, temos

$$\xi'_{R_1,R_2}(x) = \begin{cases} 0, & \forall |x| \leq R_1 \\ 0, & \forall R_1 + 1 \leq |x| \leq R_1 + R_2 \\ 0, & \forall |x| \geq R := R_1 + R_2 + 1 \end{cases} \tag{2.8}$$

e

$$|\xi'_{R_1,R_2}(x)| \leq M_1, \quad \forall x \in \mathbb{R}, (\text{ com } M_1 > 0 \text{ independente de } R_1 \text{ e } R_2).$$

Temos também

$$\xi''_{R_1,R_2}(x) = \begin{cases} 0, & \forall |x| \leq R_1 \\ 0, & \forall R_1 + 1 \leq |x| \leq R_1 + R_2 \\ 0, & \forall |x| \geq R = R_1 + R_2 + 1 \end{cases} \tag{2.9}$$

e

$$|\xi''_{R_1,R_2}(x)| \leq M_2, \quad \forall x \in \mathbb{R}, (\text{ com } M_2 > 0 \text{ independente de } R_1 \text{ e } R_2).$$

Defina também

$$\xi_R(x) := \lim_{R_2 \to \infty} \xi_{R_1,R_2}(x). \tag{2.10}$$

Devido as funções de corte, teremos regiões de integração com uma representação longa, então assumiremos a seguinte notação para regiões espaciais:

$$\begin{aligned} A_{R_1,R_2} &= \{x \in \mathbb{R}; R_1 < |x| < R_1 + R_2 + 1\} \\ A'_{R_1,R_2} &= \{x \in \mathbb{R}; R_1 \leq |x| \leq R_1 + 1 \text{ ou } R_1 + R_2 \leq |x| \leq R_1 + R_2 + 1\} \end{aligned} \tag{2.11}$$

Como hipóteses gerais temos

$$
\begin{aligned}
&(H1) \quad b \in C^0(\mathbb{R} \times [0,\infty[) \text{ com } b(\cdot,t) \in L^\infty_{loc}([0,\infty[, L^\infty(\mathbb{R}));\\
&(H2) \quad \mu \in C^0([0,\infty[) \text{ com } \mu(t) > 0 \text{ para todo } t \geq 0;\\
&(H3) \quad \int_0^\infty \mu(t)dt = \infty;\\
&(H4) \quad B_\mu \equiv \limsup_{t\to\infty} \frac{B(t)}{\mu(t)} < \infty.
\end{aligned}
\tag{2.12}
$$

Para obtermos os resultados propostos, trabalhamos com algumas definições relacionadas ao fluxo que destacamos abaixo:

$$\beta(t) := \frac{1}{2}\left(\sup_{x\in\mathbb{R}} b(x,t) + \inf_{x\in\mathbb{R}} b(x,t)\right) \tag{2.13}$$

$$B(t) := \frac{1}{2}\left(\sup_{x\in\mathbb{R}} b(x,t) - \inf_{x\in\mathbb{R}} b(x,t)\right) = \frac{1}{2}\sup_{x,y\in\mathbb{R}} |b(x,t) - b(y,t)|, \tag{2.14}$$

para $t \geq 0$. Com as definições (2.13) e (2.14) obtemos as seguintes estimativas

$$|b(x,t)| \leq \mathfrak{b}(T) \qquad \forall\, x \in \mathbb{R} \text{ e } \forall\, t \in [0,T] \tag{2.15}$$

$$|\beta(t)| \leq \mathfrak{b}(T) \qquad t \in [0,T] \tag{2.16}$$

$$|b(x,t) - \beta(t)| \leq B(t) \qquad \forall\, x \in \mathbb{R} \text{ e } \forall\, t \in [0,T] \tag{2.17}$$

A solução considerada neste trabalho é limitada, com isso temos a seguinte definição e estimativa relacionadas a solução $\mathsf{v}(x,t)$:

$$M = M(T) := \sup_{\substack{x\,\in\,\mathbb{R}\\ 0\,\leq\, t\,\leq\, T}} \mathsf{v}(x,t) = \sup_{0\leq t\leq T} \|\mathsf{v}(\cdot,t)\|_{L^\infty(\mathbb{R})}\,. \tag{2.18}$$

$$\mathsf{v}(x,t)^{r+1} = \mathsf{v}(x,t)\mathsf{v}(x,t)^r \leq \mathsf{v}(x,t)\|\mathsf{v}(x,t)\|^r_{L^\infty(\mathbb{R})} \leq \mathsf{v}(x,t)\|\mathsf{v}_0\|^r_{L^\infty(\mathbb{R})}. \tag{2.19}$$

Para cada $0 \leq t_0 \leq t < T_*$, definimos

$$
\begin{aligned}
\mathbb{V}_q(t_0;t) &:= \sup_{t_0\leq\tau\leq t} \|v(\cdot,\tau)\|_{L^q(\mathbb{R})}\,, \quad p_0 \leq q \leq \infty && (2.20)\\
\mathbb{U}_q(t_0;t) &:= \sup_{t_0\leq\tau\leq t} \|u(\cdot,\tau)\|_{L^q(\mathbb{R})}\,, \quad p_0 \leq q \leq \infty && (2.21)\\
\mathbb{B}_\mu(t_0;t) &:= \sup_{t_0\leq\tau\leq t} \frac{B(\tau)}{\mu(\tau)} && (2.22)
\end{aligned}
$$

Quando $T_* = \infty$, definimos também

$$\begin{aligned}
\mathbb{V}_q(t_0) &:= \sup_{\tau \geq t_0} \|v(\cdot,\tau)\|_{L^q(\mathbb{R})}\,, \quad t_0 \geq 0, \quad p_0 \leq q \leq \infty && (2.23)\\
\mathbb{U}_q(t_0) &:= \sup_{\tau \geq t_0} \|u(\cdot,\tau)\|_{L^q(\mathbb{R})}\,, \quad t_0 \geq 0, \quad p_0 \leq q \leq \infty && (2.24)\\
\mathbb{B}_\mu(t_0) &:= \sup_{\tau \geq t_0} \frac{B(\tau)}{\mu(\tau)}, \quad t_0 \geq 0 && (2.25)
\end{aligned}$$

e

$$\begin{aligned}
V_q(t_0) &:= \limsup_{t\to\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})}\,, \quad p_0 \leq q \leq \infty && (2.26)\\
U_q(t_0) &:= \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^q(\mathbb{R})}\,, \quad p_0 \leq q \leq \infty && (2.27)\\
B_\mu(t_0) &:= \limsup_{t\to\infty} \frac{B(t)}{\mu(\tau)}. && (2.28)
\end{aligned}$$

Nas derivações de muitos Teoremas usaremos as desigualdades de Nirenberg-Sobolev-Gagliardo (NSG):

$$\|\mathsf{w}\|_{L^{\widehat{\beta}}(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\widehat{\theta}} \|\mathsf{w}\|_{L^{\beta_0}(\mathbb{R})}^{1-\widehat{\theta}} \cdot \|\mathsf{w}_x\|_{L^2(\mathbb{R})}^{\widehat{\theta}}\,, \quad \widehat{\theta} = \frac{1-\frac{\beta_0}{\widehat{\beta}}}{1+\frac{\beta_0}{2}}, \quad \forall\, 0 < \beta_0 \leq \widehat{\beta} \quad (2.29)$$

e

$$\|\mathsf{w}\|_{L^\infty(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\theta} \|\mathsf{w}\|_{L^{\beta_0}(\mathbb{R})}^{1-\theta} \cdot \|\mathsf{w}_x\|_{L^2(\mathbb{R})}^{\theta}\,, \quad \theta = \frac{1}{1+\frac{\beta_0}{2}}, \quad \forall\, \beta_0 > 0. \quad (2.30)$$

2.3 Decrescimento de $\|u(\cdot,t)\|_{L^1}$

Iniciamos nosso estudo mostrando que uma solução suave $u(x,t)$ de

$$\begin{cases} u_t + (b(x,t)u^{k+1})_x = \mu(t)(u^\alpha u_x)_x, \\ \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \end{cases} \tag{2.31}$$

onde $\alpha > 0$, $\kappa \geq 0$ quaisquer, que inicia com $u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, permanece em L^1 e mostramos que $\|u(\cdot,t)\|_{L^1(\mathbb{R})}$ decresce. Em seguida obtemos a propriedade da conservação de massa e mostramos que se $u(x,t)$ é solução de (2.31), então $u(x,t)$ é uma função contínua em $L^1_{loc}(\mathbb{R})$.

Lema 2.1. *Seja $u(x,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$ solução suave não negativa do problema de valor inicial*

$$\begin{cases} u_t + (b(x,t)u^{k+1})_x = \mu(t)(u^\alpha u_x)_x, \quad \alpha > 0 \ e \ \kappa > 0 \\ \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}); \end{cases} \tag{2.32}$$

onde $\alpha > 0$ e $\kappa \geq 0$ são quaisquer. Então $u(\cdot,t) \in L^1(\mathbb{R})$, $\forall 0 \leq t < T_*$ e

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}, \qquad \forall 0 \leq t < T_*. \tag{2.33}$$

Prova: <u>Caso I</u>: Seja $u(x,t) > 0$ solução suave positiva. Sejam $\epsilon > 0$ dado e $R > 0$. Considere a função de corte $\zeta_R(x)$ definida em (2.1). Mutiplicamos a equação

$$u_t + (b(x,t)u^{k+1})_x = \mu(t)(u^\alpha u_x)_x, \tag{2.34}$$

por $\zeta_R(x)$ e integramos em $\mathbb{R} \times [0,t]$, com $t \in (0,T]$. Pelos Teoremas de Fubini e de Integração por Partes, obtemos

$$\int_{|x|\leq R} \zeta_R(x)u(x,t)dx = \int_{|x|\leq R} \zeta_R(x)u_0(x)dx + \int_0^t \int_{|x|\leq R} \zeta_R'(x)b(x,\tau)u^{\kappa+1}(x,\tau)dxd\tau$$
$$- \int_0^t \mu(\tau) \int_{|x|\leq R} \zeta_R'(x)u^\alpha(x,\tau)u_x(x,\tau)dxd\tau$$

Como $u^\alpha u_x = \dfrac{\partial}{\partial x}\dfrac{u^{\alpha+1}}{\alpha+1}$, usamos novamente o Teorema de Integração por Partes na última integral. Aplicando as propriedades da função modular e as estimativas (2.15) e (2.2), a última igualdade fica

$$\begin{aligned} \int_{|x|<R} \zeta_R(x)u(x,t)dx \leq & \int_{|x|<R} \zeta_R(x)u_0(x)dx+ \\ & + \int_0^t \int_{|x|<R} |\zeta_R'(x)||b(x,\tau)|u^{\kappa+1}(x,\tau)dxd\tau \\ & + \frac{1}{\alpha+1}\int_0^t \mu(\tau) \int_{|x|<R} |\zeta_R''(x)|u^{\alpha+1}(x,\tau)dxd\tau \leq \end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_{|x|<R} \zeta_R(x) u_0(x) dx + \\
& + \varepsilon \mathfrak{b}(T) \int_0^t \int_{|x|<R} e^{-\varepsilon\sqrt{1+x^2}} u^{\kappa+1}(x,\tau) dx d\tau \\
& + \frac{3\varepsilon^2}{\alpha+1} \int_0^t \mu(\tau) \int_{|x|<R} e^{-\varepsilon\sqrt{1+x^2}} u^{\alpha+1}(x,\tau) dx d\tau.
\end{aligned}
$$

Decorre de (2.19), ao fazermos $R \to \infty$,

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-\varepsilon\sqrt{1+x^2}} u(x,t) dx \leq & \int_{\mathbb{R}} e^{-\varepsilon\sqrt{1+x^2}} u_0(x) dx + \\
& + \int_0^t \left[\varepsilon \mathfrak{b}(T) M(T)^\kappa + \frac{\varepsilon^2 C}{\alpha+1} M(T)^\alpha \mu(\tau) \right] \int_{\mathbb{R}} e^{-\varepsilon\sqrt{1+x^2}} u(x,\tau) dx d\tau
\end{aligned}
$$

Pelo Teorema de Gronwall , obtemos

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-\varepsilon\sqrt{1+x^2}} u(x,t) dx \leq & \left(\int_{\mathbb{R}} e^{-\varepsilon\sqrt{1+x^2}} u_0(x) dx \right) \cdot \\
& \cdot \exp \left[\varepsilon \mathfrak{b}(T) M(T)^\kappa t + \frac{\varepsilon^2 C}{\alpha+1} M(T)^\alpha \int_0^t \mu(\tau) d\tau \right]
\end{aligned}
$$

Assim, ao $\varepsilon \to 0$, temos

$$
\int_{\mathbb{R}} u(x,t) dx \leq \int_{\mathbb{R}} u_0(x) dx < \infty.
$$

Ou seja,

$$
\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}, \quad \forall 0 < t < T_*.
$$

Portanto, as soluções suaves positivas $u(x,t)$ decrescem em $L^1(\mathbb{R})$, para todo $\forall\, t \in (0, T_*)$.

Caso II: $u(\cdot,t) \geq 0$ e limitada em $[0,T]$.
Considere $\varphi \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\varphi(x) > 0$, $\forall x \in \mathbb{R}$. Seja $\epsilon > 0$ dado e considere

$$
u_0^{[\epsilon]} := u_0 + \epsilon\varphi
$$

e

$$
u^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([0, T_*^{[\epsilon]}[, L^\infty(\mathbb{R})), \quad u^{[\epsilon]}(\cdot,t) > 0
$$

uma solução (clássica, positiva) de

$$
\begin{cases}
u_t + (b(x,t) u^{\kappa+1})_x = (u^\alpha u_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\
u(\cdot,0) = u_0 + \epsilon\varphi
\end{cases}
$$

Para $\epsilon > 0$ suficientemente pequeno, temos que $u^{[\epsilon]}(\cdot,t)$ está definida e limitada $\forall t \in [0,T]$. Pelo Caso I, considerado acima, temos

$$u^{[\epsilon]}(\cdot,t) \in L^1(\mathbb{R}), \qquad \forall\, t \in [0, T_*[,$$

com

$$\left\| u^{[\epsilon]}(\cdot,t) \right\|_{L^1(\mathbb{R})} \leq \left\| u_0^{[\epsilon]} \right\|_{L^1(\mathbb{R})}, \qquad \forall\, t \in [0,T]. \tag{2.35}$$

Como vale a propriedade da comparação

$$\left.\begin{array}{r} u(\cdot,0) \leq v(\cdot,0) \\ v(\cdot,0) > 0 \end{array}\right\} \Longrightarrow u(\cdot,t) \leq v(\cdot,t), \qquad \forall\, t \in [0,T] \tag{2.36}$$

temos

$$0 \leq u(\cdot,t) \leq u^{[\epsilon]}(\cdot,t), \qquad \forall\, t \in [0,T]. \tag{2.37}$$

Em particular, segue de (2.35) que

$$\| u(\cdot,t) \|_{L^1(\mathbb{R})} \leq \| u_0 \|_{L^1(\mathbb{R})}, \qquad \forall\, t \in [0, T_*[,$$

$\blacksquare$

Lema 2.2. *Seja* $u(x,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$ *solução suave positiva do problema de valor inicial* (2.32) *com* $\alpha > 0$ *e* $\kappa \geq 0$ *quaisquer. Então* $u(\cdot,t)$ *satisfaz a propriedade da conservação de massa e*

$$\| u(\cdot,t) \|_{L^1(\mathbb{R})} = \| u_0 \|_{L^1(\mathbb{R})}, \qquad \forall\, 0 \leq t < T_*. \tag{2.38}$$

Prova: Sejam $R > 0$ qualquer e $\psi(x)$ e $\psi_R(x)$ as funções de corte definidas em (2.3) e (2.4), respectivamente. Mutiplicamos a equação (2.34) por $\psi_R(x)$ e integramos em $\mathbb{R} \times [0,t]$, com $t \in (0,T]$. Pelos Teoremas de Fubini e de Integração por Partes, obtemos

$$\begin{aligned}
\int_{|x|\leq R+1} \psi_R(x) u(x,t) dx &= \int_{|x|\leq R+1} \psi_R(x) u_0(x) dx + \\
&\quad + \int_0^t \int_{R\leq|x|\leq R+1} \psi_R'(x) b(x,\tau) u^{\kappa+1}(x,\tau) dx d\tau \\
&\quad - \int_0^t \mu(\tau) \int_{R\leq|x|\leq R+1} \psi_R'(x) u^{\alpha}(x,\tau) u_x(x,\tau) dx d\tau \\
&= \int_{|x|\leq R+1} \psi_R(x) u_0(x) dx + \\
&\quad + \int_0^t \int_{R\leq|x|\leq R+1} \psi_R'(x) b(x,\tau) u^{\kappa+1}(x,\tau) dx d\tau \\
&\quad + \frac{1}{\alpha+1} \int_0^t \mu(\tau) \int_{R\leq|x|\leq R+1} \psi_R''(x) u^{\alpha+1}(x,\tau) dx d\tau
\end{aligned}$$

Do Lema (2.1), das hipótese sobre $b(x,t)$ e $\mu(t)$, (2.15) e (2.18), e do Teorema da Convergência Dominada, ao $R \to +\infty$, temos

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx$$

Ou seja, ao $\epsilon \searrow 0$

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}$$

Portanto, as soluções suaves positivas da equação (2.32) têm a propriedade da conservação de massa.

■

Utilizando o método de simulações numéricas denominado Método Leapfrog semi-implícito, veja [12], e o software computacional Matlab, obtemos o comportamento das soluções em L^1 do problema

$$\begin{cases} u_t + (b(x,t)u^{\kappa+1})_x = (u^\alpha u_x)_x \\ u(\cdot,0) = u_0(x) \end{cases} \tag{2.39}$$

para os valores de $\kappa > 0$ e $\alpha > 0$ satisfazendo cada uma das relaçoes consideradas neste trabalho, ou seja, $\kappa = \frac{\alpha}{2}$, $\kappa > \frac{\alpha}{2}$ e $0 \leq \kappa < \frac{\alpha}{2}$. Consideramos como fluxo as funções $b(x,t) = -\sin(x)$, $b(x,t) = -\cos(x)$ e $b(x,t) = -\tanh(x)$. Observe que o caso mais interessante é quando consideramos $b(x,t) = -\tanh(x)$, pois $b_x(x,t) < 0$. Como condição inicial tomamos

$$u_0(x) = \begin{cases} x+5, \ -5 \leq x \leq -4; \\ 1, \ -4 \leq x \leq 4; \\ 5-x, \ 4 \leq x \leq 5. \end{cases} \tag{2.40}$$

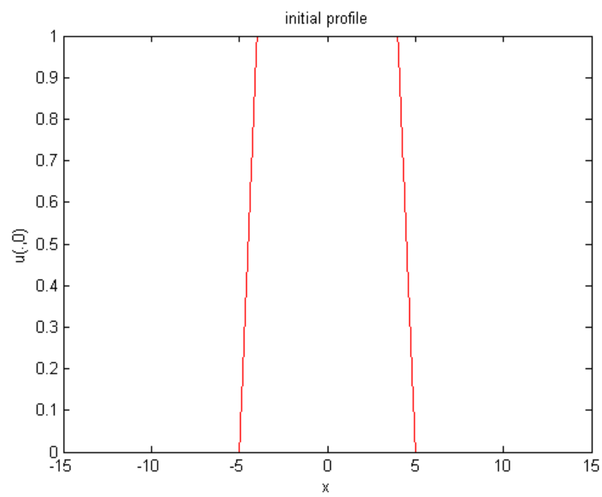

Figura 2: Perfil Inicial

Também nas simulações podemos observar a conservação de massa das soluções de (2.39), com condição inicial dada por (2.40) descritas acima. Em todos os casos particulares considerados, a saber, $\kappa = 1$ e $\alpha = 2$, $\kappa = 4$ e $\alpha = 1$, $\kappa = 1$ e $\alpha = 4$, com os fluxos discriminados acima, obtemos o mesmo resultado, que é

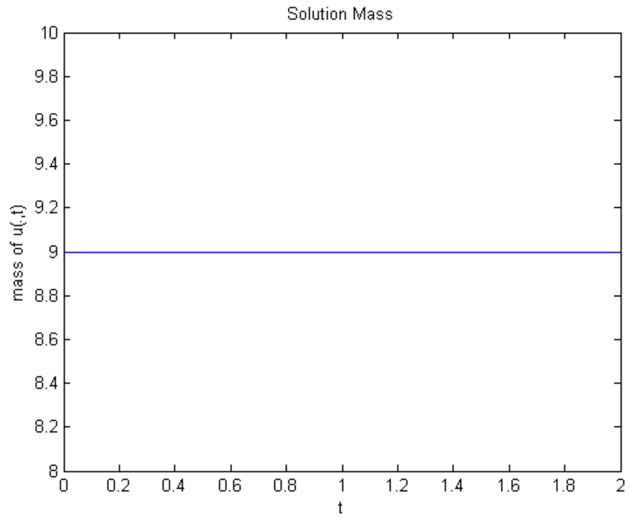

Figura 3: Massa de u(x,t)

Vemos que as simulações condizem com as estimativas encontradas no texto. Seguimos agora com a prova de que as soluções de (2.32) são contínuas em L^1.

Lema 2.3. *Seja* $u(x,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$ *solução suave positiva do problema de Cauchy* (2.31). *Então,*

$$u(x,t) \in C^0([0,T_*[, L^1(\mathbb{R})).$$

Prova: Sejam $\varepsilon > 0$, $R_1 > 0$ e $R_2 > 1$ e as funções de corte $\xi_{R_1,R_2}(x)$ e $\xi_R(x)$ definidas em (2.7) e (2.10), respecticamente. Usaremos a notação dada em (2.11) para representar a região de integração. Multiplicamos a equação (2.34) por $\xi_{R_1,R_2}(x)$ e integramos em $[t_0,t] \times \mathbb{R}$. Com isso, obtemos

$$\int_{t_0}^{t} \int_{A_{R_1,R_2}} u_\tau(x,\tau)\xi_{R_1,R_2}(x)dxd\tau + \int_{t_0}^{t} \int_{A_{R_1,R_2}} (b(x,\tau)u^{\kappa+1}(x,\tau))_x \xi_{R_1,R_2}(x)dxd\tau =$$
$$= \int_{t_0}^{t} \mu(\tau) \int_{A_{R_1,R_2}} (u^\alpha(x,\tau)u_x(x,\tau))_x \xi_{R_1,R_2}(x)dxd\tau.$$

Pelos Teoremas de Fubini e de Integração por Partes, temos

$$\int_{A_{R_1,R_2}} \xi_{R_1,R_2}(x)\left[u(x,t) - u(x,t_0)\right]dx - \int_{t_0}^{t} \int_{A'_{R_1,R_2}} b(x,\tau)u^{\kappa+1}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau =$$
$$= -\int_{t_0}^{t} \mu(\tau) \int_{A'_{R_1,R_2}} \frac{\partial}{\partial x}\left[\frac{u^{\alpha+1}}{\alpha+1}(x,\tau)\right]\xi'_{R_1,R_2}(x)dxd\tau$$

Reorganizando os termos e aplicando novamente o Teorema de Integração por Partes na última integral, temos

$$\begin{aligned}\int_{A_{R_1,R_2}} \xi_{R_1,R_2}(x)u(x,t)dx = &\int_{A_{R_1,R_2}} \xi_{R_1,R_2}(x)u(x,t_0)dx+\\ &+\int_{t_0}^{t}\int_{A'_{R_1,R_2}} b(x,\tau)u^{\kappa+1}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau+\\ &+\frac{1}{\alpha+1}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{\alpha+1}(x,\tau)\xi''_{R_1,R_2}(x)dxd\tau,\end{aligned} \tag{2.41}$$

onde A_{R_1,R_2} e A'_{R_1,R_2} são dados em (2.11). Pelas propriedades da função módulo aplicadas em (2.41) e por (2.8), (2.9), (2.15) e considerando que $|\xi_{R_1,R_2}| \leq 1$, temos

$$\begin{aligned}\int_{A_{R_1,R_2}} \xi_{R_1,R_2}(x)u(x,t)dx \leq &\int_{A_{R_1,R_2}} \xi_{R_1,R_2}(x)u(x,t_0)dx+\\ &+\mathfrak{b}(T)M_1\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u^{\kappa+1}(x,\tau)dxd\tau+\frac{M_2}{\alpha+1}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{\alpha+1}(x,\tau)dxd\tau.\end{aligned}$$

Como $u(x,t)$ é localmente limitada por hipótese, tomamos $R_3 = R_3(\varepsilon) > 0$ suficientemente grande tal que

$$\int_{A_{R_1,R_2}} \xi_{R_1,R_2}(x)u(x,t_0)dx \leq \int_{|x|\geq R_3} \xi_{R_1,R_2}(x)u(x,t_0)dx < \frac{\varepsilon}{3}. \tag{2.42}$$

Por (2.19) existe $R_4 = R_4(\varepsilon)$ suficientemente grande tal que

$$\int_{t_0}^{t}\int_{|x|\geq R_4} u(x,\tau)dxd\tau < \frac{1}{3}\frac{\varepsilon}{M_1\mathfrak{b}(T)(\|u_0\|^{\kappa}_{L^\infty(\mathbb{R})}+1)}. \tag{2.43}$$

Sabemos que $\mu(t)$ é contínua, logo existe $M_3 > 0$ tal que $\displaystyle\int_{t_0}^{t}\mu(\tau)d\tau \leq M_3$. Decorre dessa observação e de (2.19) que existe $R_5 = R_5(\varepsilon)$ suficientemente grande tal que

$$\int_{t_0}^{t}\int_{|x|\geq R_5} u(x,\tau)dxd\tau < \frac{1}{3}\frac{\varepsilon}{\frac{M_2}{\alpha+1}\mathfrak{b}(T)(\|u_0\|^{\alpha}_{L^\infty(\mathbb{R})}+1)(M_3+1)}. \tag{2.44}$$

Definindo $R_0 = \max\{R_3, R_4, R_5\}$, temos por (2.42), (2.43) e (2.44)

$$\int_{|x|\geq R} \xi_{R_1,R_2}(x)u(x,t)dx \leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon,$$

para todo $R > R_0$ e $R_2 > 1$. Ao fazer $R_2 \to +\infty$, usamos o Teorema da Convergência Dominda e obtemos

$$\int_{|x|>R} \xi_R(x)u(x,t)dx \leq \varepsilon, \quad \forall\, R \geq R_0 \quad \text{e}\ \forall\, 0 < t \leq T.$$

Logo,

$$\int_{|x|\geq R+1} \xi_R(x)u(x,t)dx \leq \int_{|x|\geq R} \xi_R(x)u(x,t)dx \leq \varepsilon, \quad \forall\, R \geq R_0 \quad \text{e}\ \forall\, 0 < t \leq T.$$

Portanto, pela definição de $\xi_R(x)$ dada em (2.10), temos

$$\int_{|x|>R+1} u(x,t)dx \leq \varepsilon, \quad \forall\, R \geq R_0 \quad \text{e} \quad \forall\, 0 < t \leq T. \tag{2.45}$$

Considere $B_{R_0+1} := B_{R_0+1}(0)$ como sendo a bola aberta centrada na origem e raio $R_0 + 1$. Logo,

$$\|u(\cdot,t) - u(\cdot,t_0)\|_{L^1(\mathbb{R})} \leq$$

$$\leq \|u(\cdot,t) - u(\cdot,t_0)\|_{L^1(\overline{B}_{R_0+1})} + \|u(\cdot,t) - u(\cdot,t_0)\|_{L^1(B^c_{R_0+1})} \quad \leq \quad 3\varepsilon.$$

De fato, $u \in C^0(\mathbb{R}\times]0,+\infty[)$ e então, ao $t \to t_0$, $\|u(\cdot,t) - u(\cdot,t_0)\|_{L^1(\overline{B}_{R_0+1})} \to 0$. Por outro lado, por (2.45), temos

$$\|u(\cdot,t) - u(\cdot,t_0)\|_{L^1(B^c_{R_0+1})} \leq \|u(\cdot,t)\|_{L^1(B^c_{R_0+1})} + \|u(\cdot,t_0)\|_{L^1(B^c_{R_0+1})} \leq 2\varepsilon.$$

Assim, para $|t - t_0| < \eta$, para $\eta > 0$ suficientemente pequeno, vale

$$\|u(\cdot,t) - u(\cdot,t_0)\|_{L^1(\mathbb{R})} \leq 3\varepsilon,$$

ou seja, $u(\cdot,t) \in C^0(]0,T_*[, L^1(\mathbb{R}))$. Considere agora $t_0 = 0$. Da desigualdade triangular segue que

$$\|u(\cdot,t) - u(\cdot,0)\|_{L^1(\mathbb{R})} \leq \|u(\cdot,t) - u_0\|_{L^1(B^c_{R_0+1})} + \|u(\cdot,t) - u_0\|_{L^1(\overline{B}_{R_0+1})}$$

Do perfil inicial temos $\|u(\cdot,t) - u_0\|_{L^1(\overline{B}_{R_0+1})} \to 0$, já que $u(x,t)$ é uma função integrável em conjuntos compactos. Por outro lado, por (2.45),

$$\|u(\cdot,t) - u_0\|_{L^1(B^c_{R_0+1})} \leq \|u(\cdot,t)\|_{L^1(B^c_{R_0+1})} + \|u_0\|_{L^1(B^c_{R_0+1})} \leq 2\varepsilon.$$

Portanto, $u(x,t)$ é uma solução contínua também em $t = 0$, ou seja,

$$u(\cdot,t) \in C^0([0,T_*[, L^1(\mathbb{R})).$$

$\blacksquare$

Nas próximas três seções provaremos que toda solução limitada do problema de Cauchy (2.31), em seu intervalo maximal de existência de solução $[0,T_*[$, é uma função no espaço de Sobolev L^q, para todo $q \geq p_0$ e $q > 1$.

2.4 Soluções Limitadas em L^q. Continuidade dos dados Iniciais - Caso $\kappa = \frac{\alpha}{2}$

Lema 2.4. *Seja $u(x,t)$ uma solução suave não negativa do problema*

$$\begin{cases} u_t + (b(x,t)u^{\frac{\alpha}{2}+1})_x = \mu(t)(u^\alpha u_x)_x; \\ u(x,0) = u_0(x) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}); \end{cases} \tag{2.46}$$

onde $\alpha > 0$, com $1 \leq p_0 < \infty$. Se $u(x,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$, $0 < T_* \leq \infty$, então $u(x,t) \in L^\infty_{loc}([0,T_*[, L^q(\mathbb{R}))$, $\forall q \in [p_0, +\infty)$ e $q > 1$.

Prova: Considere $0 \leq t \leq T$ fixo, onde $0 < T < T_*$ e $u(x,t)$ solução suave positiva de (2.46). Sejam $\epsilon > 0$ dado e $\zeta_R(x)$ a função de corte definida em (2.1). Considere $q > 1$. Multiplicando a equação

$$u_t + (b(x,t)u^{\frac{\alpha}{2}+1})_x = \mu(t)(u^\alpha u_x)_x, \tag{2.47}$$

por $qu^{q-1}(x,t)\zeta_R(x)$ e integrando em $\mathbb{R} \times [0,t]$, obtemos

$$\begin{aligned} &\int_0^t \int_{|x|\leq R} qu^{q-1}(x,\tau)u_\tau(x,\tau)\zeta_R(x)dxd\tau \\ &\qquad + q\int_0^t \int_{|x|\leq R} u^{q-1}(x,\tau)(b(x,\tau)u^{\frac{\alpha}{2}+1}(x,\tau))_x\zeta_R(x)dxd\tau \\ &\quad = q\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q-1}(x,\tau)(u^\alpha(x,\tau)_x(x,\tau))_x\zeta_R(x)dxd\tau. \end{aligned}$$

Devido a função de corte, a região de integração é compacta o que nos permite usar o Teorema de Fubini na primeira integral e o Teorema de Integração por Partes nas demais integrais. Observe, ainda, que $supp\,\zeta_R(x) \subset [-R,R]$, logo as integrais de fronteira são nulas. Então

$$\begin{aligned} &\int_{|x|\leq R} u^q(x,t)\zeta_R(x)dx - q(q-1)\int_0^t \int_{|x|\leq R} b(x,\tau)u^{q+\frac{\alpha}{2}-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau \\ &\qquad - q\int_0^t \int_{|x|\leq R} b(x,\tau)u^{q+\frac{\alpha}{2}}(x,\tau)\zeta_R'(x)dxd\tau \\ &\quad = \int_{|x|\leq R} u_0^q(x)\zeta_R(x)dx - \\ &\qquad - q(q-1)\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\ &\qquad - q\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha-1}(x,\tau)u_x(x,\tau)\zeta_R'(x)dxd\tau. \end{aligned}$$

Para obtermos um controle sobre o fluxo, adicionamos e subtraímos um termo com $\beta(t)$ definido em (2.13). Além disso, como $\dfrac{\partial}{\partial x}\left[\dfrac{u^{q+\alpha}}{q+\alpha}\right] = u^{q+\alpha-1}u_x$, usamos novamente o Teorema de Integração por Partes e obtivemos

$$
\begin{aligned}
&\int_{|x|\le R} u^q(x,t)\zeta_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau\\
&\quad= \int_{|x|\le R} u_0^q(x)\zeta_R(x)dx + q\int_0^t\int_{|x|\le R} b(x,\tau)u^{q+\frac{\alpha}{2}}(x,\tau)\zeta_R'(x)dxd\tau\\
&\qquad+\frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha}(x,\tau)\zeta_R''(x)dxd\tau\\
&\qquad+q(q-1)\int_0^t\int_{|x|\le R}(b(x,\tau)-\beta(\tau))u^{q+\frac{\alpha}{2}-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau\\
&\qquad+q(q-1)\int_0^t \beta(\tau)\int_{|x|\le R} u^{q+\frac{\alpha}{2}-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau.
\end{aligned}
$$

Note que $\zeta_R'(x)$ não se anula na fronteira. Além disso, pelas propriedades da função módulo e pelas estimativas (2.16) e (2.17), segue que

$$
\begin{aligned}
&\int_{|x|\le R} u^q(x,t)\zeta_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau\\
&\quad\le \int_{|x|\le R} u_0^q(x)\zeta_R(x)dx + q\mathfrak{b}(T)\int_0^t\int_{|x|\le R} u^q(x,\tau)u^{\frac{\alpha}{2}}(x,\tau)|\zeta_R'(x)|dxd\tau+\\
&\qquad+\frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{|x|\le R} u^q(x,\tau)u^\alpha(x,\tau)|\zeta_R''(x)|dxd\tau+\\
&\qquad+\frac{q}{q+\alpha}\int_0^t \mu(\tau)\Big[u^q(x,\tau)u^\alpha(x,\tau)|\zeta_R''(x)|\Big]_{|x|=R}d\tau+\\
&\qquad+q(q-1)\int_0^t B(\tau)\int_{|x|\le R} u^{q+\frac{\alpha}{2}-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau+\\
&\qquad+\frac{q(q-1)}{q+\frac{\alpha}{2}}\mathfrak{b}(T)\int_0^t\int_{|x|\le R} u^{q+\frac{\alpha}{2}}(x,\tau)|\zeta_R'(x)|dxd\tau.
\end{aligned}
\tag{2.48}
$$

Trabalharemos com o termo que envolve a derivada espacial de $u(x,t)$, pois não temos nenhuma informação sobre o comportamento de $u_x(x,t)$. Pela desigualdade de Young, segue que

$$
2B(\tau)\,u^{q+\frac{\alpha}{2}-1}\,|u_x|\,\zeta_R(x) \le u^{q+\frac{\alpha}{2}-2}\,u_x^2\,\zeta_R(x)\,\mu(\tau) + \frac{B(\tau)^2}{\mu(\tau)}\,\zeta_R(x)\,u^q.
$$

Substituindo este resultado em (2.48), adicionando os termos semelhantes e utilizando as estimativas sobre $\zeta_R'(x)$ e $\zeta_R''(x)$ dadas em (2.2), obtemos

$$
\int_{|x|\le R} u^q(x,t)\zeta_R(x)dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau
$$

$$
\begin{aligned}
\leq\ & \int_{|x|\leq R} u_0^q(x)\zeta_R(x)dx + q\mathfrak{b}(T)\epsilon\int_0^t\int_{|x|\leq R} u^{q+\frac{\alpha}{2}}(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& +\frac{3q}{q+\alpha}\epsilon\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha}(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& +\frac{3q}{q+\alpha}e^{-\epsilon\sqrt{1+R^2}}\epsilon\int_0^t \mu(\tau)\left[u^{q+\alpha}(R,\tau)-u^{q+\alpha}(-R,\tau)\right]d\tau \\
& +\frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{|x|\leq R} u^q(x,\tau)\zeta_R(x)dxd\tau \\
& +\frac{q(q-1)}{q+\frac{\alpha}{2}}\mathfrak{b}(T)\epsilon\int_0^t\int_{|x|\leq R} u^{q+\frac{\alpha}{2}}(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned}
$$

Assim, por (2.18),

$$
\begin{aligned}
& \int_{|x|\leq R} u^q(x,t)\zeta_R(x)dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\
& \quad \leq\ \int_{|x|\leq R} u_0^q(x)\zeta_R(x)dx + q\mathfrak{b}(T)M(T)^{\frac{\alpha}{2}}\epsilon\int_0^t\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& \quad\quad +\frac{3qM(T)^\alpha}{q+\alpha}\epsilon\int_0^t \mu(\tau)\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& \quad\quad +\frac{3qM(T)^\alpha}{q+\alpha}M(T)^{q+\alpha}(x,\tau)e^{-\epsilon\sqrt{1+R^2}}\epsilon\int_0^t \mu(\tau)d\tau \\
& \quad\quad +\frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{|x|\leq R} u^q(x,\tau)\zeta_R(x)dxd\tau \\
& \quad\quad +\frac{q(q-1)M(T)^{\frac{\alpha}{2}}}{q+\frac{\alpha}{2}}\mathfrak{b}(T)\epsilon\int_0^t\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned}
$$

Observe que $\frac{q}{q+\alpha}\leq 1$ e que $\frac{q}{q+\frac{\alpha}{2}}\leq 1$. Com isso, ao fazermos $R\to+\infty$, a estimativa acima fica

$$
\begin{aligned}
& \int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& \quad \leq\ \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + q\mathfrak{b}(T)M(T)^{\frac{\alpha}{2}}\epsilon\int_0^t\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& \quad\quad +3M(T)^\alpha\epsilon\int_0^t \mu(\tau)\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& \quad\quad +\frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& \quad\quad +(q-1)M(T)^{\frac{\alpha}{2}}\mathfrak{b}(T)\epsilon\int_0^t\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned}
\tag{2.49}
$$

Defina

$$S(\tau) := q\mathfrak{b}(T)M(T)^{\frac{\alpha}{2}} + 3M(T)^{\alpha}\mu(\tau) + (q-1)M(T)^{\frac{\alpha}{2}}\mathfrak{b}(T) \tag{2.50}$$

e

$$\overline{S}(\tau) := \frac{q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)} + \epsilon S(\tau). \tag{2.51}$$

Assim, com as definições (2.50) e (2.51), a desigualdade (2.49) obtém a forma

$$\begin{aligned}
\int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
\leq \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx \\
+ \int_0^t \left[\frac{q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)} + \epsilon S(\tau)\right]\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
= \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + \int_0^t \overline{S}(\tau)\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned} \tag{2.52}$$

De (2.52) segue, em particular, que

$$\int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx \leq \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + \int_0^t \overline{S}(\tau)\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau$$

e pelo Teorema de Gronwall, temos

$$\int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx \leq \left[\int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx\right]\exp\left[\int_0^t \frac{q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)} + \epsilon S(\tau)d\tau\right].$$

Ao fazermos $\epsilon \to 0$, obtemos a seguinte estimativa para $\|u(\cdot,t)\|_{L^q(\mathbb{R})}$:

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})}\exp^{\frac{1}{q}}\left(\frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}d\tau\right), \tag{2.53}$$

$\forall t \in [0,T]$, onde $0 < T < T_*$ e $0 < T_* \leq \infty$. Pela arbitrariedade de t e de T, podemos dizer que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} < +\infty, \quad \forall\, 0 \leq t \leq T < T_* \leq \infty \text{ e } \forall p_0 \leq q < \infty.$$

Com isso, conclui-se que as soluções clássicas localmente limitadas e positivas $u(x,t)$ em $[0,T_*[$ são também localmente limitadas em L^q, $\forall p_0 \leq q < \infty$, ou seja,

$$u(\cdot,t) \in L^{\infty}_{loc}([0,T_*[, L^q(\mathbb{R})), \quad \forall q > 1 \text{ e } q \in [p_0,\infty). \tag{2.54}$$

No caso em que $u(x,t)$ é solução suave não negativa, consideremos $\psi \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, com $\psi(x) > 0$, $\forall x \in \mathbb{R}$. Seja $\epsilon > 0$ dado e considere

$$u_0^{[\epsilon]} := u_0 + \epsilon\psi \tag{2.55}$$

e

$$u^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([0,T_*^{[\epsilon]}[, L^\infty(\mathbb{R})), \quad u^{[\epsilon]}(\cdot,t) > 0$$

uma solução (clássica, positiva) de

$$\begin{cases} u_t + (b(x,t)u^{\frac{\alpha}{2}+1})_x = (u^\alpha u_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\ \\ u(\cdot,0) = u_0 + \epsilon\psi \end{cases}$$

Para $0 < \epsilon \ll 1$, temos que $u^{[\epsilon]}(\cdot,t)$ está definida e limitada $\forall t \in [0,T]$. Pelo que mostramos acima, temos

$$u^{[\epsilon]}(\cdot,t) \in L^q(\mathbb{R}), \quad \forall t \in [0,T],$$

com

$$\left\| u^{[\epsilon]}(\cdot,t) \right\|_{L^q(\mathbb{R})} \leq K_q(T), \quad \forall t \in [0,T], \tag{2.56}$$

onde $K_q(T)$ é uma constante que independe de ϵ (veja(2.53)). Como vale a propriedade da comparação

$$\left. \begin{array}{c} u(\cdot,0) \leq v(\cdot,0) \\ v(\cdot,0) > 0 \end{array} \right\} \Longrightarrow u(\cdot,t) \leq v(\cdot,t), \qquad \forall\, t \in [0,T] \tag{2.57}$$

temos

$$0 \leq u(\cdot,t) \leq u^{[\epsilon]}(\cdot,t), \quad \forall t \in [0,T].$$

Em particular, segue de (2.56) que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq K_q(T), \quad \forall t \in [0,T],$$

O que conlcui o Lema (2.4).

■

Observação 2.1. *Pela desigualdade* (2.52) *e usando o fato que* $e^{-\epsilon\sqrt{1+x^2}} \leq 1$ *segue que*

$$\begin{aligned} &\frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\ &\qquad \leq \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + \int_0^t \frac{q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)}\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\ &\qquad\quad +\epsilon\int_0^t S(\tau)\int_{\mathbb{R}} u^q(x,\tau)dxd\tau \end{aligned}$$

Provamos no Lema (2.4) *acima que* $\|u(\cdot,t)\|_{L^q(\mathbb{R})} < +\infty$*. Logo, fazendo* $\epsilon \to 0$*, obtemos para todo* $0 \le t \le T < T_* \le \infty$*,*

$$\frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)dxd\tau \le$$
$$\le \|u_0\|_{L^q(\mathbb{R})} + \frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\|u(\cdot,t)\|^q_{L^q(\mathbb{R})}d\tau.$$

Como $\mu(t)$ *é uma função contínua e estritamente positiva, podemos assumir que* $\mu(t) \le C$*, para alguma constante* $C > 0$ *e* $\forall\, 0 \le t \le T$*. Logo, concluimos que*

$$\int_0^T \int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)dxd\tau < \infty, \quad \forall\, q \in [p_0,+\infty) \text{ e } q > 1,\ \forall\, T \in (0,T_*). \tag{2.58}$$

De (2.54) *e usando o fato de que* $u^{q+\alpha-2} \le M(T)^{q+\alpha-2}$*, segue que*

$$\int_0^T \|u_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}d\tau < \infty, \quad \forall\, q \in [p_0,+\infty) \text{ e } q > 1, \forall\, T \in (0,T_*).$$

Observe também que

$$\left|\int_0^t \int_{|x|\le R}(b(x,\tau)-\beta(\tau))u^{q+\frac{\alpha}{2}-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau\right| \le$$
$$\le \int_0^t B(\tau)\int_{\mathbb{R}} u^{q+\frac{\alpha}{2}-1}(x,\tau)|u_x(x,\tau)|dxd\tau \le$$
$$\le \frac{1}{2}\int_0^t B(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)dxd\tau + \frac{1}{2}\int_0^t B(\tau)\int_{|x|\le R} u^q(x,\tau)dxd\tau$$

e que as duas últimas integrais são finitas, por (2.53) *e* (2.58)*. Portanto, podemos concluir que*

$$\int_0^t \int_{\mathbb{R}} u^{q+\frac{\alpha}{2}-1}(x,\tau)|u_x(x,\tau)|\,dxd\tau \quad < \quad \infty. \tag{2.59}$$

As figuras a seguir foram obtidas através de simulações do comportamento das soluções na norma L^2 do problema (2.39), (2.40). Os fluxos considerados são $b(x,t) = -\sin(x)$, $b(x,t) = -\cos(x)$ e $b(x,t) = -\tanh(x)$ respectivamente. Abaixo seguem as figuras nos casos $\alpha = 2$ e $\kappa = 1$, que correspondem ao caso $\kappa = \dfrac{\alpha}{2}$ considerado nesta secção. Estas figuras nos dão evidencias visuais dos resultados obtidos no caso $q = 2$, ou seja, obtemos a norma $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$.

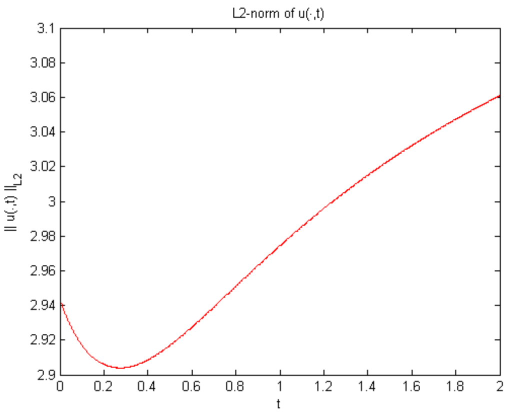

Figura 4: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$: $\alpha = 2$, $\kappa = 1$
$b(x,t) = -\sin x$

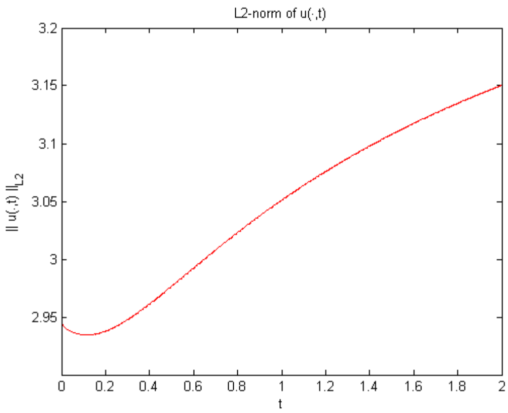

Figura 5: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$: $\alpha = 2$, $\kappa = 1$
$b(x,t) = -\cos x$

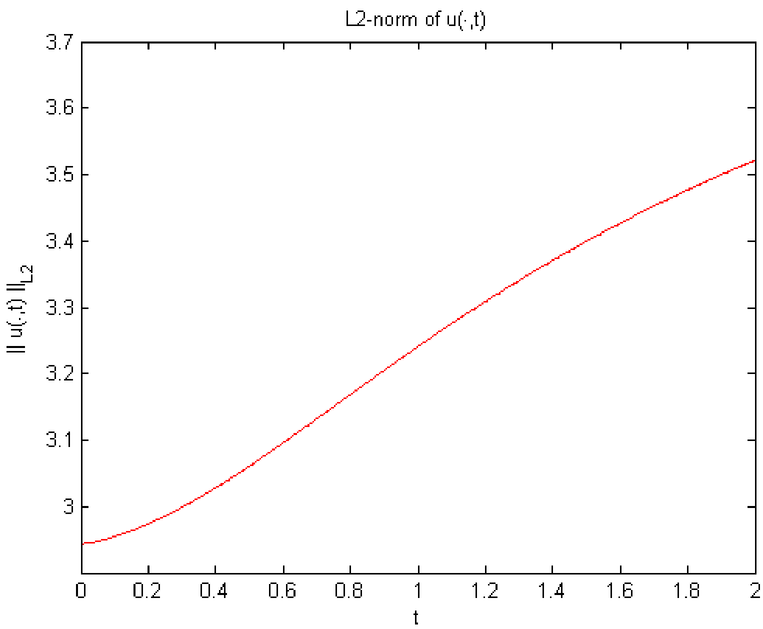

Figura 6: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$: α=2, $\kappa = 1$ $\;b(x,t) = -\tanh x$

As figuras do perfil das soluções e das interfaces em $x = -5$ e em $x = 5$ para o problema (2.39), (2.40), com $\alpha = 2$ e $\kappa = 1$ são:

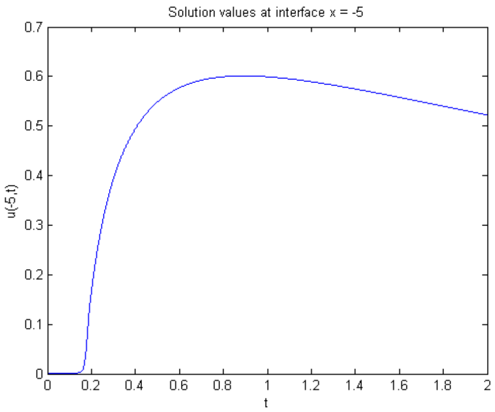

Figura 7: Interface $x = -5$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\sin x$

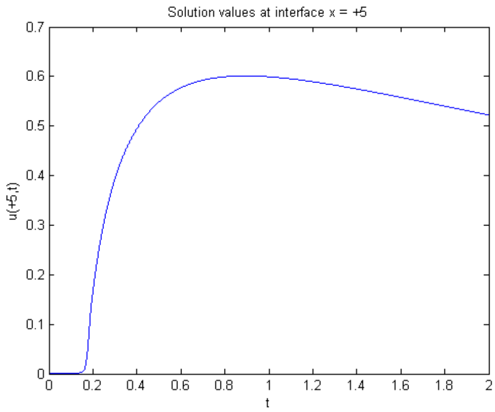

Figura 8: Interface $x = +5$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\sin x$

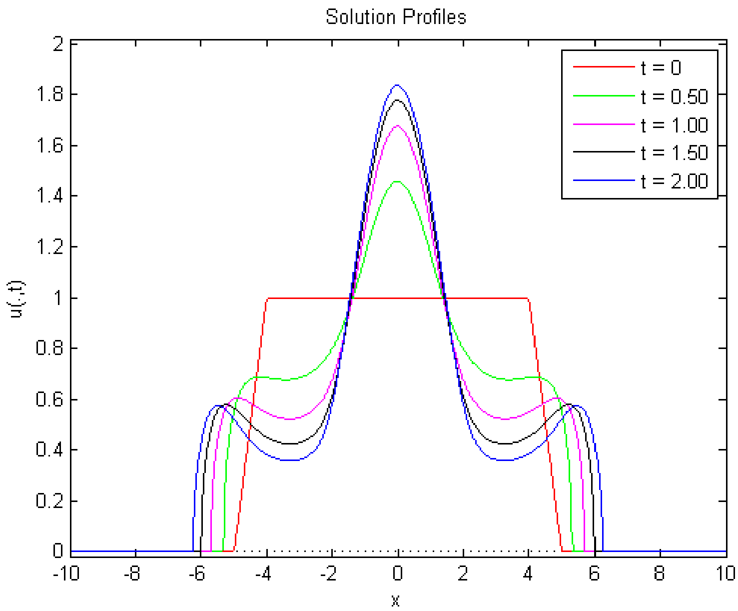

Figura 9: Perfil da Solução: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\sin x$

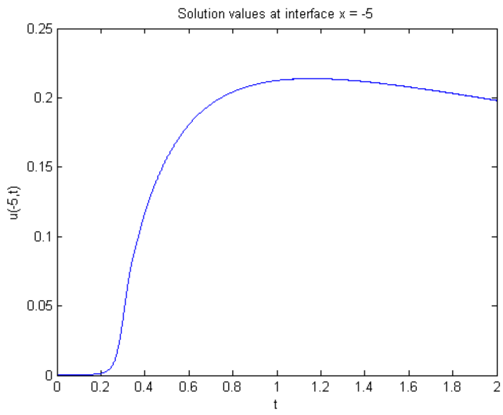

Figura 10: Interface $x = -5$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\cos x$

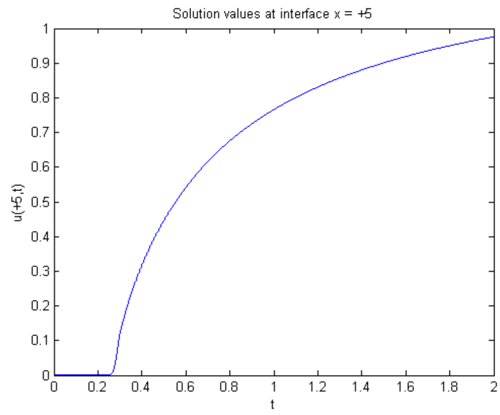

Figura 11: Interface $x = +5$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\cos x$

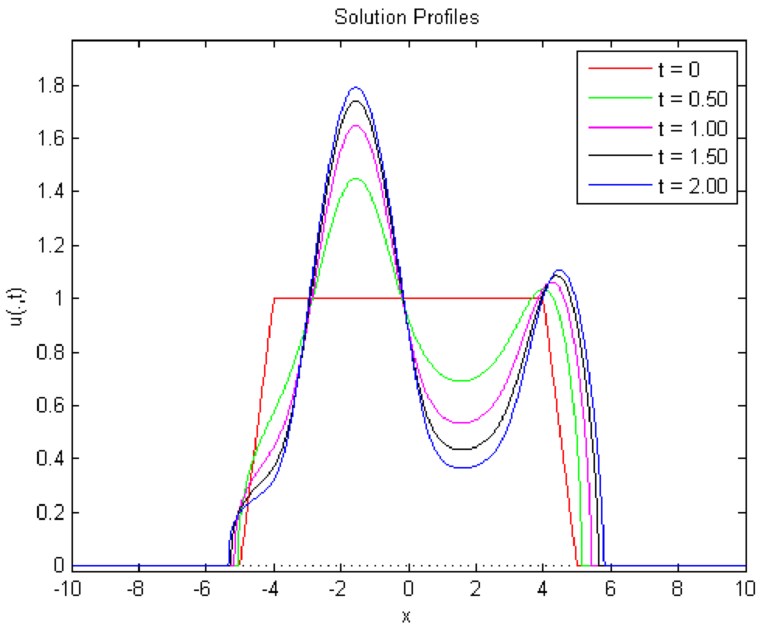

Figura 12: Perfil da Solução: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\cos x$

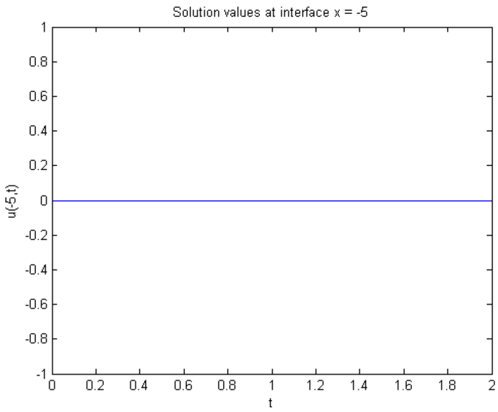

Figura 13: Interface $x = -5$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\tanh x$

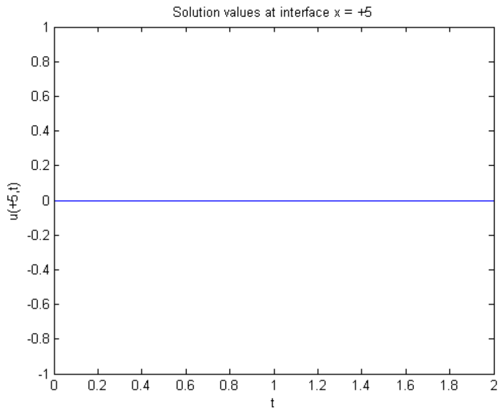

Figura 14: Interface $x = +5$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\tanh x$

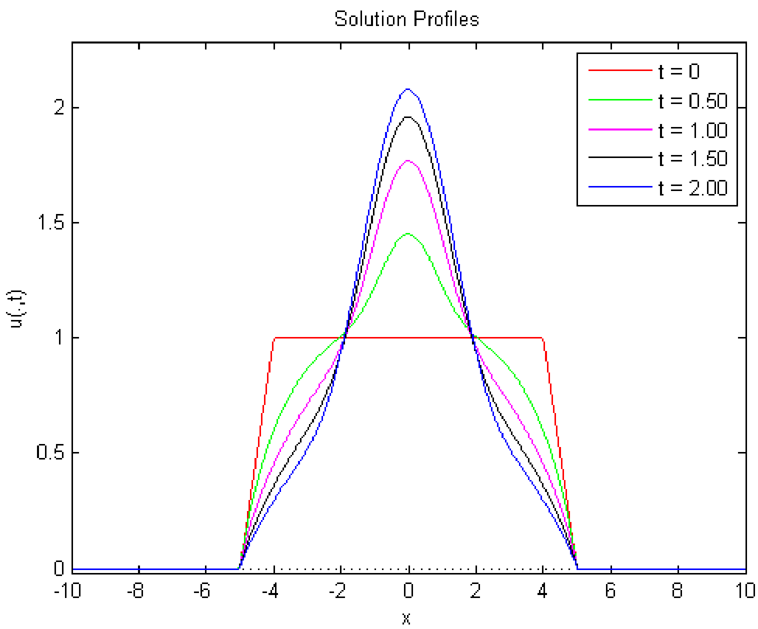

Figura 15: Perfil da Solução: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\tanh x$

Vamos mostrar agora a continuidade nos dados iniciais de uma solução positiva $u(x,t)$ de (2.46).

Lema 2.5. *Seja* $u(x,t)$ *a solução suave positiva do problema de valor inicial* (2.46). *Então,*

$$u(x,t) \in C^0([0,T_*[, L^q(\mathbb{R})), \quad \forall\, p_0 \leq q < \infty \ e\ q > 1. \tag{2.60}$$

Prova: Considere a função de corte $\xi_{R_1,R_2}(x)$ definida em (2.7), os conjuntos A_{R_1,R_2} e A'_{R_1,R_2} definidos em (2.11) e $q > 1$. Multiplicamos a equação (2.47) por $qu^{q-1}(x,t)\xi_{R_1,R_2}(x)$ e integramos em $[t_0,t] \times \mathbb{R}$. Pela compacidade da região de integração, usamos os Teoremas de Fubini de integração. Além disso, como $supp\,(\xi_{R_1,R_2}(x)) \subset [-(R+M+1), R+M+1]$, as integrais de fronteira são nulas. Assim,

$$\begin{aligned}
\int_{A_{R_1,R_2}} & u^q(x,t)\xi_{R_1,R_2}(x)dx - q\int_{t_0}^{t}\int_{A'_{R_1,R_2}} b(x,\tau)u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
&\qquad -q(q-1)\int_{t_0}^{t}\int_{A_{R_1,R_2}} b(x,\tau)u^{q+\frac{\alpha}{2}-1}(x,\tau)u_x(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \\
&= \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \\
&\qquad -q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \\
&\qquad -q\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{q+\alpha-1}(x,\tau)u_x(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau
\end{aligned}$$

Reescrevendo esta igualdade com um termo apropriado que contenha $\beta(t)$ definido em (2.13), temos

$$\begin{aligned}
\int_{A_{R_1,R_2}} & u^q(x,t)\xi_{R_1,R_2}(x)dx + q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau = \\
&= \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \\
&\quad + q\int_{t_0}^{t}\int_{A'_{R_1,R_2}} b(x,\tau)u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
&\quad + q(q-1)\int_{t_0}^{t}\int_{A_{R_1,R_2}} (b(x,\tau)-\beta(\tau))\frac{\partial}{\partial x}\left[\frac{u^{q+\frac{\alpha}{2}}(x,\tau)}{q+\frac{\alpha}{2}}\right]\xi_{R_1,R_2}(x)dxd\tau \\
&\quad + q(q-1)\int_{t_0}^{t}\beta(\tau)\int_{A_{R_1,R_2}} \frac{\partial}{\partial x}\left[\frac{u^{q+\frac{\alpha}{2}}}{q+\frac{\alpha}{2}}(x,\tau)\right]\xi_{R_1,R_2}(x)dxd\tau - \\
&\qquad -q\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} \frac{\partial}{\partial x}\left[\frac{u^{q+\alpha}}{q+\alpha}(x,\tau)\right]\xi'_{R_1,R_2}(x)dxd\tau.
\end{aligned}$$

Usamos as propriedades de módulo, as estimativas (2.15) e (2.17) e o Teorema de integração por partes em algumas integrais da identidade acima. Resultando em

$$\begin{aligned}
\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx + q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \leq \\
\leq \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \\
+q\mathfrak{b}(T)\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
-\frac{q(q-1)}{q+\frac{\alpha}{2}}\int_{t_0}^{t}B(\tau)\int_{A'_{R_1,R_2}} u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
-\frac{q(q-1)}{q+\frac{\alpha}{2}}\int_{t_0}^{t}\beta(\tau)\int_{A'_{R_1,R_2}} u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
+\frac{q}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{q+\alpha}(x,\tau)\xi''_{R_1,R_2}(x)dxd\tau.
\end{aligned}$$

Note que $q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \geq 0$. Logo,

$$\begin{aligned}
\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx \leq \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \\
+q\mathfrak{b}(T)\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
+\frac{q(q-1)}{q+\frac{\alpha}{2}}\int_{t_0}^{t}B(\tau)\int_{A'_{R_1,R_2}} u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
+\frac{q(q-1)}{q+\frac{\alpha}{2}}\int_{t_0}^{t}\beta(\tau)\int_{A'_{R_1,R_2}} u^{q+\frac{\alpha}{2}}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
+\frac{q}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{q+\alpha}(x,\tau)\xi''_{R_1,R_2}(x)dxd\tau.
\end{aligned}$$

Usando as estimativas (2.8) e (2.9) sobre as derivadas da função de corte e a estimativa (2.19) sobre $u(x,t)$, temos

$$\begin{aligned}
\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx \leq \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx+ \\
+q\mathfrak{b}(T)M_1\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{2}}\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u^q(x,\tau)dxd\tau+ \\
+\frac{q(q-1)M_1\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{2}}}{q+\frac{\alpha}{2}}\int_{t_0}^{t}B(\tau)\int_{A'_{R_1,R_2}} u^q(x,\tau)dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&-\frac{q(q-1)M_1\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{2}}}{q+\frac{\alpha}{2}}\int_{t_0}^{t}\beta(\tau)\int_{A'_{R_1,R_2}}u^q(x,\tau)dxd\tau\\
&+\frac{qM_2\|u_0\|_{L^\infty(\mathbb{R})}^{\alpha}}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}}u^q(x,\tau)dxd\tau.
\end{aligned}
$$

Seja $\epsilon > 0$ dado. Como

$$
\int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \le \int_{|x|\ge R} u^q(x,t_0)dx \;\le\; \epsilon^q,
$$

pois $u(x,t)$ é limitada em $L^q(\mathbb{R})$, logo integrável em $L^q(\mathbb{R})$. Seja $R_3 = R_3(\epsilon) > 0$ suficientemente grande tal que

$$
\int_{|x|\ge R_3} u^q(x,t_0)dx \;\le\; \frac{\epsilon^q}{4}.
$$

Seja $R_4 = R_4(\epsilon) > 0$ suficientemente grande tal que

$$
\int_{t_0}^{T}\int_{|x|\ge R_4} u^q(x,\tau)dxd\tau \;\le\; \frac{\epsilon^q}{4[qM_1(\mathfrak{b}(T)+1)(\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{2}}+1)]}.
$$

Seja $R_5 = R_5(\epsilon) > 0$ suficientemente grande tal que

$$
\int_{t_0}^{T}\int_{|x|\ge R_5} u^q(x,\tau)dxd\tau \;\le\; \frac{\epsilon^q}{4\left[\frac{q(q-1)}{q+\frac{\alpha}{2}}M_1(\mathfrak{b}(T)+1)(\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\alpha}{2}}+1)\right]}
$$

Considere $C_1 > 0$ uma constante suficientemente grande tal que $\displaystyle\int_0^T \mu(\tau)d\tau \le C_1$ e seja $R_6 = R_6(\epsilon) > 0$ suficientemete grande tal que

$$
\int_{t_0}^{T}\int_{|x|\ge R_6} u^q(x,\tau)dxd\tau \;\le\; \frac{\epsilon^q}{4\left[\frac{qM_2C_1(\|u_0\|_{L^\infty(\mathbb{R})}^{\alpha}+1)}{q+\alpha}\right]}.
$$

Definindo $R_0 := \max\{R_3, R_4, R_5, R_6\}$, temos $\forall\, R \ge R_0$,

$$
\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx \;\le\; \epsilon^q, \qquad \forall\, R_2 > 1.
$$

Logo, pelo Teorema da convergência dominada, ao $R_2 \to +\infty$, obtemos

$$
\int_{|x|\ge R} v^q(x,t)\xi_R(x)dx \;\le\; \epsilon^q, \qquad \forall\, R \ge R_0.
$$

Portanto, para $\epsilon > 0$ qualquer, existe R_0 suficientemente grande tal que

$$
\int_{|x|\ge R_0+1} u^q(x,t)dx \;\le\; \epsilon^q, \qquad \forall\, t_0 \le t < T. \tag{2.61}
$$

Assim, para $t_0 \in (0,T)$ e R_0 dado acima, temos

$$
\begin{aligned}
\|u(\cdot,t)-u(\cdot,t_0)\|_{L^q(\mathbb{R})} &\leq \|u(\cdot,t)-u(\cdot,t_0)\|_{L^q(\overline{B}_{R_0+1})}+\|u(\cdot,t)-u(\cdot,t_0)\|_{L^q(B^c_{R_0+1})}\\
&\leq \frac{\epsilon}{3}+\|u(\cdot,t)\|_{L^q(B^c_{R_0+1})}+\|u(\cdot,t_0)\|_{L^q(B^c_{R_0+1})}\\
&\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}
$$

Portanto, $u(x,t)$ é contínua em t_0, para $0<t_0<T$. Se $t_0=0$, usamos a desigualdade triangular e obtemos

$$
\begin{aligned}
\|u(\cdot,t)-u_0\|_{L^q(\mathbb{R})} &\leq \|u(\cdot,t)-u_0\|_{L^q(\overline{B}_{R_0+1})}+\|u(\cdot,t)-u_0\|_{L^q(B^c_{R_0+1})}\\
&\leq \|u(\cdot,t)-u_0\|^{1-\frac{1}{p}}_{L^\infty(\overline{B}_{R_0+1})}\|u(\cdot,t)-u_0\|^{\frac{1}{p}}_{L^1(\overline{B}_{R_0+1})}+2\epsilon\\
&\leq C\epsilon+2\epsilon,
\end{aligned}
$$

para $|t-t_0|\leq\eta$, para algum $\eta>0$ suficientemente pequeno e C uma constante positiva. Portanto, $u(x,t)$ é contínua em t, $\forall\, t\in[0,T[$.

■

2.5 Soluções Limitadas em L^q. Continuidade dos dados Iniciais - Caso $\kappa>\dfrac{\alpha}{2}$.

Lema 2.6. *Seja $u(\cdot,t)\in L^\infty_{loc}([0,T_*[,L^\infty(\mathbb{R}))$ uma solução suave não negativa do problema*

$$
\begin{cases}
u_t+(b(x,t)u^{\kappa+1})_x=\mu(t)(u^\alpha u_x)_x;\\
u(x,0)=u_0(x)\in L^{p_0}(\mathbb{R})\cap L^\infty(\mathbb{R}), \quad 1\leq p_0<\infty,
\end{cases}
\tag{2.62}
$$

onde $\alpha>0$, $\kappa>\dfrac{\alpha}{2}$. Se $u\in L^\infty_{loc}([0,T_*[,L^\infty(\mathbb{R}))$, onde $0<T_*\leq\infty$, então

$$
u\in L^\infty_{loc}([0,T_*[,L^q(\mathbb{R})), \qquad \forall\, q\in[p_0,+\infty) \quad e \quad q>1. \tag{2.63}
$$

Prova: Consideremos inicialmente $u(x,t)$ solução positiva de (4.1). Sejam $0<T<T_*$ qualquer e $0\leq t\leq T$ fixo. Sejam, ainda, $\epsilon>0$ dado, $\zeta_R(x)$ a função de corte definida por (2.1) e $q>1$. Multiplicando a equação

$$
u_t+(b(x,t)u^{\kappa+1})_x=\mu(t)(u^\alpha u_x)_x \tag{2.64}
$$

por $qu^{q-1}(x,t)\zeta_R(x)$, integrando em $\mathbb{R}\times[0,t]$, usando os Teoremas de Fubini e de Integração por Partes (pois *supp* $(\zeta_R(x))\subset[-R,R]$, o que anula as integrais de fronteira) e reorganizamos as integrais o que produz

$$
\begin{aligned}
\int_{|x|\le R} u^q(x,t)\zeta_R(x)dx - q(q-1)\int_0^t\int_{|x|\le R} & b(x,\tau)u^{q+\kappa-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau \\
& -q\int_0^t\int_{|x|\le R} b(x,\tau)u^{q+\kappa}(x,\tau)\zeta_R'(x)dxd\tau \\
= & \int_{|x|\le R} u_0^q(x)\zeta_R(x)dx \\
& -q(q-1)\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\
& -q\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-1}(x,\tau)u_x(x,\tau)\zeta_R'(x)dxd\tau
\end{aligned}
$$

Para obtermos um controle sobre o fluxo, adicionamos e subtraímos um termo com $\beta(t)$, o qual está definido em (2.13). Obtemos

$$
\begin{aligned}
\int_{|x|\le R} u^q(x,t)\zeta_R(x)dx &+ q(q-1)\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\
= & \int_{|x|\le R} u_0^q(x)\zeta_R(x)dx + q\int_0^t\int_{|x|\le R} b(x,\tau)u^{q+\kappa}(x,\tau)\zeta_R'(x)dxd\tau \\
& -q\int_0^t \mu(\tau)\int_{|x|\le R}\frac{\partial}{\partial x}\left[\frac{u^{q+\alpha}(x,\tau)}{q+\alpha}\right]\zeta_R'(x)dxd\tau \\
& +q(q-1)\int_0^t\int_{|x|\le R}(b(x,\tau)-\beta(\tau))u^{q+\kappa-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau \\
& +q(q-1)\int_0^t \beta(\tau)\int_{|x|\le R}\frac{\partial}{\partial x}\left[\frac{u^{q+\kappa}(x,\tau)}{q+\kappa}\right]\zeta_R(x)dxd\tau
\end{aligned}
\tag{2.65}
$$

Note que $\zeta_R'(x)$ não anula-se na fronteira $|x|=R$. Além disso, pelas propriedades da função módulo, por (2.16) e (2.17), segue que

$$
\begin{aligned}
\int_{|x|\le R} u^q(x,t)\zeta_R(x)dx &+ q(q-1)\int_0^t \mu(\tau)\int_{|x|\le R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \le \\
\le & \int_{|x|\le R} u_0^q(x)\zeta_R(x)dx + q\mathfrak{b}(T)\int_0^t\int_{|x|\le R} u^q(x,\tau)u^\kappa(x,\tau)|\zeta_R'(x)|dxd\tau + \\
& +\frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{|x|\le R} u^q(x,\tau)u^\alpha(x,\tau)|\zeta_R''(x)|dxd\tau + \\
& +\frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{|x|=R} u^q(x,\tau)u^\alpha(x,\tau)|\zeta_R''(x)|dxd\tau + \\
& +q(q-1)\int_0^t B(\tau)\int_{|x|\le R} u^{q+\kappa-1}(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau \\
& +\frac{q(q-1)}{q+\kappa}\mathfrak{b}(T)\int_0^t\int_{|x|\le R} u^{q+\kappa}(x,\tau)|\zeta_R'(x)|dxd\tau.
\end{aligned}
\tag{2.66}
$$

Da desigualdade de Young, temos

$$2B(\tau)\, u^{q+\kappa-1}\, |u_x|\, \zeta_R(x) \leq u^{q+\alpha-2}\, |u_x|^2\, \zeta_R(x)\, \mu(\tau) + \frac{B(\tau)^2}{\mu(\tau)}\, \zeta_R(x)\, u^{q+2\kappa-\alpha}. \quad (2.67)$$

Logo, aplicando (2.18), (2.2) e (2.67) em (2.66), temos

$$\begin{aligned}
&\int_{|x|\leq R} u^q(x,t)\zeta_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\
&\quad\leq \int_{|x|\leq R} u_0^q(x)\zeta_R(x)dx + q\mathfrak{b}(T)M(T)^\kappa\epsilon\int_0^t\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
&\qquad+\frac{3qM(T)^\alpha}{q+\alpha}\epsilon\int_0^t \mu(\tau)\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
&\qquad+\frac{3qM(T)^\alpha}{q+\alpha}M(T)^{q+\alpha}(x,\tau)e^{-\epsilon\sqrt{1+R^2}}\epsilon\int_0^t \mu(\tau)d\tau \\
&\qquad+\frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\
&\qquad+\frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{|x|\leq R} u^{q+2\kappa-\alpha}(x,\tau)\zeta_R(x)dxd\tau \\
&\qquad+\frac{q(q-1)M(T)^\kappa}{q+\frac{\alpha}{2}}\mathfrak{b}(T)\epsilon\int_0^t\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned}$$

Assim, somando os termos semelhantes desta desigualdade e aplicando (2.18) e (2.19), segue que

$$\begin{aligned}
&\int_{|x|\leq R} u^q(x,t)\zeta_R(x)dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{|x|\leq R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\zeta_R(x)dxd\tau \\
&\quad\leq \int_{|x|\leq R} u_0^q(x)\zeta_R(x)dx + q\mathfrak{b}(T)M(T)^\kappa\epsilon\int_0^t\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
&\qquad+\frac{3qM(T)^\alpha}{q+\alpha}\epsilon\int_0^t \mu(\tau)\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
&\qquad+\frac{3qM(T)^\alpha}{q+\alpha}M(T)^{q+\alpha}(x,\tau)e^{-\epsilon\sqrt{1+R^2}}\epsilon\int_0^t \mu(\tau)d\tau \\
&\qquad+\frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{|x|\leq R} u^{q+2\kappa-\alpha}(x,\tau)\zeta_R(x)dxd\tau \\
&\qquad+\frac{q(q-1)M(T)^\kappa}{q+\kappa}\mathfrak{b}(T)\epsilon\int_0^t\int_{|x|\leq R} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned}$$

Como $\dfrac{q}{q+\alpha} \leq 1$ e $\dfrac{q}{q+\kappa} \leq 1$, fazendo $R \to +\infty$ na desigualdade acima, temos

$$\int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau$$

$$\begin{aligned}
\leq\ & \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + q\mathfrak{b}(T)M(T)^\kappa\epsilon\int_0^t\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& +3M(T)^\alpha\epsilon\int_0^t \mu(\tau)\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& +\frac{q(q-1)M(T)^{2\kappa-\alpha}}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
& +(q-1)M(T)^\kappa\mathfrak{b}(T)\epsilon\int_0^t\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned} \tag{2.68}$$

Definimos

$$\begin{aligned}
S(\tau) &:= q\mathfrak{b}(T)M(T)^\kappa + 3M(T)^\alpha\mu(\tau) + (q-1)M(T)^\kappa\mathfrak{b}(T) \\
\overline{S}(\tau) &:= \frac{M(T)^{2\kappa-\alpha}q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)} + \epsilon S(\tau).
\end{aligned} \tag{2.69}$$

Assim, a desigualdade (2.68) fica

$$\begin{aligned}
& \int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
\leq\ & \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + \\
& +\int_0^t\left[\frac{q(q-1)M(T)^{2\kappa-\alpha}}{2}\frac{B(\tau)^2}{\mu(\tau)} + \epsilon S(\tau)\right]\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau \\
=\ & \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + \int_0^t \overline{S}(\tau)\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau.
\end{aligned} \tag{2.70}$$

Em particular, segue de (2.70)

$$\int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx \leq \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx + \int_0^t \overline{S}(\tau)\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau. \tag{2.71}$$

Aplicando o Teorema de Gronwall em (2.71)

$$\begin{aligned}
\int_{\mathbb{R}} u^q(x,t)e^{-\epsilon\sqrt{1+x^2}}dx \leq & \left[\int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx\right]\cdot \\
& \cdot\exp\left[\int_0^t \frac{M(T)^{2\kappa-\alpha}q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)} + \epsilon S(\tau)d\tau\right].
\end{aligned} \tag{2.72}$$

Fazendo $\epsilon \to 0$ em (2.72), obtemos

$$\|v(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|v_0\|_{L^q(\mathbb{R})} \cdot \exp^{\frac{1}{q}} \left(\frac{q(q-1)}{2} M(T)^{2\kappa-\alpha} \int_0^t \frac{B(\tau)^2}{\mu(\tau)} d\tau \right), \qquad (2.73)$$

$\forall\, t \in [0,T]$, onde $0 < T < T_*$, com $0 < T_* \leq \infty$. Pela arbitrariedade de t e de T, podemos dizer que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} < +\infty, \quad \forall\, 0 \leq t < T_* \leq \infty \qquad \text{e} \qquad \forall\, p_0 \leq q < \infty, \quad \text{com} \quad q > 1.$$

Portanto, as soluções $u(x,t)$ localmente limitadas e positivas em $[0, T_*[$ de (4.1), são também localmente limitadas em $L^q(\mathbb{R})$, $\forall\, p_0 \leq q < \infty$ e $q > 1$, ou seja,

$$u(x,t) \in L^\infty_{loc}([0,T_*[, L^q(\mathbb{R})), \quad \forall\, q \in [p_0, \infty) \qquad \text{e} \qquad q > 1. \qquad (2.74)$$

Isto que conclui a prova do Lema (2.6) para soluções $u(x,t)$ positivas. Para $u(x,t)$ solução suave não negativa, consideremos $\varphi \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\varphi(x) > 0$, $\forall x \in \mathbb{R}$. Seja $\epsilon > 0$ dado e considere

$$u_0^{[\epsilon]} := u_0 + \epsilon\varphi \qquad (2.75)$$

e

$$u^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([0, T_*^{[\epsilon]}[, L^\infty(\mathbb{R})), \quad u^{[\epsilon]}(\cdot,t) > 0$$

solução (clássica, positiva) de

$$\begin{cases} u_t + (b(x,t)u^{\kappa+1})_x = (u^\alpha u_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\ u(\cdot,0) = u_0 + \epsilon\varphi, \end{cases}$$

onde $\kappa > \dfrac{\alpha}{2}$. Para $\epsilon > 0$ suficientemente pequeno, temos que $u^{[\epsilon]}(\cdot,t)$ está definida e limitada $\forall\, t \in [0,T]$. Pelo que mostramos acima, temos

$$u^{[\epsilon]}(\cdot,t) \in L^q(\mathbb{R}), \qquad \forall\, t \in [0,T],$$

com

$$\left\| u^{[\epsilon]}(\cdot,t) \right\|_{L^q(\mathbb{R})} \leq K_q(T), \qquad \forall\, t \in [0,T], \qquad (2.76)$$

onde $K_q(T)$ é uma constante que independe de ϵ (veja(2.73)). Como vale a propriedade da comparação (2.57), temos

$$0 \leq u(\cdot,t) \leq u^{[\epsilon]}(\cdot,t), \qquad \forall\, t \in [0,T].$$

Em particular, segue de (2.76) que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq K_q(T), \qquad \forall\, t \in [0,T],$$

Concluindo o Lema (2.6).

■

Lema 2.7. *Seja* $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*[, L^{\infty}(\mathbb{R}))$ *solução suave positiva do problema* (4.1), *com* $\alpha > 0$, $\kappa > \dfrac{\alpha}{2}$. *Se* $u \in L^{\infty}_{loc}([0,T_*[, L^{\infty}(\mathbb{R}))$, *onde* $0 < T_* \leq \infty$, *então*

$$u \in C^0([0,T_*[, L^q(\mathbb{R})). \tag{2.77}$$

Prova: Considere uma função de corte $\xi_{R_1,R_1}(x)$ definida em (2.7) e as notações A_{R_1,R_2} e A'_{R_1,R_2}, definidos em (2.11), para representar as regiões de integração. Multiplicamos a equação (2.64) por $qu^{q-1}(x,t)\xi_{R_1,R_2}(x)$, $q > 1$ e integramos em $[t_0,t] \times \mathbb{R}$,

$$\begin{aligned}
&\int_{t_0}^{t}\int_{A_{R_1,R_2}} qu^{q-1}(x,\tau)u_\tau(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \\
&\qquad +q\int_{t_0}^{t}\int_{A_{R_1,R_2}} u^{q-1}(x,\tau)(b(x,\tau)u^{\kappa+1}(x,\tau))_x\xi_{R_1,R_2}(x)dxd\tau \\
&= q\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q-1}(x,\tau)(u^{\alpha}(x,\tau)u_x(x,\tau))_x\xi_{R_1,R_2}(x)dxd\tau.
\end{aligned}$$

Devido a função de corte, a região de integração é compacta. Então, usamos o Teorema de Fubini na primeira integral e o Teorema de integração por partes nas demais integrais. Observe que $supp\,\xi(x) \subset [-(R+M+1), R+M+1]$, logo as integrais de fronteira são nulas.

$$\begin{aligned}
&\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx- \\
&\qquad -q(q-1)\int_{t_0}^{t}\int_{A_{R_1,R_2}} b(x,\tau)u^{q+\kappa-1}(x,\tau)u_x(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \\
&\qquad -q\int_{t_0}^{t}\int_{A'_{R_1,R_2}} b(x,\tau)u^{q+\kappa}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau \\
&= \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \\
&\qquad -q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \\
&\qquad -q\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{q+\alpha-1}(x,\tau)u_x(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau
\end{aligned}$$

Reescrevemos $b(x,t)$ como $b(x,t) = b(x,t) - \beta(t) + \beta(t)$, onde $\beta(t)$ é dado em (2.13), com isso, é possível obter um controle sobre o fluxo $b(x,t)$. Temos

$$\begin{aligned}
&\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx+ \\
&\qquad +q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \\
&= \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \;+
\end{aligned}$$

$$
\begin{aligned}
&+q\int_{t_0}^{t}\int_{A'_{R_1,R_2}} b(x,\tau)u^{q+\kappa}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau\\
&+q(q-1)\int_{t_0}^{t}\int_{A_{R_1,R_2}} (b(x,\tau)-\beta(\tau))\frac{\partial}{\partial x}\left[\frac{u^{q+\kappa}(x,\tau)}{q+\kappa}\right]\xi_{R_1,R_2}(x)dxd\tau\\
&+q(q-1)\int_{t_0}^{t}\beta(\tau)\int_{A_{R_1,R_2}}\frac{\partial}{\partial x}\left[\frac{u^{q+\kappa}}{q+\kappa}(x,\tau)\right]\xi_{R_1,R_2}(x)dxd\tau\\
&-q\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}}\frac{\partial}{\partial x}\left[\frac{u^{q+\alpha}}{q+\alpha}(x,\tau)\right]\xi'_{R_1,R_2}(x)dxd\tau.
\end{aligned}
$$

Usamos algumas propriedades da função modular, as estimativas (2.15) e (2.17) e aplicamos novamente o Teorema de integração por partes. Assim, a identidade acima fornece

$$
\begin{aligned}
\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx+q(q-1)\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau\\
\leq \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx+\\
+q\mathfrak{b}(T)\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u^{q+\kappa}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau-\\
-\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}B(\tau)\int_{A'_{R_1,R_2}} u^{q+\kappa}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau-\\
-\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\beta(\tau)\int_{A'_{R_1,R_2}} u^{q+\kappa}(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau\\
+\frac{q}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^{q+\alpha}(x,\tau)\xi''_{R_1,R_2}(x)dxd\tau.
\end{aligned}
$$

Note que $q(q-1)\displaystyle\int_{t_0}^{t}\mu(\tau)\int_{A_{R_1,R_2}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau \geq 0$. Além disso, pelas estimativas (2.8), (2.9) e (2.19) segue que

$$
\begin{aligned}
\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx \;&\leq\; \int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx\\
&+q\mathfrak{b}(T)M_1\|u_0\|^{\kappa}_{L^\infty(\mathbb{R})}\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u^q(x,\tau)dxd\tau\\
&+\frac{q(q-1)M_1\|u_0\|^{\kappa}_{L^\infty(\mathbb{R})}}{q+\kappa}\int_{t_0}^{t}B(\tau)\int_{A'_{R_1,R_2}} u^q(x,\tau)dxd\tau\\
&+\frac{q(q-1)M_1\|u_0\|^{\kappa}_{L^\infty(\mathbb{R})}}{q+\kappa}\int_{t_0}^{t}\beta(\tau)\int_{A'_{R_1,R_2}} u^q(x,\tau)dxd\tau\\
&+\frac{qM_2\|u_0\|^{\alpha}_{L^\infty(\mathbb{R})}}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u^q(x,\tau)dxd\tau.
\end{aligned}
$$

Seja $\epsilon > 0$ dado. Segue da definição de A_{R_1,R_2} dada em (2.11) e do que provamos no item (i) que

$$\int_{A_{R_1,R_2}} u^q(x,t_0)\xi_{R_1,R_2}(x)dx \leq \int_{|x|\geq R} u^q(x,t_0)dx \leq \epsilon^q,$$

pois $u(x,t)$ é limitada em $L^q(\mathbb{R})$, logo integrável em $L^q(\mathbb{R})$. Seja $R_3 = R_3(\epsilon) > 0$ suficientemente grande tal que

$$\int_{|x|\geq R_3} u^q(x,t_0)dx \leq \frac{\epsilon^q}{4}.$$

Seja $R_4 = R_4(\epsilon) > 0$ suficientemente grande tal que

$$\int_{t_0}^{T}\int_{|x|\geq R_4} u^q(x,\tau)dxd\tau \leq \frac{\epsilon^q}{4[qM_1(\mathfrak{b}(T)+1)(\|u_0\|^{\kappa}_{L^\infty(\mathbb{R})}+1)]}.$$

Seja $R_5 = R_5(\epsilon) > 0$ suficientemente grande tal que

$$\int_{t_0}^{T}\int_{|x|\geq R_5} u^q(x,\tau)dxd\tau \leq \frac{\epsilon^q}{4[\frac{q(q-1)}{q+\kappa}M_1(\mathfrak{b}(T)+1)(\|u_0\|^{\kappa}_{L^\infty(\mathbb{R})}+1)]}.$$

Sejam $C_1 > 0$ uma constante suficientemente grande tal que $\displaystyle\int_0^T \mu(\tau)d\tau \leq C_1$ e $R_6 = R_6(\epsilon) > 0$ suficientemete grande tal que

$$\int_{t_0}^{T}\int_{|x|\geq R_6} u^q(x,\tau)dxd\tau \leq \frac{\epsilon^q}{4\left[\frac{qM_2C_1(\|u_0\|^{\alpha}_{L^\infty(\mathbb{R})}+1)}{q+\alpha}\right]}.$$

Definindo $R_0 := \max\{R_3, R_4, R_5, R_6\}$, temos $\forall\, R \geq R_0$

$$\int_{A_{R_1,R_2}} u^q(x,t)\xi_{R_1,R_2}(x)dx \leq \epsilon^q, \qquad \forall\, R_2 > 1.$$

Segue do Teorema da convergência dominada, ao $R_2 \to +\infty$, que

$$\int_{|x|\geq R} u^q(x,t)\xi_R(x)dx \leq \epsilon^q, \qquad \forall\, R \geq R_0,$$

onde $\xi_R(x)$ é a função de corte definida em (2.10). Portanto, para $\epsilon > 0$ qualquer, existe R_0 suficientemente grande tal que

$$\int_{|x|\geq R_0+1} u^q(x,t)dx \leq \epsilon^q, \qquad \forall\, t_0 \leq t < T \quad \text{e} \quad \forall\, q \in [p_0,+\infty) \tag{2.78}$$

Assim, para $t_0 \in (0,T)$ e R_0 dado acima, temos

$$
\begin{aligned}
\|u(\cdot,t)-u(\cdot,t_0)\|_{L^q(\mathbb{R})} &\leq \|u(\cdot,t)-u(\cdot,t_0)\|_{L^q(\overline{B}_{R_0+1})}+\|u(\cdot,t)-u(\cdot,t_0)\|_{L^q(B^c_{R_0+1})}\\
&\leq \frac{\epsilon}{3}+\|u(\cdot,t)\|_{L^q(B^c_{R_0+1})}+\|u(\cdot,t_0)\|_{L^q(B^c_{R_0+1})} \quad \leq \quad \epsilon.
\end{aligned}
$$

Portanto, $u(x,t)$ é contínua em t_0, para $0<t_0<T$. Se $t_0=0$, usamos a desigualdade triangular e obtemos

$$
\begin{aligned}
\|u(\cdot,t)-u_0\|_{L^q(\mathbb{R})} &\leq \|u(\cdot,t)-u_0\|_{L^q(\overline{B}_{R_0+1})}+\|u(\cdot,t)-u_0\|_{L^q(B^c_{R_0+1})}\\
&\leq \|u(\cdot,t)-u_0\|^{1-\frac{1}{p}}_{L^\infty(\overline{B}_{R_0+1})}\|u(\cdot,t)-u_0\|^{\frac{1}{p}}_{L^1(\overline{B}_{R_0+1})}+2\epsilon \quad \leq \quad \epsilon.
\end{aligned}
$$

para $|t-t_0|\leq\delta$, para algum $\delta>0$ suficientemente pequeno. Portanto, $u(x,t)$ é contínua em t, $\forall\, t\in[0,T[$. O que conclui o Lema (2.7).

■

Observação 2.2. *Pela desigualdade* (2.70) *e por* $e^{-\epsilon\sqrt{1+x^2}}\leq 1$*, segue que*

$$
\begin{aligned}
&\frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau\\
&\leq \int_{\mathbb{R}} u_0^q(x)e^{-\epsilon\sqrt{1+x^2}}dx+\int_0^t \frac{M(T)^{2\kappa-\alpha}q(q-1)}{2}\frac{B(\tau)^2}{\mu(\tau)}\int_{\mathbb{R}} u^q(x,\tau)e^{-\epsilon\sqrt{1+x^2}}dxd\tau\\
&\quad +\epsilon\int_0^t S(\tau)\int_{\mathbb{R}} u^q(x,\tau)dxd\tau
\end{aligned}
\tag{2.79}
$$

Provamos em (2.79) *que* $\|u(\cdot,t)\|_{L^q(\mathbb{R})}<+\infty$*. Logo, fazendo* $\epsilon\to 0$ *em* (2.79)*, obtemos para todo* $0\leq t\leq T<T_*\leq\infty$,

$$
\begin{aligned}
&\frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)dxd\tau \leq\\
&\qquad \leq \|u_0\|_{L^q(\mathbb{R})}+\frac{M(T)^{2\kappa-\alpha}q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\|u(\cdot,t)\|^q_{L^q(\mathbb{R})}d\tau
\end{aligned}
$$

Como $\mu(t)$ *é uma função contínua e estritamente positiva, podemos assumir que* $\mu(t)\leq C$*, para alguma constante* $C>0$ *em* $[0,T]$*. Logo, concluimos que*

$$
\int_0^T\int_{\mathbb{R}} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)dxd\tau<\infty, \qquad \forall\, q\in[p_0,+\infty) \quad e \quad q>1, \quad \forall\, T\in(0,T_*).
\tag{2.80}
$$

De (2.18), *temos* $u^{q+\alpha-2} \leq M(T)^{q+\alpha-2} < \infty$, *por* (2.74). *Assim, de* (2.79) *resulta*

$$\int_0^T \|u_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})} d\tau < \infty, \qquad \forall\, q \in [p_0, +\infty) \quad e \quad q > 1, \qquad \forall\, T \in (0, T_*).$$

Observe ainda que

$$\left| \int_0^t \int_{|x|\leq R} (b(x,\tau) - \beta(\tau)) u^{q+\kappa-1}(x,\tau) u_x(x,\tau) \zeta_R(x) dx d\tau \right| \leq$$

$$\leq \int_0^t B(\tau) \int_{\mathbb{R}} u^{q+\kappa-1}(x,\tau) |u_x(x,\tau)| dx d\tau \leq$$

$$\leq \frac{1}{2} \int_0^t B(\tau) \int_{|x|\leq R} u^{q+\alpha-2}(x,\tau) u_x^2(x,\tau) dx d\tau +$$

$$+ \frac{1}{2} \int_0^t B(\tau) \int_{|x|\leq R} u^{q+2\kappa-\alpha}(x,\tau) dx d\tau.$$

Note que as duas últimas integrais são finitas, por (2.73) *e* (2.80). *Portanto,*

$$\int_0^t \int_{\mathbb{R}} u^{q+\kappa-1}(x,\tau) |u_x(x,\tau)| dx d\tau < \infty.$$

Pelo mesmo método e software computacional citados acima, obtemos as figuras (no caso $\alpha = 1$ e $\kappa = 4$) que simulam o comportamento da solução $u(x,t)$ na norma L^2 do problema (2.39), sob a condição inicial (2.40).

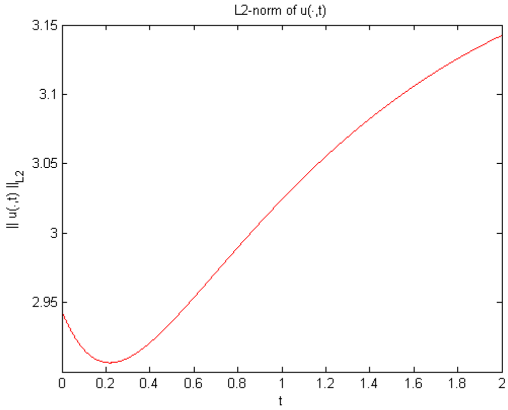

Figura 16: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\sin x$

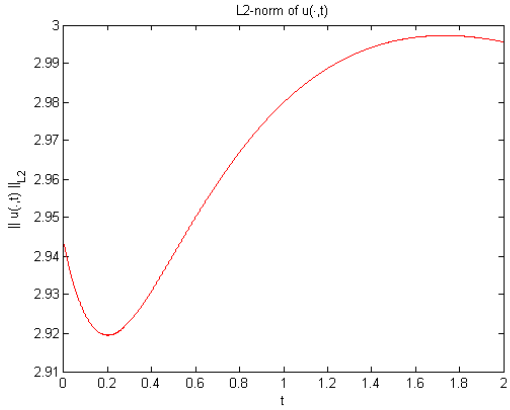

Figura 17: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\cos x$

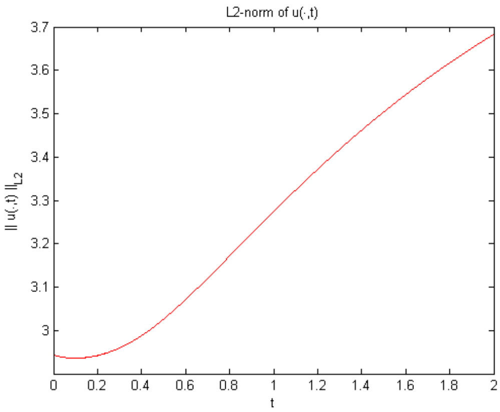

Figura 18: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\tanh x$

Abaixo seguem as simulações do perfil das soluções e das interfaces em $x = -5$ e em $x = 5$ para o problema (2.39), (2.40), onde $b(x,t) = -\sin(x)$, $b(x,t) = -\cos(x)$ e $b(x,t) = -\tanh(x)$ respectivamente, com $\alpha = 1$ e $\kappa = 4$.

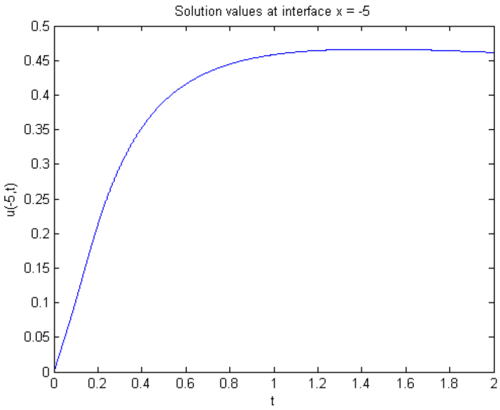

Figura 19: Interface $x = -5$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\sin x$

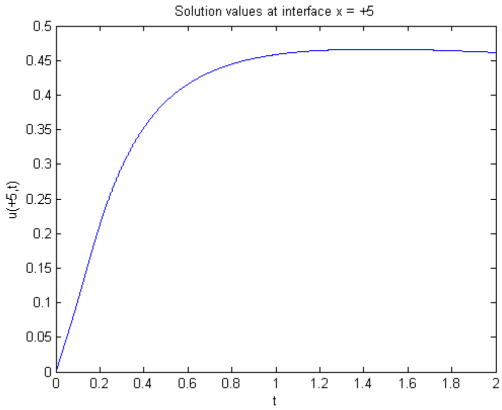

Figura 20: Interface $x = +5$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\sin x$

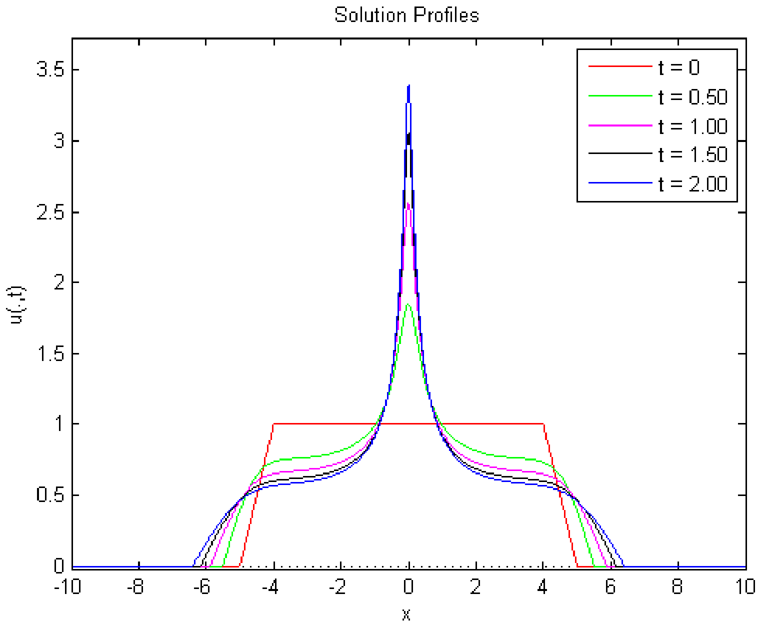

Figura 21: Perfil da Solução: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\sin x$

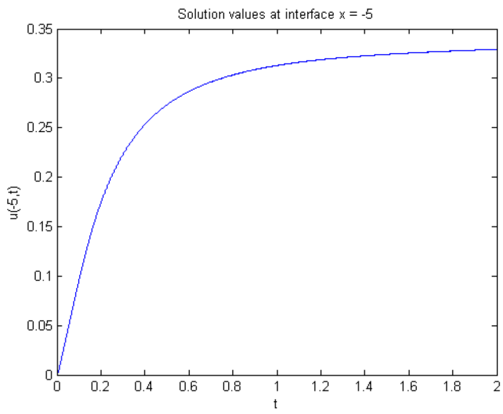

Figura 22: Interface $x = -5$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\cos x$

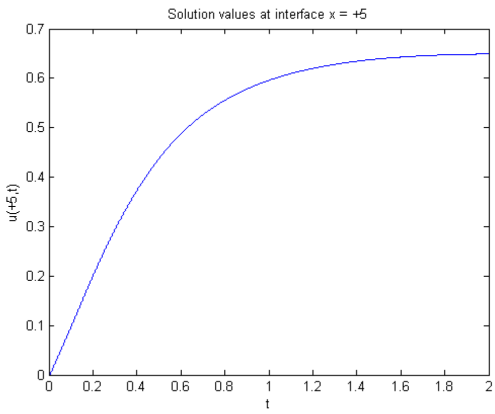

Figura 23: Interface $x = +5$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\cos x$

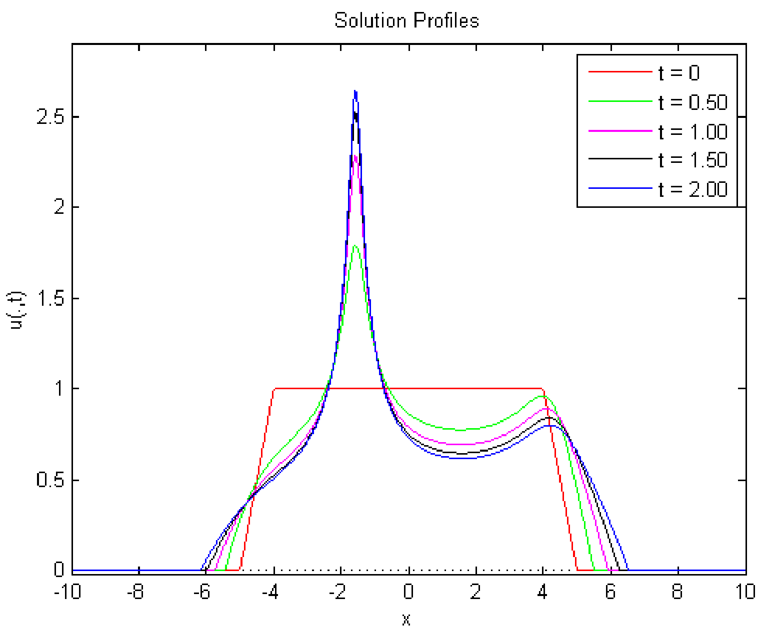

Figura 24: Perfil da Solução: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\cos x$

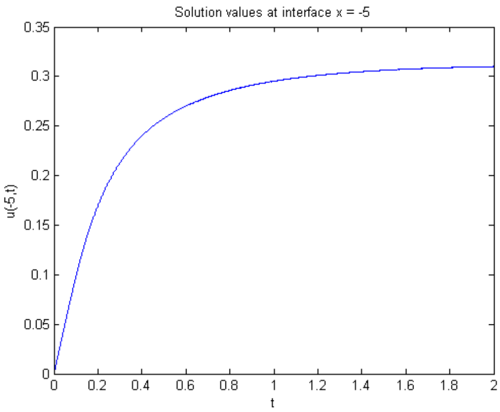

Figura 25: Interface $x = -5$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\tanh x$

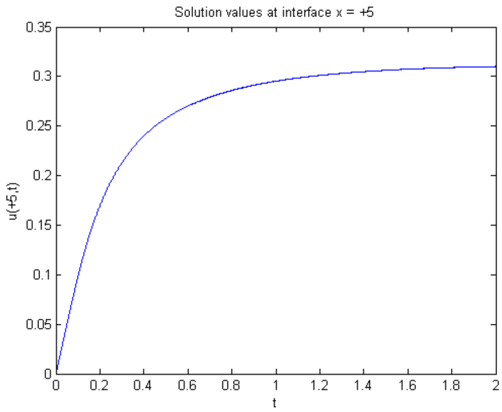

Figura 26: Interface $x = +5$: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\tanh x$

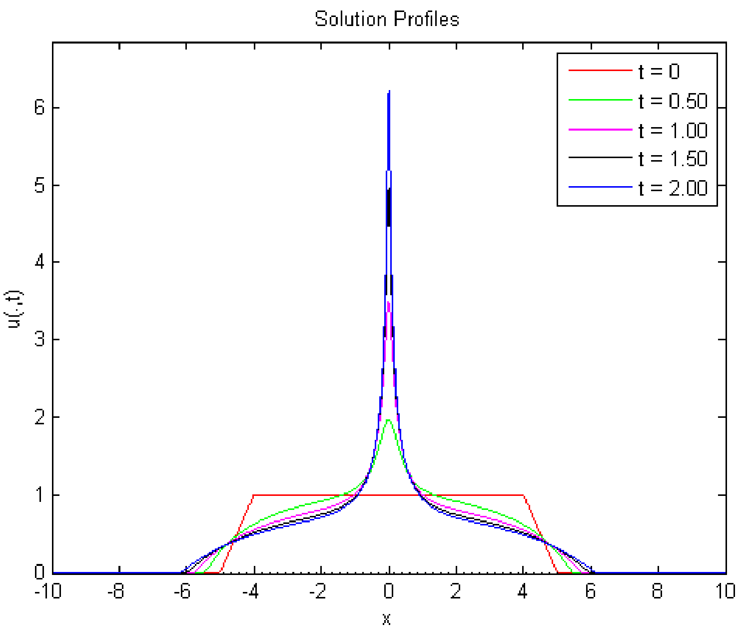

Figura 27: Perfil da Solução: $\alpha = 1$, $\kappa = 4$ $b(x,t) = -\tanh x$

2.6 Soluções Limitadas em L^q. Continuidade dos dados Iniciais - Caso $0 \leq \kappa < \frac{\alpha}{2}$

Sejam $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*[, L^{\infty}(\mathbb{R}))$ solução suave do problema de valor inicial

$$\begin{cases} u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u^{\alpha}u_x)_x; \quad x \in \mathbb{R}, \quad t > 0 \\ \\ u(x,0) = u_0(x) \in L^{p_0}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}); \quad 1 < p_0 < \infty, \end{cases} \tag{2.81}$$

onde $\alpha > 0$ e $0 \leq \kappa < \dfrac{\alpha}{2}$ são quaisquer. Iniciaremos esta secção mostrando que a solução $u(x,t)$ de (2.81) está $L^q(\mathbb{R})$, $\forall\, 0 \leq t < T_*$ com suposições distintas sobre $b(x,t)$, consideradas até o momento. Vamos supor que $b(x,t)$ satisfaz, além da hipótese $(H1)$, em (2.12), e (2.15), descritas capítulo 2, a seguinte condição sobre a derivada espacial de $b(x,t)$:

$$\left| \frac{\partial b}{\partial x}(x,\tau) \right| \leq K_1(T), \tag{2.82}$$

onde $K_1(T)$ é a constante que depende apenas de T, que é tal que $0 < T < T_*$. Neste caso, provaremos que $u(\cdot,t) \in L^q(\mathbb{R})$ para $q \geq p_0$ e $q > 1$.

Lema 2.8. *Seja $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*[, L^{\infty}(\mathbb{R}))$, $u(\cdot,t) \geq 0$, $0 < T_* \leq \infty$, solução do problema* (2.81)*. Se vale a hipótese* (2.82) *acima, então temos*

$$u(\cdot,t) \in L^q(\mathbb{R}), \qquad \forall\, 0 \leq t < T_* \tag{2.83}$$

para cada $q \in [p_0, \infty[$.

Prova: Seja $T \in]0,T_*[$ qualquer (mas fixo no que segue). Mostraremos que $u(\cdot,t) \in L^q(\mathbb{R})$, $\forall\, 0 \leq t \leq T$. Seja, $M \geq 0$ dado por (2.18).

Caso I: $u(\cdot,t) > 0$, $\forall\, 0 < t \leq T$.

Considere a função de corte $\zeta_R(x)$ definida em (2.1). Multiplicamos a equação

$$u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u^{\alpha}u_x)_x \tag{2.84}$$

por $qu(x,t)^{q-1}\zeta_R(x)$ e integramos em $\mathbb{R} \times [t_0,t]$, onde $t \in]0,T]$ e $t_0 \in]0,t[$ são dados e $0 \leq \kappa < \dfrac{\alpha}{2}$. Obtemos

$$\int_{|x|<R} u(x,t)^q \zeta_R(x)dx + q(q-1)\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha-2}u_x^2(x,\tau)\zeta_R(x)dxd\tau =$$

$$= \int_{|x|<R} u(x,t_0)^q\zeta_R(x)dx - q\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha-1}u_x(x,\tau)\zeta_R'(x)dxd\tau +$$

$$+q(q-1)\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa-1}b(x,\tau)u_x(x,\tau)\zeta_R(x)dxd\tau +$$

$$+q\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa}b(x,\tau)\zeta_R'(x)dxd\tau,$$

ou seja,

$$\begin{aligned}
&\int_{|x|<R} u(x,t)^q \zeta_R(x)dx + q(q-1)\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha-2}u_x^2(x,\tau)\zeta_R(x)dxd\tau = \\
&\quad = \int_{|x|<R} u(x,t_0)^q\zeta_R(x)dx + \frac{q}{q+\alpha}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha}\zeta_R''(x)dxd\tau \\
&\qquad -\frac{q}{q+\alpha}\int_{t_0}^{t}\left[u(x,\tau)^{q+\alpha}\zeta_R'(x)\right]_{x=-R}^{x=R}d\tau \\
&\qquad +q(q-1)\int_{t_0}^{t}\int_{|x|<R}\frac{\partial}{\partial x}\left[\frac{u(x,\tau)^{q+\kappa}}{q+\kappa}\right]b(x,\tau)\zeta_R(x)dxd\tau + \\
&\qquad +q\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa}b(x,\tau)\zeta_R'(x)dxd\tau
\end{aligned} \tag{2.85}$$

Aplicando novamente o Teorema de Integração por partes, temos

$$\begin{aligned}
&\int_{|x|<R} u(x,t)^q \zeta_R(x)dx + q(q-1)\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha-2}u_x^2(x,\tau)\zeta_R(x)dxd\tau = \\
&\quad = \int_{|x|<R} u(x,t_0)^q\zeta_R(x)dx + \frac{q}{q+\alpha}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha}\zeta_R''(x)dxd\tau \\
&\qquad -\frac{q}{q+\alpha}\int_{t_0}^{t}\left[u(x,\tau)^{q+\alpha}\zeta_R'(x)\right]_{x=-R}^{x=R}d\tau \\
&\qquad -\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa}b_x(x,\tau)\zeta_R(x)dxd\tau - \\
&\qquad -\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa}b(x,\tau)\zeta_R'(x)dxd\tau + \\
&\qquad +q\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa}b(x,\tau)\zeta_R'(x)dxd\tau
\end{aligned} \tag{2.86}$$

Pelas propriedades de módulo e pela hipótese (2.82), a identidade (2.86) fica

$$\begin{aligned}
&\int_{|x|<R} u(x,t)^q \zeta_R(x)dx + q(q-1)\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha-2}u_x^2(x,\tau)\zeta_R(x)dxd\tau \leq \\
&\quad \leq \int_{|x|<R} u(x,t_0)^q\zeta_R(x)dx + \frac{q}{q+\alpha}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha}\zeta_R''(x)dxd\tau + \\
&\qquad +\frac{q}{q+\alpha}\int_{t_0}^{t}\left[u(x,\tau)^{q+\alpha}\zeta_R'(x)\right]_{x=-R}^{x=R}d\tau \\
&\qquad +\frac{q(q-1)K_1(T)}{q+\kappa}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa}\zeta_R(x)dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa} b(x,\tau)\zeta_R'(x)dxd\tau+ \\
&+q\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa} b(x,\tau)\zeta_R'(x)dxd\tau
\end{aligned}
\tag{2.87}
$$

Observe que se ao invés da hipótese (2.82), tivéssemos

$$
\frac{\partial b}{\partial x}(x,t) \geq 0, \qquad \forall\, x \in \mathbb{R}, \quad t \geq 0, \tag{2.88}
$$

então, de (2.86) acima, teremos

$$
\begin{aligned}
&\int_{|x|<R} u(x,t)^q \zeta_R(x)dx + q(q-1)\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha-2} u_x^2(x,\tau)\zeta_R(x)dxd\tau \leq \\
&\quad\leq \int_{|x|<R} u(x,t_0)^q \zeta_R(x)dx + \frac{q}{q+\alpha}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\alpha}\zeta_R''(x)dxd\tau \\
&\quad-\frac{q}{q+\alpha}\int_{t_0}^{t}\left[u(x,\tau)^{q+\alpha}\zeta_R'(x)\right]_{x=-R}^{x=R} d\tau \\
&\quad-\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa} b(x,\tau)\zeta_R'(x)dxd\tau+ \\
&\quad+q\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa} b(x,\tau)\zeta_R'(x)dxd\tau,
\end{aligned}
\tag{2.89}
$$

pois $\int_{t_0}^{t}\int_{|x|<R} u(x,\tau)^{q+\kappa} b_x(x,\tau)\zeta_R(x)dxd\tau \geq 0$. Note que a estimativa (2.89) é tão útil quanto a estimativa (2.87) acima. Ambas produzem os mesmos resultados finais, tanto na demonstração deste resultado quanto no seguinte (Lema (2.9)). Em particular, se assumirmos a hipótese (2.88) como verdadeira, então os Lemas (2.8) e (2.9) são verdadeiros sem precisarmos da hipótese (2.82). Ou seja, os Lemas (2.8) e (2.9) são verdadeiros apenas com a hipótese (2.88).

Assim, aplicando (2.15) em (2.87) teremos

$$
\begin{aligned}
\int_{|x|\leq R} u(x,t)^q \zeta_R(x)dx \;\leq\; &\int_{|x|\leq R} u(x,t_0)^q \zeta_R(x)dx+ \\
&+\frac{q}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{|x|\leq R} u(x,\tau)^{q+\alpha}|\zeta_R''(x)|dxd\tau+ \\
&+\frac{q}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\left[u(R,\tau)^{q+\alpha}|\zeta_R'(R)| + u(-R,\tau)^{q+\alpha}|\zeta_R'(-R)|\right]d\tau \\
&+\frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\mathfrak{b}(\tau)\int_{|x|\leq R} u(x,\tau)^{q+\kappa}|\zeta_R'(x)|dxd\tau+
\end{aligned}
$$

$$+\frac{q(q-1)}{q+\kappa}K_1(T)\int_{t_0}^{t}\int_{|x|\leq R}u(x,\tau)^{q+\kappa}\zeta_R(x)dxd\tau+$$
$$+q\int_{t_0}^{t}\mathfrak{b}(\tau)\int_{|x|\leq R}u(x,\tau)^{q+\kappa}|\zeta_R'(x)|dxd\tau$$

Logo, aplicando as estimativas (2.2), (2.18) e (2.19), segue que

$$\begin{aligned}\int_{|x|\leq R}u(x,t)^{q}\zeta_R(x)dx &\leq \int_{|x|\leq R}u(x,t_0)^{q}\zeta_R(x)dx+\\ &+\frac{qM^{\alpha}\epsilon^2}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{|x|\leq R}u(x,\tau)^{q}e^{-\epsilon\sqrt{1+x^2}}dxd\tau+\\ &+\frac{2q\epsilon^2M^{q+\alpha}}{q+\alpha}e^{-\epsilon R}\int_{t_0}^{t}\mu(\tau)d\tau+\epsilon q\mathfrak{b}(T)M^{\kappa}\int_{t_0}^{t}\int_{|x|\leq R}u(x,\tau)^{q}e^{-\epsilon\sqrt{1+x^2}}dxd\tau+\\ &+\frac{\epsilon q(q-1)\mathfrak{b}(T)M^{\kappa}}{q+\kappa}\int_{t_0}^{t}\int_{|x|\leq R}u(x,\tau)^{q}e^{-\epsilon\sqrt{1+x^2}}dxd\tau+\\ &+\frac{q(q-1)K_1(T)M^{\kappa}}{q+\kappa}\int_{t_0}^{t}\int_{|x|\leq R}u(x,\tau)^{q}e^{-\epsilon\sqrt{1+x^2}}dxd\tau.\end{aligned} \tag{2.90}$$

Note que utlizamos $\zeta_R(x)\leq e^{-\epsilon\sqrt{1+x^2}}$. Tomaremos $C>0$ uma constante tal que $\int_{t_0}^{t}\mu(\tau)d\tau\leq C$. Assim,

$$\begin{aligned}\int_{|x|\leq R}u(x,t)^{q}\zeta_R(x)dx &\leq \int_{|x|\leq R}u(x,t_0)^{q}e^{-\epsilon\sqrt{1+x^2}}dx+\frac{2q\epsilon^2M^{q+\alpha}C}{q+\alpha}e^{-\epsilon R}+\\ &+K_2(M,T;\epsilon)\int_{t_0}^{t}\int_{|x|\leq R}u(x,\tau)^{q}e^{-\epsilon\sqrt{1+x^2}}dxd\tau,\end{aligned} \tag{2.91}$$

para todo $R>0$, $0<\epsilon\leq 1$, $0<t_0<t\leq T$ e $K_2(M,T;\epsilon)$ dado por

$$K_2(M,T;\epsilon)=\frac{(q-1)K_1(T)M^{\kappa+1}}{q+\kappa}+\epsilon\left(\frac{\epsilon qM^{\alpha}C}{q+\alpha}+\frac{q(q-1)\mathfrak{b}(T)M^{\kappa}}{q+\kappa}+q\mathfrak{b}(T)M^{\kappa}\right). \tag{2.92}$$

Fazendo $t_0\to 0$, obtemos

$$\begin{aligned}\int_{|x|\leq R}u(x,t)^{q}\zeta_R(x)dx &\leq \int_{|x|\leq R}u_0(x)^{q}e^{-\epsilon\sqrt{1+x^2}}dx+\frac{2q\epsilon^2M^{q+\alpha}C}{q+\alpha}e^{-\epsilon R}+\\ &+K_2(M,T;\epsilon)\int_{0}^{t}\int_{|x|\leq R}u(x,\tau)^{q}e^{-\epsilon\sqrt{1+x^2}}dxd\tau,\end{aligned} \tag{2.93}$$

para todo $R>0$, $0<\epsilon\leq 1$ e $t\in[0,T]$. Considerando $e^{-\epsilon\sqrt{1+x^2}}\leq 1$ na segunda integral e fazendo $R\nearrow+\infty$ na desiguadade (2.93), temos

$$\int_{\mathbb{R}} u(x,t)^q e^{-\epsilon\sqrt{1+x^2}}dx \le \int_{\mathbb{R}} u_0(x)^q dx + K_2(M,T;\epsilon)\int_0^t \int_{\mathbb{R}} u(x,\tau)^q e^{-\epsilon\sqrt{1+x^2}}dxd\tau, \tag{2.94}$$

para todo $t \in [0,T]$ e $0 < \epsilon \le 1$. Pelo Teorema de Gronwall, segue que

$$\int_{\mathbb{R}} u(x,t)^q e^{-\epsilon\sqrt{1+x^2}}dx \le \|u_0\|^q_{L^q(\mathbb{R})}\, e^{K_2(M,T;\epsilon)t}, \qquad \forall\, t \in [0,T]. \tag{2.95}$$

Fazendo $\epsilon \to 0$ em (2.95) obtemos

$$\int_{\mathbb{R}} u(x,t)^q dx \le \|u_0\|^q_{L^q(\mathbb{R})}\, e^{\frac{q-1}{q+\kappa}K_1(T)M^{\kappa+1}t}, \qquad \forall\, t \in [0,T]. \tag{2.96}$$

Ou seja, $u(\cdot,t) \in L^q(\mathbb{R})$, $q \in [p_0,\infty]$, $\forall\, t \in [0,T]$ e

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \le \|u_0\|_{L^q(\mathbb{R})}\, e^{\frac{q-1}{q+\kappa}K_1(T)M^{\kappa+1}t}, \qquad \forall\, t \in [0,T]. \tag{2.97}$$

Pela arbitrariedade de $T \in]0,T_*[$, temos (2.83).

Caso II: $u(\cdot,t) \ge 0$ e limitada em $[0,T]$.
Considere $\varphi \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\varphi(x) > 0$, $\forall\, x \in \mathbb{R}$. Seja $\epsilon > 0$ dado e considere

$$u_0^{[\epsilon]} := u_0 + \epsilon\varphi \tag{2.98}$$

e

$$u^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([0,T_*^{[\epsilon]}[, L^\infty(\mathbb{R})), \quad u^{[\epsilon]}(\cdot,t) > 0$$

uma solução (clássica, positiva) de

$$\begin{cases} u_t + (b(x,t)u^{\kappa+1})_x = (u^\alpha u_x)_x, & x \in \mathbb{R}, \quad t > 0 \\ u(\cdot,0) = u_0 + \epsilon\varphi \end{cases} \tag{2.99}$$

Para $\epsilon > 0$ suficientemente pequeno, temos que $u^{[\epsilon]}(\cdot,t)$ está definida e limitada $\forall t \in [0,T]$. Pelo Caso I, considerado acima, temos

$$u^{[\epsilon]}(\cdot,t) \in L^q(\mathbb{R}), \qquad \forall\, t \in [0,T],$$

com

$$\left\|u^{[\epsilon]}(\cdot,t)\right\|_{L^q(\mathbb{R})} \le K_q(T), \qquad \forall\, t \in [0,T], \tag{2.100}$$

onde $K_q(T)$ é uma constante que independe de ϵ. Como vale a propriedade da comparação (2.57), temos

$$0 \le u(\cdot,t) \le u^{[\epsilon]}(\cdot,t), \qquad \forall\, t \in [0,T]. \tag{2.101}$$

Em particular, segue de (2.100) que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq K_q(T), \qquad \forall\, t \in [0,T], \tag{2.102}$$

■

Observação 2.3. *A prova acima mostra que, para cada $T \in]0, T_*[$, temos*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})}\, e^{\frac{q-1}{q+\kappa}K_1(T)M^{\kappa+1}t}, \qquad \forall\, t \in [0,T], \tag{2.103}$$

para cada $q \in [p_0, \infty[$ (e, em particular, também para $q = \infty$), onde $K_1(T)$ é a constante dada em (2.82)*, com $M = M(T)$, definida em* (2.19)*.*

Lema 2.9. *Seja $u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$, $u(\cdot,t) \geq 0$, $0 < T_* \leq \infty$, solução suave positiva do problema* (2.81)*, com $u \in C^0(\mathbb{R}\times]0,T_*[)$ e $0 \leq \kappa < \dfrac{\alpha}{2}$. Suponhamos que vale* (2.82) *para algum $T > 0$ dado. Então, para cada $q \in [p_0, \infty[$, temos*

$$u(\cdot,t) \in C^0([0,T_*[, L^q(\mathbb{R})). \tag{2.104}$$

Prova: É suficiente mostrar que dados $T \in]0, T_*[$ e $\epsilon > 0$, existe $R_\epsilon = R(\epsilon, T)$ suficientemente grande tal que

$$\|u(\cdot,t)\|_{L^q(B^c_{R_\epsilon}(0))} \leq \epsilon, \qquad \forall\, t \in [0,T]. \tag{2.105}$$

Além disso, pela propriedade de comparação (2.57) podemos mostrar apenas o caso em que a solução é clássica positiva. Para facilitar a visualização do argumento, utilizaremos as notações A_{R_1,R_2} e A'_{R_1,R_2} dadas em (2.11). Multiplicamos a equação (2.84) por $qu(x,t)^{q-1}\xi_{R_1,R_2}$, onde $u(\cdot,t) > 0$ é uma solução suave positiva, ξ_{R_1,R_2} é a função de corte definida em (2.7), e integramos em $\mathbb{R} \times [t_0, t]$, onde $t_0 \in]0, t[$, com t e T dados arbitrariamente, mas satisfazendo $0 < t \leq T < T_*$, temos

$$\begin{aligned}
&\int_{A_{R_1,R_2}} u(x,t)^q \xi_{R_1,R_2}(x)dx + \\
&\qquad + q(q-1)\int_{t_0}^{t} \mu(\tau) \int_{A_{R_1,R_2}} u(x,\tau)^{q+\alpha-2} u_x^2(x,\tau)\xi_{R_1,R_2}(x)dxd\tau = \\
&\quad = \int_{A_{R_1,R_2}} u(x,t_0)^q \xi_{R_1,R_2}(x)dx + \\
&\qquad + \frac{q}{q+\alpha}\int_{t_0}^{t} \mu(\tau) \int_{A'_{R_1,R_2}} u(x,t)^{q+\alpha}\xi''_{R_1,R_2}(x)dxd\tau + \\
&\qquad + q(q-1)\int_{t_0}^{t}\int_{A_{R_1,R_2}} b(x,\tau)\frac{\partial}{\partial x}\left[\frac{u(x,\tau)^{q+\kappa}}{q+\kappa}\right]\xi_{R_1,R_2}(x)dxd\tau + \\
&\qquad + q\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u(x,\tau)^{q+\kappa} b(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau
\end{aligned} \tag{2.106}$$

Logo,

$$\begin{aligned}
\int_{A_{R_1,R_2}} u(x,t)^q \xi_{R_1,R_2}(x)dx \leq & \int_{A_{R_1,R_2}} u(x,t_0)^q \xi_{R_1,R_2}(x)dx+ \\
& + \frac{q}{q+\alpha}\int_{t_0}^{t}\mu(\tau)\int_{A'_{R_1,R_2}} u(x,t)^{q+\alpha}\xi''_{R_1,R_2}(x)dxd\tau+ \\
& + q\int_{t_0}^{t}\int_{A'_{R_1,R_2}} u(x,\tau)^{q+\kappa} b(x,\tau)\xi'_{R_1,R_2}(x)dxd\tau- \\
& - \frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\int_{A'_{R_1,R_2}} b(x,\tau)u(x,\tau)^{q+\kappa}\xi'_{R_1,R_2}(x)dxd\tau- \\
& - \frac{q(q-1)}{q+\kappa}\int_{t_0}^{t}\int_{A_{R_1,R_2}} \frac{\partial b}{\partial x}(x,\tau)u(x,\tau)^{q+\kappa}\xi_{R_1,R_2}(x)dxd\tau
\end{aligned} \tag{2.107}$$

Pela hipótese (2.82), pelas estimativas (2.8), (2.9) e (2.18) e considerando C uma constante positiva tal que $\mu(t) \leq C$ em $[t_0, t]$, temos

$$\begin{aligned}
\int_{A_{R_1,R_2}} u(x,t)^q \xi_{R_1,R_2}(x)dx \leq & \int_{A_{R_1,R_2}} u(x,t_0)^q \xi_{R_1,R_2}(x)dx+ \\
+ \left[\frac{qM^\alpha M_2 C}{q+\alpha} + q\mathfrak{b}(T)M^\kappa M_1 + \frac{q(q-1)\mathfrak{b}(T)M_1 M^\kappa}{q+\kappa}\right] & \int_{t_0}^{t}\int_{A_{R_1,R_2}} u^q(x,\tau)dxd\tau+ \\
+ \frac{q(q-1)M^\kappa K_1(T)}{q+\kappa}\int_{t_0}^{t}\int_{A_{R_1,R_2}} u^q(x,\tau)\xi_{R_1,R_2}(x)dxd\tau &
\end{aligned} \tag{2.108}$$

para todos $0 < t_0 < t \leq T$, $R_1 > 0$ e $R_2 > 1$. Fazendo $t_0 \to 0$ e lembrando que $\xi_{R_1,R_2} \leq 1$, obtemos

$$\begin{aligned}
\int_{A_{R_1,R_2}} u(x,t)^q \xi_{R_1,R_2}(x)dx \quad \leq \quad & \int_{A_{R_1,R_2}} u_0(x)^q \xi_{R_1,R_2}(x)dx \quad + \\
& + K(M,T)\int_{0}^{t}\int_{A_{R_1,R_2}} u^q(x,\tau)dxd\tau,
\end{aligned} \tag{2.109}$$

para todo $t \in [0, T]$, onde

$$K(M,T) = \frac{qM^\alpha M_2 C}{q+\alpha} + q\mathfrak{b}(T)M_1 M^\kappa + \frac{q(q-1)\mathfrak{b}(T)M_1 M^\kappa}{q+\kappa} + \frac{q(q-1)M^\kappa K_1(T)}{q+\kappa}. \tag{2.110}$$

Assim, em particular, segue que

$$
\begin{aligned}
\int_{A_{R_1,R_2}} u(x,t)^q \xi_{R_1,R_2}(x)dx &\leq \int_{|x|>R_1} u_0(x)^q dx + K(M,T)\int_0^t \int_{|x|>R_1} u^q(x,\tau)dxd\tau \\
&\leq \int_{|x|>R_1} u_0(x)^q dx + K(M,T)\int_0^T \int_{|x|>R_1} u^q(x,\tau)dxd\tau
\end{aligned}
$$

para todo $t \in [0,T]$ e todos $R_1 > 0$ e $R_2 > 1$. Fazendo $R_2 \nearrow \infty$, temos

$$
\int_{|x|>R_1} u(x,t)^q dx \leq \int_{|x|>R_1} u_0(x)^q dx + K(M,T)\int_0^T \int_{|x|>R_1} u^q(x,\tau)dxd\tau
$$

para todo $t \in [0,T]$ e todo $R_1 > 0$. Logo, dado $\epsilon > 0$, tomamos $R_1 \gg 1$ suficientemente grande tal que

$$
\int_{|x|>R_1} u_0(x)^q dx + K(M,T)\int_0^T \int_{|x|>R_1} u^q(x,\tau)dxd\tau \quad \leq \quad \epsilon^q, \tag{2.111}
$$

onde $K(M,T)$ está dado em (2.110). Portanto,

$$
\int_{|x|>R_\epsilon} u(x,t)^q dx \leq \epsilon^q, \qquad \forall\, t \in [0,T],
$$

onde $R_\epsilon = R_1 + 1$, com $R_1 \gg 1$ satisfazendo (2.111) acima. Concluindo, assim, a prova de (2.105) para soluções $u(x,t) > 0$. O princípio de comparação (2.36) nos garante a validade de (2.105) para soluções fracas quaisquer não-negativas, $u(x,t) \geq 0$. Temos $u \in C^0([0,T_*[, L^1_{loc}(\mathbb{R}))$, então, de (2.105), segue que

$$
u(\cdot,t) \in C^0([0,T_*[, L^q(\mathbb{R})), \qquad \forall\, q \in [p_0, \infty[.
$$

Concluindo a prova da proposição (2.9). $\blacksquare$

Apresentamos a seguir as simulações numéricas que descrevem o comportamento das soluções na norma L^2 do problema (2.39) sob a condição inicial dada em (2.40). Abaixo seguem as figuras para $\alpha = 4$ e $\kappa = 1$.

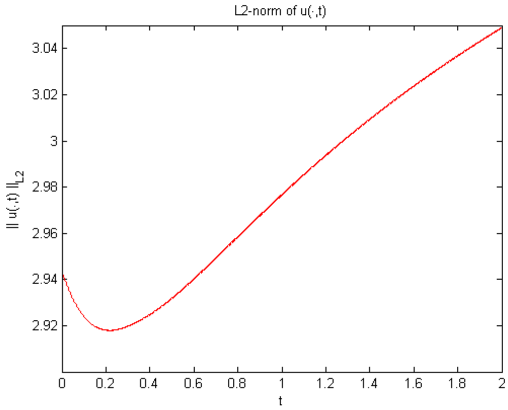

Figura 28: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$ $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\sin x$

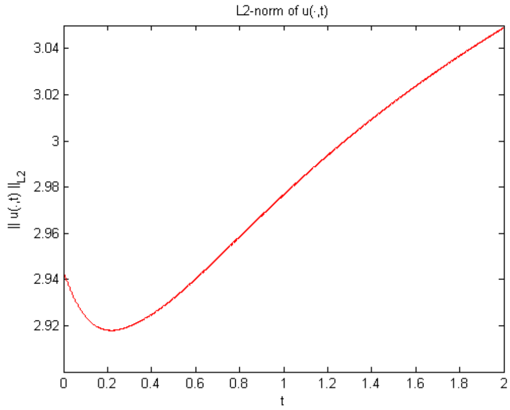

Figura 29: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$ $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\cos x$

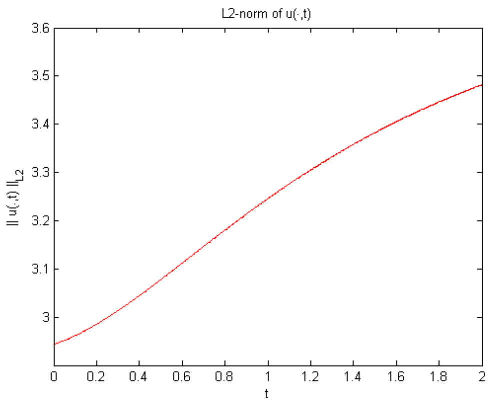

Figura 30: $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$ $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\tanh x$

As simulações correspondentes ao perfil das soluções e das interfaces em $x = -5$ e em $x = 5$ para o problema (2.39), (2.40), com $\alpha = 4$ e $\kappa = 1$ são dadas a seguir

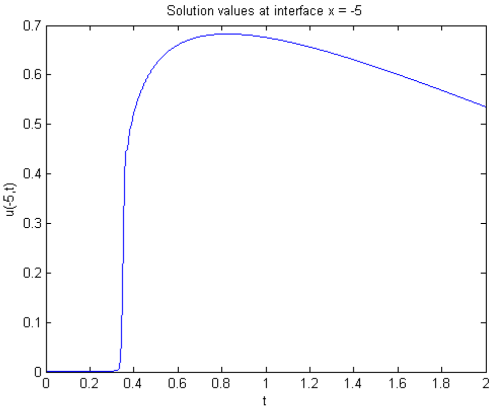

Figura 31: Interface $x = -5$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\sin x$

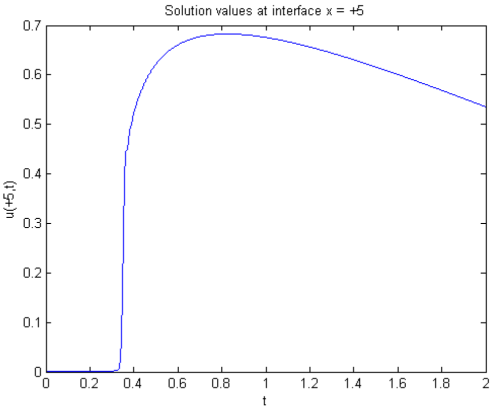

Figura 32: Interface $x = +5$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\sin x$

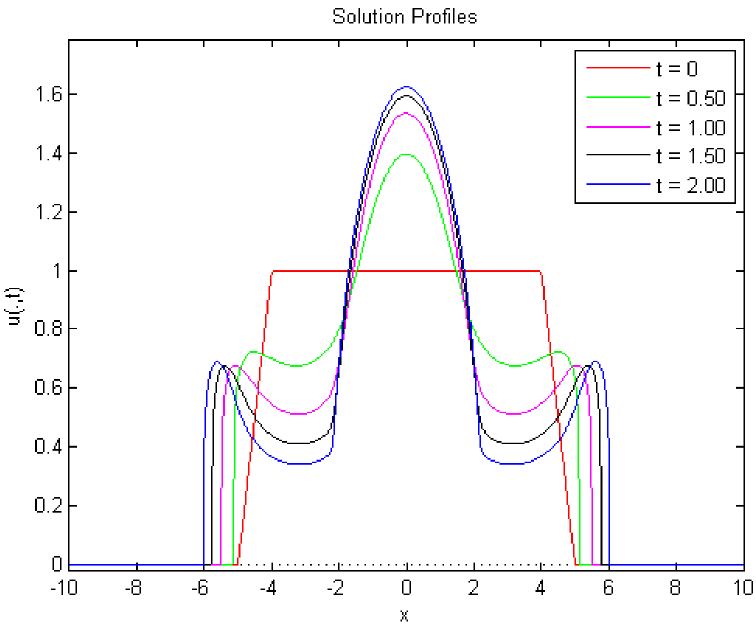

Figura 33: Perfil da Solução: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\sin x$

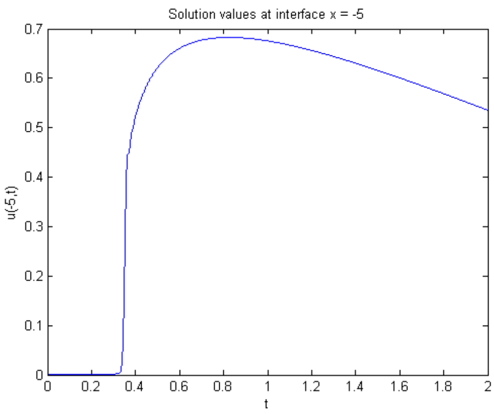

Figura 34: Interface $x = -5$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\cos x$

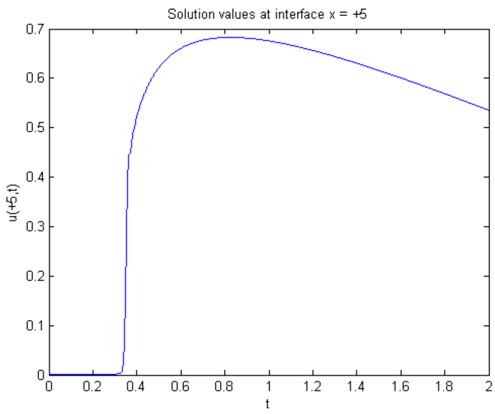

Figura 35: Interface $x = +5$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\cos x$

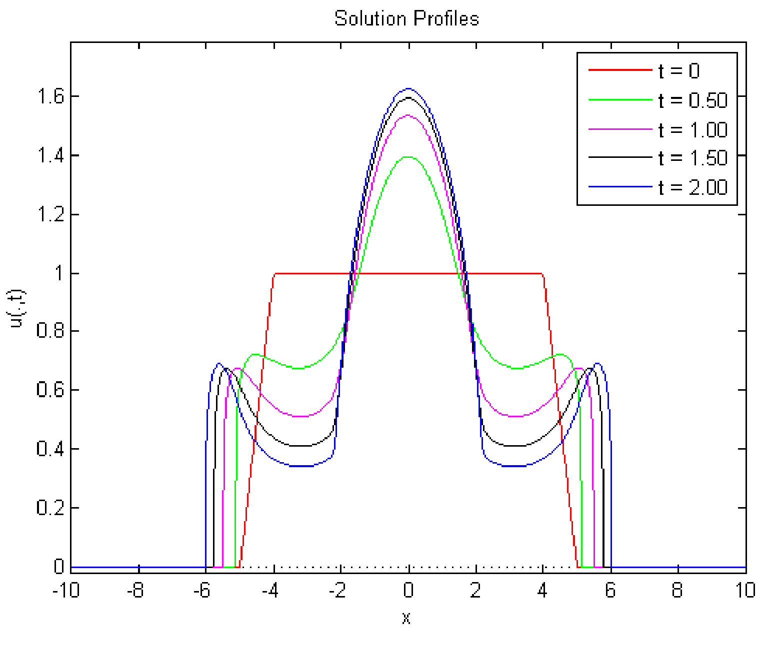

Figura 36: Perfil da Solução: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\cos x$

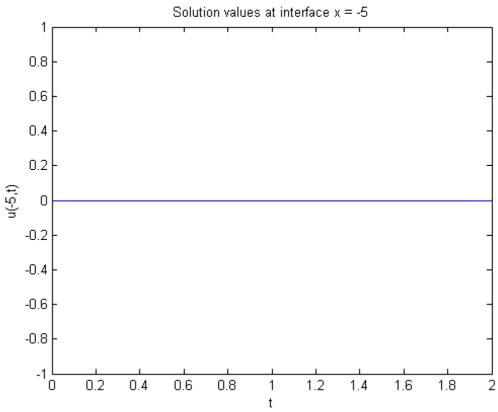

Figura 37: Interface $x = -5$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\tanh x$

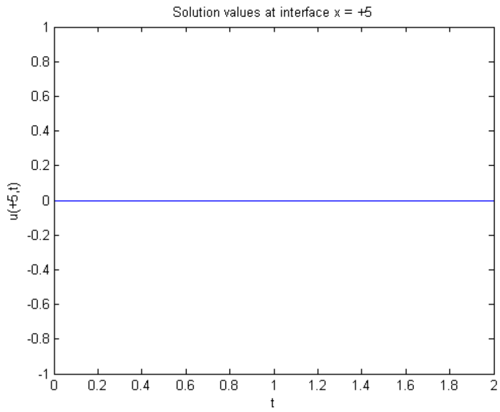

Figura 38: Interface $x = +5$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\tanh x$

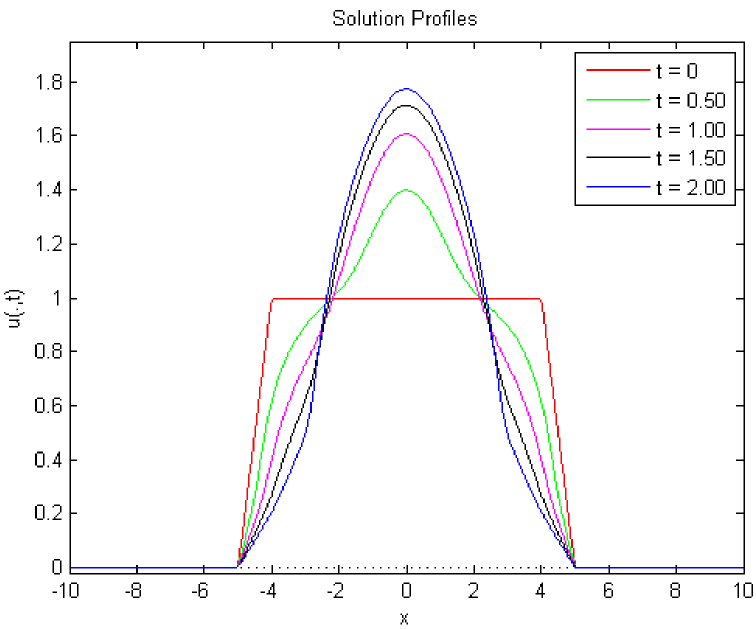

Figura 39: Perfil da Solução: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\tanh x$

2.7 Teorema da Comparação

Daremos uma prova do Teorema da Comparação para a Equação de Fluidos em Meios Porosos n-dimensional. Tal Teorema é imprescindível para provarmos os Teoremas que garantem a limitação da norma do sup de $u(x,t)$, solução de (1.1), (1.2), que nos propomos. No nosso caso, $n = 1$.

Teorema 2.1 (Teorema da Comparação)**.** *Sejam u_0, $v_0 \in L^\infty(\mathbb{R}^n)$ não negativas com $0 \le u_0(\underline{x}) \le v_0(\underline{x})$ q.t.p. $\underline{x} \in \mathbb{R}^n$ e $v_0(\underline{x}) \ge \omega(\underline{x})$ q.t.p. $\underline{x} \in \mathbb{R}^n$, com $\omega \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R})$ e $\omega(\underline{x}) > 0$, $\forall \underline{x} \in \mathbb{R}$. Sejam $u(\cdot,t)$, $v(\cdot,t)$, $0 \le t \le T$, soluções de*

$$\begin{aligned} &(i) \quad \begin{cases} u_t + div(\overrightarrow{f}(\underline{x},t,u)) = div(u^\alpha \nabla u) \\ u(\cdot,0) = u_0 \end{cases} \\ &(ii) \quad \begin{cases} v_t + div(\overrightarrow{f}(\underline{x},t,v)) = div(v^\alpha \nabla v) \\ v(\cdot,0) = v_0 \end{cases} \quad , \end{aligned} \tag{2.112}$$

respectivamente. Então,

$$v_0 \ge u_0 \quad \Longrightarrow \quad v(\underline{x},t) \ge u(\underline{x},t), \qquad \forall\, \underline{x} \in \mathbb{R}^n \quad e \quad \forall\, 0 \le t \le T. \tag{2.113}$$

Prova: Considere $\delta > 0$ pequeno, $R > 0$ e as seguintes funções

$$H_\delta(\mathsf{u}) = H\left(\frac{\mathsf{u}}{\delta}\right) \qquad \Rightarrow \qquad H'_\delta(\mathsf{u}) = \frac{1}{\delta} H\left(\frac{\mathsf{u}}{\delta}\right).$$

$$\zeta_R(\underline{x}) = \begin{cases} e^{-\sqrt{1+|\underline{x}|^2}} - e^{-\sqrt{1+R^2}}, & |\underline{x}| \le R \\ \\ 0, & |\underline{x}| > 0. \end{cases}$$

$$P(\mathsf{u}) = \begin{cases} \dfrac{\mathsf{u}^{\alpha+1}}{\alpha+1} & \mathsf{u} \ge 0 \\ \\ 0, & \mathsf{u} < 0. \end{cases}$$

$$\Psi(\underline{x},t) = \begin{cases} 0, & |\underline{x}| > R \\ \\ \zeta_R(\underline{x}) H_\delta\left[P(u(\underline{x},t)) - P(v(\underline{x},t))\right], & |\underline{x}| \le R. \end{cases}$$

Temos $P'(\mathsf{u}) = \mathsf{u}^\alpha$, $\Psi \in C^0(\mathbb{R}^n \times [0,+\infty[)$ e

$$\Psi(\underline{x},t) = \begin{cases} 0, & u(\underline{x},t) \le v(\underline{x},t) \\ \\ \zeta_R(\underline{x}) H_\delta\left[P(u(\underline{x},t)) - P(v(\underline{x},t))\right], & u(\underline{x},t) > v(\underline{x},t). \end{cases}$$

Além disso,

$$\nabla\Psi(\underline{x},t) = \nabla\zeta_R(\underline{x}) H_\delta(P(u) - P(v)) + H'_\delta(P(u) - P(v))\zeta_R(\underline{x}) \cdot \nabla(P(u) - P(v))$$

Tome $\widehat{T} > 0$ fixo e $M = M(\widehat{T})$ uma constante que limita u. Esta constante está bem definida, pois u é solução fraca limitado da equação (2.112), (i). Ou seja,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq M(\widehat{T}), \qquad 0 \leq t \leq \widehat{T}.$$

Note que

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|v_0\|_{L^\infty(\mathbb{R}^n)}, \qquad 0 \leq t \leq \widehat{T}.$$

Considere ainda T fixo, $0 < T \leq \widehat{T}$ e a seguinte região do espaço tempo $Q_{RT} = B_R(0) \times [0,T]$. Para cada $t > 0$, considere os seguintes funcionais:

$$\begin{aligned}
\langle u_t, \varphi \rangle &:= \int_{\mathbb{R}^n} f(\underline{x}, t, u(\underline{x}, t)) \cdot \nabla\varphi(\underline{x}) d\underline{x} - \\
&\quad - \int_{\mathbb{R}^n} |u(\underline{x}, t)|^\alpha \nabla u(\underline{x}, t) \cdot \nabla\varphi(\underline{x}) d\underline{x}
\end{aligned}$$

e

$$\begin{aligned}
\langle v_t, \varphi \rangle &:= \int_{\mathbb{R}^n} f(\underline{x}, t, v(\underline{x}, t)) \cdot \nabla\varphi(\underline{x}) d\underline{x} - \\
&\quad - \int_{\mathbb{R}^n} |v(\underline{x}, t)|^\alpha \nabla v(\underline{x}, t) \cdot \nabla\varphi(\underline{x}) d\underline{x}.
\end{aligned}$$

onde $\varphi \in C_0^0(\mathbb{R}^n)$, $\varphi \in H^1_{loc}(\mathbb{R}^n)$. Usaremos $\varphi(\underline{x}) = \Psi(\underline{x}, t)$, onde $t > 0$ está fixo. Então, pela definição de Ψ, temos

$$\begin{aligned}
\langle u_t, \Psi(\cdot, t) \rangle &= \int_{|\underline{x}|<R} H_\delta(P(u) - P(v)) f(\underline{x}, t, u) \cdot \nabla\zeta_R(\underline{x}) d\underline{x} \\
&\quad + \int_{|\underline{x}|<R} H'_\delta(P(u) - P(v)) \zeta_R(\underline{x}) f(\underline{x}, t, u) \cdot \nabla(P(u) - P(v)) d\underline{x} \\
&\quad - \int_{|\underline{x}|<R} H_\delta(P(u) - P(v)) |u|^\alpha \nabla u \cdot \nabla\zeta_R(\underline{x}) d\underline{x} \\
&\quad - \int_{|\underline{x}|<R} H'_\delta(P(u) - P(v)) |u|^\alpha \zeta_R(\underline{x}) \nabla u \cdot \nabla(P(u) - P(v)) d\underline{x},
\end{aligned}$$

Lembre-se que estamos considerando $v > 0$, então, pela definição de P, temos $P(v) = \dfrac{v^{\alpha+1}}{\alpha+1} > 0$. Com isso, os integrandos acima são não nulos apenas quando $u \geq v > 0$. De fato, se $u > 0$ e $u < v$, temos $P(u) - P(v) < 0$. Logo, $H_\delta(P(u) - P(v)) = 0$. Assim, quando $u > 0$, $|u|^\alpha = u^\alpha = P'(u)$. Com isso, $u^\alpha \nabla u = P'(u)\nabla u = \nabla P(u)$. Isto nos dá uma nova expressão:

$$
\begin{aligned}
\langle u_t, \Psi(\cdot,t)\rangle &= \int_{R_{u,v}} H_\delta(P(u)-P(v)) f(\underline{x},t,u)\cdot\nabla\zeta_R(\underline{x})d\underline{x} \\
&\quad + \int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x}) f(\underline{x},t,u)\cdot\nabla(P(u)-P(v))d\underline{x} \\
&\quad - \int_{R_{u,v}} H_\delta(P(u)-P(v))\nabla P(u)\cdot\nabla\zeta_R(\underline{x})d\underline{x} \\
&\quad - \int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x})\nabla P(u)\cdot\nabla(P(u)-P(v))d\underline{x}.
\end{aligned}
$$

Onde,

$$
\int_{R_{u,v}} g(\underline{x})d\underline{x} := \int_{\{|\underline{x}|<R\}\cap\{u>v>0\}} g(\underline{x})d\underline{x}.
$$

Pelo mesmo argumento

$$
\begin{aligned}
\langle v_t, \Psi(\cdot,t)\rangle &= \int_{R_{u,v}} H_\delta(P(u)-P(v)) f(\underline{x},t,v)\cdot\nabla\zeta_R(\underline{x})d\underline{x} \\
&\quad + \int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x}) f(\underline{x},t,v)\cdot\nabla(P(u)-P(v))d\underline{x} \\
&\quad - \int_{R_{u,v}} H_\delta(P(u)-P(v))\nabla P(v)\cdot\nabla\zeta_R(\underline{x})d\underline{x} \\
&\quad - \int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x})\nabla P(v)\cdot\nabla(P(u)-P(v))d\underline{x}.
\end{aligned}
$$

Fazendo a diferença dessas identidades, obtemos

$$
\begin{aligned}
\langle u_t - v_t, \Psi(\cdot,t)\rangle &= \int_{R_{u,v}} H_\delta(P(u)-P(v))[f]\cdot\nabla\zeta_R(\underline{x})d\underline{x} \\
&\quad + \int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x})[f]\cdot\nabla(P(u)-P(v))d\underline{x} \\
&\quad - \int_{R_{u,v}} H_\delta(P(u)-P(v))\nabla(P(u)-P(v))\cdot\nabla\zeta_R(\underline{x})d\underline{x} \\
&\quad - \int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x})|\nabla(P(u)-P(v))|^2 d\underline{x}.
\end{aligned}
$$

Como $\int_{R_{u,v}} H_\delta'(P(u)-P(v))\zeta_R(\underline{x})|\nabla(P(u)-P(v))|^2 d\underline{x} \geq 0$, temos

$$
\begin{aligned}
\langle u_t - v_t, \Psi(\cdot,t)\rangle \quad \leq \quad & \int_{R_{u,v}} H_\delta(P(u)-P(v))[f]\cdot +\nabla\zeta_R(\underline{x})d\underline{x} \\
& + \int_{R_{u,v}} H'_\delta(P(u)-P(v))\zeta_R(\underline{x})[f]\cdot\nabla(P(u)-P(v))d\underline{x} - \\
& - \int_{R_{u,v}} H_\delta(P(u)-P(v))\nabla(P(u)-P(v))\cdot\nabla\zeta_R(\underline{x})d\underline{x},
\end{aligned} \tag{2.114}
$$

onde $[f] = f(\underline{x},t,u)-f(\underline{x},t,v)$. Note que $\langle u_t - v_t, \Psi(\cdot,t)\rangle = \langle u_t, \Psi(\cdot,t)\rangle - \langle v_t, \Psi(\cdot,t)\rangle$. Analisaremos cada um destes termos. Da teoria de distribuições, temos

$$
\langle u_t, \Psi(\cdot,t)\rangle = \int_{R_{u,v}} u_t(\underline{x},t)H_\delta(P(u(\underline{x},t)) - P(v(\underline{x},t)))\zeta_R(\underline{x})d\underline{x}.
$$

Observe que o integrando é uma função L^1, pois H_δ anula a região onde u não é bem comportada. A intenção é eliminar a derivada no tempo de u. Então, integramos na variável tempo, com $t \in [0,T]$ e usamos Teorema de Fubini.

$$
\int_0^T \langle u_t, \Psi(\cdot,t)\rangle dt = \int_{R_{u,v}} \zeta_R(\underline{x}) \int_0^T u_t(\underline{x},t)H_\delta(P(u(\underline{x},t)) - P(v(\underline{x},t)))dt d\underline{x}.
$$

A ideia inicial era aplicar integração por partes, mas isso não nos levou a conclusão alguma. Então, note que,

$$
\begin{aligned}
& \frac{\partial}{\partial t}\int_{V_R}^{u(\underline{x},t)} H_\delta(P(z)-P(v(\underline{x},t)))dz = \\
& \qquad = H_\delta(P(z)-P(v(\underline{x},t)))u_t(\underline{x},t) - \int_{V_R}^{u(\underline{x},t)} H'_\delta(P(z)-P(v(\underline{x},t)))dz \quad P'(z)v_t
\end{aligned}
$$

onde

$$
V_R := V_{min}(R) = \inf_{\substack{0 < t \leq 0 \\ |x| < R}} V(\underline{x},t) > 0.
$$

Logo,

$$
\begin{aligned}
H_\delta(P(z)-P(v))u_t = & \frac{\partial}{\partial t}\int_{V_R}^{u(\underline{x},t)} H_\delta(P(z)-P(v))dz \\
& + \int_{V_R}^{u(\underline{x},t)} H'_\delta(P(z)-P(v))dz \quad P'(z)v_t.
\end{aligned}
$$

Substituindo, obtemos

$$
\int_0^T H_\delta(P(z)-P(v))u_t dt = \int_0^T \frac{\partial}{\partial t}\int_{V_R}^{u(\underline{x},t)} H_\delta(P(z)-P(v))dz dt +
$$

$$+\int_0^T \int_{V_R}^{u(\underline{x},t)} H'_\delta(P(z)-P(v))dzP'(z)v_t dt$$

$$= \int_{V_R}^{u(\underline{x},T)} H_\delta(P(z)-P(v(\underline{x},T)))dz - \int_{V_R}^{u(\underline{x},0)} H_\delta(P(z)-P(v(\underline{x},0)))dz$$

$$+\int_0^T \int_{V_R}^{u(\underline{x},t)} H'_\delta(P(z)-P(v))P'(z)v_t dzdt$$

Observe que o intervalo de variação de z é $[v_R, u_0]$ e, além disso, $u_0 \leq v_0$, $V_R > 0$ e $u > v$. Logo, $P(z) - P(v_0) < 0$ e com isso

$$\int_{V_R}^{u_0} H_\delta(P(z)-P(v_0))dz = 0.$$

Assim, eliminamos o termo u_t. O termo v_t não nos causa transtorno, já que v_t é bem comportada. Temos então

$$\int_0^T \langle u_t, \Psi(\cdot,t)\rangle dt = \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(\underline{x},T)} H_\delta(P(z)) - P(v(\underline{x},T)))dzd\underline{x}$$

$$+\int_{R_{u,v}} \zeta_R(\underline{x}) \int_0^T \int_{V_R}^{u(\underline{x},t)} H'_\delta(P(z)) - P(v))P'(v)v_t dzdtd\underline{x}$$

Ou seja,

$$\int_0^T \langle u_t, \Psi(\cdot,t)\rangle dt = \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(\underline{x},T)} H_\delta(P(z)) - P(v(\underline{x},T)))dzd\underline{x}$$

$$+\int_0^T \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(\underline{x},t)} H'_\delta(P(z)) - P(v))P'(v)v_t dzd\underline{x}dt$$

Agora vamos computar $\langle v_t, \Psi(\cdot,t)\rangle$. Da teoria das distribuições segue que

$$\begin{aligned}\int_0^T \langle v_t, \Psi(\cdot,t)\rangle dt &= \int_0^T \int_{R_{u,v}} \zeta_R(\underline{x}) v(\underline{x},t) H_\delta(P(u(\underline{x},t))) - P(v(\underline{x},t)))d\underline{x}dt \\ &= \int_{R_{u,v}} \zeta_R(\underline{x}) \int_0^T v_t(\underline{x},t) H_\delta(P(u(\underline{x},t))) - P(v(\underline{x},t)))dtd\underline{x}\end{aligned}$$

Note que $H'_\delta(P(z)-P(v))P'(z) = \dfrac{d}{dz}\left(H_\delta(P(z)-P(v(x,t)))\right)$. Logo,

$$\begin{aligned}\int_{V_R}^{u(x,t)} H'_\delta(P(z)-P(v))P'(z)dz &= \int_{V_R}^{u(x,t)} \frac{d}{dz}\left(H_\delta(P(z)-P(v(x,t)))\right)dz \\ &= H_\delta(P(u(x,t)) - P(v(x,t))) - H_\delta(P(V_R)-P(v)) \\ &= H_\delta(P(u(x,t)) - P(v(x,t))),\end{aligned}$$

pois

$$V_R = \inf v(x,t) \leq v(x,t) \Rightarrow P(V_R) - P(v) < 0 \Rightarrow H_\delta(P(V_R) - P(v)) = 0.$$

Assim,

$$\int_0^T \langle v_t, \Psi(\cdot,t)\rangle dt = \int_{R_{u,v}} \zeta_R(\underline{x}) \int_0^T v_t(\underline{x},t) \int_{V_R}^{u(x,t)} H_\delta'(P(z) - P(v))P'(z)dzdtd\underline{x}$$

$$= \int_0^T \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(x,t)} H_\delta'(P(z) - P(v))P'(z)v_t(\underline{x},t)dzd\underline{x}dt.$$

Como

$$\int_0^T \langle u_t - v_t, \Psi(\cdot,t)\rangle dt = \int_0^T \langle u_t, \Psi(\cdot,t)\rangle dt - \int_0^T \langle v_t, \Psi(\cdot,t)\rangle dt,$$

obtemos

$$\begin{aligned}\int_0^T \langle u_t - v_t, \Psi(\cdot,t)\rangle dt = &\int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(x,T)} H_\delta(P(z) - P(v(x,T)))dzd\underline{x}+\\ &+ \int_0^T \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(x,t)} H_\delta'(P(z) - P(v))(P'(v) - P'(z))dz \quad v_t(\underline{x},t)dtd\underline{x}.\end{aligned} \tag{2.115}$$

Considere as seguintes funções

$$\mathbb{F}_\delta(\mathrm{u},\mathrm{v}) := \int_{V_R}^{\mathrm{u}} H_\delta(P(z) - P(\mathrm{v}))dz, \qquad \forall\, \mathrm{u},\mathrm{v} \in \mathbb{R}.$$

$$\mathbb{G}_\delta(\mathrm{u},\mathrm{v}) := \int_{V_R}^{\mathrm{u}} H_\delta'(P(z) - P(\mathrm{v}))(P'(z) - P'(\mathrm{v}))dz, \qquad \forall\, \mathrm{u},\mathrm{v} \in \mathbb{R}.$$

Com estas funções e com os resultados (2.114) e (2.115), temos

$$\begin{aligned}\int_0^T \langle u_t - v_t, \Psi(\cdot,t)\rangle dt \;\leq\; &\int_0^T \int_{R_{u,v}} H_\delta(P(u) - P(v))[f] \cdot \nabla\zeta_R(\underline{x})d\underline{x}dt+\\ &+ \int_0^T \int_{R_{u,v}} H_\delta'(P(u) - P(v))\zeta_R(\underline{x})[f] \cdot \nabla(P(u) - P(v))d\underline{x}dt-\\ &- \int_0^T \int_{R_{u,v}} H_\delta(P(u) - P(v))\nabla(P(u) - P(v)) \cdot \nabla\zeta_R(\underline{x})d\underline{x}dt.\end{aligned}$$

E, por outro lado,

$$\int_0^T \langle u_t - v_t, \Psi(\cdot,t)\rangle dt = \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(x,t)} H_\delta(P(z) - P(v(x,T)))dz d\underline{x} \ +$$
$$+ \int_0^T \int_{R_{u,v}} \zeta_R(\underline{x}) \int_{V_R}^{u(x,t)} H'_\delta(P(z)-P(v))(P'(v)-P'(z))dz v_t(\underline{x},t)d\underline{x}dt =$$
$$= \int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{F}_\delta(u(x,t),v(x,t))d\underline{x} \ +$$
$$+ \int_0^T \int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{G}_\delta(u(x,t),v(x,t))v_t(\underline{x},t)d\underline{x}dt.$$

Assim,

$$\int_0^T \int_{R_{u,v}} H_\delta(P(u)-P(v))[f]\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt \ +$$
$$+ \int_0^T \int_{R_{u,v}} H'_\delta(P(u)-P(v))\zeta_R(\underline{x})[f]\cdot\nabla(P(u)-P(v))d\underline{x}dt \ -$$
$$- \int_0^T \int_{R_{u,v}} H_\delta(P(u)-P(v))\nabla(P(u)-P(v))\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt \ \geq$$
$$\geq \int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{F}_\delta(u,v)d\underline{x} + \int_0^T \int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{G}_\delta(u,v)v_t(\underline{x},t)d\underline{x}dt. \tag{2.116}$$

Vamos a análise de algumas dessas integrais. Mostraremos que $\mathbb{G}_\delta(\mathrm{u},\mathrm{v}) \to 0$, ao $\delta \to 0$. Para isto, usaremos o Teorema da Convergência Dominada. Logo, devemos mostrar também que $\mathbb{G}_\delta(\mathrm{u},\mathrm{v})$ fica limitada por uma função integrável, para cada $\delta > 0$.

$\star$ Dominação de $\mathbb{G}$. Note que estamos integrando na região onde $u > v$ e, além disso, $H'_\delta(P(z)-P(v))(P'(z)-P'(v)) \geq 0$, pois estamos considerando $z > v$, caso contrário teríamos $H'_\delta(P(z)-P(v)) = 0$. Observe também que se $P(z)-P(v) \geq 0$, então $H'_\delta(P(z)-P(v)) = 0$. Portanto, temos $H'_\delta(s) \neq 0$ apenas para $s \in (0,\delta)$. Assim, para $0 < P(z)-P(v) \leq \delta$, temos

$$0 \ \leq \ H'_\delta(P(z)-P(v))(P'(z)-P'(v)) \ \leq \ \frac{C}{\delta}P''(\xi_1)(z-v),$$

para $\xi_1 \in (v,z)$. Como $P''(\xi_1) \leq \widetilde{C}$ e $z - v \leq \frac{\delta}{V_R}$, temos

$$0 \leq H'_\delta(P(z)-P(v))(P'(z)-P'(v)) \leq \frac{C}{\delta}P''(\xi_1)(z-v) \leq \frac{C}{\delta}\widetilde{C}\frac{\delta}{V_R} = \widehat{C}$$

Lembre-se que $\mathbb{F}_\delta(\mathrm{u},\mathrm{v}) = \int_{V_R}^{\mathrm{u}} H_\delta(P(z)-P(\mathrm{v}))dz$, com $z \in [\mathrm{u},\mathrm{v}]$, $\mathrm{v} > 0$. Assim, se $z = \mathrm{v}$, então o integrando é nulo. Por outro lado, para qualquer $z > \mathrm{v} > V_R > 0$ e $\delta \ll 1$, teremos $P(z)-P(\mathrm{v}) > \delta$. Logo, $H'_\delta(P(z)-P(\mathrm{v})) = 0$. Portanto,

$$\mathbb{G}_\delta(\mathrm{u},\mathrm{v}) \to 0, \text{ ao } \delta \to 0.$$

Com essas observações acima e com o fato de que $\zeta_R(x)$ e $v_t(x,t)$ são limitadas e independem de δ, segue que

$$\int_0^T \int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{G}_\delta(u(x,t),v(x,t))v_t(\underline{x},t)d\underline{x}dt \longrightarrow 0 \text{ ao } \delta \to 0.$$

Agora analisaremos

$$\mathbb{F}_\delta(\mathrm{u},\mathrm{v}) = \int_{V_R}^{\mathrm{u}} H_\delta(P(z)-P(\mathrm{v}))dz, \quad \forall \mathrm{u},\mathrm{v} \in \mathbb{R}$$

e

$$\int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{F}_\delta(u(x,T),v(x,T))d\underline{x}.$$

Já sabemos que se $u \le v$ então $H_\delta(P(u)-P(v)) = 0$. Com isso, basta analisar o caso $u > v$. Note que

$$\int_{V_R}^{u} H_\delta(P(z)-P(v))dz = \int_v^u H_\delta(P(z)-P(v))dz.$$

Assim,

$$\mathbb{F}_\delta(u,v) = \int_v^u H_\delta(P(z)-P(v))dz \xrightarrow{\delta\to 0} \int_v^u dz = u-v,$$

pois $H_\delta(\mathsf{u}) \to 1$, quando $\delta \to 0^+$. Para não ficar carregando o fato de que $u > v$, diremos que

$$\mathbb{F}_\delta(u,v) \xrightarrow{\delta\to 0} (u-v)_+.$$

Com isso, a análise da integral fica

$$0 \le \int_{R_{u,v}} \zeta_R(\underline{x})\mathbb{F}_\delta(u(x,T),v(x,T))d\underline{x} \xrightarrow{\delta\to 0} \int_{R_{u,v}} \zeta_R(\underline{x})(u(x,T)-v(x,T))_+d\underline{x}$$

Analisaremos as demais integrais de (2.116) e veremos que

$$0 \le \int_{R_{u,v}} \zeta_R(\underline{x})(u(x,T)-v(x,T))_+d\underline{x} \le 0.$$

E como $\zeta_R(\underline{x}) > 0$, concluímos que $(u(x,T)-v(x,T))_+ = 0$. Logo, $u(x,T) = v(x,T)$, ou seja, $u = v$ em T.

$\star$ A análise da seguinte integral é feita com o Teorema da Convergência Dominada. Para usá-lo devemos mostrar uma convergência pontual e uma dominação.

$$\int_0^T \int_{R_{u,v}} H'_\delta(P(u)-P(v))\zeta_R(\underline{x})[f]\cdot\nabla(P(u)-P(v))d\underline{x}dt.$$

Note que $\zeta_R(\underline{x})[f]\cdot\nabla(P(u)-P(v))$ é constante em δ e, como já vimos, $H'_\delta(P(u)-P(v))\to 0$ ao $\delta\to 0$. Logo, temos a convergência pontual do integrando. Vamos a limitação. Analisaremos

$$|H'_\delta(P(u)-P(v))|\cdot|\zeta_R(\underline{x})|\cdot|[f]|\cdot|\nabla(P(u)-P(v))|$$

Vimos que $H'_\delta(P(u)-P(v))\neq 0$ para $u>v$. Além disso, $0<P(u)-P(v)<\dfrac{C}{\delta}$, para alguma constante C. Agora observe que

$$\begin{aligned}
|[f]| &= \sqrt{\sum_{j=1}^{n}(f_j(x,t,u)-f_j(x,t,v))^2}=\sqrt{\sum_{j=1}^{n}(\frac{\partial}{\partial \mathsf{u}}f_j(x,t,\xi)(u-v))^2}\\
&= (u-v)\sqrt{\sum_{j=1}^{n}(\frac{\partial}{\partial \mathsf{u}}f_j(x,t,\xi))^2}
\end{aligned}$$

Como $\dfrac{\partial f}{\partial \mathsf{u}}\in C^1$ por hipótese e estamos num compacto, então $\dfrac{\partial f}{\partial \mathsf{u}}$ é uniformemente contínua, portanto, limitada por uma constante C_1. Vimos anteriormente que de $0<P(u)-P(v)<\delta$ e do Teorema do Valor Médio, temos

$$\delta>P'(\xi)(u-v)=\xi^\alpha(u-v)>V_R^\alpha(u-v)\quad\Rightarrow(u-v)<\frac{\delta}{V_R^\alpha}.$$

Assim,

$$|[f]|\leq C\frac{\delta}{V_R^\alpha}.$$

Para $\xi\in(v,u)$, temos

$$|\nabla(P(u)-P(v))|=|\nabla(P'(\xi)(u-v))|\leq C,$$

pois u é bem comportada nas regiões onde $u>v$. Assim, com essas análises concluímos

$$|H'_\delta(P(u)-P(v))|\cdot|\zeta_R(\underline{x})|\cdot|[f]|\cdot|\nabla(P(u)-P(v))|\leq C\cdot C_2\cdot C_1\cdot\frac{\delta}{V_R^\alpha}\cdot\frac{C_2}{\delta}=\widetilde{C}.$$

Com essas observações e com o Teorema da Convergência Dominada, obtemos

$$\int_0^T\int_{R_{u,v}}H'_\delta(P(u)-P(v))\zeta_R(\underline{x})\,[f]\cdot\nabla(P(u)-P(v))d\underline{x}dt\longrightarrow 0 \text{ ao } \delta\to 0.$$

$\star$ Análise de

$$\int_0^T\int_{R_{u,v}}H_\delta(P(u)-P(v))\nabla(P(u)-P(v))\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt.$$

Considere o seguinte funcional

$$\mathbb{S}_\delta(\mathsf{u}) := \int_0^{\mathsf{u}} H_\delta(s)ds \xrightarrow{\delta\to 0} \mathsf{u}_+,$$

pois estamos considerando $u > v > 0$. Além disso, $\mathbb{S}_\delta(\mathsf{u}) = H_\delta(\mathsf{u})$. Com isso temos a seguinte sequência de estimativas

$$\begin{aligned}
&-\int_0^T \int_{R_{u,v}} H_\delta(P(u)-P(v))\nabla(P(u)-P(v))\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt \\
&\qquad = -\int_0^T \int_{R_{u,v}} \nabla\mathbb{S}_\delta(P(u)-P(v))\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt \\
&\qquad = \int_0^T \int_{R_{u,v}} \mathbb{S}_\delta(P(u)-P(v))\Delta\zeta_R(\underline{x})d\underline{x}dt \\
&\qquad\quad -\int_0^T \int_{R_{u,v}} \mathbb{S}_\delta(P(u)-P(v))\frac{\partial}{\partial\vec{n}}\zeta_R(\underline{x})d\sigma(\underline{x})dt.
\end{aligned}$$

Logo, ao $\delta \to 0$ obtemos

$$\begin{aligned}
&\int_0^T \int_{R_{u,v}} (P(u)-P(v))_+\Delta\zeta_R(\underline{x})d\underline{x}dt \; - \\
&-\int_0^T \int_{\{\{|x|=R\}\cap\{u>v\}\}} (P(u)-P(v))_+\frac{\partial}{\partial\vec{n}}\zeta_R(\underline{x})d\sigma(\underline{x})dt \\
\leq\; &\int_0^T \int_{R_{u,v}} (P(u)-P(v))_+|\Delta\zeta_R(\underline{x})|d\underline{x}dt \; + \\
&+\int_0^T \int_{\{\{|x|=R\}\cap\{u>v\}\}} (P(u)-P(v))_+\left|\frac{\partial}{\partial\vec{n}}\zeta_R(\underline{x})\right|d\sigma(\underline{x})dt \\
\leq\; & n\int_0^T \int_{R_{u,v}} (P(u)-P(v))_+\exp(-\sqrt{1+|x|^2})d\underline{x}dt \; + \\
&+C\exp(-\sqrt{1+|R|^2})\int_0^T \int_{\{\{|x|=R\}\cap\{u>v\}\}} (P(u)-P(v))_+d\sigma(\underline{x})dt.
\end{aligned}$$

$\star$ Análise de

$$\int_0^T \int_{R_{u,v}} H_\delta(P(u)-P(v))[\overrightarrow{f}]\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt.$$

Note que $[\overrightarrow{f}]\cdot\nabla\zeta_R(\underline{x})$ independe de δ e estamos considerando a itegral em $u > v$. Além disso, estes termos são limitados e $H_\delta \to 1$ ao $\delta \to 0$. Assim,

$$\int_0^T \int_{R_{u,v}} H_\delta(P(u)-P(v))[\overrightarrow{f}]\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt \quad \xrightarrow{\delta\to 0} \quad \int_0^T \int_{R_{u,v}} [\overrightarrow{f}]\cdot\nabla\zeta_R(\underline{x})d\underline{x}dt$$

Como

$$\int_0^T \int_{R_{u,v}} [\overrightarrow{f}] \cdot \nabla \zeta_R(\underline{x}) d\underline{x} dt \leq \int_0^T \int_{R_{u,v}} [\overrightarrow{f}] \cdot \nabla \zeta_R(\underline{x}) d\underline{x} dt$$

E, pelo Teorema do valor Médio

$$\begin{aligned}
\int_0^T \int_{R_{u,v}} [\overrightarrow{f}] \cdot \nabla \zeta_R(\underline{x}) d\underline{x} dt &\leq \int_0^T \int_{R_{u,v}} C_f |u-v| C \exp - \sqrt{1+|x|^2} d\underline{x} dt \\
&\leq C_f C \int_0^T \int_{R_{u,v}} (u-v)_+ \exp - \sqrt{1+|x|^2} d\underline{x} dt.
\end{aligned}$$

Temos,

$$\int_0^T \int_{R_{u,v}} H_\delta(P(u)-P(v)) [\overrightarrow{f}] \cdot \nabla \zeta_R(\underline{x}) d\underline{x} dt \leq C_f C \int_0^T \int_{R_{u,v}} (u-v)_+ \exp - \sqrt{1+|x|^2} d\underline{x} dt.$$

Logo, juntando as análises $\star$, a estimativa (2.116) fica:

$$\begin{aligned}
&\int_{R_{u,v}} \zeta_R(\underline{x})(u(x,T)-v(x,T))_+ d\underline{x} \leq \\
&\qquad \leq C_f C \int_0^T \int_{R_{u,v}} (u-v)_+ \exp(-\sqrt{1+|x|^2}) d\underline{x} dt \\
&\qquad + n \int_0^T \int_{R_{u,v}} (P(u)-P(v))_+ \exp(\sqrt{1+|x|^2}) d\underline{x} dt \\
&\qquad + C \exp(-\sqrt{1+R^2}) \int_0^T \int_{R_{u,v}} (P(u)-P(v))_+ d\underline{x} dt.
\end{aligned}$$

Pelo Teorema do Valor Médio, temos

$$P(u) - P(v) = P'(\xi)(u-v) = \xi^\alpha (u-v) \leq M(\widehat{T})^\alpha (u-v).$$

Assim,

$$\begin{aligned}
&\int_{R_{u,v}} (\exp(-\sqrt{1+|x|^2}) - \exp(-\sqrt{1+R^2}))(u(x,T)-v(x,T))_+ d\underline{x} \leq \\
&\qquad \leq C_f C \int_0^T \int_{R_{u,v}} (u-v)_+ \exp(-\sqrt{1+|x|^2}) d\underline{x} dt \\
&\qquad + n M(\widehat{T})^\alpha \int_0^T \int_{R_{u,v}} (P(u)-P(v))_+ \exp(\sqrt{1+|x|^2}) d\underline{x} dt \\
&\qquad + C M(\widehat{T})^\alpha \exp(-\sqrt{1+R^2}) \int_0^T \int_{R_{u,v}} (P(u)-P(v))_+ d\underline{x} dt.
\end{aligned}$$

Ao fazer $\delta \to 0$, usamos o Teorema da Convergência Monótona e obtemos

$$0 \leq \int_{\mathbb{R}^n \cap \{u>v\}} \exp(-\sqrt{1+|x|^2})(u(x,T)-v(x,T))_+ d\underline{x} \leq$$

$$
\begin{aligned}
&\leq \; C_f C \int_0^T \int_{\mathbb{R}^n \cap \{u>v\}} (u-v)_+ \exp(-\sqrt{1+|x|^2}) d\underline{x} dt \\
&\quad + nM(\widehat{T})^{\alpha} \int_0^T \int_{\mathbb{R}^n \cap \{u>v\}} (P(u)-P(v))_+ \exp(\sqrt{1+|x|^2}) d\underline{x} dt \\
&\leq \; \mathbb{K}(\widehat{T}) \int_0^T \int_{\mathbb{R}^n \cap \{u>v\}} \exp(\sqrt{1+|x|^2})(u-v))_+ d\underline{x} dt,
\end{aligned}
$$

para todo $0 < T \leq \widehat{T}$. Ou seja,

$$
\begin{aligned}
0 \leq \int_{\mathbb{R}^n \cap \{u>v\}} \exp(-\sqrt{1+|x|^2})(u(x,T)-v(x,T))_+ d\underline{x} \leq& \\
\leq \; \mathbb{K}(\widehat{T}) \int_0^T \int_{\mathbb{R}^n \cap \{u>v\}} \exp(\sqrt{1+|x|^2})(u-v))_+ d\underline{x} dt,&
\end{aligned}
$$

para todo $0 < T \leq \widehat{T}$. Agora estamos nas condições do Lema da Gronwall, onde $\Psi(t) = 0$ e

$$
\Phi(t) := \int_{\mathbb{R}^n \cap \{u>v\}} \exp(-\sqrt{1+|x|^2})(u(x,T)-v(x,T))_+ d\underline{x}.
$$

Então

$$
\Phi(t) \leq 0 \cdot \exp\left(\mathbb{K}(\widehat{T}) \int_0^T 1 dt \right) = 0.
$$

Logo,

$$
0 \;\; \leq \;\; \int_{\mathbb{R}^n \cap \{u>v\}} \exp(-\sqrt{1+|x|^2})(u(x,T)-v(x,T))_+ d\underline{x} \;\; \leq \;\; 0.
$$

Como $\exp(-\sqrt{1+|x|^2}) \neq 0$, $\forall\, \underline{x} \in \mathbb{R}^n$, devemos ter $(u(x,T)-v(x,T))_+ = 0$. Assim,

$$
u(x,T) \; \leq \; v(x,T), \qquad \forall\, \underline{x} \in \mathbb{R}^n.
$$

Note que tomamos $0 < T \leq \widehat{T}$ qualquer e fazendo o limite com $t \to 0$, obtemos

$$
u(x,t) \; \leq \; v(x,t), \qquad \forall\, \underline{x} \in \mathbb{R}^n \qquad \text{e} \qquad \forall\, 0 \leq t \leq \widehat{T}.
$$

O que conclui o Teorema da Comparação.

■

2.8 Análise de Escalas

Considere λ, L e θ constantes positivas e defina

$$\widetilde{u}(x,t) := \lambda u(Lx, \theta t).$$

Assim,

$$\widetilde{u}_t(x,t) := \lambda\theta u_t(Lx,\theta t)$$

$$\widetilde{u}_x(x,t) := \lambda L u_x(Lx,\theta t)$$

$$\widetilde{u}_{xx}(x,t) := \lambda L u_{xx}(Lx,\theta t).$$

Consideraremos soluções positivas da equação

$$u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u^\alpha u_x)_x.$$

Desenvolvendo as derivadas espaciais, temos

$$u_t + \frac{\partial}{\partial x} b(x,t)u^{\kappa+1} + (\kappa+1)b(x,t)u^\kappa u_x = \mu(t)\alpha u^{\alpha-1}u_x^2 + \mu(t)u^\alpha u_{xx}.$$

Assim,

$$\begin{aligned}
\widetilde{u}_t = \;& \lambda\theta u_t(Lx,\theta t)\\
= \;& -\lambda\theta\frac{\partial}{\partial x} b(Lx,\theta t)u^{\kappa+1}(Lx,\theta t) - (\kappa+1)b(Lx,\theta t)u^\kappa(Lx,\theta t)u_x(Lx,\theta t)\\
& +\mu(\theta t)\alpha u^{\alpha-1}(Lx,\theta t)u_x^2(Lx,\theta t) + \mu(\theta t)u^\alpha(Lx,\theta t)u_{xx}(Lx,\theta t)\\
= \;& -\lambda^{-\kappa}L^{-1}\theta\frac{\partial}{\partial x}\widehat{b}(x,t)\widetilde{u}^{\kappa+1}(x,t) - \lambda^{-\kappa}L^{-1}\theta(\kappa+1)\widehat{b}(x,t)\widetilde{u}^\kappa(x,t)\widetilde{u}_x(x,t)\\
& +\lambda^{-\alpha}L^{-2}\theta\mu(\theta t)\alpha\widetilde{u}^{\alpha-1}(x,t)\widetilde{u}_x^2(x,t) + \lambda^{-\alpha}L^{-2}\theta\mu(\theta t)\widetilde{u}^\alpha(x,t)\widetilde{u}_{xx}(x,t).
\end{aligned}$$

Onde $\widehat{b}(x,t) = b(Lx,\theta t)$. Logo, $\dfrac{\partial\widehat{b}}{\partial x}(x,t) = \dfrac{\partial b}{\partial x}(Lx,\theta t)$. Então, a equação para $\widetilde{u}$ é:

$$\widetilde{u}_t(x,t) + (\widetilde{b}(x,t)\widetilde{u}^{\kappa+1})_x = \widetilde{\mu}(t)(\widetilde{u}^\alpha\widetilde{u}_x)_x,$$

onde $\widetilde{b}(x,t) = \lambda^{-\kappa}L^{-1}\theta\widehat{b}(x,t) = \lambda^{-\kappa}L^{-1}\theta b(Lx,\theta t)$ e $\widetilde{\mu}(t) = \lambda^{-\alpha}L^{-2}\theta\mu(\theta t)$. Considere,

$$\widetilde{B}_{\widetilde{u}} = \limsup_{t\to\infty}\frac{\widetilde{B}(t)}{\widetilde{\mu}(t)}$$

$$\widetilde{B}(t) = \frac{1}{2}\left(\sup\widetilde{b}(x,t) - \inf\widetilde{b}(x,t)\right) = \lambda^{-\kappa}L^{-1}\theta\frac{1}{2}\left(\sup\widetilde{b}(Lx,\theta t) - \inf\widetilde{b}(Lx,\theta t)\right)$$

Ou seja,

$$\widetilde{B}(t) = \lambda^{-\kappa}L^{-1}\theta B(\theta t).$$

Logo,

$$\widetilde{B}_{\widetilde{u}} = \limsup_{t\to\infty} \frac{\widetilde{B}(t)}{\widetilde{\mu}(t)} = \limsup_{t\to\infty} \frac{\lambda^{-\kappa}L^{-1}\theta B(\theta t)}{\lambda^{-\alpha}L^{-2}\theta\mu(\theta t)} = \lambda^{\alpha-\kappa}L \limsup_{t\to\infty} \frac{B(\theta t)}{\mu(\theta t)}.$$

Ou seja,

$$\widetilde{B}_{\widetilde{u}} = \lambda^{\alpha-\kappa}LB_{\mu}.$$

Definindo

$$\widetilde{\mathsf{U}}_q := \limsup_{t\to\infty} \|\widetilde{u}(\cdot,t)\|_{L^q(\mathbb{R})}, \qquad p_0 \le q < \infty$$

e observando que $\|\widetilde{u}(\cdot,t)\|_{L^q(\mathbb{R})} = \lambda L^{-\frac{1}{q}}\|u(\cdot,\theta t)\|_{L^q(\mathbb{R})}$, então

$$\widetilde{\mathsf{U}}_q = \lambda L^{-\frac{1}{q}}\mathsf{U}_q.$$

Suponhamos que $\mathsf{U}_\infty \le K(p,\alpha,\kappa)B_\mu^\delta \mathsf{U}_p^\gamma$, com $0 \le B_\mu < \infty$ e $0 \le \mathsf{U}_p < \infty$. Então a única relevância é obtida quando $0 < B_\mu < \infty$ e $0 < \mathsf{U}_p < \infty$. Nestes casos, as relações acima também são válidas para $\widetilde{\mathsf{U}}_\infty$ e vale

$$\widetilde{\mathsf{U}}_\infty \le K(p,\alpha,\kappa)\widetilde{B}_\mu^\delta \widetilde{\mathsf{U}}_p^\gamma.$$

Se escolhermos argumentos de escalas $L, \theta, \lambda > 0$ tais que $\widetilde{B}_{\widetilde{\mu}} = 1$ e $\widetilde{\mathsf{U}}_p = 1$, então $\widetilde{\mathsf{U}}_\infty \le K(p,\alpha,\kappa)$ e temos a identidade

$$\lambda^{\alpha-\kappa}LB_\mu = \lambda L^{\frac{-1}{p}}\mathsf{U}_p.$$

Assim,

$$\lambda^{-1} = B_\mu^{\frac{1}{p-\kappa+\alpha}}\mathsf{U}_p^{\frac{p}{p-\kappa+\alpha}}.$$

Como

$$\widetilde{\mathsf{U}}_\infty = \limsup_{t\to\infty}\|\widetilde{u}(\cdot,t)\|_\infty = \lambda \limsup_{t\to\infty}\|u(\cdot,\theta t)\|_\infty = \lambda\mathsf{U}_\infty,$$

obtemos

$$\mathsf{U}_\infty \le K(p,\alpha,\kappa)B_\mu^{\frac{1}{p+\alpha-\kappa}} \cdot \mathsf{U}_p^{\frac{p}{p+\alpha-\kappa}}.$$

Portanto,

$$\delta = \frac{1}{p+\alpha-\kappa} \quad \text{e} \quad \gamma = \frac{p}{p+\alpha-\kappa}.$$

Note que devemos agregar mais uma condição sobre p, α e κ para que o resultado seja válido, a de que $p > \kappa - \alpha$. Além dessa condição, temos também que $p \ge p_0$.

Observação 2.4. Cabe aqui observar que é natural requerer $p > \kappa - \alpha$, pois no caso de $\kappa = 0$, temos $1 + \dfrac{\alpha}{p} > 0$.

2.9 Estimativa e decaimento no caso particular $\dfrac{\partial b}{\partial x}(x,t) \geq 0$.

O objetivo desta seção é provar a estimativa de decaimento

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(p,p_0) \cdot \|u_0\|^{\delta}_{L^{p_0}(\mathbb{R})} \cdot t^{-\gamma},$$

para todo $t > 0$, onde $\delta = \delta(p,p_0) > 0$, $\gamma = \gamma(p,p_0) > 0$, $K(p,p_0) > 0$ e $u(x,t)$ é uma solução suave positiva da equação de difusão-advecção

$$\begin{cases} u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u^\alpha u_x)_x, \qquad \alpha > 0 \qquad \text{e} \qquad \kappa \geq 0 \\ \\ u(x,0) = u_0(x) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}); \qquad 1 \leq p_0 < \infty. \end{cases} \tag{2.117}$$

Para obtermos este resultado assumiremos uma hipótese adicional às já utilizadas nas seções anteriores. Assumiremos $b(x,t)$, $\dfrac{\partial b}{\partial x}(x,t) \in C^0(\mathbb{R}\times[0,+\infty[, L^\infty(\mathbb{R}))$, onde $0 < T_* \leq \infty$ e $\dfrac{\partial b}{\partial x}(x,t) \geq 0$, $\forall\, x \in \mathbb{R}$ e $\forall\, t > 0$. Cabe ressaltar que a solução que nos interessa é, para $t > 0$, $u(\cdot,t) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, ou seja, limitada numa faixa de tempo positivo.

Antes de enunciar nosso primeiro resultado, queremos observar que usaremos a notação

$$\int_{R_x} f(x)dx := \int_{R\leq|x|\leq 2R} f(x)dx.$$

Lema 2.10. *Seja $u(x,t)$ solução suave positiva da equação* (2.117)*, então $\|u(\cdot,t)\|_{L^q(\mathbb{R})}$ é decrescente em $L^q(\mathbb{R})$, $\forall\, p_0 \leq q \leq +\infty$ e*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})}.$$

Prova: Sejam $t > 0$, $p_0 \geq 1$ e $\psi_R(x)$ a função de corte definida em (2.4) na seção 2.2. Multiplicando a equação

$$u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u^\alpha u_x)_x \tag{2.118}$$

por $qu^{q-1}(x,t)\psi_R(x)$ e integrando em $[0,t]\times\mathbb{R}$, teremos

$$\int_0^t \int_{|x|\leq 2R} \frac{\partial u^q}{\partial \tau}(x,\tau)\psi_R(x)\, dxd\tau + q\int_0^t \int_{|x|\leq 2R} u^{q-1}(x,t)(b(x,\tau)u^{\kappa+1})_x\psi_R(x)\, dxd\tau$$

$$= q\int_0^t \mu(\tau) \int_{|x|\leq 2R} u^{q-1}(x,t)(u^\alpha u_x)_x \psi_R(x)\, dxd\tau$$

Pela função de corte $\psi_R(x)$ podemos usar o Teorema de Fubini e o Teorema de integração por partes. Obtendo

$$\begin{aligned}
&\int_{|x|\leq 2R} u^q(x,t)\psi_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|\leq 2R} u^{q+\alpha-2}(x,t)u_x^2(x,t)\psi_R(x)\,dxd\tau \\
&\quad = \int_{|x|\leq 2R} u_0(x)\psi_R(x)dx + q\int_0^t \mu(\tau)\int_{R_x} \frac{\partial}{\partial x}\frac{u^{q+\alpha}}{q+\alpha}\psi_R'(x)dxd\tau \\
&\quad +q(q-1)\int_0^t\int_{|x|\leq 2R}\frac{\partial}{\partial x}\frac{u^{q+\kappa}}{q+\kappa}b(x,\tau)\psi_R(x)dxd\tau \qquad (2.119)\\
&\quad +q\int_0^t\int_{R_x} u^{q+\kappa}\psi_R'(x)dxd\tau.
\end{aligned}$$

Como $\displaystyle\int_0^t \mu(\tau)\int_{|x|\leq 2R} u^{q+\alpha-2}(x,t)u_x^2(x,t)\psi_R(x)\,dxd\tau \geq 0$ e

$$\begin{aligned}
q(q-1)\int_0^t\int_{|x|\leq 2R}\frac{\partial}{\partial x}\frac{u^{q+\kappa}}{q+\kappa}(x,\tau)b(x,\tau)\psi_R(x)dxd\tau &= \\
&= -\frac{q(q-1)}{q+\kappa}\int_0^t\int_{|x|\leq 2R} u^{q+\kappa}(x,\tau)\frac{\partial}{\partial x}b(x,\tau)\psi_R(x)dxd\tau \\
&\quad -\frac{q(q-1)}{q+\kappa}\int_0^t\int_{R_x} u^{q+\kappa}(x,\tau)b(x,\tau)\psi_R'(x)dxd\tau \\
&\leq -\frac{q(q-1)}{q+\kappa}\int_0^t\int_{R_x} u^{q+\kappa}(x,\tau)b(x,\tau)\psi_R'(x)dxd\tau,
\end{aligned}$$

(pois estamos supondo $\dfrac{\partial b}{\partial x}(x,t)\geq 0$) então a equação (2.131) fica

$$\begin{aligned}
\int_{|x|\leq 2R} u^q(x,t)\psi_R(x)dx &\leq \int_{|x|\leq 2R} u_0(x)\psi_R(x)dx \\
&\quad +\frac{q}{q+\alpha}\int_0^t\mu(\tau)\int_{R_x} u^{q+\alpha}\psi_R''(x)dxd\tau \\
&\quad -\frac{q(q-1)}{q+\kappa}\int_0^t\int_{R_x} u^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau \qquad (2.120)\\
&\quad +q\int_0^t\int_{R_x} u^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau.
\end{aligned}$$

Pelas propriedades de integrais e módulos, e, pelas estimativas (2.15), (2.18) da seção 2.2, provém de (2.132)

$$
\begin{aligned}
\int_{|x|\leq 2R} u^q(x,t)\psi_R(x)dx \quad \leq \quad & \int_{|x|\leq 2R} u_0(x)\psi_R(x)dx \\
& + \frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{R_x} M(T)^{q+\alpha}\psi_R''(x)dxd\tau \\
& - \frac{q(q-1)}{q+\kappa}\int_0^t \int_{R_x} M(T)^{q+\kappa}\mathfrak{b}(T)\psi_R'(x)dxd\tau \\
& + q\int_0^t \int_{R_x} M(T)^{q+\kappa}\mathfrak{b}(T)\psi_R'(x)dxd\tau.
\end{aligned}
$$

Assim, ao $R \nearrow \infty$, temos

$$
\int_{\mathbb{R}} u^q(x,t)dx \leq \int_{\mathbb{R}} u_0^q(x)dx < \infty,
$$

pois $u_0(x) \in L^{p_0}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, com $1 \leq p_0 < \infty$. Logo, $u(x,t) \in L^q(\mathbb{R})$, para qualquer $t > 0$ e $p_0 \leq q < +\infty$ e $q > 1$. Ou seja, $\|u(\cdot,t)\|_{L^q(\mathbb{R})}$ é decrescente em $L^q(\mathbb{R})$, para todo $t > 0$ e $p_0 \leq q < +\infty$, ou seja,

$$
\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})}, \qquad \forall\, t > 0.
$$

Fazendo $q \nearrow +\infty$, teremos

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}.
$$

Portanto, $u(\cdot,t) \in L^{\infty}(\mathbb{R})$.

∎

Teorema 2.2. *Sejam $\mu(t)$ e $b(x,t)$ satisfazendo as hipóteses $(H1)$ a $(H4)$ em (2.12), dadas na seção 2.2, com $\frac{\partial}{\partial x}b(x,t) \geq 0$. Nestas condições a solução suave positiva $u(x,t)$ de*

$$
\begin{cases}
u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u^{\alpha}u_x)_x \\
u(x,0) = u_0(x) \in L^{p_0}(\mathbb{R}), \qquad 1 \leq p_0 < \infty
\end{cases}
\tag{2.121}
$$

com $\alpha > 0$, $\kappa \geq 0$, satisfaz

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\cdot \|u_0\|^{\delta}_{L^{p_0}(\mathbb{R})}\cdot t^{-\gamma}, \qquad \forall\, t > 0,
$$

onde $\delta = \dfrac{2p_0}{2p_0+\alpha}$ e $\gamma = \dfrac{1}{2p_0+\alpha}$.

Prova: Sejam $R > 0$ e $\psi_R(x)$ a função de corte definida em (2.4). Multiplicamos a equação (2.118) por $qu^{q-1}(x,t)(t-t_0)^\gamma \psi_R(x)$, onde $\gamma > 0$ será determinado posteriormente, e integramos em $[t_0, t] \times \mathbb{R}$, $0 < t_0 < t$ é qualquer. Assim,

$$
\begin{aligned}
&\int_{t_0}^{t} \int_{|x|\leq 2R} \frac{\partial}{\partial \tau}[u^q(x,\tau)](\tau-t_0)^\gamma \psi_R(x)\, dx d\tau + \\
&\qquad + q\int_{t_0}^{t} (\tau-t_0)^\gamma \int_{|x|\leq 2R} u^{q-1}(x,\tau)(b(x,\tau)u^{\kappa+1})_x \psi_R(x)\, dx d\tau \\
&= \int_{t_0}^{t} \mu(\tau)(\tau-t_0)^\gamma \int_{|x|\leq 2R} u^{q-1}(x,\tau)(u^\alpha(x,\tau)u_x(x,\tau))_x \psi_R(x)\, dx d\tau.
\end{aligned}
$$

Aplicamos os Teoremas de Fubini e Integração por Partes, obtemos

$$
\begin{aligned}
&(t-t_0)^\gamma \int_{|x|\leq 2R} u(x,t)^q \psi_R(x)\, dx - \\
&\qquad -\gamma \int_{t_0}^{t} (\tau-t_0)^{\gamma-1} \int_{|x|\leq 2R} u^q(x,\tau)\psi_R(x)\, dx\, d\tau - \\
&\qquad -q(q-1)\int_{t_0}^{t} (\tau-t_0)^\gamma \int_{|x|\leq 2R} u^{q+\kappa-1}(x,\tau)u_x(x,\tau)b(x,\tau)\psi_R(x)\, dx\, d\tau - \\
&\qquad -q\int_{t_0}^{t} (\tau-t_0)^\gamma \int_{R_x} b(x,\tau)u^{q+\kappa}(x,\tau)\psi_R'(x)\, dx\, d\tau = \\
&= -q(q-1)\int_{t_0}^{t} \mu(\tau)(\tau-t_0)^\gamma \int_{|x|\leq 2R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\psi_R(x)\, dx\, d\tau - \\
&\qquad -q\int_{t_0}^{t} \mu(\tau)(\tau-t_0)^\gamma \int_{R_x} u^{q+\alpha-1}(x,\tau)u_x(x,\tau)\psi_R'(x)\, dx\, d\tau.
\end{aligned}
$$

Reorganizando as integrais, temos

$$
\begin{aligned}
&(t-t_0)^\gamma \int_{|x|\leq 2R} u(x,t)^q \psi_R(x)\, dx + \\
&\qquad +q(q-1)\int_{t_0}^{t} \mu(\tau)(\tau-t_0)^\gamma \int_{|x|\leq 2R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\psi_R(x)\, dx\, d\tau \\
&= \gamma \int_{t_0}^{t} (\tau-t_0)^{\gamma-1} \int_{|x|\leq 2R} u^q(x,\tau)\psi_R(x)\, dx\, d\tau \\
&\qquad +q(q-1)\int_{t_0}^{t} (\tau-t_0)^\gamma \int_{|x|\leq 2R} \frac{\partial}{\partial x}\left[\frac{u^{q+\kappa}}{q+\kappa}(x,\tau)\right] b(x,\tau)\psi_R(x)\, dx\, d\tau \\
&\qquad +q\int_{t_0}^{t} (\tau-t_0)^\gamma \int_{R_x} b(x,\tau)u^{q+\kappa}(x,\tau)\psi_R'(x)\, dx\, d\tau \\
&\qquad -q\int_{t_0}^{t} \mu(\tau)(\tau-t_0)^\gamma \int_{R_x} \frac{\partial}{\partial x}\left[\frac{u^{q+\alpha}}{q+\alpha}(x,\tau)\right] \psi_R'(x)\, dx\, d\tau.
\end{aligned}
$$

Aplicando novamente o Teorema de integração por partes, temos

$$
\begin{aligned}
&(t-t_0)^\gamma \int_{|x|\le 2R} u^q(x,t)\psi_R(x)\,dx+\\
&\qquad +q(q-1)\int_{t_0}^t \mu(\tau)(\tau-t_0)^\gamma \int_{|x|\le 2R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\psi_R(x)\,dx\,d\tau\\
&= \gamma\int_{t_0}^t (\tau-t_0)^{\gamma-1}\int_{|x|\le 2R} u^q(x,\tau)\psi_R(x)\,dx\,d\tau\\
&\qquad -\frac{q(q-1)}{q+\kappa}\int_{t_0}^t (\tau-t_0)^\gamma \int_{|x|\le 2R} u^{q+\kappa}(x,\tau)b_x(x,\tau)\psi_R(x)\,dx\,d\tau\\
&\qquad -\frac{q(q-1)}{q+\kappa}\int_{t_0}^t (\tau-t_0)^\gamma \int_{R_x} u^{q+\kappa}(x,\tau)b(x,\tau)\psi_R'(x)\,dx\,d\tau\\
&\qquad +q\int_{t_0}^t (\tau-t_0)^\gamma \int_{R_x} b(x,\tau)u^{q+\kappa}(x,\tau)\psi_R'(x)\,dx\,d\tau\\
&\qquad -\frac{q}{q+\alpha}\int_{t_0}^t \mu(\tau)(\tau-t_0)^\gamma \int_{R_x} u^{q+\alpha}(x,\tau)\psi_R''(x)\,dx\,d\tau
\end{aligned}
\tag{2.122}
$$

Por hipótese, $b_x(x,t)\ge 0$, então (2.122) fica

$$
\begin{aligned}
&(t-t_0)^\gamma \int_{|x|\le 2R} u^q(x,t)\psi_R(x)\,dx+\\
&\qquad +q(q-1)\int_{t_0}^t \mu(\tau)(\tau-t_0)^\gamma \int_{|x|\le 2R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\psi_R(x)\,dx\,d\tau\\
&\le \gamma\int_{t_0}^t (\tau-t_0)^{\gamma-1}\int_{|x|\le 2R} u^q(x,\tau)\psi_R(x)\,dx\,d\tau\\
&\qquad -\frac{q(q-1)}{q+\kappa}\int_{t_0}^t (\tau-t_0)^\gamma \int_{R_x} u^{q+\kappa}(x,\tau)b(x,\tau)\psi_R'(x)\,dx\,d\tau+\\
&\qquad +q\int_{t_0}^t (\tau-t_0)^\gamma \int_{R_x} b(x,\tau)u^{q+\kappa}(x,\tau)\psi_R'(x)\,dx\,d\tau\\
&\qquad -\frac{q}{q+\alpha}\int_{t_0}^t \mu(\tau)(\tau-t_0)^\gamma \int_{R_x} u^{q+\alpha}(x,\tau)\psi_R''(x)\,dx\,d\tau
\end{aligned}
\tag{2.123}
$$

Das estimativas sobre $b(x,t)$ em (2.15) e sobre a solução $u(x,t)$ em (2.18) e (2.19) no capítulo 2, resulta de (2.123)

$$
\begin{aligned}
&(t-t_0)^\gamma \int_{|x|\le 2R} u^q(x,t)\psi_R(x)\,dx+\\
&\qquad\qquad +q(q-1)\int_{t_0}^t \mu(\tau)(\tau-t_0)^\gamma \int_{|x|\le 2R} u^{q+\alpha-2}(x,\tau)u_x^2(x,\tau)\psi_R(x)\,dx\,d\tau\\
&\qquad\le \gamma\int_{t_0}^t (\tau-t_0)^{\gamma-1}\int_{|x|\le 2R} u^q(x,\tau)\psi_R(x)\,dx\,d\tau-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{q(q-1)M(T)^{\kappa}\mathfrak{b}(T)}{q+\kappa}\int_{t_0}^{t}(\tau-t_0)^{\gamma}\int_{R_x}u^{q}(x,\tau)\psi_R'(x)\,dx\,d\tau+\\
&+qM(T)^{\kappa}\mathfrak{b}(T)\int_{t_0}^{t}(\tau-t_0)^{\gamma}\int_{R_x}u^{q}(x,\tau)\psi_R'(x)\,dx\,d\tau-\\
&-\frac{qM(T)^{\alpha}}{q+\alpha}\int_{t_0}^{t}\mu(\tau)(\tau-t_0)^{\gamma}\int_{R_x}u^{q}(x,\tau)\psi_R''(x)\,dx\,d\tau.
\end{aligned}
\tag{2.124}
$$

Fazendo $R \nearrow +\infty$ em (2.124), obtemos seguinte estimativa de energia, a qual é muito importante no que segue,

$$
\begin{aligned}
(t-t_0)^{\gamma}\int_{\mathbb{R}}u^{q}(x,t)\,dx+q(q-1)\int_{t_0}^{t}(\tau-t_0)^{\gamma}\mu(\tau)\int_{\mathbb{R}}u^{q+\alpha-2}(x,\tau)v_x^2\,dxd\tau\\
\leq \gamma\int_{t_0}^{t}(\tau-t_0)^{\gamma-1}\int_{\mathbb{R}}u^{q}(x,\tau)\,dxd\tau.
\end{aligned}
\tag{2.125}
$$

Consideremos uma nova variável

$$
\omega(x,t)=u(x,t)^{\frac{q+\alpha}{2}}. \tag{2.126}
$$

Logo, pelas definições de norma, temos

$$
\|v(\cdot,t)\|_{L^{q}(\mathbb{R})}^{q}=\|\omega(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\beta},\ \text{onde}\ \beta=\frac{2q}{q+\alpha}. \tag{2.127}
$$

Substituindo (2.126) e (2.127) em (2.125), obtemos a desigualdade fundamental desta secção

$$
\begin{aligned}
(t-t_0)^{\gamma}\|\omega(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\beta}+\frac{4q(q-1)}{(q+\alpha)^2}\int_{t_0}^{t}(\tau-t_0)^{\gamma}\,\|\omega_x(\cdot,t)\|_{L^2(\mathbb{R})}^{2}\,d\tau\leq\\
\leq \gamma\int_{t_0}^{t}(\tau-t_0)^{\gamma-1}\,\|\omega(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\beta}\,d\tau.
\end{aligned}
\tag{2.128}
$$

Neste momento assumimos que $\mu(t)\geq 1$. Considere o funcional de energia dado por

$$
E(t):=(t-t_0)^{\gamma}\|\omega(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\beta}+\frac{4q(q-1)}{(q+\alpha)^2}\int_{t_0}^{t}(\tau-t_0)^{\gamma-1}\,\|\omega_x(\cdot,t)\|_{L^2(\mathbb{R})}^{2}\,d\tau. \tag{2.129}
$$

Pela Desigualdade de NGS, temos a seguinte desigualdade de interpolação

$$
\|\omega(\cdot,t)\|_{L^{\beta}(\mathbb{R})}\leq C\|\omega(\cdot,t)\|_{L^{r}(\mathbb{R})}^{1-\theta}\cdot\|\omega_x(\cdot,t)\|_{L^2(\mathbb{R})}^{\theta}, \tag{2.130}
$$

onde

$$r = \frac{2p_0}{q+\alpha}, \qquad \theta = \frac{1-\frac{p_0}{q}}{1+\frac{p_0}{q+\alpha}} \quad \text{e} \quad C = C(q,p_0) = \left(\frac{2}{1+\frac{p_0}{q+\alpha}}\right)^{-\theta}. \tag{2.131}$$

Substituindo (2.130) em (2.128) e utilizando o Teorema (2.10), temos

$$\begin{aligned}
(t-t_0)^{\gamma}\|\omega(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \beta^2\left(1-\frac{1}{q}\right)\int_{t_0}^{t}(\tau-t_0)^{\gamma}\|\omega_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}d\tau \leq \\
\leq \gamma C^{\beta}\int_{t_0}^{t}(\tau-t_0)^{\gamma-1}\|\omega(\cdot,\tau)\|^{(1-\theta)\beta}_{L^r(\mathbb{R})}\|\omega_x(\cdot,\tau)\|^{\theta\beta}_{L^2(\mathbb{R})}d\tau \\
\leq \gamma C^{\beta}\|\omega(\cdot,t_0)\|^{(1-\theta)\beta}_{L^r(\mathbb{R})}\int_{t_0}^{t}(\tau-t_0)^{\gamma-1}\|\omega_x(\cdot,\tau)\|^{\theta\beta}_{L^2(\mathbb{R})}d\tau,
\end{aligned} \tag{2.132}$$

onde r, θ e C são dados em (2.131). Escrevendo

$$\begin{aligned}
&(\tau-t_0)^{\gamma-1}\|\omega_x(\cdot,\tau)\|^{\theta\beta}_{L^2(\mathbb{R})} = \\
&= (\tau-t_0)^{\gamma\left(1-\frac{\theta\beta}{2}\right)-1}\beta^{-\theta\beta}\left(1-\frac{1}{q}\right)^{\frac{-\theta\beta}{2}}\left[\beta^2\left(1-\frac{1}{q}\right)(\tau-t_0)^{\gamma}\|\omega_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}\right]^{\frac{\theta\beta}{2}}
\end{aligned}$$

em (2.132) e aplicando o Teorema de Hölder, com $p' = \dfrac{2}{\theta\beta}$ e $q' = \dfrac{2}{2-\theta\beta}$, onde p' e q' satisfazem $\dfrac{1}{p'}+\dfrac{1}{q'} = 1$, temos a seguinte expressão para a desigualdade (2.132)

$$\begin{aligned}
(t-t_0)^{\gamma}\|\omega(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \beta^2\left(1-\frac{1}{q}\right)\int_{t_0}^{t}(\tau-t_0)^{\gamma}\|\omega_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}d\tau \leq \\
\leq \quad \gamma C^{\beta}\|\omega(\cdot,t_0)\|^{(1-\theta)\beta}_{L^r(\mathbb{R})}\left(\beta^2\left(1-\frac{1}{q}\right)\int_{t_0}^{t}(\tau-t_0)^{\gamma}\|\omega_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac{\theta\beta}{2}}\cdot \\
\cdot\beta^{-\theta\beta}\left(1-\frac{1}{q}\right)^{-\frac{\theta\beta}{2}}\left(\int_{t_0}^{t}(\tau-t_0)^{\gamma-\frac{2}{2-\theta\beta}}\,d\tau\right)^{1-\frac{\theta\beta}{2}}.
\end{aligned} \tag{2.133}$$

Observe que o termo $(\tau-t_0)^{\gamma-\frac{2}{2-\theta\beta}}$ é integrável desde que $\gamma - \dfrac{2}{2-\theta\beta} \geq -1$. A única suposição feita sobre γ é que γ é positivo. Então, definimos

$$\gamma := \frac{2}{2-\theta\beta}. \tag{2.134}$$

Assim, a desigualdade (2.133) fica

$$\begin{aligned}
E(t) = (t-t_0)^{\gamma}\|\omega(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \beta^2\left(1-\frac{1}{q}\right)\int_{t_0}^{t}(\tau-t_0)^{\gamma}\|\omega_x(\cdot,\tau)\|^2_{L^2(\mathbb{R})}d\tau \leq \\
\leq \quad \gamma C^{\beta}\|\omega(\cdot,t_0)\|^{(1-\theta)\beta}_{L^r(\mathbb{R})}E(t)^{\frac{\theta\beta}{2}}\cdot\beta^{-\theta\beta}\left(1-\frac{1}{q}\right)^{-\frac{\theta\beta}{2}}(\tau-t_0)^{1-\frac{\theta\beta}{2}}.
\end{aligned} \tag{2.135}$$

Supondo $E(t) \neq 0$, teremos

$$E(t)^{1-\frac{\theta\beta}{2}} \leq \gamma C^{\beta} \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{(1-\theta)\beta} \cdot \beta^{-\theta\beta} \left(1-\frac{1}{q}\right)^{-\frac{\theta\beta}{2}} (\tau-t_0)^{1-\frac{\theta\beta}{2}},$$

ou seja,

$$E(t) = (t-t_0)^{\gamma} \|\omega(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\beta} + \beta^2 \left(1-\frac{1}{q}\right) \int_{t_0}^{t} (\tau-t_0)^{\gamma} \|\omega_x(\cdot,\tau)\|_{L^2(\mathbb{R})}^2 d\tau \leq$$

$$\leq \gamma^{\gamma}\, C^{\gamma\beta} \beta^{-\theta\beta\gamma} \left(1-\frac{1}{q}\right)^{-\frac{\theta\beta\gamma}{2}} \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{(1-\theta)\beta\gamma} \cdot (t-t_0). \tag{2.136}$$

Da desigualdade 2.136 decorre, em particular,

$$\int_{t_0}^{t} (\tau-t_0)^{\gamma} \|\omega_x(\cdot,\tau)\|_{L^2(\mathbb{R})}^2 d\tau \leq \gamma^{\gamma} \cdot C^{\beta\gamma} \cdot \beta^{-(\theta\beta\gamma+2)} \cdot \left(1-\frac{1}{q}\right)^{-\left(\frac{\theta\beta\gamma}{2}+1\right)} \cdot$$

$$\cdot \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{(1-\theta)\beta\gamma} \cdot (t-t_0).$$

Agora vamos estimar por baixo. Da teoria da medida $\exists\, t_* \in [t_0,t]$, tal que

$$\|\omega_x(\cdot,t_*)\|_{L^2(\mathbb{R})}^2 \frac{(t-t_0)^{\gamma+1}}{\gamma+1} \leq \gamma^{\gamma} \cdot C^{\beta\gamma} \cdot \beta^{-(\theta\beta\gamma+2)} \cdot \left(1-\frac{1}{q}\right)^{-\left(\frac{\theta\beta\gamma}{2}+1\right)} \cdot$$

$$\cdot \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{(1-\theta)\beta\gamma} \cdot (t-t_0).$$

Logo,

$$\|\omega_x(\cdot,t_*)\|_{L^2(\mathbb{R})} \leq \widetilde{C} \cdot \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{(1-\theta)\frac{\beta\gamma}{2}} \cdot (t-t_0)^{-\frac{\gamma}{2}}, \qquad t_* \in\ [t_0,t] \tag{2.137}$$

onde r, θ e C são dados em (2.131) e γ está definida em (2.134) e

$$\widetilde{C} = \widetilde{C}(p_0,q) = \gamma^{\frac{\gamma}{2}} \cdot (\gamma+1)^{\frac{1}{2}} \cdot C^{\frac{\beta\gamma}{2}} \cdot \beta^{-\left(\frac{\theta\beta\gamma}{2}+1\right)} \cdot \left(1-\frac{1}{q}\right)^{-\frac{1}{2}\left(\frac{\theta\beta\gamma}{2}+1\right)}. \tag{2.138}$$

Da desigualdade de Young, temos

$$\|\omega(\cdot,t_*)\|_{L^{\infty}(\mathbb{R})} \leq K \cdot \|\omega(\cdot,t_*)\|_{L^r(\mathbb{R})}^{1-b} \cdot \|\omega_x(\cdot,t_*)\|_{L^2(\mathbb{R})}^{b}, \tag{2.139}$$

onde

$$b = \frac{1}{1+\frac{p_0}{q+\alpha}}, \qquad K = K(p_0,q) = (2b)^{-b}. \tag{2.140}$$

Substituindo (2.137) em (2.139), obtemos

$$\|\omega(\cdot,t_*)\|_{L^{\infty}(\mathbb{R})} \leq K\widetilde{C}^{b} \cdot \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{(1-\theta)\frac{\beta\gamma b}{2}} \cdot \|\omega(\cdot,t_*)\|_{L^r(\mathbb{R})}^{1-b} (t-t_0)^{-\frac{\gamma b}{2}}. \tag{2.141}$$

Pelo Lema 2.10, $\|\omega(\cdot,t)\|_{L^q(\mathbb{R})}$ decresce ao t crescer e como $t_* \in [t_0,t]$, temos $\|\omega(\cdot,t_*)\|_{L^r(\mathbb{R})} \leq \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}$. Por outro lado, também pelo Lema 2.10, temos $\|\omega(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|\omega(\cdot,t_*)\|_{L^\infty(\mathbb{R})}$. Com essas considerações, a desigualdade (2.141) produz

$$\|\omega(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K\widetilde{C}^b \|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})}^{b\left[(1-\theta)\frac{\beta\gamma}{2}-1\right]+1}(t-t_0)^{-\frac{\gamma b}{2}}. \tag{2.142}$$

Lembrando da definição (2.126), obtemos $\|\omega(\cdot,t)\|_{L^\infty(\mathbb{R})} = \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}^{\frac{q+\alpha}{2}}$ e $\|\omega(\cdot,t_0)\|_{L^r(\mathbb{R})} = \|u(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})}^{\frac{q+\alpha}{2}}$. Logo,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K^{\frac{2}{q+\alpha}}\widetilde{C}^{\frac{2b}{q+\alpha}}\|u(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})}^{b\left[(1-\theta)\frac{\beta\gamma}{2}-1\right]+1}(t-t_0)^{-\frac{\gamma b}{q+\alpha}}, \tag{2.143}$$

onde r, θ e C são dados em (2.131) e γ está definida em (2.134) e b em (2.140). Além disso $q \geq \max\{2, p_0\}$ é qualquer. Então, definimos $q = 2p_0$. Assim,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widehat{C}(p_0)\cdot\|u(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})}^{\frac{2p_0}{2p_0+\alpha}}\cdot(t-t_0)^{-\frac{1}{2p_0+\alpha}}, \quad \forall\, t > t_0 > 0, \tag{2.144}$$

Fazendo $t_0 \searrow 0$, temos

$$\boxed{\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widehat{C}(\alpha,p_0)\cdot\|u_0\|_{L^{p_0}(\mathbb{R})}^{\frac{2p_0}{2p_0+\alpha}}\cdot t^{-\frac{1}{2p_0+\alpha}}, \quad \forall t > 0,} \tag{2.145}$$

onde

$$\widehat{C}(\alpha,p_0) = K^{\frac{2}{q+\alpha}}\cdot\widetilde{C}^{\frac{2b}{q+\alpha}},$$

com K definido em (2.140) e $\widetilde{C}$ está dado em (2.138). Concluindo o Teorema 2.2.

■

3 Capítulo 3: $\kappa = \dfrac{\alpha}{2}$

3.1 Introdução

Neste capítulo trabalhamos com as soluções da equação dos meios porosos com termo advectivo arbitrário no caso $\kappa = \dfrac{\alpha}{2}$. Iniciamos na seção 3.2 mostrando uma estimativa de energia para soluções não-negativas e algumas desigualdades essenciais para estimarmos a norma do sup dessas soluções, a qual é desenvolvida na seção 3.3.

3.2 Estimativas de Energia

Seja $v = v(x,t)$ uma solução suave positiva do problema com valor inicial

$$\begin{cases} u_t + (b(x,t)u^{\frac{\alpha}{2}+1})_x = \mu(t)(u^\alpha \cdot u_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\ \\ u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad u_0 > 0, \end{cases} \tag{3.1}$$

onde $\alpha > 0$. Consideraremos ainda $0 < t \leq T < T_*$, sendo $[0, T_*[$ é o intervalo maximal de existência de solução e $0 < T_* \leq \infty$.

Teorema 3.1. *Seja $v(x,t)$ solução suave do problema de valor inicial* (3.1)*. Então, $\forall\, q \in [p_0, +\infty[$, tem-se*

(i) $\displaystyle\int_0^T \int_{\mathbb{R}} v(x,t)^{q+\frac{\alpha}{2}-1} |v_x(x,t)| dxdt < \infty, \qquad \forall\, T \in]0, T_*[.$

(ii) Existe um subconjunto com medida nula $E_q \subseteq [0,\infty[$ tal que

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)\, v_x^2(x,t)dx \leq \tag{3.2}$$

$$\leq q(q-1)B(t)\int_{\mathbb{R}} v^{q+\frac{\alpha}{2}-1}(x,t)\,|v_x(x,t)|dx$$

para todo $t \in [0, T_*[\setminus E_q$.

Prova: Sejam $q > 1$, $\psi(x)$ e $\psi_R(x)$ as funções de corte definidas em (2.3) e (2.4). Multiplicamos a equação

$$v_t(x,t) + (b(x,t)v^{\frac{\alpha}{2}+1}(x,t))_x = \mu(t)(v^\alpha(x,t)v_x(x,t))_x$$

por $qv^{q-1}(x,t)\psi_R(x)$ e integramos em $[0,t]\times\mathbb{R}$. Usando a compacidade da região de integração, usamos o Teorema de Fubini na primeira integral e o Teorema de Integração por Partes nas demais integrais. Logo,

$$
\begin{aligned}
\int_{|x|\leq 2R} v^q(x,t)\psi_R(x)dx - q(q-1)\int_0^t\int_{|x|\leq 2R} b(x,\tau)v^{q+\frac{\alpha}{2}-1}(x,\tau)v_x(x,\tau)\psi_R(x)dxd\tau& \\
-q\int_0^t\int_{R\leq|x|\leq 2R} b(x,\tau)v^{q+\frac{\alpha}{2}}(x,\tau)\psi_R'(x)dxd\tau& \\
= \int_{|x|\leq 2R} v_0^q(x)\psi_R(x)dx -& \\
-q(q-1)\int_0^t \mu(\tau)\int_{|x|\leq 2R} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)\psi_R(x)dxd\tau& \\
-q\int_0^t \mu(\tau)\int_{R\leq|x|\leq 2R} v^{q+\alpha-1}(x,\tau)v_x(x,\tau)\psi_R'(x)dxd\tau.&
\end{aligned} \tag{3.3}
$$

Escrevemos $b(x,t) = b(x,t) - \beta(t) + \beta(t)$, onde $\beta(t)$ é dado por (2.13) e aplicamos em (3.3). Com esta adição garantiremos um controle sobre o fluxo $b(x,t)$. Reorganizando as integrais obtidas com esta operação, obtemos a seguinte desigualdade para (3.3)

$$
\begin{aligned}
\int_{|x|\leq 2R} v^q(x,t)\psi_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|\leq 2R} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)\psi_R(x)dxd\tau& \\
= \int_{|x|\leq 2R} v_0^q(x)\psi_R(x)dx + q\int_0^t\int_{R\leq|x|\leq 2R} b(x,\tau)v^{q+\frac{\alpha}{2}}(x,\tau)\psi_R'(x)dxd\tau& \\
+\frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{R\leq|x|\leq 2R} v^{q+\alpha}(x,\tau)\psi_R''(x)dxd\tau& \\
+q(q-1)\int_0^t\int_{|x|\leq 2R}(b(x,\tau)-\beta(\tau))v^{q+\frac{\alpha}{2}-1}(x,\tau)v_x(x,\tau)\psi_R(x)dxd\tau& \\
-\frac{q(q-1)}{q+\frac{\alpha}{2}}\int_0^t \beta(\tau)\int_{R\leq|x|\leq 2R} v^{q+\frac{\alpha}{2}}(x,\tau)\psi_R'(x)dxd\tau.&
\end{aligned} \tag{3.4}
$$

Sob as estimativas (2.15), (2.16) e (2.17) do Capítulo 2 e (2.59) da Observação (2.1) no Capítulo 2, podemos fazer $R \to +\infty$ na equação (3.4). Com isso, temos

$$
\int_{\mathbb{R}} v^q(x,t)dx + q(q-1)\int_0^t \mu(\tau)\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau = \tag{3.5}
$$

$$
= \int_{\mathbb{R}} v_0^q(x)dx + q(q-1)\int_0^t\int_{\mathbb{R}}(b(x,\tau)-\beta(\tau))v^{q+\frac{\alpha}{2}-1}(x,\tau)v_x(x,\tau)dxd\tau.
$$

A identidade obtida é de grande valia no que segue. Observe agora que $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é uma função absolutamente contínua em t, pois sabemos que $v^{q+\frac{\alpha}{2}-1}v_x$ e $(b(x,\tau) - \beta(\tau))$ são funções integráveis. Pelo Teorema da Diferenciação de Lebesgue, sabemos que existe um conjunto de medida nula E_q tal que $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é diferenciável em $t \in [0, T_*[\setminus E_q$, ou seja, $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é diferenciável $q.t.p.\ t \in [0, T_*[$. Assim, a forma diferencial do problema (3.5) é

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx =$$

$$= q(q-1)\int_{\mathbb{R}}(b(x,t)-\beta(t))v^{q+\frac{\alpha}{2}-1}(x,t)v_x(x,t)dx,$$

para todo $t \in [0, T_*[\setminus E_q$. Usando as propriedades da função módulo, a limitação de $b(x,\tau)-\beta(\tau)$ e a desigualdade de Young, obtemos

$$\int_0^T\int_{\mathbb{R}} v^{q+\frac{\alpha}{2}-1}(x,\tau)|v_x(x,\tau)|dxd\tau \leq \frac{1}{2}\int_0^T\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau$$

$$+\frac{1}{2}\int_0^T\int_{\mathbb{R}} v^q(x,\tau)dxd\tau < \infty,$$

pois da Observação (2.1) do Capítulo 2, temos que $\displaystyle\int_0^T\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau <$ ∞. Portanto, $v^{q+\frac{\alpha}{2}-1}v_x$ é uma função integrável em t, $\forall\, q \in [p_0, \infty[$ e $q > 1$. Com isso, segue o resultado (i) do Teorema (3.1). Das propriedades de módulo e da estimativa sobre o fluxo, $|b(x,\tau)-\beta(\tau)| \leq B(t)$, segue que

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx \tag{3.6}$$

$$\leq q(q-1)B(t)\int_{\mathbb{R}} v^{q+\frac{\alpha}{2}-1}(x,t)|v_x(x,t)|dx, \quad t \in [0, T_*[\setminus E_q.$$

O que mostra (ii) do Teorema (3.1).

■

Observação 3.1. *Nas condições do Teorema* (3.1)*, ou seja, se* $v(x,t)$ *é uma solução suave positiva do problema de valor inicial* (3.1)*, então não haverá Blow-up em tempo finito, ou seja,* $T = +\infty$ *no caso* $\kappa = \dfrac{\alpha}{2}$*. Os detalhes desta conclusão seguem da estimativa* (3.6) *e estão expostos a seguir. Usando a desigualdade de Young em* $q(q-1)B(t)\displaystyle\int_{\mathbb{R}} v^{q+\frac{\alpha}{2}-1}(x,t)|v_x(x,t)|dx$*, obtemos*

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx \leq$$

$$\leq \frac{q(q-1)}{2}\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)\,dx +$$

$$+\frac{q(q-1)}{4}\frac{B(t)^2}{\mu(t)}\int_{\mathbb{R}} v^q(x,t)\,dx$$

Somando os termos semelhantes e observando que $\dfrac{q(q-1)}{2}\mu(t)\displaystyle\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx$ *é não negativo, obtemos*

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} \leq \frac{q(q-1)}{4}\frac{B(t)^2}{\mu(t)}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} \qquad \textit{para todo} \quad q \geq 2p_0$$

Por Gronwall segue que

$$\|v(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})} \cdot \exp\left(\frac{q-1}{4}\int_{t_0}^t \frac{B(\tau)^2}{\mu(\tau)}\,d\tau\right)$$

para todo $t \geq t_0 \geq 0$. *Tomando o* sup *de* $\|v(\cdot,t)\|_{L^q(\mathbb{R})}$ *em* $[t_0,t]$ *temos*

$$\mathbb{V}_q(t_0;t) < \infty, \qquad \forall\, t \geq t_0 \geq 0 \textit{ finito.} \tag{3.7}$$

conforme definição (2.20) *do Capítulo 2. Ou seja,* $\sup\limits_{t_0 \leq \tau \leq t}\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ *é finito, para todo* $p_0 \leq q \leq +\infty$ *e* $t \geq t_0 \geq 0$ *finito. Concluindo a Observação 3.1.*

Para facilitar os cálculos da prova da limitação da solução $v(x,t)$ do problema (3.1), introduziremos uma função auxiliar $w(x,t)$ dada por

$$w(x,t) = v^{\frac{q+\alpha}{2}}(x,t), \qquad \forall\, x \in \mathbb{R}, \quad t \in [0,T_*[. \tag{3.8}$$

$w(x,t)$ satisfaz

$$\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} = \|w(\cdot,t)\|^\beta_{L^\beta(\mathbb{R})}, \text{ onde } \beta = \frac{2q}{q+\alpha} \tag{3.9}$$

e

$$\|v(\cdot,t)\|^{\frac{q}{2}}_{L^{\frac{q}{2}}(\mathbb{R})} = \|w(\cdot,t)\|^{\beta_0}_{L^{\beta_0}(\mathbb{R})}, \text{ onde } \beta_0 = \frac{q}{q+\alpha}. \tag{3.10}$$

Além disso,

$$\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} = \frac{(q+\alpha)^2}{4}\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx.$$

Em $\int_{\mathbb{R}} v^{q+\frac{\alpha}{2}-1}|v_x|dx$ dada em (i) do Teorema (3.1), escrevemos $v^{q+\frac{\alpha}{2}-1} = v^{\frac{q+\alpha-2}{2}}v^{\frac{q}{2}}$ e aplicamos Cauchy-Schwarz na integral resultante, obtendo

$$\int_{\mathbb{R}} v^{q+\frac{\alpha}{2}-1}|v_x|dx \leq \left(\int_{\mathbb{R}} v^{q+\alpha-2}v_x^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} v^q dx\right)^{\frac{1}{2}} = \frac{2}{q+\alpha}\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}\cdot\|w(\cdot,t)\|^{\frac{\beta}{2}}_{L^\beta}.$$

Substituindo este resultado e também (3.9) e (3.10) em (3.6), obtemos para todo $t \in [0,T_*[\setminus E_q$

$$\frac{d}{dt}\|w(\cdot,t)\|^\beta_{L^\beta(\mathbb{R})} + \beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \tag{3.11}$$

$$\leq (q-1)\beta B(t)\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}\cdot\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\frac{\beta}{2}}.$$

Aplicando a desigualdade de Nirenberg-Sobolev-Gagliardo:

$$\|\mathsf{w}(\cdot,t)\|_{L^\beta(\mathbb{R})}\leq\left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}}\|\mathsf{w}(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{1-\widetilde{\theta}}\|\mathsf{w}_x(\cdot,t)\|_{L^2(\mathbb{R})}^{\widetilde{\theta}},$$

com

$$\widetilde{\theta}=\frac{1}{2+\beta_0}=\frac{q+\alpha}{3q+2\alpha}\in]0,1[\tag{3.12}$$

em (3.11), temos $\forall\, t\in[0,T_*[\setminus E_q$

$$\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}+\left(1-\frac{1}{q}\right)\beta^2\mu(t)\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2\leq \tag{3.13}$$

$$\leq (q-1)\beta B(t)\left(\frac{2+\beta_0}{4}\right)^{\frac{\beta_0}{2+\beta_0}}\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^{1+\frac{\widetilde{\theta}\beta}{2}}\cdot\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{(1-\widetilde{\theta})\frac{\beta}{2}}.$$

Note que, para $\beta=\dfrac{2q}{q+\alpha}$ e $\beta_0=\dfrac{q}{q+\alpha}$:

(i) $\dfrac{\widetilde{\theta}\beta}{2}=\dfrac{\beta_0}{2+\beta_0}=\dfrac{q}{3q+\alpha}$

(ii) $1+\dfrac{\widetilde{\theta}\beta}{2}=2\dfrac{1+\beta_0}{2+\beta_0}$

(iii) $(1-\widetilde{\theta})\dfrac{\beta}{2}=\beta_0\dfrac{1+\beta_0}{2+\beta_0}$

Teorema 3.2. *Seja $q\geq 2p$, com $p\geq p_0$. Seja $v(\cdot,t)\in L^\infty_{loc}([0,T_*[,L^\infty(\mathbb{R}))$ solução suave positiva da equação de fluidos em meios porosos com termo advectivo* (3.1)*. Então, para w definida em* (3.8)*, temos*

$$\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}+\frac{4q(q-1)}{(3q+2\alpha)(q+\alpha)}\mu(t)\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2\leq$$

$$\leq q(q-1)2^{-\frac{3q}{q+\alpha}}(q+\alpha)(3q+2\alpha)^{\frac{q}{q+\alpha}-1}\mu(t)\left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\alpha}}\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta\frac{q+\frac{\alpha}{2}}{q+\alpha}}, \tag{3.14}$$

$\forall t\in[0,T_*[\setminus E_q$, onde $E_q\subseteq[0,T_*[$ tem medida nula, $\beta_0=\dfrac{q}{q+\alpha}$ e $\beta=\dfrac{2q}{q+\alpha}$.

Prova: Usando a desigualdade de Young em (3.13), temos para todo $t \in [0,T_*[\setminus E_q$

$$\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^\beta(\mathbb{R})} + \beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \le$$

$$\le \frac{1+\beta_0}{2+\beta_0}\beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})}+$$

$$+\frac{1}{2+\beta_0}\beta^{\beta_0}\cdot\left(1-\frac{1}{q}\right)\cdot q^{2+\beta_0}\cdot B(t)^{2+\beta_0}\cdot\mu(t)^{-(1+\beta_0)}\cdot\left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}\beta_0(2+\beta_0)}\cdot$$

$$\cdot\,\|w(\cdot,t)\|^{\beta_0(1+\beta_0)}_{L^{\beta_0}(\mathbb{R})}$$

Somando os termos semelhantes e desenvolvendo os expoentes e as constantes, obtemos

$$\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^\beta(\mathbb{R})} + \frac{1}{2+\beta_0}\beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \le \tag{3.15}$$

$$\le q(q-1)2^{-\frac{3q}{q+\alpha}}(q+\alpha)(3q+2\alpha)^{-\frac{\alpha}{q+\alpha}}\cdot\left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\alpha}}\|w(\cdot,t)\|^{2\beta_0\frac{q+\frac{\alpha}{2}}{q+\alpha}}_{L^{\beta_0}(\mathbb{R})}$$

$\forall\, t\in[0,T_*[\setminus E_q$. Substituindo $\beta_0=\dfrac{q}{q+\alpha}$ e $\beta=\dfrac{2q}{q+\alpha}$, obtemos (3.14).

■

Como consequência do Teorema (3.2) temos duas estimativas úteis de $w(x,t)$:

Teorema 3.3. *Nas hipóteses do Teorema* (3.2) *acima, se existe* $t_*\in[0,T_*[\setminus E_q$ *tal que*

$$\left.\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^\beta(\mathbb{R})}\right|_{t=t_*}\ge 0, \tag{3.16}$$

então, valem as seguintes estimativas em $t=t_*$*:*

$$\|w(\cdot,t_*)\|_{L^\beta(\mathbb{R})}\le\left(\frac{q}{\beta}\right)^{\frac{1}{2}}\left(\frac{2+\beta_0}{4}\right)^{\frac{1}{2}}\left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{2}}\|w(\cdot,t_*)\|^{\frac{1}{2}(1+\beta_0)}_{L^{\beta_0}(\mathbb{R})} \tag{3.17}$$

e

$$\|w(\cdot,t_*)\|_{L^\infty(\mathbb{R})}\le\frac{q}{\beta}\left(\frac{2+\beta_0}{4}\right)\cdot\frac{B(t_*)}{\mu(t_*)}\cdot\|w(\cdot,t_*)\|^{\beta_0}_{L^{\beta_0}(\mathbb{R})}. \tag{3.18}$$

onde $\beta=\dfrac{2q}{q+\alpha}$ *e* $\beta_0=\dfrac{q}{q+\alpha}$.

Prova: Suponhamos que existe $t_*\in[0,T_*[\setminus E_q$ tal que (3.16) seja verdadeiro, então pelo Teorema (3.2) vale

$$\|w(\cdot,t_*)\|^2_{L^2(\mathbb{R})} \leq \left(\frac{q}{\beta}\right)^{2+\beta_0} \left(\frac{2+\beta_0}{4}\right)^{\beta_0} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{2+\beta_0} \|w(\cdot,t_*)\|^{\beta_0(1+\beta_0)}_{L^{\beta_0}(\mathbb{R})}. \quad (3.19)$$

Aplicando esta estimativa de $\|w(\cdot,t_*)\|_{L^2(\mathbb{R})}$ na desigualdade de SNG:

$$\|\mathsf{w}\|_{L^\beta(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\frac{1}{2+\beta_0}} \|\mathsf{w}\|^{1-\theta}_{L^{\beta_0}(\mathbb{R})} \|\mathsf{w}_x\|^{\theta}_{L^2(\mathbb{R})}, \quad (3.20)$$

onde $\theta = \dfrac{1}{2+\beta_0}$ obtemos (3.17). Por outro lado, se substituirmos (3.19) na seguinte desigualdade de SNG:

$$\|\mathsf{w}\|_{L^\infty(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\frac{2}{2+\beta_0}} \|\mathsf{w}\|^{1-\delta}_{L^{\beta_0}(\mathbb{R})} \|\mathsf{w}_x\|^{\delta}_{L^2(\mathbb{R})}, \quad (3.21)$$

onde $\delta = \dfrac{2}{2+\beta_0}$, obtemos (3.18). Note que as desigualdades de SNG são válidos para $\beta = 2\beta_0$, com $\beta_0 > 0$ qualquer.

■

Escrevemos o Teorema (3.3) em função das soluções suaves positivas $v(\cdot,t)$ de (3.1), com isso obtemos os seguintes resultados:

Teorema 3.4. *Nas hipóteses do Teorema (3.2) acima, se existe $t_* \in [0,T_*[\setminus E_q$ tal que*

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\Big|_{t=t_*} \geq 0, \quad (3.22)$$

então vale a seguinte estimativa em $t = t_*$:

$$\|v(\cdot,t_*)\|_{L^q(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{q+\alpha}} \|v(\cdot,t_*)\|^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}_{L^{\frac{q}{2}}(\mathbb{R})} \quad (3.23)$$

Teorema 3.5. *Nas hipóteses do Teorema (3.2) acima, se existe $t_* \in [0,T_*[\setminus E_q$ tal que*

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\Big|_{t=t_*} \geq 0, \quad (3.24)$$

então vale a seguinte estimativa em $t = t_*$:

$$\|v(\cdot,t_*)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\alpha}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{2}{q+\alpha}} \|v(\cdot,t_*)\|^{\frac{q}{q+\alpha}}_{L^{\frac{q}{2}}(\mathbb{R})}. \quad (3.25)$$

Até o momento obtemos estimativas pontuais de $\|v(\cdot,t_*)\|_{L^q(\mathbb{R})}$ e $\|v(\cdot,t_*)\|_{L^\infty(\mathbb{R})}$ para $t_* \in [0, T_*[\setminus E_q$, desde que $\left.\frac{d}{dt}\|v(\cdot,t_*)\|^{\beta}_{L^q(\mathbb{R})}\right|_{t=t_*} \geq 0$. As simulações numéricas a seguir correspondem ao problema de valor inicial (2.39),(2.40). Neste caso, tomamos $\alpha = 2$ e $\kappa = 1$.

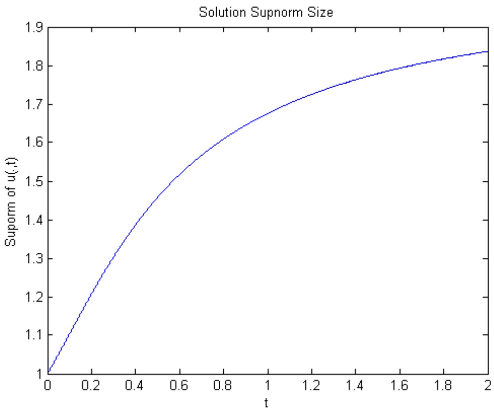

Figura 40: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\sin x$

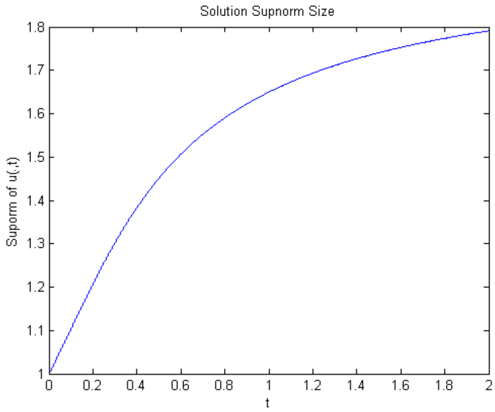

Figura 41: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: $\alpha = 2$, $\kappa = 1$ $b(x,t) = -\cos x$

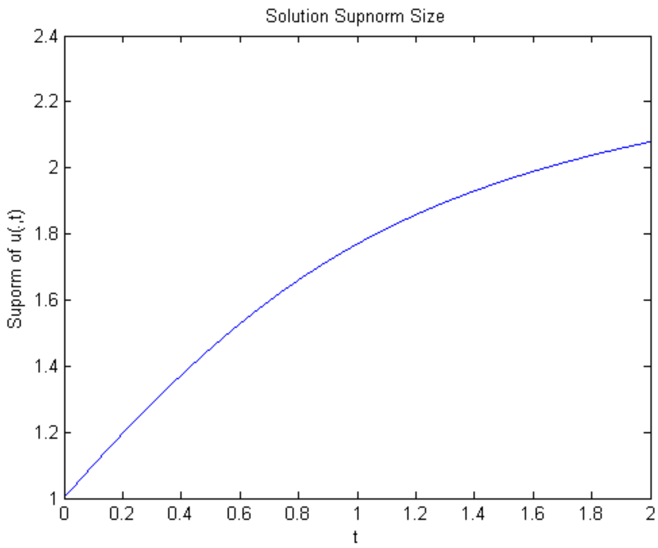

Figura 42: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: α=2, $\kappa = 1$ $b(x,t) = -\tanh x$

Observe, há um crescimento moderado das soluções, elas não explodem. O que coincide com com os nossos resultados.

O seguinte resultado é uma consequência do Teorema (3.4). Em termos de (2.20), obtemos a seguinte estimativa de $\mathbb{V}_q(t_0;t)$ em termos de $\mathbb{V}_{\frac{q}{2}}(t_0;t)$:

Teorema 3.6. *Nas hipóteses do Teorema* (3.2) *acima, tem-se, para todo* $0 \le t_0 \le t < T_*$*:*

$$\mathbb{V}_q(t_0;t) \le \max\left\{ \|v(\cdot,t_0)\|_{L^q(\mathbb{R})};\ \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\alpha}} \cdot \mathbb{V}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\alpha}{2}}{q+\alpha}} \right\}, \tag{3.26}$$

onde $\mathbb{B}_\mu(t_0;t)$ *e* $\mathbb{V}_q(t_0;t)$ *são dados em* (2.22) *e* (2.20)*, respectivamente.*

Prova: Considere $0 \le t_0 \le t < T_*$ quaisquer, mas fixos no que segue. Seja $\lambda_q \ge 0$ definida por

$$\lambda_q := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\alpha}} \cdot \mathbb{V}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}.$$

Dividimos a prova deste resultado em três casos:

<u>Caso 1</u>: Se $\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} \le \lambda_q$, então

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \le \lambda_q, \qquad \forall\, \tau \in [t_0,t]. \tag{3.27}$$

De fato, suponhamos por contradição que existe $t_2 \in]t_0,t]$ tal que $\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q$. Considere $t_1 \in [t_0,t_2[$ tal que

$$\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q, \text{ e } \|v(\cdot,t_1)\|_{L^q(\mathbb{R})} > \lambda_q, \qquad \forall\, \tau \in]t_1,t_2]. \tag{3.28}$$

Logo, se existir $t_* \in]t_1,t_2] \setminus E_q$ tal que

$$\frac{d}{d\tau}\|v(\cdot,t_1)\|^q_{L^q(\mathbb{R})}\bigg|_{\tau=t_*} \ge 0,$$

então, pelo Teorema (3.4), teremos

$$\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} \le \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\alpha}} \cdot \mathbb{V}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\alpha}{2}}{q+\alpha}} \le \lambda_q$$

contradizendo (3.28), pois $t_* \in]t_1,t_2]$. Portanto, vale (3.27).

<u>Caso 2</u>: Se $\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} > \lambda_q$ e existe $t_1 \in]t_0,t]$ tal que $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda$, então

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \le \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0,t]. \tag{3.29}$$

Mais precisamente, teremos $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \le \lambda_q$, $\forall\, \tau \in [t_0,t]$. De fato, suponhamos que $t_1 \in]t_0,t]$ seja o menor valor de τ, tal que $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} = \lambda_q$. Então, pelo mesmo argumento do Caso 1 acima, devemos ter $\dfrac{d}{d\tau}\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} < 0$, $\forall\, \tau \in [t_0,t_1[\setminus E_q$. Ou seja, $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ é (estritamente) decrescente em $[t_0,t_1]$. Assim,

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}\,, \qquad \forall\,\tau \in [t_0,t].$$

No intervalo $[t_1,t]$, repetindo o argumento do Caso 1 acima, teremos

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \lambda_q, \qquad \forall\,\tau \in [t_1,t].$$

Portanto, vale (3.29).

Caso 3: Se $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda_q$, $\forall\,\tau \in [t_0,t]$, então pelo Teorema (3.4), teremos

$$\frac{d}{d\tau}\,\|v(\cdot,t_1)\|^q_{L^q(\mathbb{R})}\bigg|_{\tau=t_*} < 0, \qquad \forall\,\tau \in [t_0,t] \setminus E_q.$$

Com isto, $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})}$ será (estritamente) descrescente em $[t_0,t]$, e assim

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}\,, \qquad \forall\,\tau \in [t_0,t].$$

O que conclui a prova do Teorema (3.6). ∎

3.3 Estimativas para $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: soluções não-negativas

Considere, agora, uma solução $u(\cdot,t) \in L^\infty_{loc}([0,T_*[,L^\infty(\mathbb{R}))$ fraca e não negativa do problema (2.46). Raciocinando por densidade, obtemos a seguinte extensão do Teorema (3.6):

Teorema 3.7. *Seja $q \geq 2p$ qualquer com $p \geq p_0$. Então,*

$$\mathbb{U}_q(t_0,t) \leq \max\left\{\|u(\cdot,t_0)\|_{L^q(\mathbb{R})}\,;\,\left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\alpha}} \cdot \mathbb{U}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}\right\}, \tag{3.30}$$

para cada $0 \leq t_0 \leq t < T_*$.

Prova: Seja $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\zeta(x) > 0$ $\forall x \in \mathbb{R}$. Considere $\epsilon > 0$ dado e seja $v^{[\epsilon]}(\cdot,t) > 0$ uma solução (suave) de

$$\begin{cases} v_t(x,t) + (b(x,t)v^{\frac{\alpha}{2}+1}(x,t))_x = \mu(t)(v(x,t)^\alpha v_x(x,t))_x, & x \in \mathbb{R}, \quad t > 0 \\ v(\cdot,0) = u_0 + \epsilon\zeta \end{cases} \tag{3.31}$$

Observe que para algum $\epsilon_0 > 0$ suficientemente pequeno, teremos $v^{[\epsilon]}(\cdot,t)$ definida em $[0,t]$, $\forall\epsilon \in]0,\epsilon_0]$ e para cada $0 \leq t_0 \leq t < T_*$. Pelo Teorema (3.6), segue que

$$\sup_{t_0\le\tau\le t}\left\|v^{[\epsilon]}(\cdot,\tau)\right\|_{L^q(\mathbb{R})}\le\max\left\{\left\|v^{[\epsilon]}(\cdot,t_0)\right\|_{L^q(\mathbb{R})};\left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}}\cdot\mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\alpha}}\cdot\right.$$
$$\left.\cdot\sup_{t_0\le\tau\le t}\left\|v^{[\epsilon]}(\cdot,\tau)\right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}\right\}$$

para todo $0<\epsilon\le\epsilon_0$. Assim, fazendo $\epsilon\to 0$, obtemos (3.30). ■

O Teorema (3.7) é de grande importância na obtenção do seguinte Lema:

Lema 3.1. *Sejam $u(\cdot,t)\in L^\infty_{loc}([0,T_*[,L^\infty(\mathbb{R}))$ solução do problema (2.46) e $p\ge p_0$. Então, para quaisquer $0\le t_0\le t<T_*$ dados, tem-se:*

$$\mathbb{U}_{2p}(t_0,t)\le\max\left\{\|u(\cdot,t_0)\|_{L^{2p}(\mathbb{R})};\left(\frac{3.2p+2\alpha}{8}\right)^{\frac{1}{2p+\alpha}}\cdot\mathbb{B}_\mu(t_0;t)^{\frac{1}{2p+\alpha}}\cdot\mathbb{U}_p(t_0;t)^{\frac{2p+\frac{\alpha}{2}}{2p+\alpha}}\right\} \tag{3.32}$$

e

$$\mathbb{U}_{4p}(t_0,t)\le\max\left\{\|u(\cdot,t_0)\|_{L^{4p}(\mathbb{R})};\left(\frac{3.4p+2\alpha}{8}\right)^{\frac{1}{4p+\alpha}}\cdot\mathbb{B}_\mu(t_0;t)^{\frac{1}{4p+\alpha}}\cdot\left(\frac{3.4p+2\alpha}{8}\right)^{\frac{1}{4p+\alpha}}\right.$$
$$\left.\cdot\left(\frac{3.2p+2\alpha}{8}\right)^{\frac{1}{2p+\alpha}\cdot\frac{4p+\frac{\alpha}{2}}{4p+\alpha}}\cdot\mathbb{B}_\mu(t_0;t)^{\frac{1}{4p+\alpha}+\frac{1}{2p+\alpha}\frac{4p+\frac{\alpha}{2}}{4p+\alpha}}\cdot\mathbb{U}_p(t_0;t)^{\frac{4p+\frac{\alpha}{2}}{4p+\alpha}}\right\}. \tag{3.33}$$

Mais geralmente, para cada $m\ge 2$:

$$\mathbb{U}_{2^mp}(t_0,t)\le\max\left\{\|u(\cdot,t_0)\|_{L^{2^mp}(\mathbb{R})};\prod_{\substack{j\,=\,l+1\\1\,\le\,l\,\le\,m}}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\alpha}\cdot\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}}\cdot\right.$$
$$\|u(\cdot,t_0)\|_{L^{2^lp}(\mathbb{R})}^{\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}}\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^m\frac{1}{2^jp+\alpha}\cdot\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}};$$
$$\left.\left[\prod_{j=1}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=1}^m\frac{1}{2^jp+\alpha}\cdot\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}}\cdot\mathbb{U}_p(t_0;t)^{\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^m\frac{\alpha}{2}}}\right\} \tag{3.34}$$

Prova: Com uma indução em m, aplicado a (3.30) para $q=2p,4p,8p,...,2^mp$. ■

Lema 3.2. *Dado $m \geq 2$, tem-se para todo $l = 1, 2, ..., m-1$:*

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}}\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}}$$

$$\leq K_m(\alpha;p)\max\left\{\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\alpha}{2}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}}\right\}, \quad (3.35)$$

onde

$$K_m(\alpha;p) = \left(\frac{3p+2\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2p+\frac{\alpha}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1+2^{-m}\widehat{\alpha}}{2^m p+\frac{\alpha}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2^j p+\frac{\alpha}{2}}}\right] \quad (3.36)$$

e $\widehat{\alpha} = \dfrac{\alpha}{2p}$.

Prova: Para cada $l = 1, 2, ..., m-1$, temos:

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}} =$$

$$= \left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}} \leq$$

$$\leq \left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{2^{-l}-2^{-m}}{1+2^{-l}\widehat{\alpha}}\frac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}}\cdot\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}}. \quad (3.37)$$

Sendo que, na última desigualdade, usamos a seguinte desigualdade de interpolação:

$$\|\mathsf{u}\|_{L^{2^l p}(\mathbb{R})} \leq \|\mathsf{u}\|_{L^p(\mathbb{R})}^{\frac{2^{-l}-2^{-m}}{1-2^{-m}}}\|\mathsf{u}\|_{L^{2^m p}(\mathbb{R})}^{\frac{1-2^{-l}}{1-2^{-m}}}.$$

Observe que $\dfrac{1-2^{-l}}{1+2^{-l}\widehat{\alpha}}\cdot\dfrac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}} < 1$. Então, podemos usar a desigualdade de Young em (3.37) e obter

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{2^{-l}-2^{-m}}{1+2^{-l}\widehat{\alpha}}\frac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}}\cdot\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}}\leq$$

$$\leq\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\left\{\frac{1-2^{-l}}{1+2^{-l}\widehat{\alpha}}\frac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}+\right.$$

$$+\frac{2^{-l}-2^{-m}}{1+2^{-l}\widehat{\alpha}}\frac{1+\widehat{\alpha}}{1-2^{-m}}\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}}\cdot$$

$$\left.\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right\}$$

$$\leq\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\max\left\{\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};\right.$$

$$\left.\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right\}$$

$$\leq\left[\prod_{j=2}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\max\left\{\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};\right.$$

$$\left.\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right\},\tag{3.38}$$

pois $\dfrac{3.2^j p+2\alpha}{8}\geq 1$, para todo $j\geq 2$. Como

$$\begin{aligned}\sum_{j=l+1}^{m}\frac{1}{2^j p+\alpha}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}} &= \frac{1}{p}\sum_{j=l+1}^{m}\frac{1}{1+2^{-j+1}\widehat{\alpha}}\cdot\frac{2^{-j}}{1+2^{-j}\widehat{\alpha}}\\ &= \frac{1}{p\widehat{\alpha}}\sum_{j=l+1}^{m}\left[\frac{1}{1+2^{-j}\widehat{\alpha}}-\frac{1}{1+2^{-j+1}\widehat{\alpha}}\right]\\ &= \frac{1}{p\widehat{\alpha}}\left(\frac{1}{1+2^{-m}\widehat{\alpha}}-\frac{1}{1+2^{-l}\widehat{\alpha}}\right)\\ &= \frac{1}{p}\frac{2^{-l}-2^{-m}}{(1+2^{-m}\widehat{\alpha})(1+2^{-l}\widehat{\alpha})},\end{aligned}$$

obtemos

$$\sum_{j=l+1}^{m}\frac{1}{2^jp+\alpha}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}=\frac{1-2^{-m}}{p+p\widehat{\alpha}}=\frac{1-2^{-m}}{p+\frac{\alpha}{2}}.$$

Substituindo este resultado em (3.38) acima, obtemos para cada $1\leq l\leq m-1$

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\alpha}\cdot\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}}\right]\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^jp+\alpha}}\cdot$$

$$\cdot\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}\cdot\|u(\cdot,t_0)\|_{L^{2^lp}(\mathbb{R})}^{\frac{2^mp+\frac{\alpha}{2}}{2^mp+2^{m-j}\frac{\alpha}{2}}}\leq$$

$$\leq K_m(\alpha;p)\cdot\max\left\{\|u(\cdot,t_0)\|_{L^{2^mp}(\mathbb{R})};\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\alpha}{2}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}}\right\}.$$

Onde

$$K_m(\alpha;p)=\prod_{j=2}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\alpha}\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}. \tag{3.39}$$

Vamos mostrar que $K_m(\alpha;p)$ obtido em (3.39) é o mesmo de (3.36). De fato,

$$\begin{aligned}
K_m(\alpha;p)^{\frac{1}{1+2^{-m}\widehat{\alpha}}} &= \prod_{j=2}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\alpha}\frac{2^j}{2^j+\widehat{\alpha}}}\\
&= \left[\prod_{j=2}^{m}\left(\frac{6.2.2^{j-1}p+2\alpha}{8}\right)^{\frac{1}{2^{j-1}+\widehat{\alpha}}}\right]^{\frac{1}{p}}\left[\prod_{j=2}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}}\right]^{-\frac{1}{p}}\\
&= \left[\prod_{j=1}^{m-1}\left(\frac{6.2^jp+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}}\right]^{\frac{1}{p}}\left[\prod_{j=2}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}}\right]^{-\frac{1}{p}}\\
&= \left(\frac{6.2^mp+2\alpha}{8}\right)^{\frac{-1}{2^m+\widehat{\alpha}}\frac{1}{p}}\left[\prod_{j=1}^{m}\left(\frac{6.2^jp+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}\frac{1}{p}}\right]^{\frac{1}{p}}\cdot\\
&\quad\cdot\left[\prod_{j=1}^{m}\left(\frac{3.2^jp+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}\frac{1}{p}}\right]^{-1}\left(\frac{3.2p+2\alpha}{8}\right)^{\frac{1}{2+\widehat{\alpha}}\frac{1}{p}}\\
&= \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{2p+\frac{\alpha}{2}}}\left(\frac{3.2^mp+\alpha}{4}\right)^{-\frac{1}{2^m+\frac{\alpha}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^jp+2\alpha}{3.2^jp+2\alpha}\right)^{\frac{1}{2^j+\frac{\alpha}{2}}}\right]
\end{aligned}$$

Usamos na segunda igualdade

$$\frac{1}{2^j p+\alpha}\frac{2^j}{2^j+\widehat{\alpha}}=\frac{1}{2p}\frac{1}{2^{j-1}+\widehat{\alpha}}\frac{2^j}{2^j+\widehat{\alpha}}=\frac{1}{p}\left(\frac{1}{2^{j-1}+\widehat{\alpha}}-\frac{1}{2^j+\widehat{\alpha}}\right).$$

Assim, $K_m(\alpha,\kappa;p)$ é dado por

$$K_m(\alpha;p)=\left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2p+\frac{\alpha}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1+2^{-m}\widehat{\alpha}}{2^m+\frac{\alpha}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1}{2^j+\frac{\alpha}{2}}}\right]^{\frac{1+2^{-m}\widehat{\alpha}}{2^m+\frac{\alpha}{2}}} \tag{3.40}$$

O que mostra (3.36).

■

Para obtermos uma estimativa fundamental para $\mathbb{U}_\infty(t_0,t)$, trabalharemos um pouco mais com os termos de (3.34). O processo é análogo ao desenvolvido na prova do Teorema (3.2). Assim, como

$$\begin{aligned}\prod_{j=1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\alpha}\frac{2^m p+\frac{\alpha}{2}}{2^m p+2^{m-j}\frac{\alpha}{2}}} &= \\ &=\left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\alpha}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1+2^{-m}\widehat{\alpha}}{2^m+\frac{\alpha}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2^j p+\frac{\alpha}{2}}}\right] \\ &=\left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\alpha}{2}}\frac{1}{2+\widehat{\alpha}}}K_m(\alpha;p),\end{aligned}$$

então, pelos Lemas (3.1) e (3.2), obtemos

$$\begin{aligned}\mathbb{U}_{2^m p}(t_0;t)\leq \max\Big\{&\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};K_m(\alpha;p)\,\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}; \\ &K_m(\alpha;p)\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\alpha}{2}}}\,\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}}; \\ &\left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\alpha}{2}}\frac{1}{2+\widehat{\alpha}}}K_m(\alpha;p)\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\alpha}{2}}}\,\mathbb{U}_p(t_0;t)^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}}\Big\}\end{aligned} \tag{3.41}$$

Onde $K_m(\alpha;p)>0$ é dado por (3.36). Como $1+\widehat{\alpha}>0$ e $\dfrac{3.2^j p+2\alpha}{8}\geq 1$, $\forall j\geq 2$, segue de (3.39) que

$$K_m(\alpha;p)>1,\qquad \forall\, m\geq 2.$$

Assim, (3.41) equivale a

$$\mathbb{U}_{2^m p}(t_0;t) \leq K_m(\alpha;p) \max\Bigg\{ \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \mathbb{B}_\mu(t_0;t)^{\dfrac{1-2^{-m}}{p+\frac{\alpha}{2}}} \cdot \|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}} ; \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\hat{\alpha}}{p+\frac{\alpha}{2}}\frac{1}{2+\hat{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\alpha}{2}}} \cdot \mathbb{U}_p(t_0;t)^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}} \Bigg\}. \tag{3.42}$$

Logo,

$$\mathbb{U}_{2^m p}(t_0;t) \leq \widetilde{K}_m(\alpha;p) \max\left\{ \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\alpha}{2}}} \mathbb{U}_p(t_0,t)^{\frac{p+2^{-m}\frac{\alpha}{2}}{p+\frac{\alpha}{2}}} \right\}, \tag{3.43}$$

onde $\widetilde{K}_m(\alpha;p) > 1$ é dado por

$$\widetilde{K}_m(\alpha;p) = K_m(\alpha;p) \max\left\{ 1; \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\hat{\alpha}}{p+\frac{\alpha}{2}}\frac{1}{2+\hat{\alpha}}} \right\}, \tag{3.44}$$

com $K_m(\alpha;p) > 1$ é dado em (3.39). Fazendo $m \to +\infty$ em (3.43), (3.44), obtemos a seguinte estimativa fundamental:

Teorema 3.8. *Seja $u(\cdot,t) \in L^\infty([0,T_*[,L^\infty(\mathbb{R}))$ solução (fraca) não negativa do problema (2.46), com $p \geq p_0$. Então, para cada $0 \leq t_0 \leq t < T_*$, tem-se*

$$\mathbb{U}_\infty(t_0;t) \leq \widetilde{K}(\alpha;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} ; \mathbb{B}_\mu(t_0;t)^{\frac{1}{p+\frac{\alpha}{2}}} \cdot \mathbb{U}_p(t_0;t)^{\frac{p}{p+\frac{\alpha}{2}}} \right\}, \tag{3.45}$$

onde $\widetilde{K}(\alpha;p) \geq 1$ é a constante dada por

$$\begin{aligned} \widetilde{K}(\alpha;p) &= \max\left\{ 1; \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}\frac{1}{2+\hat{\alpha}}} \right\} \cdot \lim_{m\to+\infty} K_m(\alpha;p) \qquad (3.46) \\ &= \max\left\{ 1; \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}\frac{1}{2+\hat{\alpha}}} \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}} \cdot \prod_{j=1}^{\infty} \left(\frac{6\cdot 2^j p + 2\alpha}{3\cdot 2^j p + 2\alpha}\right)^{\frac{1}{2^j p+\frac{\alpha}{2}}} \right\}. \end{aligned}$$

Note que $\widetilde{K}(\alpha;p) = \lim\limits_{m\to+\infty} \widetilde{K}_m(\alpha;p)$. Em particular, para cada $\alpha > 0$, temos:

$$\lim_{p\to+\infty} \widetilde{K}(\alpha;p) = 1. \tag{3.47}$$

Se tivermos $B_\mu < \infty$, $U_p < \infty$ (para $p \geq p_0$) e $T_* = \infty$, então a desigualdade (3.45), com $\widetilde{K}(\alpha;p) \geq 1$ dada por (3.46), fornece

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{K}(\alpha;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} \,; \mathbb{B}_\mu(t_0)^{\frac{1}{p+\frac{\alpha}{2}}}\, \mathbb{U}_p(t_0)^{\frac{p}{p+\frac{\alpha}{2}}} \right\}, \tag{3.48}$$

para cada $t_0 \geq 0$. Então, em particular, segue que

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{K}(\alpha;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} \,; B_\mu^{\frac{1}{p+\frac{\alpha}{2}}}\, U_p^{\frac{p}{p+\frac{\alpha}{2}}} \right\}, \tag{3.49}$$

para todo $t_0 \geq 0$. Considere uma sequência de $t_0 \to +\infty$ tal que $\|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})}$ convirja para o limite inferior, com isso

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{K}(\alpha;p) \max\left\{ \liminf_{t_0\to+\infty} \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} \,; B_\mu^{\frac{1}{p+\frac{\alpha}{2}}}\, U_p^{\frac{p}{p+\frac{\alpha}{2}}} \right\}, \tag{3.50}$$

onde $u(\cdot,t) \geq 0$ e $p \geq p_0$.

Com mais dois resultados apenas, estaremos em condições de obter o resultado final deste capítulo.

Teorema 3.9. *Suponha $V_p < \infty$ e $p \geq p_0$. Então, para cada $q \geq 2p$, temos*

$$V_q \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} B_\mu^{\frac{1}{q+\alpha}}\, V_{\frac{q}{2}}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}. \tag{3.51}$$

Prova: Considere

$$\begin{aligned} g(t) &:= \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\alpha}} \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}} \\ \text{e} \\ \lambda_q &:= \limsup_{t\to+\infty} g(t). \end{aligned} \tag{3.52}$$

Devemos mostrar que $\limsup\limits_{t\to+\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})} \leq \lambda_q$, ou seja, da definição de lim sup, devemos mostrar que

$$\limsup_{t\to+\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})} \leq \lambda_q + \eta, \qquad \forall\, \eta > 0. \tag{3.53}$$

Suponhamos por contradição que (3.53) seja falsa. Então, existe $\widehat{\eta} > 0$ tal que

$$\limsup_{t\to+\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{3.54}$$

Seja, então, $\widehat{t_0} \gg 1$ suficientemente grande tal que

$$g(t) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{3.55}$$

Assim, devemos ter

$$\|v(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{3.56}$$

De fato, suponhamos por absurdo que (3.56) seja falso, então existe $t_0 \geq \widehat{t_0}$ tal que

$$\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{3.57}$$

De (3.54), existe $t_2 > t_0$ tal que $\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}$. Logo, podemos encontrar $t_1 \in [t_0, t_2[$ tal que

$$\begin{array}{l} \|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q + \widehat{\eta} \\ \text{e} \\ \|v(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \in]t_1, t_2]. \end{array} \tag{3.58}$$

Como $\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \|v(\cdot,t_1)\|_{L^q(\mathbb{R})}$, deve existir $t_* \in [t_1,t_2] \setminus E_q$ tal que

$$\frac{d}{dt}\|v(\cdot,t_0)\|^q_{L^q(\mathbb{R})}\Big|_{t=t_*} \geq 0, \tag{3.59}$$

pois, caso contrário, teríamos $\dfrac{d}{dt}\|v(\cdot,t_0)\|^q_{L^q(\mathbb{R})} < 0$, $q.t.p.\, t \in [t_1,t_2]$, ou seja, $\|v(\cdot,t)\|_{L^q(\mathbb{R})}$ seria estritamente decrescente em $[t_1,t_2]$, contradizendo $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} < \|v(\cdot,t_2)\|_{L^q(\mathbb{R})}$. Então, pelo Teorema (3.4), teríamos

$$\begin{aligned} \|v(\cdot,t_*)\|_{L^q(\mathbb{R})} &\leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{q+\alpha}} \|v(\cdot,t_*)\|^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}_{L^{\frac{q}{2}}(\mathbb{R})} \\ &= g(t_*) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \text{ onde } t_* \in [t_1,t_2] \end{aligned} \tag{3.60}$$

A última desigualdade segue de (3.55). Mas isso contradiz (3.58) acima. Ou seja, (3.57) não pode ocorrer. Logo, (3.56) é verdadeiro. Agora, vamos provar (3.51). Para isso, consideremos

$$w(x,t) := v(x,t)^{\frac{q+\alpha}{2}}, \quad \forall x \in \mathbb{R}, \quad t \geq 0.$$

Usando a desigualdade SNG (para $\beta = 2\beta_0$, onde $\beta_0 = \dfrac{q}{q+\alpha} > 0$ qualquer e $\theta = \dfrac{1}{2+\beta_0}$),

$$\|w\|_{L^\beta(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^\theta \cdot \|w\|^{1-\theta}_{L^{\beta_0}(\mathbb{R})} \cdot \|w_x\|^\theta_{L^2(\mathbb{R})}, \tag{3.61}$$

obtemos de (3.15), para todo $t \in [\widehat{t_0}, \infty[\setminus E_q$:

$$
\begin{aligned}
\|w(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\frac{6q+4\alpha}{q+\alpha}} \leq\ & \left(\frac{3q+2\alpha}{4q+4\alpha}\right)^{2} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{4q+2\alpha}{q+\alpha}} \cdot \|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^{2} \\
\leq\ & \left(\frac{3q+2\alpha}{4q+4\alpha}\right)^{2} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{4q+2\alpha}{q+\alpha}} \left\{ \frac{(3q+2\alpha)(q+\alpha)^2}{4q(q-1)} \cdot \mu(t)^{-1} \cdot \right. \\
& \cdot \left(-\frac{d}{dt} \|w(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\beta} \right) \cdot \\
& \left. \cdot \frac{(q+\alpha)^2}{2^{\frac{5q+2\alpha}{q+\alpha}}} \cdot (3q+2\alpha)^{\frac{q}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\alpha}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0 \frac{2q+\alpha}{q+\alpha}} \right\}.
\end{aligned}
\tag{3.62}
$$

Em termos de $v(\cdot,t)$, temos

$$
\begin{aligned}
(\lambda_q+\widehat{\eta})^{3q+2\alpha} \leq\ & \|v(\cdot,t)\|_{L^q(\mathbb{R})}^{3q+2\alpha} = \|w(\cdot,t)\|_{L^{\beta}(\mathbb{R})}^{\frac{6q+4\alpha}{q+\alpha}} \\
\leq\ & \left[\left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\alpha}} \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}} \right]^{3q+2\alpha} + \\
& + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{2q+\alpha} \cdot \mu(t)^{-1} \cdot \left(-\frac{d}{dt} \|v(\cdot,t)\|_{L^q(\mathbb{R})}^{q} \right) \\
\leq\ & g(t)^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \mathbb{M}_0^{2q+\alpha} \cdot \mu(t)^{-1} \cdot \left(-\frac{d}{dt} \|v(\cdot,t)\|_{L^q(\mathbb{R})}^{q} \right) \\
\leq\ & (\lambda_q + \frac{1}{2}\widehat{\eta})^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \mathbb{M}_0^{2q+\alpha} \cdot \mu(t)^{-1} \cdot \\
& \cdot \left(-\frac{d}{dt} \|v(\cdot,t)\|_{L^q(\mathbb{R})}^{q} \right)
\end{aligned}
\tag{3.63}
$$

para todo $t \in [\widehat{t}_0, \infty[\setminus E_q$. Onde $\mathbb{M}_0 > 0$ é definido por

$$
\mathbb{M}_0 := \sup_{t \geq \widehat{t}_0} \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}. \tag{3.64}
$$

Assim, obtemos

$$
-\frac{d}{dt} \|v(\cdot,t)\|_{L^q(\mathbb{R})}^{q} \geq C(\alpha,q;\widehat{\eta})\mu(t), \qquad \forall\, t \in [\widehat{t}_0, \infty[\setminus E_q, \tag{3.65}
$$

onde $C(\alpha,q;\widehat{\eta}) > 0$ é a constante dada por

$$
C(\alpha,q;\widehat{\eta}) := \frac{64q(q-1)(q+\alpha)}{(3q+2\alpha)^3} \cdot \mathbb{M}_0^{-(2q+\alpha)} \left[(\lambda_q+\widehat{\eta})^{3q+2\alpha} - (\lambda_q + \frac{1}{2}\widehat{\eta})^{3q+2\alpha} \right]. \tag{3.66}
$$

Integrando (3.65) em $[\widehat{t}_0, \widehat{T}]$, para $\widehat{T} > \widehat{t}_0$ arbitrário, obtemos

$$-\left\|v(\cdot,\widehat{T})\right\|_{L^q(\mathbb{R})}^q+\left\|v(\cdot,\widehat{t}_0)\right\|_{L^q(\mathbb{R})}^q\geq C(\alpha,q;\widehat{\eta})\int_{\widehat{t}_0}^{\widehat{T}}\mu(t)dt,\qquad\forall\,\widehat{T}>\widehat{t}_0.\tag{3.67}$$

Assim, em particular,

$$\left\|v(\cdot,\widehat{t}_0)\right\|_{L^q(\mathbb{R})}^q\geq C(\alpha,q;\widehat{\eta})\int_{\widehat{t}_0}^{\widehat{T}}\mu(t)dt,\qquad\forall\,\widehat{T}>\widehat{t}_0,\tag{3.68}$$

o que não ocorre, pois $\displaystyle\int_{\widehat{t}_0}^{+\infty}\mu(t)dt=\infty$, por hipótese. Esta contradição nos mostra que (3.54) é falsa. Logo, (3.53) é verdadeira $\forall\,\eta>0$. Portanto,

$$\limsup_{t\to+\infty}\|v(\cdot,t)\|_{L^q(\mathbb{R})}\leq\lambda_q,\tag{3.69}$$

como afirmamos.

■

Teorema 3.10. *Suponha* $\mathsf{U}_p<\infty$ *para algum* $p\geq p_0$*. Então, para cada* $q\geq 2p$*, temos*

$$\mathsf{U}_q\leq\left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}}\cdot B_\mu^{\frac{1}{q+\alpha}}\cdot\mathsf{U}_{\frac{q}{2}}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}},\tag{3.70}$$

onde $B_\mu=\displaystyle\limsup_{t\to+\infty}\frac{B(t)}{\mu(t)}$.

Prova: Considere

$$\begin{aligned}g(t)\;&:=\;\left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}}\left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\alpha}}\|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}\\ \text{e}\qquad&\\ \lambda_q\;&:=\;\limsup_{t\to+\infty}g(t).\end{aligned}$$

Devemos mostrar que $\displaystyle\limsup_{t\to+\infty}\|u(\cdot,t)\|_{L^q(\mathbb{R})}\leq\lambda_q$, ou seja,

$$\limsup_{t\to+\infty}\|u(\cdot,t)\|_{L^q(\mathbb{R})}\leq\lambda_q+\eta,\qquad\forall\,\eta>0.\tag{3.71}$$

Novamente argumentamos por contradição. Suponhamos que (3.71) seja falsa. Então, existe $\widehat{\eta}>0$ tal que

$$\limsup_{t\to+\infty}\|u(\cdot,t)\|_{L^q(\mathbb{R})}>\lambda_q+\widehat{\eta}.\tag{3.72}$$

Para este $\widehat{\eta}>0$, seja $\widehat{t}_0\gg1$ suficientemente grande tal que

$$g(t)\leq\lambda_q+\frac{1}{4}\widehat{\eta},\qquad\forall\,t\geq\widehat{t}_0.\tag{3.73}$$

Segue, então, que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{3.74}$$

De fato, suponhamos por absurdo que (3.74) seja falso, então existe $t_0 \geq \widehat{t_0}$ tal que

$$\|u(\cdot,t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{3.75}$$

De (3.72), existe $t_2 > t_0$ tal que

$$\|u(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}.$$

Logo, podemos encontrar $t_1 \in [t_0, t_2[$ tal que

$$\begin{aligned} &\|u(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q + \widehat{\eta} \\ \text{e}\quad & \\ &\|u(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \quad \forall t \in]t_1, t_2]. \end{aligned} \tag{3.76}$$

A ideia agora é obter um contradição como na prova de (3.56) no Teorema (3.9) acima. Para isso, aproximamos $u(\cdot,t)$ por uma solução suave $v^{[\epsilon]}(\cdot,t)$. Fixamos $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\zeta(x) > 0$, $\forall x \in \mathbb{R}$. Seja, para cada $\epsilon > 0$, $v^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([\widehat{t_0}, T_*^{[\epsilon]}[, L^\infty(\mathbb{R}))$ solução suave e positiva do problema

$$\begin{cases} v_t + (b(x,t)v^{\frac{\alpha}{2}+1})_x = \mu(t)(v(x,t)^\alpha v_x)_x, & x \in \mathbb{R}, \quad t > \widehat{t_0}, \\ v(\cdot,\widehat{t_0}) = u(\cdot,\widehat{t_0}) + \epsilon\zeta \end{cases}$$

e tomamos $\epsilon > 0$ suficientemente pequeno tal que se tenha

- $v^{[\epsilon]}(\cdot,t)$ esteja definida $\forall t \in [\widehat{t_0}, t_2]$,
- $$\left\|v^{[\epsilon]}(\cdot,t_1)\right\|_{L^q(\mathbb{R})} < \left\|v^{[\epsilon]}(\cdot,t_2)\right\|_{L^q(\mathbb{R})} \tag{3.77}$$

e

- $$g^{[\epsilon]}(t) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t \in [t_1, t_2], \tag{3.78}$$

onde $g^{[\epsilon]}(t)$ é dada por

$$g^{[\epsilon]}(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\alpha}} \left\|v^{[\epsilon]}(\cdot,t)\right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}, \qquad \forall\, t \in [\widehat{t_0}, t_2].$$

Note que (3.77) é verdadeira para $\epsilon \ll 1$, pois $\|u(\cdot,t_1)\|_{L^q(\mathbb{R})} < \|u(\cdot,t_2)\|_{L^q(\mathbb{R})}$. E, (3.78), decorre de (3.73) para $\epsilon \ll 1$. De (3.77) segue que existe $t_* \in [t_1, t_2] \setminus E_q^{[\epsilon]}$, tal que

$$\frac{d}{dt}\left\|v^{[\epsilon]}(\cdot,t)\right\|_{L^q(\mathbb{R})}^q \Bigg|_{t=t_*} \geq 0,$$

pois, caso contrário, teríamos $\dfrac{d}{dt}\left\|v^{[\epsilon]}(\cdot,t)\right\|^q_{L^q(\mathbb{R})} < 0$, *q.t.p.* $t \in [t_1, t_2]$ e, em particular, $\left\|v^{[\epsilon]}(\cdot,t)\right\|_{L^q(\mathbb{R})}$ seria estritamente decrescente em $[t_1, t_2]$. Contradizendo $\left\|v^{[\epsilon]}(\cdot,t_1)\right\|_{L^q(\mathbb{R})} > \left\|v^{[\epsilon]}(\cdot,t_2)\right\|_{L^q(\mathbb{R})}$, que não condiz com (3.77) acima! Então, pelo Teorema (3.4), teríamos para este $t_* \in [t_1, t_2]$:

$$\begin{aligned}
\left\|v^{[\epsilon]}(\cdot,t_*)\right\|_{L^q(\mathbb{R})} &\leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\alpha}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{q+\alpha}} \left\|v^{[\epsilon]}(\cdot,t_*)\right\|^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}_{L^{\frac{q}{2}}(\mathbb{R})} \\
&= g^{[\epsilon]}(t_*) \leq \lambda_q + \frac{1}{2}\widehat{\eta}.
\end{aligned}$$

A última desigualdade segue de (3.78). Mas esta estimativa contradiz

$$\begin{aligned}
\left\|v^{[\epsilon]}(\cdot,t_*)\right\|_{L^q(\mathbb{R})} &\geq \|u(\cdot,t_*)\|_{L^q(\mathbb{R})}, \qquad \left(\text{pois } v^{[\epsilon]}(\cdot,t_*) \geq u(\cdot,t_*), \quad \forall\, t \in [\widehat{t_0}, t_2]\right) \\
&\geq \lambda_q + \widehat{\eta}, \qquad (\text{ por } (3.76)).
\end{aligned}$$

Esta contradição mostra que (3.75) não pode ocorrer. Logo, (3.74) é verdadeiro. Seja agora $\mathsf{U_0} \geq \mathsf{0}$, dado por

$$\mathsf{U_0} := \sup_{\mathsf{t} \geq \widehat{\mathsf{t_0}}} \|\mathsf{u}(\cdot,\mathsf{t})\|_{\mathsf{L}^{\frac{q}{2}}(\mathbb{R})},$$

e seja $C > 0$ a constante (em t) dada por

$$C := \frac{64q(q-1)(q+\alpha)}{(3q+2\alpha)^3} \cdot (1+\mathsf{U_0})^{-(2\mathsf{q}+\alpha)} \left[(\lambda_{\mathsf{q}} + \widehat{\eta})^{3\mathsf{q}+2\alpha} - (\lambda_{\mathsf{q}} + \frac{1}{2}\widehat{\eta})^{3\mathsf{q}+2\alpha}\right].. \tag{3.79}$$

Para esta constante, tomemos $\widehat{T} > \widehat{t_0}$ suficientemente grande tal que se tenha

$$\|u(\cdot,\widehat{t_0})\|^q_{L^q(\mathbb{R})} < C\int_{\widehat{t_0}}^{\widehat{T}} \mu(t)dt.$$

Note que podemos requerer esta estimativa, pois, por hipótese, $\displaystyle\int_{\widehat{t_0}}^{+\infty} \mu(t)dt = +\infty$. Considere $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ positiva, $\forall\, x \in \mathbb{R}$ e fixa no que segue. Tome, para $\epsilon > 0$ dado, a solução $v^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([\widehat{t_0}, T_*^{[\epsilon]}[, L^\infty(\mathbb{R}))$ suave positiva do problema regularizado

$$\begin{cases} v_t + (b(x,t)v^{\frac{\alpha}{2}+1})_x = \mu(t)(v(x,t)^\alpha v_x)_x, & x \in \mathbb{R}, \quad t > \widehat{t_0} \\ v(\cdot,\widehat{t_0}) = u(\cdot,\widehat{t_0}) + \epsilon\zeta. \end{cases} \tag{3.80}$$

Seja $\epsilon > 0$ suficientemente pequeno tal que

$$
\begin{aligned}
&(i) \quad v^{[\epsilon]}(\cdot,t) \quad \text{esteja definida para todo } t \in [\widehat{t}_0, \widehat{T}];\\
&(ii) \quad \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{2}}(\mathbb{R})} \leq 1 + \mathsf{U}_0, \quad \forall\, \mathsf{t} \in [\widehat{\mathsf{t}}_0, \widehat{\mathsf{T}}];\\
&(iii) \quad \left\| v^{[\epsilon]}(\cdot,\widehat{t}_0) \right\|^q_{L^q(\mathbb{R})} < C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt;\\
&(iv) \quad g^{[\epsilon]}(t) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \quad \forall\, t \in [\widehat{t}_0, \widehat{T}],
\end{aligned}
\tag{3.81}
$$

onde $C > 0$ é a constante definida em (3.79) e $g^{[\epsilon]}(\cdot)$ é dada por

$$
g^{[\epsilon]}(t) := \left(\frac{3q + 2\alpha}{8} \right)^{\frac{1}{q+\alpha}} \left(\frac{B(t)}{\mu(t)} \right)^{\frac{1}{q+\alpha}} \cdot \left\| v^{[\epsilon]}(\cdot,t) \right\|^{\frac{q+\frac{\alpha}{2}}{q+\alpha}}_{L^{\frac{q}{2}}(\mathbb{R})}
$$

Introduzimos, então, $w^{[\epsilon]}(\cdot,t)$, $\widehat{t}_0 \leq t \leq \widehat{T}$, por

$$
w^{[\epsilon]}(x,t) := v^{[\epsilon]}(x,t)^{\frac{q+\alpha}{2}}, \quad \forall x \in \mathbb{R}, \quad t \in [\widehat{t}_0, \widehat{T}]. \tag{3.82}
$$

Assim, para todo $t \in [\widehat{t}_0, \widehat{T}] \setminus E_q^{[\epsilon]}$

$$
\begin{aligned}
(\lambda_q + \widehat{\eta})^{3q+2\alpha} \leq\ & \left\| v^{[\epsilon]}(\cdot,t) \right\|^{3q+2\alpha}_{L^q(\mathbb{R})} = \left\| w^{[\epsilon]}(\cdot,t) \right\|^{\frac{6q+4\alpha}{q+\alpha}}_{L^\beta(\mathbb{R})}\\
\leq\ & g^{[\epsilon]}(t)^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \left\| v^{[\epsilon]}(\cdot,t) \right\|^{2q+\alpha}_{L^{\frac{q}{2}}(\mathbb{R})} \cdot \mu(t)^{-1} \cdot\\
& \cdot \left(-\frac{d}{dt} \left\| v^{[\epsilon]}(\cdot,t) \right\|^q_{L^q(\mathbb{R})} \right)\\
\leq\ & (\lambda_q + \frac{1}{2}\widehat{\eta})^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot (1 + \mathsf{U}_0)^{2\mathsf{q}+\alpha} \cdot \mu(\mathsf{t})^{-1} \cdot\\
& \cdot \left(-\frac{d}{dt} \left\| v^{3q+2\alpha}(\cdot,t) \right\|^q_{L^q(\mathbb{R})} \right).
\end{aligned}
$$

Assim, $\forall\, t \in [\widehat{t}_0, \widehat{T}[\setminus E_q^{[\epsilon]}$:

$$
-\frac{d}{dt} \left\| v^{[\epsilon]}(\cdot,t) \right\|^q_{L^q(\mathbb{R})} \geq C\mu(t), \tag{3.83}
$$

onde $C > 0$ é a constante (em t) definida em (3.79). Integrando (3.83) em $[\widehat{t}_0, \widehat{T}]$, para $\widehat{T} > \widehat{t}_0$ arbitrário, obtemos

$$
-\left\| v^{[\epsilon]}(\cdot,\widehat{T}) \right\|^q_{L^q(\mathbb{R})} + \left\| v^{[\epsilon]}(\cdot,\widehat{t}_0) \right\|^q_{L^q(\mathbb{R})} \geq C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt.
$$

Assim, em particular temos

$$
\left\| v^{[\epsilon]}(\cdot,\widehat{t}_0) \right\|^q_{L^q(\mathbb{R})} \geq C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \quad \forall\, \widehat{T} > \widehat{t}_0,
$$

contradizendo (iii) de (3.81). Esta contradição nos mostra que (3.72) é falsa. Logo, teremos (3.71) verdadeira, para todo $\eta > 0$. Portanto, (3.70) está provado, o que conclui a prova do Teorema (3.10).

■

Note que se aplicarmos (3.10) sucessivamente para $q = 2p, 4p, 8p, ..., 2^m p$, obtemos

$$\mathsf{U}_{2p} \leq \left(\frac{3 \cdot 2p + 2\alpha}{8} \right)^{\frac{1}{2p+\alpha}} \cdot \mathsf{B}_{\mu}^{\frac{1}{2p+\alpha}} \cdot \mathsf{U}_{p}^{\frac{2p+\frac{\alpha}{2}}{2p+\alpha}},$$

para $q = 2p$. Para $q = 4p$ e usando (3.84),

$$\begin{aligned} \mathsf{U}_{4p} &\leq \left(\frac{3 \cdot 4p + 2\alpha}{8} \right)^{\frac{1}{4p+\alpha}} \cdot B_{\mu}^{\frac{1}{4p+\alpha}} \cdot \mathsf{U}_{2p}^{\frac{4p+\frac{\alpha}{2}}{4p+\alpha}} \\ &\leq \left(\frac{3 \cdot 4p + 2\alpha}{8} \right)^{\frac{1}{4p+\alpha}} \left(\frac{3 \cdot 2p + 2\alpha}{8} \right)^{\frac{1}{2p+\alpha} \cdot \frac{4p+\frac{\alpha}{2}}{4p+\alpha}} \cdot B_{\mu}^{\frac{1}{4p+\alpha} + \frac{1}{2p+\alpha} \cdot \frac{4p+\frac{\alpha}{2}}{4p+\alpha}} \cdot \mathsf{U}_{2p}^{\frac{4p+\frac{\alpha}{2}}{4p+2\alpha}}. \end{aligned}$$

No caso geral, para $m \geq 1$ qualquer,

$$\begin{aligned} \mathsf{U}_{2^m p} &\leq \left[\prod_{j=1}^{m} \left(\frac{3 \cdot 2^j p + 2\alpha}{8} \right)^{\frac{1}{2^j p+\alpha} \cdot \frac{2^m p + \frac{\alpha}{2}}{2^m p + 2^{m-j} \frac{\alpha}{2}}} \right] \\ &\quad \cdot B_{\mu}^{\sum_{j=1}^{m} \frac{1}{2^j p+\alpha} \cdot \frac{2^m p + \frac{\alpha}{2}}{2^m p + 2^{m-j} \frac{\alpha}{2}}} \cdot \mathsf{U}_{p}^{\frac{2^m p + \frac{\alpha}{2}}{2^m p + \frac{\alpha}{2}}}, \end{aligned} \tag{3.84}$$

onde $u(\cdot, t) \geq 0$.

Teorema 3.11. *Suponha $V_q < \infty$, com $p \geq p_0$. Então, para todo $q \geq 2p$, tem-se:*

$$\liminf_{t \to +\infty} \|v(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q + 2\alpha}{8} \right)^{\frac{2}{q+\alpha}} \limsup_{t \to +\infty} \left\{ \left(\frac{B(t)}{\mu(t)} \right)^{\frac{2}{q+\alpha}} \cdot \|v(\cdot, t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{q+\alpha}} \right\}. \tag{3.85}$$

Prova: Considere a seguinte mudança de variável

$$w(x, t) = v(x, t)^{\frac{q+\alpha}{2}}.$$

Escrevendo (3.85) em termos de w, temos

$$\liminf_{t \to +\infty} \|w(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q + 2\alpha}{8} \right)^{\frac{q+\alpha}{q+\tilde{\alpha}}} \limsup_{t \to +\infty} \left\{ \left(\frac{B(t)}{\mu(t)} \right)^{\frac{q+\alpha}{q+\tilde{\alpha}}} \cdot \|w(\cdot, t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\tilde{\alpha}}} \right\}, \tag{3.86}$$

onde $\beta_0 = \frac{q}{q+\alpha}$. Definindo

$$\begin{aligned} g(t) &:= \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\alpha}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}} \\ \text{e} \\ \lambda_q &:= \limsup_{t\to+\infty} g(t), \end{aligned}$$

devemos mostrar que $\liminf\limits_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \lambda_q$, ou equivalentemente

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \lambda_q + \eta, \quad \forall \eta > 0. \tag{3.87}$$

Suponhamos por contradição que (3.87) seja falso, então deve existir algum $\widehat{\eta} > 0$, tal que

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{3.88}$$

Seja, então, $\widehat{t_0} \gg 1$ suficientemente grande tal que

$$\begin{aligned} &\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \ge \lambda_q + \widehat{\eta}, \qquad \forall\, t \ge \widehat{t_0} \\ \text{e} \\ &g(t) \le \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t \ge \widehat{t_0}. \end{aligned}$$

Pelo Teorema (3.2), obtemos para todo $t \in [\widehat{t_0}, \infty[\setminus E_q$

$$\begin{aligned} (\lambda_q + \widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} &\le \|w(\cdot,t)\|_{L^\infty(\mathbb{R})}^{\frac{3q+2\alpha}{q+\alpha}} \\ &\le \frac{1}{16}\left(\frac{3q+2\alpha}{q+\alpha}\right)^2 \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}} \cdot \|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \le \\ &\le \left[\left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\alpha}} \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}}\right]^{\frac{3q+2\alpha}{q+\alpha}} + \\ &\quad + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}} \cdot \mu(t)^{-1} \cdot \\ &\qquad \cdot \left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta\right) \le \\ &\le g(t)^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}} \mu(t)^{-1} \cdot \\ &\qquad \cdot \left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta\right) \le \\ &\le (\lambda_q + \frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}} \cdot \mu(t)^{-1} \cdot \\ &\qquad \cdot \left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta\right) \end{aligned} \tag{3.89}$$

Note que usamos a desigualdade de NSG $\|\mathsf{w}\|_{L^\infty(\mathbb{R})}^{2+\beta_0} \leq \left(\frac{2+\beta_0}{4}\right)^2 \|\mathsf{w}\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0} \|\mathsf{w}_\mathsf{x}\|_{L^2(\mathbb{R})}^2$ para $\beta_0 = \dfrac{q}{q+\alpha}$. Assim, se definirmos

$$\mathbb{M}_0 := \sup_{t \geq \widehat{t}_0} \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})},$$

teremos para todo $t \in [\widehat{t}_0, \infty[\setminus E_q$

$$(\lambda_q + \widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} \leq (\lambda_q + \frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)} \cdot \mathbb{M}_0^{\frac{q}{2}} \mu(t)^{-1} \left(-\frac{d}{dt} \|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta \right),$$

ou seja,

$$-\frac{d}{dt} \|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta \geq C(\alpha, q, \widehat{\eta})\mu(t), \qquad \forall\, t \in [\widehat{t}_0, \infty[\setminus E_q. \tag{3.90}$$

Onde $C(\alpha, q, \widehat{\eta}) > 0$ é a constante dada por

$$C(\alpha, q, \widehat{\eta}) = \frac{64q(q-1)(q+\alpha)}{(3q+2\alpha)^3} \cdot \mathbb{M}_0^{\frac{q}{2}} \left[(\lambda_q + \widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} - (\lambda_q + \frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} \right].$$

Tomando $\widehat{T} > \widehat{t}_0$ arbitrário e integrando (3.90) em $[\widehat{t}_0, \widehat{T}]$, obtemos

$$-\left\|w(\cdot,\widehat{T})\right\|_{L^\beta(\mathbb{R})}^\beta + \left\|w(\cdot,\widehat{t}_0)\right\|_{L^\beta(\mathbb{R})}^\beta \geq C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall\, \widehat{T} > \widehat{t}_0.$$

Em particular,

$$\left\|w(\cdot,\widehat{t}_0)\right\|_{L^\beta(\mathbb{R})}^\beta \geq C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \quad \forall \widehat{T} > \widehat{t}_0. \tag{3.91}$$

Em termos de $v(\cdot,t)$, podemos reescrever (3.91) como

$$\left\|v(\cdot,\widehat{t}_0)\right\|_{L^q(\mathbb{R})}^q \geq C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \quad \forall \widehat{T} > \widehat{t}_0$$

o que não é verdadeiro, pois $\displaystyle\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt = \infty$ por hipótese. Com isso, a afirmação (3.88) também não pode ser verdadeira. Logo, vale a afirmação (3.87), o que mostra que (3.86), ou equivalentemente, (3.85). Concluindo a demonstração do Teorema (3.11)

■

Observando a prova do Teorema (3.11), vemos que podemos estender este resultado para soluções mais gerais $u(\cdot,t) \geq 0$:

Teorema 3.12. *Suponha* $\mathsf{U_q} < \infty$, *com* $p \geq p_0$. *Então, para todo* $q \geq 2p$

$$\liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\alpha}} \limsup_{t\to+\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{2}{q+\alpha}} \cdot \|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{q+\alpha}} \right\}. \tag{3.92}$$

Prova: Seja $w(\cdot,t)$ definida por

$$w(x,t) = u(x,t)^{\frac{q+\alpha}{2}}, \qquad t \geq 0 \text{ e } x \in \mathbb{R}.$$

Escrevendo (3.92) em termos de w, temos

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\alpha}} \limsup_{t\to+\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\alpha}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}} \right\}, \tag{3.93}$$

onde $\beta_0 = \dfrac{q}{q+\alpha}$. Definindo

$$\begin{aligned} g(t) \quad &:= \quad \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\alpha}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}} \\ \text{e} \\ \lambda_q \quad &:= \quad \limsup_{t\to+\infty} g(t), \end{aligned}$$

devemos mostrar que $\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \lambda_q$, ou equivalentemente

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \lambda_q + \eta, \quad \forall\, \eta > 0. \tag{3.94}$$

Suponhamos por contradição que (3.94) seja falso, então deve existir algum $\widehat{\eta} > 0$, tal que

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{3.95}$$

Seja, então, $\widehat{t}_0 \gg 1$ suficientemente grande tal que

$$\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \geq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0$$

e

$$g(t) \leq \lambda_q + \frac{1}{4}\widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0.$$

Seja $\mathbb{M}_0 \geq 0$, definido por

$$\mathbb{M}_0 := \sup_{t\geq \widehat{t}_0} \|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})},$$

e seja $C > 0$ a constante dada por

$$C := \frac{64q(q-1)(q+\alpha)}{(3q+2\alpha)^3} \cdot \frac{1}{(1+\mathbb{M}_0)^{\frac{q}{2}}} \left[(\lambda_q + \widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} - (\lambda_q + \frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} \right]. \tag{3.96}$$

Considere $\widehat{T} > \widehat{t}_0$ suficientemente grande tal que

$$\left\|u(\cdot,\widehat{t_0})\right\|^q_{L^q(\mathbb{R})} < C\int_{\widehat{t_0}}^{\widehat{T}}\mu(t)dt.$$

Considere $\zeta \in C^0(\mathbb{R})\cap L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ positiva ($\zeta(x)>0, \forall x\in\mathbb{R}$). Sejam $\epsilon>0$ dado e $v^{[\epsilon]}(\cdot,t)$ solução de

$$\begin{cases} v_t + (b(x,t)v^{\frac{\alpha}{2}+1})_x = \mu(t)(v(x,t)^\alpha v_x)_x, & x\in\mathbb{R}, \quad t>\widehat{t_0} \\ \\ v(\cdot,\widehat{t_0}) = u(\cdot,\widehat{t_0}) + \epsilon\zeta(\cdot). \end{cases}$$

Defina $w^{[\epsilon]}(\cdot,t)$ por

$$w^{[\epsilon]}(x,t) = v^{[\epsilon]}(x,t)^{\frac{q+\alpha}{2}}, \quad \forall\, x\in\mathbb{R}, \quad t\in[\widehat{t_0}, T_*^{[\epsilon]}[.$$

Tomamos $\epsilon>0$ suficientemente pequeno tal que

$$\begin{aligned} &(i) && v^{[\epsilon]}(\cdot,t) \quad \text{esteja definida}, \quad \forall\, t\in[\widehat{t_0},\widehat{T}]; \\ &(ii) && \left\|v^{[\epsilon]}(\cdot,\widehat{t_0})\right\|^q_{L^q(\mathbb{R})} < C\int_{\widehat{t_0}}^{\widehat{T}}\mu(t)dt; \\ &(iii) && \left\|v^{[\epsilon]}(\cdot,t)\right\|_{L^{\frac{q}{2}}(\mathbb{R})} \le 1+\mathbb{M}_0, \qquad \forall\, t\in[\widehat{t_0};\widehat{T}] \\ &(iv) && g^{[\epsilon]} \le \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t\in[\widehat{t_0},\widehat{T}], \end{aligned} \tag{3.97}$$

onde $g^{[\epsilon]}(\cdot)$ é dada por

$$g^{[\epsilon]}(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\alpha}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\alpha}} \left\|w^{[\epsilon]}(\cdot,t)\right\|^{\frac{q}{q+\alpha}}_{L^\beta(\mathbb{R})}, \qquad \forall\, t\in[\widehat{t_0},\widehat{T}].$$

Repetindo o argumento (3.89) e (3.90), para $w^{[\epsilon]}(\cdot,t)$, obtemos para todo $t\in[\widehat{t_0},\widehat{T}]\setminus E_q^{[\epsilon]}$

$$\begin{aligned} (\lambda_q+\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} &\le \|w(\cdot,t)\|^{\frac{3q+2\alpha}{q+\alpha}}_{L^\infty(\mathbb{R})} \\ &\le \left\|w^{[\epsilon]}(\cdot,t)\right\|^{\frac{3q+2\alpha}{q+\alpha}}_{L^\infty(\mathbb{R})} \\ &\le g^{[\epsilon]}(t)^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)}\cdot\left\|w^{[\epsilon]}(\cdot,t)\right\|^{\frac{q}{2}}_{L^{\frac{q}{2}}(\mathbb{R})}\cdot\mu(t)^{-1}\cdot \\ &\qquad \cdot\left(-\frac{d}{dt}\left\|w^{[\epsilon]}(\cdot,t)\right\|^\beta_{L^\beta(\mathbb{R})}\right) \\ &\le (\lambda_q+\frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\alpha)}\cdot(1+\mathbb{M}_0)^{\frac{q}{2}}\cdot\mu(t)^{-1}\cdot \\ &\qquad \cdot\left(-\frac{d}{dt}\left\|w^{[\epsilon]}(\cdot,t)\right\|^\beta_{L^\beta(\mathbb{R})}\right). \end{aligned}$$

Assim, por (3.96), temos

$$-\frac{d}{dt}\left\|w^{[\epsilon]}(\cdot,t)\right\|_{L^{\beta}(\mathbb{R})}^{\beta} \geq C\mu(t), \quad q.t.p. \;\; t \in [\widehat{t}_0, \widehat{T}]. \tag{3.98}$$

Integrando (3.98) em $[\widehat{t}_0, \widehat{T}]$, temos

$$-\left\|w^{[\epsilon]}(\cdot,\widehat{T})\right\|_{L^{\beta}(\mathbb{R})}^{\beta} + \left\|w^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|_{L^{\beta}(\mathbb{R})}^{\beta} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt.$$

Em particular, segue que

$$\left\|w^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|_{L^{\beta}(\mathbb{R})}^{\beta} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \quad \forall\, \widehat{T} > \widehat{t}_0,$$

que, em termos de $v^{[\epsilon]}(\cdot,t)$, fica

$$\left\|v^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|_{L^{q}(\mathbb{R})}^{q} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \quad \forall\, \widehat{T} > \widehat{t}_0 \tag{3.99}$$

contradizendo (ii) de (3.97). Logo, a afirmação (3.95) não é verdadeira, o que mostra (3.94) e (3.93). Concluindo a demonstração do Teorema (3.12).

■

Como caso particular do Teorema (3.12), temos a seguinte estimativa fundamental para todo $q \geq 2p$

$$\liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\alpha}} \cdot B_{\mu}(t)^{\frac{2}{q+\alpha}} \cdot \mathsf{U}_{\frac{\mathsf{q}}{2}}^{\frac{\mathsf{q}}{\mathsf{q}+\alpha}}, \tag{3.100}$$

onde $p \geq p_0$ é tal que $\mathsf{U}_{\mathsf{p}} < \infty$. Os resultados (3.84) e (3.100) acima, nos dão condições de provar o seguinte Teorema:

Teorema 3.13. *Suponha* $\mathsf{U}_{\mathsf{p}} < \infty$*, com* $p \geq p_0$*. Então,*

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq \mathbb{K}(\alpha;p) \cdot B_{\mu}^{\frac{1}{p+\frac{\alpha}{2}}} \cdot \mathsf{U}_{\mathsf{p}}^{\frac{1}{\mathsf{p}+\frac{\alpha}{2}}}, \tag{3.101}$$

onde

$$\mathbb{K}(\alpha;p) = \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p + 2\alpha}{3\cdot 2^j p + 2\alpha}\right)^{\frac{1}{2^j p+\frac{\alpha}{2}}}. \tag{3.102}$$

Prova: De (3.50), para cada $m \geq 1$, temos

$$
\begin{aligned}
\mathsf{U}_\infty &\leq \widetilde{K}(\alpha;2^m p)\cdot \max\left\{\liminf_{t\to\infty}\|u(\cdot,t)\|\,;\,B_\mu^{\frac{1}{2^m p+\frac{\alpha}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\alpha}{2}}}\right\}\\
&\leq \widetilde{K}(\alpha;2^m p)\cdot \max\left\{\left(\frac{3\cdot 2^m p+\alpha}{4}\right)^{\frac{1}{2^m p+\frac{\alpha}{2}}}\cdot B_\mu^{\frac{1}{2^m p+\frac{\alpha}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\alpha}{2}}};\right.\\
&\qquad\qquad\left. B_\mu^{\frac{1}{2^m p+\frac{\alpha}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\alpha}{2}}}\right\}\\
&= \widetilde{\widetilde{K}}(\alpha;2^m p)\cdot B_\mu^{\frac{1}{2^m p+\frac{\alpha}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\alpha}{2}}},
\end{aligned}
$$

onde

$$
\widetilde{\widetilde{K}}(\alpha;2^m p) = \widetilde{K}(\alpha;2^m p)\cdot \max\left\{1;\left(\frac{3\cdot 2^m p+\alpha}{4}\right)^{\frac{1}{2^m p+\frac{\alpha}{2}}}\right\} \tag{3.103}
$$

Observe em particular (por (3.103) e por (3.103)) que $\widetilde{\widetilde{K}}(\alpha;2^m p)\to 1$, ao $m\to\infty$. Portanto, por (3.84), obtemos

$$
\begin{aligned}
\mathsf{U}_\infty \leq \widetilde{\widetilde{\mathsf{K}}}(\alpha;2^m\mathsf{p})\cdot \mathsf{B}_\mu^{\frac{1}{2^m\mathsf{p}+\frac{\alpha}{2}}}\cdot &\left[\prod_{\mathsf{j}=1}^{\mathsf{m}}\left(\frac{3\cdot 2^{\mathsf{j}}\mathsf{p}+2\alpha}{8}\right)^{\frac{1}{2^{\mathsf{j}}\mathsf{p}+\alpha}\frac{1+2^{-\mathsf{m}}\widehat{\alpha}}{1+2^{-\mathsf{j}}\widehat{\alpha}}}\cdot\right. \\
&\left.\cdot B_\mu^{\sum_{j=1}^m \frac{1}{2^j p+\alpha}\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot \mathsf{U}_{\mathsf{p}}^{\frac{1+2^{-\mathsf{m}}\widehat{\alpha}}{1+\widehat{\alpha}}}\right]^{\frac{2^m p}{2^m p+\frac{\alpha}{2}}}
\end{aligned} \tag{3.104}
$$

para todo $m\geq 1$. Fazendo $m\to\infty$, obtemos

$$
\mathsf{U}_\infty \leq \left[\prod_{\mathsf{j}=1}^{\mathsf{m}}\left(\frac{3\cdot 2^{\mathsf{j}}\mathsf{p}+2\alpha}{8}\right)^{\frac{1}{2^{\mathsf{j}}\mathsf{p}+\alpha}\frac{1}{1+2^{-\mathsf{j}}\widehat{\alpha}}}\right]\cdot \mathsf{B}_\mu^{\sum_{\mathsf{j}=1}^{\mathsf{m}}\frac{1}{2^{\mathsf{j}}\mathsf{p}+\alpha}\frac{1}{1+2^{-\mathsf{j}}\widehat{\alpha}}}\cdot \mathsf{U}_{\mathsf{p}}^{\frac{1}{1+\widehat{\alpha}}}.
$$

O que prova o Teorema (3.13).

■

Observação 3.2.
(i) Afirmamos que:

$$
\sum_{j=1}^{\infty}\frac{1}{2^j p+\alpha}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}} = \frac{1}{p+\frac{\alpha}{2}}. \tag{3.105}
$$

De fato, para todo $j\geq 1$, temos

$$
\frac{1}{2^j p+\alpha}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}} = \frac{1}{2^j p+\alpha}\cdot\frac{2^{j+1}p}{2^{j+1}p+\alpha} = 2\left(\frac{1}{2^j p+\alpha}-\frac{1}{2^{j+1}p+\alpha}\right).
$$

Assim,

$$\sum_{j=1}^{\infty}\frac{1}{2^jp+\alpha}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}}=2\sum_{j=1}^{\infty}\left(\frac{1}{2^jp+\alpha}-\frac{1}{2^{j+1}p+\alpha}\right)=\frac{2}{2p+\alpha}=\frac{1}{p+\frac{\alpha}{2}}.$$

(ii) Com o que mostramos em (3.105), *temos*

$$\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\alpha}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}}}=\left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}}\cdot\prod_{j=1}^{\infty}\left(\frac{6\cdot 2^jp+2\alpha}{3\cdot 2^jp+2\alpha}\right)^{\frac{1}{p+\frac{\alpha}{2}}}. \quad (3.106)$$

De fato,

$$\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\alpha}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}}}=\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^{j-1}p+\frac{\alpha}{2}}-\frac{1}{2^jp+\frac{\alpha}{2}}}=$$

$$=\left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^{j-1}p+\frac{\alpha}{2}}}\right]\cdot\left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}\right]$$

$$=\left[\prod_{j=0}^{\infty}\left(\frac{6\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}\right]\cdot\left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}\right]^{-1}$$

$$=\left(\frac{6\cdot p+2\alpha}{8}\right)^{\frac{1}{p+\frac{\alpha}{2}}}\left[\prod_{j=1}^{\infty}\left(\frac{6\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}\right]\cdot\left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^jp+2\alpha}{8}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}\right]^{-1}$$

$$=\left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}}\cdot\prod_{j=1}^{\infty}\left(\frac{6\cdot 2^jp+2\alpha}{3\cdot 2^jp+2\alpha}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}.$$

O que conclui a observação.

Fica assim mostrado o seguinte resultado fundamental:

Teorema 3.14. *Suponha* $\displaystyle\int_0^{+\infty}\mu(t)dt=\infty$, $B_\mu<\infty$ *e, para algum* $p\geq p_0$, $\mathsf{U}_\mathsf{p}<\infty$. *Então*

$$\limsup_{t\to+\infty}\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}\leq\mathbb{K}(\alpha;p)\cdot B_\mu^{\frac{1}{p+\frac{\alpha}{2}}}\cdot\mathsf{U}_\mathsf{p}^{\frac{\mathsf{p}}{p+\frac{\alpha}{2}}}, \quad (3.107)$$

onde $\mathbb{K}(\alpha;p)>0$ *é dada por*

$$\mathbb{K}(\alpha;p)=\left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\alpha}{2}}}\cdot\prod_{j=1}^{\infty}\left(\frac{6\cdot 2^jp+2\alpha}{3\cdot 2^jp+2\alpha}\right)^{\frac{1}{2^jp+\frac{\alpha}{2}}}. \quad (3.108)$$

4 Capítulo 4: $\kappa > \frac{\alpha}{2}$

4.1 Introdução

Neste capítulo trabalhamos com as soluções da equação dos meios porosos, com termo advectivo arbitrário,

$$\begin{cases} u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u(x,t)^{\alpha}u_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\ \\ u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}), \quad v_0 > 0, \end{cases} \tag{4.1}$$

com κ e α satisfazendo $\kappa > \dfrac{\alpha}{2}$. Como hipóteses consideramos as dadas na seção 2.2 do capítulo 2. Na seção 4.2 mostramos uma estimativa de energia para soluções positivas e algumas desigualdades essenciais para estimarmos a norma do sup dessas soluções. Na seção 4.3, estimamos a norma do sup de soluções positivas e, em seguida, estendemos estes resultados para soluções não negativas.

4.2 Estimativas de Energia

Teorema 4.1. *Seja $v(x,t)$ solução suave do problema de valor inicial* (4.1)*. Então, Para todo $q \in [p_0, +\infty[$, tem-se*

(i) $\displaystyle\int_0^T \int_{\mathbb{R}} v(x,t)^{q+\kappa-1}|v_x(x,t)|\,dx\,dt < \infty, \quad \forall\, T \in]0, T_*[.$

(ii) Existe $E_q \subseteq [0,\infty[$ com medida nula tal que

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)\,dx \leq$$
$$\leq q(q-1)B(t)\int_{\mathbb{R}} v^{q+\kappa-1}(x,t)|v_x(x,t)|\,dx. \tag{4.2}$$

Prova: Considere $q > 1$ e $\psi(x)$ e $\psi_R(x)$ as funções de corte definidas em (2.3) e (2.4) no Capítulo 2. Multiplicamos a equação

$$v_t + (b(x,t)v^{\kappa+1})_x = \mu(t)(v(x,t)^{\alpha}v_x)_x \tag{4.3}$$

por $qv^{q-1}(x,t)\psi_R(x)$ e integramos em $[0,T] \times \mathbb{R}$. Novamente pelos Teorema de Fubini e Teorema de Integração por Partes nas demais integrais, obtemos

$$\begin{aligned} &\int_{|x|\leq 2R} v^q(x,t)\psi_R(x)dx - q(q-1)\int_0^t \int_{|x|\leq 2R} b(x,\tau)v^{q+\kappa-1}(x,\tau)v_x(x,\tau)\psi_R(x)dxd\tau \\ &\qquad -q\int_0^t \int_{R\leq|x|\leq 2R} b(x,\tau)v^{q+\kappa}(x,\tau)\psi_R'(x)dxd\tau \\ &= \int_{|x|\leq 2R} v_0^q(x)\psi_R(x)dx - q(q-1)\int_0^t \mu(\tau)\int_{|x|\leq 2R} v^{q+\alpha-2}(x,\tau)v_x^2\psi_R(x)dxd\tau \\ &\qquad -q\int_0^t \mu(\tau)\int_{R\leq|x|\leq 2R} v^{q+\alpha-1}(x,\tau)v_x\psi_R'(x)dxd\tau. \end{aligned}$$

Considere $\beta(t) = \dfrac{1}{2}\left(\sup_{x\in\mathbb{R}} b(x,t) + \inf_{x\in\mathbb{R}} b(x,t)\right)$. Escrevemos $b(x,\tau) = b(x,\tau) - \beta(\tau) + \beta(\tau)$. Com isso, obteremos um controle sobre o fluxo $b(x,t)$.

$$\int_{|x|\le 2R} v^q(x,t)\psi_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|\le 2R} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)\psi_R(x)dxd\tau$$

$$\begin{aligned}
= \;& \int_{|x|\le 2R} v_0^q(x)\psi_R(x)dx + q\int_0^t\int_{R\le|x|\le 2R} b(x,\tau)v^{q+\kappa}(x,\tau)\psi_R'(x)dxd\tau \\
& +\frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{R\le|x|\le 2R} v^{q+\alpha}(x,\tau)\psi_R''(x)dxd\tau \\
& +q(q-1)\int_0^t\int_{|x|\le 2R}(b(x,\tau)-\beta(\tau))v^{q+\kappa-1}(x,\tau)v_x(x,\tau)\psi_R(x)dxd\tau \\
& +\frac{q(q-1)}{q+\kappa}\int_0^t \beta(\tau)\int_{R\le|x|\le 2R} v^{q+\kappa}(x,\tau)\psi_R'(x)dxd\tau
\end{aligned} \tag{4.4}$$

Sabemos que $b(x,t)-\beta(t)$ é limitado. Além disso, por (2.5), fazemos $R\to+\infty$ e obtemos

$$\begin{aligned}
&\int_{\mathbb{R}} v^q(x,t)dx + q(q-1)\int_0^t \mu(\tau)\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau = \\
&\quad = \int_{\mathbb{R}} v_0^q(x)dx + q(q-1)\int_0^t\int_{\mathbb{R}}(b(x,\tau)-\beta(\tau))v^{q+\kappa-1}(x,\tau)v_x(x,\tau)dxd\tau
\end{aligned} \tag{4.5}$$

Observe agora que $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é uma função absolutamente contínua em t, pois sabemos que $v(x,t)$, $v(x,t)^{q+\alpha-2}\cdot v_x(x,t)^2$ e $(b(x,\tau)-\beta(\tau))$ são funções integráveis. Pelo Teorema da Diferenciação de Lebesgue, sabemos que existe um conjunto de medida nula E_q, tal que $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é diferenciável em $t\in[0,T_*[\setminus E_q$. Ou seja, $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é diferenciável $q.t.p.\ t\in[0,T_*[$. Assim, a forma diferencial do problema (4.5) é

$$\begin{aligned}
&\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx = \\
&\qquad = q(q-1)\int_{\mathbb{R}}(b(x,t)-\beta(t))v^{q+\kappa-1}(x,t)v_x(x,t)dx,
\end{aligned} \tag{4.6}$$

$\forall\, t\in[0,T_*[\setminus E_q$. Como $|b(x,\tau)-\beta(\tau)|\le B(T)$, então a última integral de (4.6) nos dá

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}} v^{q+\kappa-1}(x,\tau)|v_x(x,\tau)|dxd\tau \;\le\;& \frac{1}{2}\int_0^T\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau + \\
& +\frac{1}{2}\int_0^T\int_{\mathbb{R}} v^{q+2\kappa-\alpha}(x,\tau)dxd\tau < \infty,
\end{aligned}$$

pois $\int_0^T \int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau$ é finita, pelo Lema (2.6). Além disso, como $2\kappa - \alpha \geq 0$, temos que $v^{q+2\kappa-\alpha}$ está em L^q, para algum $q \geq p_0$ e $q > 1$ o que prova o item (i) do Teorema (4.1). Das propriedades de módulo e da estimativa sobre o fluxo, obtemos

$$\begin{aligned}\frac{d}{dt}\|v(\cdot,t)\|_{L^q(\mathbb{R})}^q + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx \\ \leq q(q-1)B(t)\int_{\mathbb{R}} v^{q+\kappa-1}(x,t)|v_x(x,t)|dx, \quad t \in [0,T_*[\setminus E_q.\end{aligned} \tag{4.7}$$

O que prova (ii) do Teorema (4.1).

■

Para facilitar nos cálculos posteriores, é viável fazer a seguinte mudança de variável:

$$w(x,t) = v^{\frac{q+\alpha}{2}}(x,t), \quad \forall\, x \in \mathbb{R}, \quad t \in [0,T_*[. \tag{4.8}$$

Logo,

$$\|v(\cdot,t)\|_{L^q(\mathbb{R})}^q = \|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta, \quad \text{onde} \quad \beta = \frac{2q}{q+\alpha} \tag{4.9}$$

e

$$\|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}} = \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0}, \quad \text{onde} \quad \beta_0 = \frac{q}{q+\alpha}. \tag{4.10}$$

Além disso, temos

$$\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 = \frac{(q+\alpha)^2}{4}\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx \tag{4.11}$$

e

$$\|v(\cdot,t)\|_{L^{q+\gamma}(\mathbb{R})}^{q+\gamma} = \|w(\cdot,t)\|_{L^{\widetilde{\beta}}(\mathbb{R})}^{\widetilde{\beta}}, \tag{4.12}$$

onde $\gamma = 2\kappa + \alpha > 0$ e $\widetilde{\beta} = 2\dfrac{q+\gamma}{q+\alpha}$. Assim, escrevendo $v^{q+\kappa-1} = v^{\frac{q+\alpha-2}{2}}v^{\frac{q+2\kappa-\alpha}{2}}$ e usando Cauchy-Schwarz obtemos

$$\int_{\mathbb{R}} v^{q+\kappa-1}|v_x|dx = \left(\int_{\mathbb{R}} v^{q+\alpha-2}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} v^{q+2\kappa-\alpha}dx\right)^{\frac{1}{2}} \tag{4.13}$$

Com as identidades (4.8) a (4.13) reescrevemos (4.7) da seguinte forma

$$\begin{aligned}\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta + q(q-1)\frac{4}{(q+\alpha)^2}\mu(t)\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \leq \\ \leq q(q-1)\frac{2}{q+\alpha}B(t)\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}\|w(\cdot,t)\|_{L^{\widetilde{\beta}}(\mathbb{R})}^{\frac{\widetilde{\beta}}{2}}.\end{aligned} \tag{4.14}$$

Usando a desigualdade de Nirenberg-Sobolev-Gagliardo

$$\|\mathsf{w}(\cdot,\mathsf{t})\|_{\mathsf{L}^{\widetilde{\beta}}(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}} \|\mathsf{w}(\cdot,\mathsf{t})\|^{1-\widetilde{\theta}}_{\mathsf{L}^{\beta_0}(\mathbb{R})} \|\mathsf{w}_\mathsf{x}(\cdot,\mathsf{t})\|^{\widetilde{\theta}}_{\mathsf{L}^2(\mathbb{R})},$$

onde $\widetilde{\theta}$ é dado por

$$\widetilde{\theta} = \frac{1-\frac{\beta_0}{\widetilde{\beta}}}{1+\frac{\beta_0}{2}} = \frac{1-\frac{1}{2}\frac{q}{q+\gamma}}{1+\frac{1}{2}\frac{q}{q+\alpha}} \quad \in \quad]0,1[. \tag{4.15}$$

e β_0 e $\widetilde{\beta}$ são dados em (4.10) e (4.12), respectivamente. Reescrevemos (4.14), para todo $t \in [0,T_*[\setminus E_q$, da seguinte forma:

$$\begin{aligned} &\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + (q-1)\beta^2\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \\ &\leq (q-1)\beta B(t)\left(\frac{2+\beta_0}{4}\right)^{\frac{\widetilde{\theta}\widetilde{\beta}}{2}} \|w_x(\cdot,t)\|^{1+\frac{\widetilde{\theta}\widetilde{\beta}}{2}}_{L^2(\mathbb{R})}\|w(\cdot,t)\|^{(1-\widetilde{\theta})\frac{\widetilde{\beta}}{2}}_{L^{\beta_0}(\mathbb{R})}. \end{aligned} \tag{4.16}$$

Note que

(i) $\dfrac{\widetilde{\theta}\widetilde{\beta}}{2} = \dfrac{\widetilde{\beta}-\beta_0}{2+\beta_0} = \dfrac{q+2\gamma}{3q+2\alpha}$

(ii) Para que a desigualdade (4.16) esteja bem definida, devemos ter $1+\dfrac{\widetilde{\theta}\widetilde{\beta}}{2} < 2$, ou seja, $\widetilde{\beta}-\beta_0 < 2+\beta_0$. Substituindo os valores de β_0 e $\widetilde{\beta}$ dados em (4.10) e (4.12) respectivamente, obtemos a seguinte relação entre q, κ e α: $q > 2(\kappa-\alpha)$. Ou seja, tomamos $q = 2p$, com $p > \kappa - \alpha$. Lembre-se que já temos uma hipótese sobre p, $p \geq p_0$. De agora em diante, as hipóteses sobre p serão

$$p > \kappa - \alpha \qquad \text{e} \qquad p \geq p_0. \tag{4.17}$$

Teorema 4.2. *Seja $q \geq 2p$, com $p \geq p_0$ e $p > \kappa-\alpha$. Seja $v(\cdot,t) \in L^{\infty}_{loc}([0,T_*[,L^{\infty}(\mathbb{R}))$ solução suave positiva da equação dos meios porosos com termo advectivo* (4.1). *Então, para w definida em* (4.8), *temos*

$$\begin{aligned} &\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \frac{4q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)(q+\alpha)^2}\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \\ &\leq \frac{4q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)(q+\alpha)^2} \cdot 2^{-\frac{5q+4\gamma+2\alpha}{q+\widetilde{\alpha}}} \cdot (q+\alpha)^2(3q+2\alpha)^{\frac{q+2\gamma}{q+\widetilde{\alpha}}} \cdot \mu(t) \cdot \\ &\cdot \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\widetilde{\alpha}}} \cdot \|w(\cdot,t)\|^{\beta_0\frac{2q+\alpha+\gamma}{q+\widetilde{\alpha}}}_{L^{\beta_0}(\mathbb{R})}, \end{aligned} \tag{4.18}$$

$\forall\, t \in [0,T_*[\setminus E_q$, onde $E_q \subseteq [0,T_*[$ tem medida nula, $\gamma = 2\kappa-\alpha$, $\widetilde{\alpha} = \alpha-\gamma$, $\beta_0 = \dfrac{q}{q+\alpha}$ e $\beta = \dfrac{2q}{q+\alpha}$.

Prova: Usando a desigualdade de Young em (4.16), temos para todo $t \in [0, T_*[\diagdown E_q$

$$\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq$$

$$\begin{aligned}
&\leq \beta q\left(1-\frac{1}{q}\right)\left(\frac{2+\beta_0}{4}\right)^{\frac{\widetilde{\theta}\widetilde{\beta}}{2}} \cdot B(t)\cdot \|w_x(\cdot,t)\|^{\frac{2+\widetilde{\theta}\widetilde{\beta}}{2}}_{L^2(\mathbb{R})}\cdot \|w(\cdot,t)\|^{(1-\widetilde{\theta})\frac{\widetilde{\beta}}{2}}_{L^{\beta_0}(\mathbb{R})}\\
&\leq \left(\frac{2+\widetilde{\theta}\widetilde{\beta}}{4}\right)\beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})}+\\
&\quad +\frac{2-\widetilde{\theta}\widetilde{\beta}}{4}\beta^{-\widetilde{\theta}\widetilde{\beta}\frac{2}{2-\widetilde{\theta}\widetilde{\beta}}}\left(1-\frac{1}{q}\right)q^{2\frac{2}{2-\widetilde{\theta}\widetilde{\beta}}}\cdot\left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}\widetilde{\beta}\frac{2}{2-\widetilde{\theta}\widetilde{\beta}}}\cdot \mu(t)^{-2\frac{2+\widetilde{\theta}\widetilde{\beta}}{2-\widetilde{\theta}\widetilde{\beta}}}\cdot\\
&\qquad \cdot B(t)^{2\frac{2}{2-\widetilde{\theta}\widetilde{\beta}}}\cdot\|w(\cdot,t)\|^{(1-\widetilde{\theta})\widetilde{\beta}\frac{2}{2-\widetilde{\theta}\widetilde{\beta}}}_{L^{\beta_0}(\mathbb{R})}.
\end{aligned}$$

Somando os termos semelhantes e desenvolvendo os expoentes e as constantes, obtemos

$$\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \frac{1+\beta_0-\frac{\widetilde{\beta}}{2}}{2+\beta_0}\beta^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq$$

$$\begin{aligned}
&\leq \frac{1+\beta_0-\frac{\widetilde{\beta}}{2}}{2+\beta_0}\beta^2\left(1-\frac{1}{q}\right)\mu(t)\left(\frac{q}{\beta}\right)^{\frac{2+\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}}\left(\frac{2+\beta_0}{4}\right)^{\frac{\widetilde{\beta}-\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}}\cdot\\
&\qquad\cdot\left(\frac{B(t)}{\mu(t)}\right)^{\frac{2+\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}}\cdot\|w(\cdot,t)\|^{\beta_0\frac{1+\frac{\widetilde{\beta}}{2}}{1+\beta_0-\frac{\widetilde{\beta}}{2}}}_{L^{\beta_0}(\mathbb{R})}
\end{aligned} \tag{4.19}$$

para todo $t \in [0, T_*[\diagdown E_q$. Substituindo $\beta_0 = \dfrac{q}{q+\alpha}$, $\beta = \dfrac{2q}{q+\alpha}$ e $\widetilde{\beta} = \dfrac{2(q+\gamma)}{q+\alpha}$ obtemos (4.18).

■

Uma consequência simples, mas importante, do Teorema (4.2) é o Teorema seguinte. Com ele obteremos uma estimativa sobre $\|v(\cdot,t_*)\|_{L^{\infty}(\mathbb{R})}$, para algum $t_* \in [0, T_*[\diagdown E_q$, onde $v(x,t)$ é solução suave positiva de (4.1).

Teorema 4.3. *Nas hipóteses do Teorema (4.2) acima, se existe $t_* \in [0, T_*[\diagdown E_q$ tal que*

$$\left.\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})}\right|_{t=t_*} \geq 0, \tag{4.20}$$

então valem as seguintes estimativas em $t = t_*$:

$$\|w(\cdot,t_*)\|_{L^{\beta}(\mathbb{R})} \leq \left(\frac{q}{\beta}\right)^{\frac{\frac{1}{2}}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \left(\frac{2+\beta_0}{4}\right)^{\frac{\frac{1}{2}}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{\frac{1}{2}}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \cdot \|w(\cdot,t_*)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{1}{2}\frac{1+2\beta_0-\frac{\widetilde{\beta}}{2}}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \tag{4.21}$$

e

$$\|w(\cdot,t_*)\|_{L^{\infty}(\mathbb{R})} \leq \left(\frac{q}{\beta}\right)^{\frac{1}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \left(\frac{2+\beta_0}{4}\right)^{\frac{1}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \cdot \|w(\cdot,t_*)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}}. \tag{4.22}$$

Prova: Supondo que existe $t_* \in [0,T_*[\setminus E_q$ tal que (4.20) vale, então de (4.19) temos

$$\|w(\cdot,t_*)\|_{L^2(\mathbb{R})}^2 \leq \left(\frac{q}{\beta}\right)^{\frac{2+\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \left(\frac{2+\beta_0}{4}\right)^{\frac{\widetilde{\beta}-\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{2+\beta_0}{1+\beta_0-\frac{\widetilde{\beta}}{2}}} \cdot \|w(\cdot,t_*)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0\frac{1+\frac{\widetilde{\beta}}{2}}{1+\beta_0-\frac{\widetilde{\beta}}{2}}}. \tag{4.23}$$

Substituindo (4.23) na desigualdade de SNG:

$$\|\mathsf{w}\|_{\mathsf{L}^{\beta}(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\frac{1}{2+\beta_0}} \|\mathsf{w}\|_{\mathsf{L}^{\beta_0}(\mathbb{R})}^{1-\theta} \|\mathsf{w}_\mathsf{x}\|_{\mathsf{L}^2(\mathbb{R})}^{\theta}, \tag{4.24}$$

onde $\theta = \dfrac{1}{2+\beta_0}$, obtemos (4.21). E, subtituindo (4.23) na desigualdade de SNG

$$\|\mathsf{w}\|_{\mathsf{L}^{\infty}(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\frac{2}{2+\beta_0}} \|\mathsf{w}\|_{\mathsf{L}^{\beta_0}(\mathbb{R})}^{1-\delta} \|\mathsf{w}_\mathsf{x}\|_{\mathsf{L}^2(\mathbb{R})}^{\delta}, \tag{4.25}$$

onde $\delta = \dfrac{2}{2+\beta_0}$, obtemos (4.22). Note que as desigualdades de SNG são válidos para $\beta = 2\beta_0$, com $\beta_0 > 0$ qualquer.

■

Agora reescreveremos o Teorema (4.3) em termos da solução $v(\cdot,t)$:

Teorema 4.4. *Nas hipóteses do Teorema (4.2) acima, se existe $t_* \in [0,T_*[\setminus E_q$ tal que*

$$\frac{d}{dt}\|v(\cdot,t)\|_{L^q(\mathbb{R})}^q \bigg|_{t=t_*} \geq 0, \tag{4.26}$$

então vale a seguinte estimativa em $t = t_*$:

$$\|v(\cdot,t_*)\|_{L^q(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{q+\widetilde{\alpha}}} \|v(\cdot,t_*)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\widetilde{\alpha}}}, \tag{4.27}$$

onde $\widetilde{\alpha} = \alpha - \gamma = -2(\kappa - \alpha)$.

Teorema 4.5. *Nas hipóteses do Teorema* (4.2) *acima, se existe* $t_* \in [0, T_*[\setminus E_q$ *tal que*

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\Big|_{t=t_*} \geq 0, \tag{4.28}$$

então vale a estimativa em $t = t_*$*:*

$$\|v(\cdot,t_*)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\widetilde{\alpha}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{2}{q+\widetilde{\alpha}}} \|v(\cdot,t_*)\|^{\frac{q}{q+\widetilde{\alpha}}}_{L^{\frac{q}{2}}(\mathbb{R})}, \tag{4.29}$$

onde $\widetilde{\alpha} = \alpha - \gamma = -2(\kappa - \alpha)$.

Expomos, a seguir, as simulações numéricas do problema de valor inicial (2.40), (2.39). Consideramos $\alpha = 1$ e $\kappa = 4$.

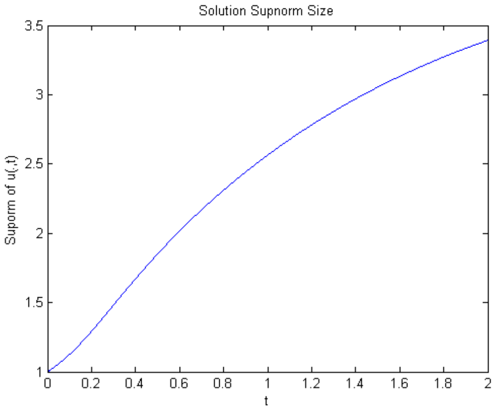

Figura 43: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: $\alpha = 1$, $\kappa = 4$
$b(x,t) = -\sin x$

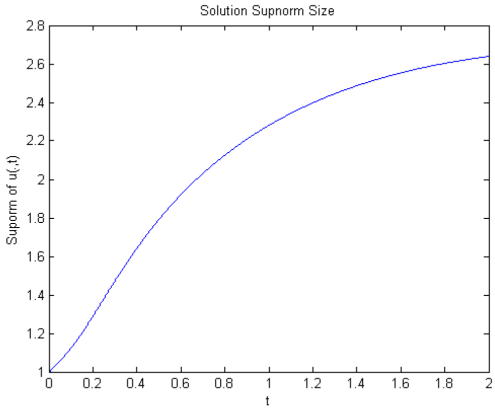

Figura 44: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: $\alpha = 1$, $\kappa = 4$
$b(x,t) = -\cos x$

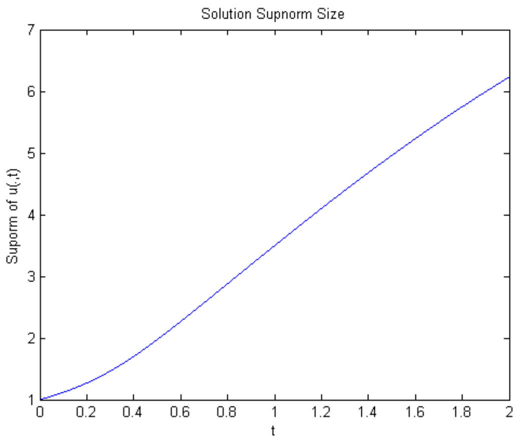

Figura 45: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: α=1, $\kappa = 4$ $b(x,t) = -\tanh x$

Seguindo na nossa análise, observamos que, até o momento, obtemos estimativas pontuais de $\|v(\cdot,t_*)\|_{L^q(\mathbb{R})}$ e $\|v(\cdot,t_*)\|_{L^\infty(\mathbb{R})}$ para $t_* \in [0,T_*[\setminus E_q$, desde que $\frac{d}{dt}\,\|v(\cdot,t_*)\|^{\beta}_{L^q(\mathbb{R})}\Big|_{t=t_*} \geq 0$. Enunciaremos a seguir uma consequência fundamental do Teorema (4.4). Tal consequência é uma estimativa de $\mathbb{V}_q(t_0;t)$ em termos de $\mathbb{V}_{\frac{q}{2}}(t_0;t)$, onde $\mathbb{V}_q(t_0;t)$ está definido em (2.20).

Teorema 4.6. *Nas hipóteses do Teorema* (4.2) *acima, temos para todo* $0 \leq t_0 \leq t < T_*$*:*

$$\mathbb{V}_q(t_0;t) \leq \max\left\{ \|v(\cdot,t_0)\|_{L^q(\mathbb{R})};\, \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{V}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\tilde{\alpha}}{2}}{q+\tilde{\alpha}}} \right\}, \tag{4.30}$$

onde $\mathbb{B}_\mu(t_0;t)$ *e* $\mathbb{V}_q(t_0;t)$ *são dados em* (2.22) *e* (2.20)*, respectivamente.* $\gamma = 2\kappa - \alpha$ *e* $\tilde{\alpha} = \alpha - \gamma = -2(\kappa-\alpha)$*.*

Prova: Considere $0 \leq t_0 \leq t < T_*$ quaisquer, mas fixos no que segue. Seja $\lambda_q \geq 0$ definida por

$$\lambda_q := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{V}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\tilde{\alpha}}{2}}{q+\tilde{\alpha}}}. \tag{4.31}$$

Dividimos a prova deste resultado em três casos.

Caso 1: Se $\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q$, então

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \lambda_q, \qquad \forall\, \tau \in [t_0,t]. \tag{4.32}$$

De fato, suponhamos por contradição que existe $t_2 \in]t_0,t]$ tal que $\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q$. Considere $t_1 \in [t_0,t_2[$ tal que

$$\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q \quad \text{e} \quad \|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda_q, \qquad \forall\, \tau \in]t_1,t_2]. \tag{4.33}$$

Logo, se existir $t_* \in]t_1,t_2] \setminus E_q$ tal que

$$\frac{d}{d\tau}\,\|v(\cdot,\tau)\|^q_{L^q(\mathbb{R})}\Big|_{\tau=t_*} \geq 0, \tag{4.34}$$

então, pelo Teorema (4.4), teremos

$$\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{V}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\tilde{\alpha}}{2}}{q+\tilde{\alpha}}} \leq \lambda_q, \tag{4.35}$$

contradizendo (4.33), pois $t_* \in]t_1,t_2]$. Portanto, vale (4.32).

Caso 2: Se $\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} > \lambda_q$ e existir $t_1 \in]t_0,t]$ tal que $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda$, então

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0,t]. \tag{4.36}$$

Mais precisamente, teremos

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \lambda_q, \qquad \forall\, \tau \in [t_0, t].$$

De fato, suponhamos que $t_1 \in]t_0, t]$ seja o menor valor de τ, tal que $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} = \lambda_q$. Então, pelo mesmo argumento do Caso 1 acima, teremos $\dfrac{d}{d\tau}\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} < 0$, $\forall\, \tau \in [t_0, t_1[\setminus E_q$. Ou seja, $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ é (estritamente) decrescente em $[t_0, t_1]$. Logo,

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0, t].$$

No intervalo $[t_1, t]$, se repetirmos o argumento do Caso 1, teremos

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \lambda_q, \qquad \forall\, \tau \in [t_1, t].$$

Portanto, vale (4.36).

<u>Caso 3</u>: Se $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda_q$, $\forall\, \tau \in [t_0, t]$, então, devido ao Teorema (4.4), teremos

$$\frac{d}{d\tau}\|v(\cdot,\tau)\|^q_{L^q(\mathbb{R})}\bigg|_{\tau=t_*} < 0, \qquad \forall\, \tau \in [t_0, t] \setminus E_q.$$

Com isto, $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ será (estritamente) descrescente em $[t_0, t]$, e assim

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0, t].$$

O que conclui a prova do Teorema (4.6).

∎

4.3 Estimativas para $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: soluções não-negativas.

Considere agora uma solução fraca, não negativa $u(\cdot,t) \in L^\infty_{loc}([0, T_*[, L^\infty(\mathbb{R}))$ do problema (4.1). Raciocinando por densidade, obtemos a seguinte extensão do Teorema (4.6):

Teorema 4.7. *Seja $q \geq 2p$ qualquer, com p satisfazendo* (4.17). *Então, para cada $0 \leq t_0 \leq t < T_*$, tem-se*

$$\mathbb{U}_q(t_0,t) \leq \max\left\{ \|u(\cdot,t_0)\|_{L^q(\mathbb{R})} \,;\, \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\widetilde{\alpha}}} \cdot \mathbb{U}_{\frac{q}{2}}(t_0;t)^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\widetilde{\alpha}}} \right\}, \tag{4.37}$$

onde $\widetilde{\alpha} = \alpha - \gamma$ e $\gamma = 2\kappa - \alpha \geq 0$.

Prova: Seja $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\zeta(x) > 0$, $\forall x \in \mathbb{R}$. Considere $\epsilon > 0$ dado e seja $v^{[\epsilon]}(\cdot,t) > 0$ solução (suave) de

$$\begin{cases} v_t(x,t) + (b(x,t)v^{\kappa+1}(x,t))_x = \mu(t)(v(x,t)^\alpha v_x(x,t))_x, \quad x \in \mathbb{R}, \quad t > 0 \\ v(\cdot,0) = u_0 + \epsilon\zeta \end{cases} \tag{4.38}$$

Para algum $\epsilon_0 > 0$ suficientemente pequeno, teremos $v^{[\epsilon]}(\cdot,t)$ definida em $[0,t]$, $\forall\, \epsilon \in]0,\epsilon_0]$ e para cada $0 \leq t_0 \leq t < T_*$. Pelo Teorema (4.6), segue

$$\sup_{t_0 \leq \tau \leq t} \left\| v^{[\epsilon]}(\cdot,\tau) \right\|_{L^q(\mathbb{R})} \leq \max \left\{ \left\| v^{[\epsilon]}(\cdot,t_0) \right\|_{L^q(\mathbb{R})} ; \left(\frac{3q+2\alpha}{8} \right)^{\frac{1}{q+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{q+\tilde{\alpha}}} \cdot \right.$$
$$\left. \cdot \sup_{t_0 \leq \tau \leq t} \left\| v^{[\epsilon]}(\cdot,\tau) \right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\frac{\tilde{\alpha}}{2}}{q+\tilde{\alpha}}} \right\},$$

para todo $0 < \epsilon \leq \epsilon_0$. Assim, fazendo $\epsilon \to 0$, obtemos (4.37).

■

Com o Teorema (4.7) obtemos o seguinte Lema:

Lema 4.1. *Sejam $u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$ solução do problema (4.1) e considere $p \geq p_0$ satisfazendo $p > \kappa - \alpha$. Então, para quaisquer $0 \leq t_0 \leq t < T_*$ dados, tem-se:*

$$\mathbb{U}_{2p}(t_0,t) \leq \max \left\{ \|u(\cdot,t_0)\|_{L^{2p}(\mathbb{R})} ; \left(\frac{3.2p+2\alpha}{8} \right)^{\frac{1}{2p+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{2p+\tilde{\alpha}}} \cdot \mathbb{U}_p(t_0;t)^{\frac{2p+\frac{\tilde{\alpha}}{2}}{2p+\tilde{\alpha}}} \right\} \tag{4.39}$$

e

$$\mathbb{U}_{4p}(t_0,t) \leq \max \left\{ \|u(\cdot,t_0)\|_{L^{4p}(\mathbb{R})} ; \left(\frac{3.4p+2\alpha}{8} \right)^{\frac{1}{4p+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{4p+\tilde{\alpha}}} \cdot \left(\frac{3.4p+2\alpha}{8} \right)^{\frac{1}{4p+\tilde{\alpha}}} \cdot \right.$$
$$\left. \cdot \left(\frac{3.2p+2\alpha}{8} \right)^{\frac{1}{2p+\tilde{\alpha}} \cdot \frac{4p+\frac{\tilde{\alpha}}{2}}{4p+\tilde{\alpha}}} \cdot \mathbb{B}_\mu(t_0;t)^{\frac{1}{4p+\tilde{\alpha}} + \frac{1}{2p+\tilde{\alpha}} \frac{4p+\frac{\tilde{\alpha}}{2}}{4p+\tilde{\alpha}}} \cdot \mathbb{U}_p(t_0;t)^{\frac{4p+\frac{\tilde{\alpha}}{2}}{4p+\tilde{\alpha}}} \right\}. \tag{4.40}$$

Mais geralmente, para cada $m \geq 2$:

$$\mathbb{U}_{2^m p}(t_0,t) \leq \max \left\{ \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \prod_{j=l+1}^{m} \left(\frac{3.2^j p+2\alpha}{8} \right)^{\frac{1}{2^j p+\tilde{\alpha}} \cdot \frac{2^m p+\frac{\tilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\tilde{\alpha}}{2}}} \cdot \right.$$
$$\cdot \|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{2^m p+\frac{\tilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\tilde{\alpha}}{2}}} \cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^m \frac{1}{2^j p+\tilde{\alpha}} \cdot \frac{2^m p+\frac{\tilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\tilde{\alpha}}{2}}};$$
$$\left. \left[\prod_{j=1}^{m} \left(\frac{3.2^j p+2\alpha}{8} \right)^{\frac{2^m p+\frac{\tilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\tilde{\alpha}}{2}}} \right] \cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=1}^m \frac{1}{2^j p+\tilde{\alpha}} \cdot \frac{2^m p+\frac{\tilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\tilde{\alpha}}{2}}} \cdot \mathbb{U}_p(t_0;t)^{\frac{2^m p+\frac{\tilde{\alpha}}{2}}{2^m p+2^m \frac{\tilde{\alpha}}{2}}} \right\} \tag{4.41}$$

Prova: Segue por indução em m, aplicando (4.37) para $q = 2p, 4p, 8p, ..., 2^m p$. ∎

Lema 4.2. *Dado $m \leq 2$, tem-se para todo $l = 1, 2, ..., m-1$:*

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}}\right]\cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}}\,\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}}$$

$$\leq K_m(\alpha,\kappa;p)\cdot\max\left\{\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\widetilde{\alpha}}{2}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}}\right\}, \quad (4.42)$$

onde

$$K_m(\alpha,\kappa;p) = \left(\frac{3p+2\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2p+\frac{\widetilde{\alpha}}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1+2^{-m}\widehat{\alpha}}{2^m p+\frac{\widetilde{\alpha}}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right] \quad (4.43)$$

e $\widehat{\alpha} = \dfrac{\widetilde{\alpha}}{2p}$.

Prova:
Para cada $l = 1, 2, ..., m-1$, temos:

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}}\right]\cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}}\cdot$$

$$\cdot\,\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}} =$$

$$= \left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\,\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}} \quad (4.44)$$

Usando a seguinte desigualdade de interpolação em (4.44)

$$\|\mathsf{w}\|_{L^{2^l p}(\mathbb{R})} \leq \|\mathsf{w}\|_{L^p(\mathbb{R})}^{\frac{2^{-l}-2^{-m}}{1-2^{-m}}}\,\|\mathsf{w}\|_{L^{2^m p}(\mathbb{R})}^{\frac{1-2^{-l}}{1-2^{-m}}},$$

obtemos

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^{2^l p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}}\leq$$

$$\leq\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{2^{-l}-2^{-m}}{1+2^{-l}\widehat{\alpha}}\frac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}}\cdot\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}}. \tag{4.45}$$

Observe que $\dfrac{1-2^{-l}}{1+2^{-l}\widehat{\alpha}}\cdot\dfrac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}<1$. Então, podemos usar a desigualdade de Young em (4.44) e obter

$$\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot \mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\cdot$$

$$\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{2^{-l}-2^{-m}}{1+2^{-l}\widehat{\alpha}}\frac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}}\cdot\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-l}\widehat{\alpha}}}\leq$$

$$\leq\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\left\{\frac{1-2^{-l}}{1+2^{-l}\widehat{\alpha}}\frac{1+2^{-m}\widehat{\alpha}}{1-2^{-m}}\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})}+\right.$$

$$\left.+\frac{2^{-l}-2^{-m}}{1+2^{-l}\widehat{\alpha}}\frac{1+\widehat{\alpha}}{1-2^{-m}}\cdot\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right\}$$

$$\leq\left[\prod_{j=l+1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\max\left\{\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};\right.$$

$$\left.\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right\}$$

$$\leq\left[\prod_{j=2}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right]\cdot\max\left\{\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})};\right.$$

$$\left.\mathbb{B}_\mu(t_0;t)^{\sum_{j=l+1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}\cdot\frac{1+2^{-l}\widehat{\alpha}}{2^{-l}-2^{-m}}\cdot\frac{1-2^{-m}}{1+\widehat{\alpha}}}\cdot\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right\}, \tag{4.46}$$

pois $\dfrac{3.2^j p + 2\alpha}{8} \geq 1$, para todo $j \geq 2$. Como

$$\begin{aligned}
\sum_{j=l+1}^{m} \frac{1}{2^j p + \widetilde{\alpha}} \cdot \frac{1}{1 + 2^{-j}\widehat{\alpha}} &= \frac{1}{p} \sum_{j=l+1}^{m} \frac{1}{1 + 2^{-j+1}\widehat{\alpha}} \cdot \frac{2^{-j}}{1 + 2^{-j}\widehat{\alpha}} \\
&= \frac{1}{p\widehat{\alpha}} \sum_{j=l+1}^{m} \left[\frac{1}{1 + 2^{-j}\widehat{\alpha}} - \frac{1}{1 + 2^{-j+1}\widehat{\alpha}} \right] \\
&= \frac{1}{p\widehat{\alpha}} \left(\frac{1}{1 + 2^{-m}\widehat{\alpha}} - \frac{1}{1 + 2^{-l}\widehat{\alpha}} \right) \\
&= \frac{1}{p} \frac{2^{-l} - 2^{-m}}{(1 + 2^{-m}\widehat{\alpha})(1 + 2^{-l}\widehat{\alpha})},
\end{aligned}$$

obtemos

$$\sum_{j=l+1}^{m} \frac{1}{2^j p + \widetilde{\alpha}} \cdot \frac{1 + 2^{-m}\widehat{\alpha}}{1 + 2^{-j}\widehat{\alpha}} \cdot \frac{1 + 2^{-l}\widehat{\alpha}}{2^{-l} - 2^{-m}} \cdot \frac{1 - 2^{-m}}{1 + \widehat{\alpha}} = \frac{1 - 2^{-m}}{p + p\widehat{\alpha}} = \frac{1 - 2^{-m}}{p + \frac{\widetilde{\alpha}}{2}}.$$

Substituindo este resultado em (4.46), obtemos para cada $1 \leq l \leq m - 1$

$$\begin{aligned}
&\left[\prod_{j=l+1}^{m} \left(\frac{3.2^j p + 2\alpha}{8} \right)^{\frac{1}{2^j p + \widetilde{\alpha}} \cdot \frac{2^m p + \frac{\widetilde{\alpha}}{2}}{2^m p + 2^{m-j} \frac{\widetilde{\alpha}}{2}}} \right] \cdot \mathbb{B}_\mu(t_0; t)^{\sum_{j=l+1}^{m} \frac{1}{2^j p + \widetilde{\alpha}} \cdot \frac{2^m p + \frac{\widetilde{\alpha}}{2}}{2^m p + 2^{m-j} \frac{\widetilde{\alpha}}{2}}} \cdot \\
&\qquad\qquad \cdot \| u(\cdot, t_0) \|_{L^{2^l p}(\mathbb{R})}^{\frac{2^m p + \frac{\widetilde{\alpha}}{2}}{2^m p + 2^{m-j} \frac{\widetilde{\alpha}}{2}}} \leq \\
&\leq K_m(\alpha, \kappa; p) \cdot \max \left\{ \| u(\cdot, t_0) \|_{L^{2^m p}(\mathbb{R})} ; \mathbb{B}_\mu(t_0; t)^{\frac{1 - 2^{-m}}{p + \frac{\widetilde{\alpha}}{2}}} \cdot \| u(\cdot, t_0) \|_{L^p(\mathbb{R})}^{\frac{p + 2^{-m} \frac{\widetilde{\alpha}}{2}}{p + \frac{\widetilde{\alpha}}{2}}} \right\}.
\end{aligned}$$

Onde

$$K_m(\alpha, \kappa; p) = \prod_{j=2}^{m} \left(\frac{3.2^j p + 2\alpha}{8} \right)^{\frac{1}{2^j p + \widetilde{\alpha}} \frac{1 + 2^{-m}\widehat{\alpha}}{1 + 2^{-j}\widehat{\alpha}}}. \tag{4.47}$$

Resta mostrar que $K_m(\alpha, \kappa; p)$ é também dado por (4.43). Observe que:

$$\frac{1}{2^j p + \widetilde{\alpha}} \frac{2^j}{2^j + \widehat{\alpha}} = \frac{1}{2p} \frac{1}{2^{j-1} + \widehat{\alpha}} \frac{2^j}{2^j + \widehat{\alpha}} = \frac{1}{p} \left(\frac{1}{2^{j-1} + \widehat{\alpha}} - \frac{1}{2^j + \widehat{\alpha}} \right).$$

Logo,

$$
\begin{aligned}
K_m(\alpha,\kappa;p)^{\frac{1}{1+2^{-m}\widehat{\alpha}}} &= \prod_{j=2}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widehat{\alpha}}\frac{2^j}{2^j+\widehat{\alpha}}} \\
&= \left[\prod_{j=2}^{m}\left(\frac{6.2.2^{j-1}p+2\alpha}{8}\right)^{\frac{1}{2^{j-1}+\widehat{\alpha}}}\right]^{\frac{1}{p}}\left[\prod_{j=2}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}}\right]^{-\frac{1}{p}} \\
&= \left[\prod_{j=1}^{m-1}\left(\frac{6.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}}\right]^{\frac{1}{p}}\left[\prod_{j=2}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}}\right]^{-\frac{1}{p}} \\
&= \left(\frac{6.2^m p+2\alpha}{8}\right)^{\frac{-1}{2^m+\widehat{\alpha}}\frac{1}{p}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}\frac{1}{p}}\right]^{\frac{1}{p}}\cdot \\
&\quad \cdot\left[\prod_{j=1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}\frac{1}{p}}\right]^{-1}\left(\frac{3.2p+2\alpha}{8}\right)^{\frac{1}{2+\widehat{\alpha}}\frac{1}{p}} \\
&= \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{2p+\frac{\widetilde{\alpha}}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1}{2^m+\frac{\widetilde{\alpha}}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1}{2^j+\frac{\widetilde{\alpha}}{2}}}\right]
\end{aligned}
$$

Assim, $K_m(\alpha,\kappa;p)$ é dado por

$$
K_m(\alpha,\kappa;p) = \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2p+\frac{\widetilde{\alpha}}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1+2^{-m}\widehat{\alpha}}{2^m+\frac{\widetilde{\alpha}}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1}{2^j+\frac{\widetilde{\alpha}}{2}}}\right]^{\frac{1+2^{-m}\widehat{\alpha}}{2^m+\frac{\widetilde{\alpha}}{2}}} \tag{4.48}
$$

O que mostra (4.43) e conclui a prova do Lema (4.2).

∎

Trabalharemos um pouco mais com os termos de (4.41) e, com isso, obteremos uma estimativa fundamental para $\mathbb{U}_\infty(t_0,t)$. O processo é análogo ao desenvolvido na prova do Teorema (4.2). Assim, como

$$
\begin{aligned}
&\prod_{j=1}^{m}\left(\frac{3.2^j p+2\alpha}{8}\right)^{\frac{1}{2^j+\widehat{\alpha}}\frac{2^m p+\frac{\widetilde{\alpha}}{2}}{2^m p+2^{m-j}\frac{\widetilde{\alpha}}{2}}} = \\
&\quad = \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\widetilde{\alpha}}{2}}}\left(\frac{3.2^m p+\alpha}{4}\right)^{-\frac{1+2^{-m}\widehat{\alpha}}{2^m+\frac{\widetilde{\alpha}}{2}}}\left[\prod_{j=1}^{m}\left(\frac{6.2^j p+2\alpha}{3.2^j p+2\alpha}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right] \\
&\quad = \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\widetilde{\alpha}}{2}}\frac{1}{2+\widehat{\alpha}}}\cdot K_m(\alpha,\kappa;p),
\end{aligned}
$$

então, pelos Lemas (4.1) e (4.2), obtemos

$$\begin{aligned}\mathbb{U}_{2^m p}(t_0;t) \quad \leq \quad \max\Bigg\{ & \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; K_m(\alpha,\kappa;p)\,\|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \\ & K_m(\alpha,\kappa;p)\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\widetilde{\alpha}}{2}}}\,\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}} ; \\ & \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\widetilde{\alpha}}{2}}\frac{1}{2+\widehat{\alpha}}} K_m(\alpha,\kappa;p)\mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\widetilde{\alpha}}{2}}}\,\mathbb{U}_p(t_0,t)^{\frac{p+2^{-m}\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}}\Bigg\}\end{aligned} \tag{4.49}$$

Onde $K_m(\alpha,\kappa;p) > 0$ é dado por (4.43). Como $1+\widehat{\alpha} > 0$ e $\dfrac{3.2^j p + 2\alpha}{8} \geq 1, \forall\, j \geq 2$, segue de (4.47) que

$$K_m(\alpha,\kappa;p) > 1, \qquad \forall\, m \geq 2.$$

Assim, (4.49) equivale a

$$\begin{aligned}\mathbb{U}_{2^m p}(t_0;t) \leq K_m(\alpha,\kappa;p)\cdot\max\Bigg\{ & \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \\ & \mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\widetilde{\alpha}}{2}}}\,\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}^{\frac{p+2^{-m}\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}} ; \\ & \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\widetilde{\alpha}}{2}}\frac{1}{2+\widehat{\alpha}}} \mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\widetilde{\alpha}}{2}}}\,\mathbb{U}_p(t_0,t)^{\frac{p+2^{-m}\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}}\Bigg\}.\end{aligned} \tag{4.50}$$

Logo,

$$\mathbb{U}_{2^m p}(t_0;t) \leq \widetilde{K}_m(\alpha,\kappa;p)\max\left\{ \|u(\cdot,t_0)\|_{L^{2^m p}(\mathbb{R})} ; \mathbb{B}_\mu(t_0;t)^{\frac{1-2^{-m}}{p+\frac{\widetilde{\alpha}}{2}}}\,\mathbb{U}_p(t_0,t)^{\frac{p+2^{-m}\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}}\right\}, \tag{4.51}$$

onde $\widetilde{K}_m(\alpha,\kappa;p) > 1$ é dado por

$$\widetilde{K}_m(\alpha,\kappa;p) = K_m(\alpha,\kappa;p)\max\left\{1; \left(\frac{3p+\alpha}{4}\right)^{\frac{1+2^{-m}\widehat{\alpha}}{p+\frac{\widetilde{\alpha}}{2}}\frac{1}{2+\widehat{\alpha}}}\right\}, \tag{4.52}$$

com $K_m(\alpha,\kappa;p) > 1$ é dado em (4.47). Fazendo $m \to +\infty$ em (4.51), (4.52), obtemos a seguinte estimativa para $\mathbb{U}_\infty(t_0;t)$:

Teorema 4.8. *Seja $u(\cdot,t) \in L^\infty([0,T_*[, L^\infty(\mathbb{R}))$ solução (fraca) não negativa do problema (4.1). Seja $\kappa \geq \dfrac{\alpha}{2}$ e $p \geq p_0$ satisfazendo $p > \kappa - \alpha$. Então, para cada $0 \leq t_0 \leq t < T_*$, tem-se*

$$\mathbb{U}_\infty(t_0;t) \leq \widetilde{K}(\alpha,\kappa;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} ; \mathbb{B}_\mu(t_0;t)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \mathbb{U}_p(t_0,t)^{\frac{p}{p+\frac{\widetilde{\alpha}}{2}}} \right\}, \quad (4.53)$$

onde $\widetilde{K}(\alpha,\kappa;p) \geq 1$ *é a constante dada por*

$$\begin{aligned} \widetilde{K}(\alpha,\kappa;p) &= \max\left\{1; \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}} \frac{1}{2+\widetilde{\alpha}}}\right\} \cdot \lim_{m\to+\infty} K_m(\alpha,\kappa;p) \qquad (4.54) \\ &= \max\left\{1; \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}} \frac{1}{2+\widetilde{\alpha}}} \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \cdot \prod_{j=1}^{\infty} \left(\frac{6\cdot 2^j p + 2\alpha}{3\cdot 2^j p + 2\alpha}\right)^{\frac{1}{2^j p + \widetilde{\alpha}/2}}\right\} \end{aligned}$$

Note que $\widetilde{K}(\alpha,\kappa;p) = \lim_{m\to+\infty} \widetilde{K}_m(\alpha,\kappa;p)$. Em particular, para cada $\alpha > 0$ e $\kappa \geq \dfrac{\alpha}{2}$ dados, temos:

$$\lim_{p\to+\infty} \widetilde{K}(\alpha,\kappa;p) = 1. \quad (4.55)$$

Se tivermos $B_\mu < \infty$ e $U_p < \infty$, (para $p \geq p_0$ satisfazendo $p > \kappa - \alpha$) e $T_* = \infty$, então a desigualdade (4.53), com $\widetilde{K}(\alpha,\kappa;p) \geq 1$ dada por (4.54), fornece

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{K}(\alpha,\kappa;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} ; \mathbb{B}_\mu(t_0)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \mathbb{U}_p(t_0)^{\frac{p}{p+\frac{\widetilde{\alpha}}{2}}} \right\}, \quad (4.56)$$

para cada $t_0 \geq 0$. Em particular, segue que

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{K}(\alpha,\kappa;p) \max\left\{ \liminf_{t\to+\infty} \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} ; B_\mu^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} U_p^{\frac{p}{p+\frac{\widetilde{\alpha}}{2}}} \right\}, \quad (4.57)$$

onde $\dfrac{\widetilde{\alpha}}{2} = \kappa - \alpha$, $\kappa \geq \alpha/2$, $u(\cdot,t) \geq 0$ e $p \geq p_0$ satisfaz $p > \kappa - \alpha$.

Com mais dois resultados, poderemos obter o resultado principal deste capítulo.

Teorema 4.9. *Suponha* $V_p < \infty$ *para* $p \geq p_0$ *com* $p > \kappa - \alpha$. *Então, para cada* $q \geq 2p$, *temos*

$$V_q \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} B_\mu^{\frac{1}{q+\widetilde{\alpha}}} V_{\frac{q}{2}}^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\widetilde{\alpha}}}, \quad (4.58)$$

onde $\widetilde{\alpha} = \alpha - \gamma$ *e* $\gamma = 2\kappa - \alpha$.

Prova: Considere

$$g(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\widetilde{\alpha}}} \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\widetilde{\alpha}/2}{q+\widetilde{\alpha}}}$$

e

$$\lambda_q := \limsup_{t\to+\infty} g(t). \tag{4.59}$$

Devemos mostrar que $\limsup_{t\to+\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})} \leq \lambda_q$, ou seja, da definição de lim sup, devemos mostrar que

$$\limsup_{t\to+\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})} \leq \lambda_q + \eta, \qquad \forall\, \eta > 0. \tag{4.60}$$

Suponhamos por contradição que (4.60) seja falsa. Então, existe $\widehat{\eta} > 0$ tal que

$$\limsup_{t\to+\infty} \|v(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{4.61}$$

Seja, então, $\widehat{t_0} \gg 1$ suficientemente grande tal que

$$g(t) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{4.62}$$

Assim, devemos ter

$$\|v(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{4.63}$$

De fato, suponhamos por absurdo que (4.63) seja falso, então existe $t_0 \geq \widehat{t_0}$ tal que

$$\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \tag{4.64}$$

De (4.61), existe $t_2 > t_0$ tal que $\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}$. Logo, podemos encontrar $t_1 \in [t_0, t_2[$ tal que

$$\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q + \widehat{\eta}$$

e

$$\|v(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \in]t_1, t_2]. \tag{4.65}$$

Como $\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \|v(\cdot,t_1)\|_{L^q(\mathbb{R})}$, deve existir $t_* \in [t_1, t_2] \setminus E_q$ tal que

$$\frac{d}{dt}\|v(\cdot,t_0)\|_{L^q(\mathbb{R})}^q \bigg|_{t=t_*} \geq 0, \tag{4.66}$$

pois, caso contrário, teríamos $\frac{d}{dt}\|v(\cdot,t_0)\|_{L^q(\mathbb{R})}^q < 0$, $q.t.p.\, t \in [t_1, t_2]$, ou seja, $\|v(\cdot,t)\|_{L^q(\mathbb{R})}$ seria estritamente decrescente em $[t_1, t_2]$, contradizendo $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} < \|v(\cdot,t_2)\|_{L^q(\mathbb{R})}$. Então, pelo Teorema (4.4), teríamos

$$\begin{aligned} \|v(\cdot,t_*)\|_{L^q(\mathbb{R})} &\leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{q+\widetilde{\alpha}}} \|v(\cdot,t_*)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\widetilde{\alpha}/2}{q+\widetilde{\alpha}}} \\ &= g(t_*) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \text{ onde } t_* \in [t_1, t_2] \end{aligned} \tag{4.67}$$

A última desigualdade segue de (4.62). Mas isso contradiz (4.65) acima. Ou seja, (4.64) não pode ocorrer. Logo, (4.63) é verdadeiro. Agora vamos provar (4.58). Para isso, introduzimos

$$w(x,t) := v(x,t)^{\frac{q+\alpha}{2}}, \qquad \forall\, x \in \mathbb{R}, \quad t \geq 0. \tag{4.68}$$

Usando a desigualdade SNG, para $\beta = 2\beta_0$, $\beta_0 = \dfrac{q}{q+\alpha} > 0$ qualquer e $\theta = \dfrac{1}{2+\beta_0}$,

$$\|\mathsf{w}\|_{L^\beta(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^\theta \cdot \|\mathsf{w}\|^{1-\theta}_{L^{\beta_0}(\mathbb{R})} \cdot \|\mathsf{w}_x\|^{\theta}_{L^2(\mathbb{R})}, \tag{4.69}$$

obtemos para todo $t \in [\widehat{t}_0, \infty[\setminus E_q$:

$$\begin{aligned}
\|w(\cdot,t)\|^{\frac{6q+4\alpha}{q+\alpha}}_{L^\beta(\mathbb{R})} \leq\ & \left(\frac{3q+2\alpha}{4q+4\alpha}\right)^2 \cdot \|w(\cdot,t)\|^{\frac{4q+2\alpha}{q+\alpha}}_{L^{\beta_0}(\mathbb{R})} \cdot \|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \\
\leq\ & \left(\frac{3q+2\alpha}{4q+4\alpha}\right)^2 \cdot \|w(\cdot,t)\|^{\frac{4q+2\alpha}{q+\alpha}}_{L^{\beta_0}(\mathbb{R})} \left\{ \frac{(3q+2\alpha)(q+\alpha)^2}{4q(q-1)(q+\widetilde{\alpha})} \cdot \mu(t)^{-1} \cdot \right. \\
& \cdot \left(-\frac{d}{dt}\|w(\cdot,t)\|^\beta_{L^\beta(\mathbb{R})}\right) \\
& \left. \cdot \frac{(q+\alpha)^2}{2^{\frac{5q+4\gamma+2\alpha}{q+\widetilde{\alpha}}}} \cdot (3q+2\alpha)^{\frac{q+2\gamma}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\widetilde{\alpha}}} \cdot \|w(\cdot,t)\|^{\beta_0 \frac{2q+\alpha+\gamma}{q+\widetilde{\alpha}}}_{L^{\beta_0}(\mathbb{R})} \right\}.
\end{aligned} \tag{4.70}$$

Em termos de $v(\cdot,t)$, temos

$$\begin{aligned}
(\lambda_q + \widehat{\eta})^{3q+2\alpha} \leq\ & \|v(\cdot,t)\|^{3q+2\alpha}_{L^q(\mathbb{R})} = \|w(\cdot,t)\|^{\frac{6q+4\alpha}{q+\alpha}}_{L^\beta(\mathbb{R})} \\
\leq\ & \left[\left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\widetilde{\alpha}}} \|v(\cdot,t)\|^{\frac{q+\widetilde{\alpha}/2}{q+\widetilde{\alpha}}}_{L^{\frac{q}{2}}(\mathbb{R})}\right]^{3q+2\alpha} \cdot \|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} + \\
& + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot \|v(\cdot,t)\|^{2q+\alpha}_{L^{\frac{q}{2}}(\mathbb{R})} \mu(t)^{-1} \left(-\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\right) \\
\leq\ & g(t)^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot \mathbb{M}_0^{2q+\alpha} \mu(t)^{-1} \left(-\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\right) \\
\leq\ & (\lambda_q + \frac{1}{2}\widehat{\eta})^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot \mathbb{M}_0^{2q+\alpha} \cdot \mu(t)^{-1} \cdot \\
& \cdot \left(-\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\right)
\end{aligned} \tag{4.71}$$

para todo $t \in [\widehat{t}_0, \infty[\setminus E_q$, onde $\mathbb{M}_0 > 0$ é definido por

$$\mathbb{M}_0 := \sup_{t \geq \widehat{t}_0} \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}.$$

Assim, obtemos

$$-\frac{d}{dt}\left\|v(\cdot,t)\right\|_{L^q(\mathbb{R})}^q \geq C(\alpha,\kappa,q;\widehat{\eta})\mu(t), \qquad \forall\, t\in[\widehat{t}_0,\infty[\setminus E_q, \tag{4.72}$$

onde $C(\alpha,\kappa,q;\widehat{\eta})>0$ é a constante dada por

$$C(\alpha,\kappa,q;\widehat{\eta}) := \frac{64q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)^3}\cdot \mathbb{M}_0^{-(2q+\alpha)}\left[(\lambda_q+\widehat{\eta})^{3q+2\alpha}-(\lambda_q+\frac{1}{2}\widehat{\eta})^{3q+2\alpha}\right].$$

Integrando (4.72) em $[\widehat{t}_0,\widehat{T}]$, para $\widehat{T}>\widehat{t}_0$ arbitrário, obtemos

$$-\left\|v(\cdot,\widehat{T})\right\|_{L^q(\mathbb{R})}^q+\left\|v(\cdot,\widehat{t}_0)\right\|_{L^q(\mathbb{R})}^q \geq C(\alpha,\kappa,q;\widehat{\eta})\int_{\widehat{t}_0}^{\widehat{T}}\mu(t)dt, \qquad \forall\,\widehat{T}>\widehat{t}_0.$$

Em particular,

$$\left\|v(\cdot,\widehat{t}_0)\right\|_{L^q(\mathbb{R})}^q \geq C(\alpha,\kappa,q;\widehat{\eta})\int_{\widehat{t}_0}^{\widehat{T}}\mu(t)dt, \qquad \forall\,\widehat{T}>\widehat{t}_0,$$

o que não ocorre, pois $\displaystyle\int_{\widehat{t}_0}^{\widehat{T}}\mu(t)dt=\infty$, por hipótese. Esta contradição nos mostra que (4.61) é falsa. Logo, (4.60) é verdadeira $\forall\,\eta>0$. Portanto,

$$\limsup_{t\to+\infty}\left\|v(\cdot,t)\right\|_{L^q(\mathbb{R})}\geq\lambda_q,$$

como afirmamos.

■

Teorema 4.10. *Suponha $\mathsf{U}_p<\infty$ para algum $p\geq p_0$ com $p>\kappa-\alpha$. Então, para cada $q\geq 2p$, temos*

$$\mathsf{U}_q \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} B_\mu^{\frac{1}{q+\widetilde{\alpha}}}\,\mathsf{U}_{\frac{q}{2}}^{\frac{q+\widetilde{\alpha}/2}{q+\widetilde{\alpha}}}, \tag{4.73}$$

onde $\widetilde{\alpha}=\alpha-\gamma$, $\gamma=2\kappa-\alpha$ e $B_\mu=\limsup\limits_{t\to+\infty}\dfrac{B(t)}{\mu(t)}$.

Prova: Considere

$$g(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}}\left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\widetilde{\alpha}}}\left\|u(\cdot,t)\right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\widetilde{\alpha}/2}{q+\widetilde{\alpha}}}$$

e

$$\lambda_q := \limsup\nolimits_{t\to+\infty} g(t).$$

Devemos mostrar que $\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \lambda_q$, ou seja,

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \lambda_q + \eta, \qquad \forall\, \eta > 0. \tag{4.74}$$

Novamente argumentamos por contradição. Suponhamos que (4.74) seja falsa. Então, existe $\widehat{\eta} > 0$ tal que

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{4.75}$$

Para este $\widehat{\eta} > 0$, seja $\widehat{t}_0 \gg 1$ suficientemente grande tal que

$$g(t) \leq \lambda_q + \frac{1}{4}\widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0. \tag{4.76}$$

Segue, então, que

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0. \tag{4.77}$$

De fato, suponhamos por absurdo que (4.77) seja falso, então existe $t_0 \geq \widehat{t}_0$ tal que

$$\|u(\cdot,t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0. \tag{4.78}$$

De (4.75), existe $t_2 > t_0$ tal que

$$\|u(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}.$$

Logo, podemos encontrar $t_1 \in [t_0, t_2[$ tal que

$$\begin{aligned} &\|u(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q + \widehat{\eta} \\ \text{e} \\ &\|u(\cdot,t)\|_{L^q(\mathbb{R})} > \lambda_q + \widehat{\eta}, \qquad \forall\, t \in]t_1, t_2]. \end{aligned} \tag{4.79}$$

A ideia agora é obter um contradição como na prova de (4.63) no Teorema (4.9) acima. Para isso, aproximamos $u(\cdot,t)$ por uma solução suave $v^{[\epsilon]}(\cdot,t)$. Fixando $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, com $\zeta(x) > 0$, $\forall\, x \in \mathbb{R}$. Seja, para cada $\epsilon > 0$, $v^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([\widehat{t}_0, T_*^{[\epsilon]}[, L^\infty(\mathbb{R}))$ solução suave e positiva do problema

$$\begin{cases} v_t + (b(x,t)v^{\kappa+1})_x = \mu(t)(v(x,t)^\alpha v_x)_x, & x \in \mathbb{R}, \quad t > \widehat{t}_0, \\ v(\cdot,\widehat{t}_0) = u(\cdot,\widehat{t}_0) + \epsilon\zeta \end{cases}$$

e tomamos $\epsilon > 0$ suficientemente pequeno tal que se tenha

$$\begin{aligned} &v^{[\epsilon]}(\cdot,t) \text{ esteja definida } \forall t \in [\widehat{t}_0, t_2], \\ &\left\|v^{[\epsilon]}(\cdot,t_1)\right\|_{L^q(\mathbb{R})} < \left\|v^{[\epsilon]}(\cdot,t_2)\right\|_{L^q(\mathbb{R})} \end{aligned} \tag{4.80}$$

e

$$g^{[\epsilon]}(t) \leq \lambda_q + \tfrac{1}{2}\widehat{\eta}, \qquad \forall\, t \in [t_1, t_2], \tag{4.81}$$

onde $g^{[\epsilon]}(t)$ é dada por

$$g^{[\epsilon]}(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\tilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\tilde{\alpha}}} \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\tilde{\alpha}/2}{q+\tilde{\alpha}}}, \quad \forall t \in [\widehat{t_0}, t_2].$$

Note que (4.80) é verdadeira para $\epsilon \ll 1$, pois $\|u(\cdot,t_1)\|_{L^q(\mathbb{R})} < \|u(\cdot,t_2)\|_{L^q(\mathbb{R})}$. E, (4.81), decorre de (4.76) para $\epsilon \ll 1$. De (4.80) segue que existe $t_* \in [t_1,t_2] \setminus E_q^{[\epsilon]}$, tal que

$$\frac{d}{dt} \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^q(\mathbb{R})}^q \Big|_{t=t_*} \geq 0,$$

pois, caso contrário, teríamos $\dfrac{d}{dt} \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^q(\mathbb{R})}^q < 0$, $q.t.p.\, t \in [t_1,t_2]$ e, em particular, $\left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^q(\mathbb{R})}$ seria estritamente decrescente em $[t_1,t_2]$. Contradizendo $\left\| v^{[\epsilon]}(\cdot,t_1) \right\|_{L^q(\mathbb{R})} > \left\| v^{[\epsilon]}(\cdot,t_2) \right\|_{L^q(\mathbb{R})}$, o que contradiz (4.80) acima! Então, pelo Teorema (4.4), teríamos para este $t_* \in [t_1,t_2]$:

$$\begin{aligned} \left\| v^{[\epsilon]}(\cdot,t_*) \right\|_{L^q(\mathbb{R})} &\leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\tilde{\alpha}}} \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\frac{1}{q+\tilde{\alpha}}} \left\| v^{[\epsilon]}(\cdot,t_*) \right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\tilde{\alpha}/2}{q+\tilde{\alpha}}} \\ &= g^{[\epsilon]}(t_*) \leq \lambda_q + \frac{1}{2}\widehat{\eta}. \end{aligned}$$

A última desigualdade segue de (4.81). Mas esta estimativa contradiz

$$\begin{aligned} \left\| v^{[\epsilon]}(\cdot,t_*) \right\|_{L^q(\mathbb{R})} &\geq \|u(\cdot,t_*)\|_{L^q(\mathbb{R})}, \text{ pois } v^{[\epsilon]}(\cdot,t_*) \geq u(\cdot,t_*), \qquad \forall\, t \in [\widehat{t_0}, t_2] \\ &\geq \lambda_q + \widehat{\eta}, \qquad \text{por (4.79)}. \end{aligned}$$

Esta contradição mostra que (4.78) não pode ocorrer. Logo, (4.77) é verdadeiro. Seja agora $\mathbb{U}_0 \geq 0$, dado por

$$\mathbb{U}_0 := \sup_{t \geq \widehat{t_0}} \|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})},$$

e seja $C > 0$ a constante (em t) dada por

$$C := \frac{64q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)^3} \cdot (1+\mathbb{U}_0)^{-(2q+\alpha)} \left[(\lambda_q + \widehat{\eta})^{3q+2\alpha} - (\lambda_q + \frac{1}{2}\widehat{\eta})^{3q+2\alpha} \right].. \tag{4.82}$$

Para esta constante, tomemos $\widehat{T} > \widehat{t_0}$ suficientemente grande tal que se tenha

$$\|u(\cdot,\widehat{t_0})\|_{L^q(\mathbb{R})}^q < C \int_{\widehat{t_0}}^{\widehat{T}} \mu(t)dt.$$

Note que podemos requerer esta estimativa, pois, por hipótese, $\displaystyle\int_{\widehat{t}_0}^{+\infty} \mu(t)dt = +\infty$. Considere $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ e positiva para todo $x \in \mathbb{R}$ e fixa no que segue. Tome, para $\epsilon > 0$ dado, uma solução $v^{[\epsilon]}(\cdot,t) \in L^\infty_{loc}([\widehat{t}_0, T_*^{[\epsilon]}[, L^\infty(\mathbb{R}))$ suave positiva do problema regularizado

$$\begin{cases} v_t + (b(x,t)v^{\kappa+1})_x = \mu(t)(v(x,t)^\alpha v_x)_x, & x \in \mathbb{R}, \quad t > \widehat{t}_0 \\ v(\cdot,\widehat{t}_0) = u(\cdot,\widehat{t}_0) + \epsilon\zeta. \end{cases} \tag{4.83}$$

Seja $\epsilon > 0$ suficientemente pequeno tal que

$$\begin{aligned} &(i) \quad v^{[\epsilon]}(\cdot,t) \qquad \text{definida para todo } t \in [\widehat{t}_0, \widehat{T}]; \\ &(ii) \quad \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{2}}(\mathbb{R})} \leq 1 + U_0, \qquad \forall\, t \in [\widehat{t}_0, \widehat{T}]; \\ &(iii) \quad \left\| v^{[\epsilon]}(\cdot,\widehat{t}_0) \right\|^q_{L^q(\mathbb{R})} < C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt; \\ &(iv) \quad g^{[\epsilon]}(t) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t \in [\widehat{t}_0, \widehat{T}], \end{aligned} \tag{4.84}$$

onde $C > 0$ é a constante definida em (4.82) e $g^{[\epsilon]}(\cdot)$ é dada por

$$g^{[\epsilon]}(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{1}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q+\widetilde{\alpha}}} \cdot \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q+\widetilde{\alpha}/2}{q+\widetilde{\alpha}}}$$

Introduzimos, então, $w^{[\epsilon]}(\cdot,t)$, $\widehat{t}_0 \leq t \leq \widehat{T}$, por

$$w^{[\epsilon]}(x,t) := v^{[\epsilon]}(x,t)^{\frac{q+\alpha}{2}}, \qquad \forall\, x \in \mathbb{R}, \quad t \in [\widehat{t}_0, \widehat{T}]. \tag{4.85}$$

Assim, para todo $t \in [\widehat{t}_0, \widehat{T}] \setminus E_q^{[\epsilon]}$

$$\begin{aligned} (\lambda_q + \widehat{\eta})^{3q+2\alpha} \leq\; & \left\| v^{[\epsilon]}(\cdot,t) \right\|^{3q+2\alpha}_{L^q(\mathbb{R})} = \left\| w^{[\epsilon]}(\cdot,t) \right\|^{\frac{6q+4\alpha}{q+\alpha}}_{L^\beta(\mathbb{R})} \\ \leq\; & g^{[\epsilon]}(t)^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot \left\| v^{[\epsilon]}(\cdot,t) \right\|^{2q+\alpha}_{L^{\frac{q}{2}}(\mathbb{R})} \cdot \mu(t)^{-1} \cdot \\ & \cdot \left(-\frac{d}{dt} \left\| v^{[\epsilon]}(\cdot,t) \right\|^q_{L^q(\mathbb{R})} \right) \\ \leq\; & (\lambda_q + \frac{1}{2}\widehat{\eta})^{3q+2\alpha} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot (1 + \sqcup_0)^{2q+\alpha} \cdot \mu(t)^{-1} \cdot \\ & \cdot \left(-\frac{d}{dt} \left\| v^{3q+2\alpha}(\cdot,t) \right\|^q_{L^q(\mathbb{R})} \right). \end{aligned}$$

Assim, $\forall t \in [\widehat{t}_0, \widehat{T}[\setminus E_q^{[\epsilon]}$:

$$-\frac{d}{dt}\left\|v^{[\epsilon]}(\cdot,t)\right\|^q_{L^q(\mathbb{R})} \geq C\mu(t), \tag{4.86}$$

onde $C > 0$ é a constante (em t) definida em (4.82). Integrando (4.86) em $[\widehat{t}_0, \widehat{T}]$, para $\widehat{T} > \widehat{t}_0$ arbitrário, obtemos

$$-\left\|v^{[\epsilon]}(\cdot,\widehat{T})\right\|^q_{L^q(\mathbb{R})} + \left\|v^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|^q_{L^q(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt. \tag{4.87}$$

Assim, em particular temos

$$\left\|v^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|^q_{L^q(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall\, \widehat{T} > \widehat{t}_0, \tag{4.88}$$

contradizendo (iii) de (4.84). Esta contradição nos mostra que (4.75) é falsa. Logo, teremos (4.74) verdadeira para todo $\eta > 0$. Portanto, (4.73) está provado, o que conclui a prova do Teorema (4.10).

∎

Note que se aplicarmos (4.10) sucessivamente para $q = 2p, 4p, 8p, ..., 2^m p$, obtemos

$$\mathsf{U}_{2p} \leq \left(\frac{3\cdot 2p + 2\alpha}{8}\right)^{\frac{1}{2p+\widetilde{\alpha}}} B_\mu^{\frac{1}{2p+\widetilde{\alpha}}}\, \mathsf{U}_p^{\frac{2p+\widetilde{\alpha}/2}{2p+\widetilde{\alpha}}}, \tag{4.89}$$

para $q = 2p$. Para $q = 4p$ e usando (4.89),

$$\begin{aligned} \mathsf{U}_{4p} &\leq \left(\frac{3\cdot 4p + 2\alpha}{8}\right)^{\frac{1}{4p+\widetilde{\alpha}}} B_\mu^{\frac{1}{4p+\widetilde{\alpha}}}\, \mathsf{U}_{2p}^{\frac{4p+\widetilde{\alpha}/2}{4p+\widetilde{\alpha}}} \\ &\leq \left(\frac{3\cdot 4p + 2\alpha}{8}\right)^{\frac{1}{4p+\widetilde{\alpha}}} \left(\frac{3\cdot 2p + 2\alpha}{8}\right)^{\frac{1}{2p+\widetilde{\alpha}}\frac{4p+\widetilde{\alpha}/2}{4p+\widetilde{\alpha}}} B_\mu^{\frac{1}{4p+\widetilde{\alpha}}+\frac{1}{2p+\widetilde{\alpha}}\cdot\frac{4p+\widetilde{\alpha}/2}{4p+\widetilde{\alpha}}}\, \mathsf{U}_{2p}^{\frac{4p+\widetilde{\alpha}/2}{4p+2\widetilde{\alpha}}}. \end{aligned} \tag{4.90}$$

No caso geral, para $m \geq 1$ qualquer,

$$\begin{aligned} \mathsf{U}_{2^m p} \;\leq\; & \left[\prod_{j=1}^{m}\left(\frac{3\cdot 2^j p + 2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{2^m p+\widetilde{\alpha}/2}{2^m p+2^{m-j}\widetilde{\alpha}/2}}\right]\cdot \\ & \cdot B_\mu^{\sum_{j=1}^m \frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{2^m p+\widetilde{\alpha}/2}{2^m p+2^{m-j}\widetilde{\alpha}/2}} \cdot \mathsf{U}_p^{\frac{2^m p+\widetilde{\alpha}/2}{2^m p+\widetilde{\alpha}/2}}, \end{aligned} \tag{4.91}$$

onde $\widetilde{\alpha} = \alpha - \gamma = -2(\kappa - \alpha)$, $\kappa \geq \dfrac{\alpha}{2}$ e $u(\cdot,t) \geq 0$. Segue agora o resultado principal deste capítulo.

Teorema 4.11. *Suponha $V_q < \infty$, com $p \geq p_0$ satisfazendo $p > \kappa - \alpha$. Então, para todo $q \geq 2p$*

$$\liminf_{t\to+\infty} \|v(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\widetilde{\alpha}}} \limsup_{t\to+\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{2}{q+\widetilde{\alpha}}} \cdot \|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}} \right\}, \tag{4.92}$$

onde $\widetilde{\alpha} = \alpha - \gamma$ *e* $\gamma = 2\kappa - \alpha$.

Prova: Considere a seguinte mudança de variável

$$w(x,t) = v(x,t)^{\frac{q+\alpha}{2}}.$$

Escrevendo (4.92) em termos de w, temos

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \limsup_{t\to+\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}} \right\}, \tag{4.93}$$

onde $\beta_0 = \dfrac{q}{q+\alpha}$. Definindo

$$g(t) := \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}}$$

e

$$\lambda_q := \limsup_{t\to+\infty} g(t),$$

devemos mostrar que $\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \lambda_q$, ou equivalentemente

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \lambda_q + \eta, \qquad \forall\, \eta > 0. \tag{4.94}$$

Suponhamos por contradição que (4.94) seja falso, então deve existir algum $\widehat{\eta} > 0$, tal que

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{4.95}$$

Seja, então, $\widehat{t_0} \gg 1$ suficientemente grande tal que

$$\begin{aligned} &\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \geq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0} \\ \text{e} \quad & \\ &g(t) \leq \lambda_q + \frac{1}{2}\widehat{\eta}, \qquad \forall\, t \geq \widehat{t_0}. \end{aligned} \tag{4.96}$$

Pelo Teorema (4.2) e pela desigualdade NSG $\|w\|_{L^\infty(\mathbb{R})}^{2+\beta_0} \leq \left(\dfrac{2+\beta_0}{4}\right)^2 \|w\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0} \|w_x\|_{L^2(\mathbb{R})}^2$, onde $\beta_0 = \dfrac{q}{q+\alpha}$, obtemos para todo $t \in [\widehat{t_0}, \infty[\setminus E_q$

$$
\begin{aligned}
(\lambda_q+\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} &\leq \|w(\cdot,t)\|_{L^\infty(\mathbb{R})}^{\frac{3q+2\alpha}{q+\alpha}} \\
&\leq \frac{1}{16}\left(\frac{3q+2\alpha}{q+\alpha}\right)^2\cdot\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}}\cdot\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2
\end{aligned}
\tag{4.97}
$$

Pela desigualdade de Young, temos a seguinte forma para a estimativa (4.97)

$$
\begin{aligned}
(\lambda_q+\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} \leq &\left[\left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}}\left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}}\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}}\right]^{\frac{3q+2\alpha}{q+\alpha}}+ \\
&+\frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})}\cdot\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}}\cdot\mu(t)^{-1}\cdot \\
&\cdot\left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}\right)
\end{aligned}
\tag{4.98}
$$

Pela definição de $g(t)$, a estimativa (4.98) resulta em

$$
\begin{aligned}
(\lambda_q+\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} &\leq g(t)^{\frac{3q+2\alpha}{q+\alpha}}+\frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})}\cdot\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\alpha}}\mu(t)^{-1} \\
&\qquad\cdot\left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}\right) \\
&\leq (\lambda_q+\frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}}+\frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})}\cdot\|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}}\cdot\mu(t)^{-1}\cdot \\
&\qquad\cdot\left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}\right).
\end{aligned}
\tag{4.99}
$$

Note que usamos (4.96) na última desigualdade. Assim, se definirmos

$$
\mathbb{M}_0 := \sup_{t\geq\widehat{t}_0}\|v(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})},
$$

teremos para todo $t\in[\widehat{t}_0,\infty[\setminus E_q$

$$
(\lambda_q+\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} \leq (\lambda_q+\frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}}+\frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})}\cdot\mathbb{M}_0^{\frac{q}{2}}\mu(t)^{-1}\left(-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}\right),
$$

ou seja,

$$
-\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^{\beta}\geq C(\alpha,\kappa,q,\widehat{\eta})\mu(t),\qquad \forall\, t\in[\widehat{t}_0,\infty[\setminus E_q. \tag{4.100}
$$

Onde $C(\alpha,\kappa,q,\widehat{\eta})>0$ é a constante dada por

$$C(\alpha,\kappa,q,\widehat{\eta}) = \frac{64q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)^3}\cdot \mathbb{M}_0^{\frac{q}{2}}\left[(\lambda_q+\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} - (\lambda_q+\frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}}\right].$$

Tomando $\widehat{T} > \widehat{t}_0$ arbitrário e integrando (4.100) em $[\widehat{t}_0, \widehat{T}]$, obtemos

$$-\left\|w(\cdot,\widehat{T})\right\|^{\beta}_{L^{\beta}(\mathbb{R})} + \left\|w(\cdot,\widehat{t}_0)\right\|^{\beta}_{L^{\beta}(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall \widehat{T} > \widehat{t}_0.$$

Em particular,

$$\left\|w(\cdot,\widehat{t}_0)\right\|^{\beta}_{L^{\beta}(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall \widehat{T} > \widehat{t}_0. \tag{4.101}$$

Em termos de $v(\cdot,t)$, podemos reescrever (4.101) como

$$\left\|v(\cdot,\widehat{t}_0)\right\|^{q}_{L^{q}(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall \widehat{T} > \widehat{t}_0.$$

o que não é verdadeiro, pois $\displaystyle\int_{\widehat{t}_0}^{+\infty} \mu(t)dt = \infty$ por hipótese. Com isso, a afirmação (4.95) também não pode ser verdadeira. Logo, vale (4.94), o que mostra (4.93), ou equivalentemente, (4.92). Concluindo a demonstração do Teorema (4.11).

■

Agora vamos estender os resultado obtidos nas seções anteriores para soluções mais gerais $u(\cdot,t) \geq 0$. Observando a prova do Teorema (4.11), vemos que podemos ter:

Teorema 4.12. *Suponha $\mathsf{U}_q < \infty$, com $p \geq p_0$ satisfazendo $p > \kappa - \alpha$. Então, para todo $q \geq 2p$*

$$\liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\widetilde{\alpha}}} \limsup_{t\to+\infty}\left\{\left(\frac{B(t)}{\mu(t)}\right)^{\frac{2}{q+\widetilde{\alpha}}}\cdot \|u(\cdot,t)\|^{\frac{q}{q+\widetilde{\alpha}}}_{L^{\frac{q}{2}}(\mathbb{R})}\right\}, \tag{4.102}$$

onde $\widetilde{\alpha} = \alpha - \gamma$ e $\gamma = 2\kappa - \alpha$.

Prova: Seja $w(\cdot,t)$, para $t \geq 0$ e $x \in \mathbb{R}$ qualquer, definida por

$$w(x,t) = u(x,t)^{\frac{q+\alpha}{2}}.$$

Escrevendo (4.102) em termos de w, temos

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \limsup_{t\to+\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}} \right\}, \tag{4.103}$$

onde $\beta_0 = \dfrac{q}{q+\alpha}$. Definindo

$$\begin{aligned} g(t) &:= \left(\frac{3q+2\alpha}{8}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}} \\ \text{e} \\ \lambda_q &:= \limsup_{t\to+\infty} g(t), \end{aligned}$$

devemos mostrar que $\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \lambda_q$, ou equivalentemente

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \lambda_q + \eta, \qquad \forall\, \eta > 0. \tag{4.104}$$

Suponhamos por contradição que (4.104) seja falso, então deve existir algum $\widehat{\eta} > 0$, tal que

$$\liminf_{t\to+\infty} \|w(\cdot,t)\|_{L^\infty(\mathbb{R})} > \lambda_q + \widehat{\eta}. \tag{4.105}$$

Seja, então, $\widehat{t}_0 \gg 1$ suficientemente grande tal que

$$\begin{aligned} &\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \geq \lambda_q + \widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0 \\ \text{e} \\ &g(t) \leq \lambda_q + \frac{1}{4}\widehat{\eta}, \qquad \forall\, t \geq \widehat{t}_0. \end{aligned}$$

Seja $\mathbb{M}_0 \geq 0$, definido por

$$\mathbb{M}_0 := \sup_{t\geq \widehat{t}_0} \|u(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})},$$

e seja $C > 0$ a constante dada por

$$C := \frac{64q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)^3} \cdot \frac{1}{(1+\mathbb{M}_0)^{\frac{q}{2}}} \left[(\lambda_q + \widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} - (\lambda_q + \frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} \right]. \tag{4.106}$$

Considere $\widehat{T} > \widehat{t}_0$ suficientemente grande tal que

$$\|u(\cdot,\widehat{t}_0)\|_{L^q(\mathbb{R})}^q < C \int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt.$$

Considere $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ positiva ($\zeta(x) > 0, \forall\, x \in \mathbb{R}$). Sejam $\epsilon > 0$ dado e $v^{[\epsilon]}(\cdot,t)$ solução de

$$\begin{cases} v_t + (b(x,t)v^{\kappa+1})_x = \mu(t)(v(x,t)^{\alpha}v_x)_x, & x \in \mathbb{R}, \quad t > \widehat{t}_0 \\ v(\cdot,\widehat{t}_0) = u(\cdot,\widehat{t}_0) + \epsilon\zeta(\cdot). \end{cases}$$

Defina $w^{[\epsilon]}(\cdot,t)$ por

$$w^{[\epsilon]}(x,t) = v^{[\epsilon]}(x,t)^{\frac{q+\alpha}{2}}, \qquad \forall\, x \in \mathbb{R}, \quad t \in [\widehat{t}_0, T_*^{[\epsilon]}[.$$

Tomamos $\epsilon > 0$ suficientemente pequeno tal que

$$\begin{array}{ll} (i) & v^{[\epsilon]}(\cdot,t) \text{ esteja definida }, \qquad \forall\, t \in [\widehat{t}_0, \widehat{T}]; \\ (ii) & \left\| v^{[\epsilon]}(\cdot,\widehat{t}_0) \right\|_{L^q(\mathbb{R})}^q < C \displaystyle\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt; \\ (iii) & \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{2}}(\mathbb{R})} \leq 1 + \mathbb{M}_0, \qquad \forall\, t \in [\widehat{t}_0; \widehat{T}] \\ (iv) & g^{[\epsilon]} \leq \lambda_q + \dfrac{1}{2}\widehat{\eta}, \qquad \forall\, t \in [\widehat{t}_0, \widehat{T}], \end{array} \tag{4.107}$$

onde $g^{[\epsilon]}(\cdot)$ é dada por

$$g^{[\epsilon]}(t) := \left(\frac{3q+2\alpha}{8} \right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)} \right)^{\frac{q+\alpha}{q+\widetilde{\alpha}}} \left\| w^{[\epsilon]}(\cdot,t) \right\|_{L^{\beta}(\mathbb{R})}^{\frac{q}{q+\widetilde{\alpha}}} \quad \forall t \in [\widehat{t}_0, \widehat{T}].$$

Repetindo o argumento de (4.97) a (4.100), para $w^{[\epsilon]}(\cdot,t)$, obtemos para todo $t \in [\widehat{t}_0, \widehat{T}] \setminus E_q^{[\epsilon]}$

$$\begin{aligned} (\lambda_q + \widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} &\leq \|w(\cdot,t)\|_{L^{\infty}(\mathbb{R})}^{\frac{3q+2\alpha}{q+\alpha}} \\ &\leq \left\| w^{[\epsilon]}(\cdot,t) \right\|_{L^{\infty}(\mathbb{R})}^{\frac{3q+2\alpha}{q+\alpha}} \\ &\leq g^{[\epsilon]}(t)^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot \left\| w^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}} \mu(t)^{-1} \\ &\qquad \cdot \left(-\frac{d}{dt} \left\| w^{[\epsilon]}(\cdot,t) \right\|_{L^{\beta}(\mathbb{R})}^{\beta} \right) \\ &\leq (\lambda_q + \frac{1}{2}\widehat{\eta})^{\frac{3q+2\alpha}{q+\alpha}} + \frac{(3q+2\alpha)^3}{64q(q-1)(q+\widetilde{\alpha})} \cdot (1+\mathbb{M}_0)^{\frac{q}{2}} \cdot \mu(t)^{-1} \cdot \\ &\qquad \cdot \left(-\frac{d}{dt} \left\| w^{[\epsilon]}(\cdot,t) \right\|_{L^{\beta}(\mathbb{R})}^{\beta} \right). \end{aligned}$$

Assim, por (4.106), temos

$$-\frac{d}{dt} \left\| w^{[\epsilon]}(\cdot,t) \right\|_{L^{\beta}(\mathbb{R})}^{\beta} \geq C\mu(t), \qquad q.t.p.\, t \in [\widehat{t}_0, \widehat{T}]. \tag{4.108}$$

Integrando (4.108) em $[\widehat{t}_0, \widehat{T}]$, temos

$$-\left\|w^{[\epsilon]}(\cdot,\widehat{T})\right\|^{\beta}_{L^{\beta}(\mathbb{R})} + \left\|w^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|^{\beta}_{L^{\beta}(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt.$$

Em particular, segue que

$$\left\|w^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|^{\beta}_{L^{\beta}(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall\,\widehat{T} > \widehat{t}_0,$$

que, em termos de $v^{[\epsilon]}(\cdot,t)$, fica

$$\left\|v^{[\epsilon]}(\cdot,\widehat{t}_0)\right\|^{q}_{L^{q}(\mathbb{R})} \geq C\int_{\widehat{t}_0}^{\widehat{T}} \mu(t)dt, \qquad \forall\,\widehat{T} > \widehat{t}_0 \tag{4.109}$$

contradizendo (ii) de (4.107). Logo, a afirmação (4.105) não é verdadeira, o que mostra (4.104) e (4.103). Concluindo a demonstração do Teorema (4.12).

■

Como caso particular do Teorema (4.12), temos a seguinte estimativa para todo $q \geq 2p$, temos

$$\liminf_{t\to+\infty} \|u(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq \left(\frac{3q+2\alpha}{8}\right)^{\frac{2}{q+\widetilde{\alpha}}} \cdot B_{\mu}(t)^{\frac{2}{q+\widetilde{\alpha}}} \cdot \mathsf{U}_{\frac{q}{2}}^{\frac{q}{q+\widetilde{\alpha}}}, \tag{4.110}$$

onde $p \geq p_0$ e $p > \kappa - \alpha$ é tal que $\mathsf{U}_p < \infty$. Os resultados (4.91) e (4.110) acima, nos dão condições de provar o seguinte resultado:

Teorema 4.13. *Suponha $\mathsf{U}_p < \infty$, com $p \geq p_0$ satisfazendo $p > \kappa - \alpha$. Então,*

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq \mathbb{K}(\alpha,\kappa;p) \cdot B_{\mu}^{\frac{1}{p-(\kappa-\alpha)}} \cdot \mathsf{U}_p^{\frac{1}{p-(\kappa-\alpha)}}, \tag{4.111}$$

onde

$$\mathbb{K}(\alpha,\kappa;p) = \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p-(\kappa-\alpha)}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p + 2\alpha}{3\cdot 2^j p + 2\alpha}\right)^{\frac{1}{2^j p-(\kappa-\alpha)}}. \tag{4.112}$$

Prova: De (4.57), para cada $m \geq 1$, temos

$$
\begin{aligned}
\mathsf{U}_\infty &\leq \widetilde{K}(\alpha,\kappa;2^m p)\max\left\{\liminf_{t\to\infty}\|u(\cdot,t)\|\,;B_\mu^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\widetilde{\alpha}}{2}}}\right\}\\
&\leq \widetilde{K}(\alpha,\kappa;2^m p)\max\left\{\left(\frac{3\cdot 2^m p+\alpha}{4}\right)^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\cdot B_\mu^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\widetilde{\alpha}}{2}}};\right.\\
&\qquad\qquad\left. B_\mu^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\widetilde{\alpha}}{2}}}\right\}\\
&= \widetilde{\widetilde{K}}(\alpha,\kappa;2^m p)B_\mu^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\cdot \mathsf{U}_{2^m p}^{\frac{2^m p}{2^m p+\frac{\widetilde{\alpha}}{2}}},
\end{aligned}
$$

onde

$$
\widetilde{\widetilde{K}}(\alpha,\kappa;2^m p)=\widetilde{K}(\alpha,\kappa;2^m p)\max\left\{1;\left(\frac{3\cdot 2^m p+\alpha}{4}\right)^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\right\} \tag{4.113}
$$

Observe, em particular (por (4.113) e por (4.113)) que $\widetilde{\widetilde{K}}(\alpha,\kappa;2^m p)\to 1$ ao $m\to\infty$. Portanto, por (4.91), obtemos

$$
\begin{aligned}
\mathsf{U}_\infty \leq \widetilde{\widetilde{K}}(\alpha,\kappa;2^m p)B_\mu^{\frac{1}{2^m p+\frac{\widetilde{\alpha}}{2}}}\cdot &\left(\prod_{j=1}^{m}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\,\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\right.\\
&\left. B_\mu^{\sum_{j=1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\,\frac{1+2^{-m}\widehat{\alpha}}{1+2^{-j}\widehat{\alpha}}}\sqcup_p^{\frac{1+2^{-m}\widehat{\alpha}}{1+\widehat{\alpha}}}\right)^{\frac{2^m p}{2^m p+\frac{\widetilde{\alpha}}{2}}}
\end{aligned}
$$

para todo $m\geq 1$. Fazendo $m\to\infty$, obtemos

$$
\mathsf{U}_\infty \leq \left[\prod_{j=1}^{m}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\,\frac{1}{1+2^{-j}\widehat{\alpha}}}\right] B_\mu^{\sum_{j=1}^{m}\frac{1}{2^j p+\widetilde{\alpha}}\,\frac{1}{1+2^{-j}\widehat{\alpha}}}\sqcup_p^{\frac{1}{1+\widehat{\alpha}}},
$$

o que prova o Teorema (4.13).

$\blacksquare$

Observação 4.1. *(i) Afirmamos que:*

$$
\sum_{j=1}^{\infty}\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}}=\frac{1}{p+\frac{\widetilde{\alpha}}{2}}=\frac{1}{p-(\kappa-\alpha)}. \tag{4.114}
$$

De fato, para todo $j\geq 1$, temos

$$\frac{1}{2^j p+\widetilde{\alpha}} \cdot \frac{1}{1+2^{-j}\widehat{\alpha}} = \frac{1}{2^j p+\widetilde{\alpha}} \cdot \frac{2^{j+1}p}{2^{j+1}p+\widetilde{\alpha}} = 2\left(\frac{1}{2^j p+\widetilde{\alpha}} - \frac{1}{2^{j+1}p+\widetilde{\alpha}}\right).$$

Assim,

$$\sum_{j=1}^{\infty} \frac{1}{2^j p+\widetilde{\alpha}} \cdot \frac{1}{1+2^{-j}\widehat{\alpha}} = 2\sum_{j=1}^{\infty}\left(\frac{1}{2^j p+\widetilde{\alpha}} - \frac{1}{2^{j+1}p+\widetilde{\alpha}}\right) = \frac{2}{2p+\widetilde{\alpha}} = \frac{1}{p+\frac{\widetilde{\alpha}}{2}}.$$

(ii) Com o que mostramos em (4.114)*, temos*

$$\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}}} = \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p+2\alpha}{3\cdot 2^j p+2\alpha}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}}. \quad (4.115)$$

De fato,

$$\begin{aligned}
\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\widetilde{\alpha}}\cdot\frac{1}{1+2^{-j}\widehat{\alpha}}} &= \prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^{j-1}p+\frac{\widetilde{\alpha}}{2}}-\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}} = \\
&= \left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^{j-1}p+\frac{\widetilde{\alpha}}{2}}}\right] \cdot \left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right]^{-1} \\
&= \left[\prod_{j=0}^{\infty}\left(\frac{6\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right] \cdot \left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right]^{-1} \\
&= \left(\frac{6\cdot p+2\alpha}{8}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}}\left[\prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right] \cdot \left[\prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j p+2\alpha}{8}\right)^{\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}}\right]^{-1} \\
&= \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p+2\alpha}{3\cdot 2^j p+2\alpha}\right)^{\frac{1}{2^j p+\frac{\widetilde{\alpha}}{2}}}
\end{aligned}$$

Fica assim mostrado o seguinte resultado fundamental:

Teorema 4.14. *Suponha* $\int_0^{+\infty}\mu(t)dt = \infty$*,* $B_\mu < \infty$ *e, para algum* $p \geq p_0$ *satisfazendo* $p > \kappa - \alpha$*, termos também* $\mathsf{U}_p < \infty$*. Então*

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathbb{K}(\alpha,\kappa;p)\cdot B_\mu^{\frac{1}{p-(\kappa-\alpha)}} \cdot \mathsf{U}_p^{\frac{p}{p-(\kappa-\alpha)}}, \quad (4.116)$$

onde $\mathbb{K}(\alpha,\kappa;p) > 0$ *é dada por*

$$\mathbb{K}(\alpha,\kappa;p) = \left(\frac{3p+\alpha}{4}\right)^{\frac{1}{p-(\kappa-\alpha)}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j p+2\alpha}{3\cdot 2^j p+2\alpha}\right)^{\frac{1}{2^j p+\widetilde{\alpha}/2}}. \quad (4.117)$$

5 Capítulo 5: $0 \leq \kappa < \frac{\alpha}{2}$

5.1 Introdução

Neste capítulo trabalharemos com $u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}))$, para $0 < T_* \leq \infty$, onde $u(\cdot,t)$ é uma solução não negativa da equação dos meios porosos com termo advectivo

$$u_t + (b(x,t)u^{\kappa+1})_x = \mu(t)(u(x,t)^\alpha u_x)_x, \qquad x \in \mathbb{R}, \quad t > 0 \tag{5.1}$$

com dado inicial

$$u(\cdot,0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{5.2}$$

onde $u_0 \geq 0$ q.s., $0 \leq \kappa < \dfrac{\alpha}{2}$, com $\alpha > 0$, e $\gamma = 2\kappa - \alpha$.

Observação 5.1. *A condição* (5.2) *é satisfeita no sentido de se ter*

$$u(\cdot,t) \to u_0 \ \textit{em} \ L^1_{loc}(\mathbb{R}) \ \textit{ao} \ t \to 0.$$

Consideramos $v(x,t)$ uma solução fraca positiva da equação nos meios porosos com termo advectivo e condição inicial dados por

$$\begin{cases} v_t + (b(x,t)v^{\kappa+1}(x,t))_x = \mu(t)(v^\alpha(x,t)v_x(x,t))_x, & x \in \mathbb{R}, \quad t > 0 \\ v(\cdot,0) = v_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad v_0 > 0 \end{cases} \tag{5.3}$$

onde $v_0 > 0$ significa que existe uma função $z \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $z(x) > 0$ para todo $x \in \mathbb{R}$ e tal que $v_0(x) \leq z(x)$, q.t.p $x \in \mathbb{R}$. Assumiremos as hipóteses descritas em (2.12), no Capítulo 2. Na secção 5.2 obtemos algumas estimativas de energia que são necessárias para obtermos as estimativas para $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$, onde $u(\cdot,t)$ é solução não negativa de (5.1) e (5.2), que esta é desenvolvida na secção 5.3.

5.2 Estimativas de Energia

Iniciamos este capítulo com soluções positivas $v(\cdot,t)$ que satisfaçam

$$v(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R})) \cap L^\infty_{loc}([0,T_*[, L^p(\mathbb{R})) \tag{5.4}$$

e provamos para esta solução alguns resultados fundamentais como a desigualdade de energia. Como segem.

Teorema 5.1. *Suponhamos que $v(\cdot,t)$ seja uma solução suave positiva do problema de valor inicial* (5.3) *e satisfaça* (5.4) *para algum $p_0 \leq p < \infty$. Então, para cada $t \in [0, T_*[$, temos*

$$\int_0^t \int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau) v_x^2(x,\tau)\, dx d\tau < \infty \tag{5.5}$$

e

$$\begin{aligned}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + \frac{q(q-1)}{2}\int_0^t \mu(\tau)\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)\,dxd\tau \leq \\ \leq \|v_0\|^q_{L^q(\mathbb{R})} + \frac{q(q-1)}{2}\int_0^t \frac{B(\tau)^2}{\mu(\tau)}\int_{\mathbb{R}} v^{q+\gamma}(x,\tau)\,dxd\tau\end{aligned} \tag{5.6}$$

para todo q satisfazendo

$$q \geq 2 \qquad e \qquad q \geq p - \gamma = p + (\alpha - 2\kappa). \tag{5.7}$$

Prova: Sejam $q \geq 2$, $R > 0$ e as funções de corte $\psi(x)$ e $\psi_R(x)$ definidas em (2.3) e (2.4), no Capítulo 2. Multiplicamos a equação (5.1) por $qv^{q-1}\psi_R(x)$ e integramos em $[0,t] \times \mathbb{R}$. Devido a função de corte $\psi_R(x)$, a região de integração é compacta, logo usamos o Teorema de Fubini na integral que envolve $v_t(x,t)$ e o Teorema de Integração por Partes nas demais integrais.

$$\begin{aligned}&\int_{|x|<2R} v(x,t)^q\psi_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|<2R} v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)\psi_R(x)dxd\tau = \\ &= \int_{|x|<2R} v_0(x)^q\psi_R(x)dx - q\int_0^t \mu(\tau)\int_{R<|x|<2R} v(x,\tau)^{q+\alpha-1}v_x(x,\tau)\psi_R'(x)dxd\tau + \\ &\quad + q(q-1)\int_0^t\int_{|x|<2R} v(x,\tau)^{q+\kappa-1}v_x(x,\tau)b(x,\tau)\psi_R(x)dxd\tau + \\ &\quad + q\int_0^t\int_{R<|x|<2R} v(x,\tau)^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau\end{aligned}$$

Escrevemos $b(x,t) = \beta(t) + b(x,t) - \beta(t)$, onde $\beta(t)$ foi definido em (2.13), Capítulo 2. Assim,

$$\begin{aligned}&\int_{|x|<2R} v(x,t)^q\psi_R(x)dx + q(q-1)\int_0^t \mu(\tau)\int_{|x|<2R} v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)\psi_R(x)dxd\tau = \\ &= \int_{|x|<2R} v_0(x)^q\psi_R(x)dx + \frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{R<|x|<2R} v(x,\tau)^{q+\alpha}\psi_R''(x)dxd\tau - \\ &\quad - \frac{q(q-1)}{q+\kappa}\int_0^t \beta(\tau)\int_{R<|x|<2R} v(x,\tau)^{q+\kappa}\psi_R'(x)dxd\tau + \\ &\quad + q(q-1)\int_0^t\int_{|x|<2R} (b(x,\tau)-\beta(\tau))v(x,\tau)^{q+\kappa-1}v_x(x,\tau)\psi_R(x)dxd\tau + \\ &\quad + q\int_0^t\int_{R<|x|<2R} v(x,\tau)^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau\end{aligned} \tag{5.8}$$

Note que $q+\kappa-1=\dfrac{q+\alpha-2}{2}+\dfrac{q+2\kappa-\alpha}{2}$. Então, pelo Teorema de Cauchy-Schwarz temos

$$\begin{aligned}
&\int_0^t\int_{|x|<2R}(b(x,\tau)-\beta(\tau))v(x,\tau)^{q+\kappa-1}v_x(x,\tau)\psi_R(x)dxd\tau\leq\\
&\qquad\leq\int_0^t\int_{|x|<2R}|b(x,\tau)-\beta(\tau)|v(x,\tau)^{q+\kappa-1}|v_x(x,\tau)|\psi_R(x)dxd\tau\\
&\qquad\leq\int_0^t B(\tau)\int_{|x|<2R}v(x,\tau)^{q+\kappa-1}|v_x(x,\tau)|\psi_R(x)dxd\tau\\
&\qquad\leq\frac{1}{2}\int_0^t\frac{B^2(\tau)}{\mu(\tau)}\int_{|x|<2R}v(x,\tau)^{q+2\kappa-\alpha}\psi_R(x)dxd\tau+\\
&\qquad\quad+\frac{1}{2}\int_0^t\mu(\tau)\int_{|x|<2R}v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)\psi_R(x)dxd\tau
\end{aligned}\tag{5.9}$$

Substituindo (5.9) em (5.8), usando que $b(x,\tau)\leq B(\tau)$ na última integral de (5.8), pois estamos integrando em $R<|x|<2R$, e somando as integrais semelhantes, tem-se que

$$\begin{aligned}
&\int_{|x|<2R}v(x,t)^q\psi_R(x)dx+\frac{q(q-1)}{2}\int_0^t\mu(\tau)\int_{|x|<2R}v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)\psi_R(x)dxd\tau\leq\\
&\quad\leq\int_{|x|<2R}v_0(x)^q\psi_R(x)dx+\frac{q}{q+\alpha}\int_0^t\mu(\tau)\int_{R<|x|<2R}v(x,\tau)^{q+\alpha}\psi_R''(x)dxd\tau-\\
&\qquad-\frac{q(q-1)}{q+\kappa}\int_0^t\beta(\tau)\int_{R<|x|<2R}v(x,\tau)^{q+\kappa}\psi_R'(x)dxd\tau+\\
&\qquad+\frac{q(q-1)}{2}\int_0^t\frac{B^2(\tau)}{\mu(\tau)}\int_{|x|<2R}v(x,\tau)^{q+2\kappa-\alpha}\psi_R(x)dxd\tau+\\
&\qquad+q\int_0^t\int_{R<|x|<2R}v(x,\tau)^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau\\
&\quad\leq\int_{|x|<2R}v_0(x)^q\psi_R(x)dx+\frac{q}{q+\alpha}\int_0^t\mu(\tau)\int_{R<|x|<2R}v(x,\tau)^{q+\alpha}|\psi_R''(x)|dxd\tau\\
&\qquad+\left(\frac{q(q-1)}{q+\kappa}+q\right)\int_0^t(\beta(\tau)+B(\tau))\int_{R<|x|<2R}v(x,\tau)^{q+\kappa}|\psi_R'(x)|dxd\tau+\\
&\qquad+\frac{q(q-1)}{2}\int_0^t\frac{B^2(\tau)}{\mu(\tau)}\int_{|x|<2R}v(x,\tau)^{q+2\kappa-\alpha}\psi_R(x)dxd\tau
\end{aligned}$$

Fazendo $R\to+\infty$ e lembrando da definição da função de corte $\psi_R(x)$ e de (2.5), obtemos

$$\int_{\mathbb{R}}v(x,t)^q dx+\frac{q(q-1)}{2}\int_0^t\mu(\tau)\int_{\mathbb{R}}v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)dxd\tau\leq$$

$$\leq \int_{\mathbb{R}} v_0(x)^q dx + \frac{q(q-1)}{2} \int_0^t \frac{B^2(\tau)}{\mu(\tau)} \int_{\mathbb{R}} v(x,\tau)^{q+\gamma} dx d\tau.$$

Ou seja,

$$\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + \frac{q(q-1)}{2} \int_0^t \mu(\tau) \int_{\mathbb{R}} v(x,\tau)^{q+\alpha-2} v_x^2(x,\tau) dx d\tau \leq \tag{5.10}$$

$$\leq \|v_0\|^q_{L^q(\mathbb{R})} + \frac{q(q-1)}{2} \int_0^t \frac{B^2(\tau)}{\mu(\tau)} \int_{\mathbb{R}} v(x,\tau)^{q+\gamma} dx d\tau.$$

Supomos (5.4) para algum $p \geq p_0$. Para que (5.10) faça sentido, devemos ter $q \geq 2$ satisfazendo $q + \gamma \geq p$. Temos assim, as seguintes condições sobre q

$$q \geq 2 \qquad \text{e} \qquad q \geq p - \gamma = p + (\alpha - 2\kappa),$$

dadas em (5.7), no Teorema (5.1). ∎

Observação 5.2. *Nas condições do Teorema* (5.1)*, ou seja, se $v(x,t)$ é uma solução suave positiva do problema de valor inicial* (5.3)*, então não haverá Blow-up em tempo finito (ou seja, $T_* = +\infty$) no caso $0 \leq \kappa < \frac{\alpha}{2}$, com $p_0 = 1$. Os detalhes seguem da identidde* (5.8) *e estão expostos a seguir. Usando as propriedades de módulo, as estimativas* (2.5) *e* (2.17) *do Capítulo 2 em* (5.8) *temos, ao $R \nearrow +\infty$,*

$$\int_{\mathbb{R}} v(x,t)^q dx + q(q-1) \int_0^t \mu(\tau) \int_{\mathbb{R}} v(x,\tau)^{q+\alpha-2} v_x^2(x,\tau) dx d\tau \leq$$

$$\leq \int_{\mathbb{R}} v_0(x)^q dx + +q(q-1) \int_0^t B(\tau) \int_{\mathbb{R}} v(x,\tau)^{q+\kappa-1} |v_x(x,\tau)| dx d\tau. \tag{5.11}$$

Usando Cauchy-Schwarz em $\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)|v_x(x,t)|dx$*, a desigualdade* (5.11) *fica*

$$\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1) \int_0^t \mu(\tau) \int_{\mathbb{R}} v(x,\tau)^{q+\alpha-2} v_x^2(x,\tau) dx d\tau \leq \|v_0\|^q_{L^q(\mathbb{R})} +$$

$$+q(q-1) \int_0^t B(\tau) \left(\int_{\mathbb{R}} v(x,\tau)^{q+\alpha-2} v_x^2(x,\tau)\, dx \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}} v(x,\tau)^{q+2\kappa-\alpha}\, dx \right)^{\frac{1}{2}} d\tau. \tag{5.12}$$

Consideremos

$$\gamma \;:=\; 2\kappa - \alpha < 0, \qquad \textit{com} \qquad \alpha < \frac{\kappa}{2}. \tag{5.13}$$

Observe que se $q = 1-\gamma$, então $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ fica limitado em L^q. Logo, não haverá blow-up em tempo finito. Se $q > 1-\gamma$, temos

$$
\begin{aligned}
\int_{\mathbb{R}} v^{q+\gamma} dx &= \|v(\cdot,t)\|^{q+\gamma}_{L^{q+\gamma}(\mathbb{R})} &\leq& \ \|v(\cdot,t)\|^{(1-\theta)(q+\gamma)}_{L^1(\mathbb{R})} \|v(\cdot,t)\|^{\theta\cdot(q+\gamma)}_{L^q(\mathbb{R})} \\
&& =& \ \|v_0\|^{(1-\theta)(q+\gamma)}_{L^1(\mathbb{R})} \left(\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\right)^{\frac{\theta(q+\gamma)}{q}},
\end{aligned} \tag{5.14}
$$

onde $\theta = \theta(q,\gamma)$. *Note que* $\theta \cdot (q+\gamma) < q$ *e que* $\mathsf{e} := \dfrac{\theta(q,\gamma)(q+\gamma)}{q} < 1$. *Logo, por interpolação*

$$
\begin{aligned}
\|v(\cdot,t)\|^{q+\gamma}_{L^{q+\gamma}(\mathbb{R})} &\leq \|v_0\|^q_{L^q(\mathbb{R})} + \frac{1}{2}q(q-1)\int_0^t \mu(\tau)\int_{\mathbb{R}} v^{q+\alpha-2} v_x^2 \, dx d\tau + \\
&+ \frac{1}{2}q(q-1)\, \|v_0\|^{(1-\theta)(q+\gamma)}_{L^1(\mathbb{R})} \int_0^t \frac{B(\tau)^2}{\mu(\tau)} \left(\|v(\cdot,\tau)\|^q_{L^q(\mathbb{R})}\right)^{\mathsf{e}} d\tau.
\end{aligned} \tag{5.15}
$$

Com isso temos $\|v(\cdot,\tau)\|^q_{L^q(\mathbb{R})}$ *fica limitado em* $[0,T]$. *Ou seja, em tempo finito, não haverá blow-up para todo* $1 \leq q < \infty$. *Além disso, se* $0 \leq \kappa < \dfrac{\alpha}{2}$ *e tivermos algum* $\widehat{p} \geq p$ *tal que* $\|v(\cdot,\tau)\|_{L^{\widehat{p}}(\mathbb{R})}$ *não explode em tempo finito, então por interpolação* $\|v(\cdot,\tau)\|_{L^p(\mathbb{R})}$ *não explode para todo* $\widehat{p} \leq p < \infty$.

Com o Teorema (5.1) obtemos o seguinte Teorema Fundamental que garante a Desigualdade de Energia na Forma Diferencial do problema (5.3).

Teorema 5.2. *Suponhamos que a solução suave positiva* $v(\cdot,t) > 0$ *de* (5.3) *satisfaça* (5.4). *Então, para cada* $q \geq 2(p-\gamma)$ *existe um subonjunto* $E_q \subseteq [0,T_*[$ *com medida nula, tal que*

$$
\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau) v_x^2(x,\tau) dx < \infty, \qquad \forall\, t \in [0,T_*[\setminus E_q \tag{5.16}
$$

e

$$
\begin{aligned}
&\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + \frac{q(q-1)(q+\widetilde{\alpha})}{3q+2\alpha}\mu(\tau)\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau) v_x^2(x,\tau) dx \leq \\
&\leq \frac{4q(q-1)(q+\widetilde{\alpha})}{3q+2\alpha}\mu(\tau) 2^{-\frac{5q+2\alpha+4\gamma}{q+\widetilde{\alpha}}} (3q+2\alpha)^{\frac{q+2\gamma}{q+\widetilde{\alpha}}} \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\widetilde{\alpha}}} \|v(\cdot,t)\|^{\frac{q}{2}\frac{2q+\alpha+\gamma}{q+\widetilde{\alpha}}}_{L^{\frac{q}{2}}(\mathbb{R})},
\end{aligned} \tag{5.17}
$$

onde $\widetilde{\alpha} := \alpha - \gamma$ *e* $\gamma = 2\kappa - \alpha < 0$.

Prova: Consideramos $\psi(x)$ e $\psi_R(x)$ as funções de corte definidas em (2.3) e (2.4), Capítulo 2. Multiplicamos a equação (5.1) por $qv^{q-1}\psi_R(x)$ e integramos em $[0,t]\times\mathbb{R}$. Pelo Teorema de Fubini e Teorema de Integração por Partes, obtemos

$$
\begin{aligned}
&\int_{|x|<2R} v(x,t)^q\psi_R(x)dx+q(q-1)\int_0^t \mu(\tau)\int_{|x|<2R} v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)\psi_R(x)dxd\tau =\\
&\quad = \int_{|x|<2R} v_0(x)^q\psi_R(x)dx - q\int_0^t \mu(\tau)\int_{R<|x|<2R} v(x,\tau)^{q+\alpha-1}v_x(x,\tau)\psi_R'(x)dxd\tau+\\
&\quad +q(q-1)\int_0^t\int_{|x|<2R} v(x,\tau)^{q+\kappa-1}v_x(x,\tau)b(x,\tau)\psi_R(x)dxd\tau+\\
&\quad +q\int_0^t\int_{R<|x|<2R} v(x,\tau)^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau.
\end{aligned}
$$

Consideramos $\beta(t)$ definido em (2.13), Capítulo 2, escrevemos $b(x,t) = \beta(t) + b(x,t) - \beta(t)$ e obtemos

$$
\begin{aligned}
&\int_{|x|<2R} v(x,t)^q\psi_R(x)dx+q(q-1)\int_0^t \mu(\tau)\int_{|x|<2R} v(x,\tau)^{q+\alpha-2}v_x^2(x,\tau)\psi_R(x)dxd\tau =\\
&\quad = \int_{|x|<2R} v_0(x)^q\psi_R(x)dx + \frac{q}{q+\alpha}\int_0^t \mu(\tau)\int_{R<|x|<2R} v(x,\tau)^{q+\alpha}\psi_R''(x)dxd\tau-\\
&\quad -\frac{q(q-1)}{q+\kappa}\int_0^t \beta(\tau)\int_{R<|x|<2R} v(x,\tau)^{q+\kappa}\psi_R'(x)dxd\tau+\\
&\quad +q(q-1)\int_0^t\int_{|x|<2R}(b(x,\tau)-\beta(\tau))v(x,\tau)^{q+\kappa-1}v_x(x,\tau)\psi_R(x)dxd\tau+\\
&\quad +q\int_0^t\int_{R<|x|<2R} v(x,\tau)^{q+\kappa}b(x,\tau)\psi_R'(x)dxd\tau.
\end{aligned}
$$

Fazendo $R\to\infty$, temos

$$
\begin{aligned}
&\|v(\cdot,t)\|_{L^q(\mathbb{R})}^q + q(q-1)\int_0^t \mu(\tau)\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dxd\tau =\\
&\qquad = \|v_0\|_{L^q(\mathbb{R})}^q + q(q-1)\int_0^t\int_{\mathbb{R}}(b(x,\tau)-\beta(\tau))v(x,\tau)^{q+\kappa-1}v_x(x,\tau)dxd\tau.
\end{aligned}
$$

Observe que a última integral está bem definida, pois usando as propriedades de módulo, obtemos

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} |b(x,\tau)-\beta(\tau)||v(x,\tau)|^{q+\kappa-1}|v_x(x,\tau)|dxd\tau \ \le&\\
\le&\ \int_0^t B(\tau)\int_{\mathbb{R}}|v(x,\tau)|^{q+\kappa-1}|v_x(x,\tau)|dxd\tau\\
\le&\ \int_0^t\int_{\mathbb{R}}|v(x,\tau)|^{q+\alpha-2}v_x^2(x,\tau)dxd\tau+\\
&+\int_0^t B^2(\tau)\int_{\mathbb{R}}|v(x,\tau)|^{q+2\kappa-\alpha}dxd\tau.
\end{aligned}
$$

O Teorema (5.1) garante

$$\int_0^t \int_{\mathbb{R}} |v(x,\tau)|^{q+\alpha-2} v_x^2(x,\tau)\, dx\, d\tau \;<\; \infty$$

e, como estamos supondo $q+2\kappa-\alpha = q+\gamma \geq 2p$, temos

$$\int_0^t B^2(\tau) \int_{\mathbb{R}} |v(x,\tau)|^{q+2\kappa-\alpha}\, dx\, d\tau \;<\; \infty.$$

Assim, $\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}$ é uma função absolutamente contínua em $[0,T_*[$ com

$$\begin{aligned} \frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx = \\ = q(q-1)\int_{\mathbb{R}} (b(x,t)-\beta(t))v(x,t)^{q+\kappa-1}v_x(x,t)dx. \end{aligned} \tag{5.18}$$

$\forall\, t \in [0,T_*[\diagdown E_q$, onde $E_q \subset [0,T_*[$ tem medida nula. Segue, em particular, que

$$\int_{\mathbb{R}} v^{q+\alpha-2}(x,\tau)v_x^2(x,\tau)dx < \infty, \qquad \forall\, t \in [0,T_*[\diagdown E_q. \tag{5.19}$$

Pela desigualdade de Young, temos

$$v^{q+\kappa-1}(x,t)v_x(x,t) \leq v^{q+\alpha-2}(x,t)v_x^2(x,t) + \frac{1}{2}v^{q+2\kappa-\alpha}(x,t),$$

com $q+2\kappa-\alpha = q+\gamma \geq 2p$, pois tomamos $q \geq 2p-\gamma$ e $\gamma = 2\kappa-\alpha < 0$. Com isso e por (5.19), segue que

$$\int_{\mathbb{R}} v^{q+\kappa-1}(x,t)|v_x(x,t)|dx < \infty, \qquad \forall\, t \in [0,T_*[\diagdown E_q. \tag{5.20}$$

Usando as propriedades de módulo e a desigualdade de Cauchy-Schwarz em (5.18), segue $\forall\, t \in [0,T_*[\diagdown E_q$

$$\begin{aligned} \frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx \leq \\ \leq\;& q(q-1)B(t)\int_{\mathbb{R}} v(x,t)^{q+\kappa-1}|v_x(x,t)|dx \leq \\ \leq\;& q(q-1)B(t)\left(\int_{\mathbb{R}} v(x,t)^{q+\alpha-2}v_x^2(x,t)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} v(x,t)^{q+\gamma}dx\right)^{\frac{1}{2}}, \end{aligned} \tag{5.21}$$

onde $\gamma = 2\kappa-\alpha \in]-\alpha, 0[$. Considere $\mathsf{w}(x,t) = \mathsf{v}(x,t)^{\frac{q+\alpha}{2}}$ e $\beta = \dfrac{2q}{q+\alpha}$ e $\widetilde{\beta} = \dfrac{2(q+\gamma)}{q+\alpha} < \beta$. Assim, temos

$$
\begin{aligned}
&(i) \quad \int_{\mathbb{R}} v^{q+\alpha-2}(x,t) v_x^2(x,t)dx = \left(\frac{2}{q+\alpha}\right)^2 \|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\\
&(ii) \quad \|v(\cdot,t)\|^q_{L^q(\mathbb{R})} = \|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} \qquad (5.22)\\
&(iii) \quad \|v(\cdot,t)\|^{\frac{q}{2}}_{L^{\frac{q}{2}}(\mathbb{R})} = \|w(\cdot,t)\|^{\beta_0}_{L^{\beta_0}(\mathbb{R})}
\end{aligned}
$$

onde β_0 é definida por

$$
\beta_0 \, = \, \frac{q}{q+\alpha}. \qquad (5.23)
$$

Assim, substituindo (5.22) e (5.23) em (5.21) temos

$$
\begin{aligned}
\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \frac{4q(q-1)}{(q+\alpha)^2}\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \\
\leq \; q(q-1)B(t)\frac{2}{q+\alpha}\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}\|w(\cdot,t)\|^{\frac{\widetilde{\beta}}{2}}_{L^{\widetilde{\beta}}(\mathbb{R})} \qquad (5.24)
\end{aligned}
$$

Pela desigualdade de SNG:

$$
\|\mathsf{w}(\cdot,t)\|_{L^{\widetilde{\beta}}(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}} \|\mathsf{w}(\cdot,t)\|^{1-\widetilde{\theta}}_{L^{\beta_0}(\mathbb{R})}\|\mathsf{w}_x(\cdot,t)\|^{\widetilde{\theta}}_{L^2(\mathbb{R})}, \qquad (5.25)
$$

onde

$$
\widetilde{\theta} \, = \, \frac{1-\frac{\beta_0}{\widetilde{\theta}}}{1+\frac{\beta_0}{2}} \, = \, \frac{1-\frac{1}{2}\frac{q}{q+\gamma}}{1+\frac{1}{2}\frac{q}{q+\alpha}}, \qquad (5.26)
$$

obtemos a seguinte expressão para (5.24)

$$
\begin{aligned}
\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \left(\frac{2q}{q+\alpha}\right)^2\left(1-\frac{1}{q}\right)\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \\
\leq \; \frac{2q}{q+\alpha}q\left(1-\frac{1}{q}\right)B(t)\left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}\frac{\widetilde{\beta}}{2}} \cdot \|w_x(\cdot,t)\|^{1+\widetilde{\theta}\frac{\widetilde{\beta}}{2}}_{L^2(\mathbb{R})} \cdot \|w(\cdot,t)\|^{(1-\widetilde{\theta})\frac{\widetilde{\beta}}{2}}_{L^{\beta_0}(\mathbb{R})}, \\
(5.27)
\end{aligned}
$$

para todo $t \in [0, T_*[\setminus E_q$. Observe que a desigualdade de SNG dada em (5.25) é verdadeira, pois $\widetilde{\theta}$, definida por (5.26), satisfaz $\widetilde{\theta} \in]0,1[$. De fato, tomamos q tal que $q \geq 2p - 2\gamma$, com isso, $\dfrac{\beta_0}{\widetilde{\beta}} = \dfrac{1}{2}\dfrac{q}{q+\gamma} < 1$, portanto vale (5.25) para $\widetilde{\theta}$ definida por (5.26). Note que:

$$
\text{(i) } 2+\beta_0 = \frac{3q+2\alpha}{q+\alpha}; \qquad \widetilde{\theta} = \frac{1-\frac{\beta_0}{\widetilde{\beta}}}{1+\frac{\beta_0}{2}}; \qquad 1-\widetilde{\theta} = \frac{\beta_0}{2+\beta_0}\left(1+\frac{2}{\widetilde{\beta}}\right)
$$

$$
\text{(ii) } \widetilde{\theta}\frac{\widetilde{\beta}}{2} = \frac{\widetilde{\beta}-\beta_0}{2+\beta_0} = \frac{q+2\gamma}{3q+2\alpha};
$$

Usaremos a desigualdade de Young para agrupar os termos semelhantes de (5.2), para isso devemos garantir que $1 + \widetilde{\theta}\dfrac{\widetilde{\beta}}{2} < 2$, o que equivale a $q + \gamma < 2q + \alpha$, mas esta estimativa é satisfeita, pois estamos supondo $q \geq 2p - 2\gamma$. Portanto, $\forall\, t \in [0, T_*[\setminus E_q$, temos

$$\frac{d}{dt}\,\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \left(\frac{2q}{q+\alpha}\right)^{2}\left(1 - \frac{1}{q}\right)\mu(t)\,\|w_x(\cdot,t)\|^{2}_{L^{2}(\mathbb{R})} \leq$$

$$\begin{aligned}
\leq\; & \frac{2q+\alpha+\gamma}{3q+2\alpha}\left(1 - \frac{1}{q}\right)\beta^{2}\mu(t)\,\|w_x(\cdot,t)\|^{2}_{L^{2}(\mathbb{R})} + \\
& + \frac{q+\alpha-\gamma}{3q+2\alpha}\cdot\left(1 - \frac{1}{q}\right)\cdot\beta^{-\frac{q+2\gamma}{q+\alpha-\gamma}}\cdot q^{\frac{3q+2\alpha}{q+\alpha-\gamma}}\cdot\left(\frac{2+\beta_0}{4}\right)^{\frac{q+2\gamma}{q+\alpha-\gamma}}\cdot \\
& \cdot\frac{B(t)^{\frac{3q+2\alpha}{q+\alpha-\gamma}}}{\mu(t)^{\frac{2q+\alpha+\gamma}{q+\alpha-\gamma}}}\cdot\|w(\cdot,t)\|^{\beta_0\frac{2q+\alpha+\gamma}{q+\alpha-\gamma}}_{L^{\beta_0}(\mathbb{R})}.
\end{aligned}$$

Ou seja,

$$\frac{d}{dt}\,\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \frac{q+\alpha-\gamma}{3q+2\alpha}\left(1 - \frac{1}{q}\right)\beta^{2}\mu(t)\,\|w_x(\cdot,t)\|^{2}_{L^{2}(\mathbb{R})} \leq$$

$$\begin{aligned}
\leq\; & \frac{q+\alpha-\gamma}{3q+2\alpha}\left(1 - \frac{1}{q}\right)\beta^{2}\mu(t)\left(\frac{q}{\beta}\right)^{\frac{3q+2\alpha}{q+\alpha-\gamma}}\left(\frac{2+\beta_0}{4}\right)^{\frac{q+2\gamma}{q+\alpha-\gamma}}\left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\alpha-\gamma}}\cdot \\
& \cdot\|w(\cdot,t)\|^{\beta_0\frac{2q+\alpha+\gamma}{q+\alpha-\gamma}}_{L^{\beta_0}(\mathbb{R})}.
\end{aligned}$$

Reescrevendo esta desigualdade em função de $v(x,t)$, com β e β_0 definimos acima, concluimos a prova do Teorema (5.2).

$\blacksquare$

O Teorema (5.2) escrito em termos de $w(\cdot,t)$, produz a seguinte Estimativa de Energia:

Teorema 5.3. *Suponhamos que a solução suave positiva $v(\cdot,t)$ de (5.3) satisfaça (5.4). Considere $w(x,t) = v(x,t)^{\frac{q+\alpha}{2}}$ e $\gamma = 2\kappa - \alpha$. Então, para cada $q \geq 2p - 2\gamma$, existe $E_q \subseteq [0, T_*[$ com medida nula, tal que*

$$\frac{d}{dt}\,\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \frac{4q(q-1)(q+\widetilde{\alpha})}{(3q+2\alpha)(q+\alpha)^{2}}\mu(t)\,\|w_x(\cdot,t)\|^{2}_{L^{2}(\mathbb{R})} \leq$$

$$\leq \frac{4q(q-1)(q+\widetilde{\alpha})}{3q+2\alpha} \cdot \mu(\tau) \cdot 2^{-\frac{5q+2\alpha+4\gamma}{q+\widetilde{\alpha}}} \cdot (3q+2\alpha)^{\frac{q+2\gamma}{q+\widetilde{\alpha}}} \cdot \left(\frac{B(t)}{\mu(t)}\right)^{\frac{3q+2\alpha}{q+\widetilde{\alpha}}} \cdot$$
$$\cdot \|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0 \frac{2q+\alpha+\gamma}{q+\widetilde{\alpha}}}, \tag{5.28}$$

onde $\widetilde{\alpha} := \alpha - \gamma$, $\beta = \dfrac{2q}{q+\alpha}$ *e* $\beta_0 = \dfrac{q}{q+\alpha}$.

Seguem algumas simulações numéricas do problema de valor inicial (2.39), (2.40). Os gráficos correspondem a norma do sup das soluções $u(\cdot,t)$, para $\alpha = 4$ e $\kappa = 1$ (caso $0 \leq \kappa < \dfrac{\alpha}{2}$).

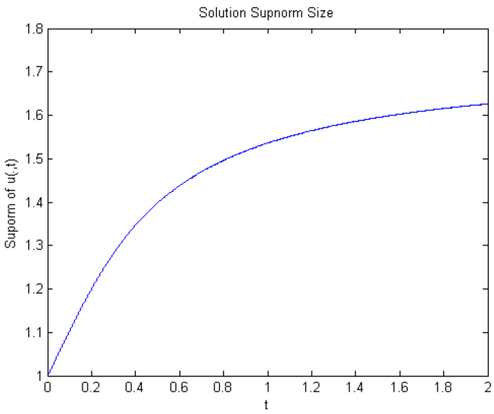

Figura 46: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\sin x$

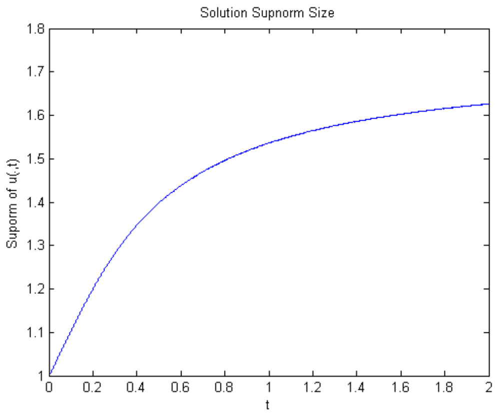

Figura 47: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: $\alpha = 4$, $\kappa = 1$ $b(x,t) = -\cos x$

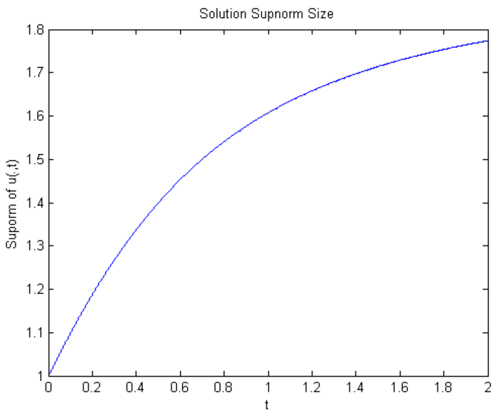

Figura 48: $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: α=4, $\kappa = 1$ $b(x,t) = -\tanh x$

5.3 Estimativas para $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$: soluções não-negativas

Repetindo a análide feita no Teorema (4.8), do Capítulo 4, para o caso $\kappa > \dfrac{\alpha}{2}$, temos os seguintes resultados:

Teorema 5.4. *Suponha que $u(\cdot,t) \in L^\infty_{loc}([0,T_*[,L^\infty(\mathbb{R})) \cap L^\infty_{loc}([0,T_*[,L^p(\mathbb{R}))$ para algum $p \geq p_0$ (finito). Considere $\gamma = 2\kappa - \alpha$, $0 \leq \kappa < \dfrac{\alpha}{2}$. Então, para cada $0 \leq t_0 \leq t < T_*$, temos*

$$\mathbb{U}_\infty(t_0;t) \leq \widetilde{K}(\widetilde{p};\alpha,\kappa) \max\left\{\|u(\cdot,t_0)\|\right\}_{L^\infty(\mathbb{R})};\mathbb{B}_\mu(t_0;t)^{\frac{1}{\widetilde{p}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{U}_{\widetilde{p}}(t_0;t)^{\frac{\widetilde{p}}{\widetilde{p}+\frac{\widetilde{\alpha}}{2}}}, \quad (5.29)$$

onde $\widetilde{p} = p - \gamma$, $\widetilde{\alpha} = \alpha - \gamma = 2(\alpha-\kappa)$ e $\widetilde{K}(\widetilde{p};\alpha,\kappa) > 1$ é dado por

$$\widetilde{K}(\widetilde{p};\alpha,\kappa) = \max\left\{1, \left(\frac{4}{3\widetilde{p}+\alpha}\right)^{\dfrac{2p}{(2p+\widetilde{\alpha})(2p+\frac{\widetilde{\alpha}}{2})}}\right\}.$$

$$\cdot \prod_{j=1}^{\infty}\left(\frac{3\cdot 2^j\widetilde{p}+2\alpha}{8}\right)^{\dfrac{2^j\widetilde{p}}{(2^j\widetilde{p}+\widetilde{\alpha})(2^j\widetilde{p}+\frac{\widetilde{\alpha}}{2})}}. \quad (5.30)$$

Em particular, para cada $\alpha > 0$, $\kappa \in \left[0, \dfrac{\alpha}{2}\right[$ dados, temos

$$\widetilde{K}(\widetilde{p};\alpha,\kappa) \longrightarrow 1, \qquad \text{ao} \qquad \widetilde{p} \to +\infty. \quad (5.31)$$

Teorema 5.5. *Seja $p \geq p_0$ tal que $\|u(\cdot,t)\|_{L^{\widetilde{p}}(\mathbb{R})}$ não exploda em tempo finito, onde $\widetilde{p} := p - \gamma$. Então, temos $T_* = \infty$ e, supondo $B_\mu < \infty$, $\mathsf{U}_{\widetilde{p}} < \infty$ e $\displaystyle\int_0^\infty \mu(t)dt = \infty$, temos*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathbb{K}_1(\widetilde{p};\alpha,\kappa)\cdot B_\mu^{\frac{1}{\widetilde{p}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_{\widetilde{p}}^{\frac{\widetilde{p}}{\widetilde{p}+\frac{\widetilde{\alpha}}{2}}}, \quad (5.32)$$

onde $\widetilde{\alpha} = \alpha - \gamma = 2(\alpha-\kappa)$ e $\mathbb{K}_1(\widetilde{p};\alpha,\kappa) > 0$ é dado por

$$\mathbb{K}_1(\widetilde{p};\alpha,\kappa) = \left(\frac{3\widetilde{p}+\alpha}{4}\right)^{\dfrac{1}{\widetilde{p}+\frac{\widetilde{\alpha}}{2}}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j\widetilde{p}+2\alpha}{3\cdot 2^j\widetilde{p}+2\alpha}\right)^{\dfrac{1}{2^j\widetilde{p}+\frac{\widetilde{\alpha}}{2}}}. \quad (5.33)$$

Nesta seção, obteremos versões para os Teoremas (5.4) e (5.5) acima, válidas mais geralmente para qualquer $p \geq p_0$. Ou seja, estimaremos $\mathsf{U}_\infty(t_0;t)$ em termos de $\|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})}$ e $\mathbb{U}_p(t_0;t)$ e estimaremos U_∞ em termos de U_p. Observe que obtemos uma estimativa de U_∞ em termos de $\mathsf{U}_{\widetilde{p}}$, então, para atingir nosso objetivo devemos obter um resultado equivalente ao Teorema (5.3) só que para $q \geq 2p$ qualquer, ao invés de $q \geq 2\widetilde{p} = 2(p-\gamma)$. Segue abaixo a segunda Estimativa Fundamental de Energia.

Teorema 5.6. *Suponhamos que $v(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R})) \cap L^\infty_{loc}([0,T_*[, L^p(\mathbb{R}))$, para algum $p \geq p_0$ finito. Sejam $\gamma = 2\kappa - \alpha$, $\widetilde{\alpha} = \alpha - \gamma$ e $\sigma = 2 - \gamma$. Então, para cada $q \geq \sigma p$, existe $E_q \subseteq [0,T_*[$ com medida nula, tal que*

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + \frac{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}{q+\frac{q}{\sigma}+\alpha} q(q-1)\mu(t)\int_{\mathbb{R}} v(x,t)^{q+\alpha-2}v_x^2(x,t)dx \leq$$

$$\leq \frac{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}{q+\frac{q}{\sigma}+\alpha} q(q-1)\mu(t)\left(\frac{q+\frac{q}{\sigma}+\alpha}{4}\right)^{\frac{q-\frac{q}{\sigma}+\gamma}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\frac{q}{\sigma}+\alpha}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\|v(\cdot,t)\|^{\frac{q}{\sigma}\frac{q+\gamma+\frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}_{L^{\frac{q}{\sigma}}(\mathbb{R})}, \tag{5.34}$$

para todo $t \in [0,T_*[\setminus E_q$.

Prova: Observe que as expressões (5.18), (5.20) e (5.21) são válidas para $q \geq 2$ e $q \geq p-\gamma$, onde $\gamma = 2\kappa - \alpha \in [-\alpha, 0[$. Assim, para este tal q, segue de (5.21)

$$\frac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} + q(q-1)\mu(t)\int_{\mathbb{R}} v^{q+\alpha-2}(x,t)v_x^2(x,t)dx \leq$$

$$\leq \ q(q-1)B(t)\left(\int_{\mathbb{R}} v(x,t)^{q+\alpha-2}v_x^2(x,t)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} v(x,t)^{q+\gamma}dx\right)^{\frac{1}{2}}, \tag{5.35}$$

Consideramos $w(x,t) := v(x,t)^{\frac{q+\alpha}{2}}$ e $\beta := \dfrac{2q}{q+\alpha}$. Para $\sigma \geq 2$, que será escolhido posteriormente, definimos

$$\beta_0 = \frac{\frac{2q}{\sigma}}{q+\alpha} = \frac{\beta}{\sigma}. \tag{5.36}$$

Observe que esta escolha de β_0 tem, como caso particular, a escolha de $\beta_0 = \dfrac{\beta}{2}$, o qual era nossa escolha anteriormente no caso de $\sigma = 2$. Assim, para $\sigma \geq 2$ fixo, temos

$$\|w(\cdot,t)\|^{\beta_0}_{L^{\beta_0}(\mathbb{R})} = \|v(\cdot,t)\|^{\frac{q}{\sigma}}_{L^{\frac{q}{\sigma}}(\mathbb{R})}. \tag{5.37}$$

Com isso, podemos reescrever (5.35) como

$$\frac{d}{dt}\|w(\cdot,t)\|^\beta_{L^\beta(\mathbb{R})} + \left(1-\frac{1}{q}\right)\beta^2\mu(t)\|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq$$

$$\leq \ q\left(1-\frac{1}{q}\right)\beta B(t)\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}\|w(\cdot,t)\|^{\frac{\widetilde{\beta}}{2}}_{L^{\widetilde{\beta}}(\mathbb{R})}, \tag{5.38}$$

$\forall\, t \in [0, T_*[\setminus E_q$, onde $\widetilde{\beta} = \dfrac{2(q+\gamma)}{q+\alpha}$ e $\gamma = 2\kappa - \alpha < 0$. Note que $\widetilde{\beta} > 0$, pois $q + \gamma \geq p$. Logo, $\widetilde{\beta} \in]0, \beta[$.

Escolheremos $\sigma \geq 2$ tal que $\beta_0 \leq \widetilde{\beta}$ e $\sigma p \geq p - \gamma$. A primeira condição nos permite estimar $\|w(\cdot,t)\|_{L^{\widetilde{\beta}}(\mathbb{R})}$ em função de $\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}$ e de $\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}$ e a segunda condição de tomar $q = \sigma p$. Da primeira condição obtemos

$$\left(1 - \frac{1}{\sigma}\right) q \geq |\gamma|, \qquad \forall\, q \geq 2. \tag{5.39}$$

Por outro lado, queremos $q \geq \sigma p$ qualquer. Então, se considerarmos $\sigma \geq 2$ satisfazendo $(1 - \dfrac{1}{\sigma})\sigma p \geq |\gamma|$, teremos

$$\sigma \geq 1 + \frac{|\gamma|}{p}.$$

Portanto, tomaremos σ dado por qualquer uma das expressões, pois qualquer uma das escolhas satisfaz as condições acima

$$\begin{array}{ll} \sigma = \max\left(2, 1 + \dfrac{|\gamma|}{p}\right) & \qquad \sigma = \max\left(2, 1 + |\gamma|\right) \\ \sigma = 2 + \dfrac{|\gamma|}{p} & \qquad \sigma = 2 + |\gamma| \end{array} \tag{5.40}$$

Por simplicidade, assumiremos σ dado por

$$\sigma := 2 + |\gamma|. \tag{5.41}$$

Logo, $q \geq \sigma p$. Usaremos a desigualdade de SNG:

$$\|\mathsf{w}\|_{L^{\widetilde{\beta}}(\mathbb{R})} \leq \left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}} (\|\mathsf{w}\|)^{1-\widetilde{\theta}}_{L^{\beta_0}(\mathbb{R})} \cdot (\|\mathsf{w_x}\|)^{\widetilde{\theta}}_{L^2(\mathbb{R})}, \tag{5.42}$$

onde

$$\widetilde{\theta} = \frac{1 - \frac{\beta_0}{\widetilde{\beta}}}{1 + \frac{\beta_0}{2}} = \frac{1 - \frac{1}{\sigma} \cdot \frac{q}{q+\gamma}}{1 + \frac{1}{\sigma} \cdot \frac{q}{q+\alpha}} \in]0, 1[, \tag{5.43}$$

em (5.38). Assim,

$$\begin{aligned} &\frac{d}{dt}\|w(\cdot,t)\|^{\beta}_{L^{\beta}(\mathbb{R})} + \left(1 - \frac{1}{q}\right)\beta^2 \mu(t)\, \|w_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \\ &\leq\ q\left(1 - \frac{1}{q}\right)\beta \left(\frac{2+\beta_0}{4}\right)^{\widetilde{\theta}\frac{\widetilde{\beta}}{2}} B(t) \cdot \|w_x(\cdot,t)\|^{1+\widetilde{\theta}\frac{\widetilde{\beta}}{2}}_{L^2(\mathbb{R})} \cdot \|w(\cdot,t)\|^{\frac{(1-\widetilde{\theta})\widetilde{\beta}}{2}}_{L^{\beta_0}(\mathbb{R})}, \end{aligned} \tag{5.44}$$

$\forall\, t \in [0, T_*[\setminus E_q$.

Note que (5.42) vale para todo $\widetilde{\beta} \geq \beta_0 > 0$. Além disso, note que

$$\widetilde{\theta} \cdot \frac{\widetilde{\beta}}{2} = \frac{1 - \frac{\beta_0}{\widetilde{\beta}}}{1 + \frac{\beta_0}{2}} \frac{\widetilde{\beta}}{2} = \frac{\widetilde{\beta} - \beta_0}{2 + \beta_0} \in]0, 1[, \tag{5.45}$$

pois $\widetilde{\beta} - \beta_0 < 2 + \beta_0$ e, por outro lado, $\widetilde{\beta} > \beta$, $\forall\, q \geq \sigma p$, com σ dado por (5.41). Em particular, temos $1 + \widetilde{\theta}\dfrac{\widetilde{\beta}}{2} < 2$, logo a estimativa (5.44) é de fato útil. Observe que:

$$\begin{aligned}
&(i) \qquad 1 + \widetilde{\theta}\frac{\widetilde{\beta}}{2} = \frac{2 + \widetilde{\beta}}{2 + \beta_0} = \frac{2q + \alpha + \gamma}{\left(1 + \frac{1}{\sigma}\right) q + \alpha} \in]1, 2[\\
&(ii) \qquad \frac{2 + \beta_0}{4} = \frac{q + \frac{q}{\sigma} + \alpha}{2(q + \alpha)} \\
&(iii) \qquad \frac{(1 - \widetilde{\theta})\widetilde{\beta}}{2} = \frac{(2q + \alpha + \gamma) \cdot q/\sigma}{(q + \frac{q}{\sigma} + \alpha) \cdot (q + \alpha)} = \frac{q + \gamma + \frac{\widetilde{\alpha}}{2}}{q + \frac{q}{\sigma} + \alpha} \cdot \frac{2q/\sigma}{q + \alpha}
\end{aligned} \tag{5.46}$$

$$(iv) \quad \widetilde{\theta}\frac{\widetilde{\beta}}{2} = \frac{\frac{\widetilde{\beta}}{2} - \frac{\beta_0}{2}}{1 + \frac{\beta_0}{2}} = \frac{q - \frac{q}{\sigma} + \gamma}{q + \frac{q}{\sigma} + \alpha} \in]0, 1[.$$

De (5.44) e de (5.3), segue que

$$\begin{aligned}
&\frac{d}{dt}\, \|w(\cdot, t)\|_{L^\beta(\mathbb{R})}^\beta + \left(1 - \frac{1}{q}\right) \beta^2 \mu(t)\, \|w_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \leq \\
&\leq \quad q \left(1 - \frac{1}{q}\right) \beta B(t) \frac{q + \frac{q}{\sigma} + \alpha}{2(q + \alpha)}^{\frac{q - \frac{q}{\sigma} + \gamma}{q + \frac{q}{\sigma} + \alpha)}}\, \|w_x(\cdot, t)\|_{L^2(\mathbb{R})}^{\frac{2q + \alpha + \gamma}{q + \frac{q}{\sigma} + \alpha}}\, \|w(\cdot, t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0 \frac{q + \gamma + \frac{\widetilde{\alpha}}{2}}{q + \frac{q}{\sigma} + \alpha}},
\end{aligned} \tag{5.47}$$

$\forall\, t \in [0, T_*[\setminus E_q$. Pela desigualdade de Young, temos

$$\begin{aligned}
&\frac{d}{dt}\, \|w(\cdot, t)\|_{L^\beta(\mathbb{R})}^\beta + \left(1 - \frac{1}{q}\right) \beta^2 \mu(t)\, \|w_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \leq \\
&\leq \quad \frac{q + \gamma + \frac{\widetilde{\alpha}}{2}}{q + \frac{q}{\sigma} + \alpha} \left(1 - \frac{1}{q}\right) \beta^2 \mu(t)\, \|w_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 + \\
&\quad + \frac{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}{q + \frac{q}{\sigma} + \alpha} \cdot \left(1 - \frac{1}{q}\right) \cdot \beta^{-\frac{q - \frac{q}{\sigma} + \gamma}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot q^{\frac{q + \frac{q}{\sigma} + \alpha}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot \left(\frac{q + \frac{q}{\sigma} + \alpha}{2(q + \alpha)}\right)^{\frac{q - \frac{q}{\sigma} + \gamma}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot \\
&\qquad\qquad \cdot \mu(t)^{-\frac{q + \gamma + \frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot B(t)^{\frac{q + \frac{q}{\sigma} + \alpha}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot \|w(\cdot, t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0 \cdot \frac{q + \gamma + \frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}}.
\end{aligned} \tag{5.48}$$

Assim,

$$\frac{d}{dt}\, \|w(\cdot, t)\|_{L^\beta(\mathbb{R})}^\beta + \frac{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}{q + \frac{q}{\sigma} + \alpha} \left(1 - \frac{1}{q}\right) \beta^2 \mu(t)\, \|w_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \leq$$

$$\leq \quad \frac{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}{q+\frac{q}{\sigma}+\alpha}\left(1-\frac{1}{q}\right)\beta^2\mu(t)\left(\frac{q}{\beta}\right)^{\frac{q+\frac{q}{\sigma}+\alpha}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\left(\frac{q+\frac{q}{\sigma}+\alpha}{2(q+\alpha)}\right)^{\frac{q-\frac{q}{\sigma}+\gamma}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\cdot$$

$$\cdot\left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\frac{q}{\sigma}+\alpha}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\cdot\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0\cdot\frac{q+\gamma+\frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}, \tag{5.49}$$

$\forall\, t\in[0,T_*[\setminus E_q$. Reescrevendo (5.49) em termos de $v(x,t)$, temos $\forall\, t\in[0,T_*[\setminus E_q$:

$$\frac{d}{dt}\|v(\cdot,t)\|_{L^q(\mathbb{R})}^q+\frac{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}{q+\frac{q}{\sigma}+\alpha}q(q-1)\mu(t)\int_{\mathbb{R}}v(x,t)^{q+\alpha-2}v_x^2(x,t)dx\leq$$

$$\leq \quad \frac{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}{q+\frac{q}{\sigma}+\alpha}\cdot q(q-1)\cdot\mu(t)\cdot 2^{-2\frac{q-\frac{q}{\sigma}+\gamma}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\cdot\left(q+\frac{q}{\sigma}+\alpha\right)^{\frac{q-\frac{q}{\sigma}+\gamma}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\cdot$$

$$\cdot\left(\frac{B(t)}{\mu(t)}\right)^{\frac{q+\frac{q}{\sigma}+\alpha}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\cdot\|v(\cdot,t)\|_{L^{\frac{q}{\sigma}}(\mathbb{R})}^{\frac{q}{\sigma}\cdot\frac{q+\gamma+\frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}, \tag{5.50}$$

onde $\widetilde{\alpha}=\alpha-\gamma$, $\gamma=2\kappa-\alpha<0$ e $\sigma=2-\gamma$.

■

Antes de enunciarmos os próximos resultados, reescreveremos (5.49) de uma forma mais conveniente. Para $\sigma=2+|\gamma|$, $\widetilde{\alpha}=\alpha-\gamma$ e $\gamma=2\kappa-\alpha$, temos $\forall\, p\in[p_0,\infty[$ e para todo $q\geq\sigma p$

$$\frac{d}{dt}\|w(\cdot,t)\|_{L^\beta(\mathbb{R})}^\beta+\frac{4q(q-1)\left(\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}\right)}{\left(q+\frac{q}{\sigma}+\alpha\right)(q+\alpha)^2}\cdot\mu(t)\cdot\|w_x(\cdot,t)\|_{L^2(\mathbb{R})}^2\leq$$

$$\leq \quad \frac{4q(q-1)\left(\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}\right)}{\left(q+\frac{q}{\sigma}+\alpha\right)(q+\alpha)^2}\cdot\mu(t)\cdot\left(q+\frac{q}{\sigma}+\alpha\right)^{\dfrac{q-\frac{q}{\sigma}+\gamma}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}2^{-2\dfrac{q+\kappa}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}(q+\alpha)^2\cdot$$

$$\cdot\left(\frac{B(t)}{\mu(t)}\right)^{\dfrac{q+\frac{q}{\sigma}+\alpha}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}\cdot\|w(\cdot,t)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0\cdot\dfrac{q+\kappa}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}}, \tag{5.51}$$

$\forall\, t\in[0,T_*[\setminus E_q$, com $\beta=\dfrac{2q}{q+\alpha}$ e $\beta_0=\dfrac{\beta}{\sigma}=\dfrac{2\frac{q}{\sigma}}{q+\alpha}$.

No próximo resultado, utilizaremos a desigualdade SNG, válida para todo $\beta\geq\beta_0>0$

$$\|\mathsf{w}\|_{L^\beta(\mathbb{R})}\leq z,\left(\frac{2+\beta_0}{4}\right)^\theta\cdot\|\mathsf{w}\|_{L^{\beta_0}(\mathbb{R})}^{1-\theta}\cdot\|\mathsf{w}\|_{L^2(\mathbb{R})}^\theta, \tag{5.52}$$

$$\theta = \frac{1 - \frac{\beta_0}{\beta}}{1 + \frac{\beta_0}{2}} = \frac{\left(1 - \frac{1}{\sigma}\right)(q + \alpha)}{q + \frac{q}{\sigma} + \alpha} \tag{5.53}$$

Observação 5.3. Note que

(i) $\dfrac{2 + \beta_0}{4} = \dfrac{1}{2}\left(1 + \dfrac{\beta_0}{2}\right) = \dfrac{q + \frac{q}{\sigma} + \alpha}{2q + 2\alpha} < 1$

(ii) $1 - \theta = \dfrac{1}{\sigma} \cdot \dfrac{2q + \alpha}{q + \frac{q}{\sigma} + \alpha}$

Teorema 5.7. *Seja $p_0 \leq p < \infty$. Suponha que a solução $v(x,t)$ de (5.3) satisfaça $v(x,t) \in L^\infty_{loc}([0, T_*[, L^\infty(\mathbb{R})) \cap L^\infty_{loc}([0, T_*[, L^p(\mathbb{R}))$. Sejam $\gamma = 2\kappa - \alpha$, $\widetilde{\alpha} = \alpha - \gamma$ e $\sigma = 2 - \gamma$, e seja $q \geq \sigma p$. Então, se $t_* \in [0, T_*[\setminus E_q$ é tal que*

$$\frac{d}{dt} \|v(\cdot, t)\|^q_{L^q(\mathbb{R})}\Bigg|_{t=t_*} \geq 0, \tag{5.54}$$

então

$$\|v(\cdot, t_*)\|_{L^q(\mathbb{R})} \leq \left(\frac{q + \frac{q}{\sigma} + \alpha}{4}\right)^{\left(1 - \frac{1}{\sigma}\right)\frac{1}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\left(1 - \frac{1}{\sigma}\right)\frac{1}{\frac{q}{\sigma} + \frac{\widetilde{\alpha}}{2}}} \cdot \|v(\cdot, t_*)\|^{\frac{q + \frac{\widetilde{\alpha}}{2}}{q + \sigma\frac{\widetilde{\alpha}}{2}}}_{L^{\frac{q}{\sigma}}(\mathbb{R})} \tag{5.55}$$

Reescrevendo o Teorema (5.7) em termos de $w(\cdot, t) = v(\cdot, t)^{\frac{q+\alpha}{2}}$, obtemos o seguinte Teorema.

Teorema 5.8. *Nas hipóteses do Teorema (5.7), se $t_* \in [0, T_*[\setminus E_q$ é tal que*

$$\frac{d}{dt} \|w(\cdot, t)\|^\beta_{L^\beta(\mathbb{R})}\Bigg|_{t=t_*} \geq 0, \tag{5.56}$$

então

$$\|w(\cdot, t_*)\|_{L^\beta(\mathbb{R})} \leq \left(\frac{q + \frac{q}{\sigma} + \alpha}{4}\right)^{\left(1 - \frac{1}{\sigma}\right)\frac{q+\alpha}{2\frac{q}{\sigma} + \widetilde{\alpha}}} \cdot \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\left(1 - \frac{1}{\sigma}\right)\frac{q+\alpha}{2\frac{q}{\sigma} + \widetilde{\alpha}}} \cdot \|w(\cdot, t_*)\|^{\frac{q + \frac{\widetilde{\alpha}}{2}}{q + \sigma\frac{\widetilde{\alpha}}{2}}}_{L^{\beta_0}(\mathbb{R})}, \tag{5.57}$$

onde $\beta = \dfrac{2q}{q + \alpha}$ e $\beta_0 = \dfrac{2\frac{q}{\sigma}}{q + \alpha}$, com $q \geq \sigma p$.

Prova: No Teorema (5.6), observe (5.51). Para $t = t_*$ e por (5.56) segue que

$$\begin{aligned} \|w_x(\cdot,t_*)\|_{L^2(\mathbb{R})} \leq & \left(q+\frac{q}{\sigma}+\alpha\right)^{\dfrac{q-\frac{q}{\sigma}+\gamma}{\frac{2q}{\sigma}+\widetilde{\alpha}}} \cdot 2^{-\dfrac{q+\kappa}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot (q+\alpha)\cdot \\ & \cdot \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\dfrac{q+\frac{q}{\sigma}+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}}} \cdot \|w(\cdot,t_*)\|_{L^{\beta_0}(\mathbb{R})}^{\beta_0\cdot\dfrac{q+\kappa}{2\frac{q}{\sigma}+\widetilde{\alpha}}}, \end{aligned} \tag{5.58}$$

Aplicando (5.58) na desigualdade de SNG (5.52) e (5.53), para $t = t_*$, temos

$$\|w(\cdot,t_*)\|_{L^{\beta}(\mathbb{R})} \leq K_0 \left(\frac{B(t_*)}{\mu(t_*)}\right)^{\theta\dfrac{q+\frac{q}{\sigma}+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}}} \cdot \|w(\cdot,t_*)\|_{L^{\beta_0}(\mathbb{R})}^{\delta}, \tag{5.59}$$

onde

$$\begin{aligned} K_0 &= \left(\frac{2+\beta_0}{4}\right)^{\theta} \cdot 2^{-\frac{q+\kappa}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot (q+\alpha)^{\theta} \cdot \left(q+\frac{q}{\sigma}+\alpha\right)^{\theta\frac{q-\frac{q}{\sigma}+\gamma}{2\frac{q}{\sigma}+\widetilde{\alpha}}} \\ &= 2^{-\theta\left(1+\frac{q+\alpha-\frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}\right)} \cdot \left(q+\frac{q}{\sigma}+\alpha\right)^{\theta\left(1+\frac{q-\frac{q}{\sigma}+\gamma}{2\frac{q}{\sigma}+\widetilde{\alpha}}\right)} \\ &= 2^{-\left(1-\frac{1}{\sigma}\right)\left(\frac{q+\alpha}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}\right)} \cdot \left(q+\frac{q}{\sigma}+\alpha\right)^{\left(1-\frac{1}{\sigma}\right)\left(\frac{q+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}}\right)} \\ &= 2^{-2\left(1-\frac{1}{\sigma}\right)\left(\frac{q+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}}\right)} \cdot \left(q+\frac{q}{\sigma}+\alpha\right)^{\left(1-\frac{1}{\sigma}\right)\left(\frac{q+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}}\right)}, \end{aligned} \tag{5.60}$$

$$\theta\cdot\frac{q+\frac{q}{\sigma}+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}} = \left(1-\frac{1}{\sigma}\right)\cdot\frac{q+\alpha}{q+\frac{q}{\sigma}+\alpha}\cdot\frac{q+\frac{q}{\sigma}+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}} = \left(1-\frac{1}{\sigma}\right)\cdot\frac{q+\alpha}{2\frac{q}{\sigma}+\widetilde{\alpha}} \tag{5.61}$$

e

$$\begin{aligned} \delta &= 1-\theta+\theta\frac{\beta_0}{2}\frac{q+\kappa}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}} \\ &= \frac{1}{\sigma}\frac{2q+\alpha}{q+\frac{q}{\sigma}+\alpha} + \left(1-\frac{1}{\sigma}\right)\cdot\frac{q+\alpha}{q+\frac{q}{\sigma}+\alpha}\cdot\frac{q}{\sigma}\cdot\frac{1}{q+\alpha}\cdot\frac{q+\alpha-\frac{\widetilde{\alpha}}{2}}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}} \\ &= \frac{1}{\sigma}\cdot\frac{1}{q+\frac{q}{\sigma}+\alpha}\cdot\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}\cdot\left[\frac{\widetilde{\alpha}}{2}\left(q+\frac{q}{\sigma}+\alpha\right)+q\left(q+\frac{q}{\sigma}+\alpha\right)\right] \\ &= \frac{1}{\sigma}\cdot\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}\cdot\left(q+\frac{\widetilde{\alpha}}{2}\right) \\ &= \frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}} \end{aligned} \tag{5.62}$$

Para obter (5.55), basta observar que $\|w(\cdot,t_*)\|^{\beta}_{L^{\beta}(\mathbb{R})} = \|v(\cdot,t_*)\|^{q}_{L^{q}(\mathbb{R})}$ e que $\|w(\cdot,t_*)\|^{\beta_0}_{L^{\beta_0}(\mathbb{R})} = \|v(\cdot,t_*)\|^{q}_{L(\mathbb{R})}$ ■

Observe que do Teorema (5.7), obtemos o seguinte resultado: Se $u(\cdot,t) \geq 0$ for uma solução de (5.1), com $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*[,L^{\infty}(\mathbb{R}))$ e $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*[,L^{p}(\mathbb{R}))$, para algum $p \in [p_0,\infty[$, então

$$\mathbb{U}_q(t_0;t) \leq \max\left\{ \|u(\cdot,t_0)\|_{L^q(\mathbb{R})}\,;\, \left(\frac{q+\frac{q}{\sigma}+\alpha}{4}\right)^{\left(1-\frac{1}{\sigma}\right)\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{B}_{\mu}(t_0;t)^{\left(1-\frac{1}{\sigma}\right)\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{U}_{\frac{q}{\sigma}}(t_0;t)^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}}} \right\}, \tag{5.63}$$

para todo $0 \leq t_0 \leq t < T_*$ dados e para cada $q \geq \sigma p$, onde $\sigma = 2-\gamma$ e $\gamma = 2\kappa - \alpha$. Como consequência de (5.63) temos que, para algum $p \in [p_0,\infty[$, $\|u(\cdot,t)\|_{L^p(\mathbb{R})} < \infty$ em tempo finito, então, se tomarmos sucessivamente $q = \sigma p$, $q = \sigma^2 p$, $q = \sigma^3 p$, ... teremos que todas as normas $\|u(\cdot,t)\|_{L^{\sigma^m p}(\mathbb{R})}$ ficarão limitadas em tempo finito, onde $m = 1,2,3,\ldots$. Logo, se usarmos a desigualdade de interpolação entre $\|u(\cdot,t)\|_{L^p(\mathbb{R})}$ e $\|u(\cdot,t)\|_{L^{\sigma^m p}(\mathbb{R})}$, para $m \geq 1$ suficientemente grande tal que $\sigma^m p > p-\gamma$, teremos $\|u(\cdot,t)\|_{L^{p-\gamma}(\mathbb{R})}$ limitado em tempo finito. Com isso e com o Teorema (5.4) teremos $T_* = \infty$ e

$$u(\cdot,t) \in L^{\infty}_{loc}([0,\infty[,L^{\infty}(\mathbb{R})) \cap L^{\infty}_{loc}([0,\infty[,L^{\infty}(\mathbb{R})).$$

Para provar (5.63), consideraremos inicialmente soluções $v(x,t)$ suaves positivas.

Teorema 5.9. *Seja $v(\cdot,t) \in L^{\infty}_{loc}([0,T_*[,L^{\infty}(\mathbb{R}) \cap L^{\infty}_{loc}([0,T_*[,L^{p}(\mathbb{R})))$, $v(\cdot,t)$ solução suave positiva do problema* (5.3)*. Então, para cada $q \geq \sigma p$, temos*

$$\mathbb{V}_q(t_0;t) \leq \max\left\{ \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}\,;\, \left(\frac{q+\frac{q}{\sigma}+\alpha}{4}\right)^{(1-\frac{1}{\sigma})\cdot\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{B}_{\mu}(t_0;t)^{(1-\frac{1}{\sigma})\cdot\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{V}_{\frac{q}{\sigma}}(t_0;t)^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}}} \right\}, \tag{5.64}$$

para todos $0 \leq t_0 \leq t < T_*$, onde $\sigma = 2-\gamma$, $\gamma = 2\kappa-\alpha$, $0 \leq \kappa < \dfrac{\alpha}{2}$ e $\widetilde{\alpha} = \alpha - \gamma$.

Prova: A prova deste Teorema se dá em três casos. Considere

$$\lambda_q := \left(\frac{q+\frac{q}{\sigma}+\alpha}{4}\right)^{(1-\frac{1}{\sigma})\cdot\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{B}_{\mu}(t_0;t)^{(1-\frac{1}{\sigma})\cdot\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{V}_{\frac{q}{\sigma}}(t_0;t)^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}}}.$$

Caso I: Suponha que $\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} \leq \lambda_q$, então $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \lambda_q$, $\forall\, \tau \in [t_0,t]$. De fato, suponhamos por contradição que exista $t_2 \in]t_0,t]$ tal que

$$\|v(\cdot,t_2)\|_{L^q(\mathbb{R})} > \lambda_q. \tag{5.65}$$

Tome $t_1 \in [t_0,t_2[$ tal que $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} = \lambda_q$ e

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda_q, \qquad \forall\, \tau \in]t_1,t_2]. \tag{5.66}$$

Nestas condições existe $t_* \in]t_1,t_2] \setminus E_q$ tal que $\left.\dfrac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})}\right|_{t=t_*} \geq 0$, pois caso contrário, teríamos $\dfrac{d}{dt}\|v(\cdot,t)\|^q_{L^q(\mathbb{R})} < 0$, *q.t.p.* $t \in [t_1,t_2]$. Logo, $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ seria decrescente em $[t_1,t_2]$, contradizendo $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_2)\|_{L^q(\mathbb{R})}$. Assim, pelo Teorema (5.7), teremos

$$\|v(\cdot,t_*)\|_{L^q(\mathbb{R})} \leq \left(\frac{q+\frac{q}{\sigma}+\alpha}{4}\right)^{\left(1-\frac{1}{\sigma}\right)\frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \|v(\cdot,t_*)\|_{L^{\frac{q}{\sigma}}}^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}}} \leq \lambda_q, \tag{5.67}$$

pela definição de λ_q, mas isso contradiz a suposição de se ter

$$\|v(\cdot,t_*)\|_{L^q(\mathbb{R})} > \lambda_q. \tag{5.68}$$

dada em (5.66). Logo, (5.65) é falso. Portanto, teremos $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda$, para todo $\tau \in [t_0,t]$, como afirmamos.

Caso II: $\|v(\cdot,t_0)\|_{L^q(\mathbb{R})} > \lambda_q$, mas existe $t_1 \in]t_0,t]$ tal que $\|v(\cdot,t_1)\|_{L^q(\mathbb{R})} \leq \lambda_q$. Seja

$$\widehat{t_1} := \inf\left\{\tau \in [t_0,t]; \|v(\cdot,\tau)\|_{L^q(\mathbb{R})} = \lambda_q\right\}. \tag{5.69}$$

Logo,

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda_q, \qquad \forall\, \tau \in [t_0,\widehat{t_1}[. \tag{5.70}$$

Assim, pelo argumento do Caso I, teremos

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \lambda_q, \qquad \forall\, \tau \in [\widehat{t_1},t]. \tag{5.71}$$

Do Teorema (5.7), segue que $\|v(\cdot,\tau)\|^q_{L^q(\mathbb{R})} \leq 0$, $\forall\, \tau \in [t_0,\widehat{t_1}] \setminus E_q$. Com isso segue que $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ é estritamente decrescente em $[t_0,\widehat{t_1}]$, logo, em particular, segue que

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0,\widehat{t_1}]. \tag{5.72}$$

De (5.71) e (5.72) resulta em

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \leq \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0,t]. \tag{5.73}$$

Caso III: $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} > \lambda_q, \qquad \forall\, \tau \in [t_0, t]$, então pelo Teorema (5.7), teremos

$$\frac{d}{dt}\,\|v(\cdot,\tau)\|^q_{L^q(\mathbb{R})} < 0, \qquad \forall\, \tau \in [t_0, t] \setminus E_q. \tag{5.74}$$

Ou seja, $\|v(\cdot,\tau)\|_{L^q(\mathbb{R})}$ é decrescente em $[t_0, t]$. Logo, em particular,

$$\|v(\cdot,\tau)\|_{L^q(\mathbb{R})} \le \|v(\cdot,t_0)\|_{L^q(\mathbb{R})}, \qquad \forall\, \tau \in [t_0, t]. \tag{5.75}$$

■

Usando argumentos de densidade, podemos estender o Teorema (5.9) para soluções fracas não negativas $u(\cdot,t) \ge 0$.

Teorema 5.10. *Seja* $u(\cdot,t) \ge 0$ *solução do problema* (5.1)*, com* $u(\cdot,t) \in L^\infty_{loc}([0,T_*[, L^\infty(\mathbb{R}) \cap L^\infty_{loc}([0,T_*[, L^p(\mathbb{R})))$*, para* $p \in [p_0, \infty[$*. Então, para cada* $q \ge \sigma p$*, temos*

$$\mathbb{U}_q(t_0;\widehat{t}) \le \max\left\{ \|u(\cdot,t_0)\|_{L^q(\mathbb{R})} \,;\, \left(\frac{q + \frac{q}{\sigma} + \alpha}{4}\right)^{\left(1-\frac{1}{\sigma}\right)\cdot \frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{B}_\mu(t_0;\widehat{t})^{\left(1-\frac{1}{\sigma}\right)\cdot \frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{U}_{\frac{q}{\sigma}}(t_0;\widehat{t})^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}}} \right\}, \tag{5.76}$$

para todos $0 \le t_0 \le \widehat{t} < T_*$*, onde* $\sigma = 2-\gamma$*,* $\gamma = 2\kappa - \alpha$*,* $0 \le \kappa < \frac{\alpha}{2}$ *e* $\widetilde{\alpha} = \alpha - \gamma$*.*

Prova: Seja $\zeta \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ uma função qualquer fixa e tal que $\zeta(x) > 0$, $\forall x \in \mathbb{R}$. Considere $\epsilon > 0$ dado e tome $v^{[\epsilon]}(\cdot,t) > 0$ a solução positiva de

$$\begin{cases} v_t + (b(x,t)v^{\kappa+1})_x = \mu(t)(v(x,t)^\alpha v_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\[2ex] v(\cdot,0) = u_0 + \epsilon\cdot\zeta \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}). \end{cases} \tag{5.77}$$

Seja $\epsilon_0 > 0$ suficientemente pequeno tal que para todo $0 < \epsilon \le \epsilon_0$:

$$\begin{array}{l} v^{[\epsilon]}(\cdot,t) \text{ esteja definida } \forall\, t \in [0,\widehat{t}] \\ \text{e} \\ v^{[\epsilon]}(\cdot,t) \in L^\infty([0,\widehat{t}], L^{\frac{q}{\sigma}}(\mathbb{R})) \cap L^\infty([0,\widehat{t}], L^\infty(\mathbb{R})). \end{array} \tag{5.78}$$

Pelo Teorema (5.9), segue que

$$\mathbb{V}^{[\epsilon]}_q(t_0;\widehat{t}) \le \max\left\{ \left\|v^{[\epsilon]}(\cdot,t_0)\right\|_{L^q(\mathbb{R})} \,;\, \left(\frac{q + \frac{q}{\sigma} + \alpha}{4}\right)^{\left(1-\frac{1}{\sigma}\right)\cdot \frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{B}_\mu(t_0;\widehat{t})^{\left(1-\frac{1}{\sigma}\right)\cdot \frac{1}{\frac{q}{\sigma}+\frac{\widetilde{\alpha}}{2}}} \cdot \mathbb{V}^{[\epsilon]}_{\frac{q}{\sigma}}(t_0;\widehat{t})^{\frac{q+\frac{\widetilde{\alpha}}{2}}{q+\sigma\frac{\widetilde{\alpha}}{2}}} \right\}, \tag{5.79}$$

$\forall\, 0 < \epsilon \leq \epsilon_0$, onde

$$\mathbb{V}_q^{[\epsilon]}(t_0;\widehat{t}) = \sup_{t_0 \leq t \leq \widehat{t}} \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^q(\mathbb{R})} \quad \text{e} \quad \mathbb{V}_{\frac{q}{\sigma}}^{[\epsilon]}(t_0;\widehat{t}) = \sup_{t_0 \leq t \leq \widehat{t}} \left\| v^{[\epsilon]}(\cdot,t) \right\|_{L^{\frac{q}{\sigma}}(\mathbb{R})}.$$

Fazendo $\epsilon \to 0$ em (5.79), obtemos (5.76). Concluindo o Teorema (5.10). ■

Resulta do Teorema acima (5.10), por argumento análogo ao do Teorema (5.4), a seguinte estimativa para $\mathbb{U}_\infty(t_0;t)$.

Teorema 5.11. *Seja $u(\cdot,t) \in L^\infty([0,T_*[,L^\infty(\mathbb{R}))$ solução (fraca) não negativa do problema (5.2), (5.1), onde $0 \leq \kappa < \dfrac{\alpha}{2}$. Então, para cada $0 \leq t_0 \leq t < T_*$, e cada $p_0 \leq p < \infty$, tem-se*

$$\mathbb{U}_\infty(t_0;t) \leq K(\alpha,\kappa;p) \max\left\{ \|u(\cdot,t_0)\|_{L^\infty(\mathbb{R})} ; \mathbb{B}_\mu(t_0;t)^{\frac{1}{p+\alpha-\kappa}} \mathbb{U}_p(t_0;t)^{\frac{p}{p+\alpha-\kappa}} \right\}, \tag{5.80}$$

onde $K(\alpha,\kappa;p) \geq 1$ é constante que depende apenas dos parâmetros α, κ, p com $K(\alpha,\kappa;p) \to 1$, ao $p \to +\infty$.

Nosso trabalho agora é estimar U_∞ em termos de U_p, para $p \in [p_0,\infty[$ qualquer. Este resultado completa o Teorema (5.5) acima.

Teorema 5.12. *Seja $p \in [p_0,\infty[$ tal que $\|u(\cdot,t)\|_{L^p(\mathbb{R})}$ não exploda em tempo finito. Então, supondo $B_\mu < \infty$, $\mathsf{U}_p < \infty$ e $\displaystyle\int_0^\infty \mu(t)dt = \infty$, temos*

$$\limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \mathbb{K}(\alpha,\kappa;p) \cdot B_\mu^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_p^{\frac{p}{p+\frac{\widetilde{\alpha}}{2}}}, \tag{5.81}$$

onde $\widetilde{\alpha} = 2(\alpha-\kappa)$ e $\mathbb{K}(\alpha,\kappa;p) > 0$ é dado por

$$\mathbb{K}(\alpha,\kappa;p) = \left(\frac{3\cdot(p-\gamma)+\alpha}{4} \right)^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \cdot \left[\prod_{j=1}^{\infty} \left(\frac{6\cdot 2^j(p-\gamma)+2\alpha}{3\cdot 2^j(p-\gamma)+2\alpha} \right)^{\frac{p-\gamma+\frac{\widetilde{\alpha}}{2}}{2^j(p-\gamma)+\frac{\widetilde{\alpha}}{2}}} \right]^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}}, \tag{5.82}$$

com $\gamma = 2\kappa - \alpha \in [-\alpha, 0[$.

Prova: Seja

$$\mathsf{U}_\infty := \limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}.$$

Pelo Teorema (5.4), temos $\mathsf{U}_\infty = \infty$. Se $\mathsf{U}_\infty = 0$, então (5.81) é trivialmente verdadeiro. Então consideramos apenas o caso $0 < \mathsf{U}_\infty < \infty$. Pelo Teorema (5.7) segue que

$$\mathsf{U}_\infty \leq \mathbb{K}_1 \cdot B_\mu^{\frac{1}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_{p-\gamma}^{\frac{p-\gamma}{p-\gamma+\frac{\widetilde{\alpha}}{2}}}, \tag{5.83}$$

onde $\gamma = 2\kappa - \alpha$ e

$$\mathbb{K}_1 = \left(\frac{3\cdot(p-\gamma)+\alpha}{4}\right)^{\frac{1}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \prod_{j=1}^{\infty}\left(\frac{6\cdot 2^j(p-\gamma)+2\alpha}{3\cdot 2^j(p-\gamma)+2\alpha}\right)^{\dfrac{1}{2^j(p-\gamma)+\frac{\widetilde{\alpha}}{2}}}. \tag{5.84}$$

Usamos a interpolação

$$\|\mathsf{u}\|_{L^{p-\gamma}(\mathbb{R})} \leq \|\mathsf{u}\|_{L^\infty(\mathbb{R})}^{-\frac{\gamma}{p-\gamma}} \cdot \|\mathsf{u}\|_{L^p(\mathbb{R})}^{\frac{p}{p-\gamma}} \tag{5.85}$$

e obtemos

$$\mathsf{U}_{p-\gamma} \leq \mathsf{U}_\infty^{-\frac{\gamma}{p-\gamma}} \cdot \mathsf{U}_p^{\frac{p}{p-\gamma}}. \tag{5.86}$$

Por (5.83) e (5.84), obtemos

$$\begin{aligned} \mathsf{U}_\infty &\leq \mathbb{K}_1 \cdot B_\mu^{\frac{1}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \left(\mathsf{U}_\infty^{-\frac{\gamma}{p-\gamma}} \cdot \mathsf{U}_p^{\frac{p}{p-\gamma}}\right)^{\frac{p-\gamma}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \\ &= \mathbb{K}_1 \cdot B_\mu^{\frac{1}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_\infty^{\frac{-\gamma}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_p^{\frac{p}{p-\gamma+\frac{\widetilde{\alpha}}{2}}}. \end{aligned} \tag{5.87}$$

Logo, como $\mathsf{U}_\infty > 0$, temos

$$\mathsf{U}_\infty^{1+\frac{\gamma}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \leq \mathbb{K}_1 \cdot B_\mu^{\frac{1}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_p^{\frac{p}{p-\gamma+\frac{\widetilde{\alpha}}{2}}},$$

ou seja,

$$\mathsf{U}_\infty^{\frac{p+\frac{\widetilde{\alpha}}{2}}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \leq \mathbb{K}_1 \cdot B_\mu^{\frac{1}{p-\gamma+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_p^{\frac{p}{p-\gamma+\frac{\widetilde{\alpha}}{2}}}.$$

Assim,

$$\mathsf{U}_\infty \leq \mathbb{K}_1^{\frac{p-\gamma+\frac{\widetilde{\alpha}}{2}}{p+\frac{\widetilde{\alpha}}{2}}} \cdot B_\mu^{\frac{1}{p+\frac{\widetilde{\alpha}}{2}}} \cdot \mathsf{U}_p^{\frac{p}{p+\frac{\widetilde{\alpha}}{2}}},$$

que é o mesmo que (5.81) com $\mathbb{K} = \mathbb{K}_1$ dado por (5.82). Concluindo o Teorema (5.12).

■

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