Chiral Bosonization

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and

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We show that the Dirac-bracket quantization of a single self-dual field leads naturally to the bosonization formulas. We find a numerical parameter describing the soliton charge and unveiling hidden soliton sectors.

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Quantization of chiral bosons is an essential ingredient in the heterotic string.[1] More generally, one is faced with the problem of quantizing self-dual fields,[2] $\dot{\psi}=\psi'$ (overdot and prime mean differentiation with respect to time t and space x, respectively; we employ the notation, terminology, and conventions of Ref. 2). Floreanini and Jackiw have offered some solutions to the problem of a single self-dual field, namely, they have constructed (i) a nonlocal Lagrangean in terms of a local field; (ii) a local Lagrangean in terms of a nonlocal field, and (iii) a local Lagrangean in terms of a local field (using the input of bosonization[3]). It was remarked in Ref. 2 that the equal-time canonical commutation relations employed are unusual, whereas in Costa and Girotti[4] it was pointed out that they are just the ones that follow from the Dirac formalism for constrained systems.[5]

The purpose of this Letter is to improve further the understanding of item iii above. We introduce the soliton field as a charge-creating field (as it should be), through its Dirac bracket with the charge-density soliton. We show then that Poincaré invariance requires the soliton field to obey one additional equation of motion. Our integrating of this equation leads to a formula which becomes identical to the "bosonization rule" after quantization, explaining the origin of this rule for self-dual fields. In our treatment there occurs a numerical parameter unveiling a hidden soliton sector.

In terms of the local charge-density soliton field, χ, the Lagrangean is

$$\mathcal{L}=\tfrac{1}{4}\int dx\,dy\,\chi(x)\epsilon(x-y)\dot{\chi}(y)-\tfrac{1}{2}\int dx\,\chi^2(x), \tag{1}$$

leading to the canonically conjugate momentum

$$\pi_\chi(x)=\tfrac{1}{4}\int dy\,\chi(y)\epsilon(y-x). \tag{2}$$

Actually this expression is a constraint since it does not depend on the velocities. It has been shown[4] that there are no further (secondary) constraints, so that the Dirac formalism will be set with the only second-class constraint

$$T(x)=\pi_\chi(x)-\tfrac{1}{4}\int dy\,\chi(y)\epsilon(y-x)\approx 0, \tag{3}$$

$$\{T(x),T(y)\}=\tfrac{1}{2}\epsilon(x-y). \tag{4}$$

As usual, Dirac brackets are defined by[4,5]

$$\{f,g\}_D=\{f,g\}-\int dz\,dz'\{f,T(z)\}Q^{-1}(z,z')\times\{T(z'),g\}, \tag{5}$$

where $Q(z,z')=\{T(z),T(z')\}$. In particular, one obtains

$$\{\chi(x),\chi(y)\}_D=\delta'(x-y). \tag{6}$$

All the symmetries (internal as well as external) are defined with Dirac brackets. Thus, the Hamiltonian which follows from (1),

$$H=\tfrac{1}{2}\int dx\,\chi^2(x), \tag{7}$$

generates the self-dual equation of motion for χ:

$$\dot{\chi}=\{\chi,H\}_D=\chi'. \tag{8}$$

The Lorentz transformations are generated by

$$M=\tfrac{1}{2}\int dx\,(t+x)\chi^2(x), \tag{9}$$

and can be written in a familiar form[2] by use of the equation of motion (8):

$$\delta_L\chi(x)=\{\chi,M\}_D=\chi(x)+t\chi'(x)+x\dot{\chi}(x). \tag{10}$$

Let us now introduce a charge-creating field u by

$$\{\chi(y),u(x)\}_D=i\gamma\delta(x-y)u(x), \tag{11}$$

where γ is the numerical parameter to which we have referred before. From (11) it follows that the Poincaré generators act on $u(x)$ as

$$\{H,u(x)\}_D = i\gamma\chi(x)u(x), \tag{12}$$

$$\{M,u(x)\}_D = i\gamma\chi(x)u(x)t + i\gamma\chi(x)u(x)x. \tag{13}$$

So besides self-duality of the soliton field, Poincaré invariance requires the following equation of motion:

$$u'(x) = -i\gamma\chi(x)u(x). \tag{14}$$

Observe also that u has zero dimension. In the following, we will show that its quantum dimension comes entirely from the normal order of (14) with respect to the quanta of the charge-density soliton.

The integration of (14) is given in terms of the momentum (2) by

$$u(x) = e^{2i\gamma\pi_\chi(x)}. \tag{15}$$

It is clear that this formula is the source of the bosonization rule in the quantum regime to which we now turn.

One approach to the quantum regime that exhibits the full symmetry of the above constrained system is the path integral.[6] One may integrate over the χ and π_χ fields subject to the δ-functional constraint $\delta(\pi_\chi(x) + \frac{1}{4}\int dy\, \epsilon(y-x)\chi(y))$. If first one integrates over the π_χ field one gets the Lagrangean (1). Alternatively, we may integrate over χ using the functional constraint $\delta(\chi(x) + 2\pi_\chi'(x))$ arriving then at the Lagrangean

$$\mathcal{L} = 2\int dx[\pi_\chi'(x)\dot{\pi}_\chi(x) - (\pi_\chi')^2]. \tag{16}$$

The second alternative is more suitable to perform the computation of the u-field correlation functions with Eq. (15). Since the Lagrangean is quadratic, these correlation functions can be easily computed, analogously to the Coulomb-gas approach to the Thirring model.[7] This calculation leads to $\gamma^2/4\pi$ for the dimension of u.

The operator quantization is obtained by the replacement of the above Dirac brackets by commutators. Then

$$[\chi(x),\chi(y)] = i\delta'(x-y), \tag{17}$$

which can be realized with the decomposition[2] (at $t=0$)

$$\chi(x) = \chi_{\rm in}(x) + \chi_{\rm in}^\dagger(x), \tag{18}$$

where

$$\chi_{\rm in}(x) = -i\int_0^\infty dk \left(\frac{k}{2\pi}\right)^{1/2} a(k)e^{-ikx} \tag{19}$$

with

$$[a(k),a^\dagger(k')] = \delta(k-k'). \tag{20}$$

In order to define the full quantum regime one has to specify the product between quantum fields. We will choose the normal order with respect to the χ field quanta:

$$:\chi(x)A(x): = \chi_{\rm in}^\dagger(x)A(x) + A(x)\chi_{\rm in}(x), \tag{21}$$

where A stands for χ itself or other fields. So, to compute the Poincaré transformation of the u field one has to normal order expressions like

$$[u(x),\chi^2(y)] = \gamma\delta(x-y)u(x)\chi(y) + \gamma\delta(x-y)\chi(y)u(x). \tag{22}$$

Using the commutator that corresponds to the bracket (11) one obtains

$$[u(x),:\chi^2(y):] = 2\gamma\delta(x-y):\chi(y)u(x): - i(\gamma^2/2\pi)\partial_y\delta(x-y)u(x), \tag{23}$$

and so

$$i[M,u(x)] \equiv (\gamma^2/4\pi)u(x) + [-i\gamma:\chi(x)u(x):](t+x). \tag{24}$$

Thus the self-dual field u has dimension $\gamma^2/4\pi$, coming entirely from quantum effects, and must fulfill the equation of motion

$$u'(x) = -i\gamma:\chi(x)u(x):. \tag{25}$$

The integral of (25) is what has been called the bosonization rule[3] or, for $\gamma^2 = 2\pi$, the fermion-boson equivalence:

$$u(x) = \exp[2i\gamma\pi_{\rm in}^\dagger(x)]\exp[2i\gamma\pi_{\rm in}(x)], \tag{26}$$

where $\pi_{\rm in}' = -\frac{1}{2}\chi_{\rm in}(x)$. We have shown that this formula follows from our demands of u being a charge-creating field and of Poincaré invariance. For $\gamma^2 = 2\pi$ a local canonical anticommuting field, $u_{1/2}$, exists and a local Lagrangean can be written in terms of it.[2] For general values of γ the situation is more complicated. Local fields with dimension $q^2/2$, where q is an integer, can be constructed as composites[8] of the $u_{1/2}$ field [they are[8] $(\partial_t+\partial_x)^q u_{1/2}(\partial_t+\partial_x)^{q-1}u_{1/2}\cdots(\partial_t+\partial_x)u_{1/2}u_{1/2}$, normal ordered with respect to the $u_{1/2}$ quanta]. So, for these particular values of the dimension, the canonical theory may be described by the $u_{1/2}$ Lagrangean. Observe that in the left-right counterpart of this theory (the Thirring model[9]) a Lagrangean can be written for all values of γ. In this case γ plays the role of coupling constant between the left- and right-handed currents. In the self-dual situation it is not known to us if it is possible to describe the theory, for general values of γ, with the $u_{1/2}$. In any case, this point raises the question of what is the elementary excitation in this model. In Ref. 2 the excitation corresponding to the field $u_{1/2}$ was considered to be the elementary one. On the other hand, for general values of γ, we have learned how to compute the correlation functions of u's from the χ (or π_χ) Lagrangean (paying attention to the charge selection rules[10]). At any rate, it will be desirable to have a better understanding of these soliton sectors.

We would like to make one further observation also concerning item ii. As we have seen, in the functional-integral approach there is a complete symmetry between the π_χ and χ field formulations. Nevertheless, the field π_χ is nonlocal which implies a non-positive-definite two-point function. At the level of the u-field correlation functions the negative-norm states are ruled out by the charge selection rule.[10]

We also add a last remark regarding the origin of the equation of motion (14). Forgetting for a moment the reasoning that led to (14), let us compute from (11) the following brackets:

$$\{\chi(y),u'(x)\}_D = i\gamma\delta'(x-y)u(x) + i\gamma\delta(x-y)u'(x), \quad (27)$$

$$\{\chi(y),i\gamma\chi(x)u(x)\}_D = -i\gamma\delta'(x-y)u(x) - \gamma^2\delta(x-y)\chi(x)u(x), \quad (28)$$

and so

$$\{\chi(y),u'(x)+i\gamma\chi(x)u(x)\}_D = i\gamma\delta(x-y)[u'(x)+i\gamma\chi(x)u(x)]. \quad (29)$$

Since we do not want to have another field creating the same charge as u creates, we have to impose $u'+i\gamma\chi u$ to be zero for all times, namely, Eq. (14). This alternative derivation of Eq. (14) goes through to the quantum case, where it means electing u to create an irreducible representation of the current algebra (17). Actually, this approach was considered by Kurak[11] in the non-Abelian case where it was shown that the irreducibility of the representation of Kac-Moody algebras[12] leads to equations analogous to Eq. (25). The non-Abelian case is relevant regarding the chiral-boson version[1] of the heterotic string and we plan to come back to this point in a future report.

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[1]J. M. F. Labastida and M. Pernici, Phys. Rev. Lett. **59**, 2511 (1987), and references therein.

[2]R. Floreanini and R. Jackiw, Phys. Rev. Lett. **59**, 1873 (1987), and references therein.

[3]S. Coleman, Phys. Rev. D **11**, 2088 (1975); S. Mandelstam, Phys. Rev. D **11**, 3026 (1975); J. A. Swieca, Fortschr. Phys. **25**, 303 (1977).

[4]M. E. V. Costa and H. O. Girotti, Phys. Rev. Lett. **60**, 1771 (1988).

[5]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva Univ. Press, New York, 1964).

[6]R. Marnelius, Acta Phys. Pol. B **13**, 669 (1982), and references therein.

[7]E. C. Marino and J. A. Swieca, Nucl. Phys. **B170**, 175 (1980).

[8]V. Kurak, Rev. Bras. Fis. **7**, 451 (1977).

[9]G. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D **6**, 988 (1972).

[10]A. S. Wightman, in *Cargese Lectures in Theoretical Physics*, edited by M. Levy (Gordon and Breach, New York, 1966).

[11]V. Kurak, "Quantum Field Theories as Representations of Infinite Dimensional Algebras" (to be published).

[12]P. Goddard and D. Olive, Int. J. Mod. Phys. A **1**, 303 (1986).