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Exact form factors of the $O(N)$ σ-model

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Abstract: A general form factor formula for the $O(N)$ σ-model is constructed and applied to several operators. The large N limits of these form factors are computed and compared with the $1/N$ expansion of the $O(N)$ σ-model in terms of Feynman graphs and full agreement is found. In particular, $O(3)$ and $O(4)$ form factors are discussed. For the $O(3)$ σ-model several low particle form factors are calculated explicitly.

Keywords: Exact S-Matrix, Bethe Ansatz, Integrable Field Theories, Sigma Models

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1 Introduction

The $O(N)$ σ-model is an asymptotically free quantum field theory which has been attracting high interest, since it exhibits some common features with Quantum Chromodynamics (QCD), the theory of strong interactions. Although very powerful, the 4-dimensional QCD still requires an adequate machinery to handle with the confinement problem. From this point of view, 2-dimensional integrable models are very useful since they serve as laboratories for investigations of those properties of quantum field theories which can not be described via standard methods, such as perturbation theory. We should also note that exact results and methods which have been developed last decades are now flourishing and finding applications in the $AdS_5 \bigotimes S^5$ theory [1]. It has also been proposed a set of consistency conditions for the worldsheet form factors for the set of off-shell operators in the $AdS_5 \bigotimes S^5$ [2]. The most important result on the AdS/CFT correspondence is the remarkable conjecture of Maldacena [3], which establishes that 10-dimensional string theory on $AdS_5 \bigotimes S^5$ could be equivalent to a 4-dimensional Super Yang Mills (SYM) theory. In order to establish this conjecture there are ongoing developments on this topic [1] employing exact methods such as the Bethe ansatz. Clearly, such new developments, among others, place to a higher level the status of the approaches employed in 2-dimensional integrable Quantum Field Theories (QFT). Consequently, integrable models in 2-dimensions are now being considered not only isolated mathematical objects; in opposite, they are universal and also coming out in higher dimensions, where enough integrals of motion necessary for integrability are being found. The nonlinear $O(N)$ σ-model is defined by the Lagrangian and the constraint

$$\mathcal{L} = \frac{1}{2}\sum_{\alpha=1}^{N}(\partial_\mu \varphi_\alpha)^2 \qquad \text{with} \quad g\sum_{\alpha=1}^{N}\varphi_\alpha^2 = 1 \tag{1.1}$$

where $\varphi_\alpha(x)$ is an isovector N-plet set of bosonic fields and g the coupling constant. This model is integrable, there exist an infinite set of conservation laws [4]. In the quantum model the infrared charge singularity leads to the disintegration of the Goldstone vacuum and to mass transmutation of particles, which form an $O(N)$ multiplet (see [5]).

In this article we construct the form factors of the model by using the solution of the $O(N)$ difference equation, derived previously [6] by generalizing Tarasov's methods [7] (see also [8]) of the algebraic Bethe ansatz. Exact form factors for the energy-momentum, the spin-field and the current are computed and compared with the $1/N$ expansion of the $O(N)$ σ- model. We should note that the form factors in $O(3)$ and $O(4)$ sigma models first were calculated by Smirnov [9, 10] (see also [11–14]). In the framework of 2-dimensional integrable QFTs the central problem is still the computation of the correlation functions or Weightman functions and the form factor program is exactly devoted to this purpose. The concept of a generalized form factor was introduced in [15, 16], where several consistency equations were formulated. Subsequently this approach was developed further and investigated in different models by Smirnov [9]. Generalized form factors are matrix elements of fields with many particle states. To construct these objects explicitly one has to solve generalized Watson's equations which are matrix difference equations. To solve these equations the so called "off-shell Bethe ansatz" is applied [6, 17–19]. The conventional Bethe ansatz

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introduced by Bethe [20] is used to solve eigenvalue problems and its algebraic formulation was developed by Faddeev and coworkers (see e.g. [21]). The off-shell Bethe ansatz has been introduced in [22] to solve the Knizhnik-Zamolodchikov equations which are differential equations. For other approches to form factors in integrable quantum field theories see also [23–31]. The main result of this paper is the general form factor formula, written as an integral representation, which provides the solution of all form factors equations and whose main idea is briefly explained below. In the $O(N)$ σ-model the particles form an isovector N-plet of $O(N)$. For a state of n particles of kind α_i with rapidities θ_i and a local operator $\mathcal{O}(x)$ the matrix element

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, \theta_1, \dots, \theta_n \,\rangle^{\text{in}}_{\underline{\alpha}} = e^{-ix(p_1+\dots+p_n)} F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$$

defines a form factor which we write as (see [15])

$$F^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = K^{\mathcal{O}}_{1\dots n}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij}) \tag{1.2}$$

where $F(\theta)$ is the minimal form factor function. We propose the following ansatz for the K-function in terms of a nested 'off-shell' Bethe ansatz written as a multiple contour integral

$$K^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = N^{\mathcal{O}}_n \int_{\mathcal{C}^1_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}^m_{\underline{\theta}}} dz_m\, \tilde{h}(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\; \tilde{\Psi}_{1\dots n}(\underline{\theta},\underline{z})\,. \tag{1.3}$$

Here $\tilde{h}(\underline{\theta},\underline{z})$ is a scalar function which depends only on the S-matrix. The dependence on the specific operator $\mathcal{O}(x)$ is encoded in the scalar p-function $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ which is in general a simple function of e^{θ_i} and e^{z_j}. The state $\tilde{\Psi}_{\underline{\alpha}}$ in (1.3) is a linear combination of the basic Bethe ansatz co-vectors (see (2.17))

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = L_{\underline{\mathring{\beta}}}(\underline{z})\, \tilde{\Phi}^{\underline{\mathring{\beta}}}_{\underline{\alpha}}(\underline{\theta},\underline{z}) \tag{1.4}$$

where summation over all $\underline{\mathring{\beta}} = (\mathring{\beta}_1, \dots, \mathring{\beta}_m)$ is assumed. The $\mathring{\beta}$ form an $(N-2)$-plet of $O(N-2)$. For $L_{\underline{\mathring{\beta}}}(\underline{z})$ we make again an ansatz like (1.3). The nested off-shell Bethe ansatz is obtained by iterating this procedure.

The article is organized as follows. In section 2 we recall some results and fix the notation concerning the $O(N)$ S-matrix, the monodromy matrix, etc. In section 3 we discuss the generalized form factors formula for the $O(N)$ σ-model. In section 4 we apply the nested off-shell Bethe ansatz to solve the $O(N)$ form factor equations. Section 5 is devoted to the computation of some examples. The appendices provide the more complicated proofs of the results we have obtained and further explicit calculations.

2 General settings

2.1 The $O(N)$ S-matrix

The 2-particle $O(N)$ S-matrix is of the form [5]

$$S = b(\theta)\mathbf{1} + c(\theta)\mathbf{P} + d(\theta)\mathbf{K} \tag{2.1}$$

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or in terms of the components

$$S^{\delta\gamma}_{\alpha\beta}(\theta) = b(\theta)\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} + c(\theta)\delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta} + d(\theta)\delta^{\delta\gamma}\delta_{\alpha\beta}$$

where θ is the rapidity difference of the particles. Crossing means

$$S^{\delta\gamma}_{\alpha\beta}(\theta) = \mathbf{C}_{\alpha\alpha'}S^{\alpha'\delta}_{\beta\gamma'}(i\pi-\theta)\mathbf{C}^{\gamma'\gamma} \tag{2.2}$$

or in terms of the amplitudes

$$b(\theta) = b(i\pi-\theta),\ d(\theta) = c(i\pi-\theta)$$

if we define the "charge conjugation matrices" as

$$\mathbf{C}_{\alpha\beta} = \delta_{\alpha\beta} \text{ and } \mathbf{C}^{\alpha\beta} = \delta^{\alpha\beta}\,. \tag{2.3}$$

The Yang-Baxter relation

$$S_{12}(\theta_{12})S_{13}(\theta_{13})S_{23}(\theta_{23}) = S_{23}(\theta_{23})S_{13}(\theta_{13})S_{12}(\theta_{12}) \tag{2.4}$$

implies [5]

$$c(\theta) = -\frac{i\pi\nu}{\theta}b(\theta),\ d(\theta) = -\frac{i\pi\nu}{i\pi-\theta}b(\theta) \tag{2.5}$$

where $\nu = 2/(N-2)$. The minimal solution is $b(\theta) = Q(\theta)Q(i\pi-\theta)$ with

$$Q(\theta) = \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2} + \frac{1}{2\pi i}\theta\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2\pi i}\theta\right)}\,. \tag{2.6}$$

This minimal solution was first constructed by Zamolodchikov and Zamolodchikov [5] and they gave arguments that it provides the $O(N)$ σ-model S-matrix

$$S^{\sigma\text{-model}}(\theta) = S^{\min}(\theta)\,.$$

The three S-matrix eigenvalues are $S_{\pm} = b \pm c$ and $S_0 = b + c + Nd$ with

$$(S_0, S_+, S_-) = \left(\frac{\theta+i\pi}{\theta-i\pi}, \frac{\theta-i\pi\nu}{\theta+i\pi\nu}, 1\right)S_-\,. \tag{2.7}$$

For later convenience we introduce

$$\tilde{S}(\theta) = S(\theta)/S_+(\theta) = \tilde{b}(\theta)\mathbf{1} + \tilde{c}(\theta)\mathbf{P} + \tilde{d}(\theta)\mathbf{K}$$

with

$$\begin{aligned}\tilde{b}(\theta) &= \frac{\theta}{\theta-i\pi\nu}\\ \tilde{c}(\theta) &= -\frac{i\pi\nu}{\theta-i\pi\nu}\\ \tilde{d}(\theta) &= -\frac{\theta}{\theta-i\pi\nu}\frac{i\pi\nu}{i\pi-\theta}\,.\end{aligned} \tag{2.8}$$

We will also need $\mathring{S}(z)$ the S-matrix for $O(N-2)$

$$\mathring{\tilde{S}}(\theta) = \mathring{S}(\theta)/\mathring{S}_+(\theta) = \mathring{\tilde{b}}(\theta)\mathbf{1} + \mathring{\tilde{c}}(\theta)\mathbf{P} + \mathring{\tilde{d}}(\theta)\mathbf{K} \tag{2.9}$$

where ν is replaced by $\mathring{\nu} = 2/(N-4)$.

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2.1.1 Complex basis

For the Bethe ansatz it is convenient to use instead of the real basis $|\alpha\rangle_r\,,\ (\alpha=1,2,\ldots,N)$ the complex basis

$$\left.\begin{array}{rcl}|\alpha\rangle &=& \frac{1}{\sqrt{2}}\left(|2\alpha-1\rangle_r+i|2\alpha\rangle_r\right)\\ |\bar{\alpha}\rangle &=& \frac{1}{\sqrt{2}}\left(|2\alpha-1\rangle_r-i|2\alpha\rangle_r\right)\end{array}\right\}\,,\quad \alpha=1,2,\ldots,[N/2]$$

and in addition $|0\rangle=|\bar{0}\rangle=|N\rangle_r$ for N odd. Below we will use the notation

$$|\theta\rangle_\alpha=|\alpha(\theta)\rangle,\ |\theta\rangle_{\bar{\alpha}}=|\bar{\alpha}(\theta)\rangle$$

for one particle and one antiparticle states with rapidity θ. The weight vectors

$$w=\left(w_1,\ldots,w_{[N/2]}\right)$$

of the one-particle states are given by

$$\begin{array}{rcll}w_k &=& \delta_{k\alpha} & \text{for } |\alpha\rangle\\ w_k &=& -\delta_{k\alpha} & \text{for } |\bar{\alpha}\rangle\\ w_k &=& 0 & \text{for } |0\rangle\,.\end{array}$$

Remark 1 *For even N this means that we consider $O(N)$ as a subgroup of $\mathrm{U}(N/2)$. For $N=3$ we may identify the particles $1,\bar{1},0$ with the pions $\pi_\pm,\pi_0$.*

The highest weight S-matrix eigenvalue is $a(\theta)=S_{11}^{11}(\theta)=S_+(\theta)$ with

$$a(\theta)=-\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2\pi i}\theta\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2\pi i}\theta\right)}\frac{\Gamma\left(1-\frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2\pi i}\theta\right)}{\Gamma\left(1+\frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2}\nu-\frac{\theta}{2\pi i}\right)} \tag{2.10}$$

$$=-\exp\left(-2\int_0^\infty\frac{dt}{t}\frac{e^{-t\nu}+e^{-t}}{1+e^{-t}}\sinh t\frac{\theta}{i\pi}\right) \tag{2.11}$$

We order the states as: $1,2,\ldots,0,\ldots,\bar{2},\bar{1}$. Then the charge conjugation matrix in the complex basis is of the form

$$\mathbf{C}^{\delta\gamma}=\delta^{\delta\bar{\gamma}}\,,\ \mathbf{C}_{\alpha\beta}=\delta_{\alpha\bar{\beta}} \tag{2.12}$$

$$\mathbf{C}=\begin{pmatrix}0&\cdots&0&\cdots&1\\ \vdots&\ddots&\vdots&.&\vdots\\ 0&\cdots&1&\cdots&0\\ \vdots&.&\vdots&\ddots&\vdots\\ 1&\cdots&0&\cdots&0\end{pmatrix}\,.$$

The annihilation-creation matrix in (2.1) may be written as

$$\mathbf{K}^{\delta\gamma}_{\alpha\beta}=\mathbf{C}^{\delta\gamma}\mathbf{C}_{\alpha\beta}\,.$$

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2.2 Nested "off-shell" Bethe ansatz

The "off-shell" Bethe ansatz is used to construct vector valued functions which have symmetry properties according to a representation of the permutation group generated by a factorizing S-matrix. In addition they satisfy matrix differential [32] or difference [17] equations. For the application to form factors we use the co-vector version $K_{1\dots n}(\underline{\theta}) \in V_{1\dots n} = (\bigotimes_{i=1}^{n} V)^{\dagger}$, $(\theta_i \in \mathbb{C},\ i = 1, \dots, n)$. We write the components of the co-vector $K_{1\dots n}$ as $K_{\underline{\alpha}}$ where $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a state of n particles. Solutions of the $O(N)$ equations

$$K_{\dots ij\dots}(\dots, \theta_i, \theta_j, \dots) = K_{\dots ji\dots}(\dots, \theta_j, \theta_i, \dots)\, \tilde{S}_{ij}(\theta_{ij})$$
$$K_{\alpha_1\alpha_2\dots\alpha_n}(\theta_1 + 2\pi i, \theta_2, \dots, \theta_n) = K_{\alpha_2\dots\alpha_n\alpha_1}(\theta_2, \dots, \theta_n, \theta_1)$$

where constructed in [6, 33] (see also [17, 19]). These equations are equivalent to the form factor equations (i) and (ii) (see (3.2) and (3.3)). The solutions have been constructed in terms of a nested $O(N)$ "off-shell" Bethe ansatz in [6, 33]. Here we need special solutions which satisfy in addition the form factor equation (iii) (see (3.4)).

Nested "off-shell" Bethe ansatz. We consider a state with n particles and write the off-shell Bethe ansatz co-vector valued function as

$$K_{\underline{\alpha}}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, \tilde{k}(\underline{\theta}, \underline{z})\, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) \tag{2.13}$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\underline{\theta} = (\theta_1, \dots, \theta_n)$ and $\underline{z} = (z_1, \dots, z_m)$. This ansatz transforms the complicated matrix equations (3.2)–(3.4) to simple equations for the scalar function $\tilde{k}(\underline{\theta}, \underline{z})$ (see [6] and below). The integration contour $\mathcal{C}_{\underline{\theta}}$ will be specified in section 4. The state $\tilde{\Psi}_{\underline{\alpha}}$ in (2.13) is the linear combination (1.4) of the basic Bethe ansatz co-vectors (2.17). For the co-vector valued function $L_{\underline{\mathring{\beta}}}(\underline{z})$ (which lies in a tensor product of smaller spaces of dimension $N-2$) we make again an ansatz like (2.13). Iterating this procedure we obtain the nested off-shell Bethe ansatz. This iteration ends up at the $O(3)$ or $O(4)$ cases which will be discussed separately.

As usual in the context of the algebraic Bethe ansatz [21, 34] the basic Bethe ansatz co-vectors $\tilde{\Phi}_{\underline{\alpha}}^{\underline{\mathring{\beta}}}$ are obtained from the monodromy matrix. We consider a state with n particles and as is usual in the context of the algebraic Bethe Ansatz we define [21, 34] the monodromy matrix by

$$\tilde{T}_{1\dots n,0}(\underline{\theta}, \theta_0) = \tilde{S}_{10}(\theta_1 - \theta_0) \cdots \tilde{S}_{n0}(\theta_n - \theta_0). \tag{2.14}$$

It is a matrix acting in the tensor product of the "quantum space" $V^{1\dots n} = V_1 \otimes \cdots \otimes V_n$ and the "auxiliary space" V_0. All vector spaces V_i are isomorphic to a space V whose basis vectors label all kinds of particles. Here $V \cong \mathbb{C}^N$ is the space of the vector representation of $O(N)$.

Suppressing the indices $1 \dots n$ we write the monodromy matrix in the complex basis as (following the notation of Tarasov [7])

$$\tilde{T}_{\alpha}^{\alpha'} = \begin{pmatrix} \tilde{A}_1 & (\tilde{B}_1)_{\mathring{\alpha}} & \tilde{B}_2 \\ (\tilde{C}_1)^{\mathring{\alpha}'} & (\tilde{A}_2)_{\mathring{\alpha}}^{\mathring{\alpha}'} & (\tilde{B}_3)^{\mathring{\alpha}'} \\ \tilde{C}_2 & (\tilde{C}_3)_{\mathring{\alpha}} & \tilde{A}_3 \end{pmatrix} \tag{2.15}$$

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where α, α' assume the values $1, 2, \dots, (0), \dots, \bar{2}, \bar{1}$ corresponding to the basis vectors of the auxiliary space $V \cong \mathbb{C}^N$ and $\mathring{\alpha}, \mathring{\alpha}'$ assume the values $2, \dots, (0), \dots, \bar{2}$ corresponding to the basis vectors of $\mathring{V} \cong \mathbb{C}^{N-2}$. We will also use the notation $\tilde{A} = \tilde{A}_1$, $\tilde{B} = \tilde{B}_1$, $\tilde{C} = \tilde{C}_1$ and $\tilde{D} = \tilde{A}_2$ which is an $(N-2) \times (N-2)$ matrix in the auxiliary space. As usual the Yang-Baxter algebra relation for the S-matrix yields the typical TTS-relation which implies the basic algebraic properties of the sub-matrices $\tilde{A}_i, \tilde{B}_i, \tilde{C}_i$.

The reference co-vector is defined as usual by

$$\Omega \tilde{B}_i = 0$$

with the solution

$$\Omega_{\underline{\alpha}} = \delta^1_{\alpha_1} \cdots \delta^1_{\alpha_n} \,. \tag{2.16}$$

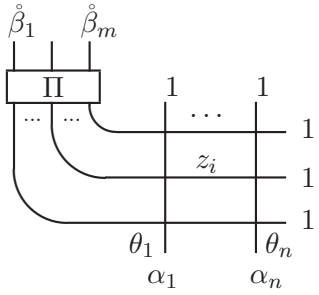
It satisfies

$$\Omega \tilde{T}(\underline{\theta}, z) = \Omega \begin{pmatrix} a_1(\underline{\theta}, z) & 0 & 0 \\ * & a_2(\underline{\theta}, z)\mathbf{1} & 0 \\ * & * & a_3(\underline{\theta}, z) \end{pmatrix} ,$$

$$a_1(\underline{\theta}, z) = 1 \,, \;\; a_2(\underline{\theta}, z) = \prod_{k=1}^{n} \tilde{b}(\theta_i - z), \;\; a_3(\underline{\theta}, z) = \prod_{k=1}^{n} \Big(\tilde{b}(\theta_i - z) + \tilde{d}(\theta_i - z) \Big) \,.$$

The basic Bethe ansatz co-vectors in (2.13) are defined as (for more details see [6])

$$\tilde{\Phi}^{\mathring{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) = \Big(\Pi^{\mathring{\beta}}_{\underline{\beta}}(\underline{z}) \Omega \tilde{T}_1^{\beta_m}(\underline{\theta}, z_m) \dots \tilde{T}_1^{\beta_1}(\underline{\theta}, z_1) \Big)_{\underline{\alpha}} =$$



(2.17)

The matrix $\Pi^{\mathring{\beta}}_{\underline{\beta}}(\underline{z})$ intertwines between the S-matrix S of $O(N)$ and $\mathring{S}$ of $O(N-2)$

$$\tilde{\mathring{S}}_{ij}(z_{ij}\mathring{\nu}/\nu) \Pi_{\dots ij \dots}(\underline{z}) = \Pi_{\dots ji \dots}(\underline{z}) \tilde{S}_{ij}(z_{ij}) \,. \tag{2.18}$$

This matrix Π is necessary[1] because for the next level Bethe ansatz the S-matrix $\mathring{S}(\theta)$ for $O(N-2)$ has to be used. The co-vectors (2.17) are generalizations of vectors introduced by Tarasov [7] for a 3-state model, the Korepin-Izergin model. The following relations for special components of Π will be used below (for more details see [6, 33])

$$\Pi^{\mathring{\beta}}_{\underline{\beta}} = \begin{cases} 0 & \text{for } \beta_1 = 1 \\ 0 & \text{for } \beta_m = \bar{1} \\ \delta^{\mathring{\beta}_1}_{\beta_1} \, \Pi^{\mathring{\beta}_2 \dots \mathring{\beta}_m}_{\beta_2 \dots \beta_m} & \text{for } \beta_1 \neq \bar{1} \\ \Pi^{\mathring{\beta}_1 \dots \mathring{\beta}_{m-1}}_{\beta_1 \dots \beta_{m-1}} \, \delta^{\mathring{\beta}_m}_{\beta_m} & \text{for } \beta_m \neq 1 \,. \end{cases} \tag{2.19}$$

[1] This matrix Π is trivial for the SU(N) Bethe ansatz because the SU(N) S-matrix amplitudes do not depend on N for a suitable normalization and parameterization.

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In particular for $n = 2$

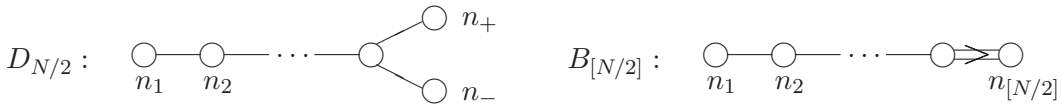
$$\Pi^{\mathring{\beta}_1\mathring{\beta}_2}_{\beta_1\beta_2}(\underline{z}) = \delta^{\mathring{\beta}_1}_{\beta_1}\delta^{\mathring{\beta}_2}_{\beta_2} + f(z_{12})\mathring{\mathbf{C}}^{\mathring{\beta}_1\mathring{\beta}_2}\delta^{\bar{1}}_{\beta_1}\delta^{1}_{\beta 2}\,, \quad f(z) = \frac{i\pi\nu}{z + i\pi\,(1-\nu)}\,. \tag{2.20}$$

Remark 2 *The* Π*-matrix is responsible for the fact that the Bethe state* $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ *is a symmetric function of the* z_i*, if the co-vector valued function* $L_{\underline{\mathring{\beta}}}(\underline{z})$ *in* (1.4) *satisfies equation* (4.22) *for level* $k = 1$*.*

It is well known (see [6]) that the 'off-shell' Bethe ansatz states are highest weight states if they satisfy certain matrix difference equations. If there are n particles, the $O(N)$ weights are

$$(w_1, \dots, w_{[N/2]}) = \begin{cases} \big(n - n_1, \dots, n_{[N/2]-1} - n_{[N/2]}\big) & \text{for } N \text{ odd} \\ \big(n - n_1, \dots, n_{[N/2]-2} - n_- - n_+, n_- - n_+\big) & \text{for } N \text{ even} \end{cases}$$

where $n_1 = m, n_2, \dots$ are the numbers of $\tilde{T}$ operators in (2.17) and the higher levels of the nesting. In particular $n_\pm$ are the numbers of positive/negative chirality spinor C-operators. For the on-shell Bethe ansatz for N even see also [35]. As is well known (see e.g. [36–38] and references therein), the various levels of the nested Bethe ansatz correspond to the nodes of the Dynkin diagrams of the corresponding Lie algebras $D_{N/2}$ for $N =$ even and $B_{[N/2]}$ for $N =$ odd:



3 Generalized form factors

For a state of n particles of kind α_i with rapidities θ_i and a local operator $\mathcal{O}(x)$ we define the form factor functions $F^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1, \dots, \theta_n)$, or using a short hand notation $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$, by

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, \theta_1, \dots, \theta_n \,\rangle^{\text{in}}_{\underline{\alpha}} = e^{-ix(p_1+\dots+p_n)} F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})\,, \quad \text{for } \theta_1 > \dots > \theta_n\,, \tag{3.1}$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\underline{\theta} = (\theta_1, \dots, \theta_n)$. For all other arrangements of the rapidities the functions $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ are given by analytic continuation. Note that the physical value of the form factor, i.e. the left hand side of (3.1), is given for ordered rapidities as indicated above and the statistics of the particles. The $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ are considered as the components of a co-vector valued function $F^{\mathcal{O}}_{1\dots n}(\underline{\theta}) \in V_{1\dots n} = \big(V^{1\dots n}\big)^\dagger$.

Now we formulate the main properties of form factors in terms of the functions $F^{\mathcal{O}}_{1\dots n}$.

3.1 Form factor equations

The co-vector valued function $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ defined by (3.1) is meromorphic in all variables $\theta_1, \dots, \theta_n$ and satisfies the following relations:

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(i) The Watson's equations describe the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces $i, j = i+1$ at the same time

$$F^{\mathcal{O}}_{\dots ij \dots}(\dots, \theta_i, \theta_j, \dots) = F^{\mathcal{O}}_{\dots ji \dots}(\dots, \theta_j, \theta_i, \dots)\, S_{ij}(\theta_{ij}) \tag{3.2}$$

for all possible arrangements of the θ's.

(ii) The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\begin{aligned} {}^{\text{out},\bar{1}}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2, \dots, p_n \,\rangle^{\text{in,conn.}}_{2\dots n} \\ = F^{\mathcal{O}}_{1\dots n}(\theta_1 + i\pi, \theta_2, \dots, \theta_n)\mathbf{C}^{\bar{1}1} = F^{\mathcal{O}}_{2\dots n1}(\theta_2, \dots, \theta_n, \theta_1 - i\pi)\mathbf{C}^{1\bar{1}} \end{aligned} \tag{3.3}$$

The charge conjugation matrix $\mathbf{C}^{\bar{1}1}$ is given by (2.12).

(iii) There are poles determined by one-particle states in each sub-channel. In particular the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\dots n}(\theta_1, \dots, \theta_n) = 2i\, \mathbf{C}_{12}\, F^{\mathcal{O}}_{3\dots n}(\theta_3, \dots, \theta_n)\, (\mathbf{1} - S_{2n} \dots S_{23}) \ . \tag{3.4}$$

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F^{\mathcal{O}}_{1\dots n}(\theta_1 + \mu, \dots, \theta_n + \mu) = e^{s\mu}\, F^{\mathcal{O}}_{1\dots n}(\theta_1, \dots, \theta_n) \tag{3.5}$$

if the local operator transforms under Lorentz transformations as $\mathcal{O} \to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

As was shown in [18] the properties (i)–(iii) follow from general LSZ-assumptions and "maximal analyticity", which means that $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ is a meromorphic function with respect to all θ's, and in the 'physical' strips $0 < \operatorname{Im}\theta_{ij} < \pi$ ($\theta_{ij} = \theta_i - \theta_j$, $i < j$) there are only poles of physical origin as for example bound state poles. In general there is also the form factor equation (iv) referring to bound states. Since there are no bound states in the $O(N)$ σ-model this equation is empty.

We will now provide a constructive and systematic way of how to solve the form factor equations for the co-vector valued function $F^{\mathcal{O}}_{1\dots n}$, once the scattering matrix is given.

Minimal form factors. The solutions of Watson's and the crossing equations (i) and (ii) for two particles

$$\begin{aligned} &\text{(i)} : F(\theta) = S(\theta)\, F(-\theta) \\ &\text{(ii)} : F(i\pi - \theta) = F(i\pi - \theta) \end{aligned}$$

with no poles in the physical strip $0 \le \operatorname{Im}\theta \le \pi$ and at most a simple zero at $\theta = 0$ are the minimal form factors. For the construction of the off-shell Bethe ansatz the minimal form factor of highest weight eigenvalue of the $O(N)$ S-matrix $a(\theta) = S_+(\theta)$ of (2.10) is essential

$$F(\theta) = c \exp\left(\int_0^\infty \frac{dt}{t \sinh t} \frac{1 - e^{-t\nu}}{1 + e^{-t}} \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi} \right) \right) \right) . \tag{3.6}$$

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For convenience we have introduced the constant c, which is defined below (4.8). The two other minimal form factors belonging to the S-matrix eigenvalues $S_-(\theta)$ and $S_0(\theta)$ (see (2.7)) are [15]

$$F_-(\theta) = \frac{i}{\sinh\frac{1}{2}\theta\,\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2\pi i}\theta\right)}\frac{\Gamma^2\left(\frac{1}{2}+\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2\pi i}\theta\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2\pi i}\theta\right)}F_+(\theta) \tag{3.7}$$

$$F_0(\theta) = \frac{\sinh\theta}{i\pi-\theta}F_-(\theta) \tag{3.8}$$

$$F_+(\theta) = \frac{1}{c}F(\theta)\,.$$

4 Nested "off-shell" Bethe ansatz for $O(N)$

4.1 The fundamental theorem

We write the general form factor $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ for n-particles following [15] as in (1.2) where $F(\theta)$ is the minimal form factor function (3.6). The K-function $K^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ contains the entire pole structure and is determined by the form factor equations (i)–(iii). We propose the ansatz (1.3) for the K-function in terms of a nested 'off-shell' Bethe ansatz (2.13)

$$K^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = N^{\mathcal{O}}_n \int_{\mathcal{C}^1_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}^m_{\underline{\theta}}} dz_m\, \tilde{h}(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\; \tilde{\Psi}_{1\dots n}(\underline{\theta},\underline{z})$$

written as a multiple contour integral. The scalar function $\tilde{h}(\underline{\theta},\underline{z})$ depends only on the S-matrix and not on the specific operator $\mathcal{O}(x)$

$$\tilde{h}(\underline{\theta},\underline{z}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\tilde{\phi}_j(\theta_i - z_j)\prod_{1\le i<j\le m}\tau_{ij}(z_i-z_j)\,. \tag{4.1}$$

The functions $\tilde{\phi}_j$ and τ_{ij} have to satisfy the shift equations

$$\tilde{\phi}_j(\theta-2\pi i) = \tilde{b}(\theta)\tilde{\phi}_j(\theta) \tag{4.2}$$

$$\tau_{ij}(z-2\pi i)/\tilde{b}(2\pi i - z) = \tau_{ij}(z)/\tilde{b}(z) \tag{4.3}$$

which follow from the form factor equation (ii) or (3.3) [6, 33]. Here, for the $O(N)$ form factors, they depend on whether i,j are even or odd

$$\tilde{\phi}_e(\theta) = \tilde{\chi}(\theta),\;\; \tilde{\phi}_o(\theta) = \tilde{\psi}(\theta) \tag{4.4}$$

$$\tau_{ee}(z) = \tau_{oo}(z) = \frac{1}{\tilde{\chi}(-z)\tilde{\chi}(z)},\;\; \tau_{eo}(z) = \tau_{oe}(-z) = \frac{1}{\tilde{\chi}(-z)\tilde{\psi}(z)} \tag{4.5}$$

where $\tilde{\psi}(z)$ and $\tilde{\chi}(z)$ are

$$\tilde{\psi}(\theta) = \frac{\Gamma\left(1-\frac{1}{2}\nu+\frac{1}{2\pi i}\theta\right)}{\Gamma\left(1+\frac{1}{2\pi i}\theta\right)}\,,\quad \tilde{\chi}(\theta) = \frac{\Gamma(-\frac{1}{2\pi i}\theta)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2\pi i}\theta)}\,. \tag{4.6}$$

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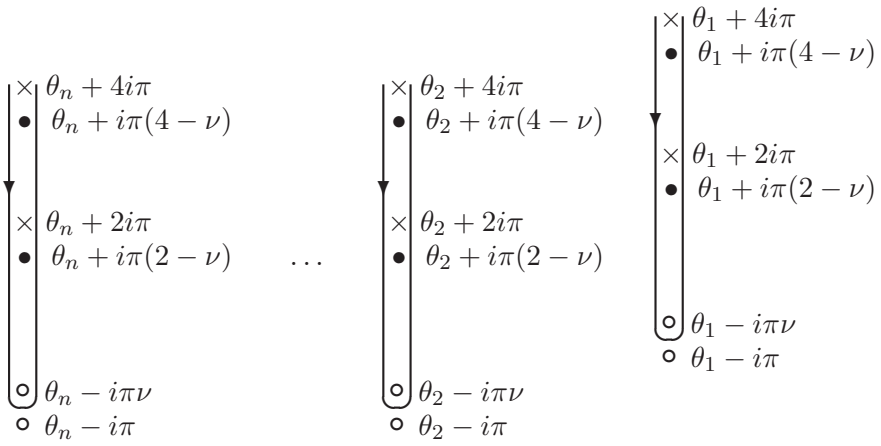

Figure 1. The integration contour $\mathcal{C}^o_{\underline{\theta}}$. The bullets and the crosses refer to poles and zeroes of the integrand resulting from $\tilde{\psi}(\theta_i - z_j)$ and the small open circles refer to poles originating from $\tilde{S}(\theta_i - z_j)$.

In addition the equation

$$F(\theta)F(\theta + i\pi)\tilde{\psi}(-\theta - i\pi + i\pi\nu)\tilde{\chi}(-\theta) = 1 \tag{4.7}$$

is satisfied. It follows from the form factor equation (iii) or (3.4) as will be discussed in appendix A.

Notice that the equations (4.6) and (4.7) also determine the normalization constant c in (3.6) as

$$c = \frac{1}{\sqrt{2}\pi}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\nu\right)\exp\left(\int_0^\infty \frac{dt}{t}\frac{1-e^{-t\nu}}{1+e^{-t}}\frac{1-\cosh\frac{1}{2}t}{\sinh t}\right). \tag{4.8}$$

The dependence on the specific operator $\mathcal{O}(x)$ is encoded in the scalar p-function $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ which is in general a simple function of e^{θ_i} and e^{z_j} (see below). By means of the ansatz (1.2) and (1.3) we have transformed the complicated form factor equations (i)–(v) (which are in general matrix equations) into much simpler scalar equations for the scalar p-function (see (4.9)).

The integration contours (corresponding to the functions $\tilde{\phi}_e(\theta) = \tilde{\chi}(\theta)$ and $\tilde{\phi}_o(\theta) = \tilde{\psi}(\theta)$) $\mathcal{C}^o_{\underline{\theta}}$ and $\mathcal{C}^e_{\underline{\theta}}$ are depicted in figure 1 and figure 2.

Theorem 3 *We make the following assumptions:*

1. *The p-function $p(\underline{\theta},\underline{z})$ satisfies the equations*

$$\left.\begin{array}{l}(\mathrm{i}') : p(\underline{\theta},\underline{z}) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j \\ (\mathrm{ii}') : p(\underline{\theta},\underline{z}) = p(\theta_1 + 2\pi i, \theta_2, \dots, \underline{z}) = p(\underline{\theta}, z_1 + 2\pi i, z_2, \dots) \\ (\mathrm{iii}') : p(\underline{\theta},\underline{z}) = p(\check{\underline{\theta}},\check{\underline{z}}) \text{ for } \theta_{12} = i\pi,\ z_1 = \theta_1 - i\pi\nu \text{ and } z_2 = \theta_2\end{array}\right\} \tag{4.9}$$

where the short notations $\check{\underline{\theta}} = (\theta_3, \dots, \theta_n)$ and $\check{\underline{z}} = (z_3, \dots, z_m)$ are used.

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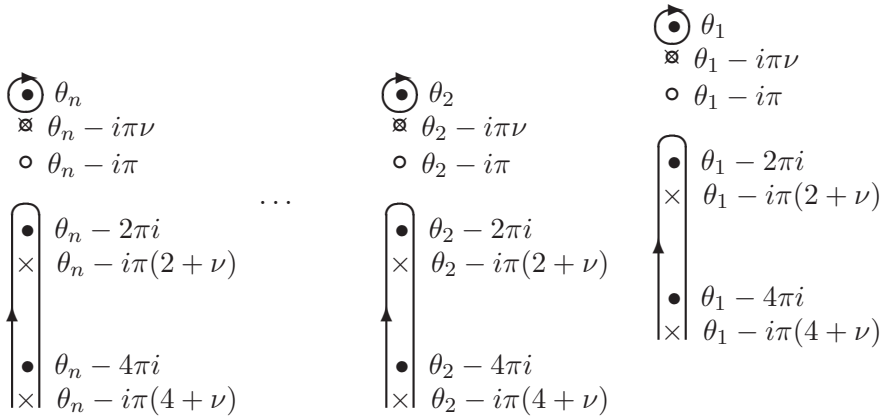

Figure 2. The integration contour $\mathcal{C}^e_{\underline{\theta}}$. The bullets and the crosses refer to poles and zeroes of the integrand resulting from $\tilde{\chi}(\theta_i - z_j)$ and the small open circles refer to poles originating from $\tilde{S}(\theta_i - z_j)$.

2. *The higher level function* $L_{\underline{\beta}}(\underline{z})$ *in* (1.4) *satisfies* $(\mathrm{i})^{(k)}$–$(\mathrm{iii})^{(k)}$ *of* (4.22)–(4.24) *for* $k = 1$

3. *The normalization constants in* (1.3) *satisfy (for* $N > 4$*)*

$$N^{\mathcal{O}}_m = \frac{1}{\left[\frac{1}{2}m\right]\left[\frac{1}{2}m+\frac{1}{2}\right]} \frac{\nu F(i\pi)}{8\pi} \frac{\tilde{\chi}(-i\pi\,(1+\nu))}{\tilde{\psi}^2(i\pi\nu)\tilde{\chi}(-i\pi)} N^{\mathcal{O}}_{m-2} \tag{4.10}$$

then the co-vector valued function $F_{\underline{\alpha}}(\underline{\theta})$ *given by the ansatz* (1.2) *and the integral representation* (1.3) *satisfies the form factor equations* (i), (ii) *and* (iii) *of* (3.2)–(3.4).

The proof of this theorem can be found in appendix A. The normalization relation (4.10), of course, depends on how the higher level K-functions are normalized. This will be discussed in subsection 4.4.

4.2 O(3) form factors

In the complex basis the three one-particle states are $1, 0, \bar{1}$. The S-matrix for $\nu = 2$ is

$$S^{O(3)}(\theta) = \frac{\theta - i\pi}{\theta + i\pi}\left(\frac{\theta}{\theta - 2i\pi}\mathbf{1} - \frac{2i\pi}{\theta - 2i\pi}\mathbf{P} - \frac{\theta}{\theta - 2i\pi}\frac{2i\pi}{i\pi - \theta}\mathbf{K}\right)$$

and the eigenvalues are

$$\begin{aligned} S^{O(3)}_+(\theta) &= \frac{\theta - i\pi}{\theta + i\pi} \\ S^{O(3)}_-(\theta) &= \frac{\theta - i\pi}{\theta + i\pi}\frac{\theta + 2\pi i}{\theta - 2\pi i} \\ S^{O(3)}_0(\theta) &= \frac{\theta + 2\pi i}{\theta - 2\pi i}\,. \end{aligned}$$

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The minimal form factors for these S-matrix eigenvalues are

$$F_+(\theta) = \frac{1}{2}(\theta - i\pi)\tanh\frac{1}{2}\theta$$
$$F_-(\theta) = \frac{1}{2}\pi^2\frac{(\theta - i\pi)}{\theta(\theta - 2\pi i)}\tanh\frac{1}{2}\theta$$
$$F_0(\theta) = -\pi^2\frac{1}{\theta(\theta - 2\pi i)}\sinh^2\frac{1}{2}\theta\,.$$

The general form factors

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})\prod_{1\le i<j\le n} F(\theta_{ij})$$

are given by (1.2) with $F(\theta) = 2F_+(\theta) = (\theta - i\pi)\tanh\frac{1}{2}\theta$ and the (one level) 'off-shell' Bethe ansatz (1.3)

$$K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = N^{\mathcal{O}}_m\int_{\mathcal{C}^1_{\underline{\theta}}} dz_1\cdots\int_{\mathcal{C}^m_{\underline{\theta}}} dz_m\,\tilde{h}(\underline{\theta},\underline{z})\,p^{\mathcal{O}}(\underline{\theta},\underline{z})\;\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) \tag{4.11}$$

with $\tilde{h}(\underline{\theta},\underline{z})$ given by (4.1). The functions $\tilde{\phi}_j(\theta)$ and $\tau_{ij}(z)$ we get from (4.6) (up to inessential constants) as

$$\tilde{\psi}(\theta) = \tilde{\chi}(\theta) = \frac{1}{\theta}$$
$$\tau(z) = z^2$$

such that (4.7) holds. It turns out that in (4.11) we have to calculate only some residues because for $\nu = 2$ many zeroes cancel poles such that we may replace the contour integrals $\int_{\mathcal{C}^j_{\underline{\theta}}} dz\ldots$ for even and odd j

$$\int_{\mathcal{C}_{\underline{\theta}}} dz\cdots \to \sum_{i=1}^{n}\left(\oint_{\theta_i} + \oint_{\theta_i - 2\pi i}\right) dz\ldots$$

where $\oint_\theta dz\ldots$ means an integral along a small circle around θ. The state $\tilde{\Psi}_{\underline{\alpha}}$ in (4.11) is here proportional to Bethe ansatz co-vectors (2.17)

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = L(\underline{z})\,\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta},\underline{z})$$

where the scalar function

$$L(\underline{z}) = \prod_{1\le i<j\le m} L(z_{ij})\,,\quad L(z) = \frac{(z - i\pi)}{z(z - 2\pi i)}\tanh\frac{1}{2}z$$

is the minimal solution of the equations (4.22), (4.23) and (4.24) with the scalar S-matrix

$$\mathring{\tilde{S}}(z\mathring{\nu}/\nu) = \mathring{\tilde{S}}(-z) = \frac{z - i\pi}{z + i\pi}\frac{z + 2i\pi}{z - 2i\pi}\,.$$

The $O(3)$ weight of the state is

$$w = n - m\,.$$

For explicit examples see section 5 and appendix E.2.

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4.3 O(4) form factors

In the complex basis the four one-particle states are $1, 2, \bar{2}, \bar{1}$. The S-matrix for $\nu = 1$ is

$$S^{O(4)}(\theta) = a^{O(4)}(\theta)\left(\frac{\theta}{\theta - i\pi}\mathbf{1} - \frac{i\pi}{\theta - i\pi}\mathbf{P} - \frac{\theta}{\theta - i\pi}\frac{i\pi}{i\pi - \theta}\mathbf{K}\right)$$

and the S-matrix eigenvalues are

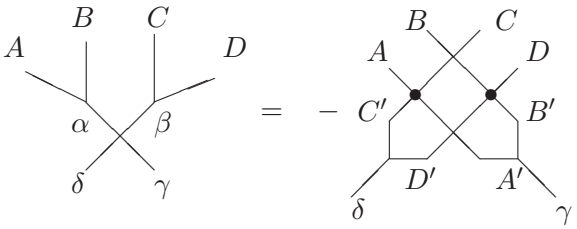
$$\begin{aligned} a^{O(4)}(\theta) = S_+^{O(4)}(\theta) &= -\left(\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right)}\right)^2 \\ S_-^{O(4)}(\theta) &= \frac{i\pi + \theta}{i\pi - \theta}\left(\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right)}\right)^2 \\ S_0^{O(4)}(\theta) &= -\left(\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{3}{2} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{3}{2} - \frac{\theta}{2\pi i}\right)}\right)^2. \end{aligned} \tag{4.12}$$

The group isomorphy $O(4) \simeq \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ reflects in terms of the S-matrices. The $O(4)$ S-matrix can be written as a tensor product of two SU(2) S-matrices [39–41]

$$\begin{aligned} S^{\mathrm{SU}(2)}(\theta) &= a^{\mathrm{SU}(2)}(\theta)\left(\frac{\theta}{\theta - i\pi}\mathbf{1} - \frac{i\pi}{\theta - i\pi}\mathbf{P}\right) \\ a^{\mathrm{SU}(2)}(\theta) = S_+^{\mathrm{SU}(2)}(\theta) &= -\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right)} \end{aligned}$$

or more precisely

$$\Gamma_\alpha^{AB}\Gamma_\beta^{CD}\left(S^{O(4)}\right)_{\delta\gamma}^{\alpha\beta} = -\left({}^{+}S^{\mathrm{SU}(2)}\right)_{C'A'}^{AC}\left({}^{-}S^{\mathrm{SU}(2)}\right)_{D'B'}^{BD}\Gamma_\delta^{C'D'}\Gamma_\gamma^{A'B'}. \tag{4.13}$$



The SU(2) S-matrices ${}^{\pm}S^{\mathrm{SU}(2)}(\theta)$ correspond to the spinor representations of $O(4)$ with positive (negative) chirality (see [42, 43]). In particular

$$a^{O(4)}(\theta) = -\left(\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right)}\right)^2 = -\left(a^{\mathrm{SU}(2)}(\theta)\right)^2. \tag{4.14}$$

The relative S-matrix for states of different chirality is trivial $S = \mathbf{1}$. The intertwiners Γ_α^{AB} have been discussed in [6, 43]. In the complex basis of the $O(4)$ states and the fundamental SU(2) representations the intertwiner matrix is diagonal and

$$\left(\Gamma_1^{\uparrow+\uparrow-}, \Gamma_2^{\uparrow+\downarrow-}, \Gamma_{\bar{2}}^{\downarrow+\uparrow-}, \Gamma_{\bar{1}}^{\downarrow+\downarrow-}\right) = \left(-1,\ 1,\ 1,\ 1\right). \tag{4.15}$$

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The minimal form factors for the S-matrix eigenvalues (4.12) are

$$
\begin{aligned}
F^{O(4)}(\theta) &= F_+^{O(4)}(\theta) = \exp\left(\int_0^\infty \frac{dt}{t\sinh t}\frac{1-e^{-t}}{1+e^{-t}}\left(1-\cosh t\left(1-\frac{\theta}{i\pi}\right)\right)\right)\\
F_-^{O(4)}(\theta) &= 2\frac{1}{\theta-i\pi}\coth\frac{1}{2}\theta\, F_+^{O(4)}(\theta)\\
F_0^{O(4)}(\theta) &= \frac{1}{i\pi-\theta}\sinh\theta\, F_-^{O(4)}(\theta)\,.
\end{aligned}
\tag{4.16}
$$

Equation (4.14) for the highest weight S-matrix amplitudes means that the highest weight minimal form factors $F = F_+$ are related by

$$
F^{O(4)}(\theta) = \frac{i}{\sinh\frac{1}{2}\theta}\left(F^{\mathrm{SU}(2)}(\theta)\right)^2. \tag{4.17}
$$

Similarly, as for S-matrices (4.13) the group isomorphy $O(4) \simeq \mathrm{SU}(2)\otimes\mathrm{SU}(2)$ reflects in terms of the form factors. The co-vector valued function

$$
F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = c_n \sum_l \prod_{i<j}\coth\frac{1}{2}\theta_{ij}\, F_{\underline{A}}^{{}^+\mathcal{O}_l}(\underline{\theta}) F_{\underline{B}}^{{}^-\mathcal{O}_l}(\underline{\theta})\Gamma_{\underline{\alpha}}^{\underline{AB}} \tag{4.18}
$$

is a candidate for an $O(4)$ form factor if $F_{\underline{A}}^{{}^+\mathcal{O}_l}$ and $F_{\underline{B}}^{{}^-\mathcal{O}_l}$ are SU(2) form factors. The SU(2) form factor equations (i) and (ii) for $F_{\underline{A}}^{{}^+\mathcal{O}_l}$ and $F_{\underline{B}}^{{}^-\mathcal{O}_l}$ imply the $O(4)$ form factor equations (i) and (ii) for $F_{\underline{\alpha}}^{\mathcal{O}}$. Moreover the double poles of $F_{\underline{A}}^{{}^+\mathcal{O}_l}(\underline{\theta})F_{\underline{B}}^{{}^-\mathcal{O}_l}(\underline{\theta})$ at $\theta_{ij} = i\pi$ are made to simple poles by the $\coth\frac{1}{2}\theta_{ij}$. However, the SU(2) form factor equations (iii) for $F_{\underline{A}}^{{}^+\mathcal{O}_l}$ and $F_{\underline{B}}^{{}^-\mathcal{O}_l}$ will in general not imply the $O(4)$ form factor equation (iii) for $F_{\underline{\alpha}}^{\mathcal{O}}$. This problem was discussed in [10, 14] and will discussed in this paper in appendix B and in terms of some examples in section 5. We write formally

$$
\mathcal{O} \equiv \sum_l {}^+\mathcal{O}_l \times {}^-\mathcal{O}_l\,. \tag{4.19}
$$

This equation is to be understood as the relation (4.18) of the form factors.

4.4 Higher level off-shell Bethe ansatz

For convenience we use the variables u and v with $\theta = i\pi\nu_k u$, $z = i\pi\nu_k v$ and $\nu_k = 2/(N-2k-2)$. Let $S^{(k)}(\theta)$ be the $O(N-2k)$ S-matrix with

$$
\begin{aligned}
\tilde{S}^{(k)}(u) &= S^{(k)}/S_+^{(k)} = \tilde{b}(u)\mathbf{1} + \tilde{c}(u)\mathbf{P} + \tilde{d}_k(u)\mathbf{K}\\
\tilde{b}(u) &= \frac{u}{u-1},\ \tilde{c}(u) = \frac{-1}{u-1},\ \tilde{d}_k(u) = \frac{u}{u-1}\frac{1}{u-1/\nu_k}
\end{aligned}
\tag{4.20}
$$

We define

$$
\begin{aligned}
K_{\underline{\alpha}}^{(k)}(\underline{u}) &= \tilde{N}_{m_k}^{(k)}\int_{\mathcal{C}_{\underline{u}}^1} dv_1\cdots\int_{\mathcal{C}_{\underline{u}}^{m_k}} dv_{m_k}\,\tilde{h}(\underline{u},\underline{v})p^{(k)}(\underline{u},\underline{v})\ \tilde{\Psi}_{\underline{\alpha}}^{(k)}(\underline{u},\underline{v})\\
\tilde{\Psi}_{\underline{\alpha}}^{(k)}(\underline{u},\underline{v}) &= L_{\underline{\mathring{\beta}}}^{(k)}(\underline{v})\left(\tilde{\Phi}^{(k)}\right)_{\underline{\alpha}}^{\underline{\mathring{\beta}}}(\underline{u},\underline{v}),\quad L_{\underline{\mathring{\beta}}}^{(k)}(\underline{v}) = K_{\underline{\mathring{\beta}}}^{(k+1)}(\underline{v}).
\end{aligned}
\tag{4.21}
$$

with $\underline{u} = u_1,\ldots,u_{n_k}$, $\underline{v} = v_1,\ldots,v_{m_k}$ and $m_k = n_{k+1}$.

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The equations (i)$^{(k)}$–(iii)$^{(k)}$ for $k > 0$ read in terms of these variables as

(i)$^{(k)}$ The symmetry property under the permutation of both, the variables u_i, u_j and the spaces $i, j = i + 1$ at the same time

$$K^{(k)}_{\dots ij\dots}(\dots, u_i, u_j, \dots) = K^{(k)}_{\dots ji\dots}(\dots, u_j, u_i, \dots)\, \tilde{S}^{(k)}_{ij}(u_{ij}) \tag{4.22}$$

for all possible arrangements of the u's.

(ii)$^{(k)}$ The periodicity property under the cyclic permutation of the rapidity variables and spaces

$$K^{(k)}_{1\dots n_k}(u_1 + 2/\nu, u_2, \dots, u_{n_k})\mathbf{C}^{\bar{1}1} = K^{(k)}_{2\dots n_k 1}(u_2, \dots, u_{n_k}, u_1)\mathbf{C}^{1\bar{1}} \tag{4.23}$$

with the charge conjugation matrix $\mathbf{C}^{\bar{1}1}$.

(iii)$^{(k)}$ The function $K^{(k)}_{1\dots n}(\underline{u})$ has a pole at $u_{12} = 1/\nu_k$ such that

$$\operatorname*{Res}_{u_{12}=1/\nu_k} K^{(k)}_{1\dots n_k}(u_1, \dots, u_{n_k}) = \prod_{j=3}^{n_k} \tilde{\psi}(u_{i1} + 1)\tilde{\chi}(u_{i2})\mathbf{C}_{12} K^{(k)}_{3\dots n_k}(u_3, \dots, u_{n_k})\,. \tag{4.24}$$

These equations are similar to the form factor equations (i)–(iii) of (3.2)–(3.4) for $O(N - 2k)$. However, there are two differences:

1. the shift in (ii)$^{(k)}$ is not the one of $O(N - 2k)$ but that of $O(N)$,
2. in (iii)$^{(k)}$ there is only one term on the right hand side.

The p-function $p^{(k)}(\underline{u}, \underline{v})$ satisfies the equations

$$\left.\begin{array}{l} (\mathrm{i}') : p^{(k)}(\underline{u}, \underline{v}) \text{ is symmetric under } u_i \leftrightarrow u_j,\ v_i \leftrightarrow v_j \\ (\mathrm{ii}') : p^{(k)}(\underline{u}, \underline{v}) = p^{(k)}(u_1 + 2/\nu, u_2, \dots, \underline{v}) = p^{(k)}(\underline{u}, v_1 + 2/\nu, v_2, \dots) \\ (\mathrm{iii}') : p^{(k)}(\underline{u}, \underline{z}) = p^{(k)}(\check{\underline{u}}, \check{\underline{v}}) \text{ for } u_{12} = 1/\nu_k,\ v_1 = u_1 - 1 \text{ and } v_2 = u_2\,. \end{array}\right\} \tag{4.25}$$

The short notations $\check{\underline{u}} = (u_3, \dots, u_{n_k})$ and $\check{\underline{v}} = (v_3, \dots, v_{m_k})$ are used. Below we will replace $p^{(k)}(\underline{u}, \underline{v})$ by 1 which will not change the results, if the $p^{(k)}$ satisfy the conditions (4.25).

Lemma 4 *The vector valued function $K^{(k)}_{\underline{\alpha}}(\underline{u})$ of (4.21) for $0 < k < \left[\frac{1}{2}(N-3)\right]$ satisfies the equations* (i)$^{(k)}$–(iii)$^{(k)}$*, if the corresponding relations are satisfied for $K^{(k+1)}$ and the normalizations satisfy*

$$\tilde{N}^{(k)}_{m_k} = \frac{1}{\left[\frac{1}{2}m_k\right]\left[\frac{1}{2}m_k + \frac{1}{2}\right]} \frac{\Gamma^2\left(1 + \frac{1}{2}\nu\right)}{4\pi^2} \frac{\tilde{\chi}(1/\nu_{k+1} - 2/\nu)}{\tilde{\chi}(1/\nu_k - 2/\nu)} \tilde{N}^{(k)}_{m_k - 2}\,, \tag{4.26}$$

where $[x]$ is the largest integer $\leq x$. The numbers $m_k = n_{k+1}$ are given by the numbers of particles $n = n_0$ and the weights of the operator $\mathcal{O}$

$$w^{\mathcal{O}} = (w_1, \dots, w_{[N/2]}) = \begin{cases} (n_0 - n_1, \dots, n_{[N/2]-1} - n_{[N/2]}) & \text{for } N \text{ odd} \\ (n_0 - n_1, \dots, n_{[N/2]-2} - n_- - n_+, n_- - n_+) & \text{for } N \text{ even}\,. \end{cases} \tag{4.27}$$

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The proof of this lemma can be found in appendix C.1. The cases $k = M = \left[\frac{1}{2}(N-3)\right]$ have to be considered separately.

Lemma 5 *For $N =$ odd the level $k = M = (N-3)/2$ means an $O(3)$ problem with $\nu_M = 2$ and $K_{\underline{\alpha}}^{(M)}(\underline{u})$ of (4.21) satisfies the equations* $(\mathrm{i})^{(k)}$–$(\mathrm{iii})^{(k)}$, *if*

$$\tilde{N}_{m_M}^{(M)} = \frac{1}{\left[\frac{1}{2}m_M\right]\left[\frac{1}{2}m_M+\frac{1}{2}\right]} \frac{\Gamma^3\left(1+\frac{1}{4}\nu\right)\Gamma\left(1+\frac{1}{2}\nu\right)}{2\pi^2\Gamma\left(1-\frac{1}{4}\nu\right)} \tilde{N}_{m_M-2}^{(M)}\,. \tag{4.28}$$

In particular for $N = 3$

$$\tilde{N}_m = \frac{1}{m\,(m-1)} \frac{1}{16\pi} \tilde{N}_{m_M-2}\,. \tag{4.29}$$

The proof of this lemma can be found in appendix C.2. Note that the shift in (4.23) is not that of $O(3)$, but that of $O(N)$.

For N even and $k = M = (N-4)/2$ we have $\nu_M = 1$ as for $O(4)$, however, the shift in (4.23) is not that of $O(4)$ but that of $O(N)$. We use the technique of subsection 4.3 and set analogously to (4.18)

$$K_{\underline{\alpha}}^{(M)}(\underline{u}) = d_{n_M} \prod_{1\le i<j\le n_M} \sin\frac{1}{2}\pi\nu\,(u_{ij}-1)\; {}^{+}K_{\underline{A}}^{\mathrm{SU}(2)}(\underline{u})\; {}^{-}K_{\underline{B}}^{\mathrm{SU}(2)}(\underline{u})\Gamma_{\underline{\alpha}}^{\underline{AB}}. \tag{4.30}$$

with

$${}^{\pm}K_{\underline{A}}^{\mathrm{SU}(2)}(\underline{u}) = {}^{\pm}\tilde{N}_{m_\pm} \int_{\mathcal{C}_{\underline{u}}^1} dv_1 \cdots \int_{\mathcal{C}_{\underline{u}}^{m_\pm}} dv_{m_\pm}\, \tilde{h}(\underline{u},\underline{v}) p_\pm(\underline{u},\underline{v})\; \tilde{\Psi}_{\underline{A}}^{\mathrm{SU}(2)}(\underline{u},\underline{v}) \tag{4.31}$$

$$\begin{aligned} \tilde{h}(\underline{u},\underline{v}) &= \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}_\nu(u_i - v_j) \prod_{1\le i<j\le m} \tau_\nu(v_{ij}) \\ \tilde{\phi}_\nu(u) &= \Gamma\left(-\frac{1}{2}\nu u\right)\Gamma\left(1-\frac{1}{2}\nu+\frac{1}{2}\nu u\right),\; \tau_\nu(u) = \frac{1}{\tilde{\phi}_\nu(u)\tilde{\phi}_\nu(-u)}\,. \end{aligned} \tag{4.32}$$

The p-functions $p_\pm$ satisfy the conditions of e.g. [19]. Note that $\tilde{\phi}_\nu(u)$ satisfies

$$\tilde{\phi}_\nu(u-2/\nu) = -\tilde{b}(u)\tilde{\phi}_\nu(u)$$

which implies the shift relation (4.23). For $N = 4$ i.e. $\nu = 1$ we obtain the $\tilde{\phi}$-function of SU(2) (see e.g. [19]).

Lemma 6 *For $N =$ even and $k = M = (N-4)/2$ the K-function of (4.30) satisfies the equations* $(\mathrm{i})^{(k)}$–$(\mathrm{iii})^{(k)}$ *if*

$$\begin{aligned} d_{n_M} &= -\frac{2}{\nu\pi^3} d_{n_M-2} \\ {}^{\pm}\tilde{N}_{m_\pm} &= \frac{1}{m_\pm} \frac{(-1)^{m_\pm}}{2\pi i\Gamma^2\left(-\frac{1}{2}\nu\right)}\, {}^{\pm}\tilde{N}_{m_\pm-1}. \end{aligned}$$

The proof of this lemma can be found in appendix C.3.

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5 Examples

5.1 Field

The fundamental field $\varphi^\alpha(x)$ in the Lagrangian (1.1) transforms as the vector representation of $O(N)$ and has therefore the weights $w = (w_1, \dots, w_{[N/2]}) = (1, 0, \dots, 0)$ (see [6, 33]) which implies with (4.27) that the numbers n_i of integrations in the various levels of the off-shell Bethe ansatz satisfy

$$\begin{cases} n-1 = n_1 = n_2 = \cdots = n_{[N/2]} & \text{for } N \text{ odd} \\ n-1 = n_1 = n_2 = \cdots = n_{[N/2]-2} = n_- + n_+, n_- = n_+ & \text{for } N \text{ even}\,. \end{cases}$$

Because the Bethe ansatz yields highest weight states we obtain the matrix elements of the highest weight component of φ^α which means in the complex basis $\alpha = 1$. We use the short notation $\varphi = \varphi^1$ and propose for the n-particle form factors ($n = m+1$ odd)

$$\langle 0|\varphi(0)|\underline{\theta}\rangle_{\underline{\alpha}} = F^\varphi_{\underline{\alpha}}(\underline{\theta}) = \prod_{i<j} F(\theta_{ij}) K^\varphi_{\underline{\alpha}}(\underline{\theta})$$

$$K^\varphi_{\underline{\alpha}}(\underline{\theta}) = N^\varphi_n \int_{\mathcal{C}^o_{\underline{\theta}}} dz_1 \dots \int_{\mathcal{C}^e_{\underline{\theta}}} dz_m\, \tilde{h}(\underline{\theta}, \underline{z}) p^\varphi(\underline{\theta}, \underline{z})\, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$$

with the p-function for $n = m+1 = \text{odd} > 1$

$$p^\varphi(\underline{\theta}, \underline{z}) = \frac{\left(\sum\limits_{\substack{j=1\\ j\,\text{odd}}}^{m-1} e^{z_j + i\pi\nu} + \sum\limits_{\substack{j=2\\ j\,\text{even}}}^{m} e^{z_j}\right)\left(\sum\limits_{\substack{j=1\\ j\,\text{odd}}}^{m-1} e^{-z_j - i\pi\nu} + \sum\limits_{\substack{j=2\\ j\,\text{even}}}^{m} e^{-z_j}\right)}{\left(\sum\limits_{j=1}^{n} e^{\theta_j}\right)\left(\sum\limits_{j=1}^{n} e^{-\theta_j}\right) - 1} \tag{5.1}$$

which satisfies (4.9). The scalar function $\tilde{h}(\underline{\theta}, \underline{z})$ is given by (4.1) and the Bethe ansatz $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ state by (1.4) and (2.17).

The one particle form factor is trivial

$$\langle 0|\varphi(0)|\theta\rangle_\alpha = F^\varphi_\alpha(\theta) = \delta^1_\alpha\,.$$

The three particle form factor is obtained by the ansatz (1.2), the integral representation (1.3) and the state (1.4) for $n = 3,\ m = 2$

$$\langle 0|\varphi(0)|\underline{\theta}\rangle_{\underline{\alpha}} = F^\varphi_{\underline{\alpha}}(\underline{\theta}) = F(\theta_{12})F(\theta_{13})F(\theta_{23}) K^\varphi_{\underline{\alpha}}(\underline{\theta}) \tag{5.2}$$

$$K^\varphi_{\underline{\alpha}}(\underline{\theta}) = N^\varphi_3 \int_{\mathcal{C}^o_{\underline{\theta}}} dz_1 \int_{\mathcal{C}^e_{\underline{\theta}}} dz_2\, \tilde{h}(\underline{\theta}, \underline{z}) p^\varphi(\underline{\theta}, \underline{z})\, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$$

with

$$\tilde{h}(\underline{\theta}, \underline{z}) = \prod_{i=1}^{3} \left(\tilde{\psi}(\theta_i - z_1)\tilde{\chi}(\theta_i - z_2)\right) \frac{1}{\tilde{\chi}(z_{12})\tilde{\psi}(-z_{12})},$$

$$p^\varphi(\underline{\theta}, \underline{z}) = \frac{\left(e^{z_1 + i\pi\nu} + e^{z_2}\right)\left(e^{-z_1 - i\pi\nu} + e^{-z_2}\right)}{\left(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\right)\left(e^{-\theta_1} + e^{-\theta_2} + e^{-\theta_3}\right) - 1},$$

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) = L_{\underline{\mathring{\beta}}}(\underline{z}) \left(\Pi^{\mathring{\beta}}_{\underline{\beta}}(\underline{z}) \Omega \tilde{T}_1^{\beta_2}(\underline{\theta}, z_2) \tilde{T}_1^{\beta_1}(\underline{\theta}, z_1)\right)_{\underline{\alpha}}.$$

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The higher level function $L_{\mathring{\beta}_1\mathring{\beta}_2}(\underline{z}) = \mathring{\mathbf{C}}_{\mathring{\beta}_1\mathring{\beta}_2} L(z_{12})$ is given by the solution of lemma 4 for the $O(N-2)$ weights $w = (0, \dots, 0)$. In appendix C.4 we get the result

$$L(z) = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2\pi i}z\right)\Gamma\left(-\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2\pi i}z\right)}{\Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2\pi i}z\right)\Gamma\left(\frac{1}{2}\nu + \frac{1}{2\pi i}z\right)}\,. \tag{5.3}$$

We could not perform the integrations[2] in (5.2) for general N, but we calculate the 3-particle form factor for $O(3)$ and $O(4)$. In addition we expand the exact expression in $1/N$-expansion to compare the result with the $1/N$-expansion of the σ-model in terms of Feynman graphs.

$O(3)$-form factors of the field $\varphi(x)$. As explained in subsection 4.2 we perform the integrations in (5.2) for $O(3)$ by calculating a finite number of residues. We obtain (see appendix E)

$$F^{\varphi}_{\alpha\beta\gamma}(\underline{\theta}) = \pi^3 g^{\varphi}_{\alpha\beta\gamma}(\underline{\theta}) G(\theta_{12}) G(\theta_{13}) G(\theta_{23})$$

with

$$\begin{aligned} g^{\varphi}_{\alpha\beta\gamma}(\underline{\theta}) &= \theta_{23}\delta^1_\alpha \mathbf{C}_{\beta\gamma} + (2\pi i - \theta_{13})\,\delta^1_\beta \mathbf{C}_{\alpha\gamma} + \theta_{12}\delta^1_\gamma \mathbf{C}_{\alpha\beta} \\ G(\theta) &= \frac{1}{\theta\,(\theta - 2\pi i)} \tanh\frac{1}{2}\theta\, F(\theta) = \frac{\theta - i\pi}{\theta\,(\theta - 2\pi i)} \tanh^2\frac{1}{2}\theta \end{aligned} \tag{5.4}$$

which agrees with the result of [11] obtained by different methods.

The 5-particle form factor of the field for $O(3)$ is determined by the same technique in appendix E.

$O(4)$-form factors of the field $\varphi(x)$. We apply the techniques of subsection 4.3 where the $O(4)$ form factor is written in terms of SU(2) ones as (4.18). For details see appendix B. We use the general formula (4.19) with

$$\begin{aligned} {}^{+}\mathcal{O}_1 &= {}^{-}\mathcal{O}_2 = \psi^A_+ \\ {}^{+}\mathcal{O}_2 &= {}^{-}\mathcal{O}_1 = \psi^B_- \end{aligned} \tag{5.5}$$

and write formally

$$\varphi^\alpha \equiv -\frac{1}{2}\left(\psi^A_+ \times \psi^B_- + \psi^A_- \times \psi^B_+\right) \Gamma^\alpha_{AB} \tag{5.6}$$

where

$$\psi^A(x) = \begin{pmatrix} \psi^A_+(x) \\ \psi^A_-(x) \end{pmatrix}$$

is the fundamental field of the SU(2) chiral Gross-Neveu model (see [19, 45]) with statistics factors (see appendix B)

$$\sigma_\pm = \pm i$$

[2]Doing one integral we obtain a generalization of Meijer's G-functions. The second integration does not yield known functions (to our knowledge). One could, of course, apply numerical integration techniques and one could determine the asymptotic behavior for large θ's which is under investigation [44].

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and "spin" $s = 1/4$ such that

$$\langle\, 0 \,|\, \psi^A(0) \,|\, \theta \,\rangle_B = \delta^A_B \begin{pmatrix} e^{-\frac{1}{4}\theta} \\ e^{+\frac{1}{4}\theta} \end{pmatrix}.$$

Because the Bethe ansatz yields highest weight states we obtain the matrix elements of the field $\varphi(x) = \varphi^1(x)$ with

$$\varphi \equiv \frac{1}{2}\left(\psi_+ \times \psi_- + \psi_- \times \psi_+\right)$$

where $\psi_\pm = \psi^\uparrow_\pm$ are the highest weight components of the SU(2) fields. The $O(4)$-weights of $\varphi(x)$ are $w^\varphi_{O(4)} = w^{\psi_+}_{O(4)} + w^{\psi_-}_{O(4)} = (1/2, 1/2) + (1/2, -1/2) = (1, 0)$ as we need. This follows because SU(2)-weights of $\psi_\pm$ are $w^{\psi_\pm}_{\mathrm{SU}(2)} = (1, 0)$ which mean the $O(4)$-weights $w^{\psi_\pm}_{O(4)} = (1/2, \pm 1/2)$ (for more details see section 3.1.3 of [33]). The 1-particle form factor is

$$\begin{aligned}\langle\, 0 \,|\, \varphi \,|\, \theta \,\rangle_\alpha &= \frac{1}{2}\left(\langle\, 0 \,|\times\langle\, 0 \,|\right)\left(\psi_+ \times \psi_- + \psi_- \times \psi_+\right)\left(|\, \theta \,\rangle \times |\, \theta \,\rangle\right) \\ &= \frac{1}{2}\delta^1_\alpha\left(e^{-\frac{1}{4}\theta}e^{\frac{1}{4}\theta} + e^{\frac{1}{4}\theta}e^{-\frac{1}{4}\theta}\right) = \delta^1_\alpha .\end{aligned}$$

The $n = 3$ particle form factor (4.18) is (up to const.)

$$F^\varphi_{\alpha\beta\gamma}(\theta_1, \theta_2, \theta_3) = \prod_{i<j} \coth\frac{1}{2}\theta_{ij}\left(F^{\psi_+}_{ABC}(\underline{\theta})F^{\psi_-}_{A'B'C'}(\underline{\theta}) + F^{\psi_-}_{ABC}(\underline{\theta})F^{\psi_+}_{A'B'C'}(\underline{\theta})\right)\Gamma^{AA'}_\alpha\Gamma^{BB'}_\beta\Gamma^{CC'}_\gamma$$

(see also (8.5) in [14]). This $O(4)$ form factor satisfies the form factor equations (i), (ii) and (iii). The three-particle SU(2) form factors $F^{\psi_\pm}_{\underline{A}}(\underline{\theta})$ have been discussed in [19, 45], they can be expressed in terms of Meijer's G-functions.

1/N expansion. For convenience we multiply the field with the Klein-Gordon operator and take

$$\mathcal{O}(x) = i(\Box + m^2)\varphi(x) .$$

We obtain (see appendix D.1)

$$F^{\mathcal{O}}_{\alpha\beta\gamma}(\underline{\theta}) = -\frac{8\pi i}{N}m^2\left(\delta^1_\alpha \mathbf{C}_{\beta\gamma}\frac{\sinh\theta_{23}}{i\pi - \theta_{23}} + \delta^1_\beta \mathbf{C}_{\alpha\gamma}\frac{\sinh\theta_{13}}{i\pi - \theta_{13}} + \delta^1_\gamma \mathbf{C}_{\alpha\beta}\frac{\sinh\theta_{12}}{i\pi - \theta_{12}}\right) + O(N^{-2}) \tag{5.7}$$

which agrees with the $1/N$ expansion using Feynman graphs (see appendix D.2).

5.2 Current

The classical Noether current (real basis)

$$J^{\alpha\beta}_\mu = \varphi^\alpha\partial_\mu\varphi^\beta - \varphi^\beta\partial_\mu\varphi^\alpha$$

transforms as the antisymmetric tensor representation of $O(N)$ and has therefore the weights $w^J = (w_1, \ldots, w_{[N/2]}) = (1, 1, 0, \ldots, 0)$ (see [6, 33]) which implies with (4.27) that the numbers n_i of integrations in the various levels of the off-shell Bethe ansatz satisfy

$$\begin{cases} n - 2 = n_1 - 1 = n_2 = \cdots = n_{[N/2]} & \text{for } N \text{ odd} \\ n - 2 = n_1 - 1 = n_2 = \cdots = n_{[N/2]-2} = n_- + n_+, n_- = n_+ & \text{for } N \text{ even} . \end{cases}$$

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Because the Bethe ansatz yields highest weight states we obtain the matrix elements of the highest weight component of $J_\mu^{\alpha\beta}$ which means in the complex basis

$$J_\mu = J_\mu^{12} = \varphi^1 \partial_\mu \varphi^2 - \varphi^2 \partial_\mu \varphi^1 .$$

The conservation law $\partial_\mu J^\mu = 0$ implies that there exists a pseudo-potential $J(x)$ with

$$J^\mu(x) = \epsilon^{\mu\nu} \partial_\nu J(x).$$

For the form factors we have

$$F_{\underline{\alpha}}^{J^\mu}(\underline{\theta}) = -i\epsilon^{\mu\nu} P_\nu F_{\underline{\alpha}}^J(\underline{\theta}), \;\; P = \sum p_i \, .$$

We propose for $n = m + 1$ even

$$\begin{aligned} \langle 0|J(0)|\underline{\theta}\rangle_{\underline{\alpha}} = F_{\underline{\alpha}}^J(\underline{\theta}) &= \prod_{i<j} F(\theta_{ij}) K_{\underline{\alpha}}^J(\underline{\theta}) \\ K_{\underline{\alpha}}^J(\underline{\theta}) &= N_n^J \int_{\mathcal{C}_{\underline{\theta}}^o} dz_1 \ldots \int_{\mathcal{C}_{\underline{\theta}}^e} dz_m \, \tilde{h}(\underline{\theta}, \underline{z}) p^J(\underline{\theta}, \underline{z}) \, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) \end{aligned} \tag{5.8}$$

with the p-function

$$p^J(\underline{\theta}, \underline{z}) = \exp\left(\sum_{i=1}^n \theta_i - \sum_{j=1}^m z_j - \frac{1}{2} n \, i\pi\nu \right) / \sum_{i=1}^n e^{\theta_i} \tag{5.9}$$

which satisfies (4.9). The scalar function $\tilde{h}(\underline{\theta}, \underline{z})$ is given by (4.1) and the Bethe ansatz state $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ by (1.4) and (2.17).

We calculate the 2-particle form factor

$$K_{\underline{\alpha}}^J(\underline{\theta}) = N_1^J \int_{\mathcal{C}_{\underline{\theta}}^o} dz \, \tilde{\psi}(\theta_1 - z) \tilde{\psi}(\theta_2 - z) \, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, z) \, e^{\theta_1 + \theta_2 - z} / \left(e^{\theta_1} + e^{\theta_2} \right) \tag{5.10}$$

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, z) = \tilde{b}(\theta_1 - z) \tilde{c}(\theta_2 - z) \delta_{\alpha_1}^1 \delta_{\alpha_2}^2 + \tilde{c}(\theta_1 - z) \delta_{\alpha_1}^2 \delta_{\alpha_2}^1 \, .$$

The integration in (5.10) can be performed using the result of Example 5.3 in [6]. We obtain for $N > 4$

$$\begin{aligned} F_{\underline{\alpha}}^J(\underline{\theta}) &= \left(\delta_{\alpha_1}^1 \delta_{\alpha_2}^2 - \delta_{\alpha_1}^2 \delta_{\alpha_2}^1 \right) \tanh \frac{1}{2} \theta_{12} F_-(\theta_{12}) \\ F_{\underline{\alpha}}^{J_\mu^{\alpha\beta}}(\underline{\theta}) &= i \left(\delta_{\alpha_1}^\alpha \delta_{\alpha_2}^\beta - \delta_{\alpha_1}^\beta \delta_{\alpha_2}^\alpha \right) (p_1 - p_2)_\mu \, F_-(\theta_{12}) \end{aligned} \tag{5.11}$$

which agrees with the result of [15] where also the $1/N$-expansion was checked.

$O(3)$-form factors of the current. The 2-particle form factor for $O(3)$ is obtained from Example 5.4 in [6] as

$$F_{\underline{\alpha}}^J(\underline{\theta}) = \left(\delta_{\alpha_1}^1 \delta_{\alpha_2}^0 - \delta_{\alpha_1}^0 \delta_{\alpha_2}^1 \right) \frac{1}{2} \pi^2 G(\theta) \tag{5.12}$$

with $G(\theta)$ given by (5.4).

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For the 4-particle form factor for $O(3)$ we use again the techniques of subsection 4.2. Performing the integrations in (5.8) by calculating a finite number of residues we obtain in appendix E for example

$$F^J_{0001}(\underline{\theta}) = \frac{1}{2}\pi^5 \left(\theta_{12}\theta_{13}\theta_{23} + 2\pi i\theta_{14}\left(\theta_{34} - 2\pi i\right) - 2i\pi^3\right) \prod_{i<j} G(\theta_{ij})$$
$$F^J_{0\bar{1}11}(\underline{\theta}) = \frac{1}{2}\pi^5 \left(\theta_{34} - 2i\pi\right)\left(\theta_{23}\theta_{24} - \theta_{12}\left(\theta_{12} - i\pi\right)\right) \prod_{i<j} G(\theta_{ij}).$$

The other components are obtained by the form factor equations (i) and (ii). These results agree with those of [11] which were obtained by different methods.

***O*(4)-form factors of the current.** The $O(4)$ form factor is written again in terms of SU(2) ones as in (4.18). We apply again the techniques of subsection 4.3 (for details see appendix B) and use the general formula (4.19) with

$$\mathcal{O} \equiv \mathcal{O}_1 \times \mathcal{O}_2 + \mathcal{O}_3 \times \mathcal{O}_4\,. \tag{5.13}$$

The p-functions of these operators are proposed to be

$$p^{\mathcal{O}_1}(\underline{\theta},\underline{z}) = \exp\left(\frac{1}{2}\sum_{i=1}^n \theta_i - \sum_{i=1}^m z_i\right) / \sum_{i=1}^n \exp\theta_i\,, \quad m = \frac{1}{2}n - 1$$
$$p^{\mathcal{O}_2}(\underline{\theta},\underline{z}) = \exp\left(-\frac{1}{2}\sum_{i=1}^n \theta_i + \sum_{i=1}^m z_i\right), \qquad m = \frac{1}{2}n$$
$$p^{\mathcal{O}_3}(\underline{\theta},\underline{z}) = \left(p^{\mathcal{O}_1}(\underline{\theta},\underline{z})\right)^2$$
$$p^{\mathcal{O}_4}(\underline{\theta},\underline{z}) = \left(p^{\mathcal{O}_2}(\underline{\theta},\underline{z})\right)^2$$

where the SU(2) weights are $w^{\mathcal{O}_1} = w^{\mathcal{O}_3} = (2,0)$, $w^{\mathcal{O}_2} = w^{\mathcal{O}_4} = (0,0)$, which means that the $O(4)$ weight vector is $w^{\mathcal{O}} = (1,1)$.

With (4.17) we get for the K-functions

$$K^J_{\underline{\alpha}}(\underline{\theta}) = \prod_{i<j} \cosh\frac{1}{2}\theta_{ij} \left(K^{\mathcal{O}_1}_{\underline{A}}(\underline{\theta}) K^{\mathcal{O}_2}_{\underline{B}}(\underline{\theta}) + K^{\mathcal{O}_3}_{\underline{A}}(\underline{\theta}) K^{\mathcal{O}_4}_{\underline{B}}(\underline{\theta})\right) \Gamma^{AB}_{\underline{\alpha}}\,. \tag{5.14}$$

The results of [19, 45] imply (for $n = 2$)

$$K^J_{\alpha_1\alpha_2}(\underline{\theta}) = \left(\delta^1_{\alpha_1}\delta^2_{\alpha_2} - \delta^2_{\alpha_1}\delta^1_{\alpha_2}\right) \frac{2}{\theta_{12} - i\pi}$$

which agrees with (5.11)

$$F^{J^{\alpha\beta}_\mu}_{\underline{\alpha}}(\underline{\theta}) = i\left(\delta^\alpha_{\alpha_1}\delta^\beta_{\alpha_2} - \delta^\beta_{\alpha_1}\delta^\alpha_{\alpha_2}\right)(p_1 - p_2)_\mu\, F_-(\theta_{12})$$

because of (4.16).

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5.3 Energy momentum

Because $T^{\mu\nu}$ is an $O(N)$ iso-scalar we have the weights $w = (w_1, \dots, w_{[N/2]}) = (0, \dots, 0)$ (see [6, 33]) which implies with (4.27) that

$$\begin{cases} n = n_1 = \cdots = n_{[N/2]} & \text{for } N \text{ odd} \\ n = n_1 = \cdots = n_{[N/2]-2} = n_- + n_+, n_- = n_+ & \text{for } N \text{ even}\,. \end{cases}$$

Following [11] we write the energy momentum tensor in terms of an energy momentum potential

$$\begin{aligned} T^{\mu\nu}(x) &= R^{\mu\nu}(i\partial_x)T(x) \\ R^{\mu\nu}(P) &= -P^\mu P^\nu + g^{\mu\nu}P^2. \end{aligned}$$

For the potential we propose the n-particle form factor as

$$\begin{aligned} \langle 0|T(0)|\underline{\theta}\rangle_{\underline{\alpha}} &= F^T_{\underline{\alpha}}(\underline{\theta}) = N^T_n \prod_{i<j} F(\theta_{ij}) K^T_{\underline{\alpha}}(\underline{\theta}) \\ K^T_{\underline{\alpha}}(\underline{\theta}) &= \int_{\mathcal{C}^1_{\underline{\theta}}} dz_1 \dots \int_{\mathcal{C}^m_{\underline{\theta}}} dz_m\, \tilde{h}(\underline{\theta}, \underline{z}) p^T(\underline{\theta}, \underline{z})\, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) \end{aligned} \tag{5.15}$$

with the p-function for $n = m =$ even

$$p^T(\underline{\theta}, \underline{z}) = 1 \tag{5.16}$$

which satisfies (4.9). The scalar function $\tilde{h}(\underline{\theta}, \underline{z})$ is given by (4.1) and the Bethe ansatz $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ state by (1.4) and (2.17). The form factor of the energy momentum tensor is then

$$\begin{aligned} \langle 0|T^{\mu\nu}(0)|\underline{\theta}\rangle_{\underline{\alpha}} &= F^{T^{\mu\nu}}_{\underline{\alpha}}(\underline{\theta}) = \prod_{i<j} F(\theta_{ij}) K^{T^{\mu\nu}}_{\underline{\alpha}}(\underline{\theta}) \\ K^{T^{\mu\nu}}_{\underline{\alpha}}(\underline{\theta}) &= \left(-P^\mu P^\nu + g^{\mu\nu}P^2\right) K^T_{\underline{\alpha}}(\underline{\theta})\,,\ P = \sum p_i\,. \end{aligned} \tag{5.17}$$

We do not calculate the integrals in (5.15) for general N, but we derive the 2 particle form factor following [15]. In addition we calculate integrals explicitly for $N = 3$, $N = 4$ and $N \to \infty$ in appendix C.4. Using the arguments of [15] we write

$$\begin{aligned} F^{T^{\mu\nu}}_{\alpha_1\alpha_2}(\underline{\theta}) &= \left(-p_1^\mu p_2^\nu - p_2^\mu p_1^\nu + g^{\mu\nu}\left(p_1 p_2 + m^2\right)\right) \mathbf{C}_{\alpha_1\alpha_2} F_0(\theta_{12}) \\ F^T_{\alpha_1\alpha_2}(\underline{\theta}) &= \frac{1}{2\cosh^2\frac{1}{2}\theta_{12}} \mathbf{C}_{\alpha_1\alpha_2} F_0(\theta_{12}) \end{aligned} \tag{5.18}$$

where $F_0(\theta)$ is the minimal form factor (3.8) in the scalar channel belonging to the S-matrix eigenvalue $S_0(\theta)$. The normalization means that the energy momentum operator satisfies for a one particle state the eigenvalue equation

$$P^\mu |\,\theta\,\rangle_\alpha = \int dx^1 T^{\mu 0}(x)\,|\,\theta\,\rangle_\alpha = |\,\theta\,\rangle_\alpha\, p^\mu(\theta).$$

Using (3.8) for general N we obtain explicitly (with $\theta = \theta_{12}$)

$$F^T_{\underline{\alpha}}(\underline{\theta}) = -\mathbf{C}_{\alpha_1\alpha_2} \frac{\left(\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right)\right)^2}{2\pi^2} \frac{\Gamma\left(-\frac{1}{2} + \frac{1}{2}\frac{\theta}{i\pi}\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\frac{\theta}{i\pi}\right)}{\Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\frac{\theta}{i\pi}\right)\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\frac{\theta}{i\pi}\right)} F(\theta)\,. \tag{5.19}$$

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***O*(3)-form factors of energy momentum.** Again we perform the integrations in (5.15) for $O(3)$ by calculating a finite number of residues.

It turns out that the leading term in the limit $\nu \to 2$ (i.e. $N \to 3$) vanishes and we have to calculate the contribution of order $(\nu - 2)$. For the 2-particle form factor we obtain

$$F^T_{\underline{\alpha}}(\underline{\theta}) = -\mathbf{C}_{\alpha_1\alpha_2}\frac{1}{2}\pi^2\frac{1}{\theta_{12} - i\pi}G(\theta_{12})$$

which agrees with (5.19) for $N = 3$.

For the 4-particle form factor for $O(3)$ we obtain in appendix E for example the component

$$F^T_{\bar{1}\bar{1}11}(\underline{\theta}) = \frac{1}{2}\pi^5\,(\theta_{12} - 2\pi i)\,(\theta_{34} - 2\pi i)\prod_{i<j}G(\theta_{ij})\,.$$

The other components are obtained by the form factor equations (i) and (ii). These results agree again with those of [11] which were obtained by different methods.

***O*(4)-form factors of energy momentum.** Applying the results of appendix C.4 we obtain from the general formula (4.18)

$$F^T_{\alpha_1\alpha_2}(\underline{\theta}) = -2\mathbf{C}_{\alpha_1\alpha_2}\left(\frac{1}{\theta_{12} - i\pi}\right)^2 F\,(\theta_{12})$$

which agrees (5.18) for $N = 4$.

1/N expansion. In appendix C.4 we also calculate the integrals in (5.15) explicitly for $n = 2$ and $N \to \infty$ and find

$$F^T_{\alpha_1\alpha_2}(\underline{\theta}) = -\mathbf{C}_{\alpha_1\alpha_2}\frac{1}{\theta_{12} - i\pi}\tanh\frac{1}{2}\theta_{12} + O(1/N)$$

which agrees with (5.18) for $N \to \infty$. This result agrees also with the one obtained by calculating Feynman graphs. This calculation is similar to that, which was done in [15] for the $O(N)$ Gross-Neveu model up to $O(1/N^2)$. Note that the leading term for $N \to \infty$ is not the free value.

6 Conclusions

In this paper the general form factor formula for the $O(N)$-sigma model is constructed. As an application, the general $O(N)$ form factors for the field, the current and the energy momentum operators are presented in terms of integral representations. The large N limits of these form factors are compared with the $1/N$-expansion of the $O(N)$-sigma model in terms of Feynman graphs and full agreement is found. Using these general results some examples of $O(3)$ and $O(4)$ form factors for low particle numbers are computed explicitly and agreement is found with previous results [11] obtained by different methods. We believe that our results may be relevant to understand the behavior of correlation functions in theories with asymptotically freedom like $4D$ QCD.

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A Proof of the main theorem 3

The co-vector valued function $K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ given by the integral representation (1.3) can be written as a sum of "Jackson-type Integrals" as investigated in [6] because of the identity

$$\int_{\mathcal{C}_a} dz \Gamma(a-z) f(z) = 2\pi i \operatorname*{Res}_{z=a} \sum_{l=-\infty}^{\infty} \Gamma(a-z-l) f(z+l) \tag{A.1}$$

where the $\mathcal{C}_a$ encircles the poles of $\Gamma(a-z)$ anti-clockwise. For these expressions symmetry properties and matrix difference equation have been proved in [6] which imply the form factor equations (i) and (ii). Therefore we only have to prove, that the assumptions of theorem 3 picks those solutions of (i) and (ii), which in addition satisfy the residue relation (iii)

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n) = 2i\,\mathbf{C}_{12}\,F^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n)\,(\mathbf{1}-S_{2n}\dots S_{23})\ .$$

Proofs. The K-function $K^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ defined by (1.2) contains the entire pole structure and is determined by the form factor equations (i)–(iii) which read in terms of $K^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ as

$$K^{\mathcal{O}}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = K^{\mathcal{O}}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\,\tilde{S}_{ij}(\theta_{ij}) \tag{A.2}$$

$$K^{\mathcal{O}}_{1\dots n}(\theta_1+2\pi i,\theta_2,\dots,\theta_n)\mathbf{C}^{\bar{1}1} = K^{\mathcal{O}}_{2\dots n1}(\theta_2,\dots,\theta_n,\theta_1)\mathbf{C}^{1\bar{1}} \tag{A.3}$$

$$\operatorname*{Res}_{\theta_{12}=i\pi} K_{1\dots n}(\underline{\theta}) = \frac{2i}{F(i\pi)}\mathbf{C}_{12}\prod_{i=3}^{n}\tilde{\psi}(\theta_{i1}+i\pi\nu)\tilde{\chi}(\theta_{i2})K_{3\dots n}(\theta_3,\dots,\theta_n)\,(\mathbf{1}-S_{2n}\dots S_{23}) \tag{A.4}$$

where (4.7) has been used. The residue of $K_{1\dots n}(\underline{\theta})$ consists of two terms

$$\operatorname*{Res}_{\theta_{12}=i\pi} K_{1\dots n}(\underline{\theta}) = \left(\overset{(1)}{\operatorname*{Res}_{\theta_{12}=i\pi}} + \overset{(2)}{\operatorname*{Res}_{\theta_{12}=i\pi}}\right) K_{1\dots n}(\underline{\theta})\,.$$

This is because for each z_j integration with j even the contours will be "pinched" at two points (see figure 2):

(1) $z_j = \theta_2 \approx \theta_1 - i\pi$

(2) $z_j = \theta_1 - 2\pi i \approx \theta_2 - i\pi$

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In appendix C.1 we prove for general level k of the off-shell Bethe ansatz the residue formulas. The general result imply for $k = 0$ that the contribution from the pinching (1) gives

$$\operatorname*{Res}^{(1)}_{\theta_{12}=i\pi} K_{1\dots n}(\underline{\theta}) = \frac{2i}{F(i\pi)}\mathbf{C}_{12}\prod_{i=3}^{n}\tilde{\psi}(\theta_{i1}+i\pi\nu)\tilde{\chi}(\theta_{i2})K_{3\dots n}(\theta_3,\dots,\theta_n)$$

if the normalization relation (4.10) holds. Therefore we have proved

$$\operatorname*{Res}^{(1)}_{\theta_{12}=+i\pi} F_{1\dots n}(\theta_1,\dots,\theta_n) = 2i\,\mathbf{C}_{12}\,F_{3\dots n}(\theta_3,\dots,\theta_n)\,.$$

To investigate $\operatorname*{Res}^{(2)}_{\theta_{12}=i\pi} F_{1\dots n}(\underline{\theta})$ due to the pinching at $z_j = \theta_1 - 2\pi i \approx \theta_2 - i\pi$ we use (ii) and (i) to write

$$\begin{aligned} F_{1\dots n}(\underline{\theta}) &= \mathbf{C}_{1\bar{1}}F_{2\dots n1}(\theta_2,\dots,\theta_n,\theta_1-2\pi i)\mathbf{C}^{1\bar{1}} \\ &= \mathbf{C}_{1\bar{1}}F_{21\dots n}(\theta_2,\theta_1-2\pi i,\dots,\theta_n)\mathbf{C}^{1\bar{1}}S_{\bar{1}n}\dots S_{\bar{1}3} \end{aligned}$$

We use the result for $\operatorname*{Res}^{(1)}_{\theta_1=\theta_2+i\pi}$ and obtain

$$\begin{aligned} \operatorname*{Res}^{(2)}_{\theta_1=\theta_2+i\pi} F_{1\dots n}(\underline{\theta}) &= -\operatorname*{Res}^{(1)}_{\theta_2=(\theta_1-2\pi i)+i\pi}\mathbf{C}_{1\bar{1}}F_{21\dots n}(\theta_2,\theta_1-2\pi i,\dots,\theta_n)\mathbf{C}^{1\bar{1}}S_{\bar{1}n}\dots S_{\bar{1}3} \\ &= -\mathbf{C}_{1\bar{1}}2i\,\mathbf{C}_{21}\,F^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n)\mathbf{C}^{1\bar{1}}S_{\bar{1}n}\dots S_{\bar{1}3} \\ &= -2i\mathbf{C}_{12}\;F^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n)S_{2n}\dots S_{23}\,. \end{aligned}$$

■

B O(4) solutions of (iii)

In order to get solutions of the form factor equation (iii) in the form of (4.18)

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = c_n\sum_l\prod_{i<j}\coth\frac{1}{2}\theta_{ij}\,F_{\underline{A}}^{\;+\mathcal{O}_l}(\underline{\theta})F_{\underline{B}}^{\;-\mathcal{O}_l}(\underline{\theta})\Gamma^{\underline{AB}}_{\underline{\alpha}} \tag{B.1}$$

we need that

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = 2i\,F^{\mathcal{O}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})\left(\mathbf{C}_{\alpha_1\alpha_2}\mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}} - \mathbf{C}_{\alpha_1\alpha_2'}\left(S^{O(4)}_{2n}\dots S^{O(4)}_{23}\right)^{\underline{\check{\alpha}}'\alpha_2'}_{\alpha_2\underline{\check{\alpha}}}\right)$$

with $\underline{\check{\theta}} = (\theta_3,\dots,\theta_n)$ etc. The fact that fields of the chiral Gross-Neveu model posses a generalized statistics implies that the form factor equations contain statistics factors σ (see [19, 45]). The residue equation (iii) for SU(2) form factors reads as

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n) = 2i\,\mathbf{C}_{12}\,F^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n)\left(\mathbf{1}-\sigma^{\mathcal{O}}\rho S_{2n}\dots S_{23}\right)\,. \tag{B.2}$$

where S is the SU(2) S-matrix. Here $\sigma^{\mathcal{O}}$ is the statistics factor of the operator $\mathcal{O}(x)$ with respect to the particle 2 and ρ is a sign factor due to the unusual crossing relation of the S-matrix, in [19, 45] was shown that

$$\sigma^{\mathcal{O}} = e^{i\pi\frac{1}{2}Q^{\mathcal{O}}}, \ \rho = (-1)^{1+\frac{1}{2}(n-Q^{\mathcal{O}})} = \pm 1$$

where $Q^{\mathcal{O}} = n \bmod 2$ is the charge of the operator $\mathcal{O}$. By charge conjugation we have for the conjugate operator $\sigma^{\bar{\mathcal{O}}} = e^{-i\pi\frac{1}{2}Q^{\mathcal{O}}}$. The condition (ii'_1) in [19] for the SU(2) p-function also contains the statistics and extra sign factors

$$p^{\mathcal{O}}(\underline{\theta}, \underline{z}) = \sigma^{\mathcal{O}}\rho(-1)^{m+1}p(\theta_1 + 2\pi i, \theta_2, \dots, \underline{z}) \,.$$

We calculate (for simplicity we skip here all constants, questions of normalization will be considered in appendix C.3)

$$\begin{aligned}
&\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) \\
&= \prod_{2<i<j} \coth\frac{1}{2}\theta_{ij} \sum_l F^{{}^+\mathcal{O}_l}_{\underline{\check{A}'}}(\underline{\check{\theta}}) F^{{}^-\mathcal{O}_l}_{\underline{\check{B}'}}(\underline{\check{\theta}}) \left(\mathbf{C}_{A_1A_2}\mathbf{1}^{\underline{\check{A}'}}_{\underline{\check{A}}} - \mathbf{C}_{A_1A'_2}\,{}^+\sigma_l\rho\,(S_{2n}\dots S_{23})^{\underline{\check{A}'}A'_2}_{A_2\underline{\check{A}}} \right) \\
&\qquad \times \left(\mathbf{C}_{B_1B_2}\mathbf{1}^{\underline{\check{B}'}}_{\underline{\check{B}}} - \mathbf{C}_{B_1B'_2}\,{}^-\sigma_l\rho\,(S_{2n}\dots S_{23})^{\underline{\check{B}'}B'_2}_{B_2\underline{\check{B}}} \right) \Gamma^{AB}_{\underline{\alpha}} \\
&\qquad\qquad = F^{\mathcal{O}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) \left(\mathbf{C}_{\alpha_1\alpha_2}\mathbf{1}^{\underline{\check{\alpha}'}}_{\underline{\check{\alpha}}} - \mathbf{C}_{\alpha_1\alpha'_2}\left(S^{O(4)}_{2n}\dots S^{O(4)}_{23}\right)^{\underline{\check{\alpha}'}\alpha'_2}_{\alpha_2\underline{\check{\alpha}}} \right)
\end{aligned}$$

if

$$\begin{aligned}
\sum_l F^{{}^+\mathcal{O}_l}_{\underline{\check{A}'}}(\underline{\check{\theta}}) F^{{}^-\mathcal{O}_l}_{\underline{\check{B}'}}(\underline{\check{\theta}}) \Big(&\mathbf{C}_{A_1A_2}\mathbf{1}^{\underline{\check{A}'}}_{\underline{\check{A}}}\mathbf{C}_{B_1B'_2}\,{}^-\sigma_l\rho\,(S_{2n}\dots S_{23})^{\underline{\check{B}'}B'_2}_{B_2\underline{\check{B}}} \\
&+ \mathbf{C}_{A_1A'_2}\,{}^+\sigma_l\rho\,(S_{2n}\dots S_{23})^{\underline{\check{A}'}A'_2}_{A_2\underline{\check{A}}}\mathbf{C}_{B_1B_2}\mathbf{1}^{\underline{\check{B}'}}_{\underline{\check{B}}} \Big) \Gamma^{AB}_{\underline{\alpha}} = 0 \qquad \text{(B.3)}
\end{aligned}$$

and

$${}^+\sigma_l\,{}^-\sigma_l = (-1)^{n-1} \,.$$

It was used that

$$\mathbf{C}_{A_1A_2}\mathbf{C}_{B_1B_2}\Gamma^{A_1B_1}_{\alpha_1}\Gamma^{A_2B_2}_{\alpha_2} = -\mathbf{C}_{\alpha_1\alpha_2}$$

and

$$\begin{aligned}
&{}^+\sigma_l\,{}^-\sigma_l\rho^2\mathbf{C}_{A_1A'_2}\mathbf{C}_{B_1B'_2}\Gamma^{A_1B_1}_{\alpha_1}(S_{2n}\dots S_{23})^{\underline{\check{A}'}A'_2}_{A_2\underline{\check{A}}}(S_{2n}\dots S_{23})^{\underline{\check{B}'}B'_2}_{B_2\underline{\check{B}}}\Gamma^{A_2B_2}_{\alpha_2}\Gamma^{\underline{\check{A}\check{B}}}_{\underline{\hat{\alpha}}} \\
&= {}^+\sigma_l\,{}^-\sigma_l(-1)^{n-2}\mathbf{C}_{A_1A'_2}\mathbf{C}_{B_1B'_2}\Gamma^{A_1B_1}_{\alpha_1}\Gamma^{A'_2B'_2}_{\alpha'_2}\Gamma^{\underline{\hat{A}'}\underline{\hat{B}'}}_{\underline{\hat{\alpha}'}}\left(S^{O(4)}_{2n}\dots S^{O(4)}_{23}\right)^{\underline{\check{\alpha}'}\alpha'_2}_{\alpha_2\underline{\check{\alpha}}} \\
&= {}^+\sigma_l\,{}^-\sigma_l(-1)^{n-1}\mathbf{C}_{\alpha_1\alpha'_2}\left(S^{O(4)}_{2n}\dots S^{O(4)}_{23}\right)^{\underline{\check{\alpha}'}\alpha'_2}_{\alpha_2\underline{\check{\alpha}}} = \mathbf{C}_{\alpha_1\alpha'_2}\left(S^{O(4)}_{2n}\dots S^{O(4)}_{23}\right)^{\underline{\check{\alpha}'}\alpha'_2}_{\alpha_2\underline{\check{\alpha}}}
\end{aligned}$$

where (4.13) has been used. The condition (B.3) holds if

$$\sum_l {}^{\pm}\sigma_l\, F^{{}^+\mathcal{O}_l}_{\underline{A}}(\underline{\theta})\, F^{{}^-\mathcal{O}_l}_{\underline{B}}(\underline{\theta})\Gamma^{AB}_{\underline{\alpha}} = 0 \,. \qquad \text{(B.4)}$$

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Scalar operators. Let us first consider an $O(4)$ iso scalar operator $\mathcal{O}$, then n is even and $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ is of the form

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = \sum_{\pi\in S'_n} f_\pi(\underline{\theta})\mathbf{C}_{\pi\alpha_1\pi\alpha_2}\dots\mathbf{C}_{\pi\alpha_{n-1}\pi\alpha_n}$$

where the set S'_n contains all $n!/(2^{n/2}(n/2)!)$ permutations of $\{1,\dots,n\}$ with the restrictions

$$\begin{aligned}&\pi\alpha_1 < \pi\alpha_2,\dots,\pi\alpha_{n-1} < \pi\alpha_n\\ &\pi\alpha_1 < \pi\alpha_3 < \dots < \pi\alpha_{n-1}\,.\end{aligned}$$

Obviously, if the special components $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ with $\alpha_i \in \{1,\bar{1}\}$ vanish, then all $f_\pi(\underline{\theta})$ vanish and $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = 0$ for all $\underline{\alpha}$.

For the case of two term of the l-sum in (B.4) and

$$\begin{aligned}{}^{+}\mathcal{O}_1 = {}^{-}\mathcal{O}_2 &= \mathcal{O}_1\\ {}^{+}\mathcal{O}_2 = {}^{-}\mathcal{O}_1 &= \mathcal{O}_2\\ \sigma_1 = -\sigma_2\,,\ \sigma_1\sigma_2 = -1 &\Rightarrow \sigma_1 = \pm 1\end{aligned}$$

we have in (B.4)

$$\sum_l {}^{\pm}\sigma_l\, F_{\underline{A}}{}^{{}^{+}\mathcal{O}_l}(\underline{\theta})\, F_{\underline{B}}{}^{{}^{-}\mathcal{O}_l}(\underline{\theta})\Gamma^{AB}_{\underline{\alpha}} = \sigma_1\left(F_{\underline{A}}{}^{\mathcal{O}_1}(\underline{\theta})\, F_{\underline{B}}{}^{\mathcal{O}_2}(\underline{\theta}) - F_{\underline{A}}{}^{\mathcal{O}_2}(\underline{\theta})\, F_{\underline{B}}{}^{\mathcal{O}_1}(\underline{\theta})\right)\Gamma^{AB}_{\underline{\alpha}} = 0$$

because for $\alpha_i \in \{1,\bar{1}\}$ the symmetry $\Gamma^{AB}_{\underline{\alpha}} = \Gamma^{BA}_{\underline{\alpha}}$ holds (see (4.15)). As an example of this construction see that for the energy momentum in subsection 5.3 and appendix C.4.

Vector operators. For the highest weight component of an iso vector $O(4)$ operator the number n is odd and the form factors are of the form

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = \sum_{\pi\in S'_n} f_\pi(\underline{\theta})\mathbf{C}_{\pi\alpha_1\pi\alpha_2}\dots\mathbf{C}_{\pi\alpha_{n-2}\pi\alpha_{n-1}}\delta^1_{\pi\alpha_n}$$

with the restrictions

$$\begin{aligned}&\pi\alpha_1 < \pi\alpha_2,\dots,\pi\alpha_{n-2} < \pi\alpha_{n-1}\\ &\pi\alpha_1 < \pi\alpha_3 < \dots < \pi\alpha_{n-2}\,.\end{aligned}$$

As above, if the special components $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ with $\alpha_i \in \{1,\bar{1}\}$ vanish, then all $f_\pi(\underline{\theta})$ vanish and $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = 0$ for all $\underline{\alpha}$. For the case of two term of the l-sum in (B.4) and

$$\begin{aligned}{}^{+}\mathcal{O}_1 = {}^{-}\mathcal{O}_2 &= \mathcal{O}_1\\ {}^{+}\mathcal{O}_2 = {}^{-}\mathcal{O}_1 &= \mathcal{O}_2\\ \sigma_1 = -\sigma_2\,,\ \sigma_1\sigma_2 = 1 &\Rightarrow \sigma_1 = \pm i\end{aligned}$$

again (B.4) holds. As an example of this construction see that for field in subsection 5.1.

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Anti-symmetric tensor operators. The construction is similar as above, however, one needs 4 operators as for the current in subsection 5.2.

The highest level off-shell Bethe ansatz for even N. For these constructions one has to apply a modification of the $O(4)$ construction above because the shift in equation (4.23) is not that of $O(4)$ but that of $O(N)$ (see appendix C.3).

C Higher level K-functions

C.1 Proof of lemma 4

Lemma 4 also holds for $k = 0$ in (4.24), if Res is replaced by $\overset{(1)}{\text{Res}}$ as explained in appendix A. For convenience we use here the variables u and v with $\theta = i\pi\nu_k u$, $z = i\pi\nu_k v$ and $\nu_k = 2/(N-2k-2)$ (for the S-matrix see (4.20).

Proofs. The relations $(\text{i})^{(k)}$ and $(\text{ii})^{(k)}$ follow as above in the proof of theorem 3 from the results of [6]. To prove $(\text{iii})^{(k)}$ we calculate

$$\operatorname*{Res}_{u_{12}=1/\nu_k} K^{(k)}_{1\dots n_k}(\underline{u}) = \operatorname*{Res}_{u_{12}=1/\nu_k} \tilde{N}^{(k)}_{m_k} \int_{\mathcal{C}^1_{\underline{u}}} dv_1 \cdots \int_{\mathcal{C}^{m_k}_{\underline{u}}} dv_{m_k}\, \tilde{h}(\underline{u},\underline{v})\, \tilde{\Psi}^{(k)}_{1\dots n_k}(\underline{u},\underline{v})\,. \tag{C.1}$$

For j even contours will be "pinched" at $v_j = u_2 \approx u_1 - 1/\nu_k$.[3] Due to symmetry it is sufficient to determine the contribution from one of the v_j and multiply the result by $\left[\frac{1}{2}m_k\right]$. We take for convenience $v_j = v_2$, then the contribution is given by the v_2 integration $\oint_{u_2} dv_2 \cdots$ along small circle around $v_2 = u_2$ (see figure 1). The S-matrix $\tilde{S}(u_2 - v_2)$ yields the permutation operator $\tilde{S}(0) = \mathbf{P}$ and the S-matrix $\tilde{S}(u_1 - v_2)$ yields $\mathbf{K}$ after taking $\text{Res}_{u_{12}=1/\nu_k} \tilde{S}(u_{12})$. In the representation of the Bethe state (2.17)

$$\left(\tilde{\Phi}^{(k)}\right)^{\mathring{\beta}}_{\underline{\alpha}}(\underline{u},\underline{v}) = \left(\left(\Pi^{(k)}\right)^{\mathring{\beta}}_{\underline{\beta}}(\underline{v})\Omega_k \left(\tilde{T}^{(k)}\right)^{\beta_{m_k}}_{k+1}(\underline{u},v_{m_k}) \dots \left(\tilde{T}^{(k)}\right)^{\beta_1}_{k+1}(\underline{u},v_1)\right)_{\underline{\alpha}}$$

we may move for generic values of the other v_j the operator $\left(\tilde{T}^{(k)}\right)^{\beta_2}_{k+1}(\underline{u},v_2)$ to the left by means of the $TTS = STT$ commutation rule (14) of [6] and (2.18) (using the short notation $\tilde{T}_k(v_i) = \tilde{T}_k(\underline{u},v_i)$)

$$\Pi_k\Omega_k\tilde{T}_k(v_m)\dots\tilde{T}_k(v_2)\tilde{T}_k(v_1) = \tilde{S}_{k+1}(v_{32})\dots\tilde{S}_{k+1}(v_{m2})\Pi_k\Omega_k\tilde{T}_k(v_2)\tilde{T}_k(v_m)\dots\tilde{T}_k(v_1).$$

Because of (2.19) $\left(\Pi^{(k)}\right)^{\dots\mathring{\beta}_{m_k}}_{\dots k+1} = 0$ we find $\text{Res}_{u_{12}=1/\nu_k}\Pi_k\Omega_k\tilde{T}_k(\underline{u},v_2) = 0$. However, the pole of $\left(L^{(k)}\right)_{1\dots m_k}(\underline{v})$ (see e.g. (4.24)) at $v_{12} = 1/\nu_{k+1}$ will produce a singular contribution from the v_i-integration $\oint_{u_1-1} dv_i \cdots$ (which is a part of $\int_{\mathcal{C}^o_u} dv_i \cdots$) for i odd. We have a $0/0$ situation which we can resolve as follows.

[3] For $k = 0$ there is a second pinching point at $v_j = u_1 - 2/\nu_k \approx u_2 - 1/\nu_k$ as explained in appendix A.

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We take $i,j=1,2$ and multiply the result by $\left[\frac{1}{2}m_k\right]\left[\frac{1}{2}m_k+\frac{1}{2}\right]$ and shift again $\tilde{T}_k(v_1)$ and $\tilde{T}_k(v_2)$ through all the other $\tilde{T}_k(v_i)$ as above

$$\begin{aligned}
&\Pi^{(k)}_{1\dots m_k}\left(\Omega_k\big(\tilde{T}_k\big)^{m_k}\dots\big(\tilde{T}_k\big)^2\big(\tilde{T}_k\big)^1\right)_{\underline{\alpha}}\\
&=\Pi^{(k)}_{1\dots m_k}\tilde{S}^{(k)}_{21}\dots\tilde{S}^{(k)}_{m_k1}\tilde{S}^{(k)}_{32}\dots\tilde{S}^{(k)}_{m_k2}\left(\Omega_k\big(\tilde{T}_k\big)^1\big(\tilde{T}_k\big)^2\big(\tilde{T}_k\big)^{m_k}\dots\big(\tilde{T}_k\big)^3\right)_{\underline{\alpha}}\\
&=\tilde{S}^{(k+1)}_{21}\dots\tilde{S}^{(k+1)}_{m_k1}\tilde{S}^{(k+1)}_{32}\dots\tilde{S}^{(k+1)}_{m_k2}\Pi^{(k)}_{3\dots m_k21}\left(\Omega_k\big(\tilde{T}_k\big)^1\big(\tilde{T}_k\big)^2\big(\tilde{T}_k\big)^{m_k}\dots\big(\tilde{T}_k\big)^3\right)_{\underline{\alpha}}.
\end{aligned}$$

Applying this to $L^{(k)}_{1\dots m_k}(\underline{v})$ using (4.22) for higher levels we get

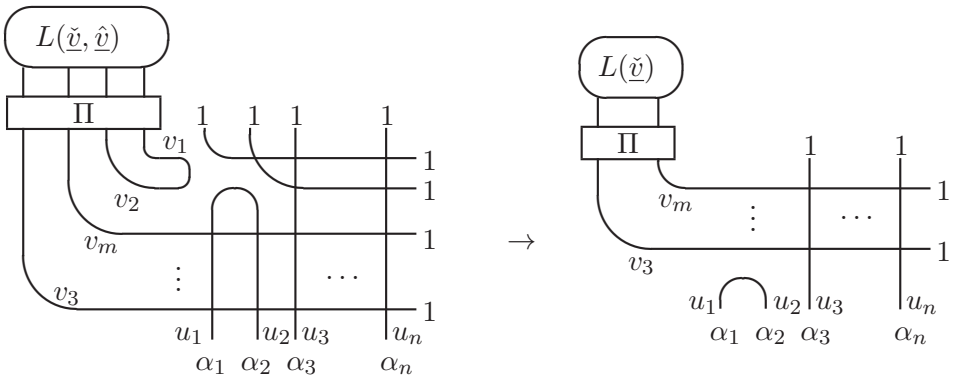
$$\tilde{\Psi}^{(k)}_{\underline{\alpha}}(\underline{u},\underline{v})=L^{(k)}_{3\dots m_k21}\Pi^{(k)}_{3\dots m_k21}\left(\Omega_k\big(\tilde{T}_k\big)^1\big(\tilde{T}_k\big)^2\big(\tilde{T}_k\big)^{m_k}\dots\big(\tilde{T}_k\big)^3\right)_{\underline{\alpha}}$$

For $u_{12}\approx 1/\nu_k$, $v_1\approx u_1-1$ and $v_2\approx u_2$ (i.e. $v_{12}\approx 1/\nu_k-1=1/\nu_{k+1}$) we may replace inside $\tilde{\Psi}^{(k)}_{\underline{\alpha}}$ the S-matrices

$$\begin{aligned}
&\tilde{S}_k(u_1-v_1)\to\tilde{c}(u_1-v_1)\mathbf{P}\\
&\tilde{S}_k(u_2-v_1)\to 1\\
&\tilde{S}_k(u_1-v_2)\to\tilde{d}_k(u_{12})\mathbf{K}\\
&\tilde{S}_k(u_2-v_2)\to\mathbf{P}\,.
\end{aligned}$$

For the first relation it has been used that $\tilde{b}(u)\approx-\tilde{c}(u)\approx 1/(u-1)$ and the only nonvanishing contribution from $\tilde{S}_k(u_1-v_1)$ is $\tilde{b}(u_1-v_1)\,(\mathbf{1}-\mathbf{P}+const\,\mathbf{K})^{\mathring{\beta}1}_{\mathring{\beta}'1}=\tilde{c}(u_1-v_1)\mathbf{P}^{\mathring{\beta}1}_{\mathring{\beta}'1}$ because $(\mathbf{1}-\mathbf{P})^{11}_{\alpha\beta}=\mathbf{K}^{11}_{\alpha\beta}=0$ and moreover $\big(\Pi^{(k)}\big)^{\dots\mathring{\beta}_{m_k}}_{\dots\overline{k+1}}=0$ holds (see (2.19)) . Therefore we may replace (see figure for $k=0$)

$$\begin{aligned}
\tilde{\Psi}^{(k)}_{\underline{\alpha}}(\underline{u},\underline{v})\to\tilde{c}(u_1-v_1)\tilde{d}_k(u_{12})L^{(k)}_{3\dots m_k21}(\underline{v}')\mathring{\mathbf{C}}^{21}\\
\times\mathbf{C}_{\hat{\alpha}}\prod_{j=3}^{m_k}\frac{1}{a_k(u_1-v_j)a_k(u_2-v_j)}\Pi^{(k)}_{3\dots m_k}\left(\Omega_k\left(\tilde{T}_k\right)^{m_k}\dots\left(\tilde{T}_k\right)^3\right)_{\check{\underline{\alpha}}}.
\end{aligned}\tag{C.2}$$



Note that unitarity and crossing imply for $u_{12} = 1/\nu_k$

$$\begin{aligned}
&\mathbf{C}_{12}\tilde{S}^{(k)}(u_1 - v_{m_k})\dots\tilde{S}^{(k)}(u_1 - v_3)\tilde{S}^{(k)}(u_2 - v_{m_k})\dots\tilde{S}^{(k)}(u_2 - v_3)\\
&= \frac{\mathbf{C}_{12}S^{(k)}(u_1 - v_{m_k})\dots S^{(k)}(u_1 - v_3)S^{(k)}(u_2 - v_{m_k})\dots S^{(k)}(u_2 - v_3)}{a_k(u_1 - v_{m_k})\dots a_k(u_1 - v_3)a_k(u_2 - v_{mk})\dots a_k(u_2 - v_3))}\\
&= \mathbf{C}_{12}\prod_{j=3}^{m_k}\frac{\mathbf{1}}{a_k(u_1 - v_j)a_k(u_2 - v_j)}\,.
\end{aligned}$$

We calculate for $v_{12} \to 1/\nu_{k+1}$

$$L^{(k)}_{3\dots m_k 21}(\underline{v}')\mathring{\mathbf{C}}^{21} = \left(\operatorname*{Res}_{v=1/\nu_{k+1}}\tilde{d}_{k+1}(v)\right)^{-1}\prod_{j=3}^{m_k}a_{k+1}(v_{1j})a_{k+1}(v_{2j})\tilde{\psi}(v_{j1}+1)\tilde{\chi}(v_{j2})L^{(k)}_{3\dots n}(\check{\underline{v}})\,.$$

It has been used that from (4.24) and (4.22) follows

$$\begin{aligned}
&\prod_{j=3}^{m_k}\tilde{\psi}(v_{j1}+1)\tilde{\chi}(v_{j2})L^{(k)}_{3\dots n}(\check{\underline{v}})\mathring{\mathbf{C}}_{12}\\
&= \operatorname*{Res}_{v_{12}=1/\nu_{k+1}}L^{(k)}_{12\dots m}(\underline{v})\\
&= L^{(k)}_{21\dots m}\operatorname*{Res}_{v_{12}=1/\nu_{k+1}}\tilde{S}^{(k+1)}_{12}(v_{12}) = L^{(k)}_{21\dots m_k}\operatorname*{Res}_{v_{12}=1/\nu_{k+1}}\tilde{d}_{k+1}(v_{12})\mathring{\mathbf{K}}_{12}\\
&= L^{(k)}_{3\dots m_k 21}(\check{\underline{v}},\hat{\underline{v}})\tilde{S}^{(k+1)}_{2m_k}\dots\tilde{S}^{(k+1)}_{23}\tilde{S}^{(k+1)}_{1m_k}\dots\tilde{S}^{(k+1)}_{13}\operatorname*{Res}_{v_{12}=1/\nu_{k+1}}\tilde{d}_{k+1}(v_{12})\mathring{\mathbf{C}}^{21}\mathring{\mathbf{C}}_{12}\\
&= L^{(k)}_{3\dots m_k 21}(\check{\underline{v}},\hat{\underline{v}})\mathring{\mathbf{C}}^{21}\prod_{j=3}^{m_k}\frac{1}{a_{k+1}(v_{1j})a_{k+1}(v_{2j})}\operatorname*{Res}_{v=1/\nu_{k+1}}\tilde{d}_{k+1}(v)\mathring{\mathbf{C}}_{12}\,.
\end{aligned}$$

where again unitarity and crossing for $v_{12} = 1/\nu_{k+1}$ has been used. We write

$$\begin{aligned}
\tilde{h}(\underline{u},\underline{v}) &= \prod_{i=1}^{n}\prod_{j=1}^{m_k}\tilde{\phi}_j(u_i - v_j)\prod_{1\le i<j\le m_k}\tau_{ij}(v_{ij})\\
&= \left(\prod_{i=1}^{2}\prod_{j=1}^{2}\tilde{\phi}_j(u_i - v_j)\right)\left(\prod_{j=3}^{m_k}\tilde{\phi}_j(u_1 - v_j)\tilde{\phi}_j(u_2 - v_j)\right)\left(\prod_{i=3}^{n_k}\tilde{\psi}(u_i - v_1)\tilde{\chi}(u_i - v_2)\right)\\
&\quad\times\tau_{12}(v_{12})\prod_{j=3}^{m_k}\left(\tau_{1j}(v_{1j})\tau_{2j}(v_{2j})\right)\tilde{h}(\check{\underline{u}},\check{\underline{v}})
\end{aligned}$$

and obtain finally

$$\begin{aligned}
&\operatorname*{Res}_{u_{12}=1/\nu_k}K^{(k)}_{\underline{\alpha}}(\underline{u}) = \tilde{N}^{(k)}_{m_k}\left[\frac{1}{2}m_k\right]\left[\frac{1}{2}m_k+\frac{1}{2}\right]\\
&\times\left(\operatorname*{Res}_{v=1/\nu_{k+1}}\tilde{d}_{k+1}(v)\right)^{-1}\operatorname*{Res}_{u_{12}=1/\nu_k}\oint_{u_1-1}dv_1\tilde{c}(u_1 - v_1)\left(-\oint_{u_2}\right)dv_2\\
&\times\tilde{d}_k(u_{12})\left(\prod_{i=1}^{2}\prod_{j=1}^{2}\tilde{\phi}_j(u_i - v_j)\right)\tau_{12}(v_{12})\prod_{i=3}^{n_k}\tilde{\psi}(u_{i1}+1)\tilde{\chi}(u_{i2})\frac{1}{\tilde{N}^{(k)}_{m_k-2}}\mathbf{C}_{\hat{\alpha}}K^{(k)}_{\check{\underline{\alpha}}}(\check{\underline{u}})
\end{aligned}$$

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It has been used that for $u_{12}=1/\nu_k,\ v_{12}=1/\nu_{k+1},\ u_2=v_2,\ u_1=v_2+1/\nu_k=v_1+1$

$$\left(\frac{a_{k+1}(v_{1j})a_{k+1}(v_{2j})}{a_k(u_1-v_j)a_k(u_2-v_j)}\tilde\psi(v_{j1}+1)\tilde\chi(v_{j2})\right)\left(\tilde\phi_j(u_1-v_j)\tilde\phi_j(u_2-v_j)\tau_{1j}(v_{1j})\tau_{2j}(v_{2j})\right)=1$$

for odd and even j, which can be shown by means of (4.2) and the formula

$$a_k(u_1)a_k(u_2)=\tilde b(-u_2)/\tilde b(u_1)\,.$$

The result is that equation (4.24) holds if

$$\frac{\tilde N^{(k)}_{m_k}}{\tilde N^{(k)}_{m_k-2}}\left[\frac12 m_k\right]\left[\frac12 m_k+\frac12\right]\left(\operatorname*{Res}_{v=1/\nu_{k+1}}\tilde d_{k+1}(v)\right)^{-1}\left(-(2\pi i)^2\right) \tag{C.3}$$

$$\times\operatorname*{Res}_{v_1=u_1-1}\tilde c(u_1-v_1)\operatorname*{Res}_{u_{12}=1/\nu_k}\tilde d_k(u_{12})\tilde\psi(1)\tilde\psi(u_{21}+1)\tilde\chi(u_{12})\operatorname*{Res}_{v_2=u_2}\tilde\chi(u_2-v_2)\tau_{12}(v_{12})$$

$$=\frac{\tilde N^{(k)}_{m_k}}{\tilde N^{(k)}_{m_k-2}}\left[\frac12 m_k\right]\left[\frac12 m_k+\frac12\right]4\pi^2\tilde\psi^2(1)\frac{\tilde\chi(1/\nu_k-2/\nu)}{\tilde\chi(1/\nu_{k+1}-2/\nu)}=1 \tag{C.4}$$

which follows from the assumption (4.26).

In particular for $k=0$ we have

$$\tilde N^{(0)}_m=\frac{1}{\left[\frac12 m\right]\left[\frac12 m+\frac12\right]}\frac{1}{4\pi^2\tilde\psi^2(1)}\frac{\tilde\chi(1/\mathring\nu-2/\nu)}{\tilde\chi(1/\nu-2/\nu)}\tilde N^{(0)}_{m-2}$$

and (A.4) with (1.3), (4.21) and (C.1) implies (4.10). ■

C.2 Proof of lemma 5

Proofs. For N odd and $k=M=(N-3)/2$ we have $\nu_k=2$ and $\nu_{k+1}=-2$ as for $O(3)$, however, the shift in (4.23) is not that of $O(3)$ but that of $O(N)$. We proceed as above for general k, for $N>3$ there is for all j even "pinching" at $v_j=u_2\approx u_1-1/2$ and we take $j=1,2$ as a pair of an odd and an even j and multiply the result with $\left[\frac12 m_M\right]\left[\frac12 m_M+\frac12\right]$. The L-function $L^{(M)}_{1\dots m_M}(\underline v)=K^{(M+1)}_{1\dots m_M}(\underline v)$ is a c-number satisfying (4.22) with

$$\tilde S_0^{(M+1)}(v)=\frac{v+1/\nu_{M+1}}{v-1/\nu_{M+1}}\frac{v+1}{v-1}=\frac{v-1/2}{v+1/2}\frac{v+1}{v-1}$$

and (4.23) with the solution (as in subsection 4.2)

$$L^{(M)}(\underline v)=\prod_{1\le i<j\le m_M}L^{(M)}(v_{ij})\,,\quad L^{(M)}(u)=\frac{\Gamma\left(\frac14\nu+\frac12\nu u\right)\Gamma\left(1+\frac14\nu-\frac12\nu u\right)}{\Gamma\left(\frac12\nu+\frac12\nu u\right)\Gamma\left(1+\frac12\nu-\frac12\nu u\right)}\,.$$

We have again (C.2) where here

$$L^{(M)}_{3\dots m_k21}(\underline v')\mathring{\mathbf{C}}^{21}\to L^{(M)}(\breve{\underline v})\prod_{j=3}^{m_M}\left(L^{(M)}(v_{j1})L^{(M)}(v_{j2})\right)L^{(M)}(v_{21})$$

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and instead of (C.3)

$$\frac{\tilde{N}^{(M)}_{m_M}}{\tilde{N}^{(M)}_{m_M-2}}\left[\frac{1}{2}m_M\right]\left[\frac{1}{2}m_M+\frac{1}{2}\right]L^{(M)}(v_{21})\left(-(2\pi i)^2\right)$$
$$\times \operatorname*{Res}_{v_1=u_1-1}\tilde{c}(u_1-v_1)\operatorname*{Res}_{u_{12}=1/\nu_M}\tilde{d}_M(u_{12})\tilde{\psi}(1)\tilde{\psi}(u_{21}+1)\tilde{\chi}(u_{12})\operatorname*{Res}_{v_2=u_2}\tilde{\chi}(u_2-v_2)\tau_{12}(v_{12})$$
$$=\frac{\tilde{N}^{(M)}_{m_M}}{\tilde{N}^{(M)}_{m_M-2}}\left[\frac{1}{2}m_M\right]\left[\frac{1}{2}m_M+\frac{1}{2}\right]\frac{2\pi^2\Gamma\left(1-\frac{1}{4}\nu\right)}{\Gamma^3\left(1+\frac{1}{4}\nu\right)\Gamma\left(1+\frac{1}{2}\nu\right)}=1\,,$$

which is (4.28). For $N=3$ there is for all j even and odd "pinching" at $v_j=u_2\approx u_1-1/2$ and $\left[\frac{1}{2}m_M\right]\left[\frac{1}{2}m_M+\frac{1}{2}\right]$ has to be replaced $m\,(m-1)$. It was used that for $u_{12}=1/2$ and $v_{12}=-1/2$

$$\frac{L^{(M)}(v_{j1})L^{(M)}(v_{j2})}{a_M(u_1-v_j)a_M(u_2-v_j)}\left(\tilde{\phi}_j(u_1-v_j)\tilde{\phi}_j(u_2-v_j)\tau_{1j}(v_{1j})\tau_{2j}(v_{2j})\right)=1$$

which follows as above because

$$L^{(M)}(-v_1)L^{(M)}(-v_2)=\tilde{\psi}(-v_1+1)\tilde{\chi}(-v_2)\tilde{b}(-v_2)/\tilde{b}(v_1)\,.$$

■

C.3 Proof of lemma 6

Proofs. For N even and $k=M=(N-4)/2$ we have $\nu_k=1$ as for $O(4)$, however, the shift in (4.23) is not that of $O(4)$ but that of $O(N)$. We use the technique of subsection 4.3 with the SU(2) S-matrix (see [19, 41, 45])

$$S^{\mathrm{SU}(2)}(u)=a^{\mathrm{SU}(2)}(u)\left(\frac{u}{u-1}\mathbf{1}-\frac{1}{u-1}\mathbf{P}\right)\,.$$

First we calculate $\operatorname*{Res}^{(1)}_{u_{12}=1}{}^{\pm}K_{\underline{A}}(\underline{u})$ which is due to the pinching at $v_1=u_2\to u_1-1$ and gives the first term in (B.2). The contribution is given by v_1-integration $\oint_{u_2}dv_1\cdots$ along small circles around $v_1=u_2$. For $u_{12}\approx 1$, $v_1\approx u_2$ we may replace the S-matrices inside of $\tilde{\Psi}^{\mathrm{SU}(2)}_{\underline{A}}$

$$\tilde{S}(u_2-v_1)\to\mathbf{P}$$
$$\tilde{S}(u_1-v_1)\to\tilde{S}(u_{12})\to\frac{1}{u_{12}-1}\left(\mathbf{1}-\mathbf{P}\right)$$

such that we may replace

$$\tilde{\Psi}^{\mathrm{SU}(2)}_{\underline{A}}(\underline{u},\underline{v})\to\frac{1}{u_{12}-1}\mathbf{C}_{A_1A_2}\prod_{j=2}^{m}\tilde{b}(u_1-v_j)\left(\Omega\tilde{C}(\breve{\underline{u}},v_m)\cdots\tilde{C}(\breve{\underline{u}},v_N)\right)_{\breve{\underline{A}}}\,.$$

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because $(\mathbf{1}-\mathbf{P})^{21}_{A_1A_2}=\mathbf{C}_{A_1A_2}$. We write

$$\tilde{h}(\underline{u},\underline{v})=\Big(\tilde{\phi}_\nu(u_1-v_1)\tilde{\phi}_\nu(u_2-v_1)\Big)\prod_{j=2}^{m_\pm}\Big(\tilde{\phi}_\nu(u_1-v_j)\tilde{\phi}_\nu(u_2-v_j)\tau_\nu(v_{1j})\Big)\prod_{i=3}^{n_M}\tilde{\phi}_\nu(u_{i2})\tilde{h}(\breve{\underline{u}},\breve{\underline{v}})$$

and get

$$\begin{aligned}
&\operatorname*{Res}_{u_{12}=1}^{(1)}{}^{\pm}K_{\underline{A}}(\underline{u})\\
&=\frac{{}^{\pm}\tilde{N}_{m_\pm}}{{}^{\pm}\tilde{N}_{m_\pm-1}}m_\pm\operatorname*{Res}_{u_{12}=1}\frac{1}{u_{12}-1}\left(\tilde{\phi}_\nu(u_{12})\oint_{u_2}dv_1\tilde{\phi}_\nu(u_2-v_1)\right)\prod_{i=3}^{n_M}\tilde{\phi}_\nu(u_{i2})\mathbf{C}_{A_1A_2}{}^{\pm}K_{\underline{\breve{A}}}(\breve{\underline{u}})\\
&=\frac{{}^{\pm}\tilde{N}_{m_\pm}}{{}^{\pm}\tilde{N}_{m_\pm-1}}m_\pm 2\pi i(-1)^{m_\pm}\Gamma^2\left(-\frac{1}{2}\nu\right)\prod_{i=3}^{n_M}\tilde{\phi}_\nu(u_{i2})\mathbf{C}_{A_1A_2}{}^{\pm}K_{\underline{\breve{A}}}(\breve{\underline{u}})
\end{aligned}$$

where we have used the identity

$$\tilde{\phi}_\nu(u_1)\tilde{\phi}_\nu(u_2)\tau_\nu(u_2)\tilde{b}(u_1)=-1$$

because of $\tilde{\phi}_\nu(u_1)\tilde{b}(u_1)=-\tilde{\phi}_\nu(-u_2)$. Finally we have

$$\operatorname*{Res}_{u_{12}=1}^{(1)}{}^{\pm}K_{\underline{A}}(\underline{u})=\prod_{i=3}^{n_M}\tilde{\phi}_\nu(u_{i2})\mathbf{C}_{A_1A_2}{}^{\pm}K_{\underline{\breve{A}}}(\breve{\underline{u}})$$

if

$${}^{\pm}\tilde{N}_{m_\pm}=\frac{(-1)^{m_\pm}}{m_\pm 2\pi i\Gamma^2\left(-\frac{1}{2}\nu\right)}{}^{\pm}\tilde{N}_{m_\pm-1}.$$

Now we take the residue of (4.30)

$$\begin{aligned}
&\operatorname*{Res}_{u_{12}=1}^{(1)}K^{(M)}_{\underline{\alpha}}(\underline{u})=d_{n_M}\left(\frac{1}{2}\pi\nu\right)\prod_{2<i<j}\sin\frac{1}{2}\pi\nu\,(u_{ij}-1)\\
&\quad\times\prod_{j=3}^{n_M}\left(\sin\frac{1}{2}\pi\nu\,(u_{1j}-1)\sin\frac{1}{2}\pi\nu\,(u_{2j}-1)\right)\sum_l\operatorname*{Res}_{u_{12}=1}^{(1)}{}^{+}K^{(l)}_{\underline{A}}(\underline{u})\operatorname*{Res}_{u_{12}=1}^{(1)}{}^{-}K^{(l)}_{\underline{B}}(\underline{u})\Gamma^{\underline{AB}}_{\underline{\alpha}}\\
&=-\frac{d_{m_M}}{d_{m_M-2}}\left(\frac{1}{2}\pi\nu\right)\pi^2\prod_{j=3}^{m_M}\tilde{\psi}(u_{i1}+1)\tilde{\chi}(u_{i2})\mathbf{C}_{\underline{\hat{\alpha}}}K^{(k)}_{\underline{\breve{\alpha}}}(\breve{\underline{u}})=\prod_{j=3}^{m_M}\tilde{\psi}(u_{i1}+1)\tilde{\chi}(u_{i2})\mathbf{C}_{\underline{\hat{\alpha}}}K^{(k)}_{\underline{\breve{\alpha}}}(\breve{\underline{u}})
\end{aligned}$$

if

$$d_{n_M}=-\frac{2}{\nu\pi^3}d_{n_M-2}\,.$$

We used that $\mathbf{C}_{A_1A_2}\mathbf{C}_{B_1B_2}\Gamma^{A_1B_1}_{\alpha_1}\Gamma^{A_2B_2}_{\alpha_2}=-\mathbf{C}_{\alpha_1\alpha_2}$ and

$$\Big(\tilde{\phi}_\nu(u)\Big)^2=\frac{\pi^2}{\sin\frac{1}{2}\pi\nu\,(u-1)\,\sin\frac{1}{2}\pi\nu u}\tilde{\psi}(u)\tilde{\chi}(u)\,.$$

■

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C.4 Two-particle higher level K-functions

For the examples of section 5 we need higher level K-functions. In particular $K^{(k)}_{\alpha_1\alpha_2}(\theta_1,\theta_2)$ (level $k=0,1,2,\dots$) belonging to $O(N-2k)$ in the iso-scalar two-particle channel (with weights $w=(0,\dots,0)$).

Lemma 7 *The vector valued functions*

$$K^{(k)}_{\alpha_1\alpha_2}(\theta_1,\theta_2)=\mathbf{C}^{(N-2k)}_{\alpha_1\alpha_2}K_k(\theta_{12})$$

with

$$K_k(\theta)=\frac{\Gamma\left(-\frac{1}{2}(1-k\nu)+\frac{1}{2}(1-k\nu)\,\frac{\theta}{\pi i}\right)\Gamma\left(1-\frac{1}{2}(1-k\nu)-\frac{1}{2}(1-k\nu)\,\frac{\theta}{\pi i}\right)}{\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}(1-k\nu)\,\frac{\theta}{\pi i}\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}(1-k\nu)\,\frac{\theta}{\pi i}\right)} \tag{C.5}$$

satisfy for $k=0,1,2,\cdots<N/2-2$ the recursion relation (for a suitable normalization)

$$K^{(k)}_{\underline{\alpha}}(\underline{\theta})=N^{(k)}_n\int_{\mathcal{C}^o_{\underline{\theta}}}dz_1\int_{\mathcal{C}^e_{\underline{\theta}}}dz_2\,\tilde{h}(\underline{\theta},\underline{z})\,L^{(k)}_{\underline{\mathring{\beta}}}(\underline{z})\tilde{\Phi}^{(k)}\,{}^{\mathring{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z})$$
$$L^{(k)}_{\underline{\mathring{\beta}}}(\underline{z})=K^{(k+1)}_{\underline{\mathring{\beta}}}(\underline{z}\nu_{k+1}/\nu_k)\,,\quad \nu_k=2/(N-2k-2)$$

with

$$\tilde{h}(\underline{\theta},\underline{z})=\prod_{i=1}^{2}\left(\tilde{\psi}(\theta_i-z_1)\tilde{\chi}(\theta_i-z_2)\right)\frac{1}{\tilde{\chi}(z_{12})\tilde{\psi}(-z_{12})}$$
$$\tilde{\Phi}^{(k)}\,{}^{\mathring{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z})=\left(\Pi^{\mathring{\beta}}_{\underline{\beta}}(\underline{z})\Omega\tilde{T}_1^{\beta_2}(\underline{\theta},z_2)\tilde{T}_1^{\beta_1}(\underline{\theta},z_1)\right)^{(k)}_{\underline{\alpha}}\,.$$

The K-function $K^{(0)}_{\underline{\alpha}}(\underline{\theta})$ belongs to an iso-scalar, spin zero operator (with p-function $p=1$). This means it belongs to the energy momentum potential (see example 5.3 formula (5.19))

$$F^T_{\underline{\alpha}}(\theta)=K^{(0)}_{\underline{\alpha}}(\underline{\theta})\,F(\theta)\,.$$

The L-function $L^{(0)}_{\underline{\mathring{\beta}}}(\underline{z})=K^{(1)}_{\underline{\mathring{\beta}}}(\underline{z}\nu_{k+1}/\nu_k)$ is for $n=2$ that of (5.3).

Proofs. We do not calculate the integrals in (4.21) for general N, but we use arguments of [15] to prove the claim. In addition we calculate integrals explicitly for $N\to\infty$, $N=3$ and $N=4$ (see below). That for all levels $n_k=2$ is follows from

$$\begin{aligned}w&=(w_1,\dots,w_{[N/2]})=(0,\dots,0)\\&=\begin{cases}(n-n_1,\dots,n_{[N/2]-1}-n_{[N/2]}) & \text{for } N \text{ odd}\\(n-n_1,\dots,n_{[N/2]-2}-n_--n_+,n_--n_+) & \text{for } N \text{ even}\,.\end{cases}\end{aligned}$$

Note that for N even we have $n_-=n_+=1$ (see below). For convenience we use here the parameterization $\theta=i\pi\nu_k u$ and $z=i\pi\nu_k v$, ($\nu_k=2/(N-2k-2)$) and the S-matrix $S^{(k)}(u)=S^{O(N-2k)}(u)$ (see (4.20)).

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Theorem 3 (with the proof in appendix A) implies that $K_k(\underline{u})$ defined by (4.21) satisfies

$$\begin{aligned}(\mathrm{i}): &\quad K_k(u) = K_k(-u)\tilde{S}_0^{(k)}(u)\\ (\mathrm{ii}): &\quad K_k(1/\nu - u) = K_k(1/\nu + u)\end{aligned}$$

with (see (2.7))

$$\tilde{S}_0^{(k)}(u) = S_0^{(k)}(u)/S_+^{(k)}(u) = \frac{u+1/\nu_k}{u-1/\nu_k}\frac{u+1}{u-1} = \frac{u+(1/\nu-k)}{u-(1/\nu-k)}\frac{u+1}{u-1}\,.$$

The minimal solution of (i) and (ii) is

$$K_k^{\min}(u) = \frac{1}{\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\nu u\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\nu u\right)\Gamma\left(1+\frac{1}{2}\left(1-k\nu\right)-\frac{1}{2}\nu u\right)\Gamma\left(\frac{1}{2}\left(1-k\nu\right)+\frac{1}{2}\nu u\right)}$$

The proof of (iii) in appendix A shows that $K_k(u)$ has a pole at $u = 1/\nu_k = 1/\nu - k$ if $K_{k+1}(u)$ has a pole at $u = 1/\nu_{k+1} = 1/\nu-(k+1)$. If there are no other poles in $0 \le \nu u \le 1$ following [15] we conclude (up to normalization)

$$K_k(u) = \frac{1}{\sin\frac{1}{2}\pi\nu\left(u-(1/\nu-k)\right)\sin\frac{1}{2}\pi\nu\left(u+1/\nu-k\right)}K_k^{\min}(u)$$

which proves (C.5). ■

$N \to \infty$. For $\nu \to 0$ with θ fixed we get from (C.5)

$$K_k(\theta) = 2\pi\frac{\tanh\frac{1}{2}\theta}{\theta - i\pi} + O(\nu) \tag{C.6}$$

for all k. We prove that this result agrees with the recursion relation (4.21).

Proofs. For convenience we use the notation $x_i = \nu u_i = \theta_i/(i\pi)$, $y_i = \nu v_i = z_i/(i\pi)$. We calculate the r.h.s. of (4.21) for the component with $\underline{\alpha} = \overline{k+1}, k+1$ (up to normalization)

$$\int_{\mathcal{C}^o_{\underline{x}}} dy_1 \int_{\mathcal{C}^e_{\underline{x}}} dy_2\, \tilde{h}(\underline{x},\underline{y})K_{k+1}(y_1-y_2)\tilde{\Phi}_k(\underline{x},\underline{y})$$

with K_{k+1} given by (C.5) and

$$\begin{aligned}\tilde{\Phi}_k(\underline{x},\underline{y}) &= \mathbf{C}^{(N-2k-2)}_{\mathring{\beta}_1\mathring{\beta}_2}\tilde{\Phi}^{(k)}{}^{\mathring{\beta}_1\mathring{\beta}_2}_{\overline{k+1},k+1}(\underline{x},\underline{y}) \qquad (\mathrm{C.7})\\ &= \mathbf{C}^{(N-2k-2)}_{\mathring{\beta}_1\mathring{\beta}_2}\mathbf{C}^{\mathring{\beta}_1\mathring{\beta}_2}_{(N-2k-2)}\left(\tilde{c}(x_1-y_2)\tilde{d}_k(x_1-y_1)+f_k(y_{12})\left(\tilde{c}(x_1-y_1)+\tilde{d}_k(x_1-y_1)\right)\right)\\ &= (N-2k-2)\frac{-\nu^2\left(y_1-y_2-\nu\right)}{\left(y_1-y_2+\nu/\nu_k-\nu\right)\left(y_1-x_1+\nu\right)\left(y_2-x_1+\nu\right)}\end{aligned}$$

where (2.8) and (2.20) have been used. We get after a lengthy calculation the result

$$\int_{\mathcal{C}^o_{\underline{x}}} dy_1 \int_{\mathcal{C}^e_{\underline{x}}} dy_2\, \tilde{h}(\underline{x},\underline{y})K_{k+1}(y_1-y_2)\tilde{\Phi}_k(\underline{x},\underline{y}) = -16\pi^2\nu^2\frac{1}{x_{12}-1}\tan\frac{1}{2}\pi\left(x_{12}\right)+O(\nu^3) \tag{C.8}$$

which agrees with (C.6) (up to const.). ■

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N = 3. For $\nu = 2$ and $k = 0, 1$ we get from (C.5)

$$K(\theta) = K_0(\theta) = 8\pi^3 \frac{\tanh \frac{1}{2}\theta}{\theta\,(\theta - i\pi)\,(\theta - 2\pi i)} \tag{C.9}$$
$$L(z) = K_1(-z) = 2\pi\,(z - i\pi)\,\frac{\tanh \frac{1}{2}z}{z\,(z - 2\pi i)}$$

The result (E.11) in appendix E proves that these functions satisfy the recursion relation (4.21) (for suitable normalization).

N = 4. For $\nu = 1$ and $k = 0$ we get from (C.5) (up to const.)

$$K(x) = \left(\frac{1}{\theta_{12} - i\pi}\right)^2 \tag{C.10}$$

We use the general formulas (4.19) and (4.18) (for details see appendix B) with

$$^{+}\mathcal{O}_1 = {}^{-}\mathcal{O}_2 = \mathcal{O}_1$$
$$^{+}\mathcal{O}_2 = {}^{-}\mathcal{O}_1 = \mathcal{O}_2$$

and we propose the p-functions

$$p^{\mathcal{O}_1}(\underline{\theta}, \underline{z}) = \sum_{i=1}^{m} \exp z_i \,/ \sum_{i=1}^{n} \exp \theta$$
$$p^{\mathcal{O}_2}(\underline{\theta}, \underline{z}) = \exp\left(\frac{1}{2}\sum_{i=1}^{n}\left(\theta_i - \frac{1}{2}i\pi\right) - \sum_{i=1}^{m} z_i\right)$$

with weights $w^{\mathcal{O}_i}_{\mathrm{SU}(2)} = (0,0) \Rightarrow w^{\mathcal{O}}_{O(4)} = (0,0) \Rightarrow n = 2m =$ even and statistics factors $\sigma_1 = -\sigma_2 = 1$ which satisfy the condition of appendix B

$$\sigma_1\sigma_2 = (-1)^{n-1}\,.$$

The results of [19, 45] imply for $n = 2$ (up to constants)

$$K^{\mathcal{O}_1}_{A_1A_2}(\underline{\theta}) = \left(\delta^1_{A_1}\delta^2_{A_2} - \delta^2_{A_1}\delta^1_{A_2}\right)\frac{1}{\cosh \frac{1}{2}\theta_{12}}\frac{1}{\theta_{12} - i\pi}$$
$$K^{\mathcal{O}_2}_{B_1B_2}(\underline{\theta}) = \left(\delta^1_{B_1}\delta^2_{B_2} - \delta^2_{B_1}\delta^1_{B_2}\right)\frac{1}{\theta_{12} - i\pi}$$

and therefore

$$K^T_{\alpha_1\alpha_2}(\underline{\theta}) = \cosh\frac{1}{2}\theta_{12} K^{\mathcal{O}_+}_{A_1A_2}(\underline{\theta}) K^{\mathcal{O}_-}_{B_1B_2}(\underline{\theta}) \Gamma^{A_1A_2}_{\alpha_1}\Gamma^{B_1B_2}_{\alpha_2}$$
$$= \mathbf{C}_{\alpha_1\alpha_2}\left(\frac{1}{\theta_{12} - i\pi}\right)^2$$

which agrees with (C.10) and (5.18).

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D 1/N expansion

D.1 1/N expansion of the exact 3-particle field form factor

For $\mathcal{O}(x) = i(\Box + m^2)\varphi(x)$ we derive for the component $F^{\mathcal{O}}_{\bar{1}11}(\underline{\theta})$

$$F^{\mathcal{O}}_{\bar{1}11}(\underline{\theta}) = -\frac{8\pi}{N}m^2\left(\frac{\sinh\theta_{12}}{\theta_{12}-i\pi}+\frac{\sinh\theta_{13}}{\theta_{13}-i\pi}\right)+O(N^{-2})$$

Proofs. The p-function of $\mathcal{O}(x)$ and three particles for $\nu = 0$ is

$$p^{\mathcal{O}} = \left(e^{z_1}+e^{z_2}\right)\left(e^{-z_1}+e^{-z_2}\right) = 4\cosh^2\frac{1}{2}\left(z_1-z_2\right)$$

We have to consider (up to const.)

$$K^{\mathcal{O}}_{\bar{1}11}(\underline{\theta}) = \int_{\mathcal{C}^o_{\underline{\theta}}}dz_1\int_{\mathcal{C}^e_{\underline{\theta}}}dz_2\prod_{i=1}^{3}\left(\tilde{\psi}(\theta_i-z_1)\tilde{\chi}(\theta_i-z_2)\right)\frac{\cosh^2\frac{1}{2}\left(z_1-z_2\right)}{\tilde{\chi}(z_{12})\tilde{\psi}(-z_{12})}\tilde{\Psi}_{\bar{1}11}(\underline{\theta},\underline{z})$$

with

$$\begin{aligned}\tilde{\Psi}_{\bar{1}11}(\underline{\theta},\underline{z}) &= L(z_{12})\mathring{\mathbf{C}}_{\mathring{\beta}_1\mathring{\beta}_2}\mathring{\mathbf{C}}^{\mathring{\beta}_1\mathring{\beta}_2}\left(\tilde{c}(\theta_1-z_2)\tilde{d}(\theta_1-z_1)+f(z_{12})\left(\tilde{c}(\theta_1-z_1)+\tilde{d}(\theta_1-z_1)\right)\right)\\ &= L(z_{12})\frac{2}{\nu}\Bigg(\frac{-i\pi\nu}{\theta_1-z_2-i\pi\nu}\frac{\theta_1-z_1}{\theta_1-z_1-i\pi\nu}\frac{i\pi\nu}{\theta_1-z_1-i\pi}\\ &\quad+\frac{i\pi\nu}{z_1-z_2+i\pi-i\pi\nu}\left(\frac{-i\pi\nu}{\theta_1-z_1-i\pi\nu}+\frac{\theta_1-z_1}{\theta_1-z_1-i\pi\nu}\frac{i\pi\nu}{\theta_1-z_1-i\pi}\right)\Bigg)\\ &= \nu\pi^3\frac{\left(z_1-z_2\right)\tanh\frac{1}{2}\left(z_1-z_2\right)}{\left(z_1-z_2-i\pi\right)\left(z_1-z_2+i\pi\right)\left(\theta_1-z_1\right)\left(\theta_1-z_2\right)}+O\left(\nu^2\right)\end{aligned}$$

for $\nu\to 0$ using $L(z)\overset{\nu\to 0}{\to}\frac{2\pi}{z-i\pi}\tanh\frac{1}{2}z$ (see (5.3)). The leading terms are given by the integrals

$$\begin{aligned}\int_{\mathcal{C}_{\underline{\theta}}}dz_1\int_{\mathcal{C}_{\underline{\theta}}}dz_2\ldots &= \left(\int_{\theta_1-i\pi\nu}+\int_{\theta_2-i\pi\nu}+\int_{\theta_3-i\pi\nu}\right)dz_1\left(\int_{\theta_1}+\int_{\theta_2}+\int_{\theta_3}\right)dz_2\ldots\\ &= \left(\int_{\theta_2}+\int_{\theta_3}\right)dz_2\oint_{\theta_1-i\pi\nu}dz_1\cdots+O(\nu) = I_1+I_2+O(\nu)\end{aligned}$$

where $\oint_\theta dz\ldots$ means an integral along a small circle around θ and $\int_\theta dz\ldots$ an integral around all the poles of the gamma function according to figures 1 and 2. Up to higher order terms in ν and with $\tilde{\psi}\overset{\nu=0}{\to}1$, $\tilde{\chi}\overset{\nu=0}{\to}1$ we get (always up to const.)

$$\begin{aligned}I_1 &= \int_{\theta_2}dz_2\tilde{\chi}(\theta_2-z_2)\oint_{\theta_1}dz_1\frac{\left(z_1-z_2\right)\tanh\frac{1}{2}\left(z_1-z_2\right)}{\left(z_1-z_2-i\pi\right)\left(z_1-z_2+i\pi\right)\left(\theta_1-z_1\right)\left(\theta_1-z_2\right)}\cos^2\frac{1}{2}z_{12}\\ &= \int_{\theta_2}dz_2\tilde{\chi}(\theta_2-z_2)\frac{\left(\theta_1-z_2\right)\sin\left(\theta_1-z_2\right)}{\left(\theta_1-z_2-i\pi\right)\left(\theta_1-z_2+i\pi\right)\left(\theta_1-z_2\right)}\\ &= -\sin\theta_{12}\left(\oint_{\theta_1-i\pi}+\oint_{\theta_1+i\pi}\right)dz_2\frac{\Gamma(-\frac{1}{2\pi i}(\theta_2-z_2))}{\Gamma(\frac{1}{2}\nu-\frac{1}{2\pi i}(\theta_2-z_2))}\frac{1}{\theta_1-z-i\pi}\frac{1}{\theta_1-z+i\pi}\\ &= \frac{1}{2i\pi}\sinh\theta_{12}\left(\frac{\Gamma(\frac{1}{2\pi i}\left(\theta_{12}-i\pi\right))}{\Gamma(\frac{1}{2}\nu+\frac{1}{2\pi i}\left(\theta_{12}-i\pi\right))}-\frac{\Gamma(\frac{1}{2\pi i}\left(\theta_{12}+i\pi\right))}{\Gamma(\frac{1}{2}\nu+\frac{1}{2\pi i}\left(\theta_{12}+i\pi\right))}\right)\\ &= \frac{\nu}{2}\sinh\theta_{12}\frac{1}{\theta_{12}-i\pi}+O\left(\nu^2\right)\ .\end{aligned}$$

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Similarly

$$I_2 = \frac{\nu}{2} \sinh\theta_{13} \frac{1}{\theta_{13} - i\pi} + O\left(\nu^2\right) ,$$

which proves (5.7). ■

D.2 1/N perturbation theory

The nonlinear $O(N)$ σ-model is defined by the Lagrangian and the constraint

$$\mathcal{L} = \frac{1}{2} \sum_{\alpha=1}^{N} (\partial_\mu \varphi_\alpha)^2 \qquad \text{with} \quad g \sum_{\alpha=1}^{N} \varphi_\alpha^2 = 1$$

The Greens's functions may be written as

$$\langle\, 0 \,|\, T\varphi_{\alpha_1}(x_1) \dots \varphi_{\alpha_n}(x_n) \,|\, 0 \,\rangle = i^{-n} \frac{\delta}{\delta J_{\alpha_1}(x_1)} \dots \frac{\delta}{\delta J_{\alpha_1}(x_n)} Z(J)$$

where α_i are $O(N)$ labels and $Z(J)$ is the generating functional of Greens's functions given by the Feynman path integral

$$Z(J) = \int d\varphi\, \delta(\varphi^2 - 1/g) \exp i\left(\mathcal{A}(\varphi) + J\varphi\right) \tag{D.1}$$

with the action $\mathcal{A}(\varphi) = \int d^2x\, \mathcal{L}(\varphi)$. The fields $\varphi_\alpha(x)$ transforms as the vector representation of $O(N)$. In eq. (D.1) and in the following we use a matrix notation of the x-integrations e.g. $J\varphi = \sum_{\alpha=1}^{N} \int d^2x J_\alpha(x)\varphi_\alpha(x)$.

For the derivation of the result below the following Feynman integral is used.

$$\begin{aligned} I(m,k) &= \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 - m^2} \frac{1}{(p+k)^2 - m^2} \\ &= \frac{i}{4\pi} \frac{1}{\sqrt{-k^2}\sqrt{4m^2 - k^2}} 2 \ln \frac{\sqrt{4m^2 - k^2} + \sqrt{-k^2}}{\sqrt{4m^2 - k^2} - \sqrt{-k^2}} \\ &= \frac{i}{4\pi m^2} \frac{\phi}{\sinh\phi} , \qquad \left(k^2 = -4m^2 \sinh^2 \frac{1}{2}\phi\right) . \end{aligned}$$

For divergent integrals we use a Pauli-Villars regularization

$$I(m) \to I(m) - I(M) \to \lim_{M\to\infty} \left(I(m) - I(M)\right) ,$$

for example

$$I_\infty = \int \frac{d^2p}{(2\pi)^2} \left(\frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2} \right) = \frac{i}{4\pi} \ln \frac{m^2}{M^2} . \tag{D.2}$$

We may introduce the bosonic field ω and rewrite eq. (D.1), equivalently as

$$Z(J) = \int d\varphi \;\; \exp i\left(\mathcal{A}(\varphi,\omega) + J\varphi\right) \tag{D.3}$$

with the action $\mathcal{A}(\varphi,\omega) = \int d^2x\, \mathcal{L}(\varphi,\omega)$ and the Lagrangian

$$\mathcal{L}(\varphi,\omega) = \frac{1}{2} \left(\sum_{\alpha=1}^{N} (\partial_\mu \varphi_\alpha)^2 - \omega \left(\sum_{\alpha=1}^{N} \varphi_\alpha^2 - 1/g \right) \right) .$$

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Performing the φ-integrations we obtain

$$Z(J) = \int d\omega \, \exp\left(i\mathcal{A}_{\text{eff}}(\omega) - \frac{1}{2} J\Delta(\omega) J \right)$$

with the propagator $\Delta_{\alpha\beta}(\omega) = i\delta_{\alpha\beta}(-\Box - \omega)^{-1}$ and the effective action

$$\mathcal{A}_{\text{eff}}(\omega) = \frac{1}{2} iN \text{ Tr } \ln(i\Delta^{-1}(\omega)) + \int d^2x \, \frac{1}{2g}\omega \, .$$

The symbol Tr means the trace with respect to x-space, the trace with respect to $O(N)$-isospin has been taken and given the factor N.

We define the vertex functions Γ by[4]

$$\mathcal{A}_{\text{eff}}(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^2x_1 \ldots d^2x_n \, \Gamma^{(n)}(x_1, \ldots, x_n) \, \omega'(x_1) \ldots \omega'(x_n),$$

where $\omega = \omega' - m^2$.

The value $\omega = m^2$ is defined by the condition $\mathcal{A}_{\text{eff}}(\omega)$ being stationary at this point, which means that the one-point vertex function vanishes $\Gamma^{(1)}(x) = \frac{\delta \mathcal{A}_{\text{eff}}}{\delta \omega} = 0$. Expanding $\mathcal{A}_{\text{eff}}$ for small ω'

$$\begin{aligned}
\mathcal{A}_{\text{eff}} &= -\frac{1}{2} iN \, \text{Tr } \ln(i\Delta^{-1}(\omega)) + \int d^2x \, \frac{1}{2g}\omega \\
&= -\frac{1}{2} iN \Big(\text{Tr } \ln(-\Box - m^2) + \text{Tr} \Big\{ (-\Box - m^2)^{-1}\omega' \\
&\quad + \frac{1}{2}(-\Box - m^2)^{-1}\omega'(-\Box - m^2)^{-1}\omega' + \ldots \Big\} \Big) + \frac{1}{2g} \int d^2x \, \left(m^2 + \omega' \right)
\end{aligned}$$

we obtain

$$\Gamma^{(1)}(x) = -\frac{1}{2} N\Delta(x,x) + \frac{1}{2g} = 0$$

with the propagator $\Delta = i(-\Box - m^2)^{-1}$. This equation defines the mass m by

$$m^2 = M^2 e^{-\frac{4\pi}{gN}}$$

where M is an UV-cutoff (see (D.2)). There is the effect of mass generation and dimensional transmutation: the dimensionless coupling g is replaced by the mass m.

The $1/N$-expansion is obtained by expanding the effective action at this stationary point. Next we calculate the two point vertex function

$$\Gamma^{(2)}(x,y) = \frac{\delta^2 \mathcal{A}_{\text{eff}}}{\delta\omega(x)\delta\omega(y)} = \int \frac{d^2k}{(2\pi)^2} e^{-i(x-y)k} \, \tilde{\Gamma}^{(2)}(k)$$

in momentum space

$$\tilde{\Gamma}^{(2)}(k) = -\frac{1}{2} iN \int \frac{d^2p}{(2\pi)^2} \, \text{tr} \left(\frac{1}{p^2 - m^2} \, \frac{1}{(p+k)^2 - m^2} \right) = \frac{N}{8\pi m^2} \, \frac{\phi}{\sinh \phi}$$

[4]Note that the $i\Gamma^{(n)}$ are the 1-particle irreducible connected graphs with n external lines.

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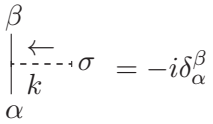

$= -i\delta_\alpha^\beta$

Figure 3. The elementary vertex for the $O(N)$ Gross-Neveu model. With respect to isospin the vertex is proportional to the unit matrix.

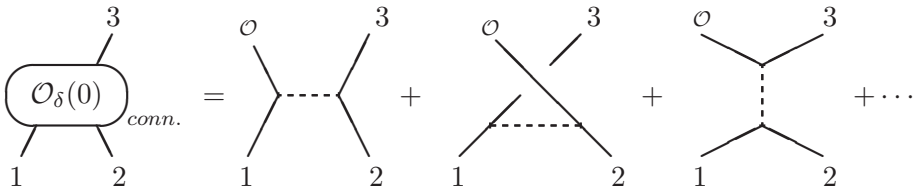

Figure 4. The connected part of the three particle form factor of the fundamental fermi field in $1/N$-expansion.

where $k^2 = -4m^2 \sinh^2(\phi/2)$. The ω-propagator is obtained by inverting the two-point vertex function $\Delta = i\Gamma^{(2)^{-1}}$

$$\tilde{\Delta}_\omega(k) = i\left(-\frac{1}{2}iNI(m,k)\right)^{-1} = \frac{8\pi i}{N}m^2\,\frac{\sinh\phi}{\phi}\ .$$

This propagator together with the simple vertex $-i\delta_\alpha^\beta$ of figure 3 yield the Feynman rules which allow to calculate general vertex functions in the $1/N$-expansion. For example the four point vertex function is

$$i\tilde{\Gamma}^{(4)\delta\gamma}_{\alpha\beta}(-p_3,-p_4,p_1,p_2) = -\delta_\alpha^\delta\delta_\beta^\gamma\,\tilde{\Delta}_\omega(p_2-p_3) - \delta_\alpha^\gamma\delta_\beta^\delta\,\tilde{\Delta}_\omega(p_3-p_1) - \delta_{\alpha\beta}\delta^{\delta\gamma}\,\tilde{\Delta}_\omega(p_1+p_2) \quad \text{(D.4)}$$

where $\alpha\beta\gamma\delta$ are isospin. We now calculate the three particle form factor of the fundamental bose field in $1/N$-expansion in lowest nontrivial order. For convenience we multiply the field with the Klein-Gordon operator and take

$$\mathcal{O}^\delta(x) = i(\Box + m^2)\varphi^\delta(x)$$

and define

$${}^{\gamma}_{\text{out}}\langle\, p_3 \,|\, \mathcal{O}^\delta(0) \,|\, p_1, p_2 \,\rangle^{\text{in}}_{\alpha\beta} = F^{\mathcal{O}^\delta\ \gamma}_{\ \alpha\beta}(\theta_3;\theta_1,\theta_2).$$

By means of LSZ-techniques one can express the connected part in terms of the 4-point vertex function.

$$F^{\mathcal{O}^\delta\ \ \gamma}_{conn.\,\alpha\beta}(\theta_3;\theta_1,\theta_2) = i\tilde{\Gamma}^{(4)\,\delta\gamma}_{\alpha\beta}(-p_3, p_3-p_1-p_2, p_1, p_2). \quad \text{(D.5)}$$

The lowest order contributions are given by the Feynman graphs of figure 4

$$F^{\mathcal{O}^\delta\ \ \gamma}_{conn.\,\alpha\beta} = -\delta_\alpha^\delta\delta_\beta^\gamma\,\tilde{\Delta}_\omega(p_2-p_3) - \delta_\alpha^\gamma\delta_\beta^\delta\,\tilde{\Delta}_\omega(p_3-p_1) - \delta_{\alpha\beta}\delta^{\delta\gamma}\,\tilde{\Delta}_\omega(p_1+p_2).$$

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Using $\tilde{\Delta}_\omega(k) = \frac{8\pi i}{N} m^2 \frac{\sinh\phi}{\phi}$ with $k^2 = -4m^2\sinh^2(\phi/2)$ we obtain up to order $1/N$

$$F^{\mathcal{O}^\delta\ \ \gamma}_{conn.\,\alpha\beta} = -\frac{8\pi i}{N} m^2 \left(\delta^\delta_\alpha \delta^\gamma_\beta \frac{\sinh(\theta_{23})}{\theta_{23}} + \delta^\gamma_\alpha \delta^\delta_\beta \frac{\sinh(\theta_{13})}{\theta_{13}} + \delta_{\alpha\beta}\delta^{\delta\gamma} \frac{\sinh(i\pi - \theta_{12})}{i\pi - \theta_{12}} \right) .$$

and by crossing in the complex basis

$$F^{\mathcal{O}^\delta}{}_{\alpha\beta\gamma}(\theta_1,\theta_2,\theta_3) = -\frac{8\pi i}{N} m^2 \left(\delta^\delta_\alpha \mathbf{C}_{\beta\gamma} \frac{\sinh\theta_{23}}{i\pi - \theta_{23}} + \delta^\delta_\beta \mathbf{C}_{\alpha\gamma} \frac{\sinh\theta_{13}}{i\pi - \theta_{13}} + \delta^\delta_\gamma \mathbf{C}_{\alpha\beta} \frac{\sinh\theta_{12}}{i\pi - \theta_{12}} \right) \tag{D.6}$$

which agrees with expansion of the exact form factor (5.7).

Now we are in the position to check whether this expression is consistent with the exact S-matrix of section 2 (see also [5]). Using LSZ-techniques we obtain

$$\begin{aligned}
&{}_{\text{out}}\langle \gamma, p_3; \delta, p_4 \,|\, \alpha, p_1; \beta, p_2 \rangle^{\text{in}} \\
&= \delta_{14}\delta^\delta_\alpha \delta_{23}\delta^\gamma_\beta + \delta_{13}\delta^\gamma_\alpha \delta_{24}\delta^\delta_\beta + (2\pi)^2\delta^{(2)}(p_1 + p_2 - p_3 - p_4)\, i\tilde{\Gamma}^{(4)\,\delta\gamma}_{\alpha\beta} \\
&= \delta_{14}\delta^\delta_\alpha \delta_{23}\delta^\gamma_\beta + \delta_{13}\delta^\gamma_\alpha \delta_{24}\delta^\delta_\beta + \frac{1}{4m^2\sinh\theta_{12}} (\delta_{14}\delta_{23} + \delta_{13}\delta_{24}) F^{\mathcal{O}_\delta\ \ \gamma}_{conn.\,\alpha\beta}
\end{aligned}$$

where $\delta_{14} = 2\pi 2p_1^0 \delta(p_1 - p_4)$ etc. and $\Gamma^{(4)}$ is the 4-point vertex function in (D.4). I has been used that

$$(2\pi)^2\delta^{(2)}(p_1 + p_2 - p_3 - p_4) = \frac{1}{4m^2\sinh\theta_{12}} (\delta_{14}\delta_{23} + \delta_{13}\delta_{24}) .$$

By means of the formula

$$\begin{aligned}
{}_{\text{out}}\langle \gamma, p_3; \delta, p_4 \,|\, \alpha, p_1; \beta, p_2 \rangle^{\text{in}} &= {}_{\text{out}}\langle \gamma, p_3; \delta, p_4 \,|\, \beta', p_2; \alpha', p_1 \rangle^{\text{out}}\, S^{\beta'\alpha'}_{\alpha\beta} \\
&= \delta_{13}\delta_{24} S^{\delta\gamma}_{\alpha\beta} + \delta_{14}\delta_{23} S^{\gamma\delta}_{\alpha\beta}
\end{aligned}$$

we obtain

$$S^{\delta\gamma}_{\alpha\beta}(\theta_{12}) = \delta^\gamma_\alpha \delta^\delta_\beta + \frac{1}{4m^2\sinh\theta_{12}} F^{\mathcal{O}^\delta\ \ \gamma}_{conn.\,\alpha\beta}(\theta_3;\theta_1,\theta_2) \quad \text{for} \quad p_4 \to p_2,\ p_3 \to p_1. \tag{D.7}$$

Equation (D.6) implies the perturbative result up to order $1/N$ or

$$S^{\delta\gamma}_{\alpha\beta}(\theta) = \delta^\gamma_\alpha \delta^\delta_\beta \left(1 - \frac{2\pi i}{N\sinh\theta}\right) + \delta^\delta_\alpha \delta^\gamma_\beta \left(-\frac{2\pi i}{N\theta}\right) + \mathbf{C}^{\delta\gamma}\mathbf{C}_{\alpha\beta} \left(-\frac{2\pi i}{N(i\pi - \theta)}\right) + O(N^{-2})$$

which agrees with the $1/N$ expansion of (2.6). The relation (D.7) is equivalent to the fact that the form factor of the field (up to $O(N^{-2})$)

$$F^\varphi{}_{\alpha\beta\gamma}(\theta_1,\theta_2,\theta_3) = \frac{\frac{\pi}{N}\left(\delta^1_\alpha \mathbf{C}_{\beta\gamma}\frac{\sinh(i\pi-\theta_{23})}{i\pi-\theta_{23}} + \delta^1_\beta \mathbf{C}_{\alpha\gamma}\frac{\sinh(i\pi-\theta_{13})}{i\pi-\theta_{13}} + \delta^1_\gamma \mathbf{C}_{\alpha\beta}\frac{\sinh(i\pi-\theta_{12})}{i\pi-\theta_{12}}\right)}{\cosh\frac{1}{2}\theta_{12}\cosh\frac{1}{2}\theta_{13}\cosh\frac{1}{2}\theta_{23}}$$

satisfies the form factor equation (iii)

$$\begin{aligned}
&\operatorname*{Res}_{\theta_{12}=i\pi} F^\varphi{}_{\alpha\beta\gamma}(\theta_1,\theta_2,\theta_3) \\
&\quad = -4\frac{\pi}{N}\left(\delta^1_\alpha \mathbf{C}_{\beta\gamma}\frac{1}{i\pi - \theta_{23}} + \delta^1_\beta \mathbf{C}_{\alpha\gamma}\frac{1}{\theta_{23}} + \delta^1_\gamma \mathbf{C}_{\alpha\beta}\frac{1}{\sinh\theta_{23}}\right) + O\left(N^{-2}\right) \\
&\quad = 2i\left(\mathbf{C}_{\alpha\beta}\delta^1_\gamma - \mathbf{C}_{\alpha\beta'}\delta^1_{\gamma'} S^{\gamma'\beta'}_{\beta\gamma}(\theta_{23})\right) .
\end{aligned}$$

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E More explicitly calculations

E.1 General formulas

The maximal number of particles of type $\alpha_i = \mathring{\alpha}_i \neq 1, \bar{1}$ in the Bethe ansatz state $\tilde{\Phi}^{\mathring{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ of (2.17) is m; the other $n-m$ particles are of type 1. This follows from the structure of the Π-matrix (see [6, 33]) and the S-matrix (2.1). We consider the corresponding component of the form factors for n particles

$$F_{\underline{\mathring{\alpha}}\,\underline{1}}(\underline{\theta}) = N_{n,m} K_{\underline{\mathring{\alpha}}\,\underline{1}}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_i - \theta_j)$$

for $\underline{\mathring{\alpha}} = \mathring{\alpha}_1, \dots, \mathring{\alpha}_m$, with $\mathring{\alpha}_i \neq 1, \bar{1}$ and $\underline{1} = 1, \dots, 1$. For convenience, we use here a different normalization compared to (1.2) and (1.3). The K-functions is given by

$$K_{\underline{\mathring{\alpha}}\,\underline{1}}(\underline{\theta}) = \prod_{k=1}^{m} \left(\frac{1}{2i\pi} \int_{\mathcal{C}_{\underline{\theta}}} dz_k \right) \tilde{h}(\underline{\theta}, \underline{z}) p(\underline{\theta}, \underline{z}) \sum_{\pi \in S_m} k_{\underline{\mathring{\alpha}}}(\underline{\theta}, \pi \underline{z}) \tag{E.1}$$

where the sum has to be taken over all permutations of the z_i and

$$\tilde{h}(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n} \prod_{j=1}^{m} \tilde{\phi}_j(\theta_i - z_j) \prod_{1\le i<j\le m} \tau_{ij}(z_{ij})$$

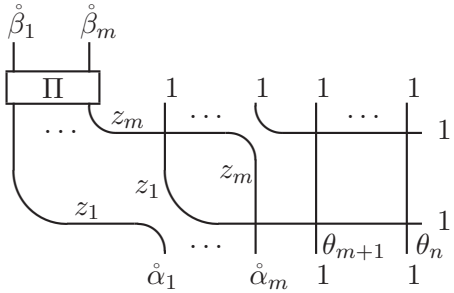
$$k_{\underline{\mathring{\alpha}}}(\underline{\theta}, \underline{z}) = L_{\underline{\mathring{\alpha}}}(\underline{z}) \prod_{1\le i<j\le m} \frac{1}{\tilde{b}(z_{ij})} \prod_{j=1}^{m} \left(\tilde{c}(\theta_j - z_j) \prod_{k=j+1}^{m} \tilde{b}(\theta_j - z_k) \right).$$

Proofs. We use formula (15) of [6]

$$\begin{aligned} \tilde{T}_{a,1\dots m}(\theta, \underline{z}) = & \prod_{j=1}^{m} \tilde{b}(\theta - z_j) \mathbf{1}_a \mathbf{1}_1 \dots \mathbf{1}_n \\ & + \sum_{i=1}^{m} \tilde{c}(\theta - z_i) \tilde{b}(\theta - z_m) \frac{\tilde{S}_{am}(z_{im})}{\tilde{b}(z_{im})} \dots \mathbf{P}_{ai} \dots \tilde{b}(\theta - z_1) \frac{\tilde{S}_{a1}(z_{i1})}{\tilde{b}(z_{i1})} \\ & + \sum_{i=1}^{m} \tilde{d}(\theta - z_i) \frac{\tilde{b}(\theta - z_m)}{\tilde{b}(\hat{z}_{mi})} \tilde{S}_{am}(\hat{z}_{mi}) \dots \mathbf{K}_{ai} \dots \frac{\tilde{b}(\theta - z_1)}{\tilde{b}(\hat{z}_{1i})} \tilde{S}_{a1}(\hat{z}_{1i}) \,. \end{aligned}$$

which implies for the Bethe state

$$\begin{aligned} & \Psi_{\underline{\mathring{\alpha}}\,\underline{1}}(\underline{\theta}, \underline{z}) \\ & = L_{\underline{\mathring{\beta}}}(\underline{z}) \Pi^{\mathring{\beta}}_{\underline{\mathring{\alpha}}}(\underline{z}) \prod_{1\le i<j\le m} \frac{1}{\tilde{b}(z_{ij})} \prod_{j=1}^{m} \left(\tilde{c}(\theta_j - z_j) \prod_{k=j+1}^{m} \tilde{b}(\theta_j - z_k) \right) + \text{ permutations of the } z_i \\ & = L_{\underline{\mathring{\beta}}}(\underline{z}) \prod_{1\le i<j\le m} \frac{1}{\tilde{b}(z_{ij})} \end{aligned}$$



$$+ \text{ permutations of the } z_i$$

$$= \sum_{\pi \in S_m} L_{\underline{\mathring{\alpha}}}(\pi \underline{z}) \prod_{1\le i<j\le m} \frac{1}{\tilde{b}(\pi z_{ij})} \prod_{j=1}^{m} \left(\tilde{c}(\theta_j - \pi z_j) \prod_{k=j+1}^{m} \tilde{b}(\theta_j - \pi z_k) \right).$$

because $\Pi^{\mathring{\beta}}_{\underline{\mathring{\alpha}}}(\underline{z}) = \delta^{\mathring{\beta}}_{\underline{\mathring{\alpha}}}$. Note that the $\tilde{d}$-terms do not contribute because of $\Pi_{\ldots\bar{1}} = 0$ (see (2.19)). It has been used that the state $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z})$ is a symmetric function of the z_i (see Remark 2). ■

Pairs of $\mathring{\alpha}_i$ may be replaced by $\bar{1}1$. For example, we obtain for $\underline{\mathring{\alpha}} = \mathring{\alpha}_1,\ldots,\mathring{\alpha}_{m-2}$, with $\mathring{\alpha}_i \neq 1,\bar{1}$

$$K_{\underline{\mathring{\alpha}}\bar{1}1\underline{1}}(\underline{\theta}) = \prod_{k=1}^{m}\left(\frac{1}{2i\pi}\int_{\mathcal{C}_{\underline{\theta}}} dz_k\right)\tilde{h}(\underline{\theta},\underline{z})p(\underline{\theta},\underline{z})\sum_{\pi\in S_m} k_{\underline{\mathring{\alpha}}}(\underline{\theta},\pi\underline{z}) \tag{E.2}$$

and

$$\begin{aligned} k_{\underline{\mathring{\alpha}}}(\underline{\theta},\underline{z}) = {} & L_{\underline{\mathring{\alpha}}\mathring{\beta}_{m-1}\mathring{\beta}_m}(\underline{z})\mathbf{C}^{\mathring{\beta}_{m-1}\mathring{\beta}_m}\prod_{1\le i<j\le m}\frac{1}{\tilde{b}(z_{ij})} \\ & \times\prod_{j=1}^{m-2}\left(\tilde{c}(\theta_j - z_j)\prod_{k=j+1}^{m-2}\tilde{b}(\theta_j - z_k)\right)f(z_{m-1}-z_m)\tilde{c}(\theta_{m-1}-z_{m-1})\tilde{b}(\theta_{m-1}-z_m) \end{aligned}$$

where (2.20) $\Pi^{\mathring{\beta}_1\mathring{\beta}_2}_{\bar{1}1}(z) = \mathbf{C}^{\mathring{\beta}_1\mathring{\beta}_2}f(z)$ has been used. Similar formulas are obtained if more pairs of $\mathring{\alpha}$'s are replaced by $\bar{1}1$.

E.2 O(3) form factors

For $O(3)$ the general formulas simplify because the integrals reduce to a finite number of residues

$$\frac{1}{2i\pi}\int_{\mathcal{C}_{\underline{\theta}}} dz\ \ldots = \frac{1}{2i\pi}\sum_{i=1}^{n}\left(\oint_{\theta_i} + \oint_{\theta_i - 2\pi i}\right)dz\ \ldots\,.$$

and we may replace

$$\tilde{\phi}_j(\theta) \to \frac{1}{\theta}\,,\quad \tau_{ij}(z) \to z^2\,.$$

Furthermore all $\mathring{\alpha}_i$ are equal to 0 and $L_{\underline{\mathring{\beta}}}(\underline{z}) = \prod_{1\le i<j\le m} L(z_{ij})$ (see section 4.2). The example (E.1) writes as

$$F_{0\cdots 0\underline{1}}(\underline{\theta}) = N_m K_{0\cdots 0\underline{1}}(\underline{\theta})\prod_{1\le i<j\le n} F(\theta_i - \theta_j) \tag{E.3}$$

$$K_{0\cdots 0\underline{1}}(\underline{\theta}) = \prod_{k=1}^{m}\left(\frac{1}{2i\pi}\sum_{i=1}^{n}\left(\oint_{\theta_i} + \oint_{\theta_i-2\pi i}\right)dz_k\right)p(\underline{\theta},\underline{z})\sum_{\pi\in S_m} k(\underline{\theta},\pi\underline{z}) \tag{E.4}$$

with

$$k(\underline{\theta},\underline{z}) = \prod_{1\le i<j\le m} l(z_i - z_j)\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{\theta_i - z_j}\prod_{j=1}^{m}\left(\frac{-2\pi i}{\theta_j - z_j - 2\pi i}\prod_{k=j+1}^{m}\frac{\theta_j - z_k}{\theta_j - z_k - 2\pi i}\right). \tag{E.5}$$

We have integrated $\tilde{h}(\underline{\theta},\underline{z})$ into $k(\underline{\theta},\underline{z})$. We have also used the amplitudes (2.8) and

$$l(z) = L(z)\tau(z)/\tilde{b}(z) = (z - i\pi)\tanh\frac{1}{2}z\,.$$

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Similarly, we obtain

$$K_{0\cdots0\bar{1}1\underline{1}}(\underline{\theta}) = \prod_{k=1}^{m}\left(\frac{1}{2i\pi}\int_{\mathcal{C}_{\underline{\theta}}} dz_k\right) p(\underline{\theta},\underline{z}) \sum_{\pi\in S_m} k(\underline{\theta},\pi\underline{z}) \tag{E.6}$$

with

$$\begin{aligned} k(\underline{\theta},\underline{z}) = & \prod_{1\le i<j\le m} l(z_i-z_j) \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}\,(\theta_i-z_j) \prod_{j=1}^{m-2}\left(\tilde{c}(\theta_j-z_j)\prod_{k=j+1}^{m}\tilde{b}(\theta_j-z_k)\right) \\ & \times f(z_{m-1}-z_m)\tilde{c}(\theta_{m-1}-z_{m-1})\tilde{b}(\theta_{m-1}-z_m) \end{aligned}$$

and

$$K_{0\cdots0\bar{1}\bar{1}111\underline{1}}(\underline{\theta}) = \prod_{k=1}^{m}\left(\frac{1}{2i\pi}\int_{\mathcal{C}_{\underline{\theta}}} dz_k\right) p(\underline{\theta},\underline{z}) \sum_{\pi\in S_m} k(\underline{\theta},\pi\underline{z})$$

with

$$\begin{aligned} k(\underline{\theta},\underline{z}) = & \prod_{1\le i<j\le m} l(z_i-z_j) \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}\,(\theta_i-z_j) \prod_{j=1}^{m-4}\left(\tilde{c}(\theta_j-z_j)\prod_{k=j+1}^{m}\tilde{b}(\theta_j-z_k)\right) \\ & \times f(z_{m-3}-z_{m-2})\tilde{c}(\theta_{m-3}-z_{m-3})\tilde{b}(\theta_{m-3}-z_{m-2})\tilde{b}(\theta_{m-3}-z_{m-1})\tilde{b}(\theta_{m-3}-z_m) \\ & \times f(z_{m-1}-z_m)\tilde{c}(\theta_{m-2}-z_{m-1})\tilde{b}(\theta_{m-2}-z_m) \end{aligned}$$

etc.

It turns out that the form factors for the field and the pseudo-potential of current are of the form (see also the explicit calculations in subsection E.2.2)

$$F_{\underline{\alpha}}(\underline{\theta}) = g_{\underline{\alpha}}(\underline{\theta})G(\underline{\theta}) \tag{E.7}$$

where $g_{\underline{\alpha}}(\underline{\theta})$ is a polynomial (cf. [11]) and

$$\begin{aligned} G(\underline{\theta}) &= \prod_{1\le i<j\le n} G(\theta_{ij}) \\ G(\theta) &= \frac{\tanh\frac{1}{2}\theta}{\theta\,(\theta-2\pi i)}F(\theta) = \frac{\theta-i\pi}{\theta\,(\theta-2i\pi)}\tanh^2\frac{1}{2}\theta\,. \end{aligned}$$

Proofs. A sketch of the proof for (E.7): the K-functions with $n = m+1$ are of the form

$$\begin{aligned} K(\underline{\theta}) &= \prod_{k=1}^{m}\left(\frac{1}{2i\pi}\sum_{i=1}^{n}\left(\oint_{\theta_i}+\oint_{\theta_i-2\pi i}\right)dz_k\right) p(\underline{\theta},\underline{z}) \sum_{\pi\in S_m} k(\underline{\theta},\pi\underline{z}) \\ &= \sum I_{i_1\ldots i_m}, \quad i_k\in\{1,\ldots,n\} \\ I_{i_1\ldots i_m}(\underline{\theta}) &= \frac{1}{2i\pi}\left(\oint_{\theta_{i_1}}+\oint_{\theta_{i_1}-2\pi i}\right)dz_1\ldots\frac{1}{2i\pi}\left(\oint_{\theta_{i_m}}+\oint_{\theta_{i_m}-2\pi i}\right)dz_m p(\underline{\theta},\underline{z}) \sum_{\pi\in S_m} k(\underline{\theta},\pi\underline{z}) \end{aligned}$$

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where $I_{i_1\dots i_m}$ is symmetric with respect to the indices and for two equal indices the integrals vanish $I_{\dots i\dots i\dots}=0$. For example for (E.4) we obtain, using (E.5)

$$I_{1\dots n-1}(\underline{\theta})=\prod_{k=1}^{n-1}\left(\frac{1}{2i\pi}\left(\oint_{\theta_k}+\oint_{\theta_k-2\pi i}\right)dz_k\right)p(\underline{\theta},\underline{z})\sum_{\pi\in S_m}k(\underline{\theta},\pi\underline{z})$$
$$=J(\underline{\theta})\left(\prod_{1\le i<j\le n-1}\tanh\frac{1}{2}\theta_{ij}\right)p(\underline{\theta},\underline{z}=\theta_1,\dots,\theta_{n-1})\,,$$

where

$$J(\underline{\theta})=\prod_{k=1}^{n-1}\left(\frac{1}{2i\pi}\left(\oint_{\theta_k}+\oint_{\theta_k-2\pi i}\right)dz_k\right)\sum_{\pi\in S_m}\mathrm{sign}(\pi)\,j(\underline{\theta},\pi\underline{z})\,,$$

$$j(\underline{\theta},\underline{z})=\prod_{1\le i<j\le m}(z_i-z_j-i\pi)\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{\theta_i-z_j}\prod_{j=1}^{m}\left(\frac{-2\pi i}{\theta_j-z_j-2\pi i}\prod_{k=j+1}^{m}\frac{\theta_j-z_k}{\theta_j-z_k-2\pi i}\right).$$

Let the p-function satisfy

$$\sum_{k=1}^{n}(-1)^{n-k}\left(\prod_{1\le i<j\le n,i,j\neq k}\tanh\frac{1}{2}\theta_{ij}\right)p(\underline{\theta},\underline{z}=\theta_1,\dots,\widehat{\theta_k},\dots,\theta_n)=\prod_{1\le i<j\le n}\tanh\frac{1}{2}\theta_{ij}$$

which holds for the p-functions p^{φ} and p^{J} of the examples below. Then, because $I_{1\dots\hat{k}\dots n}=-I_{1\dots\widehat{k+1}\dots n}$ (where $\hat{k}$ means that the index k is missing) we obtain

$$K(\underline{\theta})=J(\underline{\theta})\,n!\prod_{1\le i<j\le n}\tanh\frac{1}{2}\theta_{ij}\quad\text{and}\quad F(\underline{\theta})=g(\underline{\theta})G(\underline{\theta})$$

where $g(\underline{\theta})$ is a polynomial. ■

For the energy momentum potential formula (E.7) holds for $n>2$ (see (E.9) and (E.10). However, the prove is more complicated (see the prove of (E.9)).

The normalization factor N_m is obtained from the form factor equation (iii), which implies that[5]

$$N_m=\frac{2^{2m-3}\pi}{(m-1)\,m}N_{m-2}\,. \tag{E.8}$$

E.2.1 Examples

Form factors of the field. In particular, for the form factors of the field ($n=$ odd) with $m=n-1$ we have to apply the p-function of (5.1) for $\nu=2$

$$p^{\varphi}(\underline{\theta},\underline{z})=\frac{(\sum e^{z_j})(\sum e^{-z_j})}{(\sum e^{\theta_j})(\sum e^{-\theta_j})-1}\,.$$

To prove the form (E.7) of the field form factors one uses the identity (for $n=$ odd)

$$\sum_{i=1}^{n}\left(\left(\prod_{\substack{j=1\\j\neq i}}^{n}\coth\frac{1}{2}\theta_{ji}\right)\left(\sum_{\substack{j=1\\j\neq i}}^{n}e^{\theta_j}\right)\left(\sum_{\substack{j=1\\j\neq i}}^{n}e^{-\theta_j}\right)\right)=1-\left(\sum_{i=1}^{n}e^{\theta_i}\right)\left(\sum_{i=1}^{n}e^{-\theta_i}\right).$$

The normalizations follow from (E.8) as $N_m=\dfrac{1}{m!}\pi^{\frac{1}{2}m}2^{\frac{1}{2}m(m-1)}$.

[5]This follows from (4.29), taking into account the different normalizations here compared to that of theorem 3.

Form factors of the current. For the form factors of the pseudo-potential of the current (n = even) with $m = n - 1$ we have to apply the p-function of (5.9) for $\nu = 2$

$$p^J(\underline{\theta}, \underline{z}) = \frac{\exp\left(\sum \theta_i - \sum z_j\right)}{\sum \exp \theta_i} .$$

To prove the form (E.7) of the current form factors one uses the identity

$$\sum_{i=1}^{n} e^{\theta_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \coth \frac{1}{2}\theta_{ij} = \sum_{i=1}^{n} e^{\theta_i} .$$

The normalizations follow from (E.8) as $N_m = \frac{1}{m!} i\pi^{\frac{1}{2}(m+1)} 2^{\frac{1}{2}(m(m-1)-2)}$.

Form factors of the energy momentum potential. For the form factors for the energy momentum potential ($m = n$ = even) we have to calculate (E.1) and (E.2) with the p-function of (5.16)

$$p^T(\underline{\theta}, \underline{z}) = 1.$$

Again we have to perform the integrations for $O(3)$ by calculating a finite number of residues. It turns out that the the leading term in the limit $\nu \to 2$ (i.e. $N \to 3$) vanishes and we have to calculate the contribution of order $(\nu - 2)$.

E.2.2 Explicit calculations

n = 1, m = 0.

$$F_1^{\varphi}(\underline{\theta}) = 1$$

n = 2, m = 1.

$$F_{01}^{J}(\underline{\theta}) = \frac{1}{2} i\pi K_{01}(\underline{\theta}) = -\frac{1}{2}\pi^2 G(\theta)$$

which agrees with the result of [15] (see also (5.12).

n = 3, m = 2. After some simple calculations we obtain with $G(\theta)$ given by (5.4)

$$F_{001}^{\varphi}(\underline{\theta}) = \pi^3 \theta_{12}\, G(\underline{\theta})$$
$$F_{\bar{1}11}^{\varphi}(\underline{\theta}) = \pi^3 \left(2i\pi - \theta_{23}\right) G(\underline{\theta}) .$$

which agree with [11] which were obtained by different methods. The other components can be obtained by the form factor equations (i) and (ii)

$$F_{\alpha\beta\gamma}^{\varphi}(\underline{\theta}) = \pi^3 g_{\alpha\beta\gamma}^{\varphi}(\underline{\theta}) G(\underline{\theta})$$
$$g_{\alpha\beta\gamma}^{\varphi}(\underline{\theta}) = \theta_{23}\delta_{\alpha}^{1}\mathbf{C}_{\beta\gamma} + (2\pi i - \theta_{13})\,\delta_{\beta}^{1}\mathbf{C}_{\alpha\gamma} + \theta_{12}\delta_{\gamma}^{1}\mathbf{C}_{\alpha\beta} .$$

n = 4, m = 3. From (E.4) and (E.6) we get the results

$$F_{0001}^{J}(\underline{\theta}) = \frac{1}{2}\pi^5 \left(\theta_{12}\theta_{13}\theta_{23} + 2\pi i\theta_{12}\left(\theta_{32} - 2\pi i\right) - 2i\pi^3\right) G(\underline{\theta})$$
$$F_{\bar{0}111}^{J}(\underline{\theta}) = \frac{1}{2}\pi^5 \left(\theta_{32} - 2i\pi\right)\left(\theta_{23}\theta_{22} - \theta_{12}\left(\theta_{12} - i\pi\right)\right) G(\underline{\theta}) .$$

which again agree with [11].

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n = 5, m = 4. Again from (E.4) and (E.6) we get the results

$$F^{\varphi}_{\underline{\alpha}}(\underline{\theta}) = g_{\underline{\alpha}}(\underline{\theta})\, G(\underline{\theta})$$

with (up to normalizations)[6]

$g_{\bar{1}\bar{1}111}(\underline{\theta}) = (\theta_{12}-2\pi i)(\theta_{34}-2\pi i)(\theta_{35}-2\pi i)(\theta_{45}-2\pi i)(-4\pi^2-2i\pi\theta_{45}-3i\pi\theta_1+2i\pi\theta_3-3i\pi\theta_2+4i\pi\theta_4-2\theta_3\theta_1+\theta_3^2-2\theta_3\theta_2+2\theta_3\theta_4-\theta_4\theta_1-\theta_4\theta_2+\theta_{35}\theta_1-\theta_{35}\theta_3+\theta_{35}\theta_2-\theta_{35}\theta_4+3\theta_2\theta_1)$

$g_{00\bar{1}11}(\underline{\theta}) = -(2\pi + i(\theta_4-\theta_5))(2\ \pi^5 + 2i\pi^4(3\ \theta_1+\theta_2-2(\theta_4+\theta_5)) - i(\theta_1-\theta_2)(\theta_1^2-2\theta_1\theta_3-\theta_4\theta_5+\theta_3(\theta_4+\theta_5))(\theta_2^2-2\theta_2\theta_3-\theta_4\theta_5+\theta_3(\theta_4+\theta_5)) - 2\pi^3(4\theta_1^2-\theta_3^2+\theta_4^2+3\theta_4\theta_5+\theta_5^2-\theta_2(\theta_4+\theta_5)+\theta_3(\theta_4+\theta_5)+\theta_1(2\ \theta_2-5(\theta_4+\theta_5))) - i\pi^2((-\theta_2^2)\theta_3+\theta_1^2(5\theta_2+3\theta_3-4(\theta_4+\theta_5))+\theta_1(\theta_2^2-4\theta_3^2+\theta_4(\theta_3+4\ \theta_4)+(\theta_3+5\theta_4)\theta_5+4\theta_5^2-6\theta_2(\theta_4+\theta_5))+2(\theta_3-\theta_4-\theta_5)(\theta_4\theta_5+\theta_3(\theta_4+\theta_5))+\theta_2(5\ \theta_4\theta_5+\theta_3(\theta_4+\theta_5))) - \pi(\theta_1^3(\theta_2-\theta_3)+\theta_2^3\theta_3+2\theta_3\theta_4(2\ \theta_3-\theta_4-\theta_5)\theta_5-\theta_2^2(2\ \theta_3^2-\theta_4\theta_5+\theta_3(\theta_4+\theta_5))+\theta_2(\theta_3^2(\theta_4+\theta_5)+\theta_4\theta_5(\theta_4+\theta_5)+\theta_3(\theta_4^2-4\theta_4\theta_5+\theta_5^2))+\theta_1^2(4\theta_3^2-3\theta_4\theta_5+\theta_3(\theta_4+\theta_5)+\theta_2((-7)\theta_3+2(\theta_4+\theta_5)))+\theta_1(-\theta_2^3)+3\theta_2^2\theta_3-5\theta_3^2(\theta_4+\theta_5)+\theta_4\theta_5(\theta_4+\theta_5)+\theta_3(\theta_4^2+4\theta_4\theta_5+\theta_5^2)+2\theta_2(\theta_3^2-\theta_4^2-\theta_4\theta_5-\theta_5^2+\theta_3(\theta_4+\theta_5))))$

$g_{00001}(\underline{\theta}) = i(i(\theta_1-\theta_2)(\theta_1-\theta_3)(-\theta_2+\theta_3)(\theta_1-\theta_4)(\theta_2-\theta_4)(\theta_3-\theta_4)+2\pi^5(3\theta_1+\theta_2-5\theta_3+\theta_4)+8i\pi^4(\theta_1+\theta_2-2\theta_3)(\theta_1-\theta_5)+2\pi^3(-{\theta_1}^3+{\theta_2}^3-{\theta_3}^3+{\theta_3}^2\theta_4-3\theta_3{\theta_4}^2+{\theta_4}^3-{\theta_2}^2(3\theta_3+\theta_4)+2\theta_4(\theta_3+\theta_4)\theta_5+(3\theta_3+\theta_4){\theta_5}^2+{\theta_1}^2(-7\theta_2+\theta_3+3\theta_4+6\theta_5)+\theta_2({\theta_3}^2+\theta_3\theta_4-{\theta_4}^2-3{\theta_5}^2)+\theta_1({\theta_2}^2+{\theta_3}^2+\theta_3\theta_4-{\theta_4}^2-8(\theta_3+\theta_4)\theta_5-{\theta_5}^2+3\theta_2(\theta_3+\theta_4+2\theta_5)))-2i\pi^2(-\theta_2\theta_3(\theta_2-\theta_4)\theta_4+2{\theta_1}^3(\theta_2-\theta_5)+(2{\theta_2}^3-5{\theta_2}^2\theta_3+3\theta_2{\theta_3}^2-2{\theta_3}^3+\theta_3(4\theta_2+3\theta_3)\theta_4-(2\theta_2+7\theta_3){\theta_4}^2+2{\theta_4}^3)\theta_5+2\theta_4(-\theta_2+\theta_3+\theta_4){\theta_5}^2-{\theta_1}^2(\theta_3\theta_4-2\theta_3\theta_5-3\theta_4\theta_5-2{\theta_5}^2+\theta_2(\theta_3+2\theta_4+3\theta_5))+\theta_1(-2{\theta_2}^3+2{\theta_3}^3-3{\theta_3}^2\theta_4+6\theta_3{\theta_4}^2-2{\theta_4}^3+{\theta_2}^2(6\theta_3+\theta_4-\theta_5)-{\theta_4}^2\theta_5-2(\theta_3+2\theta_4){\theta_5}^2+\theta_2(-3{\theta_3}^2-4\theta_3\theta_4+{\theta_4}^2+4\theta_4\theta_5+2{\theta_5}^2)))+2\pi(\theta_1^3(\theta_2-\theta_5)(\theta_4-\theta_5)+\theta_2^3(\theta_4-\theta_5)\theta_5-\theta_2^2\theta_3(\theta_4-\theta_5)(\theta_4+2\theta_5)-(\theta_3-\theta_4)\theta_5(\theta_3\theta_4(\theta_3+\theta_4)-(\theta_3^2+\theta_4^2)\theta_5)-\theta_2\theta_5(-\theta_4^2\theta_5+\theta_3\theta_4(\theta_4+\theta_5)+\theta_3^2(-3\theta_4+2\theta_5))+\theta_1^2(-\theta_3(\theta_3-2\theta_4)\theta_4+\theta_3\theta_4\theta_5-(\theta_3+\theta_4)\theta_5^2+\theta_2(\theta_3^2-2\theta_4^2+4\theta_4\theta_5-\theta_5^2+\theta_3(-3\theta_4+\theta_5)))+\theta_1(\theta_3^3(\theta_4-\theta_5)+\theta_2^2(\theta_4-\theta_5)(3\theta_3+\theta_4-\theta_5)+2\theta_3^2\theta_4\theta_5+\theta_2^3(-\theta_4+\theta_5)+\theta_4^2\theta_5(\theta_4+\theta_5)-\theta_3\theta_4(\theta_4^2+4\theta_4\theta_5-\theta_5^2)+\theta_2(-\theta_3+\theta_4)(\theta_3(3\theta_4-\theta_5)-\theta_5(\theta_4+\theta_5))))))\,.$

n = 2, m = 2. For the 2-particle form factor we obtain

$$F^{T}_{\underline{\alpha}}(\underline{\theta}) = -\mathbf{C}_{\alpha_1\alpha_2}\frac{1}{2}\pi^2\frac{1}{\theta_{12}-i\pi}G(\theta_{12})\,. \tag{E.9}$$

n = 4, m = 4. For the 4-particle form factor for example a component is

$$F^{T}_{\bar{1}\bar{1}11}(\underline{\theta}) = \frac{1}{2}\pi^5\,(\theta_{12}-2\pi i)\,(\theta_{34}-2\pi i)\prod_{i<j}G(\theta_{ij})\,. \tag{E.10}$$

The other components are obtained by the form factor equations (i) and (ii). These results agree again with those of [11] which were obtained by different methods.

Proofs. We calculate (E.6) for $n = m = 2$

$$K_{\bar{1}1}(\underline{\theta}) = \prod_{k=1}^{2}\left(\frac{1}{2i\pi}\int_{\mathcal{C}^k_{\underline{\theta}}}dz_k\right)\sum_{\pi\in S_2}k(\underline{\theta},\pi\underline{z})$$

$$k(\underline{\theta},\underline{z}) = l(z_1-z_2)\prod_{i=1}^{2}\prod_{j=1}^{2}\tilde{\phi}_j\,(\theta_i-z_j)\,\tilde{c}(\theta_1-z_1)\tilde{b}(\theta_1-z_2)f(z_1-z_2)\,.$$

[6] These results have been obtained by Mathematica.

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For convenience we write $\theta = i\pi x$ and $z = i\pi y$

$$I = \int_{\mathcal{C}^o_{\underline{x}}} dy_1 \int_{\mathcal{C}^e_{\underline{x}}} dy_2 \phi(\underline{x},\underline{y},\nu)\tau_{12}(y_1-y_2)q(y_1-y_2)\varsigma(\underline{x},\underline{y},\nu)$$

with

$$\begin{aligned} q(y) &= \frac{L(y)f(y)}{\tilde{b}(y)} = \frac{\nu}{y}\frac{\Gamma\left(-\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}y\right)\Gamma\left(-\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}y\right)}{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}y\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}y\right)} = -q(-y) \\ \phi(\underline{x},\underline{y},\nu) &= \tilde{\psi}(x_1-y_1)\tilde{\psi}(x_2-y_1)\tilde{\chi}(x_1-y_2)\tilde{\chi}(x_2-y_2) \\ \varsigma(\underline{x},\underline{y},\nu) &= \tilde{c}(x_1-y_1)\tilde{b}(x_1-y_2) - \tilde{c}(x_1-y_2)\tilde{b}(x_1-y_1)\,. \end{aligned}$$

It turns out that the leading term for $\nu \to 2$ in I vanishes and we have to calculate all terms in order $O(\nu-2)$. We use

$$\begin{aligned} q(y)\tau(y) &= -\frac{1}{2}\left(\sin\frac{1}{2}\pi\,(y+\nu)\right)\frac{\nu}{\pi}\Gamma\left(-\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}y\right)\Gamma\left(-\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}y\right) \\ &= \left(\tan\frac{1}{2}\pi y\right)(1+P(y)) + O\left((\nu-2)^2\right) \\ P(y) &= O\,(\nu-2) \end{aligned}$$

and

$$\int_{\mathcal{C}^o_{\underline{x}}} dy_1 \int_{\mathcal{C}^e_{\underline{x}}} dy_2 \phi(\underline{x},\underline{y},\nu)\tan\frac{1}{2}\pi(y_1-y_2)\varsigma(\underline{x},\underline{y},\nu) = O\left((\nu-2)^2\right).$$

Therefore we obtain after some calculations up to terms $O\left((\nu-2)^2\right)$

$$\begin{aligned} I &= \int_{\mathcal{C}^o_{\underline{x}}} dy_1 \int_{\mathcal{C}^e_{\underline{x}}} dy_2 \phi(\underline{x},\underline{y},\nu)\tan\frac{1}{2}\pi\,(y_1-y_2)\,P(y_1-y_2)\varsigma(\underline{x},\underline{y},\nu) \\ &= I_{11}+I_{12}+I_{21}+I_{22} \end{aligned}$$

with

$$\begin{aligned} I_{12} &= \left(\oint_{x_1} + \oint_{x_1-2}\right) dy_1 \left(\oint_{x_2} + \oint_{x_2-2}\right) dy_2 \phi(\underline{x},\underline{y},2)\tan\frac{1}{2}\pi\,(y_1-y_2)\,P(y_1-y_2)\varsigma(\underline{x},\underline{y},\nu) \\ &= 2\,(\nu-2)\,\frac{\tan\frac{1}{2}\pi x_{12}}{x_{12}-1}\oint_{x_1} dy_1 \oint_{x_2} dy_2 \phi(\underline{x},\underline{y},2)\varsigma(\underline{x},\underline{y},\nu) \\ &= -32\,(\nu-2)\,\frac{1}{x_{12}-1}\frac{1}{x_{12}\,(x_{12}-2)}\tan\frac{1}{2}\pi x_{12} = I_{21} \end{aligned} \tag{E.11}$$

which proves (E.9) because $I_{11} = I_{22} = 0$. Similarly, one derives (E.10). ■

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